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## New delay-dependent stability conditions for linear systems with delay

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In this work, delay-dependent stability conditions for systems described by delayed differential equations are presented. The employment of a special transformation to a state space representation named Benrejeb characteristic arrow matrix permits to determine new asymptotic stability conditions. Illustrative examples are presented to show the effectiveness of the proposed approach.

**Keywords:** linear time-delay systems; stability analysis; delay-dependent stability; arrow matrix

### 1. Introduction

Many practical systems, such as biological processes, population dynamics, neural networks (Richard, 2003), feedback controlled mechanical systems (Hu, 2002), automotive power train systems (Balachandran, Kalmár-Nagy, & Gilsinn, 2009) and teleoperators (Loiseau, Michiels, Niculescu, & Sipahi, 2009), are represented by models which depend on the current states as well as on the past ones, such systems are known as time-delay systems. The obtained mathematical models of such systems can be formulated by a set of delay differential equations (DDEs). The presence of delay can degrade closed-loop system performance and even can cause instability. Difficulties are greater when these systems are nonlinear or have multiple delays, see Cepeda-Gomez and Olgac (2011), Chen (1995), Chen and Latchman (1995), Kosugi and Suyama (2011), Niculescu, Verriest, Duggard, and Dion (1997) for an excellent exposition of nonlinear DDEs. It is well known that ensuring stability is a first and an essential step in any design process. For all these reasons, there has been an extensive literature on stability analysis of time-delay systems, see, for example, Bellman and Cooke (1963), Elmadssia, Saadaoui, and Benrejeb (2011), Hale (1977), Hale, Infante, and Tsen (1985), Kamen (1980, 1982), Lewis and Anderson (1980), Malek-Zavarei and Jamshidi (1987). Many methods are employed to study stability of time-delay systems, these approaches include Lyapunov methods and their extensions, the  $\tau$ -decomposition method, the  $D$ -decomposition method (Dugard & Verriest, 1998; Neimark, 1992) and frequency-domain methods (Dugard & Verriest, 1998; Malakhovski & Mirkin, 2006).

The first step in our study is determining stability conditions for linear systems with a single delay. Then, a

generalization of these conditions to multi-delay systems is given. Our approach is based on transforming the representation of the system under consideration into another specific form and using an appropriate Lyapunov function to determine sufficient delay-dependent stability conditions. The developed criteria are mainly based on characteristic arrow form matrix representation (Borne, Vanheeghe, and Duflos, 2007; Elmadssia et al., 2011) associated with Borne and Gentina practical criterion (Gentina, Borne, & Laurent, 1972). One of the main contributions of this paper is the determination of an upper bound on the delays that guarantee the stability of the system. This upper bound is expressed as a function of the system's parameters. In Niculescu et al. (1997), a similar condition was derived for second-order systems with a single delay. In this article, more general conditions were derived for high-order systems with single and multiple delays value. The obtained stability conditions are presented explicitly and simply.

The paper is organized as follows. In Section 2, a delay-dependent stability condition for linear systems with a single delay is given. An application of this criteria to DDEs with a single delay is presented in Section 3. Section 4 is devoted to studying stability problem of linear systems with multiple delays. Finally, some illustrative examples and concluding remarks are given.

### 2. Delay-dependent stability conditions for linear systems with a single delay

Let us start by defining some notations. Let  $R^n$  denote an  $n$ -dimensional linear vector space over the reals with the norm  $\|\cdot\|$ . For any  $u = (u_i)_{1 \leq i \leq n}$ ,  $v = (v_i)_{1 \leq i \leq n} \in R^n$ , we define the scalar product of the vectors  $u$  and  $v$  as:

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$\langle u, v \rangle = \sum_{i=1}^n u_i v_i$ . Let  $R = (-\infty + \infty)$ ,  $R_+ = [0 + \infty)$ ,  $R_+^* = (0 + \infty)$ ,  $R_+^{*n} = \{v = (v_i)_{1 \leq i \leq n} \in R^n, v_i \in R_+^*, \forall i = 1, 2, \dots, n\}$ ,  $\text{sgn}(\psi) = 1$  (respectively,  $\text{sgn}(\psi) = -1$ ) if  $\psi \in R_+^*$  (respectively,  $-\psi \in R_+^*$ ) and  $\text{sgn}(\psi) = 0$  if  $\psi = 0$ ,  $\lambda(M)$  denotes the set of eigenvalues of the matrix  $M$ ,  $M'$  its transpose and  $M^{-1}$  its inverse and if  $M = (m_{ij})_{1 \leq i, j \leq n}$ , we denote  $M^* = (m_{ij}^*)_{1 \leq i, j \leq n}$  with  $m_{ij}^* = m_{ij}$  if  $i = j$  and  $m_{ij}^* = |m_{ij}|$  if  $i \neq j$  and  $|M| = (|m_{ij}|)_{1 \leq i, j \leq n}$ . Let  $C_n = C([- \tau 0], R^n)$  be the Banach space of continuous functions mapping the interval  $[- \tau 0]$  into  $R^n$  with the topology of uniform convergence. For a given  $\phi \in C_n$ , we define  $\|\phi\| = \sup_{-\tau \leq \theta \leq 0} \|\phi(\theta)\|$ ,  $\phi(\theta) \in R^n$ .

In the next, we introduce several useful tools, including Kotelyanski lemma and definition of an  $M$ -matrix.

**Kotelyanski Lemma.** (Gentina et al., 1972) *The real parts of the eigenvalues of matrix  $A$ , with non-negative off diagonal elements, are less than a real number if and only if all those of matrix  $M$ ,  $M = I_n - A$ , are positive, with  $I_n$  the  $n$  identity matrix.*

**DEFINITION 1** *The matrix  $A = (a_{ij})_{n \leq i, j \leq n}$  is called an  $M$ -matrix if the following conditions are satisfied:*

- (1)  $a_{i,i} > 0$  ( $i = 1, 2, \dots, n$ ),  $a_{i,j} \leq 0$  ( $i \neq j, i, j = 1, 2, \dots, n$ ).
- (2) *Successive principal minors of  $A$  are positive, i.e.*

$$\det \begin{pmatrix} a_{1,1} & \dots & a_{1,i} \\ \vdots & \dots & \vdots \\ a_{i,1} & \dots & a_{i,i} \end{pmatrix} > 0 \quad (i = 1, 2, 3, \dots, n).$$

**DEFINITION 2**  *$A$  is the opposite of an  $M$ -matrix if  $(-A)$  is an  $M$ -matrix.*

There are many equivalent conditions for an  $M$ -matrix, among them we used the condition given below.

$a_{i,i} > 0$  ( $i = 1, 2, \dots, n$ ),  $a_{i,j} \leq 0$  ( $i \neq j, i, j = 1, 2, \dots, n$ ), and for any positive real numbers  $\eta = (\eta_1, \eta_2, \dots, \eta_n)'$ , the algebraic equations  $Ax = \eta$  have a positive solution  $w = (w_1, w_2, \dots, w_n)$ .

*Remark 1* When successive principal minors of matrix  $(-A)$  are positive, which amounts to that successive main minors of the matrix  $A$  are alternating sign with the first is negative, the Kotelyanski lemma allows us to conclude the stability of the system characterized by  $A$ .

Now, consider a time-delay system given by the following state space representation:

$$\dot{x}(t) = \tilde{A}_0 x(t) + \mu \tilde{A}_1 x(t - \tau), \quad (1)$$

where  $\tilde{A}_i \in R^{n \times n}$ ,  $i = 0, 1$ ,  $x(t) \in R^n$  is the state vector with components  $x_i$  ( $i = 1, \dots, n$ ),  $\tau > 0$  is the time delay of the system and  $\mu$  is a real scalar parameter which is introduced

by convenience to govern the size of the delayed dynamics (De la Sen, Malaina, Soto, & Gallego, 2005).

First of all, we start by writing our system with another form. By using the Newton–Leibniz formula

$$x(t - \tau) = x(t) - \int_{t-\tau}^t \dot{x}(\theta) d\theta. \quad (2)$$

Equation (2) can be written as

$$x(t - \tau) = x(t) - \tilde{A}_0 \int_{t-\tau}^t x(\theta) d\theta - \mu \tilde{A}_1 \int_{t-\tau}^t x(\theta - \tau) d\theta. \quad (3)$$

Then, Equation (1) becomes

$$\begin{aligned} \dot{x}(t) &= (\tilde{A}_0 + \mu \tilde{A}_1)x(t) - \mu \tilde{A}_1 \tilde{A}_0 \int_{t-\tau}^t x(\theta) d\theta \\ &\quad - \mu^2 \tilde{A}_1^2 \int_{t-\tau}^t x(\theta - \tau) d\theta. \end{aligned} \quad (4)$$

The next result gives a delay-dependent stability condition for system (1).

**THEOREM 1** *The system (1) is asymptotically stable if the matrix  $T_1$ , given by*

$$T_1 = (\tilde{A}_0 + \mu \tilde{A}_1)^* + \tau (|\mu| |\tilde{A}_1 \tilde{A}_0| + \mu^2 |\tilde{A}_1^2|) \quad (5)$$

*is the opposite of an  $M$ -matrix.*

*Proof* Let  $w \in R^n$  with components  $w_i > 0$  ( $i = 1, \dots, n$ ) and let us consider the radially unbound Lyapunov functional given by

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t), \quad (6)$$

where

$$V_1(t) = \langle |x(t)|, w \rangle, \quad (7)$$

$$V_2(t) = |\mu| \left\langle |\tilde{A}_1 \tilde{A}_0| \int_{-\tau}^0 \int_{t+\theta}^t |x(s)| ds d\theta, w \right\rangle, \quad (8)$$

$$V_3(t) = \mu^2 \left\langle |\tilde{A}_1^2| \int_{-\tau}^0 \int_{t+\theta}^t |x(s - \tau)| ds d\theta, w \right\rangle \quad (9)$$

and

$$V_4(t) = \tau \mu^2 \left\langle |\tilde{A}_1^2| \int_{t-\tau}^t |x(s)| ds, w \right\rangle \quad (10)$$

it is clear that

$$V(t_0) < \infty.$$

The right Dini derivative of  $V$  along the solution of Equation (4) gives

$$\begin{aligned} D^+ V(t)|_{(4)} &= D^+ V_1(t)|_{(4)} + D^+ V_2(t)|_{(4)} + D^+ V_3(t)|_{(4)} \\ &\quad + D^+ V_4(t)|_{(4)}, \end{aligned} \quad (11)$$

we have

$$\begin{aligned} D^+V_1(t)|_{(4)} &= \left\langle \frac{d^+|x(t)|}{dt^+}, w \right\rangle \\ &= \left\langle D_x(t) \frac{d^+x(t)}{dt^+}, w \right\rangle, \end{aligned} \quad (12)$$

where

$$D_x(t) = \begin{pmatrix} \text{sgn}(x_1) & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \text{sgn}(x_n) \end{pmatrix}. \quad (13)$$

Next, we have

$$\begin{aligned} D^+V_1(t)|_{(4)} &= \left\langle D_x(t) \left( (\tilde{A}_0 + \mu\tilde{A}_1)x(t) \right. \right. \\ &\quad \left. \left. - \mu\tilde{A}_1\tilde{A}_0 \int_{t-\tau}^t x(\theta) d\theta \right), w \right\rangle \\ &\quad - \left\langle \mu^2 D_x(t) \left( \tilde{A}_1^2 \int_{t-\tau}^t x(\theta - \tau) d\theta \right), w \right\rangle. \end{aligned} \quad (14)$$

So, by overvaluing  $D^+V_1(t)|_{(4)}$ , we get

$$\begin{aligned} D^+V_1(t)|_{(4)} &\leq \left\langle \left( (\tilde{A}_0 + \mu\tilde{A}_1)^* |x(t)| \right. \right. \\ &\quad \left. \left. + |\mu| |\tilde{A}_1\tilde{A}_0| \int_{t-\tau}^t |x(\theta)| d\theta \right), w \right\rangle \\ &\quad + \left\langle \left( \mu^2 |\tilde{A}_1^2| \int_{t-\tau}^t |x(\theta - \tau)| d\theta \right), w \right\rangle, \end{aligned} \quad (15)$$

$$D^+V_2(t)|_{(4)} = |\mu| \left\langle |\tilde{A}_1\tilde{A}_0| \left( \tau|x(t)| - \int_{t-\tau}^t |x(s)| ds \right), w \right\rangle, \quad (16)$$

$$\begin{aligned} D^+V_3(t)|_{(4)} &= \mu^2 \left\langle |\tilde{A}_1^2| \left( \tau|x(t - \tau)| \right. \right. \\ &\quad \left. \left. - \int_{t-\tau}^t |x(s - \tau)| ds \right), w \right\rangle, \end{aligned} \quad (17)$$

$$D^+V_4(t)|_{(4)} = \mu^2 \langle \tau |\tilde{A}_1^2| (|x(t)| - |x(t - \tau)|), w \rangle. \quad (18)$$

Replacing  $D^+V_i(t)|_{(4)}$ ,  $i = 1, 2, 3, 4$  by the expressions found above, we get

$$D^+V(t)|_{(4)} \leq \langle T_1|x(t)|, w \rangle, \quad (19)$$

where

$$T_1 = (\tilde{A}_0 + \mu\tilde{A}_1)^* + \tau(|\mu| |\tilde{A}_1\tilde{A}_0| + \mu^2 |\tilde{A}_1^2|). \quad (20)$$

Now, suppose that  $T_1$  is the opposite of an  $M$ -matrix. Using properties of an  $M$ -matrix, we can find a vector  $\rho \in R_+^{*n}$ , i.e. with components  $\rho_k \in R_+^*$  satisfying the relation  $T^*w = -\rho$ ,  $\forall w \in R_+^{*n}$ .

Knowing that

$$\langle T_1|x(t)|, w \rangle = \langle T_1^*w, |x(t)| \rangle.$$

So, we can write

$$\langle T_1^*w, |x(t)| \rangle = \langle -\rho, |x(t)| \rangle.$$

Finally, we obtain

$$D^+V(t)|_{(4)} < - \sum_{k=1}^n \rho_k |x_k(t)| < 0.$$

Then, the system is asymptotically stable.  $\blacksquare$

### 3. Application to systems defined by delayed differential equations with a single delay

The delay systems considered in this section are governed by the linear differential–difference equation of the following form (Brierley, Chiasson, Lee, and Zak, 1982; Chiasson, 1988; Elmadssia et al., 2011; Hale et al., 1985; Kamen, 1980, 1982):

$$y^{(n)}(t) + \sum_{i=0}^{n-1} a_i y^{(i)}(t) + \mu \sum_{i=0}^{n-1} b_i y^{(i)}(t - \tau) = 0. \quad (21)$$

The presence of delay terms makes the stability study of Equation (21) very difficult. A solution is to use the following matrix representation:

$$\tilde{z}_{i+1}(t) = y^{(i)}(t) \quad (i = 0 \dots n - 1). \quad (22)$$

Equation (21) becomes

$$\begin{aligned} \dot{\tilde{z}}_i(t) &= \tilde{z}_{i+1}(t) \quad (i = 1, 2, \dots, n - 1), \\ \dot{\tilde{z}}_n(t) &= - \sum_{i=0}^{n-1} a_i \tilde{z}_{i+1}(t) - \mu \sum_{i=0}^{n-1} b_i \tilde{z}_{i+1}(t - \tau). \end{aligned} \quad (23)$$

Or under matrix form

$$\dot{\tilde{z}}(t) = A_0 \tilde{z}(t) + \mu A_1 \tilde{z}(t - \tau), \quad (24)$$

where  $\tilde{z}(t)$  is a vector of components  $\tilde{z}_i(t)$  ( $i = 1, 2, \dots, n$ ),  $A_0$  and  $A_1$  are given by

$$A_0 = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{pmatrix} \quad \text{and} \quad (25)$$

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \\ -b_0 & \cdots & -b_{n-1} \end{pmatrix}.$$



$b_{n-1} - b_{n-2}$ . Yields a new form of  $|\Delta F|$

$$|\Delta F| = \begin{pmatrix} O_{n-1,n-1} & O_{n-1,1} \\ |\sigma(\alpha_1)| \cdots |\sigma(\alpha_{n-1})| & |a_{n-1}b_{n-1} - b_{n-2}| \end{pmatrix}. \quad (42)$$

It is easy to calculate  $|\Delta^2|$

$$|\Delta^2| = \begin{pmatrix} O_{n-1,n-1} & O_{n-1,1} \\ |\delta_1||b_{n-1}| \cdots |\delta_{n-1}||b_{n-1}| & |b_{n-1}^2| \end{pmatrix}. \quad (43)$$

Taking into account the previous matrix value, we obtain the matrix  $T_1$  as follows:

$$T_1 = \begin{pmatrix} \alpha_1 & & & |\beta_1| \\ & \alpha_2 & & |\beta_2| \\ & & \ddots & \vdots \\ t_1 & t_2 & \cdots & t_{n-1} & |\beta_{n-1}| \\ & & & & t_n \end{pmatrix}, \quad (44)$$

where

$$t_i = |\gamma_i + \mu\delta_i| + \tau(|\mu|\sigma(\alpha_i)| + \mu^2|\delta_i b_{n-1}|) \quad (i = 1, 2, \dots, n-1), \quad (45)$$

$$t_n = \gamma_n - \mu b_{n-1} + \tau(|\mu||a_{n-1}b_{n-1} - b_{n-2}| + \mu^2 b_{n-1}^2). \quad (46)$$

Since the elements  $\alpha_i$  ( $i = 1, 2, \dots, n-1$ ) can be arbitrarily selected, according to Kotelyanski lemma (Gentina et al., 1972), the choice  $\alpha_i < 0$  with  $\alpha_i \neq \alpha_j$ ,  $\forall(i, j = 1, \dots, n-1)$ , permits to reduce checking that  $T_1$  is the opposite of an  $M$ -matrix to determining a simple sign test which is easier to perform.

$$\det(T_1) = \left( t_n - \sum_{i=1}^{n-1} \frac{t_i |\beta_i|}{\alpha_i} \right) \prod_{i=1}^{n-1} \alpha_i. \quad (47)$$

The exploitation of results developed in Elmadssia et al. (2011) permits us to determine a simple sufficient condition for delayed linear system (21). In fact, the condition:  $T_1$  is the opposite of an  $M$ -matrix, becomes  $t_n - \sum_{i=1}^{n-1} (t_i |\beta_i| / \alpha_i) < 0$ . This achieves the proof of theorem. ■

*Remark 2* The choice of the parameters  $\alpha_i$  plays an important role. Indeed, the delay can be considered as function of  $\alpha_i$ 's, so it can be varied until the maximum value of delay for which stability holds is found.

On the other hand, a necessary condition for having  $T_1$  the opposite of an  $M$ -matrix is that  $\text{trace}(T_1) < 0$ , which yields  $\sum_{i=1}^{n-1} \alpha_i + \gamma_n - \mu b_{n-1} + \tau(|\mu||a_{n-1}b_{n-1} - b_{n-2}| + \mu^2 b_{n-1}^2) < 0$ . Taking into account the value of  $\gamma_n$  given by Equation (35), the last inequality becomes  $-a_{n-1} - \mu b_{n-1} + \tau(|\mu||a_{n-1}b_{n-1} - b_{n-2}| + \mu^2 b_{n-1}^2) < 0$ .

Then, a necessary condition that the time delay  $\tau$  must satisfy is

$$\tau < \frac{a_{n-1} + \mu b_{n-1}}{|\mu||a_{n-1}b_{n-1} - b_{n-2}| + \mu^2 b_{n-1}^2} := \tau_{\max}^1(\mu). \quad (48)$$

For the particular case,  $n = 1$  and  $\mu = 1$ , the condition (69) is given by Niculescu et al. (1997) as a necessary and sufficient condition for stability.

Other results can be determined by varying the choice of parameters  $\alpha_i$  ( $i = 1, 2, \dots, n-1$ ). A particular choice of these  $\alpha_i$ 's can widely simplify conditions of Theorem 1. This is given in the following corollary.

**COROLLARY 1** *Let  $\mu = 1$ , if there exist  $\alpha_i < 0$  ( $i = 1, 2, \dots, n-1$ ), satisfying the following conditions:*

$$\mathcal{B}(V_0 + V_1) > 0,$$

$$b_{n-1} \mathcal{B}V_1 > 0,$$

$$\mathcal{B}(-\mathcal{A}V_1 + b_{n-1}V_0) > 0$$

are satisfied.

Where matrices  $\mathcal{A}, \mathcal{B} \in R^{(n-1) \times (n-1)}$  and  $V_i \in R^{n-1}$ ,  $i = 0, 1$ , are such as

$$\mathcal{B} = \begin{bmatrix} \beta_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \beta_{n-1} \end{bmatrix},$$

$$\mathcal{A} = \begin{bmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_{n-1} \end{bmatrix}, \quad (49)$$

$$V_i = V(p_{A_i}) = \begin{bmatrix} p_{A_i}(\alpha_1) \\ p_{A_i}(\alpha_2) \\ \vdots \\ p_{A_i}(\alpha_{n-1}) \end{bmatrix}, \quad i = 0, 1 \quad (50)$$

Table 1. Delay-dependent stability conditions.

Assumptions	Delay conditions
$a_{n-1}b_{n-1} > b_{n-2}$	$\tau < \min\left(\frac{1}{b_{n-1}}, \tau_{\max}^1(1)\right)$
$a_{n-1}b_{n-1} < b_{n-2}$	$\tau < \min\left(\frac{p_{A_0}(0) + p_{A_1}(0)}{2Q(0)(b_{n-2} - a_{n-1}b_{n-1}) + b_{n-1}(p_{A_0}(0) + p_{A_1}(0))}, \tau_{\max}^1(1)\right)$

then system (21) is asymptotically stable if the time delay satisfies the constraints in Table 1.

#### 4. Delay-dependent stability conditions for linear systems with multiple delays

In this section, a generalization of the results given in the previous section to systems with multiple delay elements is given. Consider the differential–difference equation

$$\dot{x}(t) = A_0x(t) + \mu \sum_{k=1}^m A_kx(t - \tau_k), \quad (51)$$

where  $A_k \in R^{n \times n}$  ( $i = 0, 1, \dots, m$ ). By a particular choice of a radially unbounded Lyapunov function, the following sufficient stability condition is obtained.

**THEOREM 3** *The system (51) is asymptotically stable if the matrix  $T_m$ , given by*

$$T_m = \left( A_0 + \mu \sum_{k=1}^m A_k \right)^* + \sum_{k=1}^m \tau_k (|\mu| |A_0 A_k| + \mu^2 |A_k^2|) \quad (52)$$

is the opposite of an  $M$ -matrix.

*Proof* Let  $w \in R^n$  with components  $w_i > 0$  ( $i = 1, \dots, n$ ) and let us consider the radially unbound Lyapunov functional given by

$$V_G(t) = V_{G_1}(t) + V_{G_2}(t) + V_{G_3}(t) + V_{G_4}(t), \quad (53)$$

where

$$V_{G_1}(t) = \langle |x(t)|, w \rangle, \quad (54)$$

$$V_{G_2}(t) = |\mu| \sum_{k=1}^m \left\langle |A_0 A_k| \int_{-\tau_k}^0 \int_{t+\theta}^t |x(s)| ds d\theta, w \right\rangle, \quad (55)$$

$$V_{G_3}(t) = \mu^2 \sum_{k=1}^m \left\langle |A_k^2| \int_{-\tau_k}^0 \int_{t+\theta}^t |x(s - \tau)| ds d\theta, w \right\rangle \quad (56)$$

and

$$V_{G_4}(t) = \mu^2 \sum_{k=1}^m \tau_k \left\langle |A_k^2| \int_{t-\tau_k}^t |x(s)| ds, w \right\rangle. \quad (57)$$

Using the same steps as those given in the proof of theorem 1 leads to the following condition:

$$D^+ V_G(t) = \langle T_m x, w \rangle < 0. \quad (58)$$

Finally, we conclude that the system is asymptotically stable if  $T_m$  is the opposite of an  $M$ -matrix. ■

We now determine delay-dependent stability conditions for systems having the following form:

$$\tilde{S} : \begin{cases} y^{(n)}(t) + \sum_{i=0}^{n-1} a_i y^{(i)}(t) \\ + \mu \sum_{k=1}^m \sum_{i=0}^{n-1} b_{i,k} y^{(i)}(t - \tau_k) = 0, \\ y^{(i)}(\theta) = \phi_i(\theta), \forall \theta \in \left[ -\max_{1 \leq k \leq m} \tau_k, 0 \right] \\ (i = 0, \dots, n - 1). \end{cases} \quad (59)$$

Next,  $\forall k = 1, 2, \dots, m$ , we define the new polynomial as follows:

$$p_{A_k}(s) = \sum_{i=0}^{n-1} b_{i,k} s^i$$

and

$$\sigma_{A_k}(s) = -p_{A_k}(s)s + b_{n-1,k} p_{A_0}(s) = \sum_{i=0}^{n-1} \sigma_{i,k} s^i, \quad (60)$$

the parameters  $\sigma_{i,k}$  ( $i = 0, \dots, n - 1$ ) are given by

$$\begin{aligned} \sigma_{0,k} &= b_{n-1,k} a_0, \\ \sigma_{i,k} &= b_{n-1,k} a_i - b_{i-1,k} \quad (i = 1, 2, \dots, n - 1). \end{aligned}$$

The same idea is used to determine a delay-dependent stability condition in the general case. After making the following change of variables:

$$\tilde{z}_{i+1}(t) = y^{(i)}(t) \quad (i = 0 \dots n - 1), \quad (61)$$

we get the new state space representation

$$\dot{\tilde{z}}(t) = \tilde{A}_0 \tilde{z}(t) + \mu \sum_{k=1}^m \tilde{A}_k \tilde{z}(t - \tau_k), \quad (62)$$

where  $\tilde{z}(t)$  is a vector of components  $\tilde{z}_i(t)$  ( $i = 1 \dots n$ ),  $A_0$  and  $A_k$  ( $k = 1, \dots, m$ ), are given by

$$\begin{aligned} \tilde{A}_0 &= \begin{pmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{pmatrix} \quad \text{and} \\ \tilde{A}_k &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & \dots & 0 \\ -b_{0,k} & \dots & -b_{n-1,k} \end{pmatrix}. \end{aligned} \quad (63)$$

The same variable change given by Equation (30) permits us to obtain the new state space representation given

by

$$\dot{z}(t) = Fz(t) + \mu \sum_{k=1}^m \Delta_k z(t - \tau), \quad (64)$$

where  $F$  is given by Equation (33) and the matrices  $\Delta_k$  ( $k = 1, \dots, m$ ) are given by

$$\Delta_k := P^{-1} A_k P = \begin{pmatrix} O_{n-1, n-1} & O_{n-1, 1} \\ \delta_{1, k} \cdots \delta_{n-1, k} & \delta_{n, k} \end{pmatrix} \quad (65)$$

with

$$\delta_{i, k} = -p_{A_k}(\alpha_i) \quad (i = 1, \dots, n-1) \quad (66)$$

and

$$\delta_{n, k} = -b_{n-1, k}. \quad (67)$$

Based on the result of Theorem 2, we can deduce the following delay-dependent stability condition for the system defined by Equation (59).

**COROLLARY 2** *If there exist  $\alpha_i < 0$  ( $i = 1, 2, \dots, n-1$ ) satisfying the inequality (68)*

$$\max_{1 \leq k \leq m} \tau_k < \frac{-\gamma_n + \sum_{k=1}^m \mu b_{n-1, k} + \sum_{k=1}^m \sum_{i=1}^{n-1} \alpha_i^{-1} |p_{A_0}(\alpha_i) + \mu p_{A_k}(\alpha_i)| |\beta_i|}{\sum_{k=1}^m ((\mu^2 b_{n-1, k}^2 + |\mu| |a_{n-1} b_{n-1, k} - b_{n-2, k}|) - \sum_{i=1}^{n-1} \alpha_i^{-1} (|\mu| |\sigma_{A_k}(\alpha_i)| + \mu^2 |p_{A_k}(\alpha_i) b_{n-1, k}|) |\beta_i|)} \quad (68)$$

then, system (59) is asymptotically stable.

**Remark 3** A necessary condition for having

$$T_m := \left( F + \mu \sum_{k=1}^m \Delta_k \right)^* + \sum_{k=1}^m \tau_k (|\mu| |F \Delta_k| + \mu^2 |\Delta_k^2|)$$

the opposite of an  $M$ -matrix is that

$$\text{trace} \left( \left( F + \mu \sum_{k=1}^m \Delta_k \right)^* + \sum_{k=1}^m \tau_k (|\mu| |F \Delta_k| + \mu^2 |\Delta_k^2|) \right) < 0,$$

which yields

$$\sum_{i=1}^{n-1} \alpha_i + \gamma_n - \mu \sum_{k=1}^m b_{n-1, k} + \sum_{k=1}^m \tau_k (|\mu| |a_{n-1} b_{n-1, k} - b_{n-2, k}| + \mu^2 b_{n-1, k}^2) < 0.$$

The last inequality becomes

$$-a_{n-1} - \mu \sum_{k=1}^m b_{n-1, k} + \max_{1 \leq k \leq m} \tau_k \sum_{k=1}^m (|\mu| |a_{n-1} b_{n-1, k} - b_{n-2, k}| + \mu^2 b_{n-1, k}^2) < 0.$$

Then, a necessary condition that the time delay  $\tau$  must satisfy is

$$\max_{1 \leq k \leq m} \tau_k < \frac{a_{n-1} + \mu \sum_{k=1}^m b_{n-1, k}}{|\mu| \sum_{k=1}^m |a_{n-1} b_{n-1, k} - b_{n-2, k}| + \mu^2 b_{n-1, k}^2} := \tau_{\max}^m(\mu). \quad (69)$$

For the particular case  $n = 1$  and  $\mu = 1$ , the condition (69) is given by Niculescu et al. (1997) as a necessary and sufficient condition for stability.

A particular choice of these  $\alpha_i$ 's can widely simplify the theorem's condition. This is given by the next corollary.

**COROLLARY 3** *Let  $\mu = 1$ , if there exist  $\alpha_i < 0$  ( $i = 1, 2, \dots, n-1$ ), satisfying the following conditions:*

$$\begin{aligned} \mathcal{B} \left( V_0 + \sum_{k=1}^m V_k \right) &> 0, \\ b_{n-1, k} \mathcal{B} V_k &> 0 \quad (k = 1, \dots, m), \\ \mathcal{B} (-\mathcal{A} V_k + b_{n-1, k} V_0) &> 0 \quad (k = 1, \dots, m), \end{aligned}$$

where  $V_k \in \mathbb{R}^{n-1}$  ( $k = 0, \dots, m$ ) are such as

$$V_k = V(p_{A_k}) = \begin{bmatrix} p_{A_k}(\alpha_1) \\ p_{A_k}(\alpha_2) \\ \vdots \\ p_{A_k}(\alpha_{n-1}) \end{bmatrix}, \quad (k = 1, \dots, m) \quad (70)$$

then system (59) is asymptotically stable if the time delay satisfies the constraints in Table 2.

## 5. Examples

Let us consider the system defined by the differential equation given by

$$\ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) + b_1 \dot{y}(t - \tau) + b_0 y(t - \tau) = 0.$$

By applying the method of Section 3, we get the following state representation:

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 \\ -b_0 & -b_1 \end{pmatrix} x(t - \tau).$$

After transformation of the matrix to Benrejeb's characteristic arrow form matrix, we obtain the new representation

$$\begin{aligned} \dot{z}(t) &= \begin{pmatrix} \alpha & 1 \\ -p_{A_0}(\alpha) & -a_1 - \alpha \end{pmatrix} z(t) \\ &+ \begin{pmatrix} 0 & 0 \\ -p_{A_1}(\alpha) & -b_1 \end{pmatrix} z(t - \tau). \end{aligned}$$



Table 2. Maximum delay value for stability condition.

Assumptions $\forall k = 1, 2, \dots, m$	Delay conditions $\max_{1 \leq k \leq m} \tau_k <$
$a_{n-1}b_{n-1,k} > b_{n-2,k}$	$\min \left( \frac{p_{A_0}(0) + \sum_{k=1}^m p_{A_k}(0)}{\sum_{k=1}^m b_{n-1,k}(p_{A_0}(0) + p_{A_k}(0))}, \tau_{\max}^m(1) \right)$
$a_{n-1}b_{n-1,k} < b_{n-2,k}$	$\min \left( \frac{p_{A_0}(0) + \sum_{k=1}^m p_{A_k}(0)}{\sum_{k=1}^m (2Q(0)(b_{n-2,k} - a_{n-1}b_{n-1,k}) + b_{n-1,k}(p_{A_0}(0) + p_{A_k}(0)))}, \tau_{\max}^m(1) \right)$

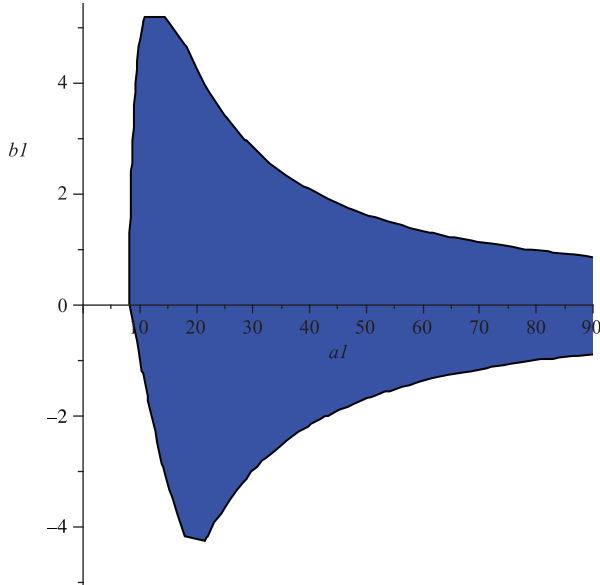


Figure 1. Parameters domains of  $(a_1, b_1)$  stabilizing for  $\tau = 0.1s$ .

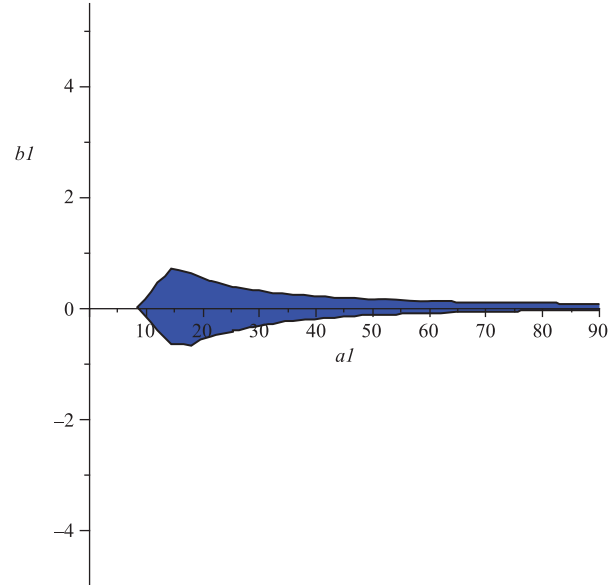


Figure 2. Parameters domains of  $(a_1, b_1)$  stabilizing for  $\tau = 1s$ .

By applying the result of Theorem 1, the following delay-dependent stability condition is determined:

$$\begin{aligned}
 & -a_1 - \alpha - b_1 - \alpha^{-1}|p_{A_0}(\alpha) + p_{A_1}(\alpha)| + \tau(b_1^2 \\
 & + |a_1b_1 - b_0| - \alpha^{-1}(|\sigma(\alpha)| + |p_{A_1}(\alpha)b_1|)) < 0.
 \end{aligned}$$

Suppose that the parameters of our system are given by  $b_0 = -1.5, a_0 = 16$  and  $a_1$  and  $b_1$  are to be determined. Choosing  $\alpha = -1$  and using the result of Theorem 2, we can determine the stability domains of the parameters  $(a_1, b_1)$  for different values of delay  $\tau$ .

We can clearly see from Figures 1–3 that stability domain depends on the value of time delay. In fact, as delay increases the stability domain will decrease. This can be explained by condition (40) of Theorem 2.

*Remark 4* As the given conditions, in the proposed approaches, are dependent on the parameters  $\alpha_i, i = 1, \dots, n - 1$ , we can determine the optimal values of the parameters  $\alpha_i$ 's that minimize or maximize stability domain.

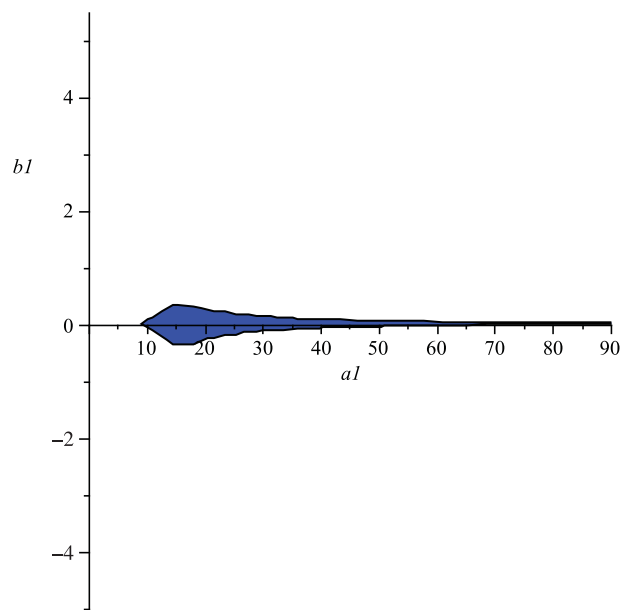


Figure 3. Parameters domains of  $(a_1, b_1)$  stabilizing for  $\tau = 2s$ .

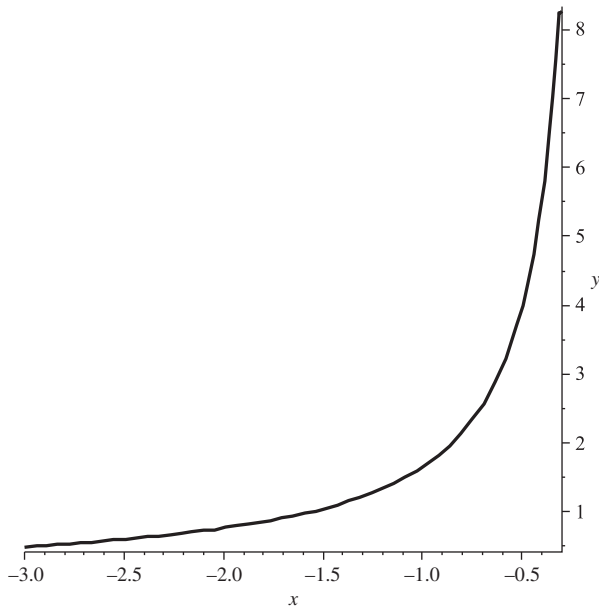


Figure 4. Value of  $\tau$  as function of  $\alpha$  for  $a_0 = 16$ ,  $b_0 = -1.5$ ,  $a_1 = 51$  and  $b_1 = 0.1$ .

In Figure 4, we plot delay as a function of the parameter  $\alpha$ . We can remark from this figure that our system remains stable for values of  $\tau$  greater than 8s for the particular choice of  $\alpha$ .

## 6. Conclusion

In this paper, we proposed some delay-dependent stability conditions for linear systems with delay. The improvement of the proposed conditions shows itself at the level of their representation as an explicit form according to the delay and the parameters of the system. This allows drawing the stability domains of the system's parameters as a function of delay. Besides, the proposed basic change permits us to find a relation between the parameters arbitrarily chosen and the upper bound of the delay which ensures stability. The proposed approaches in this article can be applied to fuzzy TSK systems with delay and they can generalize the work of Benrejeb and Abdelkrim (2003) and Benrejeb, Sakly, Ben Othmana, and Borne (2008) in the case of time-delay systems.

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