Fuzzy Information and Engineering

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/tfie20

On the Completion of Fuzzy Normed Linear Spaces in the Sense of Bag and Samanta

Conghua Yan

To cite this article: Conghua Yan (2021) On the Completion of Fuzzy Normed Linear Spaces in the Sense of Bag and Samanta, Fuzzy Information and Engineering, 13:2, 236-247, DOI: 10.1080/16168658.2021.1943876

To link to this article: https://doi.org/10.1080/16168658.2021.1943876

Published online: 24 Jun 2021.

FUZZY INFORMATION AND ENGINEERING
2021, VOL. 13, NO. 2, 236–247
https://doi.org/10.1080/16168658.2021.1943876

Taylor & Francis
Taylor & Francis Group

On the Completion of Fuzzy Normed Linear Spaces in the Sense of Bag and Samanta

Conghua Yan

Institute of Mathematics, School of Mathematical Sciences, Nanjing Normal University, Nanjing, People's Republic of China

ABSTRACT
In this paper, the completion of fuzzy normed linear space (in the sense of Bag and Samanta) is studied. First, some properties of convergence fuzzy point sequences are discussed. Specially, we give another characterisation of Q-neighbourhood base of $\theta_\lambda(\lambda \in (0,1])$ for *I*-topology introduced by Saheli. Then we show that each fuzzy normed linear space has an (up to isomorphism) unique complete fuzzy normed linear space which contains an uniformly dense in every stratum subspace isomorphic to it.

KEYWORDS
Fuzzy metric spaces; fuzzy normed linear space; Cauchy fuzzy point sequence; uniformly dense in every stratum; completion

1. Introduction

The notion of fuzzy norm on a linear space was first introduced by Katsaras [1]. Felbin [2] gave an idea of fuzzy norm on a linear space whose associated metric is Kaleva type [3]. Influenced by the work given by Karmosil and Michalek [4], Cheng and Menderson [5] introduced another definition of fuzzy norm on a linear space. Bag and Samanta [6] modified slightly the notion of fuzzy norm determined by Cheng and Menderson. The relationships between above three types of fuzzy norms are discussed by Bag and Samanta [7]. Based on the notion of fuzzy norm in the sense of Bag and Samanta, the theory of fuzzy normed linear spaces is studied systematically [8–12]. Moreover, some notions and properties of finite dimensional fuzzy cone normed linear spaces are discussed in [13–15].

Among them, Saheli [12] introduced a new *I*-topology on fuzzy normed linear space, and show that this *I*-topology is compatible with the vector structure. The Q-neighbourhood base of $\theta_\lambda(\lambda \in (0,1])$ for this *I*-vector topology is obtained. Moreover, a comparative study of *I*-topologies which obtained by Saheli in [11, 12] on fuzzy normed linear spaces is presented.

The study of completion of fuzzy metric space and fuzzy normed linear space constitutes a natural and interesting open question in the analysis of such spaces. The first effort is due to Kaleva [16] in the frame of fuzzy metric space introduced by Kaleva. From then on, many authors devoted to study the completion of fuzzy metric spaces or fuzzy normed linear spaces in the sense of Kaleva type or Felbin type, and several important results are discussed ([17–20]). The study of completion on the fuzzy metric space introduced by Karmosil

CONTACT Conghua Yan chyan@njnu.edu.cn

and Michalek is originally Gregori and Romaguera [21], an ordinary topology is considered in their study. They show that for each fuzzy metric space there is an (up to uniform isomorphism) unique complete fuzzy metric space that contains a dense subspace uniformly isomorphic to it. The original research of completion on the fuzzy normed linear spaces was Felbin' s work in [19] with help of usual classical topology. As above claimed, Saheli [12] introduced a new I-topology on fuzzy normed linear space in the sense of Bag and Samanta recently. From the properties discussed by the author, we may consider this new I-topology is more suitable for the further study in fuzzy normed linear space.

The main purpose of this paper is to study the completion of fuzzy normed linear space with respect to the I-topology determined by Saheli. At first, we study some properties of fuzzy point sequences and give another structure of Q-neighbourhood base of $\theta_\lambda(\lambda \in (0,1])$ for the I-topology. Then we prove that each fuzzy normed linear space has a completion with respect to I-vector topology.

First we fix some notations, throughout this paper, $I = [0,1]$ and I^X denotes the family of all fuzzy sets on the nonempty set X. The notation $Pt(I^X)$ denotes the set of all fuzzy points on X. For every $x_\lambda \in Pt(I^X), A \in I^X$, the notation $x_\lambda \widetilde{\in} A$ denotes the relationship $A(x) + \lambda > 1$. According to the terminology introduced by Rodabaugh [22], for $r \in [0,1]$, $\underline{r}$ denotes the fuzzy set on X which takes the constant value r.

Definition 1.1 ([6]): Let X be a vector space over $\mathbb{R}$ (real number), N a fuzzy set of $\mathbb{R}$ such that for all $x, u \in X$ and $c \in \mathbb{R}$:

(N1) $N(x,t) = 0$ for all $t \leq 0$;
(N2) $x = \theta$ if and only if $N(x,t) = 1$ for all $t > 0$;
(N3) If $c \neq 0$, then $N(cx,t) = N(x, \frac{c}{|t|})$ for all $t \in \mathbb{R}$;
(N4) $N(x+u, s+t) \geq N(x,s) \wedge N(u,t)$ for all $s,t \in \mathbb{R}$;
(N5) $N(x,\cdot)$ is nondecreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.

Then N is called a fuzzy norm on X and the pair (X,N) is called a fuzzy normed linear space.

Definition 1.2 ([23]): An I-topology on a set X is a family τ of fuzzy subsets of X satisfying the following:

(1) For each $\lambda \in [0,1]$, $\underline{\lambda} \in \tau$
(2) τ is closed under finite intersection of fuzzy subsets
(3) τ is closed under arbitrary union of fuzzy subsets.

The pair (X,τ) is called an I-topological space.

Definition 1.3 ([24]): An I-topology τ on a vector space X is said to be an I-vector topology, if the following two mappings

$$f: X \times X \to X, (x,y) \to x+y \quad \text{and} \quad g: \mathbb{K} \times X \to X,\ (k,x) \to kx$$

are continuous, where $\mathbb{K}$ is equipped with the I-topology induced by the usual topology and $X \times X$, $\mathbb{K} \times X$ are equipped with the corresponding product I-topologies. At this time, the pair (X,τ) is called an I-topological vector space.

Definition 1.4 ([24]): Let (X,τ) be an I-topological space and $x_\lambda \in Pt(I^X)$.

(1) A fuzzy set U on X is called Q-neighbourhood of x_α iff there exists $G \in \tau$ such that $x_\alpha \widetilde{\in} G \subseteq U$.
(2) A family $\mathfrak{U}_{x_\alpha}$ of Q-neighbourhoods of x_α is called Q-neighbourhood base of x_α iff for every Q-neighbourhood A of x_α, there exists $U \in \mathfrak{U}_{x_\alpha}$ such that $U \subseteq A$.
(3) (X,τ) is called first countable, if for each $x_\lambda \in Pt(I^X)$, there exists Q-neighbourhood base $\mathfrak{U}_{x_\alpha}$ of x_α such that $\mathfrak{U}_{x_\alpha}$ has countable fuzzy sets.

Definition 1.5 ([24]): An I-topological vector space is said to be a QL-type, if there exists a family $\mathfrak{U}$ of fuzzy sets on X such that for each $\lambda \in (0,1]$,

$$\mathfrak{U}_\lambda = \{U \bigcap \underline{r} \mid U \in \mathfrak{U},\ r \in (1-\lambda, 1]\}.$$

is a Q-neighbourhood base of θ_λ in (X,τ). The family $\mathfrak{U}$ is called a Q-prebase for τ.

Theorem 1.6 ([12]): *Let (X,N) be a fuzzy normed linear space. Then the family*

$$\tau_N = \{\mu \in I^X \mid \quad \forall x \in \sigma_0(\mu),\ r \in (0,\mu(x)), \quad \text{there is } \varepsilon > 0, \quad \text{s.t. } x + B_\varepsilon \bigcap \underline{r} \subseteq \mu\}$$

is an I-topology on X. Here $C_\varepsilon(x) = \bigvee\{\alpha \in (0,1] \mid N(x,\varepsilon) \geq \alpha\}$, $\forall x \in X$.

Theorem 1.7 ([12]): *Let (X,N) be a fuzzy normed linear space. Then (X,τ_N) is an I-topological vector space and for each $\lambda \in (0,1]$,*

$$\mathfrak{U}_\lambda = \{C_\varepsilon \bigcap \underline{r} \mid \varepsilon > 0, r \in (1-\lambda, 1]\}$$

is a Q-neighbourhood base of θ_λ.

2. Some Basic Properties in Fuzzy Normed Linear Spaces

Definition 2.1: Let (X,N) be a fuzzy normed linear space, $\varepsilon > 0$. The fuzzy set B_ε on X is defined as follows:

$$B_\varepsilon(x) = \bigvee\{1-\alpha \mid \bigwedge\{t>0 : N(x,t) > 1-\alpha\} < \varepsilon\}, \quad \forall x \in X.$$

Lemma 2.2: *Let (X,N) be a fuzzy normed linear space, $x_\alpha \in Pt(I^X)$. Then $x_\alpha \widetilde{\in} B_\varepsilon$ iff $\bigwedge\{t>0 : N(x,t) > 1-\alpha\} < \varepsilon$.*

Proof: Necessity. Since $x_\alpha \widetilde{\in} B_\varepsilon$, then there exists $\beta \in (0,\alpha)$ such that $\bigwedge\{t>0 : N(x,t) > 1-\beta\} < \varepsilon$. So we have $t_0 < \varepsilon$ which implies $N(x,t_0) > 1-\beta > 1-\alpha$. This deduces that $\bigwedge\{t>0 : N(x,t) > 1-\alpha\} \leq t_0 < \varepsilon$.

Sufficiency. First we want to prove

$$\lim_{\beta \to \alpha-} \bigwedge\{t>0 : N(x,t) > 1-\beta\} = \bigwedge\{t>0 : N(x,t) > 1-\alpha\}.$$

In fact, for each sequence $\{\beta_n\}$ which increases and convergence to α, since

$$\{t>0 : N(x,t) > 1-\beta_n\} \subseteq \{t>0 : N(x,t) > 1-\alpha\}, \quad \forall n \in \mathbb{N},$$

we have $\bigwedge\{t>0 : N(x,t) > 1-\beta_n\} \geq \bigwedge\{t>0 : N(x,t) > 1-\alpha\}$, $\forall n \in \mathbb{N}$. Thus $\lim_n \bigwedge\{t>0 : N(x,t) > 1-\beta_n\} \geq \bigwedge\{t>0 : N(x,t) > 1-\alpha\}$. If $\lim_n \bigwedge\{t>0 :$

$N(x,t) > 1-\beta_n\} > \bigwedge\{\ t > 0\ :\ N(x,t) > 1-\alpha\}$, then there exists $k > 0$ such that $\lim_n \bigwedge\{\ t > 0\ :\ N(x,t) > 1-\beta_n\} > k > \bigwedge\{\ t > 0\ :\ N(x,t) > 1-\alpha\}$. This implies that there exists $q \in \mathbb{N}$ such that $\bigwedge\{\ t > 0\ :\ N(x,t) > 1-\beta_n\} > k$ for all $n > p$. Hence $N(x,k) \leq 1-\beta_n$ for all $n > p$. Put $n \to \infty$, we have $N(x,k) \leq 1-\alpha$. So $\bigwedge\{\ t > 0\ :\ N(x,t) > 1-\alpha\} \geq k$, this contradicts with the fact $k > \bigwedge\{\ t > 0\ :\ N(x,t) > 1-\alpha\}$. Then $\lim_n \bigwedge\{\ t > 0\ :\ N(x,t) > 1-\beta_n\} = \bigwedge\{\ t > 0\ :\ N(x,t) > 1-\alpha\}$. Because $\bigwedge\{\ t > 0\ :\ N(x,t) > 1-\alpha\}$ is decrease for the variable α, it deduces $\lim_{\beta\to\alpha-} \bigwedge\{\ t > 0\ :\ N(x,t) > 1-\beta\} = \bigwedge\{\ t > 0\ :\ N(x,t) > 1-\alpha\}$.

From the assumption $\bigwedge\{\ t > 0\ :\ N(x,t) > 1-\alpha\} < \varepsilon$ and above proof, there exists $\beta \in (0,\alpha)$ such that $\bigwedge\{\ t > 0\ :\ N(x,t) > 1-\beta\} < \varepsilon$. Then $B_\varepsilon(x) \geq 1-\beta > 1-\alpha$. Hence $x_\alpha \widetilde{\in} B_\varepsilon$. ■

Theorem 2.3: *Let (X,N) be a fuzzy normed linear space, τ_N an I-topology determined by fuzzy norm. Then τ can be determined by the Q-neighbourhood base $\mathfrak{B}_\lambda = \{B_\varepsilon \bigcap \underline{r} \mid \varepsilon > 0, r \in (1-\lambda, 1]\}$ of θ_λ, $\lambda \in (0,1]$.*

Proof: For each $\varepsilon > 0$, we may prove the following

$$B_\varepsilon \subseteq C_\varepsilon \subseteq B_{2\varepsilon}.$$

In fact, for each $x_\alpha \widetilde{\in} B_\varepsilon$, there is $\beta \in (0,\alpha)$ such that $x_\beta \widetilde{\in} B_\varepsilon$. Then $\bigwedge\{\ t > 0\ :\ N(x,t) > 1-\beta\} < \varepsilon$. This implies $N(x,\varepsilon) > 1-\beta$. Thus $C_\varepsilon(x) \geq 1-\beta > 1-\alpha$. So $x_\alpha \widetilde{\in} C_\varepsilon$. Hence $B_\varepsilon \subseteq C_\varepsilon$.

On the other hand, for each $x_\lambda \widetilde{\in} C_\varepsilon$, there exists $\mu \in (1-\lambda, 1)$ such that $N(x,\varepsilon) \geq \mu > 1-\lambda$. Then $\bigwedge\{\ t > 0\ :\ N(x,t) > 1-\lambda\} \leq \varepsilon < 2\varepsilon$. So $x_\lambda \widetilde{\in} B_{2\varepsilon}$. This implies $C_\varepsilon \subseteq B_{2\varepsilon}$.

By Theorem 1.7, the family $\mathfrak{B}_\lambda (\lambda \in (0,1])$ of fuzzy sets is Q-neighbourhood base of θ_λ which determined the *I*-topology is equivalent to τ_N. ■

Theorem 2.4: *Let (X,N) be a fuzzy normed linear space, τ_N an I-topology determined by fuzzy norm. Then (X,τ_N) is first countable I-topological vector space.*

Proof: By Theorem 1.7 and Theorem 2.3, (X,τ_N) is an *I*-topological vector space, and for any $\lambda \in (0,1]$, $\mathfrak{U}_\lambda = \{B_\varepsilon \bigcap \underline{r} \mid r \in (1-\lambda, 1], \varepsilon > 0\}$ is a *Q*-neighbourhood base of θ_λ. Since for each $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $B_{\frac{1}{n}} \subseteq B_\varepsilon$, hence the family of fuzzy sets $\mathfrak{B}_\lambda = \{B_{\frac{1}{n}} \bigcap \underline{r} \mid r \in (1-\lambda, 1] \bigcap \mathbb{Q}, n \in \mathbb{N}\}$ is a *Q*-neighbourhood base of θ_λ and $\mathfrak{B}_\lambda$ has countable elements. So (X,τ_N) is first countable an *I*-topological vector space. ■

Theorem 2.5: *Let (X,N) be a fuzzy normed linear space, τ_N an I-topology determined by fuzzy norm. Then the fuzzy sequence $\{x^{(n)}_{\lambda_n}\}$ is convergent to x_λ with respect to τ_N if and only if for any $\varepsilon \in (0,\lambda)$, there exist $t \in (0,\varepsilon)$, $p \in \mathbb{N}$ such that $N(x^{(n)} - x, t) > 1-\lambda_n$, $\lambda_n > \lambda - \varepsilon$.*

Proof: Necessity. For any $\varepsilon \in (0,\lambda)$, $B_\varepsilon \bigcap \underline{1-\lambda+\varepsilon}$ is a *Q*-neighbourhood of θ_λ. Since $x^{(n)}_{\lambda_n} \to x_\lambda$, there exists $p \in \mathbb{N}$ such that $x^{(n)}_{\lambda_n} \widetilde{\in}\, x + B_\varepsilon \bigcap \underline{1-\lambda+\varepsilon}$ for all $n > p$. Then $(x^{(n)} - x)_{\lambda_n} \widetilde{\in}\, B_\varepsilon$ and $\lambda_n > \lambda - \varepsilon$. So we have α which satisfies $N(x^{(n)} - x, \varepsilon) \geq \alpha$ and $\alpha > 1-\lambda_n$. From the fact $N(x,t)$ holds the condition $(N7*)$, there is $t < \varepsilon$ such that $N(x,t) > 1-\lambda_n$.

Sufficiency. Let W be a *Q*-neighbourhood of x_λ, then we have $\varepsilon \in (0,\lambda)$ such that $x + B_\varepsilon \bigcap \underline{1-\lambda+\varepsilon} \subseteq W$. From the assumption of sufficiency, there exist $t \in (0,\varepsilon)$, $p \in \mathbb{N}$ such that $N(x^{(n)} - x, t) > 1 - \lambda_n, \lambda_n > \lambda - \varepsilon$ for all $n \geq p$. Then there is $\delta_n \in (0, \lambda_n)$ such that $N(x^{(n)} - x, t) > 1 - \lambda_n + \delta_n$. This implies $N(x^{(n)} - x, \varepsilon) > 1 - \lambda_n + \delta_n$, so $B_\varepsilon(x^{(n)} - x) \geq 1 - \lambda_n + \delta_n > 1 - \lambda_n$. Thus $(x^{(n)} - x)_{\lambda_n} \widetilde{\in} B_\varepsilon$. This deduces that $x^{(n)}_{\lambda_n} \widetilde{\in} x + B_\varepsilon \bigcap \underline{1 - \lambda + \varepsilon} \subseteq W$. Hence $\{x^{(n)}_{\lambda_n}\}$ converges to x_λ with respect to τ_N. ■

Definition 2.6: Let (X, N) be a fuzzy normed linear space, $\{x^{(n)}_{\lambda_n}\}$ a sequence of fuzzy points in X. Then

(1) $\{x^{(n)}_{\lambda_n}\}$ is called λ-Cauchy sequence, if for each $W \in \mathfrak{U}_\lambda$, there exists $p \in \mathbb{N}$ such that $x^{(n)}_{\lambda_n \wedge \lambda} - x^{(m)}_{\lambda_m \wedge \lambda} \widetilde{\in} W$ for all $n, m \geq p$.
(2) $\{x^{(n)}_{\lambda_n}\}$ is called Cauchy sequence, if for each $\lambda \in (0, \overline{\lim}_n \lambda_n)$, $\{x^{(n)}_{\lambda_n}\}$ is λ-Cauchy sequence.
(3) (X, N) is called fuzzy complete if for each Cauchy sequence $\{x^{(n)}_{\lambda_n}\}$, there exists $x \in X$ such that $x^{(n)}_{\lambda_n}$ converges to x_μ with respect to I-topology τ_N, where $\mu = \overline{\lim}_n \lambda_n$.

Theorem 2.7: *Let (X, N) be a fuzzy normed linear space, $\{x^{(n)}_{\lambda_n}\}$ a sequence of fuzzy points in X. Then $\{x^{(n)}_{\lambda_n}\}$ is a* Cauchy *sequence if and only if* $\lim_n \lambda_n = \mu$ *and for any* $\lambda \in (0, \mu)$, $\varepsilon \in (0, \lambda)$, *there exist* $t_0 \in (0, \varepsilon)$, $p \in \mathbb{N}$ *such that* $N(x^{(n)} - x^{(m)}, t_0) > 1 - \lambda$ *for all* $n, m \geq p$.

Proof: Necessity. Suppose that $\{x^{(n)}_{\lambda_n}\}$ is a Cauchy sequence and $\overline{\lim}_n \lambda_n = \mu$. For any $\lambda \in (0, \mu)$ and $\varepsilon \in (0, \lambda)$, $B_\varepsilon \bigcap \underline{1 - \lambda + \varepsilon} \in \mathfrak{U}_\lambda$, then there exists $p \in \mathbb{N}$ such that $x^{(n)}_{\lambda_n \wedge \lambda} - x^{(m)}_{\lambda_m \wedge \lambda} \widetilde{\in} B_\varepsilon \bigcap \underline{1 - \lambda + \varepsilon}$ for all $n, m \geq p$. This deduces that $\lambda_n > \lambda_n \wedge \lambda > \lambda - \varepsilon$ and $(x^{(n)} - x^{(m)})_{\lambda_n \wedge \lambda_m \wedge \lambda} \widetilde{\in} B_\varepsilon$. So $\lambda_n > \lambda - \varepsilon$ for all $n \geq p$ and

$$\begin{aligned} &\bigwedge \{\, t > 0 \,:\, N(x^{(n)} - x^{(m)}, t) > 1 - \lambda \} \\ &\quad \leq \bigwedge \{\, t > 0 \,:\, N(x^{(n)} - x^{(m)}, t) > 1 - (\lambda_n \wedge \lambda_m \wedge \lambda) \} < \varepsilon. \end{aligned}$$

Thus there is $t_0 \in (0, \varepsilon)$ such that $N(x^{(n)} - x^{(m)}, t_0) > 1 - \lambda$. In addition, $\underline{\lim}_n \lambda_n \geq \lambda$, since the arbitrariness of λ, we have $\underline{\lim}_n \lambda_n \geq \mu$. Hence $\lim_n \lambda_n = \mu$.

Sufficiency. If $\lim_n \lambda_n = \mu$ and for any $\lambda \in (0, \mu)$, $\varepsilon \in (0, \lambda)$, there exist $t_0 \in (0, \varepsilon)$, $p \in \mathbb{N}$ such that $N(x^{(n)} - x^{(m)}, t_0) > 1 - \lambda$ for all $n, m \geq p$. From the fact $\lim_n \lambda_n = \mu > \lambda$, we have $q \in \mathbb{N}$ with $q > p$ which implies $\lambda_n > \lambda$ for all $n \geq q$. Then $\bigwedge \{\, t > 0 \,:\, N(x^{(n)} - x^{(m)}, t) > 1 - \lambda\} \leq t_0 < \varepsilon$. So we have $(x^{(n)} - x^{(m)})_\lambda = x^{(n)}_{\lambda_n \wedge \lambda} - x^{(m)}_{\lambda_m \wedge \lambda} \widetilde{\in} B_\varepsilon \bigcap \underline{1 - \lambda + \varepsilon}$ for all $n, m \geq q$. This means that $\{x^{(n)}_{\lambda_n}\}$ is a Cauchy sequence. ■

Remark 2.8: The notion of Cauchy sequences and fuzzy complete is based on I-topology in this paper. This notions is not different from the corresponding notions introduced by Felbin [19]. In fact, the notion of Cauchy sequences and complete given by Felbin [19] is based on crisp topology, equivalently, every Cauchy sequence $\{x_n\}$ is convergent in every stratum $(X, \|\cdot\|_\alpha)$ for all $\alpha \in (0, 1]$. In addition, the notions of fuzzy normed linear spaces are not completely same. By Theorem 2.7, the notion of Cauchy sequences in this paper is for fuzzy points (not crisp points).

3. The Completion of Fuzzy Normed Linear Spaces

Definition 3.1: Let (X, N) be a fuzzy normed linear space, $A \subseteq X$ is called uniformly dense in every stratum if for any $x \in X$, there exists a sequence $\{x^n\} \subseteq A$ such that for each $\lambda \in (0,1], \varepsilon > 0$, there is $t < \varepsilon$, which deduces that $N(x^n - x, t) > 1 - \lambda$.

Definition 3.2: Let (X, N), (Y, N_1) be two fuzzy normed linear spaces, then they are called isomorphic if there exists a linear operator $T : X \to Y$ such that for any $x \in X, \lambda \in (0,1]$, the next equality holds.

$\wedge\{t > 0 : \ N(x,t) > 1 - \lambda\} = \wedge\{t > 0 : \ N_1(Tx, t) > 1 - \lambda\}$.

Definition 3.3: A complete fuzzy normed linear space $(\widetilde{X}, N_1)$ is said to a completion of the fuzzy linear space (X, N) if $(\widetilde{X}, N_1)$ has an uniformly dense subspace in every stratum (W, N_1) being isometric to (X, N).

Theorem 3.4: *Any fuzzy normed linear space has a completion.*

Proof: By the Definition 3.3, the whole proof is divided into following four steps.

Step 1. Construct a fuzzy normed linear space $(\widetilde{X}, N_1)$. At first we define the sets X_c and $\tilde{\theta}$ as follows:

$X_c = \{\{x^{(n)}\} : \ \forall\ \lambda \in (0,1], \varepsilon > 0, \ \exists\ t < \varepsilon, p \in \mathbb{N}, N(x^{(n)} - x^{(m)}, t) > 1 - \lambda (\text{for all } n, m > p)\}$.

$\tilde{\theta} = \{\{x^{(n)}\} : \ \{x^{(n)}\} \in X_c, \forall \lambda \in (0,1], \varepsilon > 0, \ \exists\, t < \varepsilon, p \in \mathbb{N}, N(x^{(n)}, t) > 1 - \lambda (\text{for any } n > p)\}$.

It is easy to find $X_c \neq \emptyset$ and $\tilde{\theta} \neq \emptyset$. The relation $\sim$ on $X_c \setminus \tilde{\theta}$ is defined as follows:

$\{x^{(n)}\} \sim \{y^{(n)}\} \iff \forall \quad \lambda \in (0,1], \varepsilon > 0, \quad \exists \quad t < \varepsilon, p \in \mathbb{N}, N(x^{(n)} - y^{(n)}, t) > 1 - \lambda (\text{for each } n > p)$.

By the definition of fuzzy norm, the above relation is equivalent. For each $\xi = \{x^{(n)}\} \in X_c \setminus \tilde{\theta}$, its equivalent class is denoted by $\tilde{\xi} = \widetilde{\{x^{(n)}\}}$. Specially, if $\{x^{(n)}\} \in \tilde{\theta}$, then we claim that $\widetilde{\{x^{(n)}\}} = \tilde{\theta}$. Denote $\widetilde{X}_0 = (X_c \setminus \tilde{\theta}) / \sim$, and $\widetilde{X} = \widetilde{X}_0 \bigcup \{\tilde{\theta}\}$. The addition and scalar multiplication in $\widetilde{X}$ are well-defined as follows:

For all $\widetilde{\{x^{(n)}\}}$, $\widetilde{\{y^{(n)}\}} \in \widetilde{X}_0$, $k \in \mathbb{K}$,

$$\widetilde{\{x^{(n)}\}} + \widetilde{\{y^{(n)}\}} = \{\widetilde{x^{(n)} + y^{(n)}}\};$$

$$\widetilde{\{x^{(n)}\}} + \tilde{\theta} = \tilde{\theta} + \widetilde{\{x^{(n)}\}} = \widetilde{\{x^{(n)}\}};$$

$$k\tilde{\theta} = \tilde{\theta};$$

$$k\widetilde{\{x^{(n)}\}} = \begin{cases} \tilde{\theta}, & k = 0 \\ \widetilde{\{kx^{(n)}\}}, & k \neq 0 \end{cases}.$$

It is easy to verify that $\widetilde{X}$ is a linear space. The mapping $N_1 : \widetilde{X} \times \mathbb{R} \to [0,1]$ is defined as follows:

$$N_1(\tilde{\xi}, t) = \begin{cases} 0, & (\tilde{\xi}, t) = (\tilde{\theta}, 0) \\ 1 - \bigwedge_{\lambda \in (0,1]} \{\lambda : \ \|\tilde{\xi}\|_\lambda^{\widetilde{X}} \leq t\}, & (\tilde{\xi}, t) \neq (\tilde{\theta}, 0) \end{cases}.$$

Here $\|\tilde{\xi}\|_\lambda^{\widetilde{X}} = \begin{cases} 0, & \tilde{\xi} = \tilde{\theta} \\ \bigwedge_{\mu<\lambda} \lim_{n\to\infty} \bigwedge\{\, t > 0 \,:\, N(x^{(n)}, t) > 1-\mu\}, & \tilde{\xi} \neq \tilde{\theta} \end{cases}$, and $\{x^{(n)}\} \in \tilde{\xi}$.

Since $\{x^{(n)}\} \in \tilde{\xi}$, the sequence $\bigwedge\{\, t > 0 \,:\, N(x^{(n)}, t) > 1-\mu\}$ is a real Cauchy sequence. Thus the limit of this sequence exists. In what follows, we must prove that $\|\tilde{\xi}\|_\lambda^{\widetilde{X}}$ is determined by ξ uniquely. In fact, if $\{x^{(n)}\} \sim \{y^{(n)}\}$, then for each $\lambda \in (0,1], \varepsilon > 0$, there exists $t < \varepsilon$ such that $N(x^{(n)} - y^{(n)}, t) > 1 - \lambda$ when $n \to \infty$. This implies $\lim_{n\to\infty} \bigwedge\{\, t > 0 \,:\, N(x^{(n)} - y^{(n)}, t) > 1 - \mu\} = 0$. On the other hand, since

$$\bigwedge\{\, t > 0 \,:\, N(x^{(n)}, t) > 1-\mu\} - \bigwedge\{\, s > 0 \,:\, N(y^{(n)}, s) > 1-\mu\}$$
$$\leq \bigwedge\{\, s+t > 0 \,:\, N(x^{(n)} - y^{(n)}, s+t) > 1-\mu\}.$$

So $\|\widetilde{\{x^{(n)}\}}\|_\lambda^{\widetilde{X}} = \|\widetilde{\{y^{(n)}\}}\|_\lambda^{\widetilde{X}}$. This means that the mappings N_1 is well-defined.

In the following, it needs to verify that $(\widetilde{X}, N_1)$ is a fuzzy normed linear space.

(N1) Clearly, $N_1(\tilde{\xi}, t) = 0$ for each $t \in (-\infty, 0]$.

(N2) If $N_1(\tilde{\xi}, t) = 1$ for each $t > 0$, then $\|\tilde{\xi}\|_\lambda^{\widetilde{X}} = 0$ for any $\lambda \in (0,1]$. For any $\varepsilon > 0, \lambda \in (0,1]$, there exists $\mu < \lambda$ such that $\lim_{n\to\infty} \bigwedge\{\, t > 0 \,:\, N(x^{(n)}, t) > 1-\mu\} < \varepsilon$. Then there is $t < \varepsilon, p \in \mathbb{N}$ such that $N(x^{(n)}, t) > 1-\mu$ for each $n > p$. So $\tilde{\xi} = \widetilde{\{x^{(n)}\}} = \tilde{\theta}$. Conversely, if $\tilde{\xi} = \tilde{\theta}$, the equality $N_1(\tilde{\theta}, t) = 1$ for each $t > 0$ holds obviously.

(N3) For any $t > 0, c \in \mathbb{R}, c \neq 0$,

$$N_1(c\tilde{\xi}, t) = 1 - \bigwedge_{\lambda\in(0,1]} \{\lambda : \|c\tilde{\xi}\|_\lambda^{\widetilde{X}} \leq t\}$$
$$= 1 - \bigwedge_{\lambda\in(0,1]} \left\{\lambda : \|\tilde{\xi}\|_\lambda^{\widetilde{X}} \leq \frac{t}{|c|}\right\} = N_1\left(\tilde{\xi}, \frac{t}{|c|}\right).$$

(N4) Suppose that $N_1(\tilde{\xi}, t) = 1 - \bigwedge_{\lambda\in(0,1]}\{\lambda : \|\tilde{\xi}\|_\lambda^{\widetilde{X}} \leq t\} = a$ and $N_1(\tilde{\eta}, s) = 1 - \bigwedge_{\lambda\in(0,1]}\{\lambda : \|\tilde{\eta}\|_\lambda^{\widetilde{X}} \leq s\} = b$. Without loss of generality, let $a \wedge b \neq 0$, then for any $r \in (0, a\wedge b)$, there exist $\alpha, \beta \in (0,1]$ with $\alpha < 1-r,\ \beta < 1-r$ such that $\|\tilde{\xi}\|_\alpha^{\widetilde{X}} \leq t$ and $\|\tilde{\eta}\|_\beta^{\widetilde{X}} \leq s$. From the definition of $\|\tilde{\eta}\|_\beta^{\widetilde{X}}$, $\|\tilde{\eta}\|_{1-r}^{\widetilde{X}} \leq \|\tilde{\eta}\|_\beta^{\widetilde{X}}$ and $\|\tilde{\xi}\|_{1-r}^{\widetilde{X}} \leq \|\tilde{\xi}\|_\alpha^{\widetilde{X}}$. Thus we have

$$\|\tilde{\xi} + \tilde{\eta}\|_{1-r}^{\widetilde{X}} \leq \|\tilde{\xi}\|_{1-r}^{\widetilde{X}} + \|\tilde{\eta}\|_{1-r}^{\widetilde{X}} \leq \|\tilde{\xi}\|_\alpha^{\widetilde{X}} + \|\tilde{\eta}\|_\beta^{\widetilde{X}} \leq t + s.$$

Hence $N_1(\tilde{\xi} + \tilde{\eta}, s+t) = 1 - \bigwedge_{\lambda\in(0,1]}\{\lambda : \|\tilde{\xi} + \tilde{\eta}\|_\lambda^{\widetilde{X}} \leq s+t\} \geq 1 - (1-r) = r$.

By the arbitrariness of r, it deduces that $N_1(\tilde{\xi} + \tilde{\eta}, s+t) \geq a \wedge b = N_1(\tilde{\xi}, t) \wedge N_1(\tilde{\eta}, s)$.

(N5) For all $t_1, t_2 \in (0, +\infty), t_1 < t_2$,

$$N_1(\tilde{\xi}, t_1) = 1 - \bigwedge_{\lambda\in(0,1]} \{\lambda : \|\tilde{\xi}\|_\lambda^{\widetilde{X}} \leq t_1\} \leq 1 - \bigwedge_{\lambda\in(0,1]} \{\lambda : \|\tilde{\xi}\|_\lambda^{\widetilde{X}} \leq t_2\} = N_1(\tilde{\xi}, t_2).$$

In addition, for any $\tilde{\xi} = \widetilde{\{x^{(n)}\}} \in \widetilde{X}, \lambda \in (0,1]$, since $\|\tilde{\xi}\|_\lambda^{\widetilde{X}} \in [0, +\infty)$ and $\|\tilde{\xi}\|_\lambda^{\widetilde{X}} \leq s$ implies $\bigwedge_{\lambda\in(0,1]}\{\lambda : \|\tilde{\xi}\|_\lambda^{\widetilde{X}} \leq s\} \leq \lambda$. Then we have $1 - \bigwedge_{\lambda\in(0,1]}\{\lambda : \|\tilde{\xi}\|_\lambda^{\widetilde{X}} \leq s\} \geq 1 - \lambda$. So $\lim_{t\to+\infty} N_1(\tilde{\xi}, t) = 1$.

Step 2. Define a mapping: $T : (X, N) \to (\widetilde{X}, N_1)$ and prove that $T(X)$ is uniformly dense in every stratum.

Let $T : X \to \widetilde{X}$ be defined as follows.

$$Tx = \begin{cases} \tilde{\theta}, & x = \theta \\ \tilde{x}, & x \neq \theta \end{cases}, \quad \forall\, x \in X.$$

Here the notation is the equivalent class of the element $\{x, x, x, x, \ldots\}$. Denote $T(X) = W$, it is easy to find T is a linear operator from X onto W. In the next we prove that T is an isometric mapping.

In fact, if $x = \theta$, both $N(\theta, t) = 1$ and $N_1(\tilde{\theta}, t) = 1$ for any $t > 0$. Then $\bigwedge\{ t > 0 : N(\theta, t) > 1 - \lambda\} = \bigwedge\{ t > 0 : N(T\theta, t) > 1 - \lambda\}$ for any $\lambda \in (0, 1]$.

If $x \neq \theta$, for each $\lambda \in (0, 1]$ and $s > \bigwedge\{ t > 0 : N_1(Tx, t) > 1 - \lambda\}$, there is $t < s$ such that $N_1(Tx, t) > 1 - \lambda$. Then we have $\mu < \lambda$ which satisfies $\|\tilde{x}\|_\mu^{\widetilde{X}} \leq t < s$. By the definition of $\|\tilde{x}\|_\mu^{\widetilde{X}}$, there exists $\nu < \mu$ and $r < s$ such that $N(x, r) > 1 - \nu > 1 - \mu > 1 - \lambda$. So $\bigwedge\{ \omega > 0 : N(x, \omega) > 1 - \lambda\} \leq r < s$. This means that $\bigwedge\{ t > 0 : N_1(Tx, t) > 1 - \lambda\} \geq \bigwedge\{ \omega > 0 : N(x, \omega) > 1 - \lambda\}$. On the other hand, for each $s > \bigwedge\{ \omega > 0 : N(x, \omega) > 1 - \lambda\}$, there is $\omega < s$ such that $N(x, \omega) > 1 - \lambda$. Further we have $\delta > 0$ with $N(x, \omega) > 1 - (\lambda - \delta)$. Then $\|\tilde{x}\|_{\lambda-\delta}^{\widetilde{X}} = \bigwedge_{\mu<\lambda-\delta} \lim_{n\to\infty} \bigwedge\{ t > 0 : N(x, t) > 1 - \mu\} \leq \omega$. Thus $N_1(\tilde{x}, \omega) \geq 1 - (\lambda - \delta) > 1 - \lambda$. This implies that $\bigwedge\{ t > 0 : N_1(Tx, t) > 1 - \lambda\} \leq \omega < s$. So we can obtain $\bigwedge\{ t > 0 : N_1(Tx, t) > 1 - \lambda\} \leq \bigwedge\{ t > 0 : N(x, t) > 1 - \lambda\}$. Therefore T is an isometric mapping.

In what follows, it needs to prove $T(X) = W$ is uniformly dense in every stratum with respect to fuzzy normed linear space $(\widetilde{X}, N_1)$. For any $\tilde{\xi} = \widetilde{\{x^{(n)}\}} \in \widetilde{X}$, it is clear $\widetilde{x^{(n)}} \in W$ for any $n \in \mathbb{N}$, here the notation $\widetilde{x^{(n)}} \in W$ is the equivalent class of $\{x^{(n)}, x^{(n)}, x^{(n)}, \ldots\}$. For all $\lambda \in (0, 1], \varepsilon > 0$, from the fact $\{x^{(n)} \in X_c$ and let $\mu_0 \in (0, \lambda)$, there exist $t < \varepsilon, p \in \mathbb{N}$ such that $N(x^{(n)} - x^{(m)}, t) > 1 - \mu_0$ for all $n, m \geq p$. Thus we have the following

$$\begin{aligned} \|\widetilde{x^{(n)}} - \tilde{\xi}\|_\lambda^{\widetilde{X}} &\leq \|\widetilde{x^{(n)}} - \widetilde{x^{(p)}}\|_\lambda^{\widetilde{X}} + \|\widetilde{x^{(p)}} - \tilde{\xi}\|_\lambda^{\widetilde{X}} \\ &= \bigwedge_{\mu<\lambda} \bigwedge\{ s > 0 : N(x^{(n)} - x^{(p)}, s) > 1 - \mu\} \\ &\quad + \bigwedge_{\mu<\lambda} \lim_{n\to\infty} \bigwedge\{ s > 0 : N(x^{(p)} - x^{(n)}, s) > 1 - \mu\} \\ &\leq \bigwedge\{ s > 0 : N(x^{(n)} - x^{(p)}, s) > 1 - \mu_0\} \\ &\quad + \lim_{n\to\infty} \bigwedge\{ s > 0 : N(x^{(p)} - x^{(n)}, s) > 1 - \mu_0\}. \end{aligned}$$

Then $\|\widetilde{x^{(n)}} - \tilde{\xi}\|_\lambda^{\widetilde{X}} \leq t$ for all $n \geq p$. So $N_1(\widetilde{x^{(n)}} - \tilde{\xi}, t) > 1 - \lambda$ for all $n \geq p$. This means that W is uniformly dense in every stratum.

Step 3. We prove that $(\widetilde{X}, N_1)$ is complete. Suppose that $\{\tilde{\xi}_{\lambda_n}^{(n)}\}$ is Cauchy fuzzy point sequence in $(\widetilde{X}, N_1)$, from Theorem 2.7, $\lim_{n\to\infty} \lambda_n = \mu > 0$. In addition, for each $\lambda \in (0, 1], \varepsilon > 0$, there exist $t < \varepsilon, p \in \mathbb{N}$ such that $N_1(\tilde{\xi}^{(m)} - \tilde{\xi}^{(n)}, t) > 1 - \lambda$ for all $n, m \geq p$.

Since W is uniformly dense in every stratum, and for any $k \in \mathbb{N}$, $\tilde{\xi}^{(k)} \in \widetilde{X}$, we have a sequence $\{\widetilde{x^{(n)_k}}\} \subseteq W$ such that for above λ and $\frac{1}{k} > 0$, there is $s_k < \frac{1}{k}$ and $q \in \mathbb{N}, q > p$, which implies that $N_1(\widetilde{x^{(n)_k}} - \tilde{\xi}^{(k)}, s_k) > 1 - \lambda$ for all $n \geq q$.

For each $k \in \mathbb{N}$, there exists $n_k \geq q$ such that $N_1(\widetilde{x^{(n_k)_k}} - \tilde{\xi}^{(k)}, s_k) > 1 - \lambda$. Denote $\widetilde{x^{(n_k)_k}} = \widetilde{x^{(k)}}$. Then for $m, n \geq q$, the following inequality holds:

$$\begin{aligned} &N_1\left(\widetilde{x^{(m)}} - \widetilde{x^{(n)}}, t + \frac{1}{m} + \frac{1}{n}\right) \\ &\geq N_1\left(\widetilde{x^{(m)}} - \tilde{\xi}^{(m)}, \frac{1}{m}\right) \bigwedge N_1(\tilde{\xi}^{(m)} - \tilde{\xi}^{(n)}, t) \bigwedge N_1\left(\tilde{\xi}^{(n)} - \widetilde{x^{(n)}}, \frac{1}{n}\right) \\ &\geq N_1(\widetilde{x^{(m)}} - \tilde{\xi}^{(m)}, s_m) \bigwedge N_1(\tilde{\xi}^{(m)} - \tilde{\xi}^{(n)}, t) \bigwedge N_1(\tilde{\xi}^{(n)} - \widetilde{x^{(n)}}, s_n) > 1 - \lambda. \end{aligned}$$

So $\{\widetilde{x^{(n)}}\}$ is Cauchy sequence in W. Notice that T is isometric, we have $\{x^{(n)}\} \in X_c$. Let $\tilde{\xi} = \widetilde{\{x^{(m)}\}}$, clearly $\tilde{\xi} \in \widetilde{X}$. We will prove $\tilde{\xi}^{(n)}_{\lambda_n} \to \tilde{\xi}_\mu$ as $n \to \infty$. In fact, for each $\varepsilon \in (0, \mu)$, $\lambda \in (\frac{\mu}{2}, \mu)$, since $\lim_{n\to\infty} \lambda_n = \mu$ and $N(x^{(m)} - x^{(n)}, t) \to 1 (m, n \to \infty)$, there is $p \in \mathbb{N}, t < \varepsilon$ such that $\lambda_n > \lambda$ and $N(x^{(m)} - x^{(n)}, t) > 1 - \frac{\mu}{2}$ for all $n, m \geq p$. Thus for any $n \geq p$,

$$\begin{aligned} \|\widetilde{x^{(n)}} - \tilde{\xi}\|^{\widetilde{X}}_\lambda &= \bigwedge_{\nu<\lambda} \lim_{m\to\infty} \bigwedge \{ s > 0 \,:\, N(x^{(n)} - x^{(m)}, s) > 1 - \nu\} \\ &\leq \lim_{m\to\infty} \bigwedge \left\{ s > 0 \,:\, N(x^{(n)} - x^{(m)}, s) > 1 - \frac{\mu}{2}\right\} \leq t. \end{aligned}$$

This implies $N_1(\widetilde{x^{(n)}} - \tilde{\xi}, t) \geq 1 - \frac{\mu}{2} > 1 - \lambda$ for all $n \geq p$. Furthermore,

$$N_1\left(\tilde{\xi}^{(n)} - \tilde{\xi}, t + \frac{1}{n}\right) \geq N_1\left(\tilde{\xi}^{(n)} - \widetilde{x^{(n)}}, \frac{1}{n}\right) \bigwedge N_1(\widetilde{x^{(n)}} - \tilde{\xi}, t) > 1 - \lambda.$$

For above $t < \varepsilon$, there is $q \in \mathbb{N}, q > p$ such that $t + \frac{1}{n} < \varepsilon$ for all $n > q$. So the proof of $\tilde{\xi}^{(n)}_{\lambda_n} \to \tilde{\xi}_\mu$ is completed. This means that $(\widetilde{X}, N_1)$ is complete fuzzy normed linear space.

Step 4. The completion of fuzzy normed linear space $(\widetilde{X}, N_1)$ is unique except for isometrics. Suppose that (Y, N_2) is also a completion of (X, N) and S is an isometric linear operator from X onto Y. Since $W = TX$ is uniformly dense in every stratum, then for each $\tilde{\xi} \in \widetilde{X}$, there exists a sequence $\{\widetilde{x^{(n)}}\} \subseteq W$ such that for each $\lambda \in (0, 1], \varepsilon > 0$, there exist $t < \frac{\varepsilon}{2}, p \in \mathbb{N}$, which deduces that $N_1(\widetilde{x^{(n)}} - \tilde{\xi}, t) > 1 - \lambda$ for all $n > p$. Thus for all $m, n > p$,

$$\begin{aligned} N_1(Tx^{(n)} - Tx^{(m)}, 2t) &= N_1(\widetilde{x^{(n)}} - \widetilde{x^{(m)}}, 2t) \\ &\geq N_1(\widetilde{x^{(n)}} - \tilde{\xi}, t) \bigwedge N_1(\tilde{\xi} - \widetilde{x^{(m)}}, t) > 1 - \lambda. \end{aligned}$$

That is to say $\{Tx^{(n)}_\alpha\}$, $(\alpha \in (0, 1])$ is a Cauchy fuzzy points sequence in W. Since T and S are isometrics, we have

$$\begin{aligned} &\bigwedge \{ s > 0 \,:\, N_1(Tx^{(n)} - Tx^{(m)}, s) > 1 - \lambda\} \\ &= \bigwedge \{ s > 0 \,:N(x^{(n)} - x^{(m)}, s) > 1 - \lambda\} \\ &= \bigwedge \{ s > 0 \,:\, N_2(Sx^{(n)} - Sx^{(m)}, s) > 1 - \lambda\}. \end{aligned}$$

Thus deduces that $\{Sx^{(n)}_\alpha\}$, $(\alpha \in (0, 1])$ is a Cauchy fuzzy point sequence in SX. From the fact Y is complete, there is a unique $y \in Y$ such that $Sx^{(n)}_\alpha \to y_\alpha$. It may be proved that the element y has nothing to do with the choice of $\{\widetilde{x^{(n)}}\}$. In reality, if there exists a sequence $\{\widetilde{z^{(n)}}\} \subseteq W$ such that for each $\lambda \in (0,1], \varepsilon > 0$, there exist $t < \frac{\varepsilon}{2}, p \in \mathbb{N}$, which deduces that $N_1(\widetilde{z^{(n)}} - \tilde{\xi}, t) > 1 - \lambda$ for all $n > p$. Taking the same method, we may prove that there exists unique $z \in Y$ such that for each $\lambda \in (0,1]$, $\varepsilon > 0$, $\exists t < \varepsilon, q > p, q \in \mathbb{N}$, which implies $N_2(Sz^{(n)} - z, t) > 1 - \lambda$ for all $n > q$. Since

$$\begin{aligned}
&\bigwedge\{ s > 0 \,:\, N_2(Sx^{(n)} - Sz^{(n)}, s) > 1 - \lambda\} \\
&\quad = \bigwedge\{ s > 0 \,:\, N(x^{(n)} - z^{(n)}, s) > 1 - \lambda\} \\
&\quad = \bigwedge\{ s > 0 \,:\, N_1(Tx^{(n)} - Tz^{(n)}, s) > 1 - \lambda\}.
\end{aligned}$$

Moreover, $N_1(Tx^{(n)} - Tz^{(n)}, 2t) \geq N_1(Tx^{(n)} - \tilde{\xi}, t) \bigwedge N_1(\tilde{\xi} - Tz^{(n)}, t) > 1 - \lambda$.

Then $N_2(Sx^{(n)} - Sz^{(n)}, 2t) > 1 - \lambda$. So we have

$$N_2(y - z, 4t) \geq N_2(y - Sx^{(n)}, t) \bigwedge N_2(Sx^{(n)} - Sz^{(n)}, 2t) \bigwedge N_2(Sz^{(n)} - z, t) > 1 - \lambda.$$

By the arbitrariness of $\lambda, y = z$. Define a mapping $\varphi : \widetilde{X} \to Y$ as follows: $\varphi(\tilde{\xi}) = y$. In the next we prove the mapping φ is isometric from $\widetilde{X}$ onto Y.

Since S and T is isometric, then for any $\lambda \in (0,1]$,

$$\begin{aligned}
&\bigwedge\{ t > 0 \,:\, N_2(Sx^{(n)}, t) > 1 - \lambda\} = \bigwedge\{ t > 0 \,:\, N(x^{(n)}, t) > 1 - \lambda\} \\
&\quad = \bigwedge\{ t > 0 \,:\, N_1(Tx^{(n)}, t) > 1 - \lambda\}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\bigwedge\{ t_1 + t_2 + t_3 > 0 \,:\, N_2(\varphi(\tilde{\xi}), t_1 + t_2 + t_3) > 1 - \lambda\} \\
&\quad \leq \bigwedge\{ t_1 > 0 \,:\, N_2(y - Sx^{(n)}, t_1) > 1 - \lambda\} \\
&\qquad + \bigwedge\{ t_2 + t_3 > 0 \,:\, N_2(Sx^{(n)}, t_2 + t_3) > 1 - \lambda\} \\
&\quad = \bigwedge\{ t_1 > 0 \,:\, N_2(y - Sx^{(n)}, t_1) > 1 - \lambda\} \\
&\qquad + \bigwedge\{ t_2 + t_3 > 0 \,:\, N_1(Tx^{(n)}, t_2 + t_3) > 1 - \lambda\} \\
&\quad \leq \bigwedge\{ t_1 > 0 \,:\, N_2(y - Sx^{(n)}, t_1) > 1 - \lambda\} \\
&\qquad + \bigwedge\{ t_2 > 0 \,:\, N_1(Tx^{(n)} - \tilde{\xi}, t_2) > 1 - \lambda\} \\
&\qquad + \bigwedge\{ t_3 > 0 \,:\, N_1(\tilde{\xi}, t_3) > 1 - \lambda\}.
\end{aligned}$$

Put $n \to \infty$, we have $\bigwedge\{ t > 0 \,:\, N_2(\varphi(\tilde{\xi}), t) > 1 - \lambda\} \leq \bigwedge\{ t > 0 \,:\, N_1(\tilde{\xi}, t) > 1 - \lambda\}$. Similarly, we may prove $\bigwedge\{ t > 0 \,:\, N_2(\varphi(\tilde{\xi}), t) > 1 - \lambda\} \geq \bigwedge\{ t > 0 \,:\, N_1(\tilde{\xi}, t) > 1 - \lambda\}$. So the mapping φ is isometric.

At last, we prove the mapping φ is surjective. For any $z \in Y$, there exists $\{z^{(n)}\} \in SX$ such that for each $\lambda \in (0,1], \varepsilon > 0, p \in \mathbb{N}, N_2(z^{(n)} - z, t) > 1 - \lambda$ holds for all $n > p$. Then we have sequence $\{x^{(n)}\}$ satisfies $S(x^{(n)}) = z^{(n)}, n = 1, 2, \ldots$. From the above proof, there is unique

$\tilde{\eta} \in \widetilde{X}$ such that for each $\lambda \in (0,1], \varepsilon > 0, q \in \mathbb{N}, N_1(\widetilde{x^{(n)}} - \tilde{\eta}, t) > 1 - \lambda$ holds for all $n > q$. Since φ is isometric, then

$$\bigwedge\{t > 0 : N_2(\varphi(Tx^{(n)}), t) > 1 - \lambda\} = \bigwedge\{t > 0 : N_1(Tx^{(n)}, t) > 1 - \lambda\}$$
$$= \bigwedge\{t > 0 : N(x^{(n)}, t) > 1 - \lambda\} = \bigwedge\{t > 0 : N_2(Sx^{(n)}, t) > 1 - \lambda\}.$$

Furthermore,

$$\begin{aligned}\bigwedge\{t > 0 : N_2(Sx^{(n)} - \varphi(\tilde{\eta}), t) > 1 - \lambda\}\\ = \bigwedge\{t > 0 : N_2(\varphi(Tx^{(n)}) - \varphi(\tilde{\eta}), t) > 1 - \lambda\}\\ = \bigwedge\{t > 0 : N_2(\varphi(Tx^{(n)} - \tilde{\eta}), t) > 1 - \lambda\}\\ = \bigwedge\{t > 0 : N_1(Tx^{(n)} - \tilde{\eta}, t) > 1 - \lambda\}\end{aligned}$$

Thus $N_2(z - \varphi(\tilde{\eta}), 2t) \geq N_2(z - z^{(n)}, t) \bigwedge N_2(Sx^{(n)} - \varphi(\tilde{\eta}), t) > 1 - \lambda$.

So $z = \varphi(\tilde{\eta})$. Therefore we complete the whole proof. ■

Acknowledgments

I am grateful to the referees for carefully reading the paper and making some critical remarks which allowed me to improve the exposition of the paper.

Disclosure Statement

No potential conflict of interest was reported by the author(s).

Funding

This project is funded by the Natural Science Foundation of China (Nos. 11571006, 12071225) and the Priority Academic Program Development (PAPD) of Jiangsu Higher Education Institutions.

Notes on contributor

Dr Conghua Yan (1966–) is a professor in school of Math. and Computer Sciences, Nanjing Normal University. He received a B.A. from Nanjing Normal University of Jiangsu Province and PhD also from Nanjing Normal University. He is a member of Fuzzy Mathematics and Systems Association of the Systems Engineering Society of China. His interest includes fuzzy analysis, fuzzy topologies. Dr C. H. Yan published numerous articles and papers concerning fuzzy analysis, fuzzy topologies in the Journal of "Fuzzy Sets and Systems", "Information Sciences", "Mathematicae Japonicae" and more.

References

[1] Katsaras AK. Fuzzy topological vector spaces II. Fuzzy Sets Syst. 1984;12:143–154.
[2] Felbin C. Finite dimensional fuzzy normed linear space. Fuzzy Sets Syst. 1992;48:239–248.
[3] Kaleva O, Seikkala S. On fuzzy metric spaces. Fuzzy Sets Syst. 1984;12:215–229.
[4] Kramosil I, Michálek J. Fuzzy metric and statistical metric spaces. Kybernetica. 1975;11:326–334.
[5] Cheng SC, Menderson JN. Fuzzy linear operators and fuzzy normed linear spaces. Bull Cal Math Soc. 1994;86:429–436.

[6] Bag T, Samanta SK. Finite dimensional fuzzy normed linear spaces. J Fuzzy Math. 2003;11: 687–705.
[7] Bag T, Samanta SK. A comparative study of fuzzy norms on a linear space. Fuzzy Sets Syst. 2008;159:670–684.
[8] Bag T, Samanta SK. Fuzzy bounded linear operator. Fuzzy Sets Syst. 2005;151:513–547.
[9] Bag T, Samanta SK. Some fixed pointed theorems in fuzzy normed linear spaces. Inform Sci. 2007;177:3271–3289.
[10] Bag T, Samanta SK. Operator's fuzzy norm and some properties. Fuzzy Inf Eng. 2015;7:151–164.
[11] Salehi M. On fuzzy topology and fuzzy norm. Ann Fuzzy Math Inform. 2015;10(4):639–647.
[12] Sadeqi M. Fuzzy topology generated by fuzzy norm. Iran J Fuzzy Syst. 2016;13(4):113–123.
[13] Bag T. Finite dimensional intuitionistic fuzzy *n*-normed linear spaces. Ann Fuzzy Math Inform. 2017;13(2):175–188.
[14] Majumder A, Bag T. Completeness and compactness in fuzzy cone metric spaces. Ann Fuzzy Math Inform. 2019;18(3):297–308.
[15] Tamang P, Bag T. Some results on finite dimensional fuzzy cone normed linear spaces. Ann Fuzzy Math Inform. 2017;13(1):123–134.
[16] Kaleva O. The completion of fuzzy metric spaces. J Math Anal Appl. 1985;109:194–198.
[17] Kaleva O. A comment on the completion of fuzzy metric spaces. Fuzzy Sets Syst. 2008;159: 2190–2192.
[18] Fang JX. A note on the completions of fuzzy metric spaces and fuzzy normed spaces. Fuzzy Sets Syst. 2002;131:399–407.
[19] Felbin C. The completion of fuzzy normed linear space. J Math Anal Appl. 1993;174:428–440.
[20] Lee BS, Lee SJ, Park KM. The completions of fuzzy metric spaces and fuzzy normed spaces. Fuzzy Sets Syst. 1999;109:469–473.
[21] Gregori V, Romaguera S. On completion of fuzzy metric spaces. Fuzzy Sets Syst. 2002;130: 399–404.
[22] Rodabaugh SE. Powerset operator foundations for Poslat fuzzy set theories and topologies. In: U. Höhle, and S.E. Rodabaugh, editors, Mathematics of fuzzy sets: logic, topology, and measure theory. Dordrecht: Kluwer Academic Publishers; 1999. p. 91–116. (The Handbooks of Fuzzy Sets Series; 3).
[23] Das NF, Das P. Fuzzy topology generated by fuzzy norm. Fuzzy Sets Syst. 1999;107:349–354.
[24] Fang JX. On *I*-topology generated by fuzzy norm. Fuzzy Sets Syst. 2006;157:2739–2750.