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An interesting method for the exponentials for some special matrices

Jing Chen[a*] and Hongfen Zou[b]

[a]*School of IoT Engineering, Jiangnan University, Wuxi 214122, People's Republic of China;* [b]*Wuxi Professional College of Science and Technology, Wuxi 214028, People's Republic of China*

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The matrix exponential e^{At} plays a central role in linear system and control theory. This paper develops a method to compute the accurate solution for the matrix exponential e^{At} with the assumption that the matrix A has an eigenvalue $s_1 = 0$. The examples show the effectiveness of the proposed method.

Keywords: matrix exponential; the eigenvalue; matrix theory; matrix equation

1. Introduction

It is well known that matrix is widely used in many areas (Dehghan & Hajarian, 2010, 2012; Hagiwara, 2011). For example, Al Zhour and Kilicman discussed some different matrix products for partitioned and non-partitioned matrices and some useful connections of the matrix products (Zhour & Kilicman, 2007). Ding and Chen defined a new operation – the block-matrix inner product – and presented a least square-based and a gradient-based iterative solutions of coupled matrix equations (Ding & Chen, 2005, 2006). Ding studied the transformations and relationships between some special matrices (Ding, 2010).

The solution $e^{At}x(0)$ of the differential equation $\dot{x}(t) = Ax(t)$ plays an important role in linear system and control theory. It is well known that e^{At} can be defined by a convergent power series $e^{At} = \sum_{i=0}^{\infty}((At)^i/i!)$. The infinite series $\sum_{i=0}^{\infty}((At)^i/i!)$ makes researchers design accurate controllers difficultly in theory and application, so it is important to develop a frame work to get the accurate solution of e^{At}.

In recent years, there exist many methods for computing e^{At} (Ben Taher & Rachidi, 2002; Bernstein & So, 1993; Cheng & Yau, 1997; Moler & Loan, 2003; Skaflestad & Wright, 2009; Wu, 2011; Zafer, 2008). Among these methods, the explicit formulas can overcome the truncation errors which are widely used (Ben Taher & Rachidi, 2002; Bernstein & So, 1993; Cheng & Yau, 1997; Wu, 2011). Based on the work in Bernstein and So (1993) and Wu (2011), the objective of this paper is to propose a method to compute the accurate solution of e^{At}, where the matrix A satisfies $A^n = \rho_1 A^{n-1} + \rho_2 A^{n-2}$. If the parameter $\rho_1 = 0$ or $\rho_2 = 0$, the matrix is the same as the matrix in Wu (2011), so our work is more widely used.

Briefly, this paper is organised as follows. Section 2 describes the main results. Section 3 provides two illustrative examples. Finally, concluding remarks are given in Section 4.

2. The main results

Let us introduce some notations first. The symbol I stands for an identity matrix of appropriate sizes, $\mathbb{C}$ denotes the set of complex number and $\mathbb{C}^{n\times n}$ denotes the set of $n \times n$ complex matrix.

As is well known, $e^{At}, A \in \mathbb{C}^{n\times n}$, can be written as the following convergent power series

$$e^{At} = I + \frac{t}{1!}A + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots + \frac{t^{n-2}}{(n-2)!}A^{n-2} + \frac{t^{n-1}}{(n-1)!}A^{n-1} + \cdots. \quad (1)$$

Bernstein gave explicit formulas for $A^2 = A$, $A^2 = \rho I_n$ and $A^3 = \rho A$ in Bernstein and So (1993). Wu gave explicit formulas for $A^{k+1} = \rho \mathbf{A}^k$, $A^{k+2} = \rho^2 A^k$ and $A^{k+3} = \rho^3 \mathbf{A}^k$ in Wu (2011). In this paper, we will propose a method for $A^n = \rho_1 A^{n-1} + \rho_2 A^{n-2} + \cdots + \rho_k A^{n-k}$, $k < n$.

First, let $A \in \mathbb{C}^{n\times n}$ and $A^n = A^{n-1} + A^{n-2}$, then $A^{n+1} = 2A^{n-1} + A^{n-2}$, $A^{n+2} = 3A^{n-1} + 2A^{n-2}, \ldots, A^{n+k} = \beta_1 A^{n-1} + \beta_2 A^{n-2}$, and we conclude

$$e^{At} = I + \frac{t}{1!}A + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots + a_1(t)A^{n-2} + a_2(t)A^{n-1}, \quad (2)$$

*Corresponding author. Email: chenjing1981929@126.com where the parameters $a_1(t)$ and $a_2(t)$ be computed as

$$a_1(t) = \frac{t^{n-2}}{(n-2)!} + \frac{t^n}{n!} + \frac{t^{n+1}}{(n+1)!} + \frac{2t^{n+2}}{(n+2)!} + \frac{3t^{n+3}}{(n+3)!} + \frac{5t^{n+4}}{(n+4)!} + \cdots, \quad (3)$$

$$a_2(t) = \frac{t^{n-1}}{(n-1)!} + \frac{t^n}{n!} + \frac{2t^{n+1}}{(n+1)!} + \frac{3t^{n+2}}{(n+2)!} + \frac{5t^{n+3}}{(n+3)!} + \frac{8t^{n+4}}{(n+4)!} + \cdots. \quad (4)$$

Equations (3) and (4) are infinite series, so it is difficult to obtain the exact figures of $a_1(t)$ and $a_2(t)$. In this paper, the solution is using the matrix theory to overcome the difficulty.

In order to compute the parameters, some mathematical preliminaries are required.

LEMMA 1 *A matrix*

$$\begin{bmatrix} 1 & s_1 & {s_1}^2 & \cdots & {s_1}^{n-1} \\ 1 & s_2 & {s_2}^2 & \cdots & {s_2}^{n-1} \\ 1 & s_3 & {s_3}^2 & \cdots & {s_3}^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & s_n & {s_n}^2 & \cdots & {s_n}^{n-1} \end{bmatrix} \quad (5)$$

is called Vandermonde matrix, and

$$\begin{vmatrix} 1 & s_1 & {s_1}^2 & \cdots & {s_1}^{n-1} \\ 1 & s_2 & {s_2}^2 & \cdots & {s_2}^{n-1} \\ 1 & s_3 & {s_3}^2 & \cdots & {s_3}^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & s_n & {s_n}^2 & \cdots & {s_n}^{n-1} \end{vmatrix} = \prod_{1\leqslant j<i\leqslant n} (s_i - s_j). \quad (6)$$

LEMMA 2 *The matrix equation $\boldsymbol{A}X = 0, X \in \mathbb{R}^{n\times 1}$, has only one solution $X = \mathbf{0}$, where $\mathbf{0}$ being a column vector whose entries are all 0 and the matrix $\boldsymbol{A}$ satisfies*

$$|\boldsymbol{A}| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0. \quad (7)$$

LEMMA 3 *The matrix $\boldsymbol{A} \in \mathbb{C}^{n\times n}$, the characteristic polynomial of $\boldsymbol{A}$ is $f(\lambda)$, then*

$$f(\boldsymbol{A}) = \boldsymbol{A}^n - \alpha_{n-1}\boldsymbol{A}^{n-1} - \alpha_{n-2}\boldsymbol{A}^{n-2} - \cdots - \alpha_1\boldsymbol{I} = 0.$$

Using Lemma 3, Equation (1) can also be simplified as

$$e^{\boldsymbol{A}t} = b_0(t)\boldsymbol{I} + b_1(t)\boldsymbol{A} + b_2(t)\boldsymbol{A}^2 + \cdots + b_{n-2}(t)\boldsymbol{A}^{n-2} + b_{n-1}(t)\boldsymbol{A}^{n-1}. \quad (8)$$

Because the matrix $\boldsymbol{A}$ satisfies $\boldsymbol{A}^n = \boldsymbol{A}^{n-1} + \boldsymbol{A}^{n-2}$, we conclude that the matrix $\boldsymbol{A}$ has three different eigenvalues: $s_1 = 0, s_2$ and s_3, and the eigenvalue $s_1 = 0$ of the matrix $\boldsymbol{A}$ is an $n-2$ eigenvalue. Comparing Equations (2) and (8), we get

$$(b_0(t) - 1)1 + \left(b_1(t) - \frac{t}{1!}\right)s_i + \left(b_2(t) - \frac{t^2}{2!}\right)s_i^2 + \cdots + (b_{n-2}(t) - a_1(t))s_i^{n-2} + (b_{n-1}(t) - a_2(t))s_i^{n-1} = 0. \quad (9)$$

The matrix $\boldsymbol{A}$ has three eigenvalues, we built the matrix equation as

$$\begin{bmatrix} 1 & s_1 & {s_1}^2 & \cdots & {s_1}^{n-1} \\ 1 & s_2 & {s_2}^2 & \cdots & {s_2}^{n-1} \\ 1 & s_3 & {s_3}^2 & \cdots & {s_3}^{n-1} \end{bmatrix} \begin{bmatrix} b_0(t) - 1 \\ b_1(t) - \dfrac{t}{1!} \\ \vdots \\ b_{n-2}(t) - a_1(t) \\ b_{n-1}(t) - a_2(t) \end{bmatrix} = 0, \quad (10)$$

where $3 < n$, the above matrix equation has two or more solutions, so we cannot obtain the exact figures of $a_1(t)$ and $a_2(t)$ by $b_i(t), i = 0, 1, \ldots, n-1$.

Substituting $s_1 = 0$ into Equation (9) gets $b_0(t) = 1$, and after the derivation of s_1 to Equation (9), we get

$$\left(b_1(t) - \frac{t}{1!}\right) + 2\left(b_2(t) - \frac{t^2}{2!}\right)s_i + \cdots + (n-2)(b_{n-2}(t) - a_1(t))s_i^{n-3} + (n-1)(b_{n-1}(t) - a_2(t))s_i^{n-2} = 0 \quad (11)$$

taking $s_1 = 0$ into Equation (11) gets $b_1(t) = t/1!$ After the $2, 3, \ldots, n-3$ derivation of s_i to Equation (9), we can get $n-4$ equations. Taking $s_1 = 0$ into these equations gets $b_2(t) = t^2/2!, b_3(t) = t^3/3!, \ldots, b_{n-3}(t) = (t^{n-3}/(n-3)!)$. Finally, we get

$$(b_{n-2}(t) - a_1(t))s_i^{n-2} + (b_{n-1}(t) - a_2(t))s_i^{n-1} = 0. \quad (12)$$

Taking s_2 and s_3 into Equation (12) gets

$$\begin{bmatrix} s_2^{n-2} & {s_2}^{n-1} \\ s_3^{n-2} & {s_3}^{n-1} \end{bmatrix} \begin{bmatrix} cb_{n-2}(t) - a_1(t) \\ b_{n-1}(t) - a_2(t) \end{bmatrix} = 0. \quad (13)$$

Because

$$\begin{vmatrix} s_2^{n-2} & {s_2}^{n-1} \\ s_3^{n-2} & {s_3}^{n-1} \end{vmatrix} \neq 0, \quad (14)$$

and based on Lemma 2, we can conclude that $b_{n-2}(t) = a_1(t)$ and $b_{n-1}(t) = a_2(t)$.

THEOREM 1 *Let* $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ *and* $\boldsymbol{A}^n = \boldsymbol{A}^{n-1} + \boldsymbol{A}^{n-2}$, *then we have*

$$\begin{aligned} e^{\boldsymbol{A}t} &= \boldsymbol{I} + \frac{t}{1!}\boldsymbol{A} + \frac{t^2}{2!}\boldsymbol{A}^2 + \cdots + \frac{t^{n-3}}{(n-3)!}\boldsymbol{A}^{n-3} \\ &\quad + b_{n-2}(t)\boldsymbol{A}^{n-2} + b_{n-1}(t)\boldsymbol{A}^{n-1}, \end{aligned} \tag{15}$$

where $b_{n-2}(t)$ *and* $b_{n-1}(t)$ *are computed by*

$$\begin{bmatrix} b_{n-2}(t) \\ b_{n-1}(t) \end{bmatrix} = \begin{bmatrix} s_2^{n-2} & {s_2}^{n-1} \\ s_3^{n-2} & {s_3}^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} e^{s_2 t} - \sum_{i=0}^{n-3} \frac{t^i s_2^i}{i!} \\ e^{s_3 t} - \sum_{i=0}^{n-3} \frac{t^i s_3^i}{i!} \end{bmatrix}, \tag{16}$$

and $s_2 = (1+\sqrt{5})/2$, $s_3 = (1-\sqrt{5})/2$.

Proof Rewritten Equations (2) and (8)

$$\begin{aligned} e^{\boldsymbol{A}t} &= \boldsymbol{I} + \frac{t}{1!}\boldsymbol{A} + \frac{t^2}{2!}\boldsymbol{A}^2 + \frac{t^3}{3!}\boldsymbol{A}^3 + \cdots + a_1(t)\boldsymbol{A}^{n-2} \\ &\quad + a_2(t)\boldsymbol{A}^{n-1}, \\ e^{\boldsymbol{A}t} &= b_0(t)\boldsymbol{I} + b_1(t)\boldsymbol{A} + b_2(t)\boldsymbol{A}^2 + \cdots + b_{n-2}(t)\boldsymbol{A}^{n-2} \\ &\quad + b_{n-1}(t)\boldsymbol{A}^{n-1}. \end{aligned}$$

Because the parameters $b_0(t) = 1, b_1(t) = t/1!, \ldots, b_{n-3}(t) = (t^{n-3}/(n-3)!)$, $a_1(t) = b_{n-2}(t)$ and $a_2(t) = b_{n-1}(t)$, we conclude

$$\begin{aligned} e^{\boldsymbol{A}t} &= \boldsymbol{I} + \frac{t}{1!}\boldsymbol{A} + \frac{t^2}{2!}\boldsymbol{A}^2 + \frac{t^3}{3!}\boldsymbol{A}^3 + \cdots + b_{n-2}(t)\boldsymbol{A}^{n-2} \\ &\quad + b_{n-1}(t)\boldsymbol{A}^{n-1}. \end{aligned}$$

The matrix $\boldsymbol{A}$ has three eigenvalues $s_1 = 0$, $s_2 = (1+\sqrt{5})/2$ and $s_3 = (1-\sqrt{5})/2$. Replacing $\boldsymbol{A}$ by $s_i, i = 2, 3$, we get

$$\begin{aligned} e^{s_i t} &= 1 + \frac{t}{1!}s_i + \frac{t^2}{2!}s_i^2 + \frac{t^3}{3!}s_i^3 + \cdots + b_{n-2}(t)s_i^{n-2} \\ &\quad + b_{n-1}(t)s_i^{n-1}. \end{aligned} \tag{17}$$

In order to compute the parameters $b_{n-2}(t)$ and $b_{n-1}(t)$, we simplify Equation (17) as

$$e^{s_i t} - \sum_{i=0}^{n-3} \frac{t^i s_i^i}{i!} = b_{n-2}(t)s_i^{n-2} + b_{n-1}(t)s_i^{n-1}. \tag{18}$$

Taking s_2 and s_3 in Equation (18) gets

$$\begin{bmatrix} b_{n-2}(t) \\ b_{n-1}(t) \end{bmatrix} = \begin{bmatrix} s_2^{n-2} & {s_2}^{n-1} \\ s_3^{n-2} & {s_3}^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} e^{s_2 t} - \sum_{i=0}^{n-3} \frac{t^i s_2^i}{i!} \\ e^{s_3 t} - \sum_{i=0}^{n-3} \frac{t^i s_3^i}{i!} \end{bmatrix}. \tag{19}$$

∎

THEOREM 2 *Let* $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ *and* $\boldsymbol{A}^n = \rho_1\boldsymbol{A}^{n-1} + \rho_2\boldsymbol{A}^{n-2}$, *then we have*

$$\begin{aligned} e^{\boldsymbol{A}t} &= \boldsymbol{I} + \frac{t}{1!}\boldsymbol{A} + \frac{t^2}{2!}\boldsymbol{A}^2 + \cdots + \frac{t^{n-3}}{(n-3)!}\boldsymbol{A}^{n-3} \\ &\quad + b_{n-2}(t)\boldsymbol{A}^{n-2} + b_{n-1}(t)\boldsymbol{A}^{n-1}, \end{aligned} \tag{20}$$

where $b_{n-2}(t)$ *and* $b_{n-1}(t)$ *are computed by*

$$\begin{bmatrix} b_{n-2}(t) \\ b_{n-1}(t) \end{bmatrix} = \begin{bmatrix} s_2^{n-2} & {s_2}^{n-1} \\ s_3^{n-2} & {s_3}^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} e^{s_2 t} - \sum_{i=0}^{n-3} \frac{t^i s_2^i}{i!} \\ e^{s_3 t} - \sum_{i=0}^{n-3} \frac{t^i s_3^i}{i!} \end{bmatrix}, \tag{21}$$

ρ_1 *and* ρ_2 *satisfy* $\rho_1^2 + 4\rho_2 > 0$, $s_2 = (\rho_1 + \sqrt{\rho_1^2 + 4\rho_2})/2$ *and* $s_3 = (\rho_1 - \sqrt{\rho_1^2 + 4\rho_2})/2$, *when* ρ_1 *and* ρ_2 *satisfy* $\rho_1^2 + 4\rho_2 < 0$, $s_2 = (\rho_1 + \sqrt{-\rho_1^2 - 4\rho_2}i)/2$ *and* $s_3 = (\rho_1 - \sqrt{-\rho_1^2 - 4\rho_2}i)/2$.

Remark 1 If ρ_1 and ρ_2 satisfy $\rho_1^2 + 4\rho_2 = 0$, then $s_2 = s_3$. Taking s_2 in Equation (9) gives

$$e^{s_2 t} - \sum_{i=0}^{n-3} \frac{t^i s_2^i}{i!} = b_{n-2}(t)s_2^{n-2} + b_{n-1}(t)s_2^{n-1}, \tag{22}$$

after the derivation of s_2 to Equation (9), we get

$$\begin{aligned} te^{s_2 t} - \sum_{i=1}^{n-3} \frac{it^i s_2^{i-1}}{i!} &= (n-2)b_{n-2}(t)s_2^{n-3} \\ &\quad + (n-1)b_{n-1}(t)s_2^{n-2}, \end{aligned}$$

then the parameters $b_{n-2}(t)$ and $b_{n-1}(t)$ are computed by

$$\begin{aligned} \begin{bmatrix} b_{n-2}(t) \\ b_{n-1}(t) \end{bmatrix} &= \begin{bmatrix} s_2^{n-2} & {s_2}^{n-1} \\ (n-2)s_2^{n-3} & (n-1)s_2^{n-2} \end{bmatrix}^{-1} \\ &\quad \times \begin{bmatrix} e^{s_2 t} - \sum_{i=0}^{n-3} \frac{t^i s_2^i}{i!} \\ te^{s_2 t} - \sum_{i=1}^{n-3} \frac{it^i s_2^{i-1}}{i!} \end{bmatrix}. \end{aligned} \tag{23}$$

THEOREM 3 *Let* $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ *and* $\boldsymbol{A}^n = \boldsymbol{A}^{n-1} + \boldsymbol{A}^{n-2} + \boldsymbol{A}^{n-3}$, *then we have*

$$\begin{aligned} e^{\boldsymbol{A}t} &= \boldsymbol{I} + \frac{t}{1!}\boldsymbol{A} + \frac{t^2}{2!}\boldsymbol{A}^2 + \cdots + \frac{t^{n-4}}{(n-4)!}\boldsymbol{A}^{n-4} \\ &\quad + b_{n-3}(t)\boldsymbol{A}^{n-3} + b_{n-2}(t)\boldsymbol{A}^{n-2} + b_{n-1}(t)\boldsymbol{A}^{n-1}, \end{aligned} \tag{24}$$

where $b_{n-3}(t)$, $b_{n-2}(t)$ and $b_{n-1}(t)$ are computed by

$$\begin{bmatrix} b_{n-3}(t) \\ b_{n-2}(t) \\ b_{n-1}(t) \end{bmatrix} = \begin{bmatrix} s_2^{n-3} & s_2^{n-2} & s_2{}^{n-1} \\ s_3^{n-3} & s_3^{n-2} & s_3{}^{n-1} \\ s_4^{n-3} & s_4^{n-2} & s_4{}^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} \mathrm{e}^{s_2 t} - \sum_{i=0}^{n-4} \frac{t^i s_2^i}{i!} \\ \mathrm{e}^{s_3 t} - \sum_{i=0}^{n-4} \frac{t^i s_3^i}{i!} \\ \mathrm{e}^{s_4 t} - \sum_{i=0}^{n-4} \frac{t^i s_4^i}{i!} \end{bmatrix}, \quad (25)$$

s_2, s_3 and s_4 are the roots of $s^3 - s^2 - s - 1 = 0$.

Theorem 4 *Let $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ and $\boldsymbol{A}^n = \rho_1 \boldsymbol{A}^{n-1} + \rho_2 \boldsymbol{A}^{n-2} + \rho_3 \boldsymbol{A}^{n-3}$, then we have*

$$\mathrm{e}^{\boldsymbol{A}t} = \boldsymbol{I} + \frac{t}{1!}\boldsymbol{A} + \frac{t^2}{2!}\boldsymbol{A}^2 + \cdots + \frac{t^{n-4}}{(n-4)!}\boldsymbol{A}^{n-4} + b_{n-3}(t)\boldsymbol{A}^{n-3} + b_{n-2}(t)\boldsymbol{A}^{n-2} + b_{n-1}(t)\boldsymbol{A}^{n-1}, \quad (26)$$

where $b_{n-3}(t)$, $b_{n-2}(t)$ and $b_{n-1}(t)$ are computed by

$$\begin{bmatrix} b_{n-3}(t) \\ b_{n-2}(t) \\ b_{n-1}(t) \end{bmatrix} = \begin{bmatrix} s_2^{n-3} & s_2^{n-2} & s_2{}^{n-1} \\ s_3^{n-3} & s_3^{n-2} & s_3{}^{n-1} \\ s_4^{n-3} & s_4^{n-2} & s_4{}^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} \mathrm{e}^{s_2 t} - \sum_{i=0}^{n-4} \frac{t^i s_2^i}{i!} \\ \mathrm{e}^{s_3 t} - \sum_{i=0}^{n-4} \frac{t^i s_3^i}{i!} \\ \mathrm{e}^{s_4 t} - \sum_{i=0}^{n-4} \frac{t^i s_4^i}{i!} \end{bmatrix}, \quad (27)$$

s_2, s_3 and s_4 are the different roots of $s^3 - \rho_1 s^2 - \rho_2 s - \rho_3 = 0$.

Remark 2 If s_2, s_3 and s_4 are not the different roots, we can also compute the parameters $b_{n-3}(t)$, $b_{n-2}(t)$ and $b_{n-1}(t)$ by using the method as in Remark 1.

Theorem 5 *Let $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ and $\mathbf{A}^n = \rho_1 \boldsymbol{A}^{n-1} + \rho_2 \boldsymbol{A}^{n-2} + \cdots + \rho_k \mathbf{A}^{n-k}, k < n$, then we have*

$$\mathrm{e}^{\boldsymbol{A}t} = \boldsymbol{I} + \frac{t}{1!}\boldsymbol{A} + \frac{t^2}{2!}\boldsymbol{A}^2 + \cdots + \frac{t^{n-k-1}}{(n-k-1)!}\boldsymbol{A}^{n-k-1} + b_{n-k}(t)\boldsymbol{A}^{n-k} + b_{n-k+1}(t)\boldsymbol{A}^{n-k+1} + \cdots + b_{n-1}(t)\boldsymbol{A}^{n-1}, \quad (28)$$

where $b_{n-k}(t), b_{n-k+1}(t), \ldots, b_{n-1}(t)$ are computed by

$$\begin{bmatrix} cb_{n-k}(t) \\ b_{n-k+1}(t) \\ \vdots \\ b_{n-1}(t) \end{bmatrix} = \begin{bmatrix} s_2^{n-k} & s_2^{n-k+1} & \cdots & s_2{}^{n-1} \\ s_3^{n-3} & s_3^{n-2} & \cdots & s_3{}^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{k+1}^{n-k} & s_{k+1}^{n-k+1} & \cdots & s_4{}^{n-1} \end{bmatrix}^{-1} \times \begin{bmatrix} \mathrm{e}^{s_2 t} - \sum_{i=0}^{n-k-1} \frac{t^i s_2^i}{i!} \\ \mathrm{e}^{s_3 t} - \sum_{i=0}^{n-k-1} \frac{t^i s_3^i}{i!} \\ \vdots \\ \mathrm{e}^{s_{k+1} t} - \sum_{i=0}^{n-k-1} \frac{t^i s_{k+1}^i}{i!} \end{bmatrix}, \quad (29)$$

$s_2, s_3, \ldots, s_{k+1}$ are the different roots of $s^k - \rho_1 s^{k-1} - \rho_2 s^{k-2} - \cdots - \rho_{k-1} s - \rho_k = 0$.

3. Examples

In this section, we will use some matrices to show the effectiveness of the proposed method.

Example 1 Let

$$\boldsymbol{A} = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}.$$

It satisfies that $\boldsymbol{A}^4 = \boldsymbol{A}^3 + 2\boldsymbol{A}^2$. For $\rho_1 = 1$, $\rho_2 = 2$ in Theorem 2, we have

$$\begin{aligned} \mathrm{e}^{\boldsymbol{A}t} &= \boldsymbol{I}_4 + t\boldsymbol{A} + b_2(t)\boldsymbol{A}^2 + b_3(t)\boldsymbol{A}^3 \\ &= \boldsymbol{I}_4 + t\boldsymbol{A} + \left(\frac{1}{12}\mathrm{e}^{2t} + \frac{2}{3}\mathrm{e}^{-t} + 0.5t - 0.75\right)\boldsymbol{A}^2 \\ &\quad + \left(\frac{1}{12}\mathrm{e}^{2t} - \frac{1}{3}\mathrm{e}^{-t} - 0.5t + 0.25\right)\boldsymbol{A}^3 \\ &= \begin{bmatrix} 1 & 2t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{2t} & 0 \\ -\mathrm{e}^{-t} + 1 & 2\mathrm{e}^{-t} + 2t - 2 & 0 & \mathrm{e}^{-t} \end{bmatrix}. \end{aligned} \quad (30)$$

Example 2 Let

$$\boldsymbol{A} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

It satisfies that $\boldsymbol{A}^4 = 2\boldsymbol{A}^3 + \boldsymbol{A}^2 - 2\boldsymbol{A}$. For $\rho_1 = 2$, $\rho_2 = 1$ and $\rho_3 = -2$ in Theorem 4, we have

$$\begin{aligned} e^{\boldsymbol{A}t} &= \boldsymbol{I}_4 + b_1(t)\boldsymbol{A} + b_2(t)\boldsymbol{A}^2 + b_3(t)\boldsymbol{A}^3 \qquad (31) \\ &= \boldsymbol{I}_4 + \left(-\frac{1}{6}e^{2t} + e^t - \frac{1}{3}e^{-t} - 0.5\right)\boldsymbol{A} \\ &\quad + (0.5e^t + 0.5e^{-t} - 1)\boldsymbol{A}^2 \\ &\quad + \left(\frac{1}{6}e^{2t} - 0.5e^t - \frac{1}{6}e^{-t} + 0.5\right)\boldsymbol{A}^3 \\ &= \begin{bmatrix} e^{2t} & 0 & 0 & 0 \\ 0 & e^t & e^t - 1 & e^t + e^{-t} - 2 \\ 0 & 0 & 1 & -2e^{-t} + 2 \\ 0 & 0 & 0 & e^{-t} \end{bmatrix}. \qquad (32) \end{aligned}$$

4. Conclusions

One method to compute the accurate solution of $e^{\boldsymbol{A}t}$ is presented in this letter. The basic idea of this method is using the matrix theory, the matrices satisfy the special case $\boldsymbol{A}^n = \rho_1\boldsymbol{A}^{n-1} + \rho_2\boldsymbol{A}^{n-2} + \cdots + \rho_k\boldsymbol{A}^{n-k}, k < n$. Furthermore, this method can be extended to the more general case $\boldsymbol{A}^k = \rho_1\boldsymbol{A}^{k-1} + \rho_2\mathbf{A}^{k-2} + \cdots + \rho_m\boldsymbol{A}^{k-m}, k < n$, $m < k$.

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