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Finite-horizon H_∞ filtering for time-varying delay systems with randomly varying nonlinearities and sensor saturations

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This paper mainly focuses on the H_∞ filtering problem for a class of discrete time-varying systems with delays and randomly varying nonlinearities and sensor saturations. Two sets of binary switching sequences taking values of 1 and 0 are introduced to account for the stochastic phenomena of nonlinearities and sensor saturations which occur and influence the dynamics of the system in a probabilistic way. To further reflect the realities of transmission failure in the measurement, missing observation case is also considered simultaneously. By appropriately constructing a time-varying Lyapunov function and utilizing the stochastic analysis technique, sufficient criteria are presented in terms of a set of recursive linear matrix inequalities (RLMIs) under which the filtering error dynamics achieves the prescribed H_∞ performance over a finite horizon. Moreover, at each time point k, the time-varying filter parameters can be solved iteratively according to the explicit solutions of the RLMIs. Finally, a numerical simulation is exploited to demonstrate the effectiveness of the proposed filter design scheme.

Keywords: H_∞ filtering; time-varying delayed systems; randomly varying sensor saturations; recursive linear matrix inequalities; finite horizon

1. Introduction

In practical engineering fields such as signal processing area, to carry out some specific design tasks, the state information or some combinations of the state information are needed to be known which however, are often unavailable. And this is one of the main backgrounds for investigating the estimation problems. Generally speaking, the aim of the estimation problem is to estimate certain system parameters or state variables by utilizing the accessible measurements, which might be with the existence of stochastic errors. In the literature, much work has been done on various estimation problems, and several filtering methodologies have been proposed (Ahmad & Namerikawa, 2013; Lu, Xie, Zhang, & Wang, 2007; Mohamed, Nahla, & Safya, 2013; Reif & Unbehauen, 1999). Among them, the Kalman filtering (Lu et al., 2007; Reif & Unbehauen, 1999; Xie, Soh, & de Souza, 1994) and the H_∞ filtering (Dong, Wang, Ho, & Gao, 2011; Li, Lam, & Shu, 2010; Zhang, Chen, & Tseng, 2005; Zhang, Feng, & Duan, 2006) are two notable ones. The main idea for the Kalman filtering is to estimate the future values of the signal by utilizing the past/current observations. When employing the H_∞ filtering, criteria are often presented in the form of Riccati difference equations (Gershon, Shaked, & Yaesh, 2001; Hung & Yang, 2003; Xie & de Souza, 1992; Zhang et al., 2006) or linear matrix inequalities (LMIs) (Li et al., 2010; Shen, Wang, & Liu, 2011; Shen, Wang, Shu, & Wei, 2010; Wang, Shen, & Liu, 2012).

It is well known that almost all the realistic systems are intrinsically time-varying, and frequently affected by the nonlinear exogenous disturbances and time delays, which markedly increase the difficulty when analyzing the system due to the complexity. In the past decades, much efforts have been devoted to the filtering and control problems for the discrete time-varying systems (Shen et al., 2011; Shen, Ding, & Wang, 2013). For instance, robust H_∞ filtering problem has been investigated in Dong et al. (2011) for the Markovian jump time-varying systems. When dealing with the time-varying systems in practice, a fundamental issue arises naturally, that is, the state performance constrains are restricted only over a finite horizon instead of the infinite one. Such kind of finite-horizon filtering problem has attracted much attention in recent years, and it is desirable to develop effective and executable algorithms to determine the filter parameters. Motivated by the novel difference linear matrix inequality method proposed in Shaked and Suplin (2001), Gershon and Shaked (2008), Gershon, Shaked, and Yaesh (2005), a new and practical recursive linear matrix inequality (RLMI) approach has been firstly presented in Shen et al. (2010) where the available state estimates have also been utilized which might decrease the conservation of

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the results obtained. On the other hand, time delays are ubiquitous in systems mainly due to the reasons such as finite capabilities of signal transmission among different parts of the systems. Numerous results pertaining to the filtering problem have been obtained in Lu et al. (2007), Chen and Zheng (2012), Wu and Wang (2009), Wei, Wang, and Shen (2013) for the delayed time-invariant systems. When referring to the time-varying delayed systems, relating results are relatively few (Basin, Shi, Alvarez, & Wang, 2009; Wei, Wang, & Shen, 2010), which might mainly due to the mathematical complexity. For example, in Basin et al. (2009), the central suboptimal H_∞ filters have been defined for the linear continuous-time systems with state or measurement delay. Particularly, a numerically appealing algorithm has been developed in Wei et al. (2010) for the error-constrained filtering problem of the discrete time-varying delay systems with bounded noise, where randomly varying nonlinearities and sensor saturations are not considered.

When the inputs are large enough, the sensors turn to be saturated rather than linear caused by physical constraints. In other words, the sensors possess the nonlinear characteristic when confronting saturations, and it may degrade the filter performance (Liu, Wang, & Yang, 2003; Wang et al., 2012; Yang & Li, 2009) when neglecting the amplitude saturation effect. Therefore, sensor saturation issue is currently an attracting and active research area. As pointed out in Shen et al. (2010), when working circumstances change suddenly or instruments abrade, this will result in the randomly changeable of the nonlinear disturbances in terms of their type and/or intensity and the missing measurement situations. Such kinds of cases are characterized and named with randomly occurring nonlinearities (RONs) and randomly occurring sensor saturations (ROSSs) in Wang et al. (2012) and Shen et al. (2010) and further studied in Shen, Wang, Shu, and Wei (2011), Wang, Wang, and Liu (2010), Wei et al. (2013), Ding, Wang, Shen, and Shu (2012). Specially, an adaptive reliable H_∞ filter method has been developed in Yang and Ye (2007) to against the sensor failure case, and the filter parameter gains are determined based on LMIs by solving two optimization problems. It should be noted that in all these references, time delay effects have not been considered.

Based on the above discussions, in this paper, we will concentrate on the H_∞ filtering problem for a class of discrete time-varying *delayed* systems with *missing measurements* and *randomly varying nonlinearities and sensor saturations over a finite horizon*. Illuminated by the ideas reported in Shen et al. (2010), Dong et al. (2011), Wang, Dong, Shen, and Gao (2013), the H_∞ filtering problem is investigated for the discrete time-varying system with delays by introducing an improved time-varying Lyapunov functional, and sufficient criteria are given which ensure the validity of the H_∞ performance constraint for the filtering error dynamics. Moreover, in the output measurement process, both possible sensor saturations and data-missing phenomena are considered, which are introduced to reflect the intricate working circumstances of the underlying system. In addition, randomly varying nonlinearities between the current and the delayed state nonlinearities are also involved, which together make the H_∞ filtering problem hard to be analyzed, not to mention the design problem for the H_∞ filter. And this is the main aim of this paper to shorten such a gap.

The rest of the paper is organized as follows. In Section 2, the discrete time-varying delayed system with randomly varying nonlinearities and sensor saturations is presented, and the H_∞ filtering problem addressed is formulated. In Section 3, by resorting to the stochastic analysis techniques, sufficient conditions are established in the form of time-varying matrix inequalities under which the output estimation error is assured to meet the constraint of the given H_∞ performance level. Furthermore, the parameters of the H_∞ filter are designed according to the feasible solutions of a set of RLMIs and a recursive filtering algorithm is developed. In Section 4, one illustrative example is given to demonstrate the effectiveness of the results derived. Finally, in Section 5 the conclusion is drawn.

Notations: The notations used here are fairly standard except where otherwise stated. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. The set of all integers is represented by $\mathbb{Z}$ and $\mathbb{R}$ means the set of all real numbers. The interval $[m, n]$ with $m, n \in \mathbb{Z}$ and $m < n$ denotes the set of integer sequence $\{m, m+1, \ldots, n\}$, and $[a, b]$ with $a, b \in \mathbb{R}$ and $a < b$ represents the set of real numbers between a and b. The notation $X \geq Y$ (respectively, $X > Y$), where X and Y are real symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). M^{T} represents the transpose of the matrix M and I is used to denote the identity matrix with compatible dimensions. $\mathrm{diag}\{\cdots\}$ stands for a block-diagonal matrix. Moreover, $\mathrm{Prob}\{X\}$ means the occurrence probability of the event X and $\mathbb{E}\{x\}$ stands for the expectation of the stochastic variable x with respect to the given probability measure Prob. The asterisk "$*$" in a matrix is used to denote a term that is induced by symmetry. Matrices, if they are not explicitly specified, are assumed to have compatible dimensions.

2. Problem formulation and preliminaries

Consider the following discrete time-varying delayed system defined on the finite horizon $k \in [0, N]$:

$$\begin{aligned} x(k+1) &= A(k)x(k) + A_1(k)x(k-d) + B(k)v(k) \\ &\quad + \alpha(k)f(k, x(k)) + (1-\alpha(k))g(k, x(k-d)), \\ y(k) &= \psi(C(k)x(k)) + D(k)v(k), \\ z(k) &= M(k)x(k), \\ x(s) &= \phi(s), \quad s = -d, -d+1, \ldots, 0, \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $y(k) \in \mathbb{R}^m$ is the measured output vector, $z(k) \in \mathbb{R}^r$ is the signal to be estimated; $A(k)$, $A_1(k)$, $B(k)$, $C(k)$, $D(k)$ and $M(k)$ are known real time-varying matrices with appropriate dimensions; $d > 0$ is an integer representing the constant delay of the system; $\phi(\cdot) \in \mathbb{R}^n$ is the initial state vector function defined on $[-d, 0]$; $v(k) \in \mathbb{R}^q$ is the exogenous disturbance signal belonging to $l_2[0, N]$ which denotes the space of square summable sequences with the norm

$$\|v\|^2_{[0,N]} = \mathbb{E}\left\{\sum_{k=0}^{N} \|v(k)\|^2\right\}.$$

The nonlinear functions $f(\cdot,\cdot)$, $g(\cdot,\cdot) : [0, N] \times \mathbb{R}^n \to \mathbb{R}^n$ are assumed to be continuous and satisfy the following sector-bounded conditions (Cao, Lin, & Chen, 2003; Wang, Liu, & Liu, 2008):

$$[f(k,x) - U_1(k)x]^{\mathrm{T}}[f(k,x) - U_2(k)x] \leq 0, \quad (2)$$

$$[g(k,x) - V_1(k)x]^{\mathrm{T}}[g(k,x) - V_2(k)x] \leq 0, \quad (3)$$

where $k \in [0, N]$ and $x \in \mathbb{R}^n$; $U_1(k)$, $U_2(k)$, $V_1(k)$ and $V_2(k)$ are known real matrices with appropriate dimensions, and $U(k) = U_1(k) - U_2(k)$, $V(k) = V_1(k) - V_2(k)$ are symmetric positive-definite matrices.

The random variable $\alpha(k) \in \mathbb{R}$ takes values of 1 and 0 with

$$\mathrm{Prob}\{\alpha(k) = 1\} = \bar{\alpha}, \quad \mathrm{Prob}\{\alpha(k) = 0\} = 1 - \bar{\alpha}, \quad (4)$$

where constant $\bar{\alpha} \in [0, 1]$ is known.

Remark 1 Most of realistic systems are subject to nonlinear disturbances which themselves might change abruptly due mainly to the reasons such as sudden changes in environment, random switching between subsystems, failure connection between part of the nodes of networks as well as asynchronous information transmission within networks. In other words, the nonlinear disturbances might occur in a probabilistic way. Such kind of phenomena has been firstly named RONs in Wang, Wang, and Liang (2009) to account for the probabilistic occurrence of different nonlinear functions. Here, illuminated by the ideas proposed in in Wang et al. (2009), $\alpha(k)$ is used just to account for the phenomena of randomly varying nonlinearities between the current state nonlinearity $f(k, x(k))$ and the delayed state nonlinearity $g(k, x(k - d)))$.

The nonlinear function $\psi(\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is given as follows:

$$\psi(C(k)x(k)) = \beta(k)\sigma(C(k)x(k)) + (1 - \beta(k))\gamma(k)C(k)x(k), \quad (5)$$

where $\sigma(\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ represents the sensor saturation function with the following form:

$$\sigma(u) = [\sigma_1^{\mathrm{T}}(u_1), \sigma_2^{\mathrm{T}}(u_2), \ldots, \sigma_m^{\mathrm{T}}(u_m)]^{\mathrm{T}}, \quad (6)$$

where $u = (u_1, \ldots, u_m)^{\mathrm{T}} \in \mathbb{R}^m$, $\sigma_i(u_i) = \mathrm{sign}(u_i) \min\{u_{i,\max}, |u_i|\}$ for $i = 1, 2, \ldots, m$ and $u_{i,\max}$ denotes the ith element of the saturation vector $u_{\max}$.

In Equation (5), the random variables $\beta(k) \in \mathbb{R}$ and $\gamma(k) \in \mathbb{R}$ are Bernoulli distributed sequences taking values of 1 and 0 with

$$\begin{aligned} &\mathrm{Prob}\{\beta(k) = 1\} = \bar{\beta}, \quad \mathrm{Prob}\{\beta(k) = 0\} = 1 - \bar{\beta}, \\ &\mathrm{Prob}\{\gamma(k) = 1\} = \bar{\gamma}, \quad \mathrm{Prob}\{\gamma(k) = 0\} = 1 - \bar{\gamma}, \end{aligned} \quad (7)$$

where $\bar{\beta}$, $\bar{\gamma} \in [0, 1]$ are known constants. Here, $\beta(k)$ is introduced to account for the phenomena of randomly varying sensor saturations caused by physical electron device constraints, while $\gamma(k)$ is used to describe the probable data-missing phenomenon caused by mutative working conditions or fluctuant signal transmission channels when the sensors fail to work. Throughout this paper, we further assume that the stochastic variables $\alpha(k)$, $\beta(k)$ and $\gamma(k)$ are mutually independent.

Remark 2 In recent years, the RONs have been extensively studied in Dong et al. (2011), Wang et al. (2010) and Wei, Wang, and Han (2013) for the Markovian jump systems and the complex networks. When such kind of phenomena occur in a sensor network, the notation of ROSSs has been firstly introduced in Wang et al. (2012) and Ding et al. (2012) by further considering the physical limitations of components. It should be noted that such kind of ideas have also been employed in earlier works such as Nahi (1969) and Chau, Qin, Sayed, Wahab, and Yang (2010), where a Markov chain has been proposed to capture the battery recovery (Chau et al., 2010), an the missing measurements (or uncertain observations) have been considered for the optimal estimation problems (Nahi, 1969).

Remark 3 The measurement output model given in Equations (1) and (5) is originated from Wang et al. (2012) where the H_∞ filtering problem has been investigated for the nonlinear sensor networks with time-invariant system matrices. As pointed out in Wang et al. (2012), the main advantage of such kind of measurement output equation is that it is capable of accounting for the phenomena of both ROSSs and missing measurements in a unified form. To be specific, if $\beta(k) = 0$ and $\gamma(k) = 0$, the output observer receives only the noise signal; if $\beta(k) = 0$ and $\gamma(k) = 1$, it means that the output observer works regularly; if $\beta(k) = 1$, whatever the value of $\gamma(k)$ is, only saturated signals are received by the output observer. In practice, the case that the sensor saturation phenomenon and the data-missing phenomenon occur simultaneously does exist. At this time, we only consider the former one since the saturated signals can also be viewed as one special form of the data-missing phenomenon.

Illuminated by the analysis method used in Dong et al. (2011) and Yang and Li (2009), we assume that there exist diagonal matrices $H_1(k)$ and $H_2(k)$ such that $0 \leq H_1(k) < I \leq H_2(k)$, and the saturation function $\sigma(C(k)x(k))$ in

Equation (5) is rewritten as follows:

$$\sigma(C(k)x(k)) = H_1(k)C(k)x(k) + h(C(k)x(k)), \quad (8)$$

where $h(C(k)x(k))$ is a nonlinear vector-valued function satisfying the following inequality:

$$h^{\mathrm{T}}(C(k)x(k))(h(C(k)x(k)) - H(k)C(k)x(k)) \leq 0 \quad (9)$$

with $H(k) = H_2(k) - H_1(k)$.

According to the above discussions, system (1) can be rewritten as follows:

$$\begin{aligned} x(k+1) &= A(k)x(k) + A_1(k)x(k-d) + B(k)v(k) \\ &\quad + \alpha(k)f(k,x(k)) + (1-\alpha(k))g(k,x(k-d)), \\ y(k) &= \beta(k)H_1(k)C(k)x(k) + \beta(k)h(C(k)x(k)) \\ &\quad + (1-\beta(k))\gamma(k)C(k)x(k) + D(k)v(k), \\ z(k) &= M(k)x(k), \\ x(s) &= \phi(s), \quad s = -d, -d+1, \ldots, 0. \end{aligned} \quad (10)$$

In this paper, we will design the following filter for the time-varying system (10):

$$\begin{aligned} \hat{x}(k+1) &= F_f(k)\hat{x}(k) + G_f(k)y(k), \\ \hat{z}(k) &= M_f(k)\hat{x}(k), \end{aligned} \quad (11)$$

where $\hat{x}(k) \in \mathbb{R}^n$ is the state vector of the filter, $\hat{z}(k) \in \mathbb{R}^r$ is the estimate of $z(k)$; $F_f(k)$, $G_f(k)$ and $M_f(k)$ are time-varying filter matrices to be designed. Here, we take $\hat{x}(k) \equiv 0$ for $k \leq 0$, which will be used when designing the filter algorithm in the sequel.

For convenience of expression, we introduce the following notions:

$$\begin{aligned} \tilde{\mathscr{A}}(e(k)) &= (A(k) - \bar{\beta}G_f(k)H_1(k)C(k) \\ &\quad - (1-\bar{\beta})\bar{\gamma}G_f(k)C(k))e(k) \\ &\quad + (B(k) - G_f(k)D(k))v(k) \\ &\quad + (A(k) - \bar{\beta}G_f(k)H_1(k)C(k) \\ &\quad - (1-\bar{\beta})\bar{\gamma}G_f(k)C(k) \\ &\quad - F_f(k))\hat{x}(k) - \bar{\beta}G_f(k)h(C(k)x(k)) \\ &\quad + \bar{\alpha}f(k,x(k)) + (1-\bar{\alpha})g(k,x(k-d)), \\ \tilde{\mathscr{B}}(e(k)) &= -G_f(k)H_1(k)C(k)e(k) - G_f(k)H_1(k)C(k)\hat{x}(k) \\ &\quad - G_f(k)h(C(k)x(k)), \\ \tilde{\mathscr{C}}(e(k)) &= -G_f(k)C(k)e(k) - G_f(k)C(k)\hat{x}(k), \\ \tilde{\mathscr{A}}_1(e(k)) &= A_1(k)e(k-d) + A_1(k)\hat{x}(k-d), \\ \tilde{\mathscr{F}}(e(k)) &= f(k,x(k)) - g(k,x(k-d)), \\ \tilde{\mathscr{M}}(e(k)) &= M(k)e(k) + (M(k) - M_f(k))\hat{x}(k). \end{aligned}$$

By letting $e(k) = x(k) - \hat{x}(k)$ and $\tilde{z}(k) = z(k) - \hat{z}(k)$, the error dynamics can be obtained as follows from Equations (10) and (11):

$$\begin{aligned} e(k+1) &= \tilde{\mathscr{A}}(e(k)) + (\beta(k) - \bar{\beta})\tilde{\mathscr{B}}(e(k)) \\ &\quad + \big((1-\beta(k))\gamma(k) - (1-\bar{\beta})\bar{\gamma}\big)\tilde{\mathscr{C}}(e(k)) \\ &\quad + (\alpha(k) - \bar{\alpha})\tilde{\mathscr{F}}(e(k)) + \tilde{\mathscr{A}}_1(e(k)), \\ \tilde{z}(k) &= \tilde{\mathscr{M}}(e(k)). \end{aligned} \quad (12)$$

The filtering problem to be addressed is as follows: design the filter (11) such that the H_∞ performance constraint (13) is satisfied. More specially, for any nonzero exogenous disturbance $v(k) \in l_2([0,N],\mathbb{R}^q)$, the estimation error $\tilde{z}(k)$ satisfies the following inequality:

$$\|\tilde{z}\|^2_{[0,N]} \leq \gamma^2 \left\{ \|v\|^2_{[0,N]} + \mathbb{E}\left\{ \sum_{k=-d}^{0} e^{\mathrm{T}}(k)S(k)e(k) \right\} \right\} \quad (13)$$

where $\gamma > 0$ is a given disturbance attenuation level and $\{S(k)\}_{-d \leq k \leq 0}$ is a known positive-definite matrix sequence.

Remark 4 In recent years, the finite-horizon filtering problem has attracted much attention for its practicability. For example, the robust H_∞ filtering problem with error variance constraints has been investigated for the discrete linear time-varying systems in Hung and Yang (2003), where the conditions are in the form of forward recursive Riccati equations which are hard to be solved in practice. Recently, novel works have been done in Shen et al. (2010) and Wei et al. (2010), respectively, for the robust H_∞ finite-horizon filtering of systems with RONs and quantization effects and the error-constrained filtering of nonlinear delayed systems with non-Gaussian noises. It should be noted that the delay effects have not been considered in Shen et al. (2010), and in Wei et al. (2010) the phenomena of ROSSs and missing measurements have not been taken into account. By taking the phenomena of time delay, ROSSs and missing measurements together, it will be hard to analyze the H_∞ filtering problem, not to mention the design problem for the time-varying H_∞ filter, which mainly motivates the present work of this article.

3. Main results

In this section, in order to design the filter (11), we first give a sufficient criterion to guarantee that the error system (12) satisfies the H_∞ performance constraint (13) via the RLMI approach, which is given by the following theorem.

Theorem 1 *Consider the error system (12) with known filter parameters $\{F_f(k)\}_{0\leq k\leq N}$, $\{G_f(k)\}_{0\leq k\leq N}$ and $\{M_f(k)\}_{0\leq k\leq N}$. Let the disturbance attenuation level $\gamma > 0$ and the positive-definite matrix sequence $\{S(k)\}_{-d\leq k\leq 0}$ be given, the estimation error $\tilde{z}(k)$ satisfies the H_∞ performance constraint (13) if there exist four families of* *positive scalars* $\{\varepsilon_1(k)\}_{0\le k\le N}$, $\{\varepsilon_2(k)\}_{0\le k\le N}$, $\{\varepsilon_3(k)\}_{0\le k\le N}$, $\{\mu(k)\}_{0\le k\le N+1}$ *and two families of positive-definite matrices* $\{P(k)\}_{0\le k\le N+1}$, $\{Q(k)\}_{-d+1\le k\le N+1}$ *satisfying the following initial condition:*

$$\mathbb{E}\left\{e^{\mathrm{T}}(0)P(0)e(0)+\sum_{k=-d}^{-1}e^{\mathrm{T}}(k)Q(k+1)e(k)\right\}+\mu(0)\le\gamma^2\mathbb{E}\left\{\sum_{k=-d}^{0}e^{\mathrm{T}}(k)S(k)e(k)\right\} \qquad (14)$$

and the RLMIs

$$\begin{bmatrix}\Xi_1(k) & \hat{\mathscr{A}}^{\mathrm{T}}(k)P(k+1) & \hat{\mathscr{B}}^{\mathrm{T}}(k)P(k+1) & \hat{\mu}\mathscr{C}^{\mathrm{T}}(k)P(k+1) & \hat{\alpha}\mathscr{F}^{\mathrm{T}}P(k+1) & \mathscr{L}^{\mathrm{T}}(k)\\ * & -P(k+1) & 0 & 0 & 0 & 0\\ * & * & -P(k+1) & 0 & 0 & 0\\ * & * & * & -\hat{\mu}P(k+1) & 0 & 0\\ * & * & * & * & -\hat{\alpha}P(k+1) & 0\\ * & * & * & * & * & -I\end{bmatrix}\le 0 \qquad (15)$$

for $0\le k\le N$, *where* $\hat{\alpha}=\bar{\alpha}(1-\bar{\alpha})$, $\hat{\beta}=\bar{\beta}(1-\bar{\beta})$, $\hat{\gamma}=(1-\bar{\beta})\bar{\gamma}-(1-\bar{\beta})^2\bar{\gamma}^2$, $\hat{\mu}=\bar{\gamma}(1-\bar{\gamma})(1-\bar{\beta})$, $\hat{m}=\hat{\beta}^{1/2}$, $\hat{n}=\hat{\beta}^{1/2}\bar{\gamma}$,

$$\Xi_1(k)=\begin{bmatrix}\Gamma_1(k) & 0 & 0 & -\varepsilon_1(k)\tilde{U}_2(k) & 0 & \Gamma_2(k) & \Gamma_3(k)\\ * & \Gamma_4(k) & 0 & 0 & -\varepsilon_2(k)\tilde{V}_2(k) & 0 & \Gamma_5(k)\\ * & * & -\gamma^2 I & 0 & 0 & 0 & 0\\ * & * & * & -\varepsilon_1(k)I & 0 & 0 & \Gamma_6(k)\\ * & * & * & * & -\varepsilon_2(k)I & 0 & \Gamma_7(k)\\ * & * & * & * & * & -\varepsilon_3(k)I & \Gamma_8(k)\\ * & * & * & * & * & * & \Gamma_9(k)\end{bmatrix},$$

$$\Gamma_1(k)=-P(k)+Q(k+1)+M^{\mathrm{T}}(k)M(k)-\varepsilon_1(k)\tilde{U}_1(k),$$

$$\Gamma_2(k)=\frac{\varepsilon_3(k)C^{\mathrm{T}}(k)H^{\mathrm{T}}(k)}{2},$$

$$\Gamma_3(k)=M^{\mathrm{T}}(k)(M(k)-M_f(k))\hat{x}(k)-\varepsilon_1(k)\tilde{U}_1(k)\hat{x}(k),$$

$$\Gamma_4(k)=-Q(k-d+1)-\varepsilon_2(k)\tilde{V}_1(k),$$

$$\Gamma_5(k)=-\varepsilon_2(k)\tilde{V}_1(k)\hat{x}(k-d),$$

$$\Gamma_6(k)=-\varepsilon_1(k)\tilde{U}_2^{\mathrm{T}}(k)\hat{x}(k),$$

$$\Gamma_7(k)=-\varepsilon_2(k)\tilde{V}_2^{\mathrm{T}}(k)\hat{x}(k-d),$$

$$\Gamma_8(k)=\frac{\varepsilon_3(k)H(k)C(k)\hat{x}(k)}{2},$$

$$\Gamma_9(k)=\mu(k+1)-\mu(k)-\varepsilon_1(k)\hat{x}^{\mathrm{T}}(k)\tilde{U}_1(k)\hat{x}(k)-\varepsilon_2(k)\hat{x}^{\mathrm{T}}(k-d)\tilde{V}_1(k)\hat{x}(k-d),$$

$$\hat{\mathscr{A}}(k)=\mathscr{A}(k)+\mathscr{A}_1(k),\quad \hat{\mathscr{B}}(k)=\hat{m}\mathscr{B}(k)-\hat{n}\mathscr{C}(k),$$

$$\mathscr{A}(k)=[\bar{A}(k)\quad 0\quad B(k)-G_f(k)D(k)\quad \bar{\alpha}I(1-\bar{\alpha})I-\bar{\beta}G_f(k)\quad (\bar{A}(k)-F_f(k))\hat{x}(k)],$$

$$\bar{A}(k)=A(k)-\bar{\beta}G_f(k)H_1(k)C(k)-(1-\bar{\beta})\bar{\gamma}G_f(k)C(k),$$

$$\mathscr{A}_1(k)=[0\ A_1(k)\ 0\ 0\ 0\ 0\ A_1(k)\hat{x}(k-d)],$$

$$\mathscr{B}(k)=[\bar{B}(k)\ 0\ 0\ 0\ 0\ -G_f(k)\ \bar{B}(k)\hat{x}(k)],$$

$$\bar{B}(k)=-G_f(k)H_1(k)C(k),\quad \mathscr{F}=[0\ 0\ 0\ I\ -I\ 0\ 0],$$

$$\mathscr{C}(k)=[-G_f(k)C(k)\ 0\ 0\ 0\ 0\ 0\ -G_f(k)C(k)\hat{x}(k)],$$

$$\mathscr{L}(k)=[0\ 0\ 0\ 0\ 0\ 0\ (M(k)-M_f(k))\hat{x}(k)],$$

$$\tilde{U}_1(k)=\frac{(U_1^{\mathrm{T}}(k)U_2(k)+U_2^{\mathrm{T}}(k)U_1(k))}{2},$$

$$\tilde{U}_2(k)=-\frac{(U_1^{\mathrm{T}}(k)+U_2^{\mathrm{T}}(k))}{2},$$

$$\tilde{V}_1(k)=\frac{(V_1^{\mathrm{T}}(k)V_2(k)+V_2^{\mathrm{T}}(k)V_1(k))}{2},$$

$$\tilde{V}_2(k)=-\frac{(V_1^{\mathrm{T}}(k)+V_2^{\mathrm{T}}(k))}{2}.$$

Proof Select a Lyapunov function for the time-varying system (12) as follows:

$$V(k,e(k))=e^{\mathrm{T}}(k)P(k)e(k)+\sum_{s=k-d}^{k-1}e^{\mathrm{T}}(s)Q(s+1)e(s)+\mu(k), \qquad (16)$$

where $k=0,1,\ldots,N+1$ and $\{P(k)\}_{0\le k\le N+1}$, $\{Q(k)\}_{-d+1\le k\le N+1}$, $\{\mu(k)\}_{0\le k\le N+1}$ are the solutions of the RLMIs (15) with the initial condition (14).

It follows from Equation (12) that

$$\begin{aligned}\mathbb{E}\{e^{\mathrm{T}}(k+1)P(k+1)e(k+1)\} &= \mathbb{E}\{\tilde{\mathscr{A}}^{\mathrm{T}}(e(k))P(k+1)\tilde{\mathscr{A}}(e(k))\\ &\quad+(\beta(k)-\bar{\beta})^2\tilde{\mathscr{B}}^{\mathrm{T}}(e(k))P(k+1)\tilde{\mathscr{B}}(e(k))\\ &\quad+((1-\beta(k))\gamma(k)-(1-\bar{\beta})\bar{\gamma})^2\tilde{\mathscr{C}}^{\mathrm{T}}(e(k))\\ &\quad\times P(k+1)\tilde{\mathscr{C}}(e(k))\end{aligned}$$

$$
\begin{aligned}
&+ (\alpha(k)-\bar{\alpha})^2\tilde{\mathscr{F}}^{\mathrm{T}}(e(k))P(k+1)\tilde{\mathscr{F}}(e(k)) \\
&+ 2\tilde{\mathscr{A}}^{\mathrm{T}}(e(k))P(k+1)[(\beta(k)-\bar{\beta})\tilde{\mathscr{B}}(e(k)) \\
&+ ((1-\beta(k))\gamma(k)-(1-\bar{\beta})\bar{\gamma})\tilde{\mathscr{C}}(e(k)) \\
&+ (\alpha(k)-\bar{\alpha})\tilde{\mathscr{F}}(e(k)) \\
&+ \tilde{\mathscr{A}}_1(e(k))] + 2(\alpha(k)-\bar{\alpha})\tilde{\mathscr{F}}^{\mathrm{T}}(e(k)) \\
&\times P(k+1)\tilde{\mathscr{A}}_1(e(k)) + 2(\beta(k) \\
&- \bar{\beta})\tilde{\mathscr{B}}^{\mathrm{T}}(e(k))P(k+1)[((1-\beta(k))\gamma(k) \\
&- (1-\bar{\beta})\bar{\gamma})\tilde{\mathscr{C}}(e(k)) \\
&+ (\alpha(k)-\bar{\alpha})\tilde{\mathscr{F}}(e(k)) + \tilde{\mathscr{A}}_1(e(k))] \\
&+ \tilde{\mathscr{A}}_1^{\mathrm{T}}(e(k))P(k+1)\tilde{\mathscr{A}}_1(e(k)) \\
&+ 2((1-\beta(k))\gamma(k)-(1-\bar{\beta})\bar{\gamma})\tilde{\mathscr{C}}^{\mathrm{T}}(e(k))P(k+1) \\
&\times [(\alpha(k)-\bar{\alpha})\tilde{\mathscr{F}}(e(k)) + \tilde{\mathscr{A}}_1(e(k))]\} \\
= {}& \mathbb{E}\{\tilde{\mathscr{A}}^{\mathrm{T}}(e(k))P(k+1)\tilde{\mathscr{A}}(e(k)) \\
&+ \bar{\beta}(1-\bar{\beta})\tilde{\mathscr{B}}^{\mathrm{T}}(e(k))P(k+1)\tilde{\mathscr{B}}(e(k)) \\
&+ \tilde{\mathscr{A}}_1^{\mathrm{T}}(e(k))P(k+1)\tilde{\mathscr{A}}_1(e(k)) + ((1-\bar{\beta})\bar{\gamma} \\
&- (1-\bar{\beta})^2\bar{\gamma}^2)\tilde{\mathscr{C}}^{\mathrm{T}}(e(k))P(k+1)\tilde{\mathscr{C}}(e(k)) \\
&+ \bar{\alpha}(1-\bar{\alpha})\tilde{\mathscr{F}}^{\mathrm{T}}(e(k))P(k+1)\tilde{\mathscr{F}}(e(k)) \\
&+ 2\tilde{\mathscr{A}}^{\mathrm{T}}(e(k))P(k+1)\tilde{\mathscr{A}}_1(e(k)) \\
&- 2\bar{\gamma}\bar{\beta}(1-\bar{\beta})\tilde{\mathscr{B}}^{\mathrm{T}}(e(k))P(k+1)\tilde{\mathscr{C}}(e(k))\}, \qquad (17)
\end{aligned}
$$

where the independence properties of $\alpha(k)$, $\beta(k)$ and $\gamma(k)$ in conditions (4) and (7) are utilized. More specifically, to derive the second equality of Equation (17), the following facts have been used:

$$
\begin{aligned}
&\mathbb{E}\{(\alpha(k)-\bar{\alpha})^2\} = \bar{\alpha}(1-\bar{\alpha}), \\
&\mathbb{E}\{(\beta(k)-\bar{\beta})^2\} = \bar{\beta}(1-\bar{\beta}), \\
&\mathbb{E}\{[(1-\beta(k))\gamma(k)-(1-\bar{\beta})\bar{\gamma}]^2\} \\
&\quad = (1-\bar{\beta})\bar{\gamma} - (1-\bar{\beta})^2\bar{\gamma}^2, \\
&\mathbb{E}\{(\beta(k)-\bar{\beta})((1-\beta(k))\gamma(k)-(1-\bar{\beta})\bar{\gamma})\} \\
&\quad = \mathbb{E}\{(\beta(k)-\bar{\beta})(\gamma(k)-\beta(k)\gamma(k) \\
&\qquad + \bar{\beta}\gamma(k) - \bar{\beta}\gamma(k) - \bar{\gamma} + \bar{\beta}\bar{\gamma})\} \\
&\quad = \mathbb{E}\{(\beta(k)-\bar{\beta})((\gamma(k)-\bar{\gamma}) + \gamma(k)(\bar{\beta}-\beta(k)) \\
&\qquad + \bar{\beta}(\bar{\gamma}-\gamma(k)))\} \\
&\quad = \mathbb{E}\{0 - \gamma(k)(\beta(k)-\bar{\beta})^2 + 0\} \\
&\quad = -\bar{\gamma}\bar{\beta}(1-\bar{\beta}).
\end{aligned}
$$

Define $\eta(k) = [e^{\mathrm{T}}(k) \quad e^{\mathrm{T}}(k-d) \quad v^{\mathrm{T}}(k) \quad f^{\mathrm{T}}(k,x(k)) \quad g^{\mathrm{T}}(k,x(k-d)) \quad h^{\mathrm{T}}(C(k)x(k)) \quad 1]^{\mathrm{T}}$, we can easily calculate, in view of Equations (12), (16) and (17) that

$$
\begin{aligned}
&\mathbb{E}\{V(k+1,e(k+1))\} - \mathbb{E}\{V(k,e(k))\} \\
&\quad + \mathbb{E}\{\|\tilde{z}(k)\|^2\} - \gamma^2\mathbb{E}\{\|v(k)\|^2\} \\
&= \mathbb{E}\{\tilde{\mathscr{A}}^{\mathrm{T}}(e(k))P(k+1)\tilde{\mathscr{A}}(e(k)) + \tilde{\mathscr{A}}_1^{\mathrm{T}}(e(k))P(k+1) \\
&\quad \times \tilde{\mathscr{A}}_1(e(k)) + \hat{\alpha}\tilde{\mathscr{F}}^{\mathrm{T}}(e(k))P(k+1)\tilde{\mathscr{F}}(e(k)) \\
&\quad + \hat{\beta}\tilde{\mathscr{B}}^{\mathrm{T}}(e(k))P(k+1)\tilde{\mathscr{B}}(e(k)) \\
&\quad + \hat{\gamma}\tilde{\mathscr{C}}^{\mathrm{T}}(e(k))P(k+1)\tilde{\mathscr{C}}(e(k)) \\
&\quad + 2\tilde{\mathscr{A}}^{\mathrm{T}}(e(k))P(k+1)\tilde{\mathscr{A}}_1(e(k)) \\
&\quad - 2\bar{\gamma}\bar{\beta}(1-\bar{\beta})\tilde{\mathscr{B}}^{\mathrm{T}}(e(k))P(k+1)\tilde{\mathscr{C}}(e(k)) \\
&\quad + e^{\mathrm{T}}(k)Q(k+1)e(k) - e^{\mathrm{T}}(k-d)Q(k-d+1) \\
&\quad \times e(k-d) + \mu(k+1) - e^{\mathrm{T}}(k)P(k)e(k) \\
&\quad - \mu(k) + \tilde{\mathscr{M}}^{\mathrm{T}}(e(k))\tilde{\mathscr{M}}(e(k)) - \gamma^2 v^{\mathrm{T}}(k)v(k)\} \\
&= \mathbb{E}\{\eta^{\mathrm{T}}(k)\mathscr{A}^{\mathrm{T}}(k)P(k+1)\mathscr{A}(k)\eta(k) + \eta^{\mathrm{T}}(k)\mathscr{A}_1^{\mathrm{T}}(k) \\
&\quad \times P(k+1)\mathscr{A}_1(k)\eta(k) + \hat{\alpha}\eta^{\mathrm{T}}(k)\mathscr{F}^{\mathrm{T}}P(k+1)\mathscr{F}\eta(k) \\
&\quad + 2\eta^{\mathrm{T}}(k)\mathscr{A}^{\mathrm{T}}(k)P(k+1)\mathscr{A}_1(k)\eta(k) \\
&\quad - 2\bar{\gamma}\bar{\beta}(1-\bar{\beta})\eta^{\mathrm{T}}(k)\mathscr{B}^{\mathrm{T}}(k)P(k+1)\mathscr{C}(k)\eta(k) \\
&\quad + \hat{\beta}\eta^{\mathrm{T}}(k)\mathscr{B}^{\mathrm{T}}(k)P(k+1)\mathscr{B}(k)\eta(k) + \hat{\gamma}\eta^{\mathrm{T}}(k) \\
&\quad \times \mathscr{C}^{\mathrm{T}}(k)P(k+1)\mathscr{C}(k)\eta(k) + \eta^{\mathrm{T}}(k)\Phi(k)\eta(k)\}, \qquad (18)
\end{aligned}
$$

where

$$
\Phi(k) = \begin{bmatrix}
\Phi_{11}(k) & 0 & 0 & 0 & 0 & 0 & M^{\mathrm{T}}(k)(M(k)-M_f(k))\hat{x}(k) \\
* & -Q(k-d+1) & 0 & 0 & 0 & 0 & 0 \\
* & * & -\gamma^2 I & 0 & 0 & 0 & 0 \\
* & * & * & 0 & 0 & 0 & 0 \\
* & * & * & * & 0 & 0 & 0 \\
* & * & * & * & * & 0 & 0 \\
* & * & * & * & * & * & \hat{\gamma}(k)
\end{bmatrix}
$$

with $\Phi_{11}(k) = -P(k) + Q(k+1) + M^{\mathrm{T}}(k)M(k)$ and $\hat{\gamma}(k) = \mu(k+1) - \mu(k) + \hat{x}^{\mathrm{T}}(k)(M(k)-M_f(k))^{\mathrm{T}} \times (M(k)-M_f(k))\hat{x}(k)$.

On the other hand, utilizing the notations defined in Equation (15), it is straightforward to show that Equations (2) and (3) infer the validity of the inequalities given below:

$$\eta^{\mathrm{T}}(k)\Phi_1(k)\eta(k) \le 0, \quad \eta^{\mathrm{T}}(k)\Phi_2(k)\eta(k) \le 0, \tag{19}$$

where

$$\Phi_1(k) = \begin{bmatrix} \tilde{U}_1(k) & 0 & 0 & \tilde{U}_2(k) & 0 & 0 & \tilde{U}_1(k)\hat{x}(k) \\ * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & I & 0 & 0 & \tilde{U}_2^{\mathrm{T}}(k)\hat{x}(k) \\ * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & \hat{x}^{\mathrm{T}}(k)\tilde{U}_1(k)\hat{x}(k) \end{bmatrix},$$

and

$$\Phi_2(k) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & \tilde{V}_1(k) & 0 & 0 & \tilde{V}_2(k) & 0 & \tilde{V}_1(k)\hat{x}(k-d) \\ * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & I & 0 & \tilde{V}_2^{\mathrm{T}}(k)\hat{x}(k-d) \\ * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & \hat{x}^{\mathrm{T}}(k-d)\tilde{V}_1(k)\hat{x}(k-d) \end{bmatrix}.$$

Similarly, Equation (9) can be transformed to an equivalent matrix inequality given as follows:

$$\eta^{\mathrm{T}}(k)\Phi_3(k)\eta(k) \le 0, \tag{20}$$

where

$$\Phi_3(k) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{C^{\mathrm{T}}(k)H^{\mathrm{T}}(k)}{2} & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & I & -\frac{H(k)C(k)\hat{x}(k)}{2} \\ * & * & * & * & * & * & 0 \end{bmatrix}.$$

By considering Equations (18)–(20) together, the inequality which guarantees the H_∞ performance of the error system turns to:

$$\begin{aligned} &\mathbb{E}\{V(k+1,e(k+1))\} - \mathbb{E}\{V(k,e(k))\} \\ &\quad + \mathbb{E}\{\|\tilde{z}(k)\|^2\} - \gamma^2\mathbb{E}\{\|v(k)\|^2\} \\ &\le \mathbb{E}\{\eta^{\mathrm{T}}(k)(\Phi(k) + \mathscr{A}^{\mathrm{T}}(k)P(k+1)\mathscr{A}(k) \\ &\quad + \mathscr{A}_1^{\mathrm{T}}(k)P(k+1)\mathscr{A}_1(k) + 2\mathscr{A}^{\mathrm{T}}(k)P(k+1)\mathscr{A}_1(k) \\ &\quad + \hat{\beta}\mathscr{B}^{\mathrm{T}}(k)P(k+1)\mathscr{B}(k) + \hat{\gamma}\mathscr{C}^{\mathrm{T}}(k)P(k+1)\mathscr{C}(k) \\ &\quad + \hat{\alpha}\mathscr{F}^{\mathrm{T}}P(k+1)\mathscr{F})\eta(k) \\ &\quad - 2\bar{\gamma}\hat{\beta}\mathscr{B}^{\mathrm{T}}(k)P(k+1)\mathscr{C}(k) - \varepsilon_1(k)\eta^{\mathrm{T}}(k)\Phi_1(k)\eta(k) \\ &\quad - \varepsilon_2(k)\eta^{\mathrm{T}}(k)\Phi_2(k)\eta(k) - \varepsilon_3(k)\eta^{\mathrm{T}}(k)\Phi_3(k)\eta(k)\} \\ &= \mathbb{E}\{\eta^{\mathrm{T}}(k)(\Phi(k) + (\mathscr{A}(k) + \mathscr{A}_1(k))^{\mathrm{T}}P(k+1)(\mathscr{A}(k) \\ &\quad + \mathscr{A}_1(k)) + \hat{\alpha}\mathscr{F}^{\mathrm{T}}P(k+1)\mathscr{F} + \hat{\beta}\mathscr{B}^{\mathrm{T}}(k) \\ &\quad \times P(k+1)\mathscr{B}(k) + \hat{\beta}\bar{\gamma}^2\mathscr{C}^{\mathrm{T}}(k)P(k+1)\mathscr{C}(k) \\ &\quad + (\hat{\gamma} - \hat{\beta}\bar{\gamma}^2)\mathscr{C}^{\mathrm{T}}(k)P(k+1)\mathscr{C}(k) \\ &\quad - 2\bar{\gamma}\hat{\beta}\mathscr{B}^{\mathrm{T}}(k)P(k+1)\mathscr{C}(k))\eta(k) \\ &\quad - \varepsilon_1(k)\eta^{\mathrm{T}}(k)\Phi_1(k)\eta(k) \\ &\quad - \varepsilon_2(k)\eta^{\mathrm{T}}(k)\Phi_2(k)\eta(k) - \varepsilon_3(k)\eta^{\mathrm{T}}(k)\Phi_3(k)\eta(k)\} \\ &= \mathbb{E}\{\eta^{\mathrm{T}}(k)(\Xi_1(k) + (\mathscr{A}(k) + \mathscr{A}_1(k))^{\mathrm{T}}P(k+1)(\mathscr{A}(k) \\ &\quad + \mathscr{A}_1(k)) + \hat{\alpha}\mathscr{F}^{\mathrm{T}}P(k+1)\mathscr{F} + \mathscr{L}^{\mathrm{T}}(k)\mathscr{L}(k) \\ &\quad + (\hat{m}\mathscr{B}(k) - \hat{n}\mathscr{C}(k))^{\mathrm{T}}P(k+1)(\hat{m}\mathscr{B}(k) - \hat{n}\mathscr{C}(k)) \\ &\quad + \hat{\mu}\mathscr{C}^{\mathrm{T}}(k)P(k+1)\mathscr{C}(k))\eta(k)\}, \end{aligned} \tag{21}$$

where $\Xi_1(k) = \Phi(k) - \varepsilon_1(k)\Phi_1(k) - \varepsilon_2(k)\Phi_2(k) - \varepsilon_3(k)\Phi_3(k) - \mathscr{L}^{\mathrm{T}}(k)\mathscr{L}(k)$. Applying the Schur complement formula (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994) and noticing Equation (15), we can easily obtain from Equation (21) that

$$\begin{aligned} &\mathbb{E}\{V(k+1,e(k+1))\} - \mathbb{E}\{V(k,e(k))\} + \mathbb{E}\{\|\tilde{z}(k)\|^2\} \\ &\quad - \gamma^2\mathbb{E}\{\|v(k)\|^2\} \le 0. \end{aligned} \tag{22}$$

Summing up both sides of Equation (22) with k varying from 0 to N yields

$$\begin{aligned} \|\tilde{z}\|^2_{[0,N]} \le \gamma^2\|v\|^2_{[0,N]} + \mathbb{E}\Big\{e^{\mathrm{T}}(0)P(0)e(0) \\ + \sum_{k=-d}^{-1} e^{\mathrm{T}}(k)Q(k+1)e(k)\Big\} + \mu(0), \end{aligned} \tag{23}$$

and the H_∞ performance constraint (13) is assured by also considering the initial condition (14). The proof of this theorem is complete. ∎

According to the H_∞ performance analysis established in Theorem 1, the design problem of the H_∞ filter for the stochastic system (1) is reduced to the one for finding feasible solutions to a set of RLMIs.

Theorem 2 *Let the disturbance attenuation level $\gamma > 0$ and the matrix sequence $\{S(k)\}_{-d\le k\le 0}$ be given. The H_∞ filtering problem is solvable for the discrete time-varying system (1) if there exist two families of positive-definite matrices $\{P(k)\}_{0\le k\le N+1}$, $\{Q(k)\}_{-d+1\le k\le N+1}$, two families of matrices $\{X(k)\}_{0\le k\le N}$, $\{Y(k)\}_{0\le k\le N}$ and four families of positive scalars $\{\varepsilon_1(k)\}_{0\le k\le N}$, $\{\varepsilon_2(k)\}_{0\le k\le N}$, $\{\varepsilon_3(k)\}_{0\le k\le N}$, $\{\mu(k)\}_{0\le k\le N+1}$ satisfying the initial condition (14) and the*

RLMIs

$$\begin{bmatrix} \Xi_1(k) & \Xi_2(k) \\ * & \Xi_3(k) \end{bmatrix} \leq 0 \tag{24}$$

for $0 \leq k \leq N$, *where* $\Xi_1(k)$ *is the same as defined in Equation* (15) *and*

$$\Xi_2(k) = \begin{bmatrix} \tilde{\Gamma}_1(k) & \tilde{\Gamma}_2(k) & -\hat{\mu}C^{\mathrm{T}}(k)Y^{\mathrm{T}}(k) & 0 & 0 \\ A_1^{\mathrm{T}}(k)P(k+1) & 0 & 0 & 0 & 0 \\ \tilde{\Gamma}_3(k) & 0 & 0 & 0 & 0 \\ \bar{\alpha}P(k+1) & 0 & 0 & \hat{\alpha}P(k+1) & 0 \\ (1-\bar{\alpha})P(k+1) & 0 & 0 & -\hat{\alpha}P(k+1) & 0 \\ -\bar{\beta}Y^{\mathrm{T}}(k) & -\hat{m}Y^{\mathrm{T}}(k) & 0 & 0 & 0 \\ \tilde{\Gamma}_4(k) & \tilde{\Gamma}_5(k) & \tilde{\Gamma}_6(k) & 0 & \hat{x}^{\mathrm{T}}(k)\left(M(k)-M_f(k)\right)^{\mathrm{T}} \end{bmatrix},$$

$$\Xi_3(k) = diag\{-P(k+1), -P(k+1), -\hat{\mu}P(k+1), -\hat{\alpha}P(k+1), -I\},$$

in which

$$\tilde{\Gamma}_1(k) = A^{\mathrm{T}}(k)P(k+1) - \bar{\beta}C^{\mathrm{T}}(k)H_1^{\mathrm{T}}(k)Y^{\mathrm{T}}(k) - (1-\bar{\beta})\bar{\gamma}C^{\mathrm{T}}(k)Y^{\mathrm{T}}(k),$$

$$\tilde{\Gamma}_2(k) = -\hat{m}C^{\mathrm{T}}(k)H_1^{\mathrm{T}}(k)Y^{\mathrm{T}}(k) + \hat{n}C^{\mathrm{T}}(k)Y^{\mathrm{T}}(k),$$

$$\tilde{\Gamma}_3(k) = B^{\mathrm{T}}(k)P(k+1) - D^{\mathrm{T}}(k)Y^{\mathrm{T}}(k),$$

$$\tilde{\Gamma}_4(k) = \hat{x}^{\mathrm{T}}(k)A^{\mathrm{T}}(k)P(k+1) - \bar{\beta}\hat{x}^{\mathrm{T}}(k)C^{\mathrm{T}}(k)H_1^{\mathrm{T}}(k)Y^{\mathrm{T}}(k) - (1-\bar{\beta})\bar{\gamma}\hat{x}^{\mathrm{T}}(k)C^{\mathrm{T}}(k)Y^{\mathrm{T}}(k) - \hat{x}^{\mathrm{T}}(k)X^{\mathrm{T}}(k) + \hat{x}^{\mathrm{T}}(k-d)A_1^{\mathrm{T}}(k)P(k+1),$$

$$\tilde{\Gamma}_5(k) = -\hat{m}\hat{x}^{\mathrm{T}}(k)C^{\mathrm{T}}(k)H_1^{\mathrm{T}}(k)Y^{\mathrm{T}}(k) + \hat{n}\hat{x}^{\mathrm{T}}(k)C^{\mathrm{T}}(k)Y^{\mathrm{T}}(k),$$

$$\tilde{\Gamma}_6(k) = -\hat{\mu}\hat{x}^{\mathrm{T}}(k)C^{\mathrm{T}}(k)Y^{\mathrm{T}}(k)$$

and the other symbols are the same as defined in Theorem 1. *Moreover, for each* $0 \leq k \leq N$, *if inequalities* (14) *and* (24) *are feasible, the desired filter is given by Equation* (11) *with the parameters as*

$$F_f(k) = P^{-1}(k+1)X(k), \quad G_f(k) = P^{-1}(k+1)Y(k), \tag{25}$$

and the filter matrix $M_f(k)$ can be obtained by solving the corresponding LMI at time point k.

Proof By substituting Equations (25) into (24) and applying Theorem 1, we can easily conclude the validity of the result. ∎

According to Theorem 2, the following RLMIs algorithm is given for the H_∞ filtering problem which is illuminated by the design ideas in Shen et al. (2010) and Dong et al. (2011). The H_∞ filtering problem can be implemented recursively as follows:

Step 1. Let the H_∞ performance index γ, the final time N, the initial matrix sequence $\{S(k)\}_{-d\leq k\leq 0}$ and the initial states $\{\phi(k)\}_{-d\leq k\leq 0}$ be given. Select appropriate positive-definite matrix $P(0)$, positive-definite matrix sequence $\{Q(s)\}_{-d+1\leq s\leq 0}$ and positive scalar $\mu(0)$ satisfying the initial condition (14) and set $k=0$;

Step 2. Solve the RLMIs (24) to obtain the positive-definite matrices $P(k+1)$, $Q(k+1)$, the positive scalars $\mu(k+1)$, $\varepsilon_1(k)$, $\varepsilon_2(k)$, $\varepsilon_3(k)$, the matrices $X(k)$, $Y(k)$ and the filter parameter matrix $M_f(k)$ by utilizing the known parameters $P(k)$, $\{Q(s)\}_{-d+1\leq s\leq k}$, $\mu(k)$ and $\hat{x}(k)$;

Step 3. From Equation (25), derive the other two filter parameter matrices $F_f(k)$ and $G_f(k)$, then get the state estimate $\hat{x}(k+1)$ according to Equation (11);

Step 4. If $k<N$, set $k=k+1$ and go to *Step* 2, else go to *Step* 5;

Step 5. Exit.

4. Numerical example

In this section, we provide a numerical example to test the proposed design algorithm. Consider a discrete time-varying delayed system described by model (1) with the following time-varying parameters:

$$A(k) = \begin{bmatrix} 0 & 0.1\sin(k) \\ \sin(6k) & 0.2 \end{bmatrix},$$

$$A_1(k) = \begin{bmatrix} 0 & 0.12 \\ -0.12 & 0.1\sin(6k) \end{bmatrix}, \quad D(k) = 1,$$

$$B(k) = \begin{bmatrix} 0.15 \\ 0.3 \end{bmatrix}, \quad C(k) = [0.12\sin(6k) \quad 0.1],$$

$$M(k) = [0.1 \quad 0.1],$$

and the nonlinear functions $f(\cdot,\cdot)$, $g(\cdot,\cdot)$ are the same as those in Shen et al. (2011) represented as follows:

$$f(k,x(k)) = \frac{1}{4}\begin{bmatrix} 0.1x_1(k) + 0.1x_2(k) + 0.25x_2(k)\sin(x_1(k)) \\ \dfrac{0.5(x_1(k)+(1/3)x_2(k))}{1+x_2^2(k)} - 0.1x_1(k) + 0.3x_2(k) \end{bmatrix},$$

$$g(k,x(k-d)) = \frac{1}{4}\begin{bmatrix} \dfrac{0.3(x_1(k-d)+x_2(k-d))}{1 +x_2^2(k-d)} + x_1^2(k-d) + 0.1x_1(k-d) + 0.1x_2(k-d) \\ 0.3x_1(k-d) + 0.3x_2(k-d) \end{bmatrix}.$$

It has been shown in Shen et al. (2011) that the above nonlinearities satisfying conditions (2) and (3) with

$$U_1(k) = V_1(k) = \begin{bmatrix} 0.1 & 0.05 \\ 0 & 0.1 \end{bmatrix},$$
$$U_2(k) = V_2(k) = \begin{bmatrix} -0.05 & 0 \\ -0.05 & 0.05 \end{bmatrix}.$$

It is assumed that the saturation function is

$$\sigma(u(k)) = \begin{cases} u(k) & \text{if} - u_{\max} \le u(k) \le u_{\max}, \\ u_{\max} & \text{if} u(k) > u_{\max}, \\ -u_{\max} & \text{if} u(k) < -u_{\max}, \end{cases}$$

where $u(k)$ is the position value of target. In this example, the saturation value $u_{\max}$ is taken as 0.08, $H_1(k)$ and $H_2(k)$ are set as $H_1(k) = 0.3(1 + |\tanh(k)|)$, $H_2(k) \equiv 1$.

In this simulation, we choose the exogenous disturbance input $v(k)$ to be $v(k) = \exp(-k/20) \times n(k)$, where $n(k)$ is uniformly distributed over $[-0.05, 0.05]$. The random variables $\alpha(k)$, $\beta(k)$ and $\gamma(k)$ are with the probabilities as $\bar{\alpha} = 0.8$, $\bar{\beta} = 0.5$, $\bar{\gamma} = 0.6$ and the constant time delay is set with $d = 1$. For illustration purposes, set $N = 20$, the H_∞ performance level $\gamma = 0.5$, $S(k) = \text{diag}\{20, 1\}$ and the initial value $x(-1) = [0.3 \quad -0.2]^{\mathrm{T}}$, $x(0) = [0.2 \quad 0]^{\mathrm{T}}$. The initial condition (14) is satisfied with $P(0) = I$, $Q(0) = \text{diag}\{2, 2\}$, $\mu(0) = 0.3$. According to the given RLMI algorithm, the time-varying LMIs in Equation (24) can be solved recursively and the desired filter matrices $F_f(k)$, $G_f(k)$ and $M_f(k)$ from time $k = 0$ to $k = 6$ are given in Table 1. It follows from Theorem 2 that the H_∞ filtering problem is solvable for the discrete time-varying system (1).

Table 1. The desired filter parameters.

k	0	1	2	3
$F_f(k)$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.0283 & 0.0654 \\ 0.0692 & 0.0195 \end{bmatrix}$	$\begin{bmatrix} 0.1794 & 0.1440 \\ 0.1702 & -0.3478 \end{bmatrix}$	$\begin{bmatrix} -0.0105 & -0.0629 \\ -0.0859 & 0.2099 \end{bmatrix}$
$G_f(k)$	$\begin{bmatrix} 0.0998 \\ 0.2057 \end{bmatrix}$	$\begin{bmatrix} 0.1029 \\ 0.2105 \end{bmatrix}$	$\begin{bmatrix} 0.1032 \\ 0.2209 \end{bmatrix}$	$\begin{bmatrix} 0.1235 \\ 0.2834 \end{bmatrix}$
$M_f(k)$	$\begin{bmatrix} 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.0524 & 0.1066 \end{bmatrix}$	$\begin{bmatrix} 0.0597 & 0.1066 \end{bmatrix}$	$\begin{bmatrix} 0.0073 & 0.0676 \end{bmatrix}$
k	4	5	6	...
$F_f(k)$	$\begin{bmatrix} 0.0186 & 0.2248 \\ 0.3614 & -0.1823 \end{bmatrix}$	$\begin{bmatrix} -0.0021 & -0.0063 \\ -0.0105 & 0.0618 \end{bmatrix}$	$\begin{bmatrix} -0.2377 & 0.0155 \\ 0.0572 & -0.1260 \end{bmatrix}$	...
$G_f(k)$	$\begin{bmatrix} 0.1103 \\ 0.2924 \end{bmatrix}$	$\begin{bmatrix} 0.1308 \\ 0.3489 \end{bmatrix}$	$\begin{bmatrix} 0.1405 \\ 0.3104 \end{bmatrix}$	...
$M_f(k)$	$\begin{bmatrix} 0.0077 & 0.0681 \end{bmatrix}$	$\begin{bmatrix} -0.0176 & -0.0417 \end{bmatrix}$	$\begin{bmatrix} -0.1006 & 0.0598 \end{bmatrix}$	...

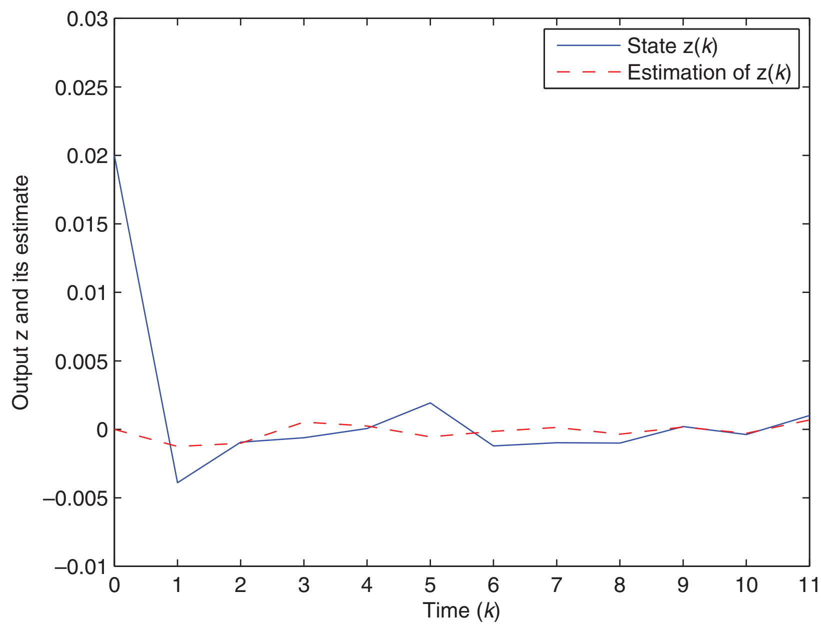

Figure 1. Trajectories of the output z and its estimate $\hat{z}$.

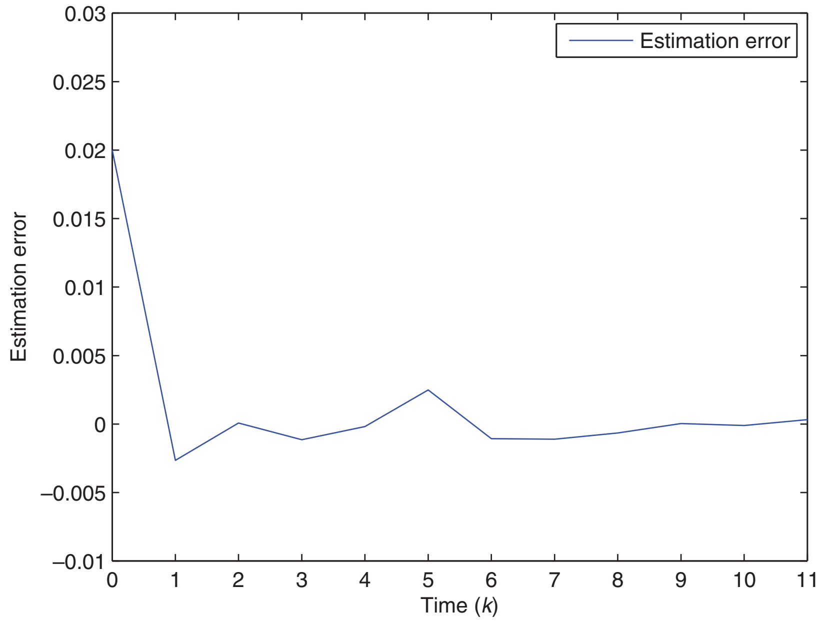

Figure 2. Trajectories of the estimation error $\tilde{z}$.

By employing the filter (11) with the parameters as given in Table 1, the estimation trajectories are shown in Figures 1 and 2. More specially, Figure 1 presents the output $z(k)$ and its estimate $\hat{z}(k)$ and Figure 2 draws the estimation error output $\tilde{z}(k)$. The simulation results further demonstrate the effectiveness of the filter design scheme.

5. Conclusions

In this paper, we have investigated the finite horizon H_∞ filtering problem for the time-varying delayed system with incomplete information such as RONs, ROSSs as well as missing measurements. A time-varying filter has been designed for the system under consideration such that the filtering error system satisfies the H_∞ performance constraints on the finite horizon. By resorting to the stochastic analysis and matrix inequality techniques, sufficient conditions have been derived in the form of RLMIs which not only guarantee the error system to preserve the H_∞ performance but also give the explicit expressions of the desired filtering parameters. Simulation results further demonstrate the feasibility of the proposed filtering methods.

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