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Review

Stability analysis of fractional-order systems with the Riemann–Liouville derivative

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In this paper, the stability of fractional-order systems with the Riemann–Liouville derivative is discussed. By applying the Mittag-Leffler function, generalized Gronwall inequality and comparison principle to fractional differential systems, some sufficient conditions ensuring stability and asymptotic stability are given.

Keywords: asymptotic stability; generalized Gronwall inequality; Riemann–Liouville derivative; fractional differential system

1. Introduction

Fraction calculus has more than 300 years history. With the development of science and engineering applications, fractional calculus has become one of the most hottest topics. Up to now, many fractional results have been presented which are very useful (Debnath, 2004; Miller & Ross, 1993; Podlubny, 1999; Samko, Kilbas, & Marichev, 1993; Zhang & Li, 2011).

Stability analysis is the most fundamental for studying fractional differential equations. Recently, many stability results of fractional-order systems are interesting in physical systems, so more and more stability results have been found, see, for instance, Ahn and Chen (2008), Ahmed, EI-Saka, and EI-Saka (2007), Deng, Li, and Liu, (2007), Li, Chen, and Podlubny (2009), Li and Zhang (2011), Miller and Ross (1993), Moze, Sabatier, and Oustaloup (2007), Odibat (2010), Qian, Li, Agarwal, and Wong (2010), Radwan, Soliman, Elwakil, and Sedeek (2009), Sabatier, Moze, and Farges (2010), Samko et al. (1993), Tavazoei and Haeri (2009), Wen, Wu, and Lu, (2008) and Zhang and Li (2011). These stability results are mainly concerned with the linear fractional differential system. For example, in Matignon (1996), a sufficient and necessary condition on asymptotic stability of linear fractional differential system with order $0 < \alpha < 1$ was first given. Then some other research on the stability of fractional-order systems appeared. Of course, there also exist fractional-order systems with order lying in (1, 2). In Zhang and Li (2011), authors dealt with the following fractional differential system:

$$D^{\alpha}_{t_0,t}x(t) = Ax(t) + B(t)x(t),$$

where $1 < \alpha < 2, D^{\alpha}_{t_0,t}$ denotes either the Caputo or the Riemann–Liouville fractional derivative operator. They analysed stability of the above fractional differential system by applying Gronwall's inequality (Corduneanu, 1971) and related results.

In this paper, three conditions about $B(t)$ are given as follows:

(I) $0 < \alpha < 1, \int_0^{\infty} \mathrm{P}\, B(t) \mathrm{Pd}t$ is bounded;
(II) $1 < \alpha < 2, \|B(t)\|$ is bounded;
(III) $1 < \alpha < 2, B(t) = O(t - t_0)^{\theta} (\theta < -\alpha, t_0 > 0)$.

Under these conditions, the stability and asymptotic stability of nonautonomous linear fractional differential systems with the Riemann–Liouville derivative are analysed by using generalized Gronwall's inequality, some properties of the Mittag-Leffler function and relevant results. From the results derived in this paper, we can also analyse the stability of these nonlinear systems in the future.

This paper is organized as follows. In Section 2 some necessary definitions and lemmas are recalled, which will be used later. The main results are presented in Section 3. Finally, some conclusions are drawn in Section 4.

2. Preliminaries

In this section, the most commonly used definitions and results are stated, which will be used later.

Definition 2.1 *The Riemann–Liouville fractional derivative with order α of function x(t) is defined as*

$${}_{RL}D^{\alpha}_{a,t}x(t) = \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \int_a^t (t-\tau)^{m-\alpha-1} x(\tau)\mathrm{d}\tau,$$

where $m - 1 \le \alpha < m, \Gamma(\cdot)$ *is the Gamma function.*

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The Laplace transform of the Riemann–Liouville fractional derivative is

$$\int_0^\infty e^{-st}{}_{\mathrm{RL}}D^\alpha_{a,t}x(t)\mathrm{d}t = s^\alpha X(s) - \sum_{k=0}^{n-1} s^k[D^{\alpha-k-1}x(t)]_{t=a} \quad (n-1 \le \alpha < n).$$

DEFINITION 2.2 *The Mittag-Leffler function with two parameters is defined as*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)},$$

where $\alpha > 0, \beta > 0, z \in C$. When $\beta = 1$, one has $E_\alpha(z) = E_{\alpha,1}(z)$, furthermore, $E_{1,1}(z) = e^z$.

The Laplace transform of the Mittag-Leffler function is

$$\int_0^\infty e^{-st}t^{k\alpha+\beta-1}E^{(k)}_{\alpha,\beta}(\pm a t^\alpha)\mathrm{d}t = \frac{k!s^{\alpha-\beta}}{(s^\alpha \mp a)^{k+1}} \quad (\Re(s) > |a|^{1/n}).$$

DEFINITION 2.3 *The zero solution of*

$${}_{RL}D^\alpha_{t_0,t}x(t) = f(t, x(t)),$$

with order $0 < \alpha \le 1 (1 < \alpha < 2)$ is said to be stable if, for any initial values $x_k (k = 0)(x_k (k = 0, 1))$, there exists $\varepsilon > 0$ such that $\|(t)\| < \varepsilon$ for all $t > t_0$. The zero solution is said to be asymptotically stable if, in addition to being stable, $\|x(t)\| \to 0$ as $t \to +\infty$.

LEMMA 1 *If $A \in C^{n\times n}$ and $0 < \alpha < 2, \beta$ is an arbitrary real number, μ satisfies $\alpha\pi/2 < \mu < \min\{\pi, \alpha\pi\}$, and $C > 0$ is a real constant, then*

$$\|E_{\alpha,\beta}(A)\| \le \frac{C}{1+\|A\|},$$

where $\mu \le |\arg(spec(A))| \le \pi$, spec(A) denotes the eigenvalues of matrix A and $\|\cdot\|$ denotes the l_2 norm.

LEMMA 2 *If $A \in C^{n\times n}$ and $0 < \alpha < 2, \beta$ is an arbitrary complex number and μ satisfies $\alpha\pi/2 < \mu < \min\{\pi, \alpha\pi\}$, then for an arbitrary integer $p \ge 1$, the following expansions hold:*

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha}z^{1-\beta/\alpha}\exp(z^{1/\alpha}) - \sum_{k=1}^{p}\frac{z^{-k}}{\Gamma(\beta-k\alpha)} + O(|z|^{-p-1}),$$

with $|z| \to \infty, |\arg(z)| \le \mu$ and

$$E_{\alpha,\beta}(z) = -\sum_{k=1}^{p}\frac{z^{-k}}{\Gamma(\beta-k\alpha)} + O(|z|^{-p-1}),$$

with $|z| \to \infty$ and $\mu < |\arg(z)| \le \pi$.

Especially, in Zhang and Li (2011) it has been obtained that the matrix $(t-t_0)^{\alpha-k-1}E_{\alpha,\alpha-k}(A(t-t_0)^\alpha)$ is bounded, i.e.

$$\|(t-t_0)^{\alpha-k-1}E_{\alpha,\alpha-k}(A(t-t_0)^\alpha)\| \le M_k,$$

for some $M_k > 0$.

LEMMA 3 *Suppose $\alpha > 0, a(t)$ is a nonnegative locally integrable function on $0 \le t < T$ (some $T \le \infty$) and $g(t)$ is a nonnegative and nondecreasing continuous function defined on $0 \le t < T, g(t) \le M$ (constant), and suppose $u(t)$ is nonnegative and locally integrable on $0 \le t < T$ with*

$$u(t) \le a(t) + g(t)\int_0^t (t-s)^{\alpha-1}u(s)\mathrm{d}s,$$

on this interval, then

$$u(t) \le a(t) + \int_0^t\left[\sum_{n=1}^{\infty}\frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}a(s)\right]\mathrm{d}s.$$

Moreover, if $a(t)$ is a nondecreasing function on $[0, T)$, then

$$u(t) \le a(t)E_\alpha(g(t)\Gamma(\alpha)t^\alpha).$$

LEMMA 4 *Suppose that $g(t)$ and $u(t)$ are continuous on $[t, t_0], g(t) \ge 0, \lambda \ge 0$ and $r \ge 0$ are two constants, if*

$$u(t) \le \lambda + \int_{t0}^{t}[g(\tau)u(\tau) + r]d\tau,$$

then

$$u(t) \le (\lambda + r(t_1 - t_0))\exp\int_{t0}^{t}g(\tau)d\tau, \quad t_0 \le t \le t_1.$$

3. Stability of nonautonomous linear fractional differential systems

3.1. Fractional-order α : $0 < \alpha < 1$

Consider the nonautonomous fractional system

$${}_{\mathrm{RL}}D^\alpha_{0,t}x(t) = Ax(t) + B(t)x(t) \quad (0 < \alpha < 1), \tag{1}$$

with the initial condition

$${}_{\mathrm{RL}}D^{\alpha-1}_{0,t}x(t)|_{t=0} = x_0, \tag{2}$$

where $x \in R^n$, matrix $A \in R^{n\times n}, B(t) : [0,\infty] \to R^{n\times n}$ is a continuous t matrix.

THEOREM 1 *Suppose $\|t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha)\| \le Me^{-\gamma t}, 0 \le t < \infty, \gamma > 0$ and $\int_0^\infty \|B(t)\|\mathrm{d}t$ is bounded, i.e. $\int_0^\infty \|B(t)\|\mathrm{d}t \le N$, where $M, N > 0$, then the solution of Equation (1) is asymptotically stable.*

Proof By the Laplace transform and the inverse Laplace transform, the solution of Equations (1) with (2) can be written as

$$x(t) = t^{\alpha-1}E_{\alpha\alpha}(At^{\alpha})x_0 + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha\alpha}(A(t-\tau)^{\alpha})B(\tau)x(\tau)\mathrm{d}\tau,$$

then we can obtain

$$\|x(t)\| \leq \|t^{\alpha-1}E_{\alpha\alpha}(At^{\alpha})\|\|x_0\| + \int_0^t \|(t-\tau)^{\alpha-1}E_{\alpha\alpha}(A(t-\tau)^{\alpha})\|\|B(\tau)\| \|x(\tau)\|\mathrm{d}\tau.$$

From the boundedness, we can obtain

$$\|x(t)\| \leq Me^{-\gamma t}\|x_0\| + \int_0^t Me^{-\gamma(t-\tau)}\|B(\tau)\|\|x(\tau)\|\mathrm{d}\tau. \tag{3}$$

Multiplying by $e^{\gamma t}$ both sides of Equation (3), we have

$$e^{\gamma t}\|x(t)\| \leq M\|x_0\| + \int_0^t Me^{\gamma\tau}\|B(\tau)\|x(\tau)\|\mathrm{d}\tau.$$

Let $e^{\gamma t}\|x(t)\| = u(t)$, then according to Lemma 4, one has

$$e^{\gamma t}\|x(t)\| \leq (M\|x_0\|)\exp\left(M\int_0^t \|B(t)\|\mathrm{d}t\right). \tag{4}$$

Multiplying by $e^{-\gamma t}$ both sides of Equation (4), we can obtain

$$\|x(t)\| \leq (M\|x_0\|)\exp\left(M\int_0^t \|B(t)\|\mathrm{d}t\right)e^{-\gamma t},$$

then $\|x(t)\| \leq (M\|x_0\|)e^{MN-\gamma t}$, so $\|x(t)\| \to 0, t \to \infty$.

That is, the solution of Equation (1) is asymptotically stable. ■

3.2. *Fractional-order* $\alpha : 1 < \alpha < 2$

Consider the following fractional-order system:

$${}_{\mathrm{RL}}D^{\alpha}_{t_0,t}x(t) = Ax(t) + B(t)x(t) \quad (1 < \alpha < 2), \tag{5}$$

with the initial conditions

$${}_{\mathrm{RL}}D^{\alpha-k}_{t_0,t}x(t)|_{t=t0} = x_{k-1} \quad (k = 1, 2), \tag{6}$$

where $x \in R^n$, matrix $A \in R^{n\times n}, B(t) : [t_0, \infty) \to R^{n\times n}$ is a continuous matrix.

Theorem 2 *If the eigenvalues of matrix* A *satisfy* $|\arg(\lambda(A))| > \alpha\pi/2$ *and* $\|B(t)\|$ *is bounded, i.e.* $\|B(t)\| \leq M$ *for some* $M > 0$, *then the zero solution of Equation (5) is asymptotically stable.*

Proof By the Laplace transform and the inverse Laplace transform, the solution of Equations (5) with (6) can be written as

$$x(t) = (t-t_0)^{\alpha-1}E_{\alpha\alpha}(A(t-t_0)^{\alpha})x_0 + (t-t_0)^{\alpha-2}E_{\alpha\alpha-1}(A(t-t_0)^{\alpha})x_1 + \int_{t_0}^t (t-\tau)^{\alpha-1}E_{\alpha\alpha}(A(t-\tau)^{\alpha})B(\tau)x(\tau)\mathrm{d}\tau,$$

then we can obtain

$$\begin{aligned}\|x(t)\| &\leq \|(t-t_0)^{\alpha-1}E_{\alpha\alpha}(A(t-t_0)^{\alpha})\|\|x_0\| + \|(t-t_0)^{\alpha-2}E_{\alpha\alpha-1}(A(t-t_0)^{\alpha})\|\|x_1\| \\ &\quad + \int_{t0}^t (t-\tau)^{\alpha-1}\|E_{\alpha\alpha}(A(t-\tau)^{\alpha})\| \times \|B(\tau)\|\|x(\tau)\|\mathrm{d}\tau \\ &\leq M_0\|x_0\| + M_1\|x_1\| + LM\int_{t_0}^t (t-\tau)^{\alpha-1} \times \|x(\tau)\|\mathrm{d}\tau,\end{aligned}$$

where $L, M, M_0, M_1 > 0$ such that

$$\|(t-t_0)^{\alpha-k-1}E_{\alpha,\alpha-k}(A(t-t_0)^{\alpha})\| \leq M_k \quad (k = 0, 1),$$
$$\|E_{\alpha\alpha}(A(t-t_0)^{\alpha})\| \leq L.$$

Based on Lemmas 2 and 3, we can obtain

$$\begin{aligned}\|x(t)\| &\leq (M_0\|x_0\| + M_1\|x_1\|)E_{\alpha}(LM\Gamma(\alpha)(t-t_0)^{\alpha}) \\ &= (M_0\|x_0\| + M_1\|x_1\|) \times \left[-\sum_{k=1}^{p}\frac{(LM\Gamma(\alpha)(t-t_0)^{\alpha})^{-k}}{\Gamma(1-k\alpha)} + O(|LM\Gamma(\alpha)t^{\alpha}|)^{-1-p}\right].\end{aligned}$$

When $t \to \infty, \|x(t)\| \to 0$. That is, the solution of Equation (5) is asymptotically stable. ■

Remark 1 Suppose the Caputo derivative takes the place of the Riemann–Liouville derivative in Equation (1) and all other assumed conditions remain the same, then the conclusions of Theorem 2 still hold.

Theorem 3 *If all eigenvalues of matrix* A *satisfy* $|\arg(\lambda(A))| > \alpha\pi/2, \|B(t)\|$ *is nondecreasing and* $B(t) = O(t-t_0)^{\theta}, (\theta < -\alpha, t_0 > 0)$, *then the zero solution is asymptotically stable.*

Proof By the Laplace transform and the inverse Laplace transform, the solution of Equations (5) with (6) can be written as

$$x(t) = (t-t_0)^{\alpha-1}E_{\alpha\alpha}(A(t-t_0)^{\alpha})x_0 + (t-t_0)^{\alpha-2}E_{\alpha\alpha-1}(A(t-t_0)^{\alpha})x_1 + \int_{t0}^{t}(t-\tau)^{\alpha-1}E_{\alpha\alpha}(A(t-\tau)^{\alpha})B(\tau)x(\tau)\mathrm{d}\tau,$$

then one can obtain

$$\|x(t)\| \le \|(t-t_0)^{\alpha-1}E_{\alpha\alpha}(A(t-t_0)^{\alpha})\|\|x_0\| + \|(t-t_0)^{\alpha-2}E_{\alpha\alpha-1}(A(t-t_0)^{\alpha})\|\|x_1\| + \int_{t_0}^{t}(t-\tau)^{\alpha-1}\|E_{\alpha\alpha}(A(t-\tau)^{\alpha})\| \|B(\tau)\|\|x(\tau)\|\mathrm{d}\tau. \tag{7}$$

Then

$$\|x(t)\| \le M_0\|x_0\| + M_1\|x_1\| + L\int_{t_0}^{t}(t-\tau)^{\alpha-1}\|B(\tau)\|\|x(\tau)\|\,\mathrm{d}\tau,$$

where $L, M_0, M_1 > 0$ such that

$$\|(t-t_0)^{\alpha-k-1}E_{\alpha,\alpha-k}(A(t-t_0)^{\alpha})\| \le M_k \quad (k=0,1),$$
$$\|E_{\alpha,\alpha}(A(t-t_0)^{\alpha})\| \le L.$$

Multiplying by $\|B(t)\|$ on both sides of Equation (7), one obtains

$$\|B(t)\|\|x(t)\| \le \|B(t)\|(M_0\|x_0\| + M_1\|x_1\|) + L\|B(t)\| \times \int_{t_0}^{t}(t-\tau)^{\alpha-1}\|B(\tau)\|\|x(\tau)\|\,\mathrm{d}\tau.$$

Applying Lemma 3 leads to

$$\|B(t)\|\|x(t)\| \le \|B(t)\|(M_0\|x_0\| + M_1\|x_1\|) \times E_{\alpha}(L\|B(t)\|\Gamma(\alpha)t^{\alpha}).$$

Then

$$\|x(t)\| \le (M_0\|x_0\| + M_1\|x_1\|)E_{\alpha}(L\|B(t)\|\Gamma(\alpha)(t-t_0)^{\alpha}) \le (M_0\|x_0\| + M_1\|x_1\|)\sum_{k=0}^{\infty}\frac{(L\Gamma(\alpha)\|B(t)\|(t-t_0)^{\alpha})^k}{\Gamma(k\alpha+1)}.$$

Since $B(t) = O(t-t_0)^{\theta}$, $(\theta < -\alpha, t_0 > 0)$, then

$\|B(t)\|\,(t-t_0)^{\alpha} \to 0$ as $t \to \infty$, so $\|x(t)\|$ is bounded, i.e. $\exists N$, such that $\|x(t)\| \le N$.

We also can obtain the following expression from the solution:

$$\|x(t)\| \le (t-t_0)^{\alpha-2}L_1\|x_0\| + (t-t_0)^{\alpha-2}L_2\|x_1\| + L_1\int_{t_0}^{t}(t-\tau)^{\alpha-2}\|B(\tau)\|\|x(\tau)\|\,\mathrm{d}\tau,$$

where

$$\|(t-t_0)E_{\alpha,\alpha}(A(t-t_0)^{\alpha})\| < L_1,$$
$$\|E_{\alpha,\alpha-1}(A(t-t_0)^{\alpha})\| < L_2.$$

Since $\|x(t)\| \le N$, then

$$\|x(t)\| \le (t-t_0)^{\alpha-2}(L_1\|x_0\| + L_2\|x_1\|) + L_1N\int_{t_0}^{t}(t-\tau)^{\alpha-2}\|B(\tau)\|\,\mathrm{d}\tau \le (t-t_0)^{\alpha-2}(L_1\|x_0\| + L_2\|x_1\|) + L_1N\frac{\Gamma(\alpha-1)\Gamma(1+\theta)}{\Gamma(\alpha+\theta)}O(t-t_0)^{\alpha+\theta-1}.$$

When $t \to \infty$, $\|x(t)\| \to 0$. So the solution of Equation (5) is asymptotically stable. ■

Remark 2 Suppose the Caputo derivative takes the place of the Riemann–Liouville derivative in Equation (5) and all other assumed conditions remain the same, then the conclusion is stable.

4. Conclusions

In this paper, we have studied the stability and asymptotic stability of the nonautonomous linear differential system with the Riemann–Liouville fractional derivative and established the corresponding stability results of its zero solution. By using the Laplace transform, Mittag-Leffler function, the generalized Gronwall inequality, some sufficient conditions ensuring the stability and asymptotic stability of the perturbed linear fractional differential system with the Riemann–Liouville fractional derivative were given.

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