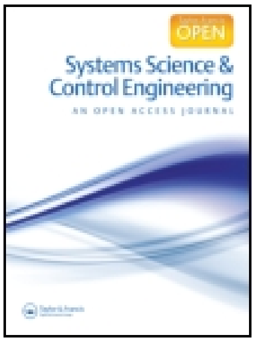

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Design of optimal output disturbance cancellation controllers via loop transfer recovery

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The authors have introduced optimal disturbance cancellation controllers as a class of controllers minimizing a non-standard quadratic performance index explicitly including disturbances. This paper discusses the application of the classical loop transfer recovery (LTR) technique to the optimal disturbance cancellation controller for step disturbances entering the plant output. The estimation error dynamics of the Kalman filter jointly estimating the plant states and the disturbances is chosen as a target of the LTR design. The weighting coefficient of the performance index is used to recover the target which has guaranteed stability margins as in the standard LTR design. It is shown by a numerical example that the proposed design provides flexible tuning of the disturbance rejection capability with sufficient stability margins.

Keywords: linear systems; optimal disturbance cancellation; step disturbances; disturbance estimation; linear-quadratic-Gaussian controllers; loop transfer recovery

1. Introduction

Recently, the authors have introduced optimal disturbance cancellation controllers (Ishihara & Guo, 2008) as an extension of linear-quadratic-Gaussian (LQG) controllers (Anderson & Moore, 1990). A quadratic performance index explicitly including disturbances is used with a stochastic extended model consisting of the plant and the disturbance model. For systematic design of the disturbance cancellation controllers, it is tempting to apply the classical loop transfer recovery (LTR) procedure (Saberi, Chen, & Sannuti, 1993; Stein & Athans, 1987). However, it turns out that the standard LTR procedure cannot directly be applied: the stabilizability assumption required in the standard LTR theory is not satisfied for the disturbance cancellation controllers. For step disturbances entering the plant input side, Guo, Ishihara, and Takeda (1996) found that the difficulty can be overcome by a slight modification of the standard LTR procedure. Some extensions of this result have been discussed in Ishihara, Guo, and Takeda (2005) and Ishihara and Guo (2008, 2011, 2012).

In this paper, the LTR design for step disturbances entering the plant output side is proposed by reformulating the results of our conference paper (Ishihara & Guo, 2009). The plant is assumed to be minimum phase without integral action. A non-standard quadratic performance index is defined such that the disturbance cancellation requirement is explicitly represented. Assuming that the plant state and the disturbance are perfectly measurable, we obtain the optimal control law by the parametric LQ approach (Makila & Toivonen, 1987). The optimal output feedback controller is constructed by the separation theorem with the use of the Kalman filter jointly estimating the plant states and the disturbances. As a target for the LTR design, we choose the estimation error dynamics of the Kalman filter. It is shown that the target has excellent stability margins and that the weighting coefficient of the performance index can be used to recover the target feedback property in the output feedback controller. This result extends the existing LTR theory to a case where the stabilizability assumption is not satisfied. Moreover, advantages of the proposed design over the conventional integrator augmentation method are pointed out through the discussion of the optimal disturbance cancellation for plants with integral action.

This paper is organized as follows: the optimal output disturbance cancellation controller is constructed in Section 2. The LTR design is discussed in Section 3. The optimal disturbance cancellation for plants with the integral action is discussed in Section 4. In Section 5, a simple numerical example is presented to illustrate the effectiveness of the proposed design. Concluding remarks are given in Section 6.

2. Problem formulation

Consider a plant given by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y_d(t) = Cx(t) + d(t), \quad (1)$$

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where $x(t) \in R^n$ is a state vector, $u(t) \in R^m$ is a control input vector, $d(t) \in R^m$ is a step disturbance vector satisfying

$$\dot{d}(t) = 0 \tag{2}$$

and $y_d(t) \in R^m$ is a disturbed output vector. In addition, the plant (1) satisfies the following assumptions:

A1: The triplet (A, B, C) is a minimal realization with no zero and pole at the origin and minimum phase.
A2: The transfer function matrix $G(s) = C(sI - A)^{-1}B$ is non-singular for almost all s.

The optimal disturbance cancellation controller for the plant (1) is constructed based on the separation theorem. First, we give the optimal disturbance cancellation control law under the assumption that the state and the disturbance are perfectly measurable.

Proposition 1 *Assume that the state $x(t)$ and the disturbance $d(t)$ in Equation (1) are perfectly measurable. Define the non-standard quadratic performance index as*

$$\bar{J}_\rho \triangleq \int_o^\infty \{\rho^2 y_d'(t) y_d(t) + [u(t) - \bar{u}]'[u(t) - \bar{u}]\}\, \mathrm{d}t, \tag{3}$$

where $t = 0$ is the time when the disturbance occurs $(d(0) \neq 0)$, ρ is a positive weighting coefficient and $\bar{u}$ is a steady-state vector of the control input $u(t)$. Consider the linear feedback control law

$$u(t) = -F_x x(t) - F_d d(t), \tag{4}$$

where $F_x \in R^{m\times n}$ and $F_d \in R^{m\times m}$ are feedback gain matrices. Then the optimal feedback gain matrices minimizing the performance index (3) are given by

$$F_x = F, \quad F_d = [C(-A + BF)^{-1}B]^{-1}, \tag{5}$$

where F is the optimal feedback gain matrix of the standard LQ optimal regulator problem for the plant (A, B, C) with the performance index

$$J_\rho \triangleq \int_o^\infty [\rho^2 x'(t) C' C x(t) + u'(t) u(t)]\, \mathrm{d}t. \tag{6}$$

Proof Assume that F_x is chosen such that $A - BF_x$ is asymptotically stable. Then, for the step disturbance $d(t) = d$ $(t \geq 0)$, the plant (1) with the control input (4) approaches the steady state as t tends infinity. Let $\bar{u}$, $\bar{x}$ and $\bar{y}_d$ denote steady-state values of the $u(t)$, $x(t)$ and $y_d(t)$, respectively, for a given disturbance vector d. The steady-state values obviously satisfy

$$A\bar{x} + B\bar{u} = 0, \quad \bar{y}_d = C\bar{x} + d, \tag{7}$$

$$\bar{u} = -F_x\bar{x} - F_d d. \tag{8}$$

It follows from Equations (7) and (8) that

$$\bar{x} = (A - BF_x)^{-1} BF_d d, \tag{9}$$

and

$$\bar{y}_d = [C(A - BF_x)^{-1} BF_d + I]d. \tag{10}$$

To guarantee that the performance index remains finite, the steady-state value $\bar{y}_d$ of the output $y_d(t)$ should be zero for arbitrary value of d. It follows from Equation (10) that

$$F_d = [C(-A + BF_x)^{-1}B]^{-1}. \tag{11}$$

Substituting Equation (11) into Equations (9) and (10), we can rewrite the steady $\bar{u}$ as

$$\begin{aligned}\bar{u} &= -\{C(-A + BF_x)^{-1}B[I - F_x(-A + BF_x)^{-1}B]^{-1}\}^{-1}d \\ &= -\{C[I - (-A + BF_x)^{-1}BF_x]^{-1}(-A + BF_x)^{-1}B\}^{-1}d \\ &= (CA^{-1}B)^{-1}d, \end{aligned} \tag{12}$$

where we have used the matrix identity $X(I - YX)^{-1} = (I - XY)^{-1}X$ with $X = (-A + BF_x)^{-1}B$ and $Y = F_x$.

From Equations (7) and (12), we can write $\bar{x}$ as

$$\bar{x} = -A^{-1}B(CA^{-1}B)^{-1}d. \tag{13}$$

Equations (12) and (13) show that the steady-state vectors $\bar{x}$ and $\bar{u}$ are independent of F_x.

Define

$$\widehat{u}(t) = u(t) - \bar{u}, \tag{14}$$

$$\widehat{x}(t) = x(t) - \bar{x}. \tag{15}$$

It follows from Equations (1) and (7) that

$$\dot{\widehat{x}}(t) = A\widehat{x}(t) + B\widehat{u}(t), \quad y_d(t) = C\widehat{x}(t). \tag{16}$$

The performance index (3) can be rewritten as

$$\bar{J}_\rho \triangleq \int_o^\infty \{\rho^2 \widehat{x}'(t) C' C \widehat{x}(t) + \widehat{u}'(t)\widehat{u}(t)\}\, \mathrm{d}t. \tag{17}$$

Using simple matrix manipulation with the non-singularity of the matrix A, we can write the disturbance feedback term in Equation (4) as

$$\begin{aligned} F_d d(t) &= [C(-A+BF_x)^{-1}B]^{-1}d \\ &= -[CA^{-1}B(I-F_xA^{-1}B)^{-1}]^{-1}d \\ &= F_xA^{-1}B(CA^{-1}B)^{-1}d-(CA^{-1}B)^{-1}d \\ &= -F_x\bar{x}-\bar{u}, \end{aligned} \tag{18}$$

where the last expression is obtained by Equations (12) and (13). Using Equation (18) in Equation (4), we obtain

$$u(t) = -F_x x(t) + F_x\bar{x} + \bar{u}. \tag{19}$$

It follows from Equations (14), (15) and (19) that

$$\hat{u}(t) = -F_x\hat{x}(t). \tag{20}$$

Consequently, the problem of finding the optimal feedback gain matrices F_x and F_d minimizing the performance index (3) is reduced to find the state feedback control law for the system (16) minimizing the standard performance index (17). From Equations (16) and (20), we can write the performance index (17) as

$$\bar{J}_\rho = \hat{x}'(0)\left[\int_o^\infty e^{(A-BF_x)'t}(\rho^2C'C+F_x'F_x)e^{(A-BF_x)t}\,dt\right] \times \hat{x}(0), \tag{21}$$

which can be expressed as

$$\bar{J}_\rho = \hat{x}'(0)P\hat{x}(0), \tag{22}$$

where the matrix $P \in R^{n\times n}$ satisfies

$$(A-BF_x)'P+P(A-BF_x) = -(\rho^2C'C+F_x'F_x). \tag{23}$$

The problem of finding the optimal feedback gain matrix F_x for the performance index (17) is reduced to find F_x minimizing the quadratic form (22) under the constraint (23). Since the initial state $\hat{x}(0)$ is independent of F_x as shown by Equations (12) and (13), the problem is same as that for the standard LQ problem formulated by the parametric LQ approach (Makila & Toivonen, 1987). Consequently, the optimal feedback gain matrix F_x is obtained as the optimal gain matrix for the plant (A, B, C) with the performance index (6). Let F denotes the optimal feedback gain matrix for the standard problem. Then the optimal feedback gain matrices F_x and F_d are given as Equation (5). ■

Figure 1 shows the structure of the above control system where the disturbance is modelled by the integrator with the non-zero initial state.

The optimal output disturbance cancellation controller is constructed based on the separation theorem. Construct the extended stochastic model of Equations (1) and (2) as

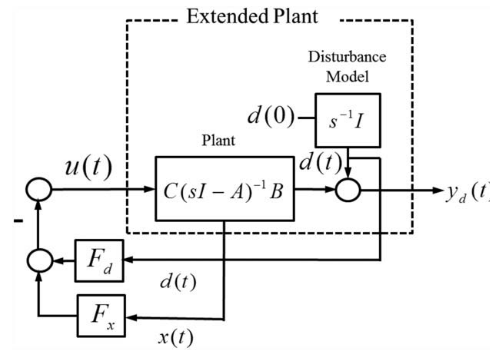

Figure 1. The structure of the control system with the perfect observation of the plant state and the disturbance.

$$\dot{\xi}(t) = \Phi\xi(t)+\Gamma u(t)+\bar{\Gamma}w(t), \quad y_d(t) = \mathrm{H}\xi(t)+v(t), \tag{24}$$

where

$$\xi(t) \triangleq \begin{bmatrix} d'(t) & x'(t) \end{bmatrix}', \tag{25}$$

$$\Phi \triangleq \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}, \quad \Gamma \triangleq \begin{bmatrix} 0 \\ B \end{bmatrix}, \quad \mathrm{H} \triangleq \begin{bmatrix} I & C \end{bmatrix}, \tag{26}$$

$\bar{\Gamma} \in R^{n\times m}$ is a matrix such that the pair $(\Phi, \bar{\Gamma})$ is controllable, $w(t)$ and $v(t)$ are mutually independent zero-mean white noise processes with the covariance matrices W and V, respectively.

It is obvious that the pair (H, Φ) is observable but (Φ, Γ) is unstabilizable. By the observability of (H, Φ) and the controllability of $(\Phi, \bar{\Gamma})$, the Kalman filter for the extended state (25) can be constructed as

$$\dot{\hat{\xi}}(t) = \Phi\hat{\xi}(t)+\Gamma u(t)+\mathrm{K}[y_d(t)-\mathrm{H}\hat{\xi}(t)], \tag{27}$$

where

$$\hat{\xi}(t) = \begin{bmatrix} \hat{d}'(t) & \hat{x}'(t) \end{bmatrix}'. \tag{28}$$

is the optimal estimate of the extended state vector (25) and

$$\mathrm{K} \triangleq \begin{bmatrix} K_d' & K_x' \end{bmatrix}' \tag{29}$$

is the Kalman filter gain matrix. It is well known that the gain matrix K is given by

$$\mathrm{K} = \Pi\mathrm{H}'V^{-1}, \tag{30}$$

where Π is a positive definite solution of the Riccati equation

$$\Phi\Pi+\Pi\Phi'-\Pi\mathrm{H}'V^{-1}\mathrm{H}\Pi+\bar{\Gamma}W\bar{\Gamma}' = 0. \tag{31}$$

The optimal output feedback controller is constructed as follows.

Proposition 2 *Consider the stochastic model* (24). *Define the performance index*

$$\tilde{J}_\rho \triangleq E\left[\lim_{T\to\infty}\frac{1}{T}\int_o^T \{\rho^2 y_d'(t)y_d(t) + [u(t)-\bar{u}]'[u(t)-\bar{u}]\}\,\mathrm{d}t\right], \tag{32}$$

which is a stochastic version of the performance index (3) *for the deterministic state feedback case. Then, the optimal output feedback control law minimizing the performance index can be obtained as*

$$u(t) = -\Psi\hat{\xi}(t), \tag{33}$$

where Ψ *is defined by use of the optimal feedback gain matrices defined in Equation* (5) *as*

$$\Psi \triangleq \begin{bmatrix} F_d & F_x \end{bmatrix} = \begin{bmatrix} [C(-A+BF)^{-1}B]^{-1} & F \end{bmatrix}, \tag{34}$$

$\hat{\xi}(t)$ *is the optimal estimate of* $\xi(t)$ *obtained by the Kalman filter* (27). *In addition, the transfer function matrix of the output feedback controller from* $y(t)$ *to* $-u(t)$ *can be expressed in the right factorization form as*

$$C(s) \triangleq \Psi(sI-\Phi+\Gamma\Psi)^{-1} \times \mathrm{K}[I+\mathrm{H}(sI-\Phi+\Gamma\Psi)^{-1}\mathrm{K}]^{-1}. \tag{35}$$

Proof The optimality of the control law (33) is proved by the separation theorem as in the standard LQG controller. The proof of the separation theorem can be found in Anderson and Moore (1990). The expression (35) for the controller transfer function matrix is obtained by straightforward matrix manipulations. ■

The structure of the output feedback optimal output cancellation control system is shown in Figure 2. We can easily obtain the following result.

Proposition 3 *Consider the control system consisting of the plant* (24) *and the output feedback controller* (33). *The sensitivity matrix at the plant output side (the point marked* P_o *in Figure 2) can be expressed as*

$$\Sigma(s) \triangleq [I+\mathrm{H}(sI-\Phi+\Gamma\Psi)^{-1}\mathrm{K}]S(s), \tag{36}$$

where

$$S(s) \triangleq [I+\mathrm{H}(sI-\Phi)^{-1}\mathrm{K}]^{-1} \tag{37}$$

is the sensitivity matrix for the estimation error dynamics of the Kalman filter.

Proof The sensitivity matrix at the plant output side is defined as

$$\Sigma(s) = [I+C(sI-A)^{-1}BC(s)]^{-1}, \tag{38}$$

where $C(s)$ is the controller transfer function matrix (35). Note that the plant transfer function matrix is written as

$$C(sI-A)^{-1}B = \mathrm{H}(sI-\Phi)^{-1}\Gamma, \tag{39}$$

where Φ, Γ and H are defined in Equation (26). Using Equations (35) and (39) in Equation (38), we can easily obtain the expression (36). ■

Using the submatrices in Equations (26) and (29), we can write Equation (37) as

$$S(s) = \left[I + C(sI-A)^{-1}K_x + \frac{1}{s}K_d\right]^{-1}. \tag{40}$$

It follows from Equations (36) and (40) that the sensitivity matrix (36) for the optimal output disturbance cancellation controller has a zero at the origin provided $K_d \neq 0$. Since the plant has no pole at the origin by the assumption, the existence of the zero at the origin implies that the controller (35) has integral action.

Remark: It is well known that the advanced H^2 and H^∞ theory can be used to design optimal or suboptimal integral controllers (Zhou, Doyle, & Glover, 1996) by using appropriate frequency domain weighting matrices including integral action. It should be noted that the integral action of the optimal controller proposed in this section emerges as a result of the minimization of the performance index (32) without assuming integral action. In addition, the proposed controller has a lucid controller structure that can provide efficient tuning of the disturbance cancellation capability with sufficient stability margins.

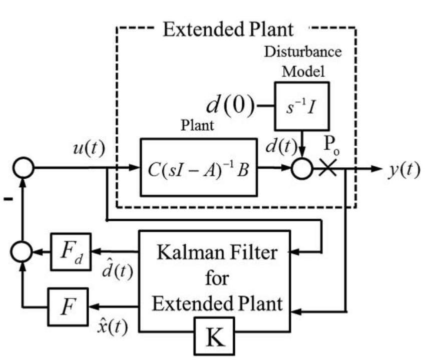

Figure 2. The structure of the output feedback optimal disturbance cancellation controller.

3. LTR design

In this section, the LTR design of the optimal disturbance cancellation controller (33) is discussed. Since the output disturbance is considered, the design goal is to achieve desirable feedback property at the plant output side. Note that, in the framework of the LTR design, the weighting coefficient ρ in the performance index (32) and the matrices

$\bar{\Gamma}$, W and V for the stochastic model (24) are used as tuning parameters for achieving desired feedback property.

3.1. Asymptotic sensitivity property

Using the expression (36) given in Proposition 3, we can obtain the following result on the asymptotic sensitivity property.

PROPOSITION 4 *Consider the control system consisting of the plant* (1) *and the optimal disturbance cancellation controller* (35) *with the Kalman filter gain matrix* (30) *and the optimal feedback gain matrix* (34) *determined by the performance index* (32). *Then the sensitivity matrix* $\Sigma(s)$ *at the plant output side satisfies*

$$\Sigma(s) \to S(s) \quad \text{pointwise in } s \text{ as } \rho \to \infty, \tag{41}$$

where ρ *is the weighting parameter in the performance index* (32) *and* $S(s)$ *is the matrix defined by Equation* (37).

Proof Note that the optimal feedback gain matrix (34) is determined by the optimal feedback gain matrix F for the standard quadratic performance index (6). It is known (Anderson & Moore, 1990) that, for sufficiently large ρ, the optimal gain matrix F can be written as

$$F = \rho C. \tag{42}$$

Let us define the feedback gain matrix obtained by substituting Equation (42) into Equation (34) as

$$\Psi_\rho \triangleq \left[[C(-A+\rho BC)^{-1}B]^{-1} \quad \rho C \right]. \tag{43}$$

Applying the well-known formula for block matrix inversion (Anderson & Moore, 1990) with the submatrices in Equations (26), (29) and (42), we can write the matrix $\mathrm{H}(sI - \Phi + \Gamma\Psi_\rho)^{-1}\mathrm{K}$ in the right side of Equation (36) with Ψ replaced by Ψ_ρ as

$$\begin{aligned}\mathrm{H}(sI - \Phi + \Gamma\Psi_\rho)^{-1}\mathrm{K} &= C(sI - A + \rho BC)^{-1}K_x + \frac{1}{s}K_d \\ &\quad - \frac{1}{s}C(sI - A + \rho BC)^{-1} \\ &\quad \times B[C(-A+\rho BC)^{-1}B]^{-1}K_d.\end{aligned} \tag{44}$$

Consider asymptotic behaviour of the above matrix as ρ tends to infinity. For the first and the third matrices in Equation (44), we find by simple matrix manipulations that

$$\begin{aligned}&C(sI - A + \rho BC)^{-1}K_x \\ &= [I + \rho C(sI - A)^{-1}B]^{-1}C(sI - A)^{-1} \\ &\times K_x \to 0 \quad (\rho \to \infty),\end{aligned} \tag{45}$$

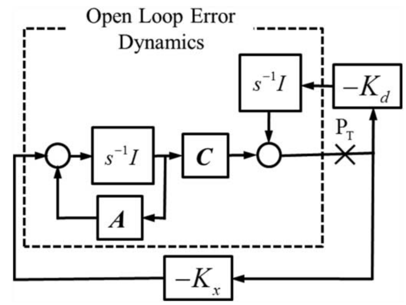

Figure 3. The structure of the target system.

$$\begin{aligned}&\frac{1}{s}C(sI - A + \rho BC)^{-1}B[C(-A+\rho BC)^{-1}B]^{-1}K_d \\ &= \frac{1}{s}[I + \rho C(sI-A)^{-1}B]^{-1}C(sI-A)^{-1}B \\ &\times \left\{[I - \rho CA^{-1}B]^{-1}(-CA^{-1}B)\right\}^{-1} \\ &\times K_d \to \frac{1}{s}K_d \quad (\rho \to \infty),\end{aligned} \tag{46}$$

where the matrix A is non-singular by the assumption and the convergence is pointwise in s. It follows from Equations (45) and (46) that

$$\mathrm{H}(sI - \Phi + \Gamma\Psi_\rho)^{-1}\mathrm{K} \to 0 \quad (\rho \to \infty). \tag{47}$$

From Equations (36) and (47), we can conclude the asymptotic property (41). ■

3.2. Design procedure

Proposition 4 shows that the LTR procedure similar to that used for the standard LQG controllers can be used for the output feedback optimal disturbance cancellation controller (33). The target of the design is the estimation error dynamics of the Kalman filter (27), the structure of which is shown in Figure 3. The target has the sensitivity matrix (37) defined at the point P_T in Figure 3. As in the standard LTR design, the target has the large stability margins.

The LTR design consists of the following two steps:

(1) By using the matrices $\bar{\Gamma}$, W and V for the stochastic model (24) as tuning parameters, determine the Kalman filter gain matrix K such that the target has appropriate feedback property.
(2) By increasing the weighting parameter ρ in the performance index (32), determine the feedback gain matrix F such that the output feedback disturbance cancellation controller provides satisfactory feedback property.

4. Plants with integral action

4.1. Disturbance and initial state

In the preceding discussions, we have assumed that the plant is free of the integral action, i.e. the matrix A of the plant is non-singular. For plants with integral action, the design procedure proposed in the previous section cannot be applied: although the state feedback control law (4) can be defined for plants with the integral action, the optimality cannot be guaranteed since the steady-state vectors $\bar{u}$ and $\bar{x}$ defined in Equations (12) and (13), respectively, fail to exist. In addition, the pair (H, Φ) for the extended plant (24) becomes unobservable for plants with the integral action so that the optimal estimator for the plant states and the disturbances required for the output feedback configuration cannot be constructed.

Despite the above facts, the LTR design of the optimal output disturbance cancellation controller is possible for plants with the integral action. Assume that the plant transfer function matrix is in the form of

$$G(s) = \frac{1}{s}\bar{G}(s), \tag{48}$$

where $\bar{G}(s)$ is a proper transfer function matrix. Let $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ denote a realization of $\bar{G}(s)$. The assumptions A1 and A2 in Section 2 are replaced by the following $\bar{\text{A}}1$ and $\bar{\text{A}}2$, respectively.

- $\bar{\text{A}}1$: The quadruple $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is a minimal realization with no zero at the origin and minimum phase.
- $\bar{\text{A}}2$: The transfer function matrix $\bar{G}(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D}$ is non-singular for almost all s.

Consider a realization (A, B, C) of Equation (48) given by

$$A = \begin{bmatrix} O_m & \bar{C} \\ O_m & \\ \vdots & \bar{A} \\ O_m & \end{bmatrix}, \quad B = \begin{bmatrix} \bar{D} \\ \bar{B} \end{bmatrix},$$

$$C = \begin{bmatrix} I_m & O_m & \dots & O_m \end{bmatrix}, \tag{49}$$

where O_m is an $m \times m$ null matrix, I_m is an $m \times m$ identity matrix, $\bar{A} \in R^{(n-m)\times(n-m)}$, $\bar{B} \in R^{(n-m)\times m}$, $\bar{C} \in R^{m\times(n-m)}$ and $\bar{D} \in R^{m\times m}$.

The structure of the realization (49) with the output disturbance model is illustrated in Figure 4 where the initial condition of the disturbance model $d(0)$ is explicitly

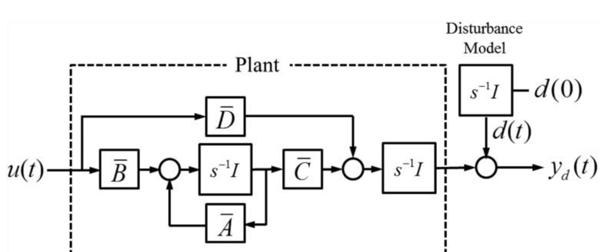

Figure 4. The plant with the integral action and the disturbance model.

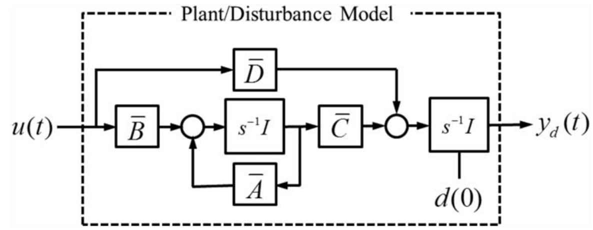

Figure 5. The plant model integrated with the disturbance model.

included. Note that we have assumed that the time when the disturbance occurs is defined as $t = 0$.

Noting that the output step disturbance in Figure 4 can be regarded as an initial condition for the rightmost integrator of the plant as in Figure 5, we can convert the optimal output disturbance cancellation problem into the optimal regulation problem for an impulse disturbance generating the initial condition in Figure 5.

For an arbitrary minimal realization of Equation (48), it can be readily shown that there exists an initial condition for the realization such that the output of the realization is same as that of Equation (49) for $t \geq 0$.

From the above observations, we can conclude the following.

PROPOSITION 5 *Consider the plant with the transfer function matrix in the form of Equation* (48)*. The standard LQG controller for an arbitrary minimal realization* (*A, B, C*) *of Equation* (48) *with the quadratic performance index*

$$J_{\text{LQG}} \triangleq E\left\{\lim_{T\to\infty}\frac{1}{T}\int_o^T [\rho^2 x'(t)C'Cx(t) + u'(t)u(t)]\,\mathrm{d}t\right\} \tag{50}$$

can be regarded as the optimal disturbance cancellation controller for the step disturbances entering the plant output side.

Note that the existence of the disturbances is implicit in the above LQG controller design. The standard LTR procedure for the plant output side can be used as a systematic design procedure. The choice of the minimal realization affects the design efficiency in determining the target with sufficient disturbance cancellation capability. It appears that the realization in the form (49) provides the easiest tuning of the target.

4.2. *Integrator augmentation*

For plants without the integral action, it is possible to construct an optimal disturbance cancellation controller by applying Proposition 5 to an extended plant obtained by augmenting artificial integrators to the input side of the plant.

This approach is simple but has the following issues.

- The performance index for the extended plant cannot help but use the input vector of the augmented integrators instead of the actual control input vector so that the effect of the weighting coefficient ρ on the actual control input is obscure.
- The state vector of the augmented integrators is usually measurable. However, to ensure the integral action of the extended plant, the state vector of the augmented integrators should be estimated together with the plant state vector.

Compared with the integrator augmentation approach, the design of the optimal output disturbance cancellation controller proposed in the previous sections has the following advantages:

- Tuning of the target for improving the disturbance cancellation capability is easier for the optimal output disturbance cancellation controller as is apparent from the target sensitivity matrix (40). Target of the integrator augmentation approach lacks transparency for tuning the disturbance cancellation capability.
- The optimal disturbance cancellation controller utilizes the performance index which represents the disturbance cancellation requirement more explicitly than the performance index used in the integrator augmentation approach.
- The disturbance estimates, which are sometimes useful for monitoring the control process, can directly be obtained from the optimal output disturbance cancellation controller. On the other hand, the integral augmentation approach requires additional computation to obtain the disturbance estimates.

5. Illustrative example

Using a simple numerical example, we show that the design procedure proposed in Section 3 provides flexible tuning of feedback property with sufficient stability margins by a small number of tuning parameters. Consider a single-input-single-output plant described by

$$A = \begin{bmatrix} -2.0 & -1.0 & -0.5 \\ 2.0 & 0 & 0 \\ 0 & 1.0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 \\ 0 \\ 0 \end{bmatrix},$$
$$C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \qquad (51)$$

which is a minimal realization of the transfer function given by

$$G(s) = \frac{1}{(s+1)(s^2+s+1)}. \qquad (52)$$

It is obvious that the plant satisfies the basic assumptions A1 and A2 given in Section 2. Using the plant model (51), we illustrate design freedom provided by the design procedure given in Subsection 3.2.

5.1. *Target design*

In the first step, we determine the Kalman filter gain matrix K such that the target has appropriate feedback property. To determine the Kalman filter gain matrix K, we introduce the scalar tuning parameters λ and σ in the stochastic model (24) as

$$\bar{\Gamma} = \begin{bmatrix} \lambda \\ \sigma B \end{bmatrix} \qquad (53)$$

with $W = I$ and $V = 1$. For given λ and σ, the Kalman filter gain matrix K is determined by solving the Riccati equation (31). Note that the choice of the tuning parameters is essential for the efficient tuning of the target. This issue is unique in the proposed LTR design. There is room for finding more efficient tuning parameters than those in Equation (53).

For $\lambda = 0.01$, 5 and 10 with $\sigma = 10$, the magnitude characteristics of the target sensitivity function (37) is shown in Figure 6. Note that the sensitivity matrix is the transfer function matrix from the disturbance to the disturbed output. The disturbance cancellation capability of the target can be evaluated in the time domain by the computer simulation injecting the step disturbance at the point P_T in Figure 3. The time–response of the target is shown in Figure 7, where the unit step disturbance is injected at $t = 5.0$. It is seen from the figures that the disturbance cancellation capability of the target is improved by increasing λ. For $\lambda = 0.01$, the integral action in the estimation error dynamics is almost lost. The stability margins are summarized in Table 1, which confirms that the target has the infinite gain margin and phase margin more than 60 degrees irrespective of λ. Let us choose the Kalman filter gain matrix for $\sigma = 10$ and $\lambda = 5$ as the target design.

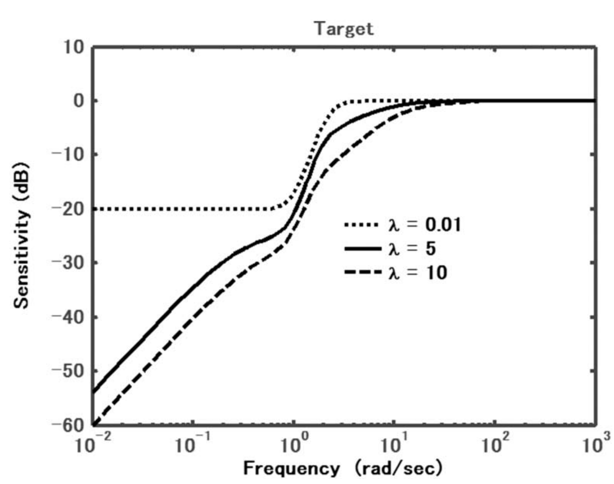

Figure 6. The target sensitivity function.

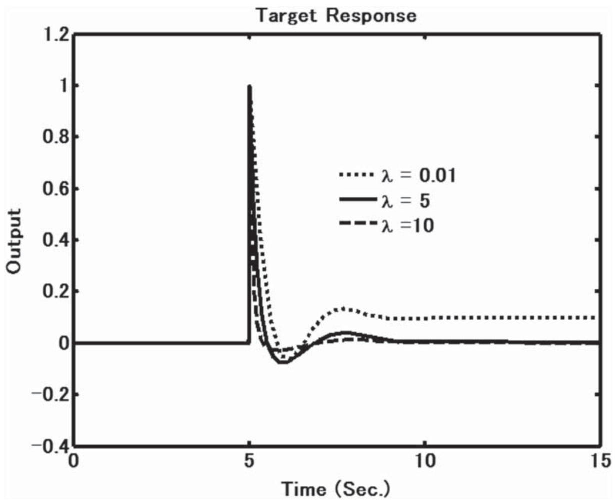

Figure 7. The time–response of the target.

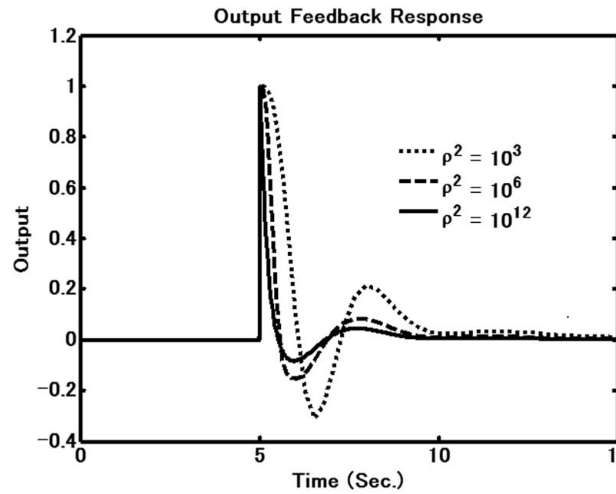

Figure 9. The time–response of the output feedback controller.

Table 1. Target stability margins.

λ	Gain margin (dB)	Phase margin (deg)
0.001	∞	66.1
5	∞	85.4
10	∞	89.4

Table 2. Stability margins of the output feedback controller.

ρ^2	Gain margin (dB)	Phase margin (deg)
10^3	6.83	37.7
10^6	10.1	51.8
10^{12}	15.9	79.3

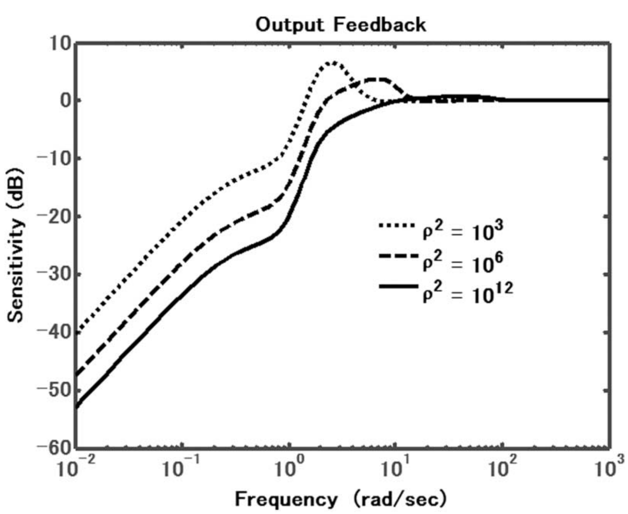

Figure 8. The sensitivity function for the output feedback controller.

5.2. *Target recovery*

The second step determines the feedback gain matrix F by the formal procedure using the weighting parameter ρ in the performance index (32). Consider the output feedback disturbance cancellation controller (33) using the Kalman filter gain matrix determined in the first step and the optimal feedback gain matrix F for $\rho^2 = 10^3,\ 10^6$ and 10^{12}. The magnitude characteristics of the sensitivity function (36) and the time–response for the unit step disturbance injected at $t = 5.0$ are shown in Figures 8 and 9, respectively. It is confirmed numerically that the sensitivity characteristics and the time–response approach those of the target as the weighting parameter ρ increases. The stability margins of the output feedback controller are summarized in Table 2. It is seen that the stability margins are improved as the weighting parameter ρ increases.

The above results have demonstrated that the proposed LTR design provides flexible tuning of the achievable feedback property with sufficient stability margins. For real world problems, more elaborated case studies are required taking account of practical constraints which are not considered in this illustrative example.

6. Conclusions

The application of the classical LTR technique for designing the optimal output disturbance cancellation controllers has been discussed. The optimality of the controller is shown by the parametric LQ approach. Although the stabilizability assumption in the standard LTR theory is not satisfied, it has been shown that the recovery procedure similar to that used in the standard LTR achieves the recovery. A numerical design example has been presented to illustrate the effectiveness of the proposed design.

Extension of this paper to non-minimum phase plant has been presented in the conference (Ishihara & Guo, 2011). The optimal output disturbance cancellation for more general class of disturbances will be discussed elsewhere.

It is known that the LTR procedure can be used for tuning of predictive controllers (Bitmead, Gevers, & Wertz, 1990; Maciejowski, 2002). An interesting future topic is to construct a new type of predictive controllers by considering the discrete-time version of the proposed design in finite-time interval with additional constraints.

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