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To cite this article: Mojtaba Borza & Azmin Sham Rambely (2021) A New Method to Solve Multi-Objective Linear Fractional Problems, Fuzzy Information and Engineering, 13:3, 323-334, DOI: [10.1080/16168658.2021.1938868](https://doi.org/10.1080/16168658.2021.1938868)

To link to this article: <https://doi.org/10.1080/16168658.2021.1938868>



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Published online: 05 Jul 2021.



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A New Method to Solve Multi-Objective Linear Fractional Problems

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ABSTRACT

Background: In the literature, there exists several approaches to address the multi-objective linear fractional programming problem (MOLFPP). However, there is a drawback to these methods.

Aim: This paper presents an efficient method treating the MOLFPP.

Methodology: To construct our approach, the membership functions of the objectives, suitable non-linear variable transformations, and max-min technique are used.

Results: In our proposed method, the MOLFPP is finally changed into a linear programming problem (LPP). It is proven that the optimal solution of the LPP is an efficient solution for the MOLFPP.

Conclusion: Numerical examples are solved, and the results demonstrate that our method with less computational expenses and cost reach the efficient solutions.

ARTICLE HISTORY

Received 23 December 2020

Revised 9 May 2021

Accepted 1 June 2021

KEYWORDS

Efficient solution;
membership function;
max-min technique; linear
programming; fractional
programming

1. Introduction

The concept of a multi-objective programming problem (MOPP) arises when a decision maker is going to consider more than one objective over a common set of restrictions. A number of real-world problems in transportation, finance, engineering, commercials, house planning, energy systems, etc. have appropriately been modelled as MOPPs [1]. For this class of problems, the concept of *efficient* solution is considered instead of exact optimal solution. A solution is *efficient* if moving to another solution does not improve all the objectives. In MOPP, if the objectives are linear fractional functions and the constraints are affine, then this model represents the multi-objective linear fractional programming problem (MOLFPP). In Ref. [2], applications of a linear fractional programming problem (LFPP) in economy, business, engineering, management, etc. were demonstrated. Radhakrishnan and Anukokila [3] addressed a solid transportation problem with interval cost by the use of a fractional goal programming method. Wang et al. [4] developed a framework of bi-level MOLFPP to optimise a water consumption structure. Ahmad et al. [5] investigated the fractional-order tumour-immune-vitamin model through fixed point results. Das et al. [6] presented an application of the LFPP with fuzzy nature in industry sector.

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Because of the importance of the LFPP and also MOLFPF, many studies have been accomplished to come out with efficient methods and techniques for these optimisation problems. Chakraborty and Gupta [7] developed a method to address MOLFPF. In their method, the multi-objective problem is transformed into a multi-objective linear programming problem (MOLPP). Subsequently, the membership functions are specified after identifying the fuzzy aspiration levels of the linear objectives. Finally, the MOLPP is changed into a linear programming problem (LPP) using a max–min technique. Motivated by Chakraborty and Gupta’s methodology, Veeramani and Sumathi [8] and De and Deb [9] introduced approaches to deal with LFPP with fuzzy coefficients and MOLFPF, respectively. Following the methodology of Dinkelbach [10], Güzel [11] and Nayak and Ojha [12] developed approaches to MOLFPF. In fact, Nayak and Ojha attempted to improve the results of Güzel’s approach by employing the ε -constrain technique. However, applying the ε -constrain technique encompasses some difficulties in practice when the decision maker is trying to specify the value of ε for each constraint. Pal et al. [13] transformed the MOLFPF into a LPP using a fuzzy goal programming approach in addition to suitable variable transformations. Toksari [14] introduced an approach to tackle the MOLFPF where the membership functions of the objectives are defined and then linearised using the first-order Taylor series about the individual optimal solutions. For some examples, Borza et al. [15] reported that the results of using the first-order Taylor series proposed by Toksari are to some extent more accurate than the results of the fuzzy goal programming used by Pal et al. Nayak and Ojha [16] introduced a method dealing with the MOLFPF with fuzzy coefficients where the fuzzy problem is altered into interval valued LFPP using the concept of α -cuts. In their method, the fuzzy problem is reduced into the MOLFPF. Afterwards, they reach a MOLPP employing the first-order Taylor series. Finally, weighted sum technique is utilised to transform the MOLPP into a LPP. Borza and Rambely [17] designed a non-iterative method to obtain the global optimal solution of the sum of the linear fractional programming problem (S-LFPP) by the use of variable transformation. Liu et al. [18] constructed an iterative algorithm for the large-scale S-LFPP using a branch and bound technique.

In the literature, many researchers have tried to transform the MOLFPF into a LPP using different methodology and techniques such as the first-order Taylor series method, Dinkelbach’s methodology, and Chakraborty and Gupta’s approach. However, there are drawbacks regarding these methods and methodologies. Using the first-order Taylor expansion reduces the accuracy of the method automatically. The method of Chakraborty and Gupta was designed in such a way that it has not been possible to prove their methodology results in efficient solutions. Following the methodology of Dinkelbach, a fractional programme is changed into a parametric non-fractional programming problem. However, in the existing methods, a non-parametric model of Dinkelbach’s methodology has been used, which reduces the accuracy. In this paper, we aim to present a new efficient and straightforward method with less computational expenses and appropriate accuracy to transform the MOLFPF into a LPP. In addition, we use the membership functions of the objectives to construct our approach in order to a wide range of problems be covered. Therefore, the membership functions of the objectives are specified after identifying the maxima and minima of the objectives and then a new MOLFPF is designed. This new problem is changed into a MOLPP by the use of non-linear variable transformations. Finally, the max–min approach is used to tackle the MOLPP. It is proven that the solution resulted is efficient for the MOLFPF. Numerical examples are given to illustrate the method in addition to make comparison to some existing methods.

This article is organised in four sections. Following the introduction, in Section 2, some preliminaries are given. In Section 3, the main result and outcome of this survey are released. In Section 3, numerical examples are solved to illustrate the method and make comparison. Finally, Section 4 concludes the study.

2. Preliminaries

2.1. Linear Fractional Programming

Consider the general form of the LFPP as follows:

$$\begin{aligned} & \text{Maximize } \frac{C^T X + \alpha}{D^T X + \beta} \\ & \text{s.t. } AX \leq b, X \geq 0, D^T X + \beta > 0. \end{aligned} \quad (1)$$

According to the method introduced by Charnes and Cooper [19], Equation (1) is changed into the following linear problem by the use of variable transformations $t = 1/(D^T X + \beta)$, $Y = tX$.

$$\begin{aligned} & \text{Maximize } C^T Y + \alpha t \\ & \text{s.t. } AY - bt \leq 0, D^T Y + \beta t = 1, Y, t \geq 0. \end{aligned} \quad (2)$$

Theorem 1: (Ref. [19]). Let (Y^*, t^*) be the optimal solution of (2), then $X^* = (Y^*/t^*)$ is optimum for (1).

2.2. Multi-Objective Programming

Let us consider the general form of the MOPP as follows:

$$\text{Maximize } \{F_1(X), \dots, F_k(X)\} \quad \text{s.t. } X \in S. \quad (3)$$

Definition 1: (Ref. [20]). For (3), a solution $X^* \in S$ is called efficient if and only if $\nexists X \in S$ such that $F_j(X^*) \leq F_j(X)$, $j = 1, \dots, k$, and $\exists i \in \{1, \dots, k\}$ such that $F_j(X^*) < F_j(X)$.

3. Main Results

In this section, an approach is introduced in order to change the MOLFP into a LPP such that the optimal solution of the LPP becomes an efficient solution for the MOLFP.

Consider the general type of the MOLFP as follows:

$$\begin{aligned} & \text{Maximize } \left\{ Z_i(X) = \frac{N_i^T X + m_i}{P_i^T X + q_i} \quad \text{for } i = 1, \dots, k \right\} \\ & \text{s.t. } \mathbf{S} = \{AX \leq b, X \geq 0\}, \end{aligned} \quad (4)$$

where \mathbf{S} is a regular set (non-empty and bounded set). Furthermore, for $X = (X_1, \dots, X_n) \in \mathbf{S}$, it is assumed $P_i^T X + q_i > 0$ for $i = 1, \dots, k$.

Our aim is to design a method so as to come out with efficient solution for (4). To do this, we need $N_i^T X + m_i \geq 0, \forall X \in S, i = 1, \dots, k$. But, these conditions are restrictive. To

overcome this difficulty, an equivalent problem to (4) is constructed in which numerators are non-negative. Therefore, the membership functions of the objectives are specified and then are utilised instead of the objectives.

Let $\max_{X \in S} Z_i = z_i^{\max}$ and $\min_{X \in S} Z_i = z_i^{\min}, i = 1, \dots, k$. Thus, $\mu_i(X) = (C_i^T X + d_i) / (P_i^T X + q_i)$ is the membership function for objective Z_i , where $X \in S, C_i = (1 / (z_i^{\max} - z_i^{\min})) N_i - z_i^{\min} P_i$, and $d_i = (m_i / z_i^{\max} - z_i^{\min}) - z_i^{\min} q_i$. Accordingly, $C_i^T X + d_i \geq 0$ because $\mu_i(X) \in [0, 1]$, and $P_i^T X + q_i > 0$ for $i = 1, \dots, k$.

The equivalent of (4) in terms of the membership functions is

$$\begin{aligned} & \text{Maximize } \left\{ \frac{C_i^T X + d_i}{P_i^T X + q_i} \text{ for } i = 1, \dots, k \right\} \\ & \text{s.t. } X \in S = \{AX \leq b, X \geq 0\}. \end{aligned} \tag{5}$$

Let us define new variables λ and Y as the functions of variable X as follows:

$$\lambda = \min\{\lambda_i = (1 / P_i^T X + q_i), i = 1, \dots, k\} \text{ and } \lambda X = Y. \tag{6}$$

Thus, (5) is transformed into

$$\begin{aligned} & \text{Maximize } \{C_i^T Y + \lambda d_i \text{ for } i = 1, \dots, k\} \\ & \text{s.t. } F = \{AY - \lambda b \leq 0, Y, \lambda \geq 0, P_i^T Y + \lambda q_i \leq 1 \text{ for } i = 1, \dots, k\}. \end{aligned} \tag{7}$$

Lemma 1: In (7), variable $\lambda \neq 0, \forall (Y, \lambda) \in F$.

Proof: If $(\hat{Y}, 0) \in F$, then $A\hat{Y} \leq 0$. Now, if $\hat{X} \in S$, then $A(\hat{X} + \beta\hat{Y}) = A\hat{X} + \beta(A\hat{Y}) \leq A\hat{X} \leq b$ for all $\beta \geq 0$; this means $\hat{X} + \beta\hat{Y} \in S, \forall \beta \geq 0$. This results that the feasible region S is an unbounded set, which is a contradiction to the regularity of S . ■

Lemma 2: If $(\bar{Y}, \bar{\lambda}) \in F$, then $(\bar{Y} / \bar{\lambda}) \in S$.

Proof: Since $(\bar{Y}, \bar{\lambda}) \in F$, then $\bar{Y} \geq 0, \bar{\lambda} > 0$, and $A\bar{Y} - \bar{\lambda}b \leq 0$. Therefore, $(1 / \bar{\lambda})(A\bar{Y} - \bar{\lambda}b) = A(\bar{Y} / \bar{\lambda}) - b \leq 0$. ■

By setting $\beta = \min_{X \in F} \{C_i^T Y + \lambda d_i, i = 1, \dots, k\}$, (7) is altered into

$$\begin{aligned} & \text{Maximize } \beta \\ & \text{s.t. } \phi = \{Y, \lambda, \beta \geq 0, AY - \lambda b \leq 0, \beta \leq C_i^T Y + \lambda d_i, P_i^T Y + \lambda q_i \leq 1 \text{ for } i = 1, \dots, k\}. \end{aligned} \tag{8}$$

Theorem 2: The optimal solution of (8) is unique.

Proof: Let $(Y^*, \lambda^*, \beta^*)$ be the optimal solution and is not unique; this means constraint $\beta \geq 0$ is active at the optimum, i.e. $\beta^* = 0$. In other words, if $(Y, \lambda, \beta) \in \phi$, then $\beta = 0$. Therefore, $\exists j \in \{1, \dots, k\}$ such that $C_j^T Y + \lambda d_j = 0$ for all $(Y, \lambda, 0) \in \phi$. Since $\lambda > 0$, then $C_j^T X + d_j = 0$ for all $X \in S$; this means $\mu_j(X) = 0$ for all $X \in S$. As the consequence, (5) is reduced into $(k - 1)$ objective LFPP. This is a contradiction. ■

Theorem 3: If $(Y^*, \lambda^*, \beta^*)$ is optimal for (8), then $X^* = (Y^* / \lambda^*)$ is an efficient solution for (5).

Proof: Let $X^* = (Y^*/\lambda^*)$ not be an efficient solution for (5). Therefore, $\exists \bar{X} \in S$ such that

$$\frac{C_i^T(X^*) + d_i}{P_i^T(X^*) + q_i} \leq \frac{C_i^T\bar{X} + d_i}{P_i^T\bar{X} + q_i} \quad \text{for } i = 1, \dots, k, \text{ and}$$

$$\exists j \in \{1, \dots, k\} \quad \text{such that} \quad \frac{C_j^T(X^*) + d_j}{P_j^T(X^*) + q_j} < \frac{C_j^T\bar{X} + d_j}{P_j^T\bar{X} + q_j}. \quad (9)$$

Consider

$$(Y^*, \lambda^*, \beta^*) \in \phi \Rightarrow \lambda^* \leq \lambda_i^* = \frac{1}{P_i^T(X^*) + q_i} \quad \text{and} \quad 0 \leq \beta^* \leq C_i^T Y^* + \lambda^* q_i, \quad i = 1, \dots, k. \quad (10)$$

Let us define $\bar{\theta} = \max\{\bar{\lambda}_i = (1/(P_i^T\bar{X} + q_i)), i = 1, \dots, k\}$ and $\bar{\lambda} = \bar{\theta} - \epsilon$, where

$$\bar{\theta} - \bar{\lambda}_i \leq \epsilon \leq \bar{\theta} - \lambda^* \left(\frac{C_i^T X^* + d_i}{C_i^T \bar{X} + d_i} \right), \quad i = 1, \dots, k. \quad (11)$$

We need to show that (11) is well defined. In other words, there must exist ϵ satisfying (11). To do this, two below conditions must hold true.

- (I) $C_i^T \bar{X} + d_i \neq 0, i = 1, \dots, k.$
- (II) $\bar{\theta} - \bar{\lambda}_i \leq \bar{\theta} - \lambda^* \left(\frac{C_i^T X^* + d_i}{C_i^T \bar{X} + d_i} \right), i = 1, \dots, k.$

Since $\mu_i(X) = ((C_i^T X + d_i)/(P_i^T X + q_i)) \in [0, 1], P_i^T X + q_i > 0$, then $C_i^T X + d_i \geq 0, \forall X \in S$. Now, let $\exists j \in \{1, \dots, k\}$ such that $C_j^T \bar{X} + d_j = 0$. Due to (9), it is possible that $((C_j^T(X^*) + d_j)/(P_j^T(X^*) + q_j)) < ((C_j^T \bar{X} + d_j)/(P_j^T \bar{X} + q_j))$; this means $\mu_j(X^*) < 0$. This contradicts the non-negativity of membership functions. Therefore, (I) is verified.

It follows directly from (9) and (10) that

$$\lambda^*(C_i^T(X^*) + d_i) \leq \lambda_i^*(C_i^T X^* + d_i) = \frac{C_i^T \bar{X} + d_i}{P_i^T \bar{X} + q_i} \leq \frac{C_i^T \bar{X} + d_i}{P_i^T \bar{X} + q_i} = \bar{\lambda}_i(C_i^T \bar{X} + d_i), \quad (12)$$

$$\lambda^* \left(\frac{C_i^T(X^*) + d_i}{C_i^T \bar{X} + d_i} \right) \leq \bar{\lambda}_i. \quad (13)$$

(13) $\Rightarrow \bar{\theta} - \bar{\lambda}_i \leq \bar{\theta} - \lambda^*(C_i^T(X^*) + d_i/C_i^T \bar{X} + d_i), i = 1, \dots, k.$ Therefore, (II) is demonstrated.

It is time to show:

- (III) $\bar{\lambda}(P_i^T \bar{X} + q_i) \leq 1, i = 1, \dots, k.$
- (IV) $\lambda^*(C_i^T(X^*) + d_i) \leq \bar{\lambda}(C_i^T \bar{X} + d_i), i = 1, \dots, k.$

To do this:

(11) implies $\bar{\theta} - \epsilon \leq \bar{\lambda}_i$. Furthermore, according to the definitions $\bar{\theta} = \max\{\bar{\lambda}_i, \text{ for } i = 1, \dots, k\}$, $\bar{\lambda} = \bar{\theta} - \epsilon$, and $\bar{\lambda}_i = (1/P_i^T \bar{X} + q_i), i = 1, \dots, k$, it is concluded that $\bar{\lambda}(P_i^T \bar{X} + q_i) = (\bar{\theta} - \epsilon)(P_i^T \bar{X} + q_i) \leq \bar{\lambda}_i(P_i^T \bar{X} + q_i) = 1, i = 1, \dots, k$. Thus, (III) is demonstrated.

Since $\leq \bar{\theta} - \lambda^*(C_i^T X^* + d_i/C_i^T \bar{X} + d_i)$, then $\lambda^*(C_i^T X^* + d_i/C_i^T \bar{X} + d_i) \leq \bar{\theta} - \epsilon \Rightarrow \lambda^*(C_i^T X^* + d_i) \leq (\bar{\theta} - \epsilon)(C_i^T \bar{X} + d_i) = \bar{\lambda}(C_i^T \bar{X} + d_i), i = 1, \dots, k$. Thus, (IV) is verified.

Now, let us define $\bar{Y} = \bar{\lambda}\bar{X}$. To show $(\bar{Y}, \bar{\lambda}) \in F$, the followings must be true:

(a) $\bar{\lambda} \geq 0$.

Due to (11), let us set $\max \epsilon = \max\{\bar{\theta} - \lambda^*(C_i^T X^* + d_i/C_i^T \bar{X} + d_i), i = 1, \dots, k\} = \bar{\theta} - \lambda^*(C_i^T X^* + d_i/C_i^T \bar{X} + d_i)$. Thus, $\bar{\lambda} \geq \bar{\theta} - \max \epsilon = \bar{\theta} - (\bar{\theta} - \lambda^*(C_i^T X^* + d_i/C_i^T \bar{X} + d_i)) = \lambda^*(C_i^T X^* + d_i/C_i^T \bar{X} + d_i) \geq 0$.

(b) $\bar{Y} \geq 0$.

Since $\bar{X} \in S$, then $\bar{X} \geq 0$. Consequently, $\bar{Y} = \bar{\lambda}\bar{X} \geq 0$.

(c) $(P_i^T \bar{Y} + \bar{\lambda}q_i) \leq 1$ for $i = 1, \dots, k$.

Considering $\bar{Y} = \bar{\lambda}\bar{X}$ and (III) proves c.

(d) $A\bar{Y} - \bar{\lambda}b \leq 0$.

$\bar{X} \in S \Rightarrow A\bar{X} - b \leq 0$. Therefore, $A\bar{Y} - \bar{\lambda}b = \bar{\lambda}(A\bar{X} - b) \leq 0$.

In what follows, we create $\bar{\beta}$ such that $\bar{\beta} \geq \beta^*$ and $(\bar{Y}, \bar{\lambda}, \bar{\beta}) \in \phi$.

(IV) \Rightarrow

$$C_i^T Y^* + \lambda^* d_i = \lambda^*(C_i^T (X^*) + d_i) \leq \bar{\lambda}(C_i^T \bar{X} + d_i) = C_i^T \bar{Y} + \bar{\lambda}d_i, i = 1, \dots, k. \quad (14)$$

(10) and (14) \Rightarrow

$$0 \leq \beta^* \leq C_i^T \bar{Y} + \bar{\lambda}d_i, i = 1, \dots, k. \quad (15)$$

Let us set

$$\gamma = \min\{C_i^T \bar{Y} + \bar{\lambda}d_i - \beta^*, i = 1, \dots, k\} \text{ and } \bar{\beta} = \beta^* + \gamma. \quad (16)$$

(15) and (16) \Rightarrow

$$\gamma \geq 0, \text{ and subsequently } \beta^* \leq \bar{\beta}. \quad (17)$$

It follows directly from (16) that $0 \leq \gamma + \beta^* \leq C_i^T \bar{Y} + \bar{\lambda}d_i, i = 1, \dots, k$. Thus,

$$0 \leq \bar{\beta} \leq C_i^T \bar{Y} + \bar{\lambda}d_i, i = 1, \dots, k. \quad (18)$$

Equation (18) in addition to $(\bar{Y}, \bar{\lambda}) \in F$ results $(\bar{Y}, \bar{\lambda}, \bar{\beta}) \in \phi$.

In brief, we found $(\bar{Y}, \bar{\lambda}, \bar{\beta}) \in \phi$ such that $\beta^* \leq \bar{\beta}$. This contradicts the unique optimality of $(Y^*, \lambda^*, \beta^*)$ for (8). The proof is then complete. \blacksquare

4. Numerical Example

In this section, four examples are considered taken from different references in order to illustrate and evaluate this method. The third and fourth examples are mathematical models of the real-world organisations problems.

4.1. Example 1 (Ref. [14])

$$\text{Maximize } \left\{ Z_1(X) = \frac{12X_1 + 13X_2}{40X_1 + 55X_2 + 500}, Z_2(X) = \frac{12X_1 + 13X_2}{1.5X_3 + 1.6X_4} \right\}$$

$$\text{s.t. } S = \{2X_1 + X_2 \leq 250, 5X_1 + 4X_2 \leq 500, 45X_1 + 30X_2 \leq 1500,$$

$$0.1X_1 + 0.1X_2 - X_3 - X_4 \leq 0, 0.1X_1 - X_3 \leq 0, 0.05X_2 - X_4 \leq 0,$$

$$-X_1 + X_3 \leq 0, -X_2 + X_4 \leq 0, X_1, X_2, X_3, X_4 \geq 0\}. \quad (19)$$

First, the values of z_i^{\max} and z_i^{\min} for $i = 1, 2$ are individually determined by the use of Ref. [19] so as to define the membership functions: $z_1^{\max} = 0.2182$, $z_1^{\min} = 0$, $z_2^{\max} = 83.6735$, $z_2^{\min} = 8$. Thus, $\mu_{Z_1}(X) = (54.9954X_1 + 59.579X_2)/(40X_1 + 55X_2 + 500)$ and $\mu_{Z_2}(X) = (0.1586X_1 + 0.1718X_2 - 0.1586X_3 - 0.1691X_4)/(1.5X_3 + 1.6X_4)$.

Equation (8) is formulated for (19) as follows:

$$\text{Maximize } \beta$$

$$\text{s.t. } \{2Y_1 + Y_2 - 250\lambda \leq 0, 5Y_1 + 4Y_2 - 500\lambda \leq 0,$$

$$45Y_1 + 30Y_2 - 1500\lambda \leq 0, 0.1Y_1 + 0.1Y_2 - Y_3 - Y_4 \leq 0,$$

$$0.1Y_1 - Y_3 \leq 0, 0.05Y_2 - Y_4 \leq 0,$$

$$-Y_1 + Y_3 \leq 0, -Y_2 + Y_4 \leq 0,$$

$$40Y_1 + 55Y_2 + 500\lambda \leq 1, 1.5Y_3 + 1.6Y_4 \leq 1,$$

$$\beta \leq 54.9954Y_1 + 59.579Y_2, \beta \leq 0.1586Y_1 + 0.1718Y_2 - 0.1586Y_3 - 0.1691Y_4,$$

$$Y_1, Y_2, Y_3, Y_4, \lambda, \beta \geq 0\}. \quad (20)$$

Equation (20) is solved and the unique solution obtained is $(Y^*, \lambda^*, \beta^*) = (0.0008, 0.0146, 0.0008, 0.0007, 0.0003, 0.0029)$. Furthermore, the solution for (19) is $X^* = (Y^*/\lambda^*) = (2.5642, 46.1537, 2.5641, 2.3077)$.

At the solution X^* , $Z_1(X) = 0.2008$, $Z_2(X) = 83.6735$, $\mu_{Z_1}(X) = 0.9203$, and $\mu_{Z_2}(X) = 1$. The average of $\mu_{Z_1}(X)$ and $\mu_{Z_2}(X)$ is 0.9602.

4.1.1. Comparison

The solution resulted by Toksari is $\hat{X} = (0, 50, 0, 5)$.

At the solution \hat{X} , $Z_1(X) = 0.2$, $Z_2(X) = 81.25$, $\mu_{Z_1}(X) = 0.9166$, and $\mu_{Z_2}(X) = 0.9669$.

The average of $\mu_{Z_1}(X)$ and $\mu_{Z_2}(X)$ is 0.9417.

As we see, the solution of Toksari is dominated by our proposed solution, i.e.

$$Z_1(\hat{X}) < Z_1(X^*), Z_2(\hat{X}) < Z_2(X^*).$$

4.2. Example 2 (Ref. [7])

$$\text{Maximize } \left\{ Z_1(X) = \frac{-3X_1 + 2X_2}{X_1 + X_2 + 3}, Z_2(X) = \frac{7X_1 + X_2}{5X_1 + 2X_2 + 1} \right\}$$

$$\text{s.t. } S = \{-X_1 + X_2 \leq -1, 2X_1 + 3X_2 \leq 15, -X_1 \leq -3, X_1, X_2 \geq 0\}. \quad (21)$$

For (21), $Z_1^{\max} = -0.6087$, $Z_1^{\min} = -2.1429$, and $Z_2^{\max} = 1.3636$, $Z_2^{\min} = 1.148$. Accordingly, $\mu_{Z_1}(X) = (-0.5587X_1 + 2.7004X_2 + 4.1903/X_1 + X_2 + 3)$ and $\mu_{Z_2}(X) = (5.8473X_1 - 6.041X_2 - 5.3482/5X_1 + 2X_2 + 1)$.

Equation (8) is formulated for (21) as follows:

$$\begin{aligned}
 &\text{Maximize } \beta \\
 &\text{s.t. } \phi = \{-Y_1 + Y_2 + \lambda \leq 0, 2Y_1 + 3Y_2 - 15\lambda \leq 0, -Y_1 + 3\lambda \leq 0, \\
 &Y_1 + Y_2 + 3\lambda \leq 1, 5Y_1 + 2Y_2 + \lambda \leq 1, \\
 &\beta \leq -0.5587Y_1 + 2.7004Y_2 + 4.1903\lambda, \beta \leq 5.8473Y_1 - 6.041Y_2 - 5.3482\lambda, \\
 &Y_1, Y_2, \lambda, \beta \geq 0\}. \tag{22}
 \end{aligned}$$

Equation (22) is solved and the unique optimal solution obtained is (0.1647, 0.0608, 0.0549, 0.3022). Thus, the solution proposed for problem (21) is

$$X^* = \frac{Y^*}{\lambda^*} = (3, 1.1073).$$

At the solution X^* ,

$$Z_1(X) = -0.9547, Z_2(X) = 1.2137, \mu_{Z_1}(X) = 0.7746, \text{ and } \mu_{Z_2}(X) = 0.3022.$$

The average of $\mu_{Z_1}(X)$ and $\mu_{Z_2}(X)$ is 0.5384.

4.2.1. Comparison

The solution of Chakraborty and Gupta is $\hat{X} = (3, 2)$.

At the solution \hat{X} ,

$$Z_1(X) = -0.625, Z_2(X) = 1.15, \mu_{Z_1}(X) = 0.9894, \text{ and } \mu_{Z_2}(X) = 0.0056.$$

The average of $\mu_{Z_1}(X)$ and $\mu_{Z_2}(X)$ is 0.4975.

As we observe, the solution of Chakraborty and Gupta does not dominate our proposed solution and vice versa. However, the average of membership functions shows that our proposed method has a better efficiency and function.

4.3. Example 3 (Ref. [21])

Maximize $\{Z_1(X), Z_2(X)\}$

$$\begin{aligned}
 &= \left\{ \left(\frac{59890X_1 + 23390X_2 + 30750X_3 + 59750X_4 + 40700X_5 + 59435X_6}{35345X_1 + 13420X_2 + 18455X_3 + 39455X_4 + 23840X_5 + 24070X_6 + 500000} \right), \right. \\
 &\quad \left. \times \left(\frac{59890X_1 + 23390X_2 + 30750X_3 + 59750X_4 + 40700X_5 + 59435X_6}{96X_1 + 120X_2 + 144X_3 + 144X_4 + 84X_5 + 120X_6 + 480} \right) \right\}
 \end{aligned}$$

$$\text{s.t. } S = \{0.3X_1 + 0.4X_2 + 0.4X_3 + 0.98X_4 + 0.97X_5 + 0.98X_6 \leq 600,$$

$$2280000X_1 + 9200X_2 + 16000X_3 + 22500X_4 + 20000X_5 + 20000X_6 \leq 20000000,$$

$$650X_1 + 630X_2 + 320X_3 + 660X_4 + 360X_5 + 640X_6 \leq 500000,$$

$$20X_1 + 22X_2 + 20X_3 + 18X_4 + 20X_5 + 17X_6 \leq 15000,$$

$$\begin{aligned}
11400X_1 + 3220X_2 + 1800X_3 + 12750X_4 + 3250X_5 + 3000X_6 &\leq 6000000, \\
148X_1 + 238X_4 + 135X_6 &\leq 50000, \\
180X_1 + 220X_2 + 200X_3 + 150X_4 + 100X_5 + 160X_6 &\leq 120000, \\
60X_1 + 40X_2 + 35X_3 + 50X_4 + 30X_5 + 45X_6 &\leq 30000, \\
30X_1 + 32X_2 + 28X_3 + 35X_4 + 26X_5 + 20X_6 &\leq 200000, \\
15X_1 + 18X_2 + 16X_3 + 14X_4 + 17X_5 + 18X_6 &\leq 10000, \\
42X_1 + 38X_2 + 36X_3 + 40X_4 + 37X_5 + 35X_6 &\leq 25000, \\
X_i \geq 0, \quad i = 1, \dots, 6.
\end{aligned} \tag{23}$$

For (23), $Z_1^{\min} = 0, Z_1^{\max} = 2.3381, Z_2^{\min} = 0, Z_2^{\max} = 491.5151,$

$$\mu_{Z_1}(X) = \frac{25615X_1 + 10004X_2 + 13152X_3 + 25555X_4 + 17407X_5 + 25420X_6}{35345X_1 + 13420X_2 + 18455X_3 + 39455X_4 + 23840X_5 + 24070X_6 + 500000},$$

$$\mu_{Z_2}(X) = \frac{121.8477X_1 + 47.5876X_2 + 62.5617X_3 + 121.5629X_4 + 82.8052X_5 + 120.992X_6}{96X_1 + 120X_2 + 144X_3 + 144X_4 + 84X_5 + 120X_6 + 480}.$$

Equation (8) is formed for the above problem as follows:

Maximize β

$$\begin{aligned}
\text{s.t. } \phi &= \{0.3Y_1 + 0.4Y_2 + 0.4Y_3 + 0.98Y_4 + 0.97Y_5 + 0.98Y_6 - 600\lambda \leq 0, \\
2280000Y_1 + 9200Y_2 + 16000Y_3 + 22500Y_4 + 20000Y_5 + 20000Y_6 - 20000000\lambda &\leq 0, \\
650Y_1 + 630Y_2 + 320Y_3 + 660Y_4 + 360Y_5 + 640Y_6 - 500000\lambda &\leq 0, \\
20Y_1 + 22Y_2 + 20Y_3 + 18Y_4 + 20Y_5 + 17Y_6 - 15000\lambda &\leq 0, \\
11400Y_1 + 3220Y_2 + 1800Y_3 + 12750Y_4 + 3250Y_5 + 3000Y_6 - 6000000\lambda &\leq 0, \\
148Y_1 + 238Y_4 + 135Y_6 - 50000\lambda &\leq 0, \\
180Y_1 + 220Y_2 + 200Y_3 + 150Y_4 + 100Y_5 + 160Y_6 - 120000\lambda &\leq 0, \\
60Y_1 + 40Y_2 + 35Y_3 + 50Y_4 + 30Y_5 + 45Y_6 - 30000\lambda &\leq 0, \\
30Y_1 + 32Y_2 + 28Y_3 + 35Y_4 + 26Y_5 + 20Y_6 - 200000\lambda &\leq 0, \\
15Y_1 + 18Y_2 + 16Y_3 + 14Y_4 + 17Y_5 + 18Y_6 - 10000\lambda &\leq 0, \\
42Y_1 + 38Y_2 + 36Y_3 + 40Y_4 + 37Y_5 + 35Y_6 - 25000\lambda &\leq 0, \\
35345Y_1 + 13420Y_2 + 18455Y_3 + 39455Y_4 + 23840Y_5 + 24070Y_6 + 500000\lambda &\leq 1, \\
96Y_1 + 120Y_2 + 144Y_3 + 144Y_4 + 84Y_5 + 120Y_6 + 480\lambda &\leq 1, \\
\beta &\leq 25615Y_1 + 10004Y_2 + 13152Y_3 + 25555Y_4 + 17407Y_5 + 25420Y_6, \\
\beta &\leq 121.8477Y_1 + 47.5876Y_2 + 62.5617Y_3 + 121.5629Y_4 + 82.8052Y_5 + 120.0492Y_6, \\
Y_i \geq 0, \quad i = 1, \dots, 6, \lambda, \beta \geq 0.
\end{aligned} \tag{24}$$

Equation (24) is solved and the solution $X^* = (Y^*/\lambda^*) = (0, 0, 0, 0, 0, 370)$ is obtained as an efficient solution for (23).

At the solution X^* ,

$$Z_1(X) = 2.3380, Z_2(X) = 489.9944, \mu_{Z_1} = 0.9999, \text{ and } \mu_{Z_2} = 0.9897.$$

The average of $\mu_{Z_1}(X)$ and $\mu_{Z_2}(X)$ is 0.9948.

4.3.1. Comparison

The solution proposed by Pramy and Islam is $\hat{X} = (0, 0, 0, 0, 196.078, 370.37)$.

At the solution \hat{X} ,

$$Z_1(X) = 2.1288, Z_2(X) = 488.531, \mu_{Z_1} = 0.9105, \text{ and } \mu_{Z_2} = 0.9887.$$

The average of $\mu_{Z_1}(X)$ and $\mu_{Z_2}(X)$ is 0.9496.

The results show that our solution X^* dominates the solution \hat{X} proposed by Pramy and Islam due to the fact that

$$Z_1(\hat{X}) < Z_1(X^*), Z_2(\hat{X}) < Z_2(X^*).$$

4.4. Example 4 (Ref. [22])

In this section, a real life production planning in Taiwan is considered. The original problem is modelled as a LFPP with fuzzy coefficients and fuzzy decision variables. In order to be able to solve the problem with the method provided, we change the fuzzy numbers into the intervals using the concept of α -cuts. Moreover, the decision variables are set to be non-fuzzy. Therefore, we transformed the problem into the MOLFPF by the use of interval operations as follows:

Maximize $\{Z_1(X), Z_2(X)\}$

$$= \left\{ \begin{array}{l} \frac{9.2X_1 + 21.4X_2 + 9.2X_3 + 19.5X_4 + 14.6X_5 + 19.3X_6 + 11.2X_7 + 7.2X_8 + 19.4X_9 + 11X_{10} + 9.1X_{11} + 14.6X_{12}}{2.2X_1 + 5.4X_2 + 2.2X_3 + 4.4X_4 + 3.4X_5 + 3.4X_6 + 3.4X_7 + 2.2X_8 + 4.4X_9 + 3.4X_{10} + 2.5X_{11} + 3.4X_{12}}, \\ \frac{9.7 + 22.8X_2 + 9.76X_3 + 20.8X_4 + 15.4X_5 + 20.8X_6 + 12.4X_7 + 8.32X_8 + 20.4X_9 + 12.4X_{10} + 10.32X_{11} + 15.4X_{12}}{1.8X_1 + 4.6X_2 + 1.72X_3 + 3.6X_4 + 2.8X_5 + 2.6X_6 + 2.7X_7 + 1.8X_8 + 3.6X_9 + 2.6X_{10} + 1.8X_{11} + 2.6X_{12}} \end{array} \right\}$$

s.t. $S = \{X_1 + X_2 + X_3 + X_4 \leq 8.32, X_5 + X_6 + X_7 + X_8 \leq 14.8,$

$X_9 + X_{10} + X_{11} + X_{12} \leq 12.72,$

$X_1 + X_5 + X_9 \geq 7.32, X_2 + X_6 + X_{10} \geq 10.44, X_3 + X_7 + X_{11} \geq 8.6, X_4 + X_8 + X_{12} \geq 9.48,$

$X_i \geq 0, \quad i = 1, \dots, 12\}.$ (25)

Equation (25) is solved by the proposed method and the solution obtained is

$$X^* = (0.92, 0, 7.4, 1.56, 13.24, 0, 0, 4.2, 0, 0, 8.52).$$

At the solution X^* ,

$$Z_1(X) = 4.8271, Z_2(X) = 6.6052,$$

$$\mu_{Z_1}(X) = 1, \mu_{Z_2}(X) = 0.9660.$$

The average of $\mu_{Z_1}(X)$ and $\mu_{Z_2}(X)$ is 0.983.

As we observe, our proposed method addressed (25) in an excellent way since the average of the membership functions is very close to one.

It is noticeable that the genetic algorithm of the global optimisation toolbox of MATLAB R2016 failed to reach a solution for this example.

5. Conclusion

In this paper, a new method was presented to solve the MOLFPF. In the approach, the MOLFPF was changed finally into a LPP using suitable non-linear variable transformations. It was proven that the optimal solution of the LPP is unique and is efficient for the MOLFPF. We need to mention that the proposed method is easy and straightforward with less computational complexities compared to the other existing methods. Moreover, this approach can be applied to address the LFPP with fuzzy coefficients if the fuzzy coefficients are changed into intervals using the concept of α -cuts. In this case, the fuzzy problem is further changed into a bi-objective LFPP.

Four examples were solved to illustrate the approach in addition to make comparisons. For numerical examples, our proposed solutions gave better outcomes compared to Toksari, Chakraborty and Gupta, and Pramy and Islam. Furthermore, the results demonstrate that the method of Chakraborty and Gupta is reliable, but we cannot consider the methods of Toksari and Pramy and Islam as the effective approaches since their solutions proposed for Examples 1 and 3 were completely dominated by our proposed solutions.

As a future research, one can employ the results of this study to cope with multi-level MOLFPF.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

This work was supported by Universiti Kebangsaan Malaysia [Grant Number ST-2019-016].

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