

## Robust and resilient state-dependent control of discrete-time nonlinear systems with general performance criteria

Xin Wang, Edwin E. Yaz & James Long

To cite this article: Xin Wang, Edwin E. Yaz & James Long (2014) Robust and resilient state-dependent control of discrete-time nonlinear systems with general performance criteria, *Systems Science & Control Engineering: An Open Access Journal*, 2:1, 48-54, DOI: [10.1080/21642583.2013.877858](https://doi.org/10.1080/21642583.2013.877858)

To link to this article: <https://doi.org/10.1080/21642583.2013.877858>



© 2014 The Author(s). Published by Taylor & Francis.



Published online: 31 Jan 2014.



[Submit your article to this journal](#)



Article views: 740



[View related articles](#)



[View Crossmark data](#)



Citing articles: 1 [View citing articles](#)

## Robust and resilient state-dependent control of discrete-time nonlinear systems with general performance criteria

Xin Wang<sup>a\*</sup>, Edwin E. Yaz<sup>b</sup> and James Long<sup>c</sup>

<sup>a</sup>Department of Electrical and Computer Energy Engineering, Southern Illinois University Edwardsville, Edwardsville, IL 62026, USA;

<sup>b</sup>Department of Electrical and Computer Engineering, Marquette University, Milwaukee, WI 53201, USA; <sup>c</sup>Computer Systems Engineering, Oregon Institute of Technology, Klamath Falls, OR 97601, USA

(Received 19 November 2013; final version received 18 December 2013)

A novel state-dependent control approach for discrete-time nonlinear systems with general performance criteria is presented. This controller is robust for unstructured model uncertainties, resilient against bounded feedback control gain perturbations in achieving optimality for general performance criteria to secure quadratic optimality with inherent asymptotic stability property together with quadratic dissipative type of disturbance reduction. For the system model, unstructured uncertainty description is assumed, which incorporates commonly used types of uncertainties, such as norm-bounded and positive real uncertainties as special cases. By solving a state-dependent linear matrix inequality (LMI) at each time step, sufficient condition for the control solution can be found which satisfies the general performance criteria. The results of this paper unify existing results on nonlinear quadratic regulator,  $H_\infty$  and positive real control to provide a novel robust control design. The effectiveness of the proposed technique is demonstrated by simulation of the control of inverted pendulum.

**Keywords:** nonlinear control; robust control; linear matrix inequality

### 1. Introduction

Optimal control of nonlinear systems is traditionally characterized in terms of Hamilton Jacobi Equations (HJEs). The solution of the HJEs provides the necessary and sufficient optimal control condition for nonlinear systems. Furthermore, when the controlled system is linear time invariant and the performance index is linear quadratic regulator (LQR), the HJEs are reduced to Algebraic Riccati Equations (AREs). As for  $H_\infty$  nonlinear control problem, the optimal control solution is equivalent to solving the corresponding Hamilton Jacobi Inequalities (HJIs). However, HJEs and HJIs, which are first-order partial differential equations and inequalities, cannot be solved for more than a few state variables. In the past few years, it has been shown that the problems of quadratic regulation and  $H_\infty$  nonlinear control can be effectively solved by state-dependent Riccati equation (SDRE) and nonlinear matrix inequality (NLMI) techniques (Huang & Lu, 1996). The state-dependent LMI control of nonlinear systems, as pointed out in Wang and Yaz (2009), Wang, Yaz, and Jeong (2010), and Wang, Yaz, and Yaz (2010, 2011), synthesizes a controller to achieve mixed nonlinear quadratic regulator (NLQR) and  $H_\infty$  control.

Dissipative control for linear systems has also received considerable attention over the past two decades. The concept of dissipative system was first introduced by Willems (1972a, 1972b), and further generalized by Hill and Moylan

(1976, 1980), playing an important role in systems, circuits and controls. The theory of dissipative systems generalizes the basic tools including the passivity theorem, bounded real lemma, Kalman–Yakubovich lemma and circle criterion. Dissipativity performance includes  $H_\infty$  performance, passivity, positive realness and sector-bounded constraint as special cases. Research addressing the problems of  $H_\infty$  and positive real control systems can be found in Safonov, Jonckheere, Verma, and Limebeer (1987), Doyle, Glover, Khargonekar, and Francis (1989), Haddad and Bernstein (1991), Sun, Khargonekar, and Shim (1994), and Shim (1996). Control of uncertain linear systems with  $l_2$ -bounded structured uncertainty satisfying  $H_\infty$  and passivity criteria has been tackled in Petersen (1987) and Khargonekar, Petersen, and Zhou (1990). More recent development involving the quadratic dissipative control for linear systems problem has been tackled in Xie, Xie, and De Souza (1998) and Tan, Soh, and Xie (2000).

In this paper, we further consider the problem of optimal, robust and resilient LMI control of discrete-time nonlinear systems with general performance criteria. The controller is robust for model uncertainties and resilient for gain perturbations. As for the uncertain nonlinear systems, we consider a general form of  $l_2$ -bounded uncertainty description, without any standard structure, incorporating commonly used types of uncertainty, such as norm-bounded

\*Corresponding author. Email: [xwang@siue.edu](mailto:xwang@siue.edu)

and positive real uncertainties as special cases. The purpose behind this novel approach is to convert a nonlinear system control problem into a convex optimization problem which is solved by state-dependent LMI. The recent development in convex optimization provides very efficient algorithms for solving LMIs. If a solution can be expressed in an LMI form, then there exist optimization means providing efficient global numerical solutions (Boyd, Ghaoui, Feron, & Balakrishnan, 1994). Therefore if the LMI is feasible, then the LMI control technique provides asymptotically stable solutions satisfying the general performance criteria. We further propose to employ general performance criteria to design the controller guaranteeing the quadratic sub-optimality with inherent stability property in combination with dissipativity type of disturbance attenuation. The general performance criteria is a generalization of the NLQR,  $H_\infty$ , positive realness and sector-bounded constraint. The results of the paper unify existing results on NLQR,  $H_\infty$  and positive real control and provide a novel robust control design. The paper is organized as follows: Section 2 covers the general performance criteria including the performance of NLQR,  $H_\infty$ , positive realness and sector-bounded constraint. Section 3 presents state-dependent LMI-based control for nonlinear systems achieving general performance criteria. In the final section, an inverted pendulum on a cart system is used to demonstrate the effectiveness and robustness of the new approach.

## 2. System model and general performance criteria

The following notation is used in this work:  $\mathfrak{R}_+$  stands for the set of non-negative real numbers,  $\mathfrak{R}^n$  stands for the  $n$ -dimensional Euclidean space.  $x_k \in \mathfrak{R}^n$  denotes  $n$ -dimensional real vector with norm  $\|x_k\| = (x_k^T x_k)^{1/2}$ , where  $(\cdot)^T$  indicates transpose.  $\mathfrak{R}^{n \times m}$  is the set of  $n \times m$  real matrices.  $I_n$  is the  $n \times n$  identity matrix.  $A \geq 0$  for a symmetric matrix denotes a positive semi-definite matrix.  $l_2$  is the space of finite-dimensional vectors with finite energy:  $\sum_{k=0}^{\infty} \|x_k\|^2 < \infty$ . The inner product on  $\mathfrak{R}^n$  is defined by  $\langle u, v \rangle = \sum_{i=1}^n u_i v_i$ .

Consider the nonlinear dynamical system and performance output equation as following:

$$\begin{aligned} x_{k+1} &= f(x_k, u_k, w_k) \\ &= (A(x_k) + \Delta_A(x_k))x_k + (B(x_k) + \Delta_B(x_k))u_k \\ &\quad + (E(x_k) + \Delta_E(x_k))w_k \\ &= (A + \Delta_A)x_k + (B + \Delta_B)u_k + (E + \Delta_E)w_k, \quad (1) \\ z_k &= g(x_k, u_k) = C_k \cdot x_k + D_k \cdot w_k, \quad (2) \end{aligned}$$

where  $x_k \in \mathfrak{R}^n$  is the state of the dynamical system;  $u_k \in \mathfrak{R}^m$  the applied input;  $w_k \in \mathfrak{R}^p$  the  $l_2$  type of disturbance;  $z_k \in \mathfrak{R}^r$  the performance output;  $f, g$  the nonlinear vector functions;  $A_k \in \mathfrak{R}^{n \times n}, B_k \in \mathfrak{R}^{n \times m}, E_k \in \mathfrak{R}^{n \times p}, C_k \in \mathfrak{R}^{r \times n}$  and  $D_k \in \mathfrak{R}^{r \times p}$  the state-dependent coefficient (SDC)

matrices and  $\Delta_A \in \mathfrak{R}^{n \times n}, \Delta_B \in \mathfrak{R}^{n \times m}$  and  $\Delta_E \in \mathfrak{R}^{n \times p}$  the state-dependent uncertainty matrices.

Note that the discrete-time, state feedback, input affine and autonomous nonlinear system must be fully controllable and state observable. The way of finding the nominal system parameter matrices  $A_k, B_k, E_k, C_k$  and  $D_k$  is a process of factorizing the nonlinear system into a linear-like structure which contains SDC matrices, so-called *mathematical factorization*. The  $l_2$ -bounded perturbation matrices  $\Delta_A, \Delta_B$  and  $\Delta_E$  are the unstructured uncertainty matrices, which can also be time-varying state-dependent matrices. Without any standard structures, the uncertainty matrices provide us a general framework to compensate for effect of the unmodeled system dynamics, external disturbances, perturbation and noise. The commonly used types of uncertainty, such as norm-bounded, structured uncertainties and positive real uncertainties are special cases of the uncertainty matrices description in this work.

It is assumed that the full state is available for feedback and the state feedback control input is given by

$$u_k = (K(x_k) + \Delta_k(x_k))x_k = (K_k + \Delta_k)x_k, \quad (3)$$

where there is additive (possibly state dependent) perturbation on the feedback gain.

Introducing the quadratic energy supply function  $E$  associated with the system equations, defined by Hill and Moylan (1976, 1980) as

$$E(z_k, w_k) = \langle z_k, Qz_k \rangle + 2 \langle z_k, Sw_k \rangle + \langle w_k, R w_k \rangle, \quad (4)$$

where  $Q \in \mathfrak{R}^{r \times r}, S \in \mathfrak{R}^{r \times p}, R \in \mathfrak{R}^{p \times p}$  are the chosen weighting matrices. Next, from the definition of dissipativity, we have

**DEFINITION 2.1** Given matrices  $Q \in \mathfrak{R}^{r \times r}, S \in \mathfrak{R}^{r \times p}, R \in \mathfrak{R}^{p \times p}$  with  $Q, R$  symmetric, the system (1) and (2) with energy function (4) is said to be  $(Q, S, R)$  dissipative if for some real function  $\beta(\cdot)$  with  $\beta(0) = 0$ ,

$$E(z_k, w_k) + \beta(x_0) \geq 0, \forall w \in l_2, \forall k \geq 0. \quad (5)$$

Furthermore, if for some scalar  $\alpha > 0$ ,

$$E(z_k, w_k) + \beta(x_0) \geq \alpha \langle w_k, w_k \rangle, \forall w \in l_2, \forall k \geq 0. \quad (6)$$

The system (1) and (2) is said to be strictly  $(Q, S, R)$  dissipative.

**THEOREM 1** Consider the quadratic function  $V_k = x_k^T P_k x_k > 0$ , matrices  $Q \in \mathfrak{R}^{r \times r}, S \in \mathfrak{R}^{r \times p}, R \in \mathfrak{R}^{p \times p}$  with  $Q, R$  symmetric,  $M \in \mathfrak{R}^{n \times n}, M > 0, N \in \mathfrak{R}^{m \times m}, N > 0$  with  $M, N$  symmetric, the system (1) and (2) control will achieve mixed NLQR and dissipative performance if the following condition holds:

$$\begin{aligned} V_{k+1} - V_k + x_k^T M x_k + u_k^T N u_k \\ - (z_k^T Q z_k + 2z_k^T S w_k + w_k^T R w_k) < 0, \forall k \geq 0. \quad (7) \end{aligned}$$

*Proof* Note that upon summation over  $k$ , we have

$$\begin{aligned} & \sum_{i=0}^{N-1} [z_k^T Q z_k + 2z_k^T S w_k + w_k^T R w_k] \\ & > \sum_{i=0}^{N-1} [x_k^T M x_k + u_k^T N u_k] + V_N - V_0. \end{aligned} \quad (8)$$

Let  $\beta(x_0) = V_0$ ,  $V_k(x) = x_k^T P_k x_k$ ,  $V_N \geq 0$ , Equation (8) implies

$$\sum_{i=0}^{N-1} (z_k^T Q z_k + 2z_k^T S w_k + w_k^T R w_k) + \beta(x_0) > 0, \quad (9)$$

which is the condition for  $(Q, S, R)$  dissipativity. ■

*Remark 1* By adding the terms  $x_k^T M x_k + u_k^T N u_k$ , we include the NLQR control performance into the original  $(Q, S, R)$  dissipative criteria.

*Remark 2* Note that both  $H_\infty$  and passivity are special cases of  $(Q, S, R)$  dissipativity.

The special cases are summarized as follows:

*Case 1*  $Q = -I, S = 0, R = \gamma^2 I$ , the strict  $(Q, S, R)$  dissipativity reduces  $H_\infty$  design (Doyle et al., 1989). The overall control design satisfies mixed NLQR– $H_\infty$  performance.

*Case 2*  $Q = 0, S = I, R = 0$ , the strict  $(Q, S, R)$  dissipativity reduces to strict positive realness (Sun et al., 1994). The overall control design satisfies mixed NLQR–strict positive realness performance.

*Case 3*  $Q = -\theta I, S = (1 - \theta)I, R = \theta\gamma^2 I$ , the strict  $(Q, S, R)$  dissipativity reduces to mixed  $H_\infty$  and positive real performance design, when  $\theta \in (0, 1)$ . The overall control design satisfies mixed NLQR– $H_\infty$ –positive real performance.

*Case 4*  $Q = -I, S = \frac{1}{2}(K_1 + K_2)^T, R = -\frac{1}{2}(K_1^T K_2 + K_2^T K_1)^T$ , where  $K_1$  and  $K_2$  are constant matrices of appropriate dimensions, the strict  $(Q, S, R)$  dissipativity reduces to a sector-bounded constraint (Gupta & Joshi, 1994). The overall control design satisfies mixed NLQR–sector-bounded constraint performance.

Before introducing the main result of the paper, the following model of uncertainties is introduced.

**ASSUMPTION 1** *The following general form of  $l_2$ -bounded unstructured uncertainties is considered:*

$$\begin{aligned} \Delta_A \Delta_A^T &\leq \gamma_A I, \\ \Delta_B \Delta_B^T &\leq \gamma_B I, \end{aligned}$$

$$\Delta_E \Delta_E^T \leq \gamma_E I,$$

$$\Delta_K \Delta_K^T \leq \gamma_K I, \quad (10)$$

for  $\forall x_k \in \mathfrak{R}^n$  and  $k \geq 0$ .

### 3. State-dependent LMI control

LEMMA 1

$$AB^T + BA^T \leq \alpha AA^T + \alpha^{-1} BB^T. \quad (11)$$

*This can be proven easily by considering*

$$(\alpha^{1/2} A - \alpha^{-1/2} B)(\alpha^{1/2} A - \alpha^{-1/2} B)^T \geq 0. \quad (12)$$

*Also, by choosing  $A$  and  $B$  matrices as  $A = \begin{bmatrix} a^T \\ 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 0 \\ b^T \end{bmatrix}$ , we have*

$$\begin{bmatrix} 0 & a^T b \\ b^T a & 0 \end{bmatrix} \leq \begin{bmatrix} \xi a^T a & 0 \\ 0 & \xi^{-1} b^T b \end{bmatrix}. \quad (13)$$

The following theorem summarizes the main results of the paper:

**THEOREM 2** *Given the system equation (1), performance output (2) and control input (3), if there exist matrices  $X_k = P_k^{-1} > 0$  and  $Y_k$  for all  $k > 0$ , such that the following state-dependent LMI holds:*

$$\text{If } Q < 0, \begin{bmatrix} X_k & \Upsilon_{12} & \Upsilon_{13} & Y_k^T & \Upsilon_{15} & X_k \\ * & \Upsilon_{22} & E^T & 0 & 0 & 0 \\ * & * & \Upsilon_{33} & 0 & 0 & 0 \\ * & * & * & \Upsilon_{44} & 0 & 0 \\ * & * & * & * & \Upsilon_{55} & 0 \\ * & * & * & * & * & \Upsilon_{66} \end{bmatrix} > 0, \quad (14)$$

$$\text{If } Q = 0, \begin{bmatrix} X_k & \Upsilon_{12} & \Upsilon_{13} & Y_k^T & X_k \\ * & \Upsilon_{22} & E^T & 0 & 0 \\ * & * & \Upsilon_{33} & 0 & 0 \\ * & * & * & \Upsilon_{44} & 0 \\ * & * & * & * & \Upsilon_{66} \end{bmatrix} > 0, \quad (15)$$

$$\begin{aligned} \Upsilon_{12} &= X_k C_k^T Q D_k + X_k C_k^T S, \\ \Upsilon_{13} &= X_k A_k^T + Y_k^T B_k^T, \\ \Upsilon_{15} &= X_k C_k^T, \\ \Upsilon_{22} &= D_k^T S + S^T D_k + D_k^T Q D_k + R + I, \\ \Upsilon_{33} &= X_k + (2\gamma_B + \gamma_E + 1)I + B_k B_k^T, \\ \Upsilon_{44} &= N^{-1}, \\ \Upsilon_{55} &= -Q^{-1}, \\ \Upsilon_{66} &= M^{-1} - (\gamma_A + 2\gamma_K)^{-1} I. \end{aligned} \quad (16)$$

Then the inequality (7) to guarantee mixed NLQR and dissipative performance is satisfied. The nonlinear feedback control gain is given by

$$K_k = Y_k \cdot P_k. \quad (17)$$

*Proof* In the proof below, the time and state argument will be dropped for notational simplicity. By applying system and performance output equations (1) and (2), and state feedback input equation (3), the performance index can be formed as follows:

$$\begin{aligned} & \{x_k^T [A_k + \Delta_A + (B_k + \Delta_B)(K_k + \Delta_K)]^T + w_k^T [E_k + \Delta_E]^T\} \\ & \cdot P_{k+1} \cdot \{[A_k + \Delta_A + (B_k + \Delta_B)(K_k + \Delta_K)]x_k \\ & + [E_k + \Delta_E]w_k\} - x_k^T P_k x_k \\ & + x_k^T M x_k + x_k^T [K_k + \Delta_K]^T N [K_k + \Delta_K] x_k \\ & - [C_k x_k + D w_k]^T Q [C_k x_k + D w_k] - 2[C_k x_k + D_k w_k]^T \\ & S w_k - w_k^T R w_k < 0. \end{aligned} \quad (18)$$

By grouping the terms, we have

$$\begin{aligned} [x_k^T \quad w_k^T] \Psi [x_k \quad w_k]^T &= [x_k^T \quad w_k^T] \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ * & \Psi_{22} \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix} \\ &< 0, \end{aligned} \quad (19)$$

where

$$\begin{aligned} \Psi_{11} &= \{(A_k + \Delta_A) + (B_k + \Delta_B)(K_k + \Delta_K)\}^T \cdot P_{k+1} \\ & \cdot \{(A_k + \Delta_A) + (B_k + \Delta_B)(K_k + \Delta_K)\} \\ & + M - P_k + [K_k + \Delta_K]^T N [K_k + \Delta_K] - C_k^T Q C_k, \\ \Psi_{12} &= \{(A_k + \Delta_A) + (B_k + \Delta_B)(K_k + \Delta_K)\}^T \\ & P_{k+1} [E_k + \Delta_E] - C_k^T Q D_k - C_k^T S, \\ \Psi_{22} &= [E_k + \Delta_E]^T P_{k+1} [E_k + \Delta_E] - D_k^T Q D_k \\ & - (D_k^T S + S^T D_k) - R. \end{aligned} \quad (20)$$

Denote the following terms:

$$\begin{aligned} A &= (A_k + \Delta_A) + (B_k + \Delta_B)(K_k + \Delta_K), \\ K &= K_k + \Delta_K, \\ E &= E_k + \Delta_E. \end{aligned} \quad (21)$$

Then Equation (19) is equivalent to

$$\begin{aligned} & \begin{bmatrix} A^T P_{k+1} A - P_k & A^T P_{k+1} E \\ * & E^T P_{k+1} E \end{bmatrix} \\ & + \begin{bmatrix} M + K^T N K - C_k^T Q C_k & -C_k^T Q D_k - C_k^T S \\ * & -D_k^T S - S^T D_k - D_k^T Q D_k - R \end{bmatrix} \\ & < 0. \end{aligned} \quad (22)$$

By adding and subtracting  $P_k$  term, we have

$$\begin{aligned} & \begin{bmatrix} A^T \\ E^T \end{bmatrix} (P_{k+1} - P_k + P_k) [A \quad E] - \begin{bmatrix} I \\ 0 \end{bmatrix} P_k [I \quad 0] \\ & + \begin{bmatrix} M + K^T N K - C_k^T Q C_k & -C_k^T Q D_k - C_k^T S \\ * & -D_k^T S - S^T D_k - D_k^T Q D_k - R \end{bmatrix} \\ & < 0. \end{aligned} \quad (23)$$

Imposing the property  $P_{k+1} \leq P_k$ , the sufficient condition for Equation (23) is given as follows:

$$\begin{aligned} & \begin{bmatrix} A^T \\ E^T \end{bmatrix} P_k [A \quad E] - \begin{bmatrix} I \\ 0 \end{bmatrix} P_k [I \quad 0] \\ & + \begin{bmatrix} M + K^T N K - C_k^T Q C_k & -C_k^T Q D_k - C_k^T S \\ * & -D_k^T S - S^T D_k - D_k^T Q D_k - R \end{bmatrix} \\ & < 0. \end{aligned} \quad (24)$$

Equivalently, we obtain

$$\begin{aligned} & \begin{bmatrix} P_k - M - K^T N K + C_k^T Q C_k & C_k^T Q D_k + C_k^T S \\ * & D_k^T S + S^T D_k + D_k^T Q D_k + R \end{bmatrix} \\ & - \begin{bmatrix} A^T \\ E^T \end{bmatrix} P_k [A \quad E] > 0. \end{aligned} \quad (25)$$

Applying the Schur complement (Boyd et al., 1994), we have

$$\begin{bmatrix} \begin{pmatrix} P_k - M - K^T N K \\ + C_k^T Q C_k \end{pmatrix} & C_k^T Q D_k + C_k^T S & A^T \\ * & \begin{pmatrix} D_k^T S + S^T D_k \\ + D_k^T Q D_k + R \end{pmatrix} & E^T \\ * & * & P_k^{-1} \end{bmatrix} > 0. \quad (26)$$

Taking  $Q < 0$  (the case where  $Q = 0$  will be considered later), we apply the Schur complement twice to Equation (26), then

$$\begin{bmatrix} P_k - M & C_k^T Q D_k + C_k^T S & A^T & K^T & C_k^T \\ * & D_k^T S + S^T D_k + D_k^T Q D_k + R & E^T & 0 & 0 \\ * & * & P_k^{-1} & 0 & 0 \\ * & * & * & N^{-1} & 0 \\ * & * & * & * & -Q^{-1} \end{bmatrix} > 0. \quad (27)$$

Let  $X_k = P_k^{-1}$ , by pre- and post-multiplying the above matrix inequality by  $\text{diag}\{X_k \quad I \quad I \quad I \quad I\}$ , we have

$$\begin{bmatrix} X_k - X_k M X_k & X_k C^T Q D_k + X_k C^T S & X_k A^T & X_k K^T & X_k C_k^T \\ * & \begin{pmatrix} D_k^T S + S^T D_k + \\ D_k^T Q D_k + R \end{pmatrix} & E^T & 0 & 0 \\ * & * & X_k & 0 & 0 \\ * & * & * & N^{-1} & 0 \\ * & * & * & * & -Q^{-1} \end{bmatrix} > 0 \quad (28)$$

By applying the Schur complement again, we have

$$\begin{bmatrix} X_k & X_k C_k^T Q D_k + X_k C_k^T S & X_k A^T & X_k K^T & X_k C^T & X_k \\ * & \begin{pmatrix} D_k^T S + S^T D_k + \\ D_k^T Q D_k + R \end{pmatrix} & E^T & 0 & 0 & 0 \\ * & * & X_k & 0 & 0 & 0 \\ * & * & * & N^{-1} & 0 & 0 \\ * & * & * & * & -Q^{-1} & 0 \\ * & * & * & * & * & M^{-1} \end{bmatrix} > 0 \quad (29)$$

Denote

$$Y_k = K_k X_k \quad (30)$$

By replacing the variables with Equation (21) and applying Lemma 1 and Assumption 1, the sufficient condition for inequality (29) is given below

$$\begin{bmatrix} X_k & \begin{pmatrix} X_k C_k^T Q D_k + \\ X_k C_k^T S \end{pmatrix} & \begin{pmatrix} X_k A^T + \\ Y_k^T B^T \end{pmatrix} & X_k K_k^T & X_k C_k^T & X_k \\ * & \begin{pmatrix} D_k^T S + S^T D_k + \\ D_k^T Q D_k + R \end{pmatrix} & E_k^T & 0 & 0 & 0 \\ * & * & X_k & 0 & 0 & 0 \\ * & * & * & N^{-1} & 0 & 0 \\ * & * & * & * & -Q^{-1} & 0 \\ * & * & * & * & * & M^{-1} \end{bmatrix} + \begin{bmatrix} \Omega_{11} & 0 & 0 & 0 & 0 & 0 \\ * & \Omega_{22} & 0 & 0 & 0 & 0 \\ * & * & \Omega_{33} & 0 & 0 & 0 \\ * & * & * & \alpha_1^{-1} I & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix} > 0 \quad (31)$$

where

$$\begin{aligned} \Omega_{11} &= (\alpha_1 \gamma_K + \alpha_1 \gamma_A + \alpha_4 \gamma_K) X_k X_k + \alpha_2 Y_k^T Y_k, \\ \Omega_{22} &= \alpha_2 I, \\ \Omega_{33} &= \alpha_1^{-1} \gamma_B I + \alpha_2^{-1} (\gamma_B + \gamma_E) I + \alpha_3^{-1} I + \alpha_4^{-1} B_k B_k^T. \end{aligned} \quad (32)$$

Finally, by applying the Schur complement twice, we have

$$\begin{bmatrix} X_k & \Upsilon_{12} & \Upsilon_{13} & Y_k^T & \Upsilon_{15} & X_k \\ * & \Upsilon_{22} & E^T & 0 & 0 & 0 \\ * & * & \Upsilon_{33} & 0 & 0 & 0 \\ * & * & * & \Upsilon_{44} & 0 & 0 \\ * & * & * & * & \Upsilon_{55} & 0 \\ * & * & * & * & * & \Upsilon_{66} \end{bmatrix} > 0, \quad (33)$$

where

$$\begin{aligned} \Upsilon_{12} &= X_k C_k^T Q D_k + X_k C_k^T S, \\ \Upsilon_{13} &= X_k A_k^T + Y_k^T B_k^T, \\ \Upsilon_{15} &= X_k C_k^T, \\ \Upsilon_{22} &= D_k^T S + S^T D_k + D_k^T Q D_k + R + \alpha_2 I, \\ \Upsilon_{33} &= X_k + \alpha_1^{-1} \gamma_B I + \alpha_2^{-1} (\gamma_B + \gamma_E) I + \alpha_3^{-1} I \\ &\quad + \alpha_4^{-1} B_k B_k^T, \\ \Upsilon_{44} &= N^{-1} + (\alpha_1^{-1} - \alpha_2^{-1}) I, \\ \Upsilon_{55} &= -Q^{-1}, \\ \Upsilon_{66} &= M^{-1} - (\alpha_1 \gamma_A + \alpha_2 \gamma_K + \alpha_4 \gamma_K)^{-1} I. \end{aligned} \quad (34)$$

Note that Equation (33) is derived under the condition that  $Q < 0$ . However, when strict positive realness criteria are chosen for control design, then condition  $Q = 0$  must be satisfied. In this case, matrix inequality (33) should be replaced by

$$\begin{bmatrix} X_k & \Upsilon_{12} & \Upsilon_{13} & Y_k^T & X_k \\ * & \Upsilon_{22} & E^T & 0 & 0 \\ * & * & \Upsilon_{33} & 0 & 0 \\ * & * & * & \Upsilon_{44} & 0 \\ * & * & * & * & \Upsilon_{66} \end{bmatrix} > 0. \quad (35)$$

Since positive constants  $\alpha_1, \dots, \alpha_5$  are arbitrary, choosing all of them as 1, we obtain Equations (14) and (15). Therefore, if LMI e (14) or (15) holds under different conditions on  $Q$ , the inequality (7) is satisfied. By solving the LMI at each step, the values of  $P_k, Y_k$  can be obtained. The nonlinear feedback control gain can be found by  $K_k = Y_k \cdot P_k$ . This concludes the proof. ■

*Remark 3* At this point, it is to be noted that other choices of constants  $\alpha_1, \dots, \alpha_4$  are possible and can be tried if the value 1 for all these constants does not work.

#### 4. Application to the inverted pendulum on a cart

We test the novel robust and resilient state-dependent LMI approach with the inverted pendulum on a cart (Wang et al. 2010) to compare the performance of different controllers. Using the Euler-Lagrange Equation technique, the complete equations of motion for the inverted pendulum on a cart are found to be

$$\begin{aligned} (M + m)\ddot{x} + b\dot{x} + mL\ddot{\theta} \cos(\theta) - mL\dot{\theta} \sin(\theta) &= F, \\ (I + mL^2)\ddot{\theta} + mgL \sin(\theta) + mL\ddot{x} \cos(\theta) &= 0. \end{aligned} \quad (36)$$

The following system parameters are assumed

$$M = 0.5 \text{ kg}, m = 0.5 \text{ kg}, b = 0.1 \text{ N s/m}, L = 0.3 \text{ m},$$

$$I = 0.06 \text{ kg m}^2.$$

$$\text{Sampling time : } T = 0.01 \text{ s}.$$



Denote the following state variables:

$$x_{1,k} = x(kT), x_{2k} = \dot{x}(kT), x_{3k} = \theta(kT), x_{4k} = \dot{\theta}(kT).$$

The following initial conditions are assumed:

$$x_1 = 1, x_2 = 0, x_3 = \pi/4, x_4 = 0.$$

The following design parameters are chosen to satisfy different mixed criteria:

Mixed NLQR– $H_\infty$  design (predominant NLQR)

$$C = [0.01 \ 0.01 \ 0.01 \ 0.01], D = [0.01], M = I_4, \\ N = 1, Q = -1, S = 0, R = 5.$$

Mixed NLQR– $H_\infty$  design (predominant  $H_\infty$ )

$$C = [1 \ 1 \ 1 \ 1], D = [1], M = 0.01 \times I_4, N = 0.01, \\ Q = -1, S = 0, R = 5.$$

Mixed NLQR– $H_\infty$ –positive real design (NLQR passivity)

$$C = [1 \ 1 \ 1 \ 1], D = [1], M = I_4, N = 1, \\ Q = -0.01, \\ S = 0.5, R = 0.01.$$

All of the above-mixed criteria control performance results are shown in the Figures 1–5, in comparison with the traditional LQR technique based on linearization. From these figures, we find that the novel state-dependent LMI control has better performance compared with the traditional LQR technique based on linearization. Especially, Figures 1 and 2 show that the traditional LQR technique loses control of the position and velocity of the cart, respectively, while the state-dependent LMI approach effectively stabilizes the position and the velocity of the cart. It should also be noted that predominant NLQR and predominant  $H_\infty$  control techniques lead to faster response times than

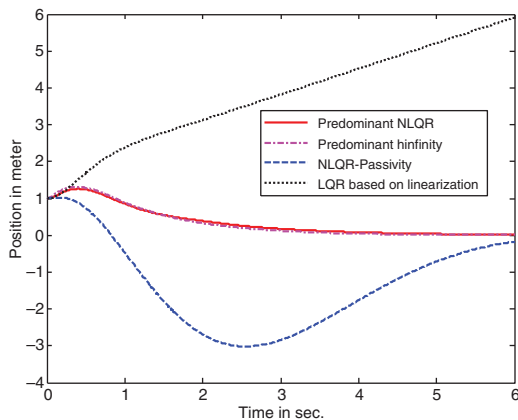


Figure 1. Position trajectory of the inverted pendulum.

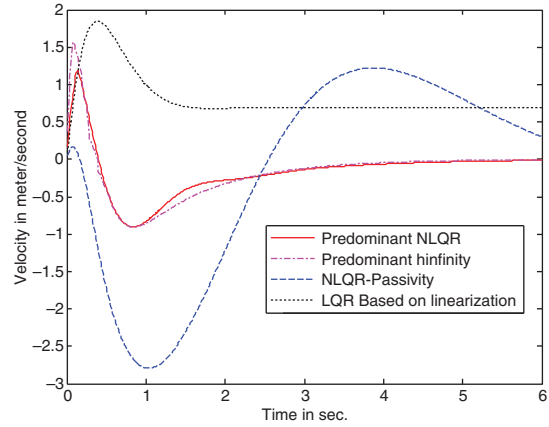


Figure 2. Velocity trajectory of the inverted pendulum.

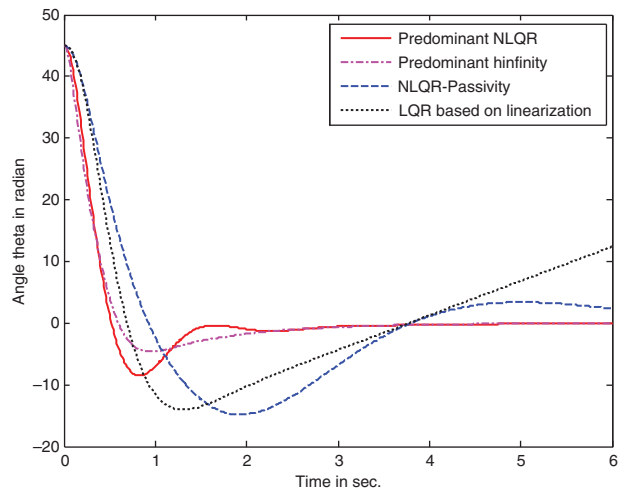


Figure 3. Angle “theta” trajectory of the inverted pendulum.

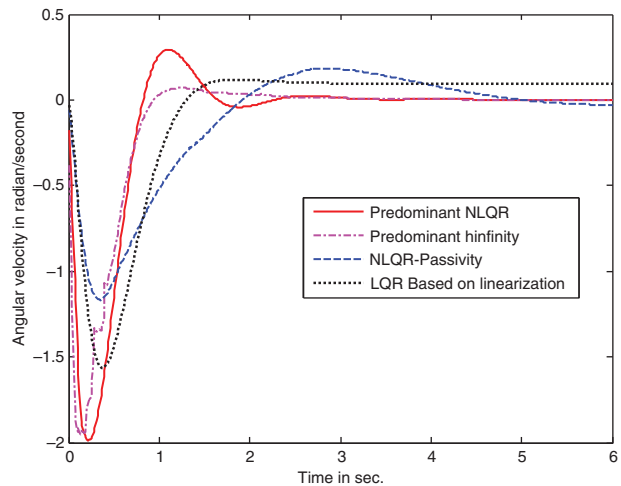


Figure 4. Angular velocity trajectory of the inverted pendulum.

the NLQR-passivity technique. We observe that predominant  $H_\infty$  control shows the fastest response. Figure 5 shows that the highest magnitude of control is needed by the

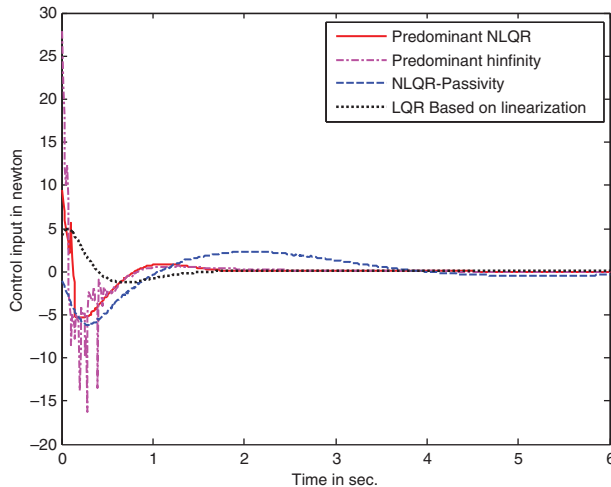


Figure 5. Control input.

predominant  $H_\infty$  control and the lowest control magnitude is needed by the linearization-based LQR technique.

## 5. Conclusions

This paper has addressed discrete-time nonlinear control system design with general NLQR and quadratic dissipative criteria to achieve asymptotic stability, quadratic optimality and strict quadratic dissipativeness. For systems with unstructured but bounded uncertainty, the LMI-based sufficient conditions are derived for the control solution. These results unify the existing results on SDRE control, robust  $H_\infty$  and positive real control. The relative weighting matrices of these criteria can be achieved by choosing different coefficient matrices. The optimal control can be obtained by solving LMI at each time step. The inverted pendulum on a cart is used as an example to demonstrate the effectiveness and robustness of the proposed method. The numerical simulation studies show that the proposed method provides a satisfactory alternative to the existing nonlinear control approaches.

## Acknowledgements

We thank the reviewers for their remarks, which have improved the quality of the paper. This research has been financially supported by a grant from NITC (National Institute for Transportation and Communities), which is gratefully acknowledged.

## References

- Boyd, S., Ghaoui, L. E., Feron, E., & Balakrishnan, V. (1994). *Linear matrix inequalities in system and control theory*. Philadelphia, PA: SIAM Studies in Applied Mathematics.
- Doyle, J. C., Glover, K., Khargonekar, P. P., & Francis, B. A. (1989). State space solution to the standard  $H_2$  and  $H_\infty$  control problems. *IEEE Transactions on Automatic Control*, 34, 831–847.

- Gupta, S., & Joshi, S. M. (1994). *Some properties and stability results for sector bounded LTI systems*. Proceedings of the 33rd IEEE conference on decision and control, Orlando, FL, pp. 2973–2978.
- Haddad, W. M., & Bernstein, D. S. (1991). Robust stabilization with positive real uncertainty, beyond the small gain theorem. *Systems and Control Letters*, 17, 191–208.
- Hill, D. J., & Moylan, P. J. (1976). Stability of nonlinear dissipative systems. *IEEE Transactions on Automatic Control*, 21, 708–711.
- Hill, D. J., & Moylan, P. J. (1980). Dissipative dynamical systems: Basic input-output and state properties. *Journal of Franklin Institute*, 309, 327–357.
- Huang, Y., & Lu, W.-M. (1996). *Nonlinear optimal control: Alternatives to Hamilton-Jacobi equation*. Proceedings of 35th IEEE conference on decision and control, Kobe, Japan, pp. 3942–3947.
- Khargonekar, P. P., Petersen, I. R., & Zhou, K. (1990). Robust stabilization of uncertain linear systems: Quadratic stabilizability and  $H_\infty$  control theory. *IEEE Transactions on Automatic Control*, 35, 356–361.
- Petersen, I. R. (1987). A stabilization algorithm for a class of uncertain systems. *Systems and Control Letters*, 8, 181–188.
- Safonov, M. G., Jonckheere, E. A., Verma, M., & Limebeer, D. J. N. (1987). Synthesis of positive real multivariable feedback systems. *International Journal of Control*, 45, 817–842.
- Shim, D. (1996). Equivalence between positive real and norm-bounded uncertainty. *IEEE Transactions on Automatic Control*, 41, 1190–1193.
- Sun, W., Khargonekar, P. P., & Shim, D. (1994). Solution to the positive real control problem for linear time-invariant systems. *IEEE Transactions on Automatic Control*, 39, 2034–2046.
- Tan, Z., Soh, Y. C., & Xie, L. (2000). Dissipative control of linear discrete-time systems with dissipative uncertainty. *International Journal of Control*, 73 (4), 317–328.
- Wang, X., & Yaz, E. E. (2009). *The state dependent control of continuous time nonlinear systems with mixed performance criteria*. Proceedings of IASTED international conference on identification and control applications, Honolulu, HI, pp. 98–102.
- Wang, X., Yaz, E. E., & Jeong, C. S. (2010). *Robust nonlinear feedback control of discrete-time systems with mixed performance criteria*. Proceedings of American control conference, Baltimore, MD, pp. 6357–6362.
- Wang, X., Yaz, E. E., & Yaz, Y. I. (2010). *Robust and resilient state dependent control of continuous time nonlinear systems with general performance criteria*. Proceedings of the 49th IEEE conference on decision and control, Atlanta, GA, pp. 603–608.
- Wang, X., Yaz, E. E., & Yaz, Y. I. (2011). *Robust and resilient state dependent control of discrete time nonlinear systems with general performance criteria*. Proceedings of the 18th IFAC congress, Milano, Italy.
- Willems, J. C. (1972a). Dissipative dynamical systems-part 1: General theory. *Archive for Rational Mechanics and Analysis*, 45, 321–351.
- Willems, J. C. (1972b). Dissipative dynamical systems-part 2: Linear systems with quadratic supply rates. *Archive for Rational Mechanics and Analysis*, 45, 352–393.
- Xie, S., Xie, L., & De Souza, C. E. (1998). Robust dissipative control for linear systems with dissipative uncertainty. *International Journal of Control*, 70 (2), 169–191.