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On the Primary Decomposition of k -Ideals and Fuzzy k -Ideals in Semirings

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ABSTRACT

Observing that every k -irreducible ideal of a semiring R is a k -primary ideal, if R is an additively cancellative, yoked and commutative Noetherian semiring, we establish the primary decomposition and uniqueness of the primary decomposition of k -ideals of such semirings. Finally, the primary decomposition and uniqueness of primary decomposition proved for k -ideals is also generalised for fuzzy k -ideals of these semirings.

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1. Introduction

Ideals play an important role in both ring theory and semiring theory. But in the absence of additive inverses in semirings, the structure of ideals in semirings differs from that of ring theory. The ideals in semirings possessing the very obvious property of ideals of rings, 'if $x + y \in I, x \in I$, then $y \in I$ ' are known as k -ideals and the role of these ideals in semirings becomes significant in the absence of additive inverses. The results which are true for ideals in rings have also been established for k -ideals in semirings by various authors (cf. [1–13]). In view of these facts various researchers attempted the primary decomposition for k -ideals in semirings analogous to the primary decomposition of ideals in rings: In a commutative Noetherian ring, every ideal can be decomposed as a finite intersection of primary ideals (Lasker–Noether Theorem [14]). For a deep study of primary ideals in rings and semirings one can refer to [8–12, 15–19].

The above result of ring theory is not true for arbitrary ideals in semirings as noticed in [20]. Atani and Atani [20, Theorem 4] had proved that in a commutative Noetherian semiring, every proper k -ideal can be represented as a finite intersection of k -primary ideals. But it was observed by Lescot [21] that there were some errors in the results used to prove the aforementioned result. For example, $I + Ra^n$ is not a k -ideal, even if I is a k -ideal. But in [20], it is taken for granted that the ideal $I + Ra^n$ is a k -ideal. Lescot [21] found these errors after observing in Example 6.2 that $\{0\}$ ideal may not be a finite intersection of k -primary ideals in a commutative Noetherian semiring. With these observations, he developed

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the theory of weak primary decomposition for semirings of characteristic 1. But still the question of settling the primary decomposition for a proper k -ideal (other than $\{0\}$ and semiring R) remained unsolved. In this direction, Kar et al. [22, Theorem 4.4] proved that every proper k -ideal of a commutative Noetherian semiring R can be expressed as a finite intersection of k -irreducible ideals (Definition 2.1). They tried to prove the primary decomposition by observing that in a commutative Noetherian semiring, every k -irreducible ideal is a k -primary ideal [22, Theorem 4.6]. But the proof of this result has the same errors again and the problem still remains unsolved. This motivated us to establish the primary decomposition of k -ideals in commutative Noetherian semirings.

In this paper, we provide a correct proof of Theorem 4.6 of [22], which settles the primary decomposition for k -ideals in commutative Noetherian semirings. In order to establish the uniqueness of primary decomposition for semirings, first we prove some required basic results for uniqueness and finally establish the uniqueness of primary decomposition for k -ideals in commutative Noetherian semirings: If $I = \bigcap_{i=1}^n Q_i$, where each Q_i is a primary ideal and $\sqrt{Q_i} = P_i$, then the set $\{P_1, P_2, \dots, P_n\}$ is independent of particular choice of decomposition of I .

Fuzzy k -ideals play an important role for the study of different classes of semirings. These ideals have been studied by many authors (cf. [1, 7, 18, 23–27]). From the primary decomposition of k -ideals in semirings, the fuzzy primary decomposition of fuzzy k -ideals follows as given in [22]. But the uniqueness theorem for fuzzy k -ideals [22, Theorem 5.5], is also incorrect. This motivated us to reinvestigate the uniqueness of fuzzy k -primary decomposition in semirings.

2. Primary Decomposition of k -Ideals in Semirings

Throughout this paper, R is a commutative semiring with identity. First, we recall some definitions and results which are necessary to prove the primary decomposition and uniqueness theorems for k -ideals in commutative Noetherian semirings.

Definition 2.1: A proper k -ideal I of a semiring R is called k -irreducible ideal if for any two k -ideals J, K of R , $I = J \cap K$ implies that either $I = J$ or $I = K$.

Definition 2.2: Let I be a k -ideal of a semiring R . Then I is said to have a primary decomposition if I can be expressed as $I = \bigcap_{i=1}^n Q_i$, where each Q_i is a primary ideal of R . Also, a primary decomposition of the type $I = \bigcap_{i=1}^n Q_i$ with $\sqrt{Q_i} = P_i$, is called a reduced primary decomposition of I , if P_i 's are distinct and I cannot be expressed as an intersection of a proper subset of ideals Q_i in the primary decomposition of I . A reduced primary decomposition can be obtained from any primary decomposition by deleting those Q_j that contains $\bigcap_{i \neq j} Q_i$ and grouping together all distinct $\sqrt{Q_i}$'s-primary ideal.

The main aim of this section is to prove the existence and uniqueness of primary decomposition for k -ideals in a commutative Noetherian semiring. First, we prove the existence part as stated below:

Theorem 2.3 (Primary Decomposition of k -Ideals): Let I be a k -ideal of an additively cancellative, yoked and commutative Noetherian semiring R . Then I can be represented as a finite intersection of primary ideals of R , that is, I has a reduced primary decomposition.

The decomposition of k -ideals in terms of k -irreducible ideals has already been proved in [22] as follows:

Theorem 2.4 ([22], Theorem 4.4): *Every proper k -ideal of a commutative Noetherian semiring R can be expressed as a finite intersection of k -irreducible ideals.*

Therefore, primary decomposition for k -ideals in semirings follows immediately if we prove

Theorem 2.5: *Let R be an additively cancellative, yoked and commutative Noetherian semiring. Then every k -irreducible ideal of R is a primary ideal of R .*

Kar et al. [22, Theorem 4.6], made an attempt to prove the same without considering additively cancellative and yoked semiring, but the proof of the result has some errors as mentioned in the introduction. For the proof of the required result, we first analyse some common errors committed by many authors regarding k -ideals, and prove some facts about k -ideals. First, we note that any ideal $I = \langle a \rangle$ is not a k -ideal in semirings as observed by Golan in [28, Example 6.17]. That is, if $I = \langle (1 + x) \rangle$ is an ideal of $\mathbb{N}[x]$ (semiring of polynomials over non-negative integers \mathbb{N} in the indeterminate x), then $(1 + x)^3 = (x^3 + 1) + 3x(1 + x) \in I$, $3x(1 + x) \in I$, but $(x^3 + 1) \notin I$ implies that I is not a k -ideal of $\mathbb{N}[x]$. Any ideal generated by a single element becomes a k -ideal, if we impose some conditions on a semiring as shown below:

Lemma 2.6: *Let R be an additively cancellative, yoked and zerofree semiring. Then for any $a \in R$, the ideal $I = \langle a \rangle$ is a k -ideal of R .*

Proof: Let $a_1 + a_2, a_1 \in I$. Then there exist some $r_1, r_2 \in R$ such that $a_1 + a_2 = ar_1$ and $a_1 = ar_2$. As R is yoked, there exists some $r_3 \in R$ such that $r_1 + r_3 = r_2$ or $r_2 + r_3 = r_1$. If $r_1 + r_3 = r_2$, then $ar_1 = ar_2 + a_2 = ar_1 + ar_3 + a_2$ implies that $ar_3 + a_2 = 0$, as R is additively cancellative. Also, $a_2 = ar_3 = 0 \in I$, as R is zerofree.

Further, if $r_2 + r_3 = r_1$, then $ar_2 + a_2 = ar_1 = a(r_2 + r_3) = ar_2 + ar_3$ implies that $a_2 = ar_3 \in I$, since R is additively cancellative. Thus, $I = \langle a \rangle$ is a k -ideal of R . ■

The semiring considered in above example is additively cancellative, zerofree, but it is not yoked, for let

$$f(x) = 5x^2 + 9x + 2 \quad \text{and} \quad g(x) = 11x^2 + 3x + 5$$

be two polynomials in $\mathbb{N}[x]$, then there exists no $h(x) \in \mathbb{N}[x]$ such that either $f(x) + h(x) = g(x)$ or $g(x) + h(x) = f(x)$.

Similar to an ideal generated by a single element, the sum of two k -ideals may not be a k -ideal in a semiring. There are plenty of k -ideals in the semiring \mathbb{N} , but their sum is not a k -ideal. However, the sum of two k -ideals is a k -ideal in a lattice ordered semiring (cf. [28, Corollary 21.22]). While proving Theorem 2.5, the authors wrongly used that the ideals $(Q + \langle a^k \rangle)$ and $(Q + \langle b \rangle)$ are k -ideals. In view of the above observations, Theorem 2.5 follows verbatim as proved in [22, Theorem 4.6] for additively cancellative, yoked, zerofree and lattice ordered semirings, because in this case, both an ideal generated by a single

element and sum of two k -ideals are k -ideals. Here, we give a proof of Theorem 2.5 without resorting to the restrictions: lattice ordered and zerosumfree semiring.

Proof of Theorem 2.5.: Let Q be a k -irreducible ideal of a Noetherian semiring R . Let $ab \in Q$ be such that $b \notin Q$. Now, we construct two ideals I and J of R as follows:

$$I = \langle a^k \rangle + Q \quad \text{and} \quad J = \langle b \rangle + Q.$$

Then, clearly, $Q \subseteq I \cap J$.

Let $y \in I \cap J$. Then $y = a^n z + q$ for some $z \in R$ and $q \in Q$. Again $aJ \subseteq Q$ (since $ab \in Q$) and so $ay \in Q$ (since $y \in J$).

Therefore, $ay = a^{n+1}z + aq$. Thus, we get that $a^{n+1}z + aq \in Q$. Also, $aq \in Q$, since $q \in Q$ and Q is an ideal of R . It follows that $a^{n+1}z \in Q$, since Q is a k -ideal of R . Construct a set $A_n = \{x \in R \mid a^n x \in Q\}$. It is easy to check that A_n is an ideal of R and $A_1 \subseteq A_2 \subseteq \dots$ is an ascending chain of ideals. Since R is Noetherian, so $A_n = A_{n+1} = \dots$ for some $n \in \mathbb{Z}^+$. Again $a^{n+1}z \in Q$ implies that $z \in A_{n+1} = A_n$. It demonstrates that $a^n z \in Q$ which implies that $y \in Q$ and hence $I \cap J = Q$.

Let $rad(R)$ denote the Jacobson Bourne radical of a semiring R , that is, the intersection of all maximal k -ideals of R , as R is assumed to be additively cancellative and yoked (cf. [29, Proposition 23]). Assume that A is an ideal of R and $rad(A)$ denotes the intersection of all maximal k -ideals of R containing A .

Let \bar{A} denotes the k -closure of an ideal A of R , i.e. $\bar{A} = \{a \in A \mid a + b = c, \text{ for some } b, c \in A\}$ and $\sqrt{I} = \{x \in R \mid a^n \in I \text{ for some positive integer } n\}$.

Obviously, \bar{I} is a k -ideal of R and every Noetherian semiring is weakly Noetherian (a semiring R is weakly Noetherian if every ascending chain of k -ideals of R is ultimately stationary). Then by Lescot [21, Corollary 6.6], there are prime k -ideals P_1, P_2, \dots, P_n of R such that $\sqrt{I} = P_1 \cap P_2 \cap \dots \cap P_n$. Thus, we have $\sqrt{I} \subseteq rad(R)$ and by Lescot [21, Lemma 2.2], $\bar{I} \cap \bar{J} \subseteq \sqrt{\bar{I} \cap \bar{J}} = \sqrt{\bar{I} \cap \bar{J}} \subseteq rad(\bar{I} \cap \bar{J})$. Therefore, $rad(\bar{I} \cap \bar{J}) \subseteq rad(\bar{I} \cap \bar{J})$. Again, $\bar{I} \cap \bar{J} \subseteq \bar{I} \cap \bar{J}$ implies that $rad(\bar{I} \cap \bar{J}) \subseteq rad(\bar{I} \cap \bar{J})$. Thus, $rad(\bar{I} \cap \bar{J}) = rad(\bar{I} \cap \bar{J}) = rad(\bar{I}) \cap rad(\bar{J})$. So, we have $rad(\bar{I} \cap \bar{J}) = rad(\bar{I}) \cap rad(\bar{J})$.

Now, $Q = I \cap J$ which implies that $\bar{Q} = \bar{I} \cap \bar{J} \Rightarrow Q = \bar{I} \cap \bar{J}$, since Q is a k -ideal of R . This shows that $rad(Q) = rad(\bar{I} \cap \bar{J}) = rad(\bar{I}) \cap rad(\bar{J})$, that is, $rad(Q) = rad(\bar{I}) \cap rad(\bar{J})$.

Again, Q is a k -irreducible ideal, which implies that $rad(Q)$ is k -irreducible. Also $rad(Q) = Q \cap rad(R)$, but $rad(Q) \neq rad(R)$. Thus, $rad(Q) = Q$, since $rad(Q)$ is k -irreducible. Accordingly, $Q = rad(\bar{I}) \cap rad(\bar{J})$, where each of $rad(\bar{I})$ and $rad(\bar{J})$ are k -ideals of R . Now $b \in J$ implies that $b \in rad(\bar{J})$, but $b \notin Q$, that is, $Q \neq rad(\bar{J})$. So $Q = rad(\bar{I})$, since Q is k -irreducible. Further, $a^n \in I$ leads to $a^n \in rad(\bar{I}) = Q$. Hence, Q is a primary ideal. ■

Remark 2.7: Now Theorem 2.3, follows by combining Theorems 2.4 and 2.5. It is important to note here that all Q_i 's in the primary decomposition of a k -ideal I have an additional property that these are also k -ideals. Now it only remains to prove the uniqueness of primary decomposition of a k -ideal I .

The following lemma will be used to prove the uniqueness of reduced primary decomposition of k -ideals in semirings.

Lemma 2.8: Let R be a semiring and Q a P -primary ideal of R . Then we have

- (i) If $x \in R$, $(Q : x) = \{r \in R \mid rx \in Q\}$ is an ideal of R ;
- (ii) If $x \in Q$, then $(Q : x) = R$;
- (iii) If $x \notin P$, then $(Q : x) = Q$;
- (iv) If $x \notin Q$, then $(Q : x)$ is P -primary.

Proof: (i) Let $r_1, r_2 \in (Q : x)$ and $a \in R$. Then $r_1x, r_2x \in Q$ implies that $(r_1 + r_2)x \in Q$ and $r_1ax \in Q$, as Q is an ideal of R . Hence $(Q : x)$ is an ideal of R .

(ii) If $x \in Q$, then $Rx \subseteq Q$, as Q is an ideal of R which implies that $R \subseteq (Q : x)$. Also, $(Q : x)$ is an ideal of R and so, $(Q : x) = R$.

(iii) Assume that $x \notin P$. If $a \in Q$, then $ax \in Q$ implies that $a \in (Q : x)$. For the converse part, suppose that $b \notin Q$ and $xb \in Q$. Then $x \in \sqrt{Q} = P$, as Q is P -primary ideal of R which contradicts that $x \notin P$. This infers that $xb \notin Q$ and so $b \notin (Q : x)$.

(iv) Suppose that $x \notin Q$. If $y \in (Q : x)$, then $xy \in Q$ implies that $y \in \sqrt{Q} = P$, as Q is P -primary ideal of R . Thus, $Q \subseteq (Q : x) \subseteq P$ implies that $P = \sqrt{Q} \subseteq \sqrt{(Q : x)} \subseteq \sqrt{P}$. Also, $\sqrt{P} = P$, as P is a prime ideal of R and so $P = \sqrt{(Q : x)}$. Now, we show that $(Q : x)$ is a primary ideal of R . Clearly, the ideal $(Q : x)$ is a proper as $x \notin Q$ and so $1 \notin (Q : x)$. Assume that $ab \in (Q : x)$ and $b \notin \sqrt{(Q : x)}$ for $a, b \in R$. Then $abx \in Q$ and Q is a P -primary ideal of R which implies that either $ax \in Q$ or $b \in P = \sqrt{(Q : x)}$. Thus, $a \in (Q : x)$ as $b \notin \sqrt{(Q : x)}$. ■

We now prove the uniqueness of the reduced primary decomposition of a k -ideal of a semiring as follows:

Theorem 2.9 (Uniqueness of Primary Decomposition): Let R be an additively cancellative, yoked and commutative Noetherian semiring and I be a k -ideal of R . If $I = \bigcap_{i=1}^n Q_i$ is a reduced primary decomposition of I with $\sqrt{Q_i} = P_i$ for $i = 1, 2, \dots, n$, then

$$\{P_1, P_2, \dots, P_n\} = \{\text{Prime ideals } P \mid \text{there exists } x \in R \text{ such that } P = \sqrt{(I : x)}\}.$$

The set $\{P_1, P_2, \dots, P_n\}$ is independent of the particular reduced primary decomposition chosen for I .

Proof: Let $x \in R$. Then by Lemma 2.8 (iv),

$$\sqrt{(I : x)} = \sqrt{\left(\bigcap_{i=1}^n Q_i : x\right)} = \bigcap_{i=1}^n \sqrt{(Q_i : x)} = \bigcap_{i, x \notin Q_i} P_i$$

and therefore, $\sqrt{(I : x)} \subseteq P_i$, for all $i = 1, 2, \dots, n$. Also, if $\sqrt{(I : x)}$ is prime, then $\bigcap_{i, x \notin Q_i} P_i \subseteq \bigcap_{i, x \notin Q_i} P_i = \sqrt{(I : x)}$ implies that $P_i \subseteq \sqrt{(I : x)}$ for some $i = 1, 2, \dots, n$.

Thus, we have

$$\{P_1, P_2, \dots, P_n\} \subseteq \{\text{Prime ideals } P \mid \text{there exists } x \in R \text{ such that } P = \sqrt{(I : x)}\}.$$

On the other hand, for $i \in \{1, 2, \dots, n\}$, we have $\bigcap_{j \neq i} Q_j \not\subseteq Q_i$, as the primary decomposition is reduced. So there exists some $x_i \in \bigcap_{j \neq i} Q_j$ and $x_i \notin Q_i$. If $y \in (Q_i : x_i)$, then $yx_i \in Q_i$ and $yx_i \in (\bigcap_{j \neq i} Q_j) \cap Q_i = I$ which implies that $y \in (I : x_i)$.

Thus, $(Q_i : x_i) \subseteq (I : x_i) \subseteq (Q_i : x_i)$, as $I \subseteq Q_i$.

So, $(Q_i : x_i) = (I : x_i)$ and by Lemma 2.8(iv), $\sqrt{(I : x_i)} = \sqrt{(Q_i : x_i)} = P_i$. Hence, $\{P_1, P_2, \dots, P_n\} = \{\text{Prime ideals } P \mid \exists x \in R \text{ such that } P = \sqrt{(I : x)}\}$. ■

3. Fuzzy Primary Decomposition of Fuzzy k -Ideals in Semirings

In this section, we prove the existence and uniqueness of fuzzy primary decomposition of fuzzy k -ideals of semirings. We first recall some definitions from [22] which are required to prove the uniqueness

Definition 3.1: Let μ be a fuzzy ideal of a semiring R . Then the radical of μ , denoted by $\sqrt{\mu}$ is defined by $\sqrt{\mu}(x) = \sup_{n \geq 1} \{\mu(x^n)\}$, for all $x \in R$.

Example 3.2: Consider the semiring $R = (\mathbb{N}, +, \cdot)$ of non-negative integers with respect to usual addition and multiplication of integers. Define a fuzzy ideal μ of \mathbb{N} as follows:

$$\mu(x) = \begin{cases} 1 - \frac{1}{n}, & \text{if } x \in \langle 2^{n-1} \rangle \setminus \langle 2^n \rangle, n = 1, 2, \dots \\ 1, & \text{if } x = 0 \end{cases}$$

Observe that $im \mu = \{0, 1, \frac{1}{2}, \frac{2}{3}, \dots, \frac{n}{n+1}, \dots\}$ and clearly,

$$\sqrt{\mu}(x) = \begin{cases} 1, & \text{if } x \in \langle 2 \rangle \\ 0, & \text{if } x \in \mathbb{Z} \setminus \langle 2 \rangle \end{cases}$$

Definition 3.3: Let μ be a non-constant fuzzy k -ideal of a semiring R . Then μ is said to be a fuzzy k -irreducible ideal of R if for any two fuzzy k -ideals θ and η of R , $\mu = \theta \cap \eta$ implies either $\mu = \theta$ or $\mu = \eta$.

Example 3.4 ([22]): Let R be a semiring as in Example 3.2 and μ be a fuzzy ideal of R defined as follows:

$$\mu(x) = \begin{cases} 1, & \text{if } x \in \langle 16 \rangle \\ 0.4, & \text{otherwise} \end{cases}$$

Then, by Kar et al. [22, Theorem 4.2], one can easily check that μ is a fuzzy k -irreducible ideal of R .

Definition 3.5: A fuzzy ideal μ of a semiring R is said to be fuzzy prime ideal of R if it is non-constant (i.e. $|im \mu| \geq 2$) and any two fuzzy ideals θ and η of R , $\theta \cap \eta \subseteq \mu$ implies that either $\theta \subseteq \mu$ or $\eta \subseteq \mu$.

Definition 3.6: A fuzzy ideal μ of a semiring R is said to be fuzzy primary ideal of R if it is non-constant (i.e. $|im \mu| \geq 2$) and any two fuzzy ideals θ and η of R , $\theta \cap \eta \subseteq \mu$ implies that either $\theta \subseteq \mu$ or $\eta \subseteq \sqrt{\mu}$. If $\sqrt{\mu} = \nu$, then μ is called fuzzy ν -primary ideal of R .

Definition 3.7: Let μ and θ be two fuzzy subsets of a semiring R . Then fuzzy colon ideal $(\mu : \theta)$ is defined by $(\mu : \theta)(x) = \sup_{\lambda \in IF_5(R)} \{\lambda(x) \mid \lambda \circ \theta \subseteq \mu\}$, where $IF_5(R)$ denotes the set of all fuzzy subsets of R .

Definition 3.8: Let μ be a fuzzy ideal of a semiring R . Then μ_0 is defined as, $\mu_0 = \{x \in R \mid \mu(x) = \mu(0)\}$.

Definition 3.9: If μ is a fuzzy k -ideal of a semiring R , then μ is said to have a fuzzy primary decomposition if μ can be expressed as $\mu = \bigcap_{i=1}^n \mu_i$, where each μ_i is a fuzzy primary ideal of R . A fuzzy primary decomposition $\mu = \bigcap_{i=1}^n \mu_i$ is said to be reduced if $\sqrt{\mu_i}$'s are distinct and $\bigcap_{j \neq i} \mu_j \not\subseteq \mu_i$, for all $i = 1, 2, \dots, n$. By Kar et al. [22, Theorem 5.3], a reduced fuzzy primary decomposition can be obtained from a fuzzy primary decomposition $\mu = \bigcap_{i=1}^n \mu_i$, where each μ_i is a fuzzy primary ideal of R .

The primary decomposition proved for k -ideals in a commutative Noetherian semiring (Theorem 2.5) can be generalised to fuzzy k -ideals as in [22, Theorem 5.2]. Thus we have

Theorem 3.10 (Fuzzy Primary Decomposition of Fuzzy k -Ideals): Let R be an additively cancellative, yoked and commutative Noetherian semiring and μ a fuzzy k -ideal of R such that $\text{im} \mu = \{1, \alpha\}$, where $\alpha \in [0, 1)$. Then μ can be represented as a finite intersection of fuzzy primary ideals of R , i.e., μ has a primary decomposition.

Similar to the primary decomposition of k -ideals in semirings, all μ_i 's in the fuzzy primary decomposition of μ have an additional property that these are also fuzzy k -ideals.

There are some errors (as mentioned below) in the result used for the proof of uniqueness of fuzzy primary decomposition of fuzzy k -ideals in semirings. So the question of establishing the result still remains unsolved. First, we state the result used for uniqueness.

Theorem 3.11 ([22], Theorem 5.4): Let μ be a fuzzy k -ideal of a semiring R and $\mu = \bigcap_{i=1}^n \mu_i$ be a reduced fuzzy primary decomposition of μ . Let λ be a fuzzy k -prime ideal of R . Then $\lambda = \sqrt{\mu_i}$ for some $i = 1, 2, \dots, n$ if and only if there exists a fuzzy k -ideal θ of R such that $\theta \not\subseteq \mu$ and $\sqrt{(\mu : \theta)} = \lambda$, i.e. $(\mu : \theta)$ is a fuzzy λ -primary ideal of R .

In the proof of above result, $\mu = \bigcap_{i=1}^n \mu_i$ is a reduced fuzzy primary decomposition of fuzzy k -ideal μ , so $\sqrt{\mu_i}$'s are distinct for each $i = 1, 2, \dots, n$. But the authors have used $\sqrt{\mu_i} = \lambda$, for each $i = 1, 2, \dots, n$ in the step

$$\sqrt{\bigcap_{i=1}^n (\mu_i : \theta)} = \bigcap_{i=1}^n \sqrt{(\mu_i : \theta)} = \bigcap_{i=1}^n \sqrt{\mu_i} = \bigcap_{i=1}^n \lambda = \lambda$$

contrary to the assumption that the decomposition is reduced.

In order to establish a correct proof of the uniqueness of fuzzy primary decomposition of a fuzzy k -ideal in a semiring, first we need the following result:

Lemma 3.12: Let μ be a fuzzy v -primary ideal of a semiring R and θ a fuzzy ideal of R . Then the following hold:

- (i) If $\theta \subseteq \mu$, then $(\mu : \theta) = \chi_R$, where χ_R is the chai function on R ;
- (ii) If $\theta \not\subseteq \mu$, then $(\mu : \theta)$ is a fuzzy v -primary ideal of R , i.e. $\sqrt{(\mu : \theta)} = v$.

Proof: (i) Let $x \in R$ and $\theta \subseteq \mu$. Then we have

$$(\mu : \theta)(x) = \sup_{\lambda \in IFS(R)} \{ \lambda(x) | \lambda \circ \theta \subseteq \mu \}.$$

Also,

$$\begin{aligned} (\chi_R \circ \theta)(x) &= \sup \{ \min(\chi_R(y), \theta(z)) | x = yz \} \\ &= \sup \{ \theta(z) | x = yz \} \\ &\leq \theta(x) \\ &\leq \mu(x) \end{aligned}$$

as $\theta \subseteq \mu$. Thus, by the definition of colon ideal and $(\chi_R \circ \theta) \subseteq \mu$, we get $(\mu : \theta) = \chi_R$.

(ii) Let

$$\theta \not\subseteq \mu \quad \text{and} \quad \eta_1 \circ \eta_2 \subseteq (\mu : \theta)$$

for some fuzzy ideals η_1 and η_2 of R . Then by the definition of fuzzy colon ideal, $\eta_1 \circ \eta_2 \circ \theta \subseteq \mu$. Since μ is a fuzzy v -primary ideal of R and $\theta \not\subseteq \mu$, we have $\eta_1 \circ \eta_2 \subseteq \sqrt{\mu}$ which further implies that either $\eta_1 \subseteq \sqrt{\mu}$ or $\eta_2 \subseteq \sqrt{\mu}$, as $\sqrt{\mu}$ is a fuzzy prime ideal of R .

If $\eta_1 \subseteq \sqrt{\mu}$, then by Kar et al. [22, Lemma 3.2 (iii) and Lemma 5.2 (i)], $\sqrt{\mu} \subseteq \sqrt{(\mu : \theta)}$ implies that $\eta_1 \subseteq \sqrt{(\mu : \theta)}$. Similarly, if $\eta_2 \subseteq \sqrt{\mu}$, then we get $\eta_2 \subseteq \sqrt{(\mu : \theta)}$. Thus, $(\mu : \theta)$ is a fuzzy primary ideal of R .

We now show that $\sqrt{(\mu : \theta)} = v$. For this, let $x \in R$. Then

$$\begin{aligned} \sqrt{(\mu : \theta)}(x) &= \sup_{n \geq 1} \{ (\mu : \theta)(x^n) \} \\ &= \sup_{n \geq 1} \left\{ \sup_{\lambda \in IFS(R)} \{ \lambda(x^n) | \lambda \circ \theta \subseteq \mu \} \right\} \\ &= \sup_{n \geq 1} \left\{ \sup_{\lambda \in IFS(R)} \{ \lambda(x^n) | \lambda \subseteq \sqrt{\mu} \} \right\} \\ &\leq \sup_{n \geq 1} \{ \sqrt{\mu}(x^n) \} \\ &= \sqrt{\sqrt{\mu}}(x) = \sqrt{\mu}(x) \end{aligned}$$

as $\theta \not\subseteq \mu$ and $\sqrt{\mu}$ is a fuzzy prime ideal of R . Thus, $\sqrt{(\mu : \theta)} \subseteq \sqrt{\mu}$. Also by Kar et al. [22, Lemma 3.2 (iii) and Lemma 5.2 (i)], $\sqrt{\mu} \subseteq \sqrt{(\mu : \theta)}$ implies that $\sqrt{(\mu : \theta)} = \sqrt{\mu} = v$. ■

Now, we give the correct proof of Theorem 3.11, which is required for the establishment of uniqueness of fuzzy primary decomposition of a fuzzy k -ideal of a semiring as follows:

Theorem 3.13: *Let R be an additively cancellative, yoked, zerosumfree and commutative Noetherian semiring, and μ a fuzzy k -ideal of R such that $\text{im } \mu = \{1, \alpha\}$, where $\alpha \in [0, 1)$. Let $\mu = \bigcap_{i=1}^n \mu_i$ be a reduced fuzzy primary decomposition of μ and λ be a fuzzy prime ideal of R . Then $\lambda = \sqrt{\mu_i}$, for some $i = 1, 2, \dots, n$ if and only if there exists a fuzzy k -ideal θ of R such that $\theta \not\subseteq \mu$ and $\sqrt{(\mu : \theta)} = \lambda$, i.e. $(\mu : \theta)$ is a fuzzy λ -primary ideal of R .*

Proof: Let $\lambda = \sqrt{\mu_i}$, for some $i = 1, 2, \dots, n$. As $\mu = \bigcap_{i=1}^n \mu_i$ is a reduced fuzzy k -primary decomposition of μ , so $\bigcap_{j=1, j \neq i}^n \mu_j \not\subseteq \mu_i$, for any $i \neq j$. Therefore, there exists some $x_i \in R$ such that $\bigcap_{j=1, j \neq i}^n \mu_j(x_i) > \mu_i(x_i)$. Assume that $\bigcap_{j=1, j \neq i}^n \mu_j(x_i) = a$, where $a \in [0, 1)$. We now establish a fuzzy subset θ of R as follows:

$$\theta(x) = \begin{cases} a, & \text{if } x \in \langle x_i \rangle \\ 0, & \text{otherwise} \end{cases}.$$

By Lemma 3.8, the ideal $\langle x_i \rangle$ is a k -ideal as R is additively cancellative, yoked and zerosum-free, which clearly implies that θ is a fuzzy k -ideal of R . Also,

$$\theta(x_i) = a = \bigcap_{j=1, j \neq i}^n \mu_j(x_i) > \mu_i(x_i) > \bigcap_{i=1}^n \mu_i(x_i) = \mu(x_i)$$

implies that $\theta \not\subseteq \mu$ and $\theta \not\subseteq \mu_i$, for $i \neq j$. Further,

$$\theta(x_i) = a = \bigcap_{j=1, j \neq i}^n \mu_j(x_i) = \inf\{\mu_1(x_i), \mu_2(x_i), \dots, \mu_{i-1}(x_i), \mu_{i+1}(x_i), \dots, \mu_n(x_i)\} \leq \mu_j(x_i)$$

for all $j \neq i$ and $j = 1, 2, \dots, n$. Now, we claim that $\theta \subseteq \mu_j$, for all $j \neq i$ and $j = 1, 2, \dots, n$. Let $x = x_i y_i \in \langle x_i \rangle$, for some $y_i \in R$. Then

$$\theta(x) = \theta(x_i) \leq \mu_j(x_i) \leq \mu_j(x_i y_i) = \mu_j(x)$$

implies that $\theta \subseteq \mu_j$ for all $j \neq i$ and $j = 1, 2, \dots, n$. Thus, by Lemma 3.12 (i), we have

$$\begin{aligned} (\mu_j : \theta) &= \chi_R, \quad \text{for all } j \neq i, j = 1, 2, \dots, n \text{ and} \\ \sqrt{(\mu_i : \theta)} &= \sqrt{\mu_i} = \lambda. \end{aligned}$$

Finally, by Kar et al. [22, Theorem 3.2 (v) and Lemma 5.2 (iii)], we get

$$\sqrt{(\mu : \theta)} = \sqrt{\left(\bigcap_{i=1}^n \mu_i : \theta\right)} = \sqrt{\bigcap_{i=1}^n (\mu_i : \theta)} = \bigcap_{i=1}^n \sqrt{(\mu_i : \theta)} = \sqrt{(\mu_i : \theta)} = \lambda.$$

Also by Lemma 3.12 (ii), $(\mu : \theta)$ is a fuzzy λ -primary ideal of R .

Conversely, assume that there exists a fuzzy k -ideal θ of R such that $\theta \not\subseteq \mu$ and $\sqrt{(\mu : \theta)} = \lambda$. As

$$\theta \not\subseteq \mu = \bigcap_{i=1}^n \mu_i, \quad \text{we get } \theta \not\subseteq \mu_j \text{ for some } j \in \{1, 2, \dots, n\}$$

and by Lemma 3.12 (ii), we have $\sqrt{(\mu_j : \theta)} = \sqrt{\mu_j}$.

Also,

$$\sqrt{(\mu : \theta)} = \bigcap_{i=1}^n \sqrt{(\mu_i : \theta)} = \lambda$$

implies that

$$\bigcap_{\substack{i=1 \\ i \neq j}}^n \sqrt{(\mu_i : \theta)} \cap \sqrt{\mu_j} = \lambda.$$

Further,

$$\left(\bigcap_{\substack{i=1 \\ i \neq j}}^n \sqrt{(\mu_i : \theta)} \circ \sqrt{\mu_j} \right) \subseteq \left(\bigcap_{\substack{i=1 \\ i \neq j}}^n \sqrt{(\mu_i : \theta)} \right) \cap \sqrt{\mu_j} = \lambda$$

and

$$\lambda = \bigcap_{i=1}^n \sqrt{(\mu_i : \theta)} \subseteq \bigcap_{\substack{i=1 \\ i \neq j}}^n \sqrt{(\mu_i : \theta)}$$

implies that $\sqrt{\mu_j} \subseteq \lambda$, as λ is prime.

Also,

$$\lambda = \sqrt{(\mu : \theta)} = \sqrt{\bigcap_{i=1}^n (\mu_i : \theta)} = \bigcap_{i=1}^n \sqrt{(\mu_i : \theta)} \subseteq \sqrt{(\mu_j : \theta)} = \sqrt{\mu_j}.$$

Thus, we get $\lambda = \sqrt{\mu_j}$ for some, $j \in \{1, 2 \dots .n\}$. ■

Finally, we are in a position to give the correct proof of uniqueness theorem of reduced fuzzy k -primary decomposition of a fuzzy k -ideal in semirings as follows:

Theorem 3.14 (Uniqueness of Fuzzy Primary Decomposition): *Let R be an additively cancellative, yoked, zerosumfree and commutative Noetherian semiring, and μ a fuzzy k -ideal of R such that $im\mu = \{1, \alpha\}$, where $\alpha \in [0, 1)$. Let $\mu = \bigcap_{i=1}^n \mu_i$ with $\sqrt{\mu_i} = v_i$ for $i = 1, 2, \dots .n$ and $\mu = \bigcap_{i=1}^m \xi_i$ with $\sqrt{\xi_i} = \eta_i$ for $i = 1, 2, \dots .m$ be two reduced fuzzy k -primary decompositions of μ . Then, $n = m$ and $\{v_1, v_2 \dots .v_n\} = \{\eta_1, \eta_2, \dots, \eta_m\}$.*

Proof: Let $v_i \in \{v_1, v_2 \dots .v_n\}$. Then by Theorem 3.13, there exists a fuzzy k -ideal θ of R such that

$$\theta \not\subseteq \mu \quad \text{and} \quad \sqrt{(\mu : \theta)} = \sqrt{u_i} = v_i \text{ for some, } i \in \{1, 2, \dots, n\}.$$

Also, since $\mu = \bigcap_{i=1}^m \xi_i$ with $\sqrt{\xi_i} = \eta_i$ for $i = 1, 2, \dots .m$ is another reduced fuzzy k -primary decomposition of μ , therefore there exists some $j \in \{1, 2 \dots .m\}$ such that $v_i = \sqrt{(\mu : \theta)} = \sqrt{\xi_j} = \eta_j$. This implies that

$$\{v_1, v_2 \dots .v_n\} \subseteq \{\eta_1, \eta_2, \dots, \eta_m\} \quad \text{and so } n \leq m.$$

By reversing the role of v_i and ξ_j , we get

$$\{\eta_1, \eta_2, \dots, \eta_m\} \subseteq \{v_1, v_2 \dots .v_n\} \quad \text{and } m \leq n.$$

Thus,

$$\{v_1, v_2 \dots .v_n\} = \{\eta_1, \eta_2, \dots, \eta_m\} \quad \text{and } n = m. \quad \blacksquare$$

4. Conclusions

All the results that hold good for ideals in rings may not be true for the ideals in semirings and not even for the k -ideals in semirings. But this fact was ignored by various mathematicians while generalising the Lasker-Noether's Theorem for semirings. This made Lescot [21] to establish weak primary decomposition for k -ideals in semirings in 2015. But the question of settling the primary decomposition for k -ideals still remained unsolved. In this paper, we prove:

Let I be a k -ideal of an additively cancellative, yoked and commutative Noetherian semiring R . Then I can be uniquely represented as a finite intersection of primary ideals of R , i.e. I has a unique reduced primary decomposition. The above said result is also generalised for fuzzy k -ideals of same class of semirings.

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No potential conflict of interest was reported by the author(s).

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