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To cite this article: Mario Lefebvre & Foued Zitouni (2014) Analytical solutions to LQG homing problems in one dimension, Systems Science & Control Engineering: An Open Access Journal, 2:1, 41-47, DOI: [10.1080/21642583.2013.878886](https://doi.org/10.1080/21642583.2013.878886)

To link to this article: <https://doi.org/10.1080/21642583.2013.878886>



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Published online: 16 Dec 2014.



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Analytical solutions to LQG homing problems in one dimension

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(Received 21 October 2013; final version received 22 December 2013)

The problem of optimally controlling one-dimensional diffusion processes until they leave a given interval is considered. By linearizing the Riccati differential equation satisfied by the derivative of the value function in the so-called linear quadratic Gaussian homing problem, we are able to obtain an exact expression for the solution to the general problem. Particular problems are solved explicitly.

Keywords: optimal stochastic control; diffusion processes; first-passage time; survival time optimization

AMS Subject Classification: 93E20

1. Introduction

Consider the following problem in one dimension: let $\{X(t), t \geq 0\}$ be a controlled diffusion process that satisfies the stochastic differential equation

$$dX(t) = f[X(t)] dt + b[X(t)]u[X(t)] dt + v^{1/2}[X(t)] dB(t) \quad (1)$$

in which $u(\cdot)$ is the control variable, $b(\cdot)$, $f(\cdot)$ and $v(\cdot) > 0$ are Borel measurable functions, and $\{B(t), t \geq 0\}$ is a standard Brownian motion. The set of admissible controls consists of Borel measurable functions. We assume that the solution of this equation exists for all $t \in [0, \infty)$ and is weakly unique.

We look for the control that minimizes the mathematical expectation of the cost function

$$J(x) = \int_0^{T(x)} \left(\frac{1}{2} q[X(t)] u^2[X(t)] + \lambda \right) dt, \quad (2)$$

where $q(\cdot)$ is a positive Borel measurable function, λ is a real parameter and $T(x)$ is the first-passage time defined by $T(x) = \inf\{t > 0 : X(t) = d_1 \text{ or } d_2 \mid X(0) = x \in (d_1, d_2)\}$. (3)

Whittle (1982, p. 289) has termed this type of problem linear quadratic Gaussian (LQG) homing. Actually, Whittle considered the case of n -dimensional processes. In the general formulation, $T(x_1, \dots, x_n)$ is the first time $(X_1(t), \dots, X_n(t), t)$ enters a stopping set $D \subset \mathbb{R}^n \times (0, \infty)$. Moreover, there can be a termination cost $K[X_1(T(x)), \dots, X_n(T(x)), T(x)]$.

Lefebvre has written a series of papers on LQG homing problems; see, for instance, Lefebvre (2011) and the references therein. Kuhn (1985) and Makasu (2009) solved homing problems with a risk-sensitive cost criterion; see also Whittle (1990, p. 222). Recent papers written on homing problems include the ones published by Makasu Lefebvre (2012a, 2012b).

A practical application of LQG homing problems is an optimal landing problem: assume that $X(t)$ denotes the height of an aircraft at time t . The optimizer controls the aircraft until the time $T(x)$ it touches the runway. Because of the noise in the system, $T(x)$ is a random variable. This problem was considered by Lefebvre (1998).

Another possible application would be to find the control that enables a dam manager to release water in an optimal way when there is a risk of flooding. Suppose that $X(t)$ is the flow of a certain river at time t , and let $T(x)$ be defined as in Equation (3). The constant d_2 would be the value of the flow from which flooding takes place, while d_1 would be a flow value that is considered to be safe. In this application, we would give a very large termination cost if $X(T(x)) = d_2$. Then, the optimal control would be such that the flow will never reach d_2 . In practice, the dam manager does not want to release too much water, because of the economic losses due to the decrease in electricity production it entails.

Now, to obtain the optimal control, we can try to find the value function $F(x)$ defined by

$$F(x) = \inf_{u\{X(t)\}, 0 \leq t \leq T(x)} E[J(x)]. \quad (4)$$

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We assume that this function exists and is twice differentiable. It then satisfies the dynamic programming equation

$$\inf_u H(u) = 0, \quad (5)$$

where $u := u(x)$ and

$$H(u) := \frac{1}{2}q(x)u^2 + \lambda + [f(x) + b(x)u]F'(x) + \frac{1}{2}v(x)F''(x).$$

The optimal control can be expressed as

$$u^* = -\frac{b(x)}{q(x)}F'(x). \quad (6)$$

Hence, we have

$$H(u^*) = \lambda + f(x)F'(x) - \frac{b^2(x)}{2q(x)}[F'(x)]^2 + \frac{1}{2}v(x)F''(x) = 0. \quad (7)$$

That is,

$$\lambda + f(x)G(x) - \frac{b^2(x)}{2q(x)}G^2(x) + \frac{1}{2}v(x)G'(x) = 0, \quad (8)$$

with $G(x) := F'(x)$. Notice that this last equation is a particular Riccati equation.

Next, if the relation

$$\alpha v[X(t)] = \frac{b^2[X(t)]}{q[X(t)]} \quad (9)$$

holds for some positive constant α , then, setting

$$\phi(x) = e^{-\alpha F(x)}, \quad (10)$$

Whittle (1982) has shown that the differential equation (7) satisfied by the value function is transformed into the linear equation

$$\frac{1}{2}v(x)\phi''(x) + f(x)\phi'(x) = \alpha\lambda\phi(x). \quad (11)$$

Since $F(x) = 0$ if $x = d_1$ or d_2 , the boundary conditions are

$$\phi(x) = 1 \quad \text{if } x = d_1 \text{ or } d_2. \quad (12)$$

Now, not only is the differential equation (11) linear, it is actually the Kolmogorov backward equation satisfied by the mathematical expectation (that is, the moment-generating function)

$$L(x; \theta) := E[e^{-\theta\tau(x)}],$$

where $\theta := \alpha\lambda$ and $\tau(x)$ is the same as the first-passage time $T(x)$, but for the uncontrolled process $\{\eta(t), t \geq 0\}$ obtained by setting $u[X(t)] \equiv 0$ in Equation (1). That is,

$$d\eta(t) = f[\eta(t)]dt + v^{1/2}[\eta(t)]dB(t). \quad (13)$$

Moreover, the boundary conditions (12) are the appropriate ones.

Thus, if $b^2[X(t)]/(q[X(t)]v[X(t)])$ is a constant, it is possible to transform the optimal stochastic control problem into a purely probabilistic problem. Notice, however, that obtaining an explicit expression for the function $\phi(x)$ defined in Equation (11) is itself often a difficult problem.

In Section 2, we will show that even if $b^2[X(t)]/(q[X(t)]v[X(t)])$ is not a constant, we can obtain an analytical solution to the LQG problem set up above, as long as $b[X(t)] \neq 0$ in the interval $[d_1, d_2]$. Particular problems will be solved explicitly in Section 3. We will end this paper with a few concluding remarks in Section 4.

2. Analytical solutions to LQG homing problems in one dimension

The transformation $\phi(x)$ defined in Equation (10) enables us to linearize the differential equation satisfied by the function $F(x)$, if the relation (9) is satisfied. Actually, as is well known, it is always possible to linearize the Riccati equation (8). Indeed, we can transform this first-order nonlinear ordinary differential equation into a second-order linear differential equation. However, we then need two boundary conditions to determine the value of the two arbitrary constants that will appear in the general solution to the second-order equation.

It is also sometimes possible to directly obtain an explicit solution to the Riccati equation. The problem is that, in general, we do not have a boundary condition for the function $G(x)$. Therefore, we cannot easily determine the value of the arbitrary constant in the expression obtained for $G(x)$. We must then try to integrate $G(x)$ to obtain $F(x)$ and make use of the boundary conditions $F(d_1) = F(d_2) = 0$ to find out the two arbitrary constants. Unfortunately, this integral is often very difficult to perform. Thus, we cannot find the optimal control explicitly.

The Lefebvre and Zitouni, in a paper published in 2012, showed that it is sometimes possible to use the symmetry present in the problem to determine the value of x_0 for which the function $F(x)$ should have either a maximum or a minimum, so that $G(x_0)$ is equal to zero. Then, if we are indeed able to solve the Riccati equation explicitly, we can obtain an exact expression for the optimal control u^* . However, in the general case, finding the exact value of x_0 that corresponds to an extremal point of the value function is not an easy problem.

Let

$$z(x) = \exp \left\{ - \int \frac{b^2(x)}{q(x)v(x)} G(x) dx \right\}. \quad (14)$$

If $b^2(x)/[q(x)v(x)]$ is a constant, then the function $z(x)$ is equivalent to $\phi(x)$. Assuming that $b(x) \neq 0$ in the interval $[d_1, d_2]$, we can write that

$$G(x) = -\frac{z'(x)q(x)v(x)}{z(x)b^2(x)}. \quad (15)$$

We find that the function $z(x)$ satisfies the following differential equation:

$$\lambda - \frac{f(x)q(x)v(x)}{b^2(x)} \frac{z'(x)}{z(x)} - \frac{1}{2} \frac{q(x)v^2(x)}{b^2(x)} \frac{z''(x)}{z(x)} - \frac{1}{2} \frac{v(x)[q'(x)v(x) + q(x)v'(x)]}{b^2(x)} \frac{z'(x)}{z(x)} + \frac{b'(x)q(x)v^2(x)}{b^3(x)} \frac{z'(x)}{z(x)} = 0. \tag{16}$$

Simplifying, we obtain the second-order linear differential equation

$$z''(x) + \left\{ 2 \frac{f(x)}{v(x)} + \frac{[q'(x)v(x) + q(x)v'(x)]}{q(x)v(x)} - 2 \frac{b'(x)}{b(x)} \right\} z'(x) - 2\lambda \frac{b^2(x)}{q(x)v^2(x)} z(x) = 0. \tag{17}$$

Let

$$z(x) = c_1 z_1(x) + c_2 z_2(x)$$

be the general solution of Equation (17), where c_1 and c_2 are arbitrary constants. We have

$$z'(x) = -z(x) \frac{b^2(x)}{q(x)v(x)} G(x). \tag{18}$$

Since $F(d_1) = F(d_2) = 0$, we can state that there exists a point $x_0 \in (d_1, d_2)$ for which $G(x_0) = 0$. We then deduce from the previous equation that $z'(x_0) = 0$ as well. Hence, we may write that

$$c_2 = -c_1 \frac{z_1'(x_0)}{z_2'(x_0)}.$$

We assume that $z_i'(x_0) \neq 0$ for $i = 1, 2$. It follows that both c_1 and c_2 must be different from zero. Thus, from Equation (15), we obtain that

$$G(x) = - \frac{z_2'(x_0)z_1'(x) - z_1'(x_0)z_2'(x)}{z_2'(x_0)z_1(x) - z_1'(x_0)z_2(x)} \frac{q(x)v(x)}{b^2(x)}.$$

Next, we set

$$F(x) = \int_{d_1}^x G(y) dy.$$

As we mentioned above, in general it is very difficult to obtain an exact expression for the point x_0 . However, we can estimate x_0 by making use of the condition

$$0 = F(d_2) = - \int_{d_1}^{d_2} \frac{z_2'(x_0)z_1'(y) - z_1'(x_0)z_2'(y)}{z_2'(x_0)z_1(y) - z_1'(x_0)z_2(y)} \frac{q(y)v(y)}{b^2(y)} dy. \tag{19}$$

Indeed, if we denote the integral in the previous equation by $I(x_0)$, then we can use a mathematical software to compute

$I(x_0)$ for $x_0 \in (d_1, d_2)$. It is not difficult to estimate x_0 quite precisely.

Summing up, we can state the following proposition.

PROPOSITION 2.1 *The control $u^*(x)$ that minimizes the expected value of the cost function $J(x)$ defined in Equation (2) is given by*

$$u^*(x) = \frac{z_2'(x_0)z_1'(x) - z_1'(x_0)z_2'(x)}{z_2'(x_0)z_1(x) - z_1'(x_0)z_2(x)} \frac{v(x)}{b(x)},$$

in which $z_1(x)$ and $z_2(x)$ are two linearly independent solutions of Equation (17). Furthermore, x_0 is such that $G(x_0) = 0$ and can be obtained from the condition (19).

Remarks

- (i) If we can determine the exact value of x_0 with the help of a symmetry argument, for instance, then of course we do not have to make use of the condition (19).
- (ii) The proposition provides an expression for the optimal control $u^*(x)$, which only depends on the derivative of the value function $F(x)$. If one needs $F(x)$ as well, then one must be able to integrate the function $G(x)$. In general, as we mentioned above, this is not an easy task.
- (iii) We have assumed above that $b(x) \neq 0$ in the interval $[d_1, d_2]$. If there exists a point x_1 in this interval for which $b(x_1) = 0$, then we deduce from Equation (18) that $z'(x_1) = 0$, which yields the following expression for $G(x)$:

$$G(x) = - \frac{z_2'(x_1)z_1'(x) - z_1(x_1)z_2'(x)}{z_2(x_1)z_1(x) - z_1(x_1)z_2(x)} \frac{q(x)v(x)}{b^2(x)},$$

in which there is no unknown. Hence, we cannot satisfy the boundary condition $F(d_2) = 0$ (respectively, $F(d_1) = 0$) by setting

$$F(x) = \int_{d_1}^x G(y) dy \times \left(\text{respectively, } F(x) = \int_x^{d_2} G(y) dy \right).$$

Actually, there could be some points in $[d_1, d_2]$ at which $b(x)$ is equal to zero, but we should then allow the functions v and q to be non-negative (rather than strictly positive) and the ratio $b^2(x)/[q(x)v(x)]$ should always be positive in the interval $[d_1, d_2]$. For instance, with $d_1 = 0$ and $d_2 = 1$, we could have: $b(x) = x$, $q(x) = q_0 > 0$ and $v(x) = x^2$, in which case we can use the result in Whittle (1982).

In the next section, two particular problems will be solved explicitly.

3. Particular examples

3.1. Example 1

In the first example that we present, we assume that $f[X(t)] = X(t)$ and $b[X(t)] = v[X(t)] = X^2(t) + 1$, so that the stochastic differential equation (1) becomes

$$dX(t) = X(t) dt + [X^2(t) + 1]u[X(t)] dt + \{X^2(t) + 1\}^{1/2} dB(t).$$

Moreover, we set $d_1 = -1$ and $d_2 = 1$, and we choose the cost function

$$J(x) = \int_0^{T(x)} \left(\frac{1}{2} u^2[X(t)] + \lambda \right) dt. \quad (20)$$

That is, we take $q[X(t)] \equiv 1$. We assume that the parameter λ is positive. Therefore, the aim is to minimize the (expected) time spent by the process in the interval $(-1, 1)$, taking the quadratic control costs into account.

Remark We can choose the parameter λ as large as we want. However, when λ is negative, there is a minimum value that it can take. Otherwise, the value function will become infinite.

With the choices above, Equation (17) becomes

$$z''(x) - 2\lambda z(x) = 0. \quad (21)$$

The general solution of this equation can be written as

$$z(x) = c_1 e^{-\sqrt{2\lambda}x} + c_2 e^{\sqrt{2\lambda}x},$$

where c_1 and c_2 are arbitrary constants.

Now, by symmetry, it is clear that the value function $F(x)$ takes on its maximum value at $x = 0$. Hence (see Equation (18)),

$$z'(0) = -z(0)b(0)G(0) = 0.$$

It follows that

$$z(x) = 2c_1 \cosh(\sqrt{2\lambda}x)$$

and

$$G(x) = -\sqrt{2\lambda} \frac{\sinh(\sqrt{2\lambda}x)}{\cosh(\sqrt{2\lambda}x)} \frac{1}{x^2 + 1}. \quad (22)$$

The functions $G(x)$ and $F(x)$ and the optimal control $u^*(x) = -b(x)G(x)$ are shown in Figures 1–3, respectively, in the case when $\lambda = 1$.

Remarks

- (i) If one did not deduce from the symmetry in the problem that the value of x_0 for which $G(x_0) = 0$ is $x_0 = 0$, then one can plot (with $\lambda = 1$)

$$F(1) = \sqrt{2} \int_{-1}^1 \frac{e^{\sqrt{2}x_0} e^{-\sqrt{2}x} - e^{-\sqrt{2}x_0} e^{\sqrt{2}x}}{e^{\sqrt{2}x_0} e^{-\sqrt{2}x} + e^{-\sqrt{2}x_0} e^{\sqrt{2}x}} \frac{1}{x^2 + 1} dx$$

for $x_0 \in (-1, 1)$. We easily deduce from Figure 4 that x_0 is indeed equal to 0, since $F(1) = 0$ for this value.

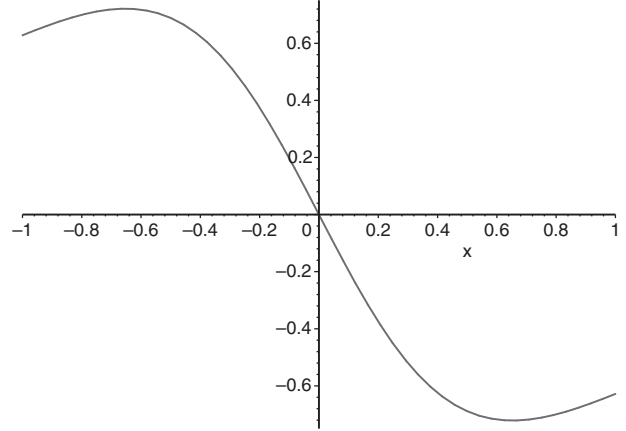


Figure 1. Derivative $G(x)$ of the value function when $d_1 = -1$, $d_2 = 1$, $f(x) = x$, $b(x) = v(x) = x^2 + 1$, $q(x) \equiv 1$ and $\lambda = 1$.

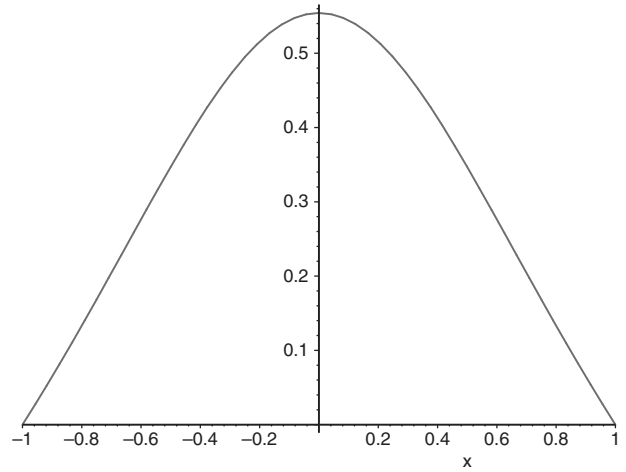


Figure 2. Value function $F(x)$ when $d_1 = -1$, $d_2 = 1$, $f(x) = x$, $b(x) = v(x) = x^2 + 1$, $q(x) \equiv 1$ and $\lambda = 1$.

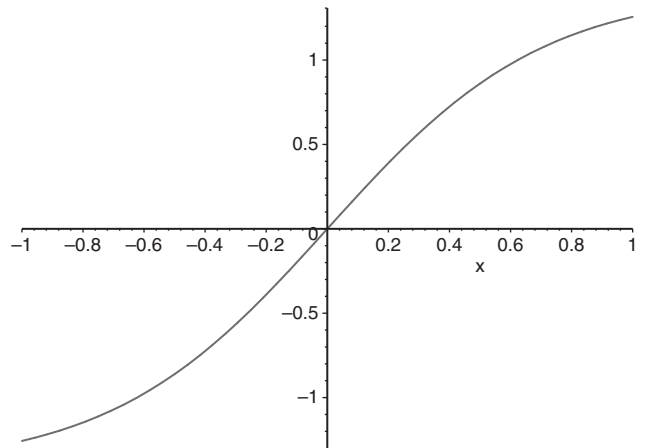


Figure 3. Optimal control $u^*(x)$ when $d_1 = -1$, $d_2 = 1$, $f(x) = x$, $b(x) = v(x) = x^2 + 1$, $q(x) \equiv 1$ and $\lambda = 1$.

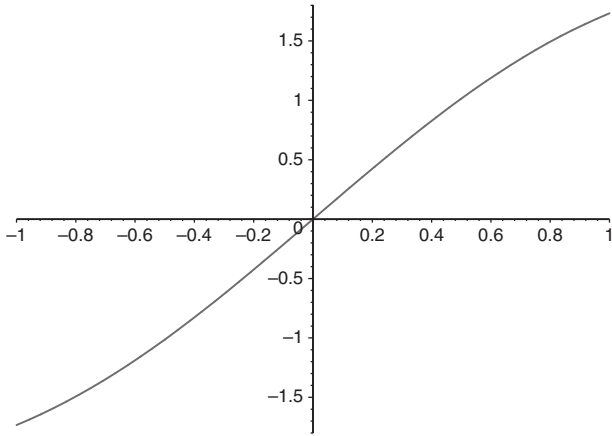


Figure 4. The value of $F(1)$ as a function of $x_0 \in (-1, 1)$ in Example 1.

Notice that $G(x)$ is an odd function when $x_0 = 0$, which implies that x_0 is exactly equal to 0.

- (ii) It is interesting to compare the value of $F(x)$, which is obtained by using the optimal control above, and the expected value of the cost function $J(x)$ if the optimizer chooses $u[X(t)] \equiv 0$. We then have

$$E[J(x)] = E \left[\int_0^{T(x)} \lambda dt \right] = \lambda E[T(x)].$$

Let $m(x)$ denote $E[T(x)]$. It is well known that this function satisfies (here) the ordinary differential equation

$$\frac{1}{2}(x^2 + 1)m''(x) + xm'(x) = -1,$$

subject to the boundary conditions $m(-1) = m(1) = 0$; see, for instance, [Lefebvre \(2007\)](#). We find that

$$m(x) = \ln \left(\frac{2}{x^2 + 1} \right).$$

We show in Figure 5 the difference $D(x) := F(x) - m(x)$.

- (iii) If we consider the Riccati differential equation satisfied by $G(x)$, namely

$$\lambda + xG(x) - \frac{(x^2 + 1)^2}{2}G^2(x) + \frac{1}{2}(x^2 + 1)G'(x) = 0,$$

we find that

$$G(x) = \frac{\sqrt{2\lambda i} \tan(\sqrt{2\lambda i}x + c)}{x^2 + 1}.$$

It is not obvious to determine the constant c , which (contrary to x_0) can take any value. If we apply the condition $G(0) = 0$, we retrieve the solution given in Equation (22).

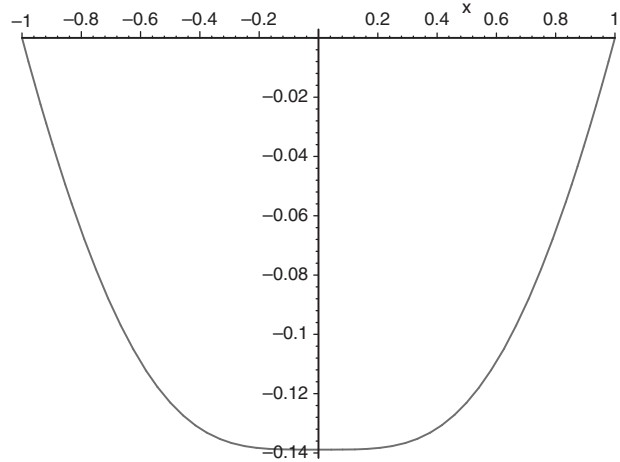


Figure 5. Difference between the value function $F(x)$ and the expected value of the cost function $J(x)$ when $u[X(t)] \equiv 0$ in Example 1.

3.2. Example 2

We now consider the case when $f[X(t)] \equiv 0$, $b[X(t)] = X(t)$ and $q[X(t)] = v[X(t)] \equiv 1$. The stochastic differential equation (1) is thus

$$dX(t) = X(t)u[X(t)] dt + dB(t)$$

and the cost function is the same as in the previous example. Again, we assume that the parameter λ is positive. Finally, we take $d_1 = 1$ and $d_2 = 2$. Notice that the uncontrolled process in this example is a standard Brownian motion.

The function $z(x)$ satisfies the ordinary differential equation

$$z''(x) - \frac{2}{x}z'(x) - 2\lambda x^2 z(x) = 0,$$

whose general solution can be written as follows:

$$z(x) = x^{3/2} [c_1 I_{3/4}(\sqrt{\lambda/2}x^2) + c_2 \sqrt{x} K_{3/4}(\sqrt{\lambda/2}x^2)],$$

where $I_{3/4}$ and $K_{3/4}$ are modified Bessel functions; see [Abramowitz and Stegun \(1965, p. 374\)](#).

Making use of the function $z(x)$ and choosing $\lambda = 1$, we obtain that

$$G(x) = -\frac{\sqrt{2}}{x} \frac{K_{1/4}(x_0^2/\sqrt{2})I_{-1/4}(x^2/\sqrt{2}) - I_{-1/4}(x_0^2/\sqrt{2})K_{1/4}(x^2/\sqrt{2})}{K_{1/4}(x_0^2/\sqrt{2})I_{3/4}(x^2/\sqrt{2}) + I_{-1/4}(x_0^2/\sqrt{2})K_{3/4}(x^2/\sqrt{2})}.$$

Contrary to Example 1, we cannot easily determine the value of x_0 for which $G(x_0) = 0$. Intuitively, this value should be near 1.5, namely near the middle of the interval $[1, 2]$.

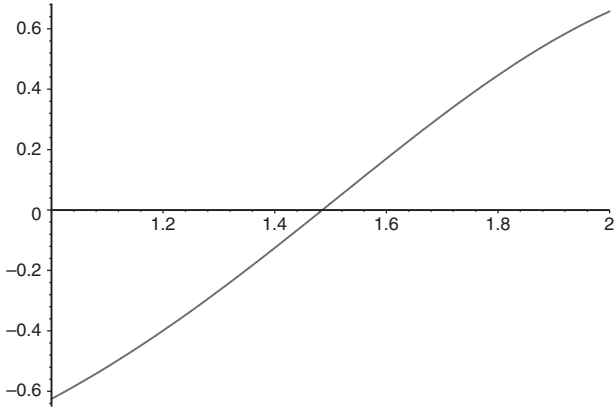


Figure 6. The value of $F(2)$ as a function of $x_0 \in (1, 2)$ in Example 2.

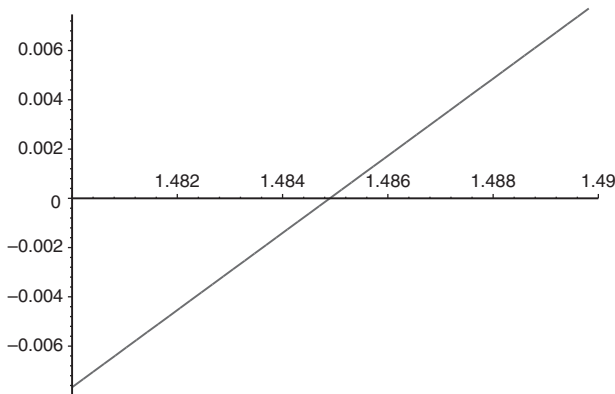


Figure 7. The value of $F(2)$ as a function of $x_0 \in [1.48, 1.49]$ in Example 2.

To obtain an approximate value for x_0 , we plot

$$F(2) = \int_1^2 G(x) dx$$

as a function of $x_0 \in (1, 2)$. We obtain the curve shown in Figure 6. We see that x_0 is slightly larger than 1.48. Therefore, we then plot $F(2)$ for $x_0 \in [1.48, 1.49]$; see Figure 7. We can now state that $x_0 \simeq 1.485$, which should be precise enough.

Next, we present the value function $F(x)$ and the optimal control $u^*(x)$ in Figures 8 and 9, respectively.

Finally, we compute the difference between $F(x)$ and the expected value of $J(x)$ when $u[X(t)] \equiv 0$. Proceeding as in Example 1, we easily find that this expected value is given by

$$E[J(x)] = E[T(x)] = -x^2 + 3x - 2.$$

See Figure 10.

Remark In this example, $G(x)$ satisfies the Riccati differential equation

$$1 - \frac{x^2}{2} G^2(x) + \frac{1}{2} G'(x) = 0.$$

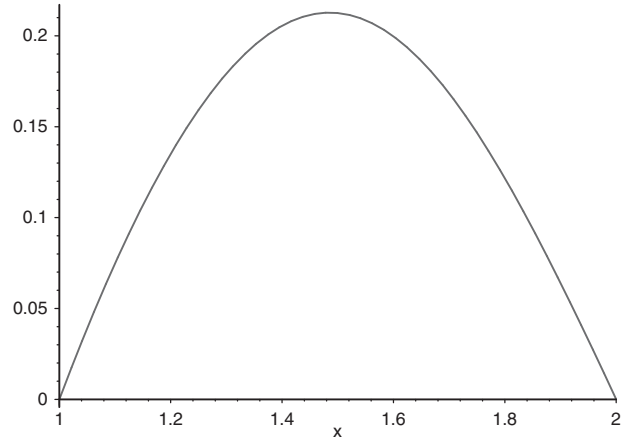


Figure 8. Value function $F(x)$ when $d_1 = 1, d_2 = 2, f(x) \equiv 0, b(x) = x, v(x) = q(x) \equiv 1$ and $\lambda = 1$.

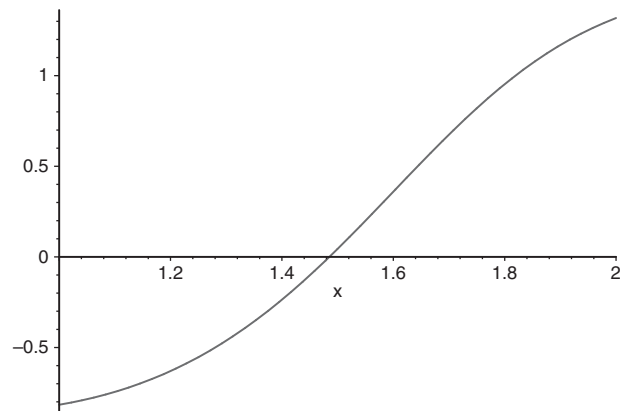


Figure 9. Optimal control $u^*(x)$ when $d_1 = 1, d_2 = 2, f(x) \equiv 0, b(x) = x, v(x) = q(x) \equiv 1$ and $\lambda = 1$.

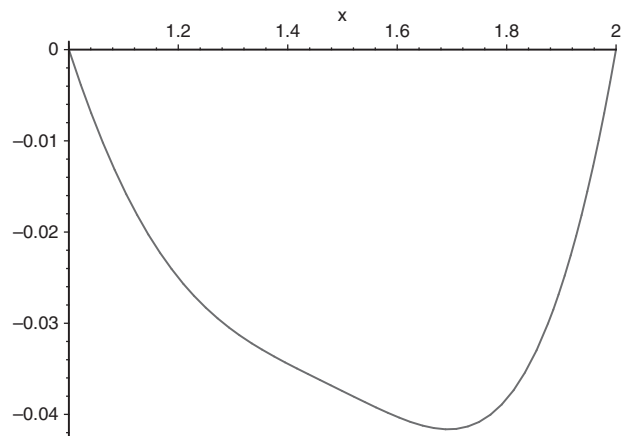


Figure 10. Difference between the value function $F(x)$ and the expected value of the cost function $J(x)$ when $u[X(t)] \equiv 0$ in Example 2.

Its solution can be written as

$$G(x) = -\frac{\sqrt{2} I_{-1/4}(x^2/\sqrt{2}) - cK_{1/4}(x^2/\sqrt{2})}{x I_{3/4}(x^2/\sqrt{2}) + cK_{3/4}(x^2/\sqrt{2})}.$$

As in the previous example, it is more difficult to find the value of the constant c than to determine x_0 .

4. Conclusion

By defining the function $z(x)$ in Equation (14) in terms of the derivative $G(x)$ of the value function, we were able to linearize the Riccati differential equation satisfied by $G(x)$. Moreover, making use of the fact that $z'(x_0) = 0$ for a certain $x_0 \in (d_1, d_2)$, we obtained analytical solutions to much more general LQG homing problems than the ones that can be solved when the relation in Equation (9) holds.

In Section 3, we presented two particular problems that we were able to solve explicitly, even though Equation (9) does not hold. In the first problem, we deduced from symmetry that $x_0 = 0$, while in the second one we showed that we can easily obtain a very good approximation for x_0 .

Next, we could try to extend our results to the n -dimensional case. We could at least find problems in two or more dimensions for which symmetry arguments can be used to obtain explicit (and exact) expressions for the optimal control.

Finally, LQG problems can be modified in various ways: we can assume that the noise term is *uniform white noise* rather than *Gaussian white noise*, the formulation of the problem with linear state dynamics and quadratic control costs can be modified, a parameter that takes the risk-sensitivity of the optimizer into account can be introduced, etc.

Acknowledgements

The authors are grateful to the anonymous reviewers whose constructive comments helped to improve their paper.

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