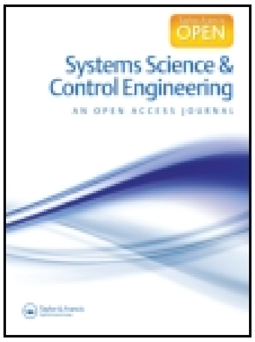

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Stability analysis for closed-loop management of a reservoir based on identification of reduced-order nonlinear model

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This objective of this study is to analyze the stability of an oil producing reservoir under closed-loop control. Given a five-spot pattern reservoir as an example, a nonlinear reduced-order model is identified and an asymptotically stabilizing controller is proposed based on the circle criterion.

Keywords: reservoir model; stability analysis; NARX.

1. Introduction

Oil consumption worldwide has experienced dramatic increases due to unprecedented growth in more than one emerging country. As a consequence, the management of oil production is becoming increasingly complex. An oil producing reservoir system is difficult to manage because of the unknown and complex geology (Gu & Olive, 2004; Jensen, 2007; Nævdal, Mannseth, & Vefring, 2002; Nævdal, Johnsen, Aanonsen, & Vefring, 2003). Several studies have appeared that uses a closed-loop optimal control framework to manage the reservoir production. However, none of these studies consider the issue of closed-loop stability when a complex nonlinear system is in a feedback loop with a model-based optimal controller. One of the most important reasons is the nonlinear behavior of the oil reservoir system, which confounds the stability analysis of the oil production management system.

Many implementation studies of the design of the model-based, optimal controllers reveal that it is reasonable to develop a regulator based on a first-principles, mechanistic model that describes the process. Here, the first-principles model of an oil production process is nonlinear, of high dimension, and is represented by a system of partial differential algebraic equations (Nævdal, Brouwer, & Jansen, 2006; Sarma, Aziz, & Durlofsky, 2005; Sarma, Durlofsky, & Aziz, 2005). Intuitively, these features detract from efficient real-time applications. Many studies have overcome this limitation, by replacing the first-principles, mechanistic model with a reduced-order model (ROM), identified from real and simulated data (taken from the mechanistic model) (Chen Hoo, 2011a, 2011b; van Doren, Markovinović, and Jansen, 2006; Sarma *et al.*, 2006).

One advantage of a nonlinear reduced-order model (NLROM) is that it can represent the primary nonlinear dynamic behavior to some prescribed satisfactory degree. Thus, applications that involve the analysis of a model, such as closed-loop model-based control, become tractable without ignoring the nonlinear behavior of the original system. In this work, a NLROM will be identified. The stability of the nonlinear process in closed-loop with a nonlinear model-based regulator should be guaranteed for the known operating modes.

Stabilization of nonlinear dynamic systems has been studied widely (Kurtz & Henson, 1998; Michalska & Mayne, 1993; Wu, He, Liu, & She, 2005). However, because of the existence of both model and parameter uncertainties, the management of an oil reservoir focuses on reducing the uncertainty between the current data and historical data (referred to as history matching) without the consideration of the objectives of the controller or the stability of the closed-loop management structure (Brouwer, 2004; Brouwer, Nævdal, Jansen, Vefring, & van Kruijsdijk, 2004; Chen & Hoo, 2012). Section 2 introduces an oil producing reservoir. The highly nonlinear first-principles model is provided. An NLROM is identified, based on the first-principles model, for model-based optimal controller design and for closed-loop stability analysis. Next, Section 3 will introduce the circle criterion used to establish closed-loop stability. Section 4 analyzes the closed-loop stability of the reservoir as represented by the NLROM. Finally, Section 5 summarizes the contributions of the study.

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2. Reservoir and NLROM identification

2.1. Reservoir

Consider the following two-dimensional and two-phase flow (oil and water) porous media oil producing reservoir system as depicted in Figure 1.

To represent the reservoir, the ubiquitous five-spot pattern is used (Latil, 1980). The water injection wells are located at the four corners of the reservoir and the oil well is located in the middle of the reservoir. The region is modeled by a $9 \times 9 \times 1$ horizontal two-dimensional grid blocks. The fluid system consists of two phases, with 0.1 connate water saturation and 0.3 residual oil saturation. Following Giles (2008), the porosity and permeability distributions are in the intervals $[0.04, 0.33]$ and $[0.006, 0.6] \times 10^{-13}\ \mathrm{m}^2$ ($[6, 600]$ milliDarcy), respectively. The relative permeability curve used to solve the reservoir model is shown in Figure 2.

The following mathematical representation is a first-principles model for the above two-dimensional reservoir based on the assumptions of immiscible porous media flow,

$$\begin{aligned} \nabla\left(\frac{k_{\mathrm{ro}}k}{\mu_{\mathrm{o}}B_{\mathrm{o}}}\nabla p_{\mathrm{o}}\right) &= \frac{\partial}{\partial t}\left(\frac{\phi S_{\mathrm{o}}}{B_{\mathrm{o}}}\right) + q_{\mathrm{o}}, \\ \nabla\left(\frac{k_{\mathrm{rw}}k}{\mu_{\mathrm{w}}B_{\mathrm{w}}}\nabla p_{\mathrm{w}}\right) &= \frac{\partial}{\partial t}\left(\frac{\phi S_{\mathrm{w}}}{B_{\mathrm{w}}}\right) + q_{\mathrm{w}}, \\ S_{\mathrm{o}} + S_{\mathrm{w}} &= 1, \\ p_{\mathrm{o}} - p_{\mathrm{w}} &= P_{\mathrm{c}}(S_{\mathrm{w}}), \end{aligned} \tag{1}$$

where subscripts o,w stand for oil and water, respectively. The definition of the variables and parameters can be found in Table 1.

2.2. Identification of a NLROM

The use of a nonlinear and high-dimension mechanistic model constrains real-time applications because of the potential lengthy computational burden. A means to overcome this limitation is to identify a suitable low-order model using system identification methods. In this example, a nonlinear autoregressive exogenous (NARX) model (De Nicolao, Magni, & Scattolini, 1997; Zhang & Ljung, 2004), which is a type of NLROM is identified.

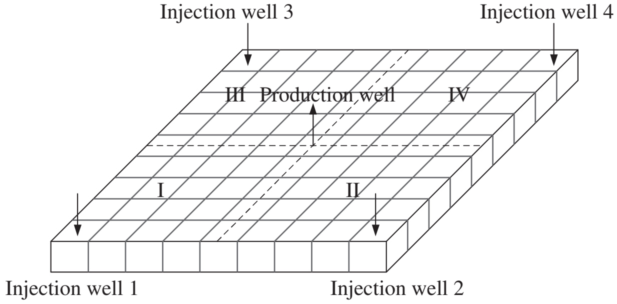

Figure 1. Schematic of a two-dimensional reservoir and wells. ↓, water injection well; ↑, oil production well.

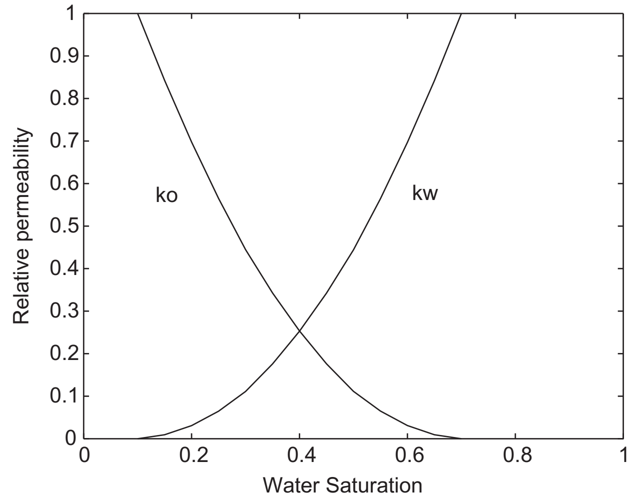

Figure 2. Relative permeability curve.

Table 1. Definition of reservoir variables and parameters.

Variable	Definition	Unit
∇	Gradient operator	
k	Absolute permeability	m^2
k_{r}	Relative permeability	–
μ	Viscosity	Pa·s
B	Formation volume factor	RB/STB (real barrels/standard barrels)
p	Pressure	Pa
P_{c}	Capillary pressure	Pa
ϕ	Porosity	%
q_{o}	Oil production rate	m^3/day
q_{w}	Water injection rate	m^3/day
S	Fluid phase saturation	–
S_{o}	Oil saturation	%
S_{w}	Water saturation	%
t	Time	day

The form of the NARX is given by The MATH WORKS, Inc.,

$$\begin{aligned} F(\boldsymbol{X}) = (\boldsymbol{X}-\boldsymbol{r})\boldsymbol{Pl} + \sum_i \boldsymbol{a}_i^{\mathrm{s}} f(\boldsymbol{b}_i^{\mathrm{s}}((\boldsymbol{X}-\boldsymbol{r})\boldsymbol{Q}-\boldsymbol{c}_i^{\mathrm{s}})) \\ + \sum_j \boldsymbol{a}_j^{\mathrm{w}} g(\boldsymbol{b}_j^{\mathrm{w}}((\boldsymbol{X}-\boldsymbol{r})\boldsymbol{Q}-\boldsymbol{c}_j^{\mathrm{w}})) + d, \end{aligned} \tag{2}$$

where $\boldsymbol{X}$ is a $1 \times m$ vector containing regressors; f is a scaling function, $\boldsymbol{z}$ is a $1 \times q$ row vector

$$f(\boldsymbol{z}) = \exp(-\tfrac{1}{2}\boldsymbol{z}\boldsymbol{z}^{\mathrm{T}})$$

and g is a wavelet function

$$g(\boldsymbol{z}) = (q - \boldsymbol{z}\boldsymbol{z}^{\mathrm{T}})\exp(-\tfrac{1}{2}\boldsymbol{z}\boldsymbol{z}^{\mathrm{T}}).$$

The other variable definitions and sizes are listed in Table 2.

Assume there is a pseudo-oil producing well located in each of the four parts of the well (Figure 1). Let the true oil

Table 2. Definition of NARX variables.

Variable	Definition	Size
$\boldsymbol{r}$	Regressor mean	$1 \times m$
$\boldsymbol{P}$	Linear subspace	$m \times p$
$\boldsymbol{l}$	Linear coefficients	$p \times l$
$\boldsymbol{Q}$	Nonlinear subspace	$m \times q$
$\boldsymbol{a}^{\mathrm{s}}$	Scaling coefficients	$n_{\mathrm{S}} \times 1$
$\boldsymbol{b}^{\mathrm{s}}$	Scaling dilation	$n_{\mathrm{S}} \times 1$
$\boldsymbol{c}^{\mathrm{s}}$	Scaling translation	$n_{\mathrm{S}} \times q$
$\boldsymbol{a}^{\mathrm{w}}$	Wavelet coefficients	$n_{\mathrm{W}} \times 1$
$\boldsymbol{b}^{\mathrm{w}}$	Wavelet dilation	$n_{\mathrm{W}} \times 1$
$\boldsymbol{c}^{\mathrm{w}}$	Wavelet translation	$n_{\mathrm{W}} \times q$
d	Output offset	Scalar

production be represented by a summation of what is produced by the four-pseudo wells. How much water to inject is the decision variable determined by the optimizer. Using system identification methods, an NARX model is identified from data taken from the simulation of the mechanistic model for each part of the reservoir. There is one decision variable and one control variable associated with each NARX model. In the NARX reservoir model (2), $F(\boldsymbol{X})(k)$ is the amount of oil produced by the addition of water. The vector $\boldsymbol{X}$ consists of the regressors, including oil and water rates at time $(k-1)$, that is the pair, $\{\boldsymbol{x}(k-1), \boldsymbol{u}(k-1)\}$. In this model, the water injection flow rates to the four water wells are the manipulated variables, while the oil production rates of the four pseudo-oil wells are the model's outputs.

The length of the simulation covers 30 time steps. The step interval used to calculate the production rate is 10 days. Figure 3 is the water cut (ratio of water production to the sum of water and oil production from a well) from the production well. After 300 days, only 14% of the product is oil. The top graph in Figure 4 compares the oil production rate of each part of the reservoir between the predictions of the first-principles model and the NARX model. The bottom graph of Figure 4 compares the total oil production rates. From this figure, it can be observed that the oil production rate from the identified model is a satisfactory fit to that of the results from the first-principles model. The maximum error is within 20% and the average error is 1.8%.

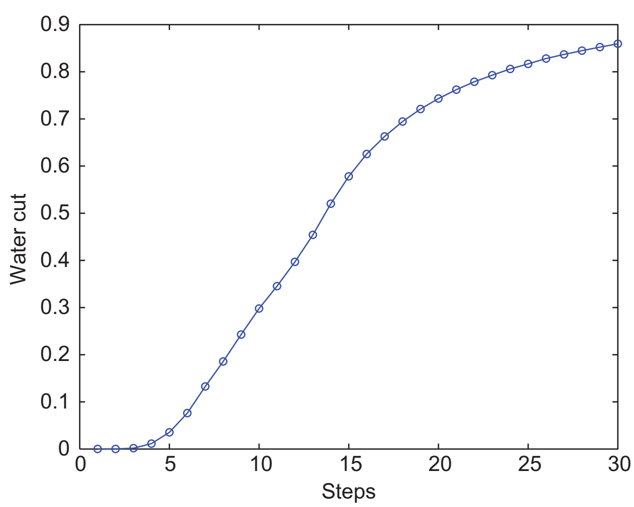

Figure 3. Water cut from the production well.

To check the accuracy of the identified NARX model to that of the first-principles model ±10% change in the water injection flow rates are made. Figure 5 shows the comparison. The average errors between the results of the first-principles model and the NARX model are 1.18% for a 10% decrease and 1.92% for 10% increase in the water injection flow rates.

3. Circle criterion for stability of closed-loop systems

Consider a nonlinear system described by

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}), \\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u} \end{aligned} \tag{3}$$

and its related linear form,

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}, \\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}, \end{aligned} \tag{4}$$

where $\boldsymbol{x}$ and $\boldsymbol{u}$ are the functions of $t, t \in \mathbb{R}^+$, $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{u} \in \mathbb{R}^p$, $\boldsymbol{y} \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$, $\boldsymbol{B} \in \mathbb{R}^{n\times p}$, $\boldsymbol{C} \in \mathbb{R}^{m\times n}$, $\boldsymbol{D} \in \mathbb{R}^{m\times P}$. The system is assumed to be controllable with $\boldsymbol{A}$ Hurwitz and the pair $(\boldsymbol{A}, \boldsymbol{C})$ assumed to be observable. The function $\boldsymbol{F}$ $(\mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n)$ represents the nonlinearity of the system.

3.1. *Definitions*

Definition 1 (Benabdallah & Hammami, 2006) *A nonlinearity $\boldsymbol{\varphi}$: $\mathbb{R}^+ \times \mathbb{R}^m \to \mathbb{R}^m$ is said to belong to a closed section $[\boldsymbol{0}, \boldsymbol{K}]$ if*

$$\boldsymbol{\varphi}(\boldsymbol{y})^{\mathrm{T}}[\boldsymbol{\varphi}(\boldsymbol{y}) - \boldsymbol{K}\boldsymbol{y}] \leq 0, \quad \forall t \geq 0, \forall \boldsymbol{y} \in \mathbb{R}^m$$

for some symmetric positive-definite matrix $\boldsymbol{K}$.

Definition 2 *A $(p \times p)$ matrix $\boldsymbol{Z}(s)$ composed of functions of complex variables s is called positive real if*

- *$\boldsymbol{Z}(s)$ has elements that are analytic for* $\mathrm{Re}[s] > 0$,
- *$\boldsymbol{Z}(s)$ is Hermitian, $\boldsymbol{Z}^*(s) = \boldsymbol{Z}(s^*)$, for* $\mathrm{Re}[s] > 0$, *and*
- *$\boldsymbol{Z}^{\mathrm{T}}(s^*) + \boldsymbol{Z}(s)$ is positive semi-definite for* $\mathrm{Re}[s] > 0$,

where $*$ denotes complex conjugation.

The matrix $\boldsymbol{Z}(s)$ is called strictly positive real if $\boldsymbol{Z}(s-\varepsilon)$ is positive real for some $\varepsilon > 0$.

For closed-loop control, the following controller is proposed,

$$\boldsymbol{u}(\boldsymbol{y}(t)) = -\boldsymbol{\phi}(\boldsymbol{y}(t)) = -\boldsymbol{\varphi}(\boldsymbol{y}(t)) - \boldsymbol{v}(\boldsymbol{y}(t)), \tag{5}$$

where vector function $\boldsymbol{\varphi}(\boldsymbol{y})$ is a k-Lipschitz function $(\|\boldsymbol{\varphi}(\boldsymbol{y}) - \boldsymbol{\varphi}(\boldsymbol{z})\| \leq k\|\boldsymbol{y} - \boldsymbol{z}\|, \forall t \geq 0, \forall \boldsymbol{y}, \forall \boldsymbol{z})$ which belongs

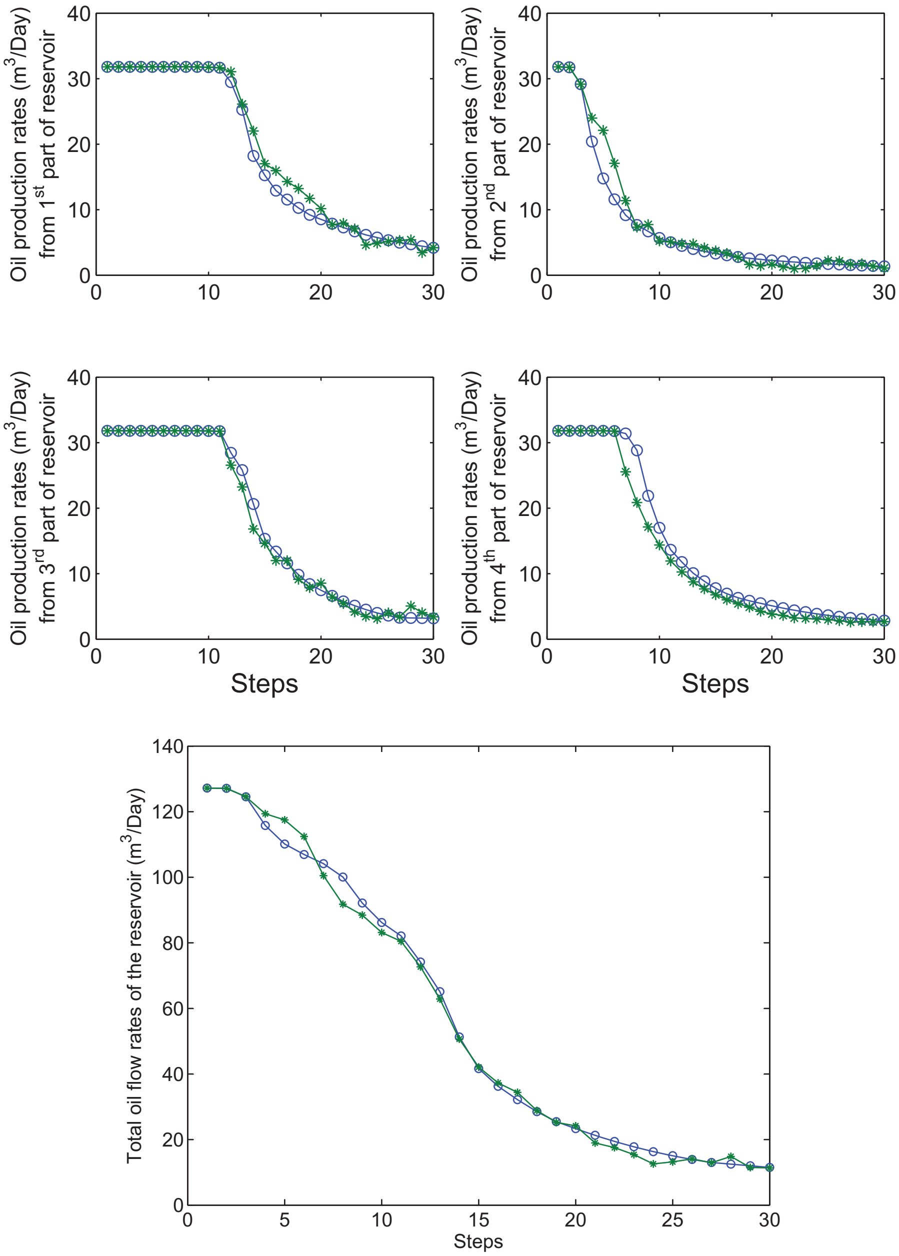

Figure 4. Comparison of oil production flow rates between first-principles model and NARX. ∘, results from first-principles model; ∗, results from NARX.

to a section $[\mathbf{0}, \boldsymbol{K}]$, $\boldsymbol{K}$ is a symmetric positive-definite matrix, and $\boldsymbol{v}(\boldsymbol{y})$ is a nonlinear vector function of $\boldsymbol{y}$ whose proposed form is given by

$$\boldsymbol{v}(\boldsymbol{y}) = \alpha(\boldsymbol{y})\boldsymbol{K}^{-1}\boldsymbol{y}, \tag{6}$$

with $\alpha(\boldsymbol{y})$, a function to be selected to stabilize the closed-loop system given in Equation (3).

3.2. *Theorem*

To investigate the stability of the nonlinear closed-loop system given in Equation (3), the following quadratic Lyapunov function is proposed,

$$V(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}, \quad \boldsymbol{P} = \boldsymbol{P}^{\mathrm{T}} > \mathbf{0},$$

where $\boldsymbol{P}$ is a symmetric positive-definite matrix.

The circle criterion design depends on the Kalman–Yakubovich–Popov lemma (Khalil, 1996).

Lemma 1 *Let* $\boldsymbol{Z}(s) = \boldsymbol{I} + \boldsymbol{K}C(s\boldsymbol{I} + \boldsymbol{A})^{-1}\boldsymbol{B}$ *be a* $p \times p$ *transfer function matrix where* $\boldsymbol{A}$ *is Hurwitz* $(\boldsymbol{A}, \boldsymbol{B})$ *is controllable, and* $(\boldsymbol{A}, \boldsymbol{C})$ *is observable. Then,* $\boldsymbol{Z}(\cdot)$ *is strictly positive real if and only if there exist a symmetric positive*

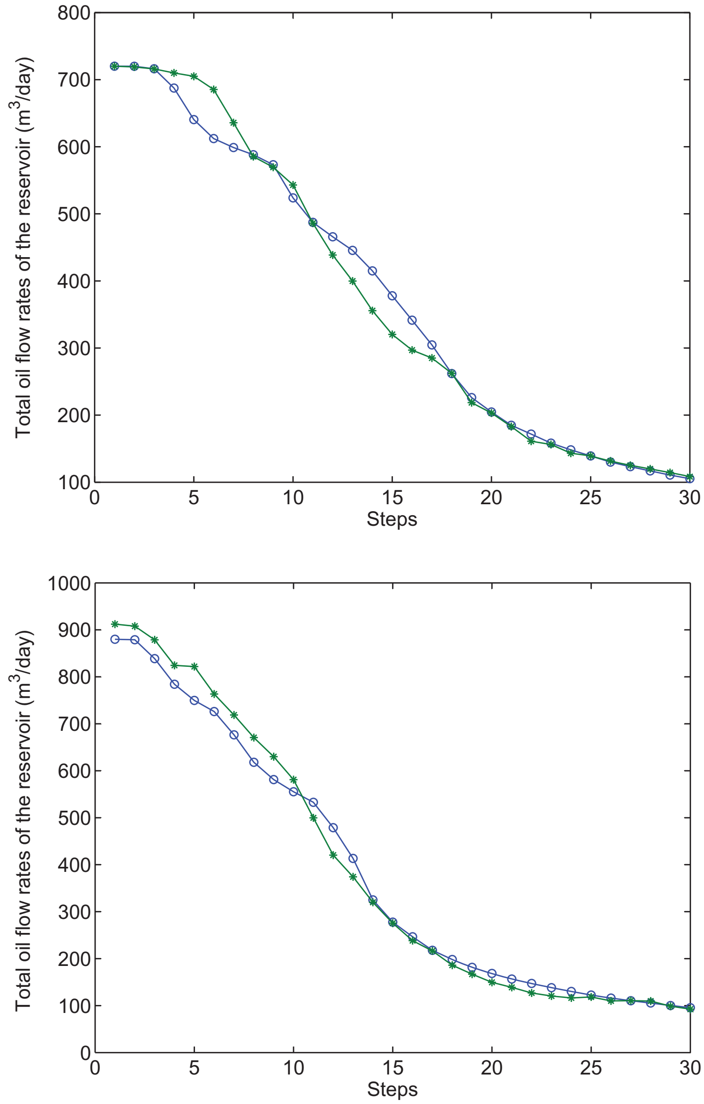

Figure 5. Comparison of oil production flow rates between first-principles model and NARX. Top: decrease 10% of injected water. Bottom: increase 10% of injected water. ∘, results from first-principles model; ∗, results from NARX.

matrix $\boldsymbol{P}$, *a matrix* $\boldsymbol{L}$, *and a positive constant* ε *such that*

$$\begin{aligned} \boldsymbol{PA} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} &= -\boldsymbol{L}^{\mathrm{T}}\boldsymbol{L} - \varepsilon \boldsymbol{P}, \\ \boldsymbol{PB} &= \boldsymbol{C}^{\mathrm{T}}\boldsymbol{K} - \sqrt{2}\boldsymbol{L}^{\mathrm{T}}. \end{aligned} \tag{7}$$

(A1) The $p \times p$ matrix $\boldsymbol{Z}(s)$ defined by

$$\boldsymbol{Z}(s) = \boldsymbol{I} + \boldsymbol{K}C(s\boldsymbol{I} + \boldsymbol{A})^{-1}\boldsymbol{B}$$

is strictly positive real. Then, $\boldsymbol{u}(\boldsymbol{y}) = -\boldsymbol{\varphi}(\boldsymbol{y})$ stabilizes exponentially and globally the nominal system (4). To achieve stabilization of the uncertain system in Equation (3) subject to the controller proposed in Equation (5), assume A1 and the following assumptions are satisfied (Benabdallah & Hammami, 2006):

(A2) There exists a mapping $\boldsymbol{h} : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ satisfying

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}) = \boldsymbol{P}^{-1}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u})$$

where $\boldsymbol{P}$ is the positive-definite matrix obtained from Equation (7).

(A3) There exists a nonnegative continuous function $\rho(\cdot)$ such that $\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u})$ is bounded

$$\|\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u})\|_2 \le \rho(\boldsymbol{y}).$$

(A4) There exists a nonnegative function $\rho_0(\cdot)$ such that

$$\rho(\boldsymbol{y}) \le \rho_0(\boldsymbol{y})\|\boldsymbol{y}\|_2$$

with

$$\rho_0(\boldsymbol{y}) < \frac{(2k - \lambda_{\min}(\boldsymbol{K}))^2}{4},$$

where $\lambda_{\min}(\boldsymbol{K})$ denotes the minimum eigenvalue of the matrix $\boldsymbol{K}$ and k is the Lipschitz constant.

According to Benabdallah and Hammami (2006), the following theorem is proposed.

THEOREM 1 *Consider the nonlinear system described by Equation* (3) *and assume that* (A1)–(A4) *are satisfied. Then, there exists a function* $\alpha(\boldsymbol{y})$ *such that the closed-loop system* (3)–(5) *is globally exponentially stable.*

Proof Consider the quadratic Lyapunov function,

$$V(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}.$$

The time derivative of V with reference to system (3) is given by,

$$\dot{V} = 2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{A}\boldsymbol{x} + 2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{B}\boldsymbol{u} + 2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{F}(\boldsymbol{x},\boldsymbol{u}).$$

Due to the fulfillment of (A1), Equation (7) can be applied to the first and second terms in the above expression to give,

$$2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{A}\boldsymbol{x} = -\|\boldsymbol{L}\boldsymbol{x}\|_2^2 - \varepsilon\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}$$

and

$$2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{B}\boldsymbol{u} = 2\boldsymbol{y}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{u} - 2\sqrt{2}(\boldsymbol{L}\boldsymbol{x})^{\mathrm{T}}\boldsymbol{u}.$$

Substitution of these expression into the derivative of V gives,

$$\begin{aligned}\dot{V} &= -\|\boldsymbol{L}\boldsymbol{x}\|_2^2 - \varepsilon\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x} + 2\boldsymbol{y}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{u} - 2\sqrt{2}(\boldsymbol{L}\boldsymbol{x})^{\mathrm{T}}\boldsymbol{u}\\ &\quad + 2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{F}(\boldsymbol{x},\boldsymbol{u})\\ &= -\|\boldsymbol{L}\boldsymbol{x} + \sqrt{2}\boldsymbol{u}\|_2^2 - \varepsilon\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x} + 2\boldsymbol{y}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{u} + 2\|\boldsymbol{u}\|_2^2\\ &\quad + 2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{F}(\boldsymbol{x},\boldsymbol{u}),\end{aligned}$$

The above implies that

$$\dot{V} \le -\varepsilon\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x} + 2\boldsymbol{y}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{u} + 2\|\boldsymbol{u}\|_2^2 + 2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{F}(\boldsymbol{x},\boldsymbol{u}).$$

Based on Definition 1 and Equations (5) and (6), the second and third terms above can be analyzed to arrive at,

$$\begin{aligned}&2\boldsymbol{y}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{u} + 2\|\boldsymbol{u}\|_2^2\\ &\quad= 2\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{y})(\boldsymbol{\varphi}(\boldsymbol{y}) - \boldsymbol{K}\boldsymbol{y}) - 2\boldsymbol{y}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{v}(\boldsymbol{y}) + 2\|\boldsymbol{v}(\boldsymbol{y})\|_2^2\\ &\qquad + 4\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{y})\boldsymbol{y}\boldsymbol{v}(\boldsymbol{y})\\ &\quad\le -2\boldsymbol{y}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{v}(\boldsymbol{y}) + 2\|\boldsymbol{v}(\boldsymbol{y})\|_2^2 + 4\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{y})\boldsymbol{v}(\boldsymbol{y})\\ &\quad= -2\alpha\|\boldsymbol{y}\|_2^2 + 2\alpha^2\|\boldsymbol{K}^{-1}\boldsymbol{y}\|_2^2 + 4\alpha\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{y})\boldsymbol{K}^{-1}\boldsymbol{y}\\ &\quad\le -2\alpha\|\boldsymbol{y}\|_2^2 + 2\alpha^2\|\boldsymbol{K}^{-1}\|_2^2\|\boldsymbol{y}\|_2^2 + 4\alpha\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{y})\|\boldsymbol{K}^{-1}\|_2\|\boldsymbol{y}\|_2^2\\ &\quad\le \left(-2\alpha + 2\frac{\alpha^2}{\lambda_{\min}^2(\boldsymbol{K})} + \frac{4k\alpha}{\lambda_{\min}(\boldsymbol{K})}\right)\|\boldsymbol{y}\|_2^2,\end{aligned}$$

where λ is the spectral radius of $\boldsymbol{K}$.

The establishment of the above inequality is based on the fact that the *spectral radius* of $\boldsymbol{K}$ is no larger than the *spectral norm* of $\boldsymbol{K}$ (Rugh, 1996). That is,

$$|\lambda(\boldsymbol{K})| \le \|\boldsymbol{K}\|_2$$

The square of the spectral norm can be written in terms of the dot product,

$$\|\boldsymbol{K}\boldsymbol{a}\|_2^2 = \boldsymbol{K}\boldsymbol{a}\cdot\boldsymbol{K}\boldsymbol{a} = (\boldsymbol{K}\boldsymbol{a})^{\mathrm{T}}\boldsymbol{K}\boldsymbol{a} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{K}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{a}.$$

Thus,

$$\|\boldsymbol{K}\|_2^2 = \max_{\|\boldsymbol{a}\|_2=1}\boldsymbol{a}^{\mathrm{T}}(\boldsymbol{K}^{\mathrm{T}}\boldsymbol{K})\boldsymbol{a},$$

where the entries of $\boldsymbol{a}$ and $\boldsymbol{K}$ are real.

Let $\boldsymbol{Q} \equiv \boldsymbol{K}^{\mathrm{T}}\boldsymbol{K}$ be a symmetric matrix. For any symmetric matrix, there is an orthonormal basis consisting of unit length eigenvectors that are pairwise perpendicular. Let the order of the basis set be q, where q is also the order of $\boldsymbol{K}$. Let λ_i be the corresponding eigenvalues such that $\boldsymbol{Q}\boldsymbol{q}_i = \lambda_i\boldsymbol{q}_i$. Since $\|\boldsymbol{a}\|_2 = 1$, then $\sum_{i=1}^{q} a_i^2 = 1$.

We can express the square of the norm of $\boldsymbol{K}$ as

$$\boldsymbol{a}^{\mathrm{T}}(\boldsymbol{K}^{\mathrm{T}}\boldsymbol{K})\boldsymbol{a} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{a} = \sum_{i=1}^{q}\lambda_i a_i^2.$$

It then follows that

$$\|\boldsymbol{K}\|_2^2 = \max_{\sum a_i^2=1}\sum_{i=1}^{q}\lambda_i a_i^2$$

and

$$\begin{aligned}\|\boldsymbol{K}\|_2^2 &= \max(\lambda_i),\\ |\lambda(\boldsymbol{K})| &\le \|\boldsymbol{K}\|_2.\end{aligned}$$

By (A2)–(A4),

$$\begin{aligned}2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{F}(\boldsymbol{x},\boldsymbol{u}) &= 2\boldsymbol{y}^{\mathrm{T}}\boldsymbol{h}(\boldsymbol{x},\boldsymbol{u}) \le 2\|y\|_2\,\|\boldsymbol{h}(\boldsymbol{x}\boldsymbol{u})\|_2\\ &\le 2\|\boldsymbol{y}\|_2\rho(\boldsymbol{y}) \le 2\rho_0(\boldsymbol{y})\|\boldsymbol{y}\|_2^2\end{aligned}$$

It then follows that

$$\begin{aligned}\dot{V} &\le -\varepsilon\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x} + 2\left(\frac{1}{\lambda_{\min}^2(K)}\alpha^2(\boldsymbol{y}) + \left(\frac{2k}{\lambda_{\min}(K)} - 1\right)\alpha(\boldsymbol{y})\right.\\ &\quad\left. + \rho_0(\boldsymbol{y})\right)\|\boldsymbol{y}\|_2^2.\end{aligned}$$

If the choice of $\alpha(\boldsymbol{y})$ is such that

$$\frac{1}{\lambda_{\min}^2(\boldsymbol{K})}\alpha^2(\boldsymbol{y}) + \left(\frac{2k}{\lambda_{\min}(\boldsymbol{K})} - 1\right)\alpha(\boldsymbol{y}) + \rho_0(\boldsymbol{y}) \le 0 \quad (8)$$

then $\dot{V} \le 0$ and global exponential stability of the closed-loop system is achieved.

Remark 1 An analysis of the above equation reveals that the coefficients of $\alpha^2(\boldsymbol{y})$ are positive. Thus, the minimum value of this quadratic equation can be calculated, which is $\rho_o(\boldsymbol{y}) - (k - \lambda_{\min}(\boldsymbol{K})/2)^2$. According to (A4), the minimum value is less than 0.

The solution to

$$\frac{1}{\lambda^2_{\min}(\boldsymbol{K})}\alpha^2(\boldsymbol{y}) + \left(\frac{2k}{\lambda_{\min}(\boldsymbol{K})} - 1\right)\alpha(\boldsymbol{y}) + \rho_0 = 0$$

can be calculated. Let

$$\Delta = (2k - \lambda_{\min}(\boldsymbol{K}))^2 - 4\rho_0(\boldsymbol{y}).$$

According to (A4), $\Delta > 0$, thus two, distinct real solutions exist,

$$\alpha(\boldsymbol{y}) = \frac{\lambda_{\min}(\boldsymbol{K})}{2}[\lambda_{\min}(\boldsymbol{K}) - 2k \pm \sqrt{\Delta}].$$

Because the quadratic equation has a minimum value, the selection of $\alpha(\boldsymbol{y})$ can satisfy the inequality in Equation (8),

$$\frac{\lambda_{\min}(\boldsymbol{K})(\lambda_{\min}(\boldsymbol{K}) - 2k - \sqrt{\Delta})}{2} \le \alpha(\boldsymbol{y}) \le \frac{\lambda_{\min}(\boldsymbol{K})(\lambda_{\min}(\boldsymbol{K}) - 2k + \sqrt{\Delta})}{2} \tag{9}$$

■

Remark 2 The existence of $\alpha(\boldsymbol{y})$ in Equation (9) is sufficient for the nonlinear closed-system to be globally asymptotically stable.

4. Closed-loop stability of reservoir NARX

In the NARX reservoir model (2), the oil production is controlled by the injection of water flow rates. According to the above stability analysis, to guarantee the closed-loop control stability, it is required that each part should satisfy (A1)–(A4) and the circle criterion.

Initially, the NARX form (2) is transformed to the form of Equation (3),

$$\begin{aligned} x(k) &= ([x(k-1)u(k-1)]^{\mathrm{T}} - \boldsymbol{r})\boldsymbol{Pl} \\ &\quad + \sum_i \boldsymbol{a}_i^{\mathrm{s}} f(\boldsymbol{b}_i^{\mathrm{s}}(([x(k-1)u(k-1)]^{\mathrm{T}} - \boldsymbol{r})\boldsymbol{Q} - \boldsymbol{c}_i^{\mathrm{s}})) \\ &\quad + \sum_j \boldsymbol{a}_j^{\mathrm{w}} g(\boldsymbol{b}_j^{\mathrm{w}}(([x(k-1)u(k-1)]^{\mathrm{T}} - \boldsymbol{r}) \\ &\qquad \times \boldsymbol{Q} - \boldsymbol{c}_j^{\mathrm{w}})) + d \\ &= [x(k-1)u(k-1)]^{\mathrm{T}}\boldsymbol{Pl} - \boldsymbol{rPl} + F_1(x,u,k) \\ &= A_1 x(k-1) + B_1 u(k-1) + F_2(x,u,k), \\ \dot{x} &= \Delta t(A_1^{-1}A_1 - x + A_1^{-1}B_1 u + A_1^{-1}F_2). \end{aligned}$$

The form of Equation (3) can be obtained with

$$\begin{aligned} A &= \Delta t A_1^{-1}(A_1 - 1), \\ B &= \Delta t A_1^{-1} B, \\ F(x,u) &= \Delta t A_1^{-1}\left(\sum_i \boldsymbol{a}_i^{\mathrm{s}} fi + \sum_j \boldsymbol{a}_j^{\mathrm{s}} g_j - \boldsymbol{rPl} + d\right). \end{aligned}$$

In Equation (3), y is the oil production rate; thus, $C = 1$. Because each model has one manipulated variable and one control variable, u, x and y are scalars; thus, A, B and C. are scalar. The k-Lipschitz function is simply,

$$\|\varphi(y) - \varphi(z)\| = \|u(y) - u(z) + v(y) - v(z)\| \le k\|y - z\|.$$

The relationship between the amount of water added and the amount of oil produced is

$$y = u(1 - \mathrm{wc}),$$

where wc is the water cut value, which is the ratio between the water produced to the sum of oil and water produced,

$$k \ge \frac{1}{1 - \mathrm{wc}}.$$

For part I of the reservoir: $A = -0.5818$, $B = -0.5840$. To satisfy (A1), assume $P = 1$ and thus $0 < \varepsilon < 1.1636$, $|L| < 1.0787$ and $0 < K < 0.9415$. For condition (A2), $h(t,x,u) = f(t,x,u)$. Based on the NARX model, it can be found that values of $\rho = 33.0462$ and $\rho_0 = 0.1957$ satisfy (A3). Also, $k \ge 1/(1 - wc)$ and $0 < K < 0.9415$ satisfy (A4).

For part II of the reservoir: $A = -0.1658$, $B = 2.1516$. Assume $P = 1$; then $0 < \varepsilon < 0.3316$, $|L| < 0.5758$ and $1.3373 < K < 2.9659$ to satisfy (A1). Values of $\rho = 27.9713$ and $\rho_0 = 0.1401$ satisfy (A3). And $k \ge 1/(1 - \mathrm{wc})$ and $0 < K < 1.2548$ or $2.7520 < K < 2.9659$ satisfy (A4).

For part III of the reservoir: $A = -0.5752$, $B = -0.4143$. Assume $P = 1$; then $0 < \varepsilon < 1.1504$, $|L| < 1.0726$ and $0 < K < 1.1028$ to satisfy (A1). Values of $\rho = 47.9380$ and $\rho_0 = 0.2410$ satisfy (A3). And $k \ge 1/(1 - \mathrm{wc})$ and $0 < K < 1.0290$ satisfy (A4).

For part IV of the reservoir: $A = -0.2904$, $B = -0.0409$. Assume $P = 1$; $0 < \varepsilon < 0.5808$, $|L| < 0.7621$ and $0 < K < 1.0369$ to satisfy condition (A1). Values of $\rho = 5.3165$ and $\rho_0 = 0.0269$ satisfy condition (A3). And $k \ge 1/(1 - \mathrm{wc})$ and $0 < K < 1.0369$ satisfy (A4).

It was already demonstrated that with the satisfaction of (A1)–(A4), there exists an $\alpha(\cdot)$ function to stabilize the closed-loop system given by Equation (3) (Theorem 1). Thus, the closed-loop stability of the reservoir example is established.

5. Summary

The primary contribution of this work is the analysis of closed-loop stability of a nonlinear dynamic system (3) in feedback with an optimal model-based controller design. An original theorem was developed and proven about the existence of a model-based controller that guarantees that the closed-loop system is globally exponentially stable (Section 3). An approximate nonlinear reduced-model of a two-phase oil producing reservoir was introduced to demonstrate the design of the stabilizing controller. The controller parameters that satisfy the condition for closed-loop stability were analyzed to establish this state. Because of the varying but natural geological conditions, any mathematical model of an oil reservoir model will contain uncertainty. In the future, it will be interesting to consider the main sources of uncertainty and their effect on the closed-loop stability analysis for the designed controller and oil reservoir system.

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