Dissipative Decomposition and Feedback Stabilization of Nonlinear Control Systems

by

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Dedicated to my late mother.

Abstract

This dissertation considers the problem of approximate dissipative potentials construction and their use in smooth feedback stabilization of nonlinear control systems. For mechanical systems, dissipative potentials, usually a generalized Hamiltonian function, can be derived from physical intuition. When a dissipative Hamiltonian is not available, one can rely on dissipative Hamiltonian realization techniques, as proposed recently by Cheng and coworkers. Extensive results are available in the literature for (robust) stabilization based on the obtained potential.

For systems of interest in chemical engineering, especially systems with mass action kinetics, energy is often ill-defined. Moreover, realization techniques are difficult to apply, due to the nonlinearities associated with the reaction terms. Approximate dissipative realization techniques have been considered by many researchers for analysis and feedback design of controllers in the context of chemical processes. The objective of this thesis is to study the construction of local dissipative potentials and their application to solve stabilization problems.

The present work employs the geometric stabilization approach proposed by Jurdjevic and Quinn, refined by Faubourg and Pomet, and by Malisoff and Mazenc, for the design of stabilizing feedback laws. This thesis seeks to extend and apply the Jurdjevic–Quinn stabilization method to nonlinear stabilization problems, assuming no *a priori* knowledge of a Lyapunov function.

A homotopy-based local decomposition method is first employed to study the dissipative

Hamiltonian realization problem, leading to the construction of locally defined dissipative potentials. If the obtained potential satisfies locally the weak Jurdjevic–Quinn conditions, it is then shown how to construct feedback controllers using that potential, and under what conditions a Lyapunov function can be constructed locally for time-independent control affine systems. The proposed technique is then used for the construction of state feedback regulators and for the stabilization of periodic orbits based on a construction proposed by Bacciotti and Mazzi. In the last chapter of the thesis, stabilization of time-dependent control affine systems is considered, and the main result is used for the stabilization of periodic solutions using asymptotic feedback tracking.

Low-dimensional examples are used throughout the thesis to illustrate the proposed techniques and results.

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Chapter 1

Introduction

This thesis studies the local construction of smooth damping stabilizers for continuous-time control affine nonlinear systems. The proposed approach decomposes locally the drift vector field to obtain a dissipative potential. Under some conditions, the obtained potential is used for feedback stabilization. The proposed problem is motivated by recent constructions that appeared in the literature for passivity-based control of mechanical systems (or more generally, for systems that can be expressed as dissipative Hamiltonian systems). Recent applications of this approach to chemical engineering systems, and in particular mass-balance systems, illustrated the fact that representation of non-mechanical systems in terms of generalized Hamiltonian systems is still a challenging task (Bao and Lee, 2007; Hangos et al., 2004). Moreover, representing a given nonlinear system in terms of a dissipative Hamiltonian system, even if it leads to strong results on stability, gives very little insight in practice for feedback stabilization when one needs to rely on an approximate Hamiltonian realization. In the present thesis, a locally-defined homotopy-based decomposition approach for approximate realization of (dissipative) Hamiltonian systems is proposed. The design of feedback stabilizing controllers based on this decomposition is then considered.

This introductory chapter is divided as follows. General background and motivating examples for the research presented here are summarized in Section 1.1. Specific research objectives are given in Section 1.2. A summary of the contributions of the research presented here is provided in Section 1.3. Finally, the thesis organization is outlined in Section 1.4.

1.1 Context and Motivations

The theory of generalized Hamiltonian systems is a central approach for stability studies and controller design for nonlinear control systems (van der Schaft, 2000) and several physical problems were studied using this special class of control systems. However, one limitation associated with the study of non-mechanical nonlinear systems using dissipative Hamiltonian systems is to derive a suitable Hamiltonian function for the problem (Hangos et al., 2004). As discussed in (Johnsen and Allgöwer, 2007) and (Ortega et al., 1999), applications of Interconnection and Damping Assignment Passivity-Based Control (IDA-PBC) techniques prove to be difficult in practice for process control applications where the concept of "energy" is ill-defined, for example when mass balances are considered. One example in the context of chemical engineering was given recently in Otero-Muras et al. (2008, Section 4.2.1) where the stability of a reaction network was studied using its dissipative Hamiltonian representation. They considered the reaction network

$$
\begin{aligned}
P_1 + P_2 &\underset{k_2}{\overset{k_1}{\rightleftharpoons}} P_3 \\
P_2 + P_3 &\underset{k_4}{\overset{k_3}{\rightleftharpoons}} 2P_3,
\end{aligned}
$$

where P_i are chemical complexes and k_i denote reaction rates. Carrying the mass balances, with $x_1(t)$, $x_2(t)$, and $x_3(t)$ denoting the concentration of chemical complexes P_1, P_2 and

P_3, respectively, one obtains

$$\begin{aligned}\dot{x}_1 &= -k_1x_1x_2 + k_2x_3 \\ \dot{x}_2 &= -k_1x_1x_2 + k_2x_3 - k_3x_2x_3 + k_4x_3^2 \\ \dot{x}_3 &= k_1x_1x_2 - k_2x_3 + k_3x_2x_3 - k_4x_3^2.\end{aligned}$$

For $x_i > 0$, it was shown, using a logarithmic state transformation based on thermodynamical arguments, that the system can be expressed, in new coordinates, as a dissipative Hamiltonian system of the form

$$\dot{z} = (J(z) - R(z))\nabla H(z) \tag{1.1}$$

where $J(z)$ is skew-symmetric, $R(z)$ is positive-definite, and $\nabla H(z)$ is the gradient of a Hamiltonian function. Moreover, stability of an equilibrium x^* was demonstrated, following the argument given, for example, in (Ortega et al., 2002). It should be noted that the transformation considered in (Otero-Muras et al., 2008) is typical of the transformations given in the literature for a wide variety of mass-action systems, in particular the Lotka–Volterra system considered throughout this thesis as an application example. Hamiltonian representation of Lotka–Volterra dynamics was presented for example in (Evans and Findley, 1999) and (Szederkényi and Hangos, 2004).

In some applications, computing a coordinate transformation can be difficult. An example is the HIV-1 nonlinear dynamics presented in (Chang and Astolfi, 2008),

$$\begin{aligned}
\dot{x} &= \lambda - dx - \eta\beta xy \\
\dot{y} &= \eta\beta xy - ay - p_1 z_1 y - p_2 z_2 y \\
\dot{z}_1 &= c_1 z_1 y - b_1 z_1 y - b_1 z_1 \\
\dot{w} &= c_2 xyw - c_2 qyw - b_2 w \\
\dot{z}_2 &= c_2 qyw - hz_2,
\end{aligned}$$

where x, y, z_1, w, z_2 denote the uninfected CD4 T-cell, the infected CD4 T-cell, the helper-independent Cytotoxic T Lymphocyte (CTL), the CTL precursor, and the helper-dependent CTL, respectively. The drug effect is denoted by η, with $\eta = 1 - \eta^* u$, and u the injected drug concentration.

In practice, designing a passivity-based controller for this class of systems could be of interest, especially if robustness properties, such as those developed in (van der Schaft, 2000), are desired. Clearly, the concept of an ”energy function” in the context of the last example has no physical meaning.

In (Cheng et al., 2005), it was shown that a nonlinear system of the form

$$\dot{x} = f(x) + g(x)u, \tag{1.2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $g(x)$ full rank, is transformable to a stable Port-Controlled Hamiltonian (PCH) system

$$\dot{x} = F(x)\nabla H(x), \quad F(x) = [J(x) - R(x)] \tag{1.3}$$

if there exists a feedback $\beta : \mathbb{R}^n \to \mathbb{R}^m$ such that the **matching equation**

$$f(x) + g(x)\beta(x) = F(x)\nabla H(x) \tag{1.4}$$

holds. In particular, for a fixed structure matrix $F(x)$ and a free Hamiltonian function $H(x)$, the problem leads to a set of Partial Differential Equations (PDEs) parameterized by the structure matrix and the feedback controller $\beta(x)$. The key advantage of this representation is that the closed-loop system inherits the "Universal stabilizing property of IDA-PBC" (Ortega et al., 2002). For Lotka–Volterra systems, it was shown in (Ortega et al., 1999) that solutions to the matching equations can be computed. However, for the HIV-1 control system, even if the reaction terms are represented by polynomials, the extra couplings between the species render the computation of the function $\beta(x)$ difficult. Moreover, the "Universal stabilizing property of IDA-PBC" relies on convexity properties at the equilibrium point, assumed to be the maximal invariant set, which can be of limited usage in applications such as drug infusion dynamics, where periodic behavior might be of interest from a therapeutical point of view.

A non-exact matching IDA-PBC approach was recently developed and applied to chemical reactor process stabilization (Ramírez et al., 2009). However, the authors showed subsequently in (Batlle et al., 2009) that their original derivations were wrong and that in the best scenario, one has to rely on a linearized stability argument at the desired equilibrium. The problem of approximate matching could then be solved using Linear Matrix Inequalities (LMIs).

Another way to look at the matching is to consider an approximate dissipative Hamiltonian realization instead of solving the matching equations. In (Cheng et al., 2000), conditions for approximate Hamiltonian realizations were given in terms of a normal form. Sufficient conditions and a constructive algorithm for a generalized Hamiltonian realization for time-invariant nonlinear systems were presented in (Wang et al., 2003). In particular, the method proposed in (Wang et al., 2003) relied on a vector field decomposition along the gradient direction $\nabla H(x)$ and the tangential direction on constant energy surfaces of $H(x)$, for a regular positive-definite function $H(x)$. Following the work in (Maschke et al., 2000) which related port-controlled Hamiltonian systems to the construction of Lyapunov functions, it was shown in (Wang et al., 2007) how k-th degree approximate dissipative Hamiltonian systems can be used to solve the realization problem and how associated k-th degree approximate Lyapunov functions can be used to study the stability of such systems. In Chapter 3, this approach to decomposition is reviewed and then, the approximate Hamiltonian decomposition problem is studied using a differential one-form associated with the drift vector field. In particular, the problem of computing a local dissipative potential is studied following the original contribution of Edelen (1973). Using a coordinate transformation between the exact part of the dynamics and a pre-defined Hamiltonian dissipative realization, viewed as a reference system, stability of the original system can be assessed.

Building on this result, the design of smooth stabilizing feedback controllers is considered in Chapters 4 and 5. In essence, the knowledge of an approximate dissipative function enables one to derive conditions for which this potential can be used to construct a damping feedback controller, as proposed originally by Jurdjevic and Quinn (1978), and to compute a Lyapunov function to show stability of the closed-loop dynamics. This approach to feedback stabilization was originally given by (Faubourg and Pomet, 2000) (see also Faubourg (2001)) for homogeneous systems. More recently, Malisoff and Mazenc (2009) presented numerous applications of this design method for different classes of nonlinear systems. A review of Jurdjevic–Quinn damping controllers is given in Section 2.4. Applications of this approach are presented in Chapter 4 for time-independent systems and in Chapter 5 for time-dependent nonlinear systems. Regulator design is considered in 4.1 by using the technique for an extended time-independent system. Asymptotic stabilization (Section 4.3) and asymptotic tracking (Section 5.2) of periodic solutions are also considered.

1.2 Problem Statement and Objectives

In this thesis, and as advocated recently in (Johnsen and Allgöwer, 2007) and (Favache and Dochain, 2009), it is desired to apply the theory of generalized Hamiltonian systems for the analysis of stability and for feedback stabilization of chemical process control problems. In particular, the representation of systems where the notion of "energy" is ill-defined from a physical point of view is of interest. At term, the ability to apply the theory of dissipative systems to the control of chemical processes could eventually lead to more robust control design, especially if one considers the results on control of dissipative and passive systems presented for example in (Willems, 1972), (Byrnes et al., 1991), (Sepulchre et al., 1997), and (van der Schaft, 2000).

However, the matching problem associated to IDA-PBC techniques are difficult to apply in practice, since it leads typically to the solution of nonlinear partial differential equations. The knowledge of a desired dissipative potential is required in practice, which is not a straightforward task, even for the relatively simple systems studied in (Hangos et al., 2004). On the other hand, approximate matching conditions, as presented in (Ramírez et al., 2009), might lead to erroneous stability conclusions (Batlle et al., 2009). Finally, one should note that, as given in (Ortega et al., 2002), the "Universal stabilization property" of IDA-PBC is limited to analysis around an isolated equilibrium, and does not admit extensions, in its actual form, to cycle stabilization.

The focus of the present thesis is to study stabilization problems, based on an approximate decomposition of the nonlinear system. The first objective of the thesis is to present a decomposition approach of the drift dynamics, and illustrate how the locally-defined potential can be used to study the representation of the drift dynamics as a dissipative Hamiltonian system. Then, the objective is to show that under some conditions, the obtained dissipative potential can be used as a basis for smooth damping feedback design for different classes of stabilization problems. Finally, the approach is extended to time-dependent feedback stabilization problems. In terms of potential applications, this last extension is desirable since, for the drug infusion dynamics example presented in Section 1.2, parameters might vary with respect to time, as drug tolerance develops over the course of treatment, or simply vary periodically over some fixed period.

To summarize, the three principal objectives of this dissertation are:

Objective 1 Construct a decomposition-based approach to approximate dissipative realization problems.

Objective 2 Derive conditions for closed-loop stability of the control-affine system in closed-loop with a damping state feedback controller using the obtained dissipative potential.

Objective 3 Extend the approach to the stabilization of time-dependent control affine systems.

1.3 Summary of Contributions

The contributions of this thesis are now summarized, with the original publication or submission reference noted, when applicable.

Contribution 1 Computation of approximate dissipative Hamiltonian realization using a homotopy-based decomposition (Chapter 3). Originally published in (Hudon et al., 2008).

Contribution 2 Construction of Lyapunov functions for time-independent control affine systems based on a dissipative potential (Section 4.1). Originally published in (Hudon and Guay, 2009b).

Contribution 3 Construction of feedback regulator using Jurdjevic–Quinn techniques in an extended space (Section 4.2). Submitted for publication in (Hudon and Guay, 2010d).

Contribution 4 Computation of first integral for stabilization of time-independent control-affine systems to a desired cycle (Section 4.3).

Contribution 5 Construction of time-dependent damping feedback law for the stabilization of control affine time-dependent systems (Section 5.1). Accepted for publication in (Hudon and Guay, 2010a).

Contribution 6 Construction of smooth tracking feedback controller to periodic trajectories (5.2). Submitted for publication in (Hudon and Guay, 2010c).

All contributions were written with the author as the principal investigator. Collaboration of Prof. Martin Guay for Contributions 2–6, and of K. Höffner and Prof. Martin Guay for Contribution 1 is acknowledged.

1.4 Thesis Organization

This dissertation is organized as follows. A review of stability, stabilizability and stabilization is presented in Chapter 2, following (Bacciotti, 1992), with elements from (Khalil, 2002), (Coron, 2007, Chapters 11 & 12), (Bullo and Lewis, 2005, Chapter 6), and (Nijmeijer and van der Schaft, 1990, Chapter 10).

The dissipative Hamiltonian realization problem is covered in Chapter 3. The approach that is proposed here relies on the construction of a radial homotopy operator, used in the context of feedback linearization by Banaszuk (1995) (see also Banaszuk and Hauser (1996)). Presentation of this operator follows the treatment from (Edelen, 2005, Chapter 5). Similar decomposition approaches (Sira-Ramírez and Angulo-Núñez, 1997) and the dissipative Hamiltonian realization technique from (Wang et al., 2007) are also reviewed.

Chapter 4 presents the main contribution of the present thesis, *i.e.*, the application of the above decomposition for the design of feedback damping controllers of Jurdjevic–Quinn type. The construction of Lyapunov functions for control-affine systems based on the dissipative potential obtained by application of the homotopy decomposition is given in Section 4.1. This result is used in Section 4.2 for the design of state feedback regulator by computing a potential in an extended space. Section 4.3 presents the stabilization of periodic orbits in $\mathbb{R}^n$, following the construction originally proposed by Bacciotti and Mazzi (1995). Potential extensions are discussed in Section 4.4.

Chapter 5 presents an extension of Jurdjevic–Quinn method for the stabilization of time-dependent control affine systems, based on the computation of an integrating factor for the non-exact one-form associated to the system (Section 5.1). This result is applied to periodic trajectory tracking in Section 5.2. Potential areas for future investigation are illustrated in Section 5.3.

Conclusions and some future areas of studies are discussed in Chapter 6.

Throughout Chapters 3, 4, and 5, low-dimensional examples are presented to illustrate the application of the proposed constructions. In particular, a two dimensional Lotka–Volterra system is used as an illustration of the motivating examples discussed in Section 1.2. The application of some elements presented in the present thesis were considered for higher dimensional systems and systems with non-polynomial nonlinearities. An example is the feedback stabilization of a wastewater plant of dimension $n = 4$ with Monod and Haldane kinetics given in (Hudon and Guay, 2010e). The results presented here can be extended in that sense, with additional computational burden, which is expected for the particular choice of feedback controller design technique favored here.

Chapter 2

Review of State Feedback Stabilization

This chapter reviews elements of nonlinear control systems stability and stabilization in $\mathbb{R}^n$ to be used in the sequel. The presentation follows the presentation given by Bacciotti (1992) and by Coron (2007, Chapters 11 and 12), with some elements adapted from (Khalil, 2002), (Nijmeijer and van der Schaft, 1990, Chapter 10), and (Bullo and Lewis, 2005, Chapter 6 and Section 10.1).

Generalities on nonlinear control systems and control problems considered in this thesis are given in Section 2.1. Definitions and classical results related to stability, time-dependent stability, and orbital stability are reviewed in Section 2.2. In Section 2.3, elements related to the problem of state feedback stabilization, including necessary conditions for stabilization are summarized. In Section 2.4, damping feedback stabilization, that is used in the present thesis, is reviewed.

2.1 Introduction

Stabilization problems are central in control theory and control applications. A review of the main problems and contributions to the development of modern nonlinear stabilization techniques were presented in (Bacciotti, 1992). An interesting expository review was also given by Kokotović and Arcak (2001).
In this section, the class of systems and stabilization problems covered in the present thesis are presented. Consider a time-independent control system

$$S: \quad \dot{x} = F(x,u), \quad x \in \mathcal{X} \tag{2.1}$$

where $\mathcal{X}$ is an open connected region of $\mathbb{R}^n$. In the present thesis, the controls u are taken as elements in $\mathbb{R}^m$. Let the equilibrium of S be denoted by $(x^*, u^*) \in \mathcal{X} \times \mathbb{R}^m$.
The general problem of state feedback stabilizability is concerned with finding conditions for the existence of a state feedback control $u = u(x)$ defined in a neighborhood of x^* such that the closed loop system

$$\dot{x} = F(x, u(x)) = F_{\mathrm{CL}}(x) \tag{2.2}$$

has a stable equilibrium position at $x = x^*$. The function $u(x)$ is called a **static stabilizing feedback law or a stabilizer**. More generally, S is said to be **dynamically stabilizable** at x_0 if there exists an integer ν, a point $\xi_0 \in \mathbb{R}^\nu$ and a function $\phi : \mathcal{X} \times \mathbb{R}^\nu \to \mathbb{R}^\nu$ such that

$$\dot{x} = F(x,u) \tag{2.3}$$

$$\dot{\xi} = \phi(x,\xi) \tag{2.4}$$

is stabilizable at (x^*, ξ^*) by means of a feedback of the form $u = u(x, \xi)$. The system $\dot{\xi} = \phi(x,\xi)$ is called a **compensator** for S. The compensator together with the feedback

$u = u(x, \xi)$ is called a **dynamic stabilizer**.

In particular, the present thesis is, except for some discussions on potential generalizations, concerned with control affine systems

$$\dot{x} = X_0(x) + \sum_{i=1}^{m} u_i(x) X_i(x) \tag{2.5}$$

where $X_0, X_1, \ldots, X_m$ are smooth vector fields on $\mathbb{R}^n$. In the sequel, X_0 is called the drift vector field. The design of stabilizing static feedback law is considered in Section 4.1. An extension for stabilizer design for time-dependent control affine systems of the form

$$\dot{x}(t) = X_0(t, x) + \sum_{i=1}^{m} u_i(t, x) X_i(t, x) \tag{2.6}$$

is presented in Section 5.1. The compensator design problem is considered in Section 4.2. In the sequel, the Lie derivative of X_j along X_i is denoted by

$$\mathcal{L}_{X_i} X_j(x) := \left(\frac{\partial X_j}{\partial x} X_i \right)(x). \tag{2.7}$$

The Lie bracket $[X_i,\ X_j](x)$ is given in coordinates as

$$[X_i,\ X_j](x) = \frac{\partial X_j}{\partial x} X_i(x) - \frac{\partial X_i}{\partial x} X_j(x). \tag{2.8}$$

Finally, $\mathrm{ad}^k_{X_0} X_i \in \mathcal{C}^\infty$ is defined by induction for $k \in \mathbb{N}$ as

$$\mathrm{ad}^0_{X_i} X_j = X_j \tag{2.9}$$

$$\mathrm{ad}^k_{X_i} X_j = [X_i, \mathrm{ad}^{k-1}_{X_i} X_j]. \tag{2.10}$$

2.2 Stability

This section reviews elements of stability for dynamical systems of the form

$$\dot{x} = X_0(x), \quad x(0) = x_0, \tag{2.11}$$

with $x \in \mathbb{R}^n$. Lyapunov stability is reviewed in Section 2.2.1. Lasalle's Invariance Principle is reviewed in Section 2.2.2. Section 2.2.3 gives a review of stability of time-varying systems of the form

$$\dot{x} = X_0(t, x), \tag{2.12}$$

where $x \in \mathbb{R}^n$, $t \in I \subset \mathbb{R}_+$, and with initial condition $x(0) = x_0$, equilibrium $x^*(t, 0)$. Lasalle's Invariance Principle for time-varying systems is also given, following the discussion in (Sastry, 1999). Finally, the notion of orbital stability is reviewed in Section 2.2.4 following the presentation of (Bacciotti and Mazzi, 1995).

2.2.1 Lyapunov Stability

This section reviews elements of Lyapunov stability theory for systems of the form (2.11). First consider the following definition from (Khalil, 2002).

Definition 2.2.1 (Lyapunov Stability). *The equilibrium point $x^* = 0$ of* (2.11) *is*

- ***stable*** *if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that*

$$\|x(0)\| < \delta \quad \Rightarrow \quad \|x(t)\| < \epsilon, \quad \forall t \geq 0. \tag{2.13}$$

- ***unstable*** *if it is not stable.*

- ***asymptotically stable*** *if it is stable and* δ *can be chosen such that*

$$\|x(0)\| < \delta \quad \Rightarrow \quad \lim_{t\to\infty} x(t) = 0. \tag{2.14}$$

Following (Khalil, 2002), let $V : \mathcal{O} \to \mathbb{R}$ be a continuously differentiable function defined in a domain $\mathcal{O} \subset \mathbb{R}^n$ that contains the origin. The derivative of V along the trajectories of (2.11), denoted $\dot{V}(x)$, is given by

$$\dot{V}(x) \quad = \quad \mathcal{L}_{X_0} V. \tag{2.15}$$

The following theorem is proved in (Khalil, 2002, Chapter 4).

Theorem 2.2.2 (Lyapunov Stability Theorem)**.** *Let* $x^* = 0$ *be an equilibrium point of* (2.11) *and* $\mathcal{O} \subset \mathbb{R}^n$ *be a domain containing the origin. Let* $V : \mathcal{O} \to \mathbb{R}$ *be a continuously differentiable function such that*

- $V(0) = 0$*,*
- $V(x) > 0$ *for* $x \in \mathcal{O} \setminus \{0\}$*,*
- $\dot{V} \leq 0$ *for* $x \in \mathcal{O}$*.*

Then $x^* = 0$ *is stable. Moreover if*

- $\dot{V}(x) < 0$ *for* $x \in \mathcal{O} \setminus \{0\}$*,*

then $x^* = 0$ *is asymptotically stable.*

In the sequel, a function V such that the conditions of the above theorem hold will be called a Lyapunov function.

2.2.2 Lasalle's Invariance Principle

The discussion on Lasalle's Invariance Principle follows (Khalil, 2002, section 4.2). Let $x(t)$ denote a solution of (2.11). A point p is said to be a **positive limit point** of $x(t)$ if there is a sequence $\{t_n\}$, with $t_n \to \infty$ as $n \to \infty$ such that $x(t_n) \to p$ as $n \to \infty$. The set of all limit points of $x(t)$ is called the **positive limit set** of $x(t)$.

A set M is said to be an **invariant set** with respect to (2.11) if

$$x(0) \in M \quad \Rightarrow \quad x(t) \in M, \quad \forall t \in \mathbb{R}.$$

A set M is said to be a **positively invariant set** if

$$x(0) \in M \quad \Rightarrow \quad x(t) \in M, \quad \forall t \geq 0.$$

A solution of (2.11) is said to approach a set M as t approaches infinity, if for $\epsilon > 0$, there is a $T > 0$ such that

$$\text{dist}(x(t), M) < \epsilon, \quad \forall t > T, \tag{2.16}$$

where $\text{dist}(p, M)$ denotes the smallest distance from a point p to any point in the set M, *i.e.*,

$$\text{dist}(p, M) = \inf_{x \in M} \|p - x\|. \tag{2.17}$$

First consider the following property of limit sets, proved in (Khalil, 2002).

Lemma 2.2.3. *If a solution $x(t)$ of (2.11) is bounded and belongs to $\mathcal{O}$ for $t \geq 0$, then its positive limit set L^+ is a nonempty, compact, invariant set. Moreover, $x(t)$ approaches L^+ as $t \to \infty$.*

The following statement of the Lasalle's Invariance Principle is given in (Khalil, 2002).

Theorem 2.2.4. *Let $\Gamma \subset \mathcal{O}$ be a compact set that is positively invariant with respect to* (2.11). *Let $V : \mathcal{O} \to \mathbb{R}$ be a continuously differentiable function such that $\dot{V}(x) \leq 0$ in Γ. Let E be the set of all points in Γ where $\dot{V} = 0$. Let M be the largest invariant set in E. Then every solution starting in Γ approaches M as $t \to \infty$.*

The following corollary to Lasalle's Invariance Principle is used in Chapters 4 and 5.

Corollary 2.2.5 (Barbashin and Krasovskiĭ)**.** *Let $x^* = 0$ be an equilibrium for* (2.11)*. Let $V : \mathcal{O} \to \mathbb{R}$ be a continuously differentiable function on a domain $\mathcal{O}$ containing the origin, such that $\dot{V}(x) \leq 0$ in $\mathcal{O}$. Let $S = \{x \in \mathcal{O} | \dot{V}(x) = 0\}$ and suppose that no solution can stay identically in S, other than the trivial solution $x(t) \equiv 0$. Then, the origin is asymptotically stable.*

2.2.3 Stability of Time-Varying Systems

This section presents elements of stability theory for time-dependent (also referred to time-varying systems). Consider the following time-dependent system

$$\dot{x}(t) = X_0(t, x), \tag{2.18}$$

where $X_0(t, x)$ is assumed to be smooth in x, continuous and bounded over bounded intervals I in $\mathbb{R}_+$. Assume that $X_0(t, 0) = 0$. The following definitions of stability for time-dependent systems are taken from (Khalil, 2002, Section 4.5).

Definition 2.2.6. *The equilibrium point $x^* = 0$ of* (2.18) *is*

- ***stable*** *if, for each $\epsilon > 0$, there is $\delta(\epsilon, t_0) > 0$ such that*

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall t \geq t_0 \geq 0;$$

- ***uniformly stable*** *if, for each* $\epsilon > 0$*, there is* $\delta(\epsilon) > 0$*, independent of* t_0 *such that*

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall t \geq t_0 \geq 0$$

  is satisfied;

- ***asymptotically stable*** *if it is stable and there is a* $c = c(t_0) > 0$ *such that* $x(t) \to 0$ *as* $t \to \infty$*, for all* $\|x(t_0)\| < c$*;*

- ***uniformly asymptotically stable*** *if it is uniformly stable and there is a* $c > 0$*, independent of* t_0*, such that for all* $\|x(t_0)\| < c$*,* $x(t) \to 0$ *as* $t \to \infty$*, uniformly in* t_0*; that is for each* $\epsilon > 0$*, there is* $T(\epsilon) > 0$ *such that*

$$\|x(t)\| < \epsilon, \quad \forall t \geq t_0 + T(\epsilon), \ \forall \|x(t_0)\| < c;$$

- ***globally uniformly asymptotically stable*** *if it is uniformly stable and, for each pair of positive numbers* ϵ *and* c*, there is* $T(\epsilon, c) > 0$ *such that*

$$\|x(t)\| < \epsilon, \ \forall t \geq t_0 + T(\epsilon, c), \ \forall \|x(t_0)\| < c.$$

The next theorems, proved in (Khalil, 2002), summarizes the extension of Lyapunov Stability Theorem to the time-dependent case.

Theorem 2.2.7. *Let* $x^* = 0$ *be an equilibrium point for* (2.18) *and* $\mathcal{O} \subset \mathbb{R}^n$ *be a domain containing the origin. Let* $V : [0, \infty) \times \mathcal{O} \to \mathbb{R}$ *be a continuously differentiable function such that*

$$W_1(x) \leq V(t, x) \leq W_2(x) \tag{2.19}$$

$$\dot{V}(t, x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} X_0(t, x) \leq 0 \tag{2.20}$$

for $t \geq 0$ and for all $x \in \mathcal{O}$, where $W_1(x)$ and $W_2(x)$ are continuous positive functions on $\mathcal{O}$. Then $x^* = 0$ is uniformly stable.

Theorem 2.2.8. *Suppose the assumptions of theorem 2.2.7 are satisfied with inequality (2.20) strengthened to*

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} X_0(t, x) \leq -W_3(x) \tag{2.21}$$

for $t \geq 0$ and for all $x \in \mathcal{O}$, where $W_3(x)$ is a continuous positive definite function on $\mathcal{O}$. Then $x^* = 0$ is uniformly asymptotically stable.

The following presents Lasalle's Invariance Principle for periodic time-dependent system (2.18), following the exposition in (Sastry, 1999).

Theorem 2.2.9 (Lasalle's Invariance Principle for Periodic Systems)**.** *Assume that the system $\dot{x} = X_0(t, x)$, with $x(0) = x_0$ is periodic, i.e.,*

$$X_0(t, x) = f(t + T, x), \quad \forall t \in \mathbb{R}_+, \ \forall x \in \mathbb{R}^n. \tag{2.22}$$

Further, let $V(t, x)$ be a positive definite function which is periodic in t also with period T. Define

$$S = \{x \in \mathbb{R}^n | \dot{V}(t, x) = 0, \quad \forall t \geq 0\}. \tag{2.23}$$

Then if $\dot{V}(t, x) \leq 0$, for all $t \geq 0$, $\forall x \in \mathcal{O}$ and the largest invariant set in S is the origin, then the origin is uniformly stable.

2.2.4 Orbital Stability

Stabilization of periodic orbits is considered in Section (4.3), based on a construction proposed in (Bacciotti and Mazzi, 1995). To complete the discussion, orbital stability is considered here. The following definition of orbital stability is adapted from (Bacciotti and Mazzi, 1995).

Definition 2.2.10. *Let $x(t)$ denote a solution of* (2.11). *Let $M \subset \mathbb{R}^n$ be a compact nonempty set. M is said to be:*

- ***orbitally attractive*** *with respect to* (2.11) *if there exists a neighborhood $\mathcal{O}_0$ of M such that for each $x \in U_0$,*

$$\lim_{t \to \infty} \operatorname{dist}(x(t), M) = 0.$$

- ***orbitally stable*** *with respect to* (2.11) *if, for each neighborhood $\mathcal{O}_0$ of M, there exists a neighborhood $\mathcal{O}$ of M such that for each $t \geq 0$,*

$$x(t) \in \mathcal{O}_0.$$

- ***orbitally asymptotically attractive*** *if it is orbitally stable and orbitally attractive.*

2.3 Feedback Stabilization

This section presents basic definitions on the topic of feedback stabilization. In Section 2.3.1, the concepts of closed-loop stability are reviewed while Section 2.3.2 reviews some results on obstructions to smooth feedback stabilization.

2.3.1 Closed-Loop Stability

Consider the C^∞ control affine system

$$\dot{x} = X_0(x) + \sum_{i=1}^{m} u_i X_i(x) \tag{2.24}$$

The main elements of feedback and stabilization are given in the following definition from (Bullo and Lewis, 2005), adapted to the context of this thesis, *i.e.*, for stabilization on $\mathcal{X} \subset \mathbb{R}^n$ with controls on $\mathcal{U} \subset \mathbb{R}^m$.

Definition 2.3.1 (Feedback and Stabilization)**.** *(i) A **controlled equilibrium point** for* (2.24) *is a pair $(x^*, u^*) \in \mathcal{X} \times \mathcal{U}$ with the property that*

$$X_0(x^*) + \sum_{i=1}^{m} u_i^* X_i(x^*) = 0. \tag{2.25}$$

*(ii) A **state feedback** (resp. **time-dependent state feedback**) for* (2.24) *is a map $u : \mathcal{X} \to \mathcal{U}$ (resp. $u : \bar{\mathbb{R}}_+ \times \mathcal{X} \to \mathcal{U}$).*

*(iii) Given a **state feedback** (resp. **time-dependent state feedback**) u for* (2.24)*, the closed-loop system, is the vector field (resp. time-dependent vector field) defined by*

$$x \mapsto X_0(x) + \sum_{i=1}^{m} u_i(x) X_i(x) \quad \left(\textit{resp. } (t,x) \mapsto X_0(x) + \sum_{i=1}^{m} u_i(t,x) X_i(x) \right). \tag{2.26}$$

(iv) For $r \in \mathbb{Z}_+ \cup \{\infty\} \cup \{\omega\}$, a state feedback (resp. time-dependent state feedback) is C^r if the corresponding closed-loop system is of class C^r.

(v) For $r \in \mathbb{Z}_+ \cup \{\infty\} \cup \{\omega\}$ and $x^* \in \mathbb{R}^n$, a state feedback is almost C^r about x^* if there exists a neighborhood $\mathcal{O}$ of x^* such that the corresponding closed-loop system is C^r on $\mathcal{O} \setminus \{x^*\}$.

(vi) A controlled equilibrium point (x^*, u^*) is ***stabilizable by state feedback*** *(resp.* ***stabilizable by time-dependent state feedback****) if there exists a state feedback (resp. time-dependant state feedback) u for* (2.24) *with the property that the closed-loop system has x^* as a a stable equilibrium point.*

(vii) A controlled equilibrium point (x^*, u^*) is ***locally asymptotically stabilizable*** *by state feedback (resp. by time-dependent state feedback) if there exists a state feedback (resp. time-dependent state feedback) and a neighborhood $\mathcal{O}$ of x^* with the properties that*

- *(a) the closed-loop system leaves $\mathcal{O}$ invariant, and*
- *(b) the restriction of the closed-loop system to $\mathcal{O}$ possesses x^* as an asymptotically stable equilibrium point.*

(viii) A controlled equilibrium point (x^*, u^*) is globally asymptotically stabilizable by state feedback (resp. by time-dependent state feedback) if in part (vii) one can take $\mathcal{O} = \mathbb{R}^n$.

An important concept for feedback stabilization is the concept of control Lyapunov function.

Definition 2.3.2 (Control Lyapunov Function)**.** *Consider the $\mathcal{C}^\infty$ control affine system for which $(x^*, 0)$ is a controlled equilibrium point.*

- *A control Lyapunov triple for* (2.24) *at x^* is a triple $(V, \phi, \mathcal{O})$, where*
  - *(a) $\mathcal{O}$ is a neighborhood of x^*,*
  - *(b) $V : \mathcal{O} \to \bar{\mathbb{R}}_+$ is continuous, proper, and locally positive-definite about x^*,*
  - *(c) $\phi : \mathcal{O} \to \bar{\mathbb{R}}_+$ is continuous and positive-definite about x^*,*
  - *(d) for each compact subset $K \subset \mathcal{O}$, there exists a compact subset $\mathcal{U} \in \mathbb{R}^m$ such that, for all $x \in K$, there exists $u(x) \in \mathcal{U}$ such that*

$$dV(x) \cdot \left(X_0(x) + \sum_{i=1}^{m} u_i X_i(x) \right) \leq -\phi(x). \tag{2.27}$$

- *A control Lyapunov function for* (2.24) *at* x^* *is a continuous function* $V : M \to \mathbb{R}$ *for which there exists a neighborhood* $\mathcal{O}$ *and a continuous function* $\phi : \mathcal{O} \to \bar{\mathbb{R}}_+$ *for which* $(\psi|_{\mathcal{O}}, \phi, \mathcal{O})$ *is a control Lyapunov triple.*

Theorem 2.3.3 (Artstein's Theorem)**.** *For a controlled equilibrium point* $(x^*, 0)$ *for the* $\mathcal{C}^\infty$ *control affine system* (2.24)*, the following statements are equivalent:*

- *(i)* x^* *is locally asymptotically stabilizable using almost* $\mathcal{C}^\infty$ *state feedback;*
- *(ii) there exists a* $\mathcal{C}^1$ *control Lyapunov function for* (2.24) *at* x^**.*

The stabilization problem for time-varying systems

$$\dot{x}(t) = X_0(t, x) + \sum_{i=1}^{m} u_i X_i(t, x) \tag{2.28}$$

can be summarized in the following definition from Moulay and Perruquetti (2005).

Definition 2.3.4. *System* (2.28) *is almost stabilizable (resp. almost* $\mathcal{C}^k$*-stabilizable) if there exists a feedback control law* $u : \mathbb{R} \times \mathcal{X} \to U$ *continuous (resp.* $\mathcal{C}^k$*) on* $I \times \mathcal{X} \setminus \{0\}$ *such that*

(i) $u(t, 0) = 0$ *for all* $t \in I$*,*

(ii) the origin is a uniformly asymptotically stable equilibrium of the closed-loop system.

Moreover, if u *is continuous (resp.* $\mathcal{C}^k$*) on* $\mathbb{R} \times \mathcal{X}$*, then the system* (2.28) *is stabilizable (resp.* $\mathcal{C}^k$*-stabilizable). If the system* (2.28) *is globally defined, it is globally stabilizable if there exists a continuous control law* $u : I \times \mathbb{R}^n \to U$ *satisfying the two previous conditions for all* $I = \mathbb{R}_+$ *and* $\mathcal{X} = \mathbb{R}^n$*.*

Finally, orbital stabilization of a periodic orbit can be defined following (Bacciotti and Mazzi, 1995):

Definition 2.3.5. *Suppose that* Γ *is an isolated periodic orbit of the unforced dynamics* $\dot{x} = X_0$*. Let the state space* $\mathcal{X}$ *be a neighborhood of* Γ*. A feedback control law of the* *form $u = \alpha(x)$ is said to locally asymptotically orbitally stabilizes the nonlinear system $\dot{x} = F(x,u)$ if Γ is also a periodic orbit of the closed-loop system*

$$\dot{x} = F(x, \alpha(x)), \tag{2.29}$$

and is orbitally asymptotically stable.

2.3.2 Obstructions to Stationary Feedback Stabilization

The important result of Brockett (Brockett, 1983) is reviewed here. First, local asymptotic controllability is defined.

Definition 2.3.6. *A control affine system* (2.24) *is **locally asymptotically controllable** to $x^* \in \mathbb{R}^n$ if there exists a neighborhood $\mathcal{O}$ of x^* with the property that, for each $x \in \mathcal{O}$, there exists a map $u : \bar{\mathbb{R}}_+ \to \mathbb{R}^m$ for which the solution to the initial value problem*

$$\gamma\prime(t) = X_0(\gamma(t)) + \sum_{i=1}^{m} u_i(t,x) X_i(\gamma(t)), \quad \gamma(0) = x, \tag{2.30}$$

has the property that $\lim_{t\to\infty} \gamma(t) = x^*$.

The following expression of Brockett's necessary condition is taken from (Sontag, 1998).

Theorem 2.3.7 (Brockett's Necessary Condition)**.** *Assume that the $\mathcal{C}^1$ continuous time system*

$$\dot{x} = X(x,u) \tag{2.31}$$

is locally $\mathcal{C}^1$ stabilizable with respect to x^*. Then the image of the map

$$f : \mathcal{X} \times \mathcal{U} \to \mathbb{R}^n \tag{2.32}$$

contains some neighborhood of x^*.

Theorem 2.3.8. *Let $(x^*, 0)$ be a controlled equilibrium point for the* (2.24)*. Then $(x^*, 0)$ is locally asymptotically stabilizable by state feedback if and only if* (2.24) *is locally asymptotically controllable at x_0.*

This last theorem is illustrated with the following classical example. Consider $\mathcal{X} = \mathbb{R}^3$ and $\mathcal{U} = \mathbb{R}^2$ with the dynamics given by

$$\dot{x}_1 = u_1 \tag{2.33}$$

$$\dot{x}_2 = u_2 \tag{2.34}$$

$$\dot{x}_3 = x_2 u_1 - x_1 u_2 \tag{2.35}$$

and equilibrium state at the origin. No point of the form

$$\begin{bmatrix} 0 & 0 & \epsilon \end{bmatrix}, \quad (\epsilon \neq 0) \tag{2.36}$$

is in the image of f, so there is no $\mathcal{C}^1$ feedback stabilizing the system even if it is controllable. A related result in the context of dynamic feedback law design was presented in (Pomet, 1992).

Theorem 2.3.9. *Consider a control system of the form*

$$\dot{x} = \sum_{k=1}^{m} u_k X_k, \;\; x \in \mathbb{R}^n, \;\; u_k \in \mathbb{R}. \tag{2.37}$$

In $m < n$ and

$$\operatorname{rank}\{X_1(0), \ldots, X_m(0)\} = m \tag{2.38}$$

then there exists no continuous feedback law

$$u_1(x), \ldots, x_m(x) \tag{2.39}$$

making the origin a locally asymptotically stable equilibrium point of the closed-loop system. There does not exist either any continuous dynamic feedback law $u_1(x,\xi),\ldots,x_m(x,\xi)$*, and*

$$\dot{\xi} = g(x,\xi), \tag{2.40}$$

such that $(x,\xi) = 0 \in \mathbb{R}^{2n}$ *is a locally asymptotically stable equilibrium point of the closed-loop system.*

2.4 Damping Feedback Stabilization

This section reviews some elements of the damping feedback stabilization, that will be used throughout the thesis. The discussion here follows (Nijmeijer and van der Schaft, 1990, Chapter 10). The approach is also presented in (Coron, 2007, Section 12.2), in (Malisoff and Mazenc, 2009, Chapter 4), and in (Bacciotti, 1992, Section 10).

Given a control affine system

$$\dot{x} = X_0 + \sum_{k=1}^{m} X_k u_k, \tag{2.41}$$

with X_i smooth, $i = 0,\ldots,k$, with $x \in \mathcal{X} \subset \mathbb{R}^n$, $u \in \mathcal{U} \subset \mathbb{R}^m$, an equilibrium point x^* for X_0 and a Lyapunov function $V : \mathcal{X} \to \mathbb{R}$ for X_0 at x^*, define the functions $u_{k,\mathrm{diss}} : \mathbb{R}^n \to \mathbb{R}$ by

$$u_{k,\mathrm{diss}}(x) = -(\mathcal{L}_{X_k}V)(x), \quad k \in \{1,\ldots,m\}. \tag{2.42}$$

The control functions $(u_{1,\mathrm{diss}},\ldots,u_{m,\mathrm{diss}})$ are called **dissipative feedbacks**. An immediate computation shows that

$$\mathcal{L}_{(X_0+\sum_{k=1}^{m} u_{k,\mathrm{diss}}\cdot X_k)}V(x) = \mathcal{L}_{X_0}V(x) - \sum_{k=1}^{m}((\mathcal{L}_{X_k}V)(x))^2 \leq 0, \tag{2.43}$$

for all x in the neighborhood of x^*, since with V a Lyapunov function, $(\mathcal{L}_{X_0}V)(x)$ is locally negative semi-definite. Sufficient conditions for stabilization are given by the following result.

Theorem 2.4.1 (Dissipative Control)**.** *Let the control affine system* (2.41) *be a* $\mathcal{C}^\infty$ *control affine system. Let* x^* *be an equilibrium point for* X_0*, and let* $V : \mathbb{R}^n \to \mathbb{R}$ *be a class* $\mathcal{C}^\infty$ *Lyapunov function for* X_0 *at* x^* *for which* x^* *is an isolated local minimum. Also define*

$$\mathcal{W} = \{x \in \mathcal{O} | \mathcal{L}_{X_0}V(x) = 0, \mathcal{L}_{X_k}V(x) = 0,\ k = 1, \ldots, m\}, \tag{2.44}$$

and suppose that $\{x^*\}$ *is the largest* X_0*-invariant subset of* $\mathcal{W}$*. Then the equilibrium point* x^* *is locally asymptotically stable for the closed-loop control system with dissipative feedback control*

$$u_{k,\mathrm{diss}}(x) = -(\mathcal{L}_{X_k}V)(x), \quad k \in \{1, \ldots, m\}. \tag{2.45}$$

Proof: Because the Lie derivative of V along the closed-loop system is

$$\mathcal{L}_{X_0}V(x) - \sum_{k=1}^{m}((\mathcal{L}_{X_k}\psi)(x))^2 \leq 0, \tag{2.46}$$

then V is a Lyapunov function for the closed-loop system at x^*. Note that, since $\mathcal{L}_{X_k}V(x) = 0$ for all $x \in \mathcal{W}$ and $k \in \{1, \ldots, m\}$, any subset of $\mathcal{W}$ that is invariant under the closed-loop system will also be invariant under X_0. The largest subset of $\mathcal{W}$ invariant under the closed-loop system is contained in the largest subset of $\mathcal{W}$ invariant under X_0. From this observation, the result follows from Lasalle's invariance principle. ■

As mentioned in (Nijmeijer and van der Schaft, 1990), the condition that the largest X_0 invariant subset of $\mathcal{W}$ is often impractical to verify for the last theorem. Therefore, one would like to have stronger, but checkable, conditions from which this hypothesis follows. One such condition was given by the following result, given originally by (Lee and Arapostathis, 1988).

Lemma 2.4.2. *Let* (2.41)*,* x^**,* V*,* $\mathcal{O}$*, and* $\mathcal{W}$ *be as in the last theorem. If*

$$\text{span}\{X_0, \text{ad}^k_{X_0}X_i | i \in \{1, \dots, m\}, k \in \mathbb{Z}_+\} = \mathbb{R}^n, \tag{2.47}$$

Then $\{x^*\}$ *is the largest* X_0*-invariant subset of* $\mathcal{W}$*.*

Proof: Note that the hypotheses ensures that

$$\text{span}\{\text{ad}^k_{X_0}X_i | i \in \{1, \dots, m\}, k \in \mathbb{Z}_+\} = \mathbb{R}^n \tag{2.48}$$

for x in a neighborhood of x^*. Since V is a Lyapunov function for X_0, it follows that, if $x \in \mathcal{W}$, then x is a maximum for $\mathcal{X}_l V$. Therefore, $d(\mathcal{L}_{X_0}V)(x) = 0$ for $x \in \mathcal{W}$. Let γ be the integral curve of X_0 through $x \in \mathcal{W}$. Note that $\gamma(t) \in \mathcal{W}$ for all t, since V is a Lyapunov function for X_0 and since $V^{-1}(0) \subset \mathcal{W}$. For $i \in \{1, \dots, m\}$, define $V_i(t) = \mathcal{L}_{X_i}V(\gamma(t))$, noting that $V_i(t)$ is necessarily zero for all t. Therefore,

$$\frac{d^k}{dt^k}|_{t=0}V_i(t) = \mathcal{L}^k_{X_0}\mathcal{L}_{X_i}V(x) = 0, \tag{2.49}$$

for $k \in \mathbb{Z}_+$. Since $d(\mathcal{L}_{X_0}V)(x) = 0$, it follows that $\mathcal{L}^k_{X_0}\mathcal{L}_{X_i}V(x) = \mathcal{L}_{\text{ad}^k_{X_0}X_i}V(x)$ $k \in \mathbb{Z}_+$. By hypothesis, the only way that $\mathcal{L}_{\text{ad}^k_{X_0}X_a}\psi(x) = 0$, $k \in \mathbb{Z}_+$, $a \in \{1, \dots, m\}$, is that $dV(x) = 0 \in \mathcal{U}$ implies that $x = x^*$. ∎

In particular, the smooth feedback laws

$$u_k(x) = -(\nabla^T V \cdot X_k)(x), \tag{2.50}$$

$k = 1, \dots, m$, locally stabilizes the system to the invariant x^*.

The knowledge of a Lyapunov function $V(x)$ to design the damping feedback can be relaxed, as presented in (Malisoff and Mazenc, 2009, Section 2.2.3), where the (global) stabilization problem is considered using weak Jurdjevic–Quinn conditions, defined in the following.

Definition 2.4.3. *The control affine system* (2.41) *is said to satisfy the (weak) Jurdjevic-Quinn conditions provided there exists a smooth function* $\psi : \mathbb{R}^n \to \mathbb{R}$ *satisfying:*

(i) ψ *is positive definite and radially unbounded;*

(ii) for all $x \in \mathbb{R}^n$, $\mathcal{L}_{X_0}\psi(x) \leq 0$*;*

(iii) there exists an integer l *such that the set*

$$W(\psi) = \{x \in \mathbb{R}^n | \mathcal{L}_{X_0}\psi(x) = \mathcal{L}_{ad^i_{X_0} X_k}\psi(x) = 0, k = 1, \ldots, m, i \in \{0, 1, \ldots, l\}\} \quad (2.51)$$

is $\{0\}$*.*

If (2.41) satisfies the weak Jurdjevic-Quinn conditions, then it is globally asymptotically stabilized by any feedback

$$u = -\xi(x)(\mathcal{L}_{X_k}\psi)(x), \quad (2.52)$$

where $\xi(x)$ is any everywhere positive function of class $\mathcal{C}^1$. This result will be used in the sequel, dropping the radially unbounded assumption, hence limiting the analysis to local stabilization using damping feedback controls.

Before considering some stabilization problems in Chapters 4 and 5, the next chapter considers the problem of computing a dissipative function for nonlinear systems, where the problem is first considered from the point of view of dissipative Hamiltonian realization.

Chapter 3

Decomposition and Dissipative Realization

This chapter considers the problem of deriving a generalized Hamiltonian potential for autonomous dynamical systems. For a given vector field, the objective is to construct a locally defined dissipative generating function for the system. The proposed approach consists in studying the deviation of the given vector field from a canonically defined Hamiltonian vector field. First, a one-form is obtained by taking the interior product of an arbitrary non-vanishing two-form with respect to the vector field. A radial homotopy operator is then constructed on a star-shaped region to decompose the system into an exact part and an anti-exact one. A coordinate transformation between the exact part and an exact one-form generated from a known dissipative Hamiltonian system is used to compute a locally-defined dissipative potential for the original system. Examples are presented to illustrate the proposed method.

After a brief introduction to the problem considered in this chapter, dissipation-based decomposition and control is reviewed in Section 3.2, with an emphasis on dissipative Hamiltonian realization. In Section 3.3, the locally defined radial homotopy operator is introduced, following the presentation in (Edelen, 2005). In the context of feedback linearization, this operator was introduced in Banaszuk and Hauser (1996) (see also Banaszuk and Hauser (1994)). In Section 3.4, it is shown how a dissipative potential can be obtained from the exact part of a certain one-form associated with the dynamics. Examples illustrating the approach are presented in Section 3.5. Summary and related extensions of the material from this chapter are outlined in Section 3.6.

3.1 Introduction

The general problem of deriving a generalized Hamiltonian realization for a known system was considered from a feedback equivalence point of view in (Tabuada and Pappas, 2003) for control-affine systems. Feedback equivalence conditions to port-controlled Hamiltonian systems was presented by Cheng et al. (2005). In (Cheng et al., 2000), conditions for approximate Hamiltonian realizations were given in terms of a normal form. Sufficient conditions and a constructive algorithm for a generalized Hamiltonian realization for time-invariant nonlinear systems were presented in (Wang et al., 2003). In particular, the method proposed in (Wang et al., 2003) which is reviewed briefly in Section 3.2, seeks to decompose the vector field along the gradient direction $\nabla H(x)$ and the tangential direction of the equivalue surfaces of $H(x)$, for a regular positive-definite function $H(x)$. Extensions to port-controlled time-varying systems were carried out in (Fujimoto et al., 2003) using an error dynamic system and in (Wang et al., 2005) using Poisson structures. The relationship between the concepts of Lyapunov stability and Hamiltonian with dissipation was discussed in (McLachlan et al., 1998) using Morse theory. Recently, following the work in (Maschke et al., 2000) which related port-controlled Hamiltonian systems to the construction of Lyapunov functions, it was shown in (Wang et al., 2007) how k-th degree approximate dissipative Hamiltonian systems can be used to solve the realization problem and how associated k-th degree approximate Lyapunov functions can be used to study the stability of such systems.

In this chapter, a different approach, used originally in the context of non-equilibrium thermodynamics, is proposed to approximate dissipative realization, following the work of Edelen (1973). A dissipative potential is computed locally using a differential one-form associated to the vector field and a homotopy operator. The approach to compute a potential presented here differs slightly from the approach given recently in (Yap, 2009), where only closed one-forms are considered together with the Poincaré lemma and where the solution to partial differential equations are required. However, the contribution found in (Yap, 2009) is close in spirit to an early controller design procedure published in (Hudon and Guay, 2009a), where the anti-exact part obtained by application of the homotopy operator is canceled by feedback. This approach is briefly discussed in Section 3.6. Finally, it should be noted that a related approach to decomposition was developed and applied to Liénard systems in (Demongeot et al., 2007a,b; Glade et al., 2007; Forest et al., 2007).

3.2 Dissipative Hamiltonian Realization

The problem of dissipative Hamiltonian realization is reviewed in the present section, based on the contributions from Wang et al. (2003, 2005, 2007) and Cheng et al. (2002). However, to illustrate the idea of decomposing the drift dynamics in terms of a dissipative and a non-dissipative part, a passivity-based approach to drift dynamics decomposition proposed by Sira-Ramírez and Angulo-Núñez (1997) is reviewed. This last paper is inspired by the original paper from Willems (1972) and the contribution from Byrnes et al. (1991).

3.2.1 Passivity-Based Decomposition

Consider the single-input single-output control affine system

$$\dot{x} = f(x) + g(x)u \tag{3.1}$$

$$y = h(x) \tag{3.2}$$

where $x \in \mathcal{X} \subset \mathbb{R}^n$ is the state vector, $u \in \mathcal{U} \subset \mathbb{R}$ is the control input and the scalar function $y \in \mathcal{Y} \subset \mathbb{R}$ is the output function of the system. The vector fields $f(x)$ and $g(x)$ are assumed to be smooth vector fields on $\mathcal{X}$ and such that there exists an isolated non-zero state of interest $x = x_e \in \mathcal{X}$, s.t. $f(x_e) + g(x_e)\bar{u} = 0$. In the following, the term $(\mathcal{L}_\phi V)$ denotes the Lie derivative of a function V in the direction of a smooth vector field $\phi(x)$. In local coordinates, it is given as $\mathcal{L}_\phi V(x) = \frac{\partial V}{\partial x} \cdot \phi(x)$.

In (Sira-Ramírez and Angulo-Núñez, 1997), the case where the drift vector field $f(x)$ has a **natural decomposition** with respect to the known storage function V was considered. The key idea was to express the drift part of system, $f(x)$, as the sum of three components:

$$f(x) = f_{\mathrm{d}}(x) + f_{\mathrm{nd}}(x) + f_{\mathrm{I}}(x) \tag{3.3}$$

such that

$$\mathcal{L}_{f_{\mathrm{d}}} V(x) \leq 0, \quad \forall x \in \mathcal{X} \tag{3.4}$$

$$\mathcal{L}_{f_{\mathrm{nd}}} V(x) \quad \begin{cases} \text{is either sign-undefined in } \mathcal{X} \\ \text{or else it is non-negative in } \mathcal{X} \end{cases} \tag{3.5}$$

$$\mathcal{L}_{f_{\mathrm{I}}} V(x) = 0, \quad \forall x \in \mathcal{X}. \tag{3.6}$$

Following (Sira-Ramírez and Angulo-Núñez, 1997), $f_{\mathrm{d}}(x)$ is called the **dissipative component** of $f(x)$. Similarly, $f_{\mathrm{nd}}(x)$ is the **non-dissipative component** of $f(x)$, and $f_{\mathrm{I}}(x)$

is the **invariant component** of $f(x)$. The vector fields $(f_{\mathrm{d}}(x), f_{\mathrm{nd}}(x), f_{\mathrm{I}}(x))$ are called the **natural components** of $f(x)$ with respect to $V(x)$.

The main contribution from (Sira-Ramírez and Angulo-Núñez, 1997) was to use this decomposition for the design of feedback passifiable controllers. Consider the system (f, g, h) with V being, locally in $\mathcal{X}$, a strict relative degree one function, *i.e.*, $\mathcal{L}_g V(x) \neq 0$, $\forall x \in \mathcal{X}$. Then,

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) + \left(\frac{\partial V}{\partial x} g(x)\right) u = \mathcal{L}_f V(x) + \mathcal{L}_g V(x) u. \tag{3.7}$$

Suppose that the vector field $f(x)$ has natural components $f_{\mathrm{d}}(x)$, $f_{\mathrm{nd}}(x)$, $f_{\mathrm{I}}(x)$ with respect to the storage function $V(x)$. Then, the time-derivative of the storage function can be re-written as

$$\dot{V} = \mathcal{L}_{f_{\mathrm{d}}} V(x) + \mathcal{L}_{f_{\mathrm{nd}}} V(x) + \mathcal{L}_g V(x) u \tag{3.8}$$
$$= \mathcal{L}_{f_{\mathrm{d}}} V(x) + \mathcal{L}_g V(x) \left(\frac{\mathcal{L}_{f_{\mathrm{nd}}} V(x)}{\mathcal{L}_g V(x)} + u\right). \tag{3.9}$$

One can define the following state dependent input coordinate transformation:

$$u(x) = \frac{1}{\mathcal{L}_g V(x)} \left(h(x) v - L_{f_{\mathrm{nd}}} V(x) - \gamma h^2(x)\right) \tag{3.10}$$

with γ an arbitrary strictly positive scalar, and as a result

$$\dot{V} = \mathcal{L}_{f_{\mathrm{d}}} V(x) + h(x) v - \gamma h^2(x) \leq y v - \gamma y^2 \leq y v. \tag{3.11}$$

The main result from (Sira-Ramírez and Angulo-Núñez, 1997) is the following.

Proposition 3.2.1. *The system (f, g, h) is locally strictly output passifiable with respect to the storage function $V(x)$, by means of affine feedback of the form $u = \alpha(x) + \beta(x) v$ if and* *only if*

$$\mathcal{L}_g V(x) \neq 0, \quad \forall x \in \mathcal{X}. \tag{3.12}$$

The affine feedback law, or state dependent input coordinate transformation, which achieves strict output passivation, is given by

$$u = \frac{1}{\mathcal{L}_g V(x)} \left(h(x)v - \mathcal{L}_{f_{\text{nd}}} V(x) - \gamma h^2(x) \right). \tag{3.13}$$

The proof relies essentially on the classical construction from Byrnes et al. (1991), making use of the following definition.

Definition 3.2.2. *A system (f, g, h) has the Kalman-Yacubovich-Popov (KYP) property if there exists a continuously differentiable non-negative function $V : \mathcal{X} \to \mathbb{R}$ with $V(0) = 0$ such that*

$$\begin{aligned} L_f V(x) &\leq 0 & (3.14) \\ L_g V(x) &= h(x) & (3.15) \end{aligned}$$

for all $x \in \mathcal{X}$.

Then the following statement from (Byrnes et al., 1991) is used to prove Proposition 3.2.1.

Proposition 3.2.3. *A system which has the KYP property is passive with storage function $V(x)$. Conversely, a passive system having a continuously differentiable storage function has the KYP property.*

With respect to the topics covered in the present thesis, the following geometric interpretation of this decomposition, provided by Sira-Ramírez and Angulo-Núñez (1997), has to be considered. In the transformed input coordinates the system is given by:

$$\dot{x} = f_{\mathrm{d}}(x) + f_{\mathrm{I}} + \left(I - g(x)\frac{\partial V(x)}{\partial x}\frac{1}{\mathcal{L}_g V(x)}\right) f_{\mathrm{nd}} + \frac{h}{\mathcal{L}_g V(x)} g(x) v - \gamma \frac{h^2(X)}{\mathcal{L}_g V(x)} g(x) \quad (3.16)$$

where the terms of the right hand side are interpreted as follows: first term: $f_{\mathrm{d}}(x)$ is naturally dissipative, f_{I} is the invariant term, $\left(I - g(x)\frac{\partial V(x)}{\partial x}\frac{1}{\mathcal{L}_g V(x)}\right) f_{\mathrm{nd}}$ is the workless term, $\frac{h(x)}{\mathcal{L}_g V(x)} g(x) v$ is the supply rate, and $\gamma \frac{h^2(x)}{\mathcal{L}_g V(x)} g(x)$ is the artificially induced dissipation term making use of nonlinear (quadratic) output feedback.

Note that the matrix:

$$M(x) = \left(I - g(x)\frac{\partial V(x)}{\partial x}\frac{1}{\mathcal{L}_g V(x)}\right) \quad (3.17)$$

is a projection operator onto the tangent space of the level surface $V(x) = c$ along the distribution span$\{g\}$. This projection operator "hides" all destabilizing components of $f_{\mathrm{nd}}(x)$ by making the vector $M(x) f_{\mathrm{nd}}$ tangent to the level surfaces of constant stored energy $\{x : V(x) = c\}$. Thus, any unstable behavior contained in f_{nd} does not increment, nor diminish, the value of the energy function $V(x)$ along the controlled trajectories of the transformed system.

This decomposition approach for control affine systems is similar to the one used throughout this thesis. However, three differences should be pointed out. In the proposed approach, the components equivalent to $f_{\mathrm{nd}}(x)$ and $f_{\mathrm{I}}(x)$ are not distinguished. However, as noted in (Sira-Ramírez and Angulo-Núñez, 1997), those two components might not be easily distinguishable and therefore, can be lumped together in practice. Also, the approach taken by Sira-Ramírez and Angulo-Núñez (1997) for feedback passivation is based on an inversion of $\mathcal{L}_g V(x)$ for the controller computation. This inversion is avoided in the constructions of Chapters 4 and 5. Finally, the decomposition approach considered in the present thesis does not assume *a priori* knowledge of a potential (or of a Lyapunov function).

The decomposition technique proposed in the present chapter is oriented toward dissipative Hamiltonian realization. Elements of generalized Hamiltonian systems are reviewed in the next section. Connections between the approach from (Sira-Ramírez and Angulo-Núñez, 1997) and the material that follows are given in (van der Schaft, 2000, Chapter 4).

3.2.2 Generalized Hamiltonian Decomposition

This section reviews the contributions from (Wang et al., 2003, 2005, 2007), beginning with the following definitions, given originally in (Cheng et al., 2000).

Definition 3.2.4. *A dynamic system*

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \tag{3.18}$$

is said to have a ***generalized Hamiltonian realization*** *(GHR) if there exists a suitable coordinate chart and a Hamiltonian function $H(x)$ such that* (3.18) *can be expressed as*

$$\dot{x} = T(x)\nabla H(x), \tag{3.19}$$

where $T(x)$ is a $n \times n$ matrix called the ***structure matrix*** *and $\nabla H(x) = \frac{\partial H}{\partial x}$ is a $n \times 1$ vector. If the structure matrix can be expressed as $T(x) = J(x) - R(x)$, with a skew-symmetric $J(x)$ and a symmetric positive semi-definite $R(x)$, then system* (3.18) *is called a* ***dissipative Hamiltonian realization****. Furthermore, if $R(x) > 0$,* (3.18) *is called a* ***strict dissipative Hamiltonian realization****.*

This definition is extended to control affine systems in the following definition.

Definition 3.2.5. *A controlled dynamic system*

$$\dot{x} = f(x) + g(x)u \tag{3.20}$$

*is said to have a **state feedback Hamiltonian realization** if there exists a suitable state feedback* $u(x) = \alpha(x) + v$ *such that the closed-loop system can be expressed as*

$$\dot{x} = T(x)\nabla H(x) + g(x)v. \tag{3.21}$$

If $T(x)$ *can be expressed as* $T(x) = J(x) - R(x)$, *with a skew-symmetric* $J(x)$ *and* $R(x) \geq 0$, (> 0), *then system* (3.20) *is called a **feedback (strict) dissipative Hamiltonian realization**. Furthermore, if* $R(x) > 0$, (3.20) *is called a **strict dissipative Hamiltonian realization**.*

The realization problem was tackled in the following way by Wang et al. (2003). First, let J_f denote the Jacobian matrix. The approach proposed in (Wang et al., 2003) is based on the following proposition (see also Cheng et al. (2002)):

Proposition 3.2.6. *If the Jacobian matrix* J_f *is invertible, then*

$$\dot{x} = J_f^{-T}\frac{\partial H}{\partial x}, \quad H = \frac{1}{2}\sum_{i=1}^{n} f_i^2 \tag{3.22}$$

is a GHR of system (3.18).

The decomposition technique is summarized in the following result:

Theorem 3.2.7. *If* $J_f^T + J_f$ *is negative definite, then system* (3.18) *has a strict dissipative Hamiltonian realization as follows:*

$$\dot{x} = (J(x) - R(x))\nabla H \tag{3.23}$$

where $J(x)$ *is some* $n \times n$ *skew-symmetric matrix* $R(x)$ *is some* $n \times n$ *positive definite matrix and* $H(x) = \frac{1}{2}\sum_{i=1}^{n} f_i^2(x)$.

To keep the exposition complete, it is noted that the proof of this theorem given in (Wang et al., 2003), makes use of the following lemma.

Lemma 3.2.8. *Assume $J(x)$ is an $n \times n$ skew-symmetric matrix and $R(x)$ is an $n \times n$ positive (semi)-definite matrix. If $J(x) - R(x)$ is non-singular, then there exists a skew-symmetric matrix $J_1(x)$ and a positive (semi)-definite matrix $R_1(x)$ such that*

$$(J(x) - R(x))^{-1} = J_1(x) - R_1(x). \tag{3.24}$$

Remark 3.2.9. *In the case where $J(x)$ is singular, a regularization technique is presented in (Wang et al., 2003).*

In (Wang et al., 2005), the generalized Hamiltonian realization concept was generalized to time-varying systems in the following way.

Definition 3.2.10. *A time-varying dynamical system*

$$\dot{x} = f(t, x), \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}^+ \tag{3.25}$$

is said to have a ***generalized Hamiltonian realization*** *(GHR) if there exists a suitable coordinate chart and a Hamiltonian function $H(t, x)$ such that* (3.25) *can be expressed as*

$$\dot{x} = T(t, x)\nabla H(t, x), \tag{3.26}$$

where $T(t, x)$ is the structure matrix. Furthermore, if $T(t, x)$ can be decomposed as $M(t, x) = J(t, x) - R(t, x)$, with $J(t, x)$ skew-symmetric and $R(t, x) \geq 0$, then (3.26) *is called a* ***dissipative Hamiltonian realization****.*

The extension to control affine systems is given in the following definition.

Definition 3.2.11. *A controlled dynamical system*

$$\dot{x} = f(t, x) + \sum_{i=1}^{m} g_i(t, x)u_i \tag{3.27}$$

is said to have a ***feedback generalized Hamiltonian realization*** *if there exists a feedback law* $u(t,x) = \alpha(t,x) + v$ *such that the closed-loop system can be expressed as*

$$\dot{x} = T(t,x)\nabla H(t,x) + g(t,x)v, \tag{3.28}$$

where $g(t,x) = (g_1(t,x), \dots, g_m(t,x))$ *and* $u = (u_1, \dots, u_m)^T$

The following results were presented in (Wang et al., 2005).

Proposition 3.2.12. *Consider the following time-varying nonlinear system*

$$\dot{x} = f(t,x) + g(t,x)u, \quad f(t,0) = 0 \tag{3.29}$$

where $x \in \mathbb{R}^n$*,* $t \in \mathbb{R}^+$*,* $u \in \mathbb{R}^m$*. For arbitrary positive definite function* $H(x)$*,* (3.29) *can be expressed as*

$$\dot{x} = (J(t,x) + P(t,x))\frac{\partial H}{\partial x}(x) + g(t,x)u, \tag{3.30}$$

where

$$P(t,x) = \begin{cases} \frac{\langle f(t,x), \nabla^T H(x)\rangle}{\|\nabla H\|^2} I, & x \neq 0 \\ 0, & x = 0 \end{cases} \tag{3.31}$$

is symmetric,

$$J(t,x) = \begin{cases} \frac{1}{\|\nabla H\|^2}\left[f_{\text{td}}(t,x)\frac{\partial H^T}{\partial x}(x) - \frac{\partial H}{\partial x}(x) f_{\text{td}}(t,x)\right], & x \neq 0 \\ 0, & x = 0 \end{cases} \tag{3.32}$$

is skew-symmetric, $\langle \cdot, \cdot \rangle$ *denotes the inner product and*

$$f_{\mathrm{td}}(t,x) = f(t,x) - f_{\mathrm{gd}}(t,x) \tag{3.33}$$

$$f_{\mathrm{gd}}(t,x) = \frac{\langle f(t,x), \nabla^T H(x) \rangle}{\|\nabla H\|^2} \nabla H, \quad x \neq 0. \tag{3.34}$$

The connection between generalized Hamiltonian realization and the stability (and stabilization) of nonlinear control affine systems was established, from a less restrictive point of view, in (Wang et al., 2007), where the concepts of approximate dissipative Hamiltonian realization and stability were introduced.

Definition 3.2.13. *System* (3.18) *is said to have a* k***-th degree approximate DHR*** *(*$k \geq 1$*) if there exists a suitable coordinate chart and a Hamiltonian function* $H(x)$ *such that (3.18) can be expressed as*

$$\dot{x} = (J(x) - R(x)) \nabla H(x) + O(\|x\|^{k+1}), \tag{3.35}$$

with a skew-symmetric $J(x)$ *and a symmetric positive semi-definite* $R(x)$*.*

Definition 3.2.14. *System* (3.20) *is said to have a* ***state feedback*** k***-th degree approximate DHR*** *(*$k \geq 1$*) if there exists a feedback law* $u(x) = \alpha(x) + v$ *such that the closed-loop system can be expressed as*

$$\dot{x} = (J(x) - R(x)) \nabla H + O(\|x\|^{k+1}) + g(x)v, \tag{3.36}$$

with a skew-symmetric $J(x)$ *and a symmetric positive semi-definite* $R(x)$*.*

The stability arguments are given in the following.

Proposition 3.2.15. *If the Hamiltonian function* $H(x)$ *has a local minimum at the origin and if* $R(x)\nabla H \sim O(\|x\|^l)$ *as* $x \to 0$*,* $0 \leq l \leq k$*, then* (3.35) *is locally stable.*

Proof: Choose $H(x)$ as the Lyapunov function, then

$$\dot{H} = -dHR(x)\nabla H + dH \cdot O(\|x\|^{k+1}), \quad dH = \nabla^T H. \tag{3.37}$$

it follows that $\dot{H} \leq 0$ in some neighborhood of the origin. ■

Definition 3.2.16. *A scalar function $V(x)$ is called a **k-th degree approximate Lyapunov function** for* (3.18) *if*

(1) $V(x)$ is positive definite,

(2) $\dot{V} + O(\|x\|^{k+1}) = \delta_k(x) \leq 0$ holds along the trajectory of the system, where $\delta_k(x) \sim O(\|x\|^l)$, as $x \to 0$, $0 \leq l \leq k$.

Finally, from definition 3.2.16, one obtains (Wang et al., 2007):

Proposition 3.2.17. *If $V(x)$ is a k-th degree approximate Lyapunov function for the system above, then $V(x)$ is a local Lyapunov function for the system.*

The approximate dissipative realization approach presented in the remainder of the present chapter relies on differential one-forms, reviewed in Appendix A. The essential element of the approach proposed here is to use a radial homotopy operator. It is used to achieve a decomposition similar to the ones presented in Section 3.2.1 and the approach of Wang et al. (2005) without an *a priori* knowledge of a Hamiltonian function or a storage function. The homotopy operator is used to compute a local dissipative potential. The construction of this operator is given in the next section.

3.3 Homotopy Decomposition

This section presents the construction of a homotopy operator $\mathbb{H}$, *i.e.*, a linear operator on elements of $\Lambda(\mathbb{R}^n)$ that satisfies the identity

$$\omega = d(\mathbb{H}\omega) + \mathbb{H}d\omega, \tag{3.38}$$

for a given differential form $\omega \in \Lambda(\mathbb{R}^n)$. The review presented here follows (Edelen, 2005, Chapter 5). The first step in the construction of a homotopy operator is to define a star-shaped domain on $\mathbb{R}^n$.

An open subset S of $\mathbb{R}^n$ is said to be star-shaped with respect to a point $p^0 = (x_1^0, \ldots, x_n^0) \in S$ if the following conditions hold:

- S is contained in a coordinate neighborhood U of p^0.
- The coordinate functions of U assign coordinates $(x_1^0, \ldots, x_n^0)$ to p^0.
- If p is any point in S with coordinates $(x_1, \ldots, x_n)$ assigned by functions of U, then the set of points $(x + \lambda(x - x^0))$ belongs to S, $\forall \lambda \in [0, 1]$.

A star-shaped region S has a natural associated vector field $\mathfrak{X}$, defined in local coordinates by

$$\mathfrak{X}(x) = (x_i - x_i^0)\partial_{x_i}, \quad \forall x \in S. \tag{3.39}$$

Without loss of generality, except when noted, the star-shaped domain is centered at the origin, hence $\mathfrak{X}(x) = x_i \partial_{x_i}$.

For a differential form ω of degree k on a star-shaped region S centered at the origin, the homotopy operator is defined, in coordinates, as

$$(\mathbb{H}\omega)(x) = \int_0^1 \mathfrak{X}(x) \lrcorner \omega(\lambda x)\lambda^{k-1} d\lambda, \tag{3.40}$$

where $\omega(\lambda x)$ denotes the differential form evaluated on the star-shaped domain in the local coordinates defined above.

The important properties of the homotopy operator that are used in the sequel are the following:

1. $\mathbb{H}$ maps $\Lambda^k(S)$ into $\Lambda^{k-1}(S)$ for $k \geq 1$ and maps $\Lambda^0(S)$ identically to zero.

2. $d\mathbb{H} + \mathbb{H}d =$ identity for $k \geq 1$ and $(\mathbb{H}df)(x) = f(x) - f(x_0)$ for $k = 0$.

3. $(\mathbb{H}\mathbb{H}\omega)(x_i) = 0, \quad (\mathbb{H}\omega)(x_i^0) = 0.$

4. $\mathfrak{X} \lrcorner \mathbb{H} = 0, \quad \mathbb{H}\mathfrak{X} \lrcorner = 0.$

The first part of the right hand side of (3.38), $d(\mathbb{H}\omega)$, is a closed form, since $d \circ d(\mathbb{H}\omega) = 0$. Since by property (1) of the homotopy operator, for $\omega \in \Lambda^k(S)$, we have $(\mathbb{H}\omega) \in \Lambda^{k-1}(S)$, $d(\mathbb{H}\omega)$ is also exact on S. We denote the exact part of ω by $\omega_e = d(\mathbb{H}\omega)$ and the anti-exact part by $\omega_a = \mathbb{H}d\omega$. It is possible to show that ω vanishes on $\mathbb{R}^n$ if and only if ω_e and ω_a vanish together (Edelen, 2005).

In the sequel, the homotopy operator is applied on one-forms. Since ω_e is an exact one-form, $(\mathbb{H}\omega_e)$ computed by homotopy is a dissipative potential for the system (Edelen, 1973). A non-dissipative potential is associated with the anti-exact part, but on the star-shaped domain S, ω_a does not contribute to the dissipative part of the system. In other words, ω_a belongs to the kernel of $\mathbb{H}$, which can be seen by applying property (3) from above to the definition of $\omega_a = \omega - \omega_e$.

A one-form for the system (3.18) is constructed in the following way. Let $X = \sum_{i=1}^{n} f_i \partial_{x_i}$ denote the vector field associated with (3.18). Define a non-vanishing closed two-form Ω on $\mathbb{R}^n$ as

$$\Omega = \sum_{1 \leq i < j \leq n} dx_i \wedge dx_j. \tag{3.41}$$

Here, the non-vanishing two-form Ω is not necessarily defined in a canonical way, since the objective is ultimately to compute an admissible potential (and not a minimal one). For example, if $n = 3$, Ω could be given as

$$\Omega = dx_1 \wedge dx_2 + dx_1 \wedge dx_3 + dx_2 \wedge dx_3. \tag{3.42}$$

The orientation of the two-form will be fixed, if necessary, by checking the sign of the obtained dissipative function. Then, a (possibly non-closed) one-form for the system is computed by taking the interior product of Ω with respect to X, *i.e.*,

$$\omega = X \lrcorner \Omega. \tag{3.43}$$

Different approaches to compute a characteristic one-form were explored in (Banaszuk, 1995) and presented in (Edelen, 2005), however those approaches are not considered in this thesis. The one-form ω defined above will serve as the basis for the computations presented below. The choice of Ω is arbitrary, and gives some degree of freedom for the constructions given in this thesis, to ensure that the desired properties on the obtained one-form and the dissipative potential. This certainly limits the conclusions given below, but from a constructive point of view, it gives to the user some freedom in applications, as presented in Chapters 4 and 5. In the present chapter, ω_e obtained from

$$\omega_e = d(\mathbb{H}\omega) \tag{3.44}$$

will be used to compute a coordinate transformation and to study the dissipative Hamiltonian realization problem.

3.4 Approximate Representation

The objective of this section is to show how to compute a change of coordinates to express the exact one-form ω_e obtained by application of the homotopy operator on $\omega(x) = (X \lrcorner \Omega)(x)$ in new coordinates. To set the problem, let the vector field $X(x) = \sum_{i=1}^{n} f_i(x)\partial_{x_i}$ be known, $i = 1, \ldots, n$. Assume that X is of class $\mathcal{C}^k$ with $k \geq 2$. It is also assumed that X has an equilibrium point at the origin. First, define a non vanishing closed two-form $\Omega = \sum_{1 \leq i < j \leq n} dx_i \wedge dx_j$ on $\mathbb{R}^n$.

Remark 3.4.1. *The choice of a two-form is arbitrary. Depending on the problem and the structure of the vector field $X(x)$, a better choice for Ω than the one considered here might be possible. As an example, for the four-tanks system considered in (Hudon and Guay, 2009a) and mentioned in Section 3.6, an educated guess, based on the tanks coupling was used. A different approach to obtain a characteristic one-form for the system was used in (Banaszuk and Hauser, 1996).*

Taking the interior product of Ω with respect to the vector field X, the desired one-form $\omega(x)$ is computed as follows

$$\begin{aligned} \omega &= X \lrcorner \Omega && (3.45) \\ &= \sum_{1 \leq i < j \leq n} \left(f_i dx_j - f_j dx_i \right). && (3.46) \end{aligned}$$

Given a star-shaped region centered at the origin, with associated vector field $\mathfrak{X}(x) = x_i \partial_{x_i}$,

$$(\mathbb{H}\omega)(x) = \int_0^1 \left(\mathfrak{X} \lrcorner \omega(\lambda x) \right) d\lambda. \quad (3.47)$$

Letting $\tilde{f}_i$ denote the values of the components of f after integration with respect to λ, the dissipative potential is given as

$$(\mathbb{H}\omega)(x) = \sum_{1\le i<j\le n} \left(\tilde{f}_i \cdot x_j - \tilde{f}_j \cdot x_i\right) := \tilde{F}(x). \tag{3.48}$$

Taking the exterior derivative, the exact one-form ω_e is as follows

$$\omega_e(x) = \sum_{i=1}^{n} \frac{\partial \tilde{F}}{\partial x_i} dx_i. \tag{3.49}$$

As a result, the anti-exact form ω_a is then given by

$$\begin{aligned} \omega_a &= \omega - \omega_e \\ &= \sum_{1\le i<j\le n} \left(f_i - \frac{\partial \tilde{F}}{\partial x_j}\right) dx_j - \left(f_j + \frac{\partial \tilde{F}}{\partial x_i}\right) dx_i. \end{aligned} \tag{3.50}$$

Remark 3.4.2. *As a special case, if one defines Ω to be the canonical symplectic two-form, i.e.,*

$$\Omega = \sum_{i=1}^{n} dx_i \wedge dp_i, \tag{3.51}$$

and the vector field X_H as the vector field generated by a known Hamiltonian H,

$$\dot{x}_i = \frac{\partial H}{\partial x_i} \tag{3.52}$$

$$\dot{p}_i = -\frac{\partial H}{\partial p_i}, \tag{3.53}$$

for $i = 1, \ldots, n$, then the one-form ω obtained by taking the interior product $X_H \lrcorner \Omega$ is closed, hence $\omega = \omega_e = -dH$ (see Farber (2004, Chapter 2)).

To study the approximate dissipative Hamiltonian realization problem, a particular target system is now defined[1]. Consider the following dissipative Hamiltonian system

$$\dot{z} = (J(z) - R(z))\nabla H(z) \tag{3.54}$$

with $z \in \mathbb{R}^n$ and H, the dissipative Hamiltonian. Following the argument given in (Cheng et al., 2002), it can be shown that if $f(0) = 0$, the Hamiltonian function

$$H(z) = \frac{1}{2}\sum_{i=1}^{n} z_i^2 \tag{3.55}$$

is a suitable locally defined dissipative potential for the system. Assuming for simplicity that n is even, the simplest form for $J = -J^T$ in suitable dimensions is

$$J = \begin{pmatrix} 0 & -I_{\frac{n}{2}\times\frac{n}{2}} \\ I_{\frac{n}{2}\times\frac{n}{2}} & 0 \end{pmatrix} \tag{3.56}$$

where I denotes the identity matrix. In the case where n is odd, J can be complemented with an extra column and an extra row of 0. The dissipative component in the targeted system is set as $R = I_{n\times n}$. In this particular case, it can be shown that $(\mathbb{H}\omega)(z) = -H(z)$ and the closed one-form obtained by the procedure depicted in the last section is given by

$$\bar{\omega}_e = \sum_{i=1}^{n} -z_i dz_i. \tag{3.57}$$

It can be observed that the obtained anti-exact part is

$$\bar{\omega}_a = (J\nabla H)dz \tag{3.58}$$

which corresponds to the tangential component from (Wang et al., 2003).

The problem of expressing the original system in the form of the reference dissipative

[1]The construction presented here obviously relies on the particular choice of dissipative Hamiltonian realization described here. This choice could be different, depending on the system to be considered.

Hamiltonian system is now considered. The idea exploited here consists in finding conditions under which there exists a diffeomorphism preserving the exact form, $i.e.$, a diffeomorphism between the reference exact one-form $\bar{\omega}_e(z)$ defined in (3.57) and the exact one-form for the system of interest $\omega_e(x)$ given in (3.57). To guarantee that the closed one-form is preserved, the following condition has to be fulfilled:

$$\bar{\omega}_e = \Phi(\omega_e) \tag{3.59}$$

$$-\sum_{i=1}^{n} z_i dz_i = \Phi\left(\sum_{i=1}^{n} \frac{\partial \tilde{F}}{\partial x_i} dx_i\right). \tag{3.60}$$

First, the terms $-z_i$ can be identified directly with $\frac{\partial \tilde{F}}{\partial x_i}$. Then to identify dz_i and dx_i, the problem is to ensure that

$$d\Phi = I_{n\times n}. \tag{3.61}$$

Hence, the transformation considered here is such that $d\Phi = dz \cdot \left(d\left(\frac{\partial \tilde{F}}{\partial x_i}\right)\right)^{-1} = I_{n\times n}$. Since ω_e is closed, $i.e.$,

$$\frac{\partial^2 \tilde{F}}{\partial x_i \partial x_j} = \frac{\partial^2 \tilde{F}}{\partial x_j \partial x_i}, \tag{3.62}$$

the particular coordinate transformation is such that

$$d\Phi = \left(\sum_{i=1}^{n} \frac{\partial^2 \tilde{F}}{\partial x_i^2}\right)^{-1}. \tag{3.63}$$

The transformation is hence given by

$$z_i = -\frac{\partial \tilde{F}}{\partial x_i}\left(\sum_{i=1}^{n} \frac{\partial^2 \tilde{F}}{\partial x_i^2}\right)^{-1}. \tag{3.64}$$

Hence, for this particular choice of coordinate transformation to be admissible, the following condition of the potential $\tilde{F}$ has to be fulfilled

$$\sum_{i=1}^{n} \frac{\partial^2 \tilde{F}}{\partial x_i^2} \neq 0. \tag{3.65}$$

The transformation between closed one-forms can be related to the Poincaré lemma, as discussed in (Yap, 2009). This particular form of transformation could also be related to the computations involved in the realization of Brayton–Moser equations and the existence of a dissipative potential in the context of power shaping where the Poincaré lemma is used, as presented in (García-Canseco et al., 2010).

The examples in the next section will show the application of this particular transformation is the context of approximate dissipative Hamiltonian realization.

3.5 Applications

3.5.1 Two Simple Examples

In this section, two simple examples are considered to illustrate the proposed construction. First, consider the system given by

$$\dot{x}_1 = x_1^2 x_2 - x_1^3 := f_1(x) \tag{3.66}$$

$$\dot{x}_2 = -x_1 x_2^2 - x_2^3 := f_2(x). \tag{3.67}$$

The vector field associated with this system is expressed as $X|_x = f_1(x)\partial_{x_1} + f_2(x)\partial_{x_2}$. Let the two-form Ω be given by

$$\Omega = dx_1 \wedge dx_2. \tag{3.68}$$

Computing $\omega = X \lrcorner \Omega$ gives

$$\omega = -f_2 dx_1 + f_1 dx_1. \tag{3.69}$$

One can check that ω is not closed ($i.e.$, $d\omega \neq 0$ since $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \neq 0$).

A homotopy operator $\mathbb{H}$ centered at the origin is constructed by letting

$$\mathfrak{X}(x) = x_1 \partial_{x_1} + x_2 \partial_{x_2} \tag{3.70}$$

and by evaluating the one-form ω on the star-shaped domain. The potential is given by

$$(\mathbb{H}\omega)(x) = \int_0^1 \left(-\lambda^3 x_1 x_2^2(-\lambda x_1 - \lambda x_2) + \lambda^3 x_1^3(\lambda x_2 - \lambda x_1)\right) d\lambda \tag{3.71}$$

$$\tilde{F} = \frac{1}{4}(-x_1^3 x_2 + 2x_1^2 x_2^2 + x_1 x_2^3). \tag{3.72}$$

The exact part ω_e of the one-form ω is given by

$$\begin{aligned} \omega_e(x) &= d(\mathbb{H}\omega)(x) \\ &= x_2 \left(\frac{1}{4}x_2^2 + x_1 x_2 - \frac{3}{4}x_1^2\right) dx_1 + x_1 \left(-\frac{1}{4}x_1^2 + x_1 x_2 + \frac{3}{4}x_2^2\right) dx_2. \end{aligned} \tag{3.73}$$

The anti-exact part is given by

$$\omega_a = x_2(x_2 - \frac{3}{2}x_1)dx_1 + x_1(x_1 + \frac{3}{2}x_2)dx_2. \tag{3.74}$$

One locally admissible dissipative potential for the system is given by $\tilde{F}$, as noted in Section 3.3. However, the interest here is to use a change of coordinates to define a potential that is easier to use, such as the one of Section 3.4. Using the state transformation proposed in the last section, the new coordinates are given by

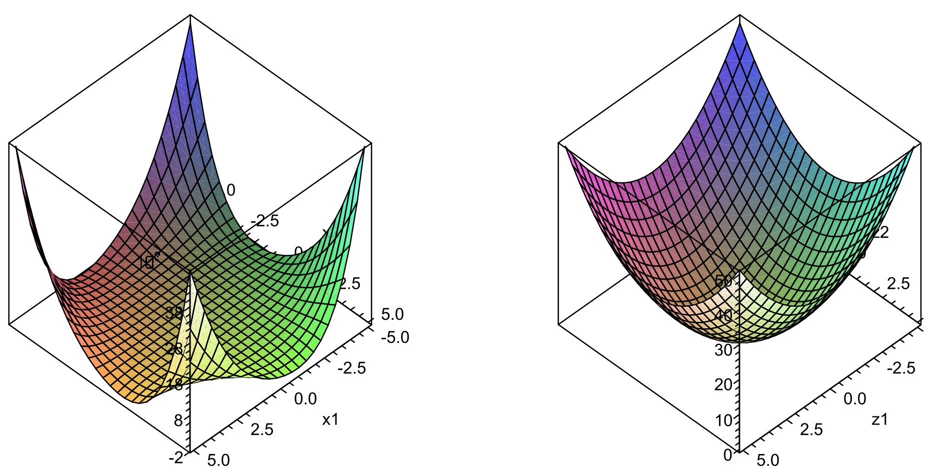
$$z_1 = -x_2 \frac{-\frac{3}{4}x_1^2 + x_1 x_2 + \frac{x_2^2}{4}}{x_1^2 + x_2^2} \tag{3.75}$$

$$z_2 = -x_1 \frac{-\frac{1}{4}x_1^2 + x_1 x_2 + \frac{3x_2^2}{4}}{x_1^2 + x_2^2}. \tag{3.76}$$

The transformation maps the origin of (x_1, x_2) to the origin of (z_1, z_2). As noted above, a suitable dissipative potential for the system is given by

$$H(z) = \frac{1}{2}(z_1^2 + z_2^2). \tag{3.77}$$

In the neighborhood of the origin, a regular positive function that can be used as a dissipative potential for the system is obtained, as depicted in Figure 3.1a.

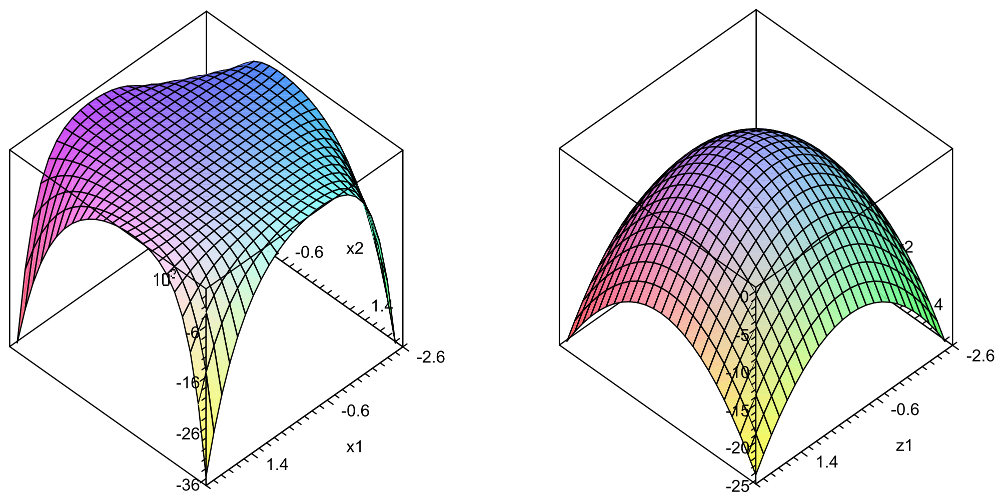


(a) x-coordinates (b) z-coordinates

Figure 3.1: Dissipative potential — first example

In the new coordinates,

$$\bar{\omega}_e(z) = \frac{1}{2}(-z_1 dz_1 - z_2 dz_2), \tag{3.78}$$



(a) x-coordinates (b) z-coordinates

Figure 3.2: Time-derivative of the dissipative potential — first example

from which a Lyapunov function can be computed using $\bar{X}(z) = (-z_1 - z_2)\partial_{z_1} + (z_1 - z_2)\partial_{z_2}$, as

$$V(z) = \bar{X} \lrcorner \bar{\omega}_e \tag{3.79}$$

$$= \frac{1}{2}(z_1^2 + z_2^2). \tag{3.80}$$

Taking the derivative with respect to time, we have

$$\dot{V} = -(z_1^2 + z_2^2). \tag{3.81}$$

The original system is therefore locally dissipative in the z-coordinates. Moreover, a suitable dissipative potential for the system is given by

$$H(x) = \frac{1}{2}(z_1^2(x) + z_2^2(x)). \tag{3.82}$$

Since the nonsingular transformation is mapping the origin in x-coordinates to the origin of the z-coordinates, the origin is asymptotically locally stable by Lyapunov theory.

Consider, as a second example, a spring-mass system with damping, given by

$$\dot{x}_1 = x_2 := f_1(x) \tag{3.83}$$

$$\dot{x}_2 = -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2 := f_2(x) \tag{3.84}$$

with positive parameters g, l, m and k. The vector field associated with this system is expressed as $X(x) = f_1(x)\partial_{x_1} + f_2(x)\partial_{x_2}$. Let the two-form Ω be given by

$$\Omega = dx_1 \wedge dx_2. \tag{3.85}$$

Computing $\omega(x) = X \lrcorner \Omega$ leads to

$$\omega = -f_2 dx_1 + f_1 dx_1. \tag{3.86}$$

One can check that $\omega(x)$ is not closed.

Constructing the homotopy operator $\mathbb{H}$ centered at the origin by letting

$$\mathfrak{X}(x) = x_1\partial_{x_1} + x_2\partial_{x_2} \tag{3.87}$$

and evaluating the one-form $\omega(x)$ on the star-shaped domain, leads to the potential

$$\tilde{F} = (\mathbb{H}\omega)(x) = \frac{1}{2}\frac{2gm(1-\cos(x_1) + klx_1x_2 + lmx_2^2}{lm}. \tag{3.88}$$

The exact part $\omega_e(x)$ of the one-form $\omega(x)$ is given by

$$\begin{aligned} \omega_e(x) &= d(\mathbb{H}\omega)|_x \\ &= \frac{1}{2lm}\left((2gm\sin(x_1) + klx_2)dx_1 + (klx_1 + 2lmx_2)dx_2\right). \end{aligned} \tag{3.89}$$

The anti-exact part is given as

$$\omega_a(x) = \frac{1}{2km}\left(x_2 dx_1 - x_1 dx_2\right). \tag{3.90}$$

Using the construction proposed in the last section, the transformation (3.64) is applied using $\tilde{F}$ and leads to:

$$z_1 = -\frac{g\sin(x_1) + \frac{x_2}{2}}{g\cos(x_1) + 1} \tag{3.91}$$

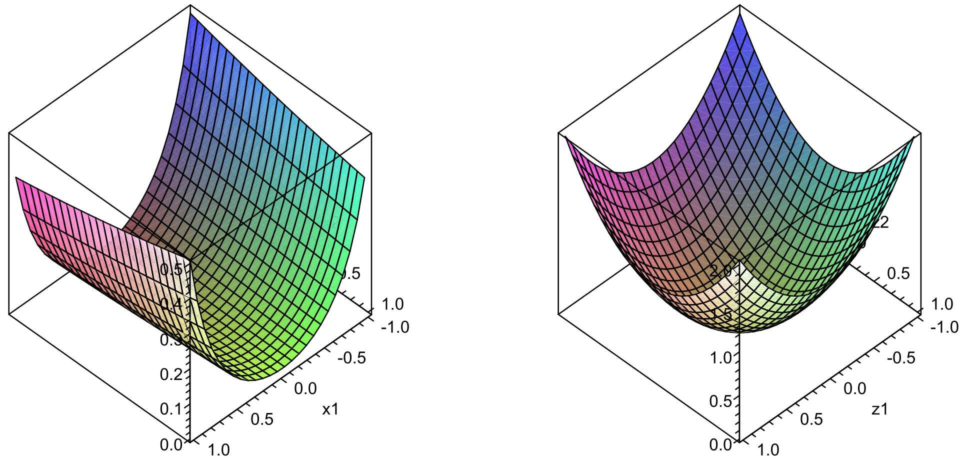
$$z_2 = -\frac{\frac{x_1}{2} + x_2}{g\cos(x_1) + 1}. \tag{3.92}$$

The transformation maps the origin of (x_1, x_2) to the origin of (z_1, z_2). As noted above, a suitable dissipative potential for the system is given by

$$H(z) = \frac{1}{2}(z_1^2 + z_2^2). \tag{3.93}$$

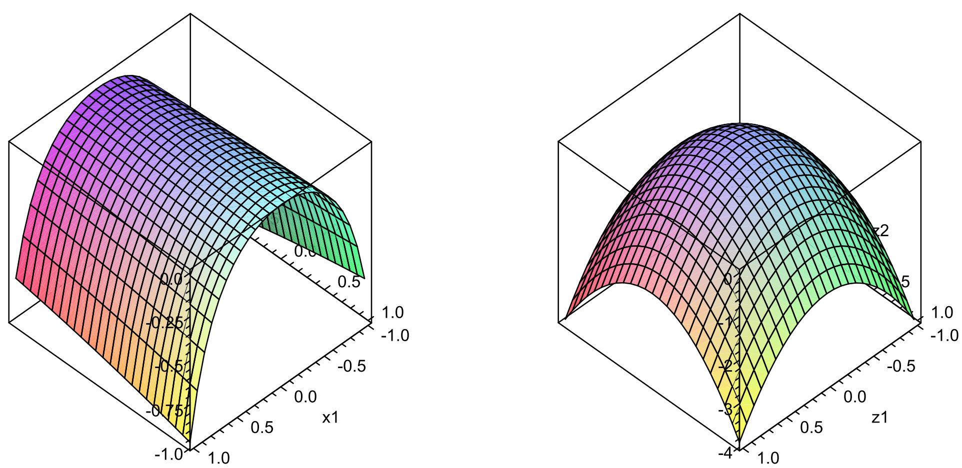
Fixing the parameters to be $g = 9.8$, $l = 1$, $m = k = 10$, a regular positive function is obtained in a neighborhood of the origin, as presented in Figure 3.3b, with negative derivative $\dot{V}(z)$ in a neighborhood of the origin (Figure 3.4b). However, expressing the potential in the x-coordinates, see Figures 3.3a and 3.4a, it is clear that the obtained potential function is not strictly convex at the origin.

As discussed in (Khalil, 2002) and (Coron, 2007), stability analysis for that particular case relies on the Lasalle's invariance principle, or more precisely on the corollary result of Barbashin and Krasovskiĭ (see for example Khalil (2002, Chapter3)). What is important to realize at this point is that the maximal invariant set of the computed dissipative potential is the origin, since $\omega = \omega_e + \omega_a$ vanishes at the origin and only there, *i.e.*, the anti-exact part vanishes only at the origin. This observation will be used in the next chapter.



(a) x-coordinates (b) z-coordinates

Figure 3.3: Dissipative potential — mass-spring system



(a) x-coordinates (b) z-coordinates

Figure 3.4: Time derivative of the dissipative potential — mass-spring system

3.5.2 Representation of a Lotka–Volterra System

The construction presented in this chapter is here applied to a special class of predator-prey systems, the Lotka–Volterra dynamics, that will be used throughout Chapters 4 and 5. Representation of Lotka–Volterra as Hamiltonian systems was presented in (Hernández-Bermejo and Fairén, 1998) and (Szederkényi and Hangos, 2004) in the context of control, see also (Evans and Findley, 1999) and references therein for a historical account and some special solutions. Results on stability and global behavior of Lotka–Volterra systems are summarized in (Gouzé, 1993), whereas multi-stability was considered recently in (Efimov, 2009).

The general form of a Lotka–Volterra ecology (Ortega et al., 1999) is given as

$$\begin{aligned} \dot{x}_i &= x_i\left(k_i + \sum_{j\neq i} a_{ij}x_j\right), \quad i = 1,\dots,n-1 \\ \dot{x}_n &= x_n\left(k_n + \sum_{j\neq n} a_{nj}x_j\right) \end{aligned}$$

with k_i, the net birth/mortality rate coefficients, $a_{ij} = -a_{ji}$, $\forall i \neq j$, the predation coefficients, $x \in \mathbb{R}^n_+$. Following (Ortega et al., 1999), consider a two-dimensional Lotka–Volterra system:

$$\dot{x}_1 = ax_1 - bx_1x_2 \tag{3.94}$$

$$\dot{x}_2 = -cx_2 + bx_1x_2, \tag{3.95}$$

with $x_1 \geq 0$ and $x_2 \geq 0$ and where a, b, c are known positive constants. The uncontrolled case has 2 equilibria, a saddle equilibrium at the origin and a center equilibrium surrounded by stable periodic orbits at $x^* := [x_1^*,\ x_2^*]^T = [\frac{c}{b},\ \frac{a}{b}]^T$.

Using the above discussion, it follows that the system (3.94)-(3.95) has a dissipative representation as in the previous section, with the local change of coordinates given by

$$z_1 = \frac{1}{4}\frac{x_1(-2bx_1 - 4bx_2 + 3c + 3a}{b(x_1 + x_2)} \tag{3.96}$$

$$z_2 = \frac{1}{4}\frac{x_2(-4bx_1 - 2bx_2 + 3c + 3a}{b(x_1 + x_2)} \tag{3.97}$$

in a neighborhood of the equilibrium at $(x_1^*, x_2^*)^T = (\frac{c}{b},\ \frac{a}{b})^T$.

3.6 Summary and Extensions

In this chapter, a procedure to study autonomous systems using local dissipative Hamiltonian realization for nonlinear dynamical systems has been derived. Taking the interior product of a non-vanishing two-form with respect to the vector field defining the system, a (possibly non-closed) one-form was obtained. Constructing a locally defined homotopy operator on a star-shaped domain, it was shown how to locally decompose the obtained form into an exact and an anti-exact one-form. A coordinate transformation between the dissipative potential and a known potential was defined using the exact form. The coordinate transformation enables one to explicitly write the dissipative potential for the original system as an approximate dissipative Hamiltonian realization. The obtained anti-exact form is associated to a non-dissipative potential that do not contribute locally to the value of the dissipative potential on the star-shaped domain. Since the approach is local, an interesting study would be to compute the domain of application of the decomposition outlined here. From the discussion in Chapters 4 and 5, this could lead to an approach to compute the domain of attraction of an equilibrium, see for example (Genesio et al., 1985) for a summary of the existing techniques for this problem.

One aspect not addressed in this thesis is the potential use of the proposed decomposition to energy shaping analysis. An example of application in chemical engineering for application of energy shaping methods is the four-tank process considered by Johnsen and Allgöwer (2007). The dynamic model for the four-tank system is given as a control affine nonlinear system of the form

$$\dot{x} = f(x) + g(x)u \tag{3.98}$$

where $x \in \mathbb{R}^4$ are the levels in the respective tanks and $u \in \mathbb{R}^2$ are the manipulated flows. Using the model proposed in (Johnsen and Allgöwer, 2007), $f(x)$ and $g(x)$ are given by

$$f(x) = \begin{pmatrix} \frac{-a_1}{A_1}\sqrt{2gx_1} + \frac{a_3}{A_1}\sqrt{2gx_3} \\ \frac{-a_2}{A_2}\sqrt{2gx_2} + \frac{a_4}{A_2}\sqrt{2gx_4} \\ \frac{-a_3}{A_3}\sqrt{2gx_3} \\ \frac{-a_4}{A_4}\sqrt{2gx_4} \end{pmatrix}, \tag{3.99}$$

$$g(x) = \begin{pmatrix} \frac{\gamma_1}{A_1} & 0 \\ 0 & \frac{\gamma_2}{A_2} \\ 0 & \frac{1-\gamma_2}{A_3} \\ \frac{1-\gamma_1}{A_4} & 0 \end{pmatrix}. \tag{3.100}$$

The parameters A_i represent the cross sections of the respective tanks $i = 1, \ldots, 4$, such that the volumes are given by $V_i = A_i x_i$. The parameters a_i are the cross section of the outlet flows. The gravitational acceleration is denoted by g. The parameters $\gamma_1,\ \gamma_2 \in [0,\ 1]$ are the valve parameters that determined how much of the flows u_i are re-directed in bottom tanks $i = 1, 2$. If the levels of tanks 1 and 2 are the only measured states, it was shown in (Johansson, 2000) that the condition for stable zero dynamics is that $\gamma_1 + \gamma_2 \neq 1$.

The problem of stabilizing the quadruple-tank process using an approximate dissipative

Hamiltonian realization was considered in (Hudon and Guay, 2009a). To stabilize the system at a desired admissible steady-state, (x^*, u^*), a controller of the form

$$u_1(t) = k_{11}(x) \cdot x_1(t) + k_{12}(x) \cdot x_2(t) \quad (3.101)$$

$$u_2(t) = k_{21}(x) \cdot x_1(t) + k_{22}(x) \cdot x_2(t), \quad (3.102)$$

was proposed, assuming that all tanks levels are measured. The idea in (Hudon and Guay, 2009a) was to cancel the deviation of the system from a Hamiltonian system using feedback. The stabilizing feedback controller was designed by canceling the anti-exact part of the one-form obtained by the procedure outlined in this chapter. For this example, it was shown that the restriction $\omega_a \equiv 0$, using the controller given above, led to a set of algebraic equations to be solved for $k_{ij}(x)$, $i, j = 1, 2$. One advantage of this approach, as suggested in (Ramírez et al., 2009), is that inversion of the dynamics is avoided, $i.e.$, the approach is applicable even for cases where $\gamma_1 + \gamma_2 = 1$.

However, this approach to stabilization of nonlinear systems is limited. For example, applying the method to non-isothermal reactors, such as the example considered in Ramírez et al. (2009), it appears that the anti-exact form might not contain any information on the desired controller. For this reason, the classical approach of Jurdjevic and Quinn for stabilization by damping is considered in Chapters 4 and 5, using the dissipative potential computed in the present Chapter as a basis for construction of stabilizing feedback controllers.

Chapter 4

Stabilization of Time-Independent Systems

This chapter presents the main result of the thesis. It is shown that the locally-defined potential obtained from application of the homotopy operator, as demonstrated in Chapter 3, can be used to design smooth feedback stabilizers for time-independent control affine systems following the procedure proposed originally in (Jurdjevic and Quinn, 1978). Under some restrictions on the anti-exact part, it is shown that the dissipative potential can be shaped locally to compute a Lyapunov function for the closed-loop system, a problem studied by Faubourg and Pomet (2000) and Mazenc and Malisoff (2006).

The construction of a Lyapunov function based on the dissipative potential is demonstrated in Section 4.1. In Section 4.2, this result is used for the design of dynamic state feedback controllers, where a dissipative potential is used to stabilize the origin in an extended space. Finally, as an extension of Jurdjevic–Quinn technique, it is shown in Section 4.3, following (Bacciotti and Mazzi, 1995), that the homotopy-based decomposition approach can be used for the stabilization of closed orbits. The summary in Section 4.4 presents some possible extensions of the proposed approach.

4.1 Control Affine Systems Stabilization

4.1.1 Introduction

Consider control affine systems of the form

$$\dot{x} = X_0(x) + \sum_{k=1}^{m} u_k X_k(x) \tag{4.1}$$

with states $x \in \mathbb{R}^n$, and control $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$. It is assumed that the vector fields X_k are of class $\mathcal{C}^\infty$ and that $X_k(0) = 0$, for $k = 0, 1, \ldots, m$. The original works of Jacobson (1977) and Jurdjevic and Quinn (1978) showed, under certain conditions and assuming the knowledge of a Lyapunov function $V(x)$, that a feedback law

$$u_k = -(X_k \cdot \nabla^T V)(x), \quad k = 1, \ldots, m \tag{4.2}$$

asymptotically stabilizes system (4.1) (see for example Bacciotti (1992), Sontag (1998), Coron (2007), Malisoff and Mazenc (2009) and references therein). A well-known limitation of this approach is that there is no systematic way to build the required Lyapunov function *a priori*.

A connection between mechanical systems and the construction of Lyapunov functions for this damping feedback approach was pointed out in Mazenc and Malisoff (2006) using a Hamiltonian function (see also Malisoff and Mazenc (2009)). As illustrated in Ortega et al. (2002), the determination of an admissible Hamiltonian function for a general nonlinear system remains an open problem. In Wang et al. (2007), approximate dissipative Hamiltonian realization techniques were developed with connections to stabilization. It was shown how k-th degree approximate dissipative Hamiltonian systems can be used to solve the realization problem, and how an associated k-th degree approximate Lyapunov function can be used to study the stability of such systems. Part of their argument, presented in

Section 3.2, was based on a decomposition of the dynamics into a gradient part (generated by a potential) and a tangential part. This observation motivates the approach considered in the present chapter to construct a local dissipative function $\psi(x)$ to stabilize control affine nonlinear systems of the form (4.1) using a damping feedback controller of the form $u_k = -X_k \cdot \nabla^T \psi(x)$, $k = 1, \ldots, m$.

In Chapter 3, it was shown that a radial homotopy operator can be used to decompose the drift dynamics into a dissipative (exact) part and a non-dissipative (anti-exact) one. More precisely, a one-form for the system was obtained by taking the interior product of a non vanishing two-form with respect to the drift vector field[1]. Then, a radial homotopy operator centered at the desired equilibrium point for the system was designed. Applying this linear operator on the aforementioned one-form for the system, an exact one form generated by the desired potential function and an anti-exact form that generates the tangential dynamics were obtained. In this chapter, it is shown how this procedure, using the interior product of a non-vanishing two-form with the drift vector field $X_0(x)$ of (4.1), leads to the construction of a first function $\psi(x)$ (an auxiliary scalar field following the nomenclature proposed in Malisoff and Mazenc (2009) and Faubourg and Pomet (2000)) that can be used locally by a second application of the decomposition method to obtain a Lyapunov function for stability characterization of the closed-loop system.

The idea of using an auxiliary scalar function $\psi(x)$, with the property that $(\nabla^T \psi \cdot X_0 < 0)(x)$, to obtain a Lyapunov function for the control affine system (4.1) under damping feedback control was studied in Faubourg and Pomet (2000) for homogeneous system (see for example Hermes (1991)) using the flow of a vector field constructed using a linear combination of elements of $\{\mathrm{ad}^j_{X_0} X_k,\ j \in \mathbb{N},\ k = 1, \ldots, m\}$. Another solution to the problem was presented in Mazenc and Malisoff (2006).

[1]The choice of the two-form is arbitrary, however in the construction that follows, the choice of the two-form will be such that the obtained potential has the desired properties.

4.1.2 Problem Formulation

Consider the control affine system (4.1) in a neighborhood $\mathcal{O}$ of the origin in $\mathbb{R}^n$.

Definition 4.1.1 (Assignable Lyapunov Functions)**.** *Let $\psi(x)$ be a positive definite and radially unbounded function. A continuous $u : \mathbb{R}^n \to \mathbb{R}^m$ is said to assign $\psi(x)$ to be a Lyapunov function for the closed loop system* (4.1) *if the derivative of $\psi(x)$ along the trajectories of the closed loop system is negative definite, i.e., that if for all $x \in \mathbb{R}^n \setminus \{0\}$,*

$$\left(\mathcal{L}_{X_0} \cdot \psi + \sum_{k=1}^{m} u_k \mathcal{L}_{X_k} \cdot \psi\right)(x) < 0. \tag{4.3}$$

Following (Coron, 2007), the statement of Artstein's theorem Artstein (1983) is first recalled.

Theorem 4.1.2. *Let $\psi(x)$ be a positive radially unbounded function. There exists a continuous feedback that assigns $\psi(x)$ to be a Lyapunov function for the closed loop system* (4.1) *if and only if*

(a) It is a control Lyapunov function, i.e., for all $x \in \mathbb{R}^n \setminus \{0\}$,

$$(\mathcal{L}_{X_k})\psi(x) = 0 \Rightarrow (\mathcal{L}_{X_0}\psi)(x) < 0, \quad k = 1, \ldots, m. \tag{4.4}$$

(b) It satisfies the small control property, i.e., for any $\epsilon > 0$, there is a $\delta > 0$ such that for all $x \in \mathbb{R}^n \setminus \{0\}$,

$$\|x\| < \delta \Rightarrow \exists u \quad \begin{cases} \|u\| < \epsilon \\ \mathcal{L}_{X_0} + \sum_{k=1}^{m} u_k \mathcal{L}_{X_k} \psi(x) < 0. \end{cases} \tag{4.5}$$

In the sequel, stabilization by damping feedback is considered, *i.e.*, feedback law $u = (u_1, \ldots, u_m)^T$ defined by

$$u_k = -X_k(x) \cdot \nabla\psi(x), \quad \forall\ k \in 1, \ldots, m. \tag{4.6}$$

In that sense, the ultimate objective is to compute assignable Lyapunov functions for Jurdjevic–Quinn feedback controllers built using the dissipative potential constructed as above.

With this choice of damping feedback, one has for all $x \in \mathbb{R}^n \setminus \{0\}$

$$\frac{d\psi}{dt} = X_0(x) \cdot \nabla\psi(x) - \sum_{k=1}^{m}(X_k(x) \cdot \nabla\psi(x))^2 < 0. \tag{4.7}$$

Therefore, $0 \in \mathbb{R}^n$ is a stable point for the closed-loop system $\dot{x} = X(x, u(x))$. By Lasalle's invariance principle, $0 \in \mathbb{R}^n$ is globally asymptotically stable if for every $x(t) \in \mathcal{C}^\infty(\mathbb{R}; \mathbb{R}^n)$, for all $t \in \mathbb{R}$,

$$\begin{aligned} \dot{x}(t) &= X_0(x(t)), & (4.8)\\ X_k(x(t)) \cdot \nabla V_0(x(t)) &= 0, \quad \forall k \in 0, \ldots, m, & (4.9) \end{aligned}$$

one obtains $x(t) = 0$. In the present thesis, the result is limited to a neighborhood of the equilibrium, denoted $\mathcal{O} \subset \mathbb{R}^n$.

An alternate formulation of this result is given by the following theorem (a proof of this statement can be found in Coron (2007)).

Theorem 4.1.3. *Given the smooth control affine system* (4.1) *and a function* $\psi(x)$ *such that* $\mathcal{L}_{X_0}\psi < 0$ *for every* $x \in \mathbb{R} \setminus \{0\}$. *Suppose moreover that*

$$\text{span}\{X_0(x), \text{ad}_{X_0}^k X_i(x) : i = 1, \ldots, m, k \in \mathbb{N}\} = \mathbb{R}^n \tag{4.10}$$

on $\mathbb{R}^n \setminus \{0\}$. *Then the feedback law*

$$u_i(x) = -(\mathcal{L}_{X_i}\psi)(x), \forall i \in 1, \ldots, m \tag{4.11}$$

globally asymptotically stabilizes the control affine system (4.1).

As noted in (Coron, 2007), the function $\psi(x)$ is not a control Lyapunov function in general. Other Lyapunov functions construction methods based on the prior knowledge of a function $\psi(x)$ satisfying the weak Jurdjevic–Quinn conditions were given in (Faubourg and Pomet, 2000) and (Mazenc and Malisoff, 2006). In the next section, it is shown under which conditions a Lyapunov function for the closed-loop system can be computed using dissipative potentials obtained as in Chapter 3.

4.1.3 Damping Feedback and Deformation using Homotopy Operator

Define a non-vanishing closed two-form Ω on $\mathbb{R}^n$ as[2]

$$\Omega = \sum_{1\leq i<j\leq n} dx_i \wedge dx_j. \tag{4.12}$$

A first one-form associated to the system is obtained by contracting this two-form with respect to the drift vector field,

$$\omega_0 = X_0 \lrcorner \Omega. \tag{4.13}$$

From Section 3.3, it is known that a locally defined homotopy operator on $\mathbb{R}^n$ can be constructed such that $\omega_0 = \omega_{0,e} + \omega_{0,a}$. Since $\omega_{0,e}$ is exact, it is given as the exterior derivative of a potential function ψ and ω_0 is re-written as

$$\omega_0 = -d\psi + \omega_{0,a}. \tag{4.14}$$

Remark 4.1.4. *The negative sign for $-d\psi$ is set to comply with the notation introduced in* (Byrnes and Brockett, 2010)*, where a known closed positive one-form ω_0 (i.e., $\omega_{0,a} \equiv 0$)*

[2] As discussed in Chapter 3, the choice for Ω is not unique and might be simplified by inspecting the dynamics of the system. Moreover, the two-form proposed here is not a minimal one. The interested reader is referred to (Roels, 1974) for a related discussion on the choice of a two-form in the particular case where n is even.

was used for stability arguments. The positive closed one-forms, i.e., $\langle \omega_0, X_0 \rangle > 0$*, was generated as* $\omega_0 = -dV$*. In the present case, it will be clear when the negative sign should be used, depending on the orientation of the basis for* (4.12)*.*

Assume that $\psi(x)$, obtained after application of the locally-defined homotopy operator, *i.e.*,

$$\psi(x) = (\mathbb{H}\omega_{0,e})(x), \tag{4.15}$$

is such that $(\mathcal{L}_{X_0}\psi)(x) < 0$ for $x \in \mathcal{O} \subset \mathbb{R}^n \setminus \{0\}$. If it is not the case (for dissipative systems), it is always possible to apply a change of coordinates (see for example Bacciotti (1992)) to ensure that the above contraction with respect to the drift vector field generates a one-form with the desired properties. The damping feedback controller is designed as

$$u_k(x) = -\kappa_k(\mathcal{L}_{X_k}\psi)(x). \tag{4.16}$$

By the first property of the Lie derivative from Appendix (A), $\mathcal{L}_{X_k}\psi = X_k \lrcorner d\psi$, which is given as $X_k \lrcorner \omega_{0,a} - X_k \lrcorner \omega_0$ by the homotopy decomposition. Consider the affine system under the feedback law, *i.e.*, the system

$$X_0 - \sum_{k=1}^{m} \kappa_k \mathcal{L}_{X_k}\psi \cdot X_k. \tag{4.17}$$

Taking the interior product of this closed-loop vector field with respect to the non-vanishing two-form Ω, one obtains the one-form

$$\omega = \left(X_0 - \sum_{k=1}^{m} \kappa_k (X_k \lrcorner \omega_{0,a} - X_k \lrcorner \omega_0) X_k \right) \lrcorner \Omega, \tag{4.18}$$

which is, following above, a one-form for the closed loop system, *i.e.*, $\omega = -dV + \omega_a$. Since $(X_k \lrcorner \omega_{0,a} - X_k \lrcorner \omega_0) \in \Gamma^\infty(\mathbb{R}^n)$ and by the property (4) of the interior product given in

Appendix A, this one-form is re-written as

$$\omega = X_0 \lrcorner \Omega - \sum_{k=1}^{m} \kappa_k (X_k \lrcorner \omega_{0,a} - X_k \lrcorner \omega_0) \cdot (X_k \lrcorner \Omega) \quad (4.19)$$

$$= \omega_0 + \sum_{k=1}^{m} \kappa_k (X_k \lrcorner \omega_{0,e})(X_k \lrcorner \Omega) \quad (4.20)$$

$$= \omega_0 - \sum_{k=1}^{k} \xi_k (X_k \lrcorner \Omega), \quad (4.21)$$

where the functions ξ_k account for the deformation of the potential function with respect to the controlled vectors. The one-form ω will serve as the basis for the computation of a local Lyapunov function using the homotopy operator defined in Section 3.3. To simplify the computations and to follow the construction from Byrnes and Brockett (2010), it is assumed in the sequel that the dissipative potential $\psi(x)$ computed above is such that $\omega_{0,a} \equiv 0$. This is equivalent to assume that $\psi(x)$ is a first integral for the system. If this assumption is not met, one can try to compute an integrating factor $\gamma(x)$ such that $(\mathbb{H}(d(\gamma\omega_0)))\,(x) \equiv 0$. Such a computation was carried out in the context of feedback linearization in (Banaszuk, 1995). A local form of the result from Section 4.1.2 is now restated in terms of differential forms, using the dissipative potential constructed as above.

Theorem 4.1.5. *Assume that for all $x \in \mathcal{O} \subset \mathbb{R}^n \setminus \{0\}$, one can construct $\psi(x)$ such that ω_0 is exact, i.e., $\omega_0 = -d\psi \neq 0$ and that $d\psi$ vanishes at the origin, an isolated equilibrium for the system* (4.1)*. Then, the damping feedback $u_k = -\kappa_k X_k \lrcorner d\psi$ locally asymptotically stabilizes* (4.1) *at the origin. Moreover, a local Lyapunov function for the closed-loop system can be computed using $V = (\mathbb{H}\omega)$, with ω given by* (4.21)*, if the anti-exact part $\omega_a = \mathbb{H}d\omega$ vanishes only at the origin.*

Proof: The first part of the theorem is a re-statement of the original result. Consider the closed loop system under damping feedback control with ψ as required:

$$X_0 - \sum_{k=1}^{m} \kappa_k \mathcal{L}_{X_k}\psi \cdot X_k. \tag{4.22}$$

Then

$$\dot{\psi} = X_0 \cdot \nabla\psi - \sum_{k=1}^{m} \kappa_k (X_k \cdot \nabla\psi)^2. \tag{4.23}$$

In terms of differential forms, the last expression is re-written as

$$\dot{\psi} = X_0 \lrcorner d\psi - \sum_{k=1}^{m} \kappa_k \left(X_k \lrcorner d\psi\right)^2. \tag{4.24}$$

The second term of the right hand side is obviously negative definite for $\kappa_k > 0$. Since $X_0 \lrcorner d\psi = X_0 \lrcorner (\omega_{0,a} - \omega_0)$, which can be decomposed by the properties of the interior product as

$$\begin{aligned} X_0 \lrcorner d\psi &= X_0 \lrcorner \omega_{0,a} - X_0 \lrcorner \omega_0 && (4.25)\\ &= X_0 \lrcorner \omega_{0,a} - X_0 \lrcorner X_0 \lrcorner \Omega && (4.26)\\ &= X_0 \lrcorner \omega_{0,a}. && (4.27) \end{aligned}$$

By assumption, $\omega_{0,a} \equiv 0$, and hence

$$\dot{\psi} = -\sum_{k=1}^{m} \kappa_k \left(X_k \lrcorner d\psi\right)^2 < 0, \quad \forall\, x \in \mathbb{R}^n \setminus \{0\}, \tag{4.28}$$

under the assumption that $d\psi \neq 0,\ \forall\, x \in \mathbb{R}^n \setminus \{0\}$. It is now to be shown that the origin is the only invariant for the closed-loop system.

Consider the deformation of ψ to compute a Lyapunov function using the homotopy operator on a one-form ω. The one-form ω is given by

$$\omega = \omega_0 - \sum_{k=1}^{k} \kappa_k (X_k \lrcorner \omega_{0,a} - X_k \lrcorner \omega_0) \cdot (X_k \lrcorner \Omega). \tag{4.29}$$

By assumption $\omega_{0,a} \equiv 0$ and the second term of the right hand side is re-written as $(-X_k \lrcorner \omega_0)$, which is given, for all $k = 1, \ldots, m$, as

$$\begin{aligned} -X_k \lrcorner \omega_0 &= -X_k \lrcorner X_0 \lrcorner \Omega & (4.30)\\ &= X_0 \lrcorner X_k \lrcorner \Omega. & (4.31) \end{aligned}$$

Hence, we have

$$\omega = \omega_0 - \sum_{k=1}^{k} \kappa_k (X_0 \lrcorner X_k \lrcorner \Omega) \cdot (X_k \lrcorner \Omega). \tag{4.32}$$

Applying the homotopy operator on ω, one obtains $\omega = \omega_e + \omega_a$. From Section 3.3, it is known that $\omega = 0$ if and only if ω_a and ω_e vanish at the same point. By assumption, ω_a vanishes only at the origin, at least on the star-shaped domain where the homotopy operator is defined. Hence, it is to be shown that the exact part of ω, *i.e.*, dV vanishes at the origin, which is given, using the expression (4.21), as the requirement

$$\begin{aligned} \omega_e(0) = -dV(0) &= \omega_0(0) - \sum_{k=1}^{m} \xi_k(0)(X_k(0) \lrcorner \Omega) = 0, \quad x = 0 & (4.33)\\ &= \omega_0(0) - \sum_{k=1}^{m} (X_k(0) \lrcorner \omega_0(0)) \cdot (X_k(0) \lrcorner \Omega). & (4.34) \end{aligned}$$

Since $\omega_0(0) = X_0(0) \lrcorner \Omega$, and $X_0(0)$ is assumed to be an isolated equilibrium, it can be concluded that $X_0(0) = 0$, implying $\omega_0(0) = 0$ and that $\omega_e(0) = 0$, as required. Hence, under the assumption that $dV_0 \neq 0$, $\forall\, x \in \mathbb{R}^n \setminus \{0\}$ and that ω_a vanishes only at the origin, the maximal invariant set is $\{0\}$. By the usual Lasalle's arguments, damping feedback stabilizes asymptotically the origin of the system. ■

Remark 4.1.6. *The exactness condition on ω_0 could be relaxed, provided some convexity conditions on $(\mathbb{H}\omega_0)$, which is possible to compute only if a structure is assumed for the drift vector field. The same remark on "classical" Jurdjevic–Quinn conditions were discussed in* (Faubourg and Pomet, 2000)*.*

Remark 4.1.7. *The domain where the the obtained $V(x)$ is assignable by Jurdjevic–Quinn damping feedback can be computed in the following way. Restating condition* (4.4)*, $V(x)$ is an assignable Lyapunov function provided that $\forall\ x \in \mathcal{O} \subset \mathbb{R}^n \setminus \{0\}$,*

$$X_k \lrcorner dV = 0 \Rightarrow X_0 \lrcorner dV < 0, \quad k = 1, \ldots, m. \tag{4.35}$$

Consider

$$\begin{aligned} dV &= \omega_a - \omega & (4.36)\\ &= -\omega_0 + \sum_{k=1}^{m} (\xi_k \cdot (X_k \lrcorner \Omega)) + \omega_a. & (4.37) \end{aligned}$$

The restriction $X_k \lrcorner dV = 0$ leads to

$$X_k \lrcorner dV = -X_k \lrcorner \omega_0 + \sum_{k=1}^{m} \xi_k \left(X_k \lrcorner X_k \lrcorner \Omega \right) + X_k \lrcorner \omega_a = 0. \tag{4.38}$$

By definition, $X_k \lrcorner X_k \equiv 0$, hence the condition is restated as

$$X_k \lrcorner \omega_0 = X_k \lrcorner \omega_a. \tag{4.39}$$

Computing

$$X_0 \lrcorner dV = -X_0 \lrcorner \omega_0 + \sum_{k=1}^{m} \xi_k \cdot (X_0 \lrcorner X_k \lrcorner \Omega) + X_0 \lrcorner \omega_a, \tag{4.40}$$

and using the definition $\omega_0 = X_0 \lrcorner \Omega$ *leads to*

$$X_0 \lrcorner dV = -\sum_{k=1}^{m} \xi_k \cdot (X_k \lrcorner X_0 \lrcorner \Omega) + X_0 \lrcorner \omega_a \tag{4.41}$$

$$X_0 \lrcorner dV = -\sum_{k=1}^{m} \xi_k \cdot (X_k \lrcorner \omega_0) + X_0 \lrcorner \omega_a, \tag{4.42}$$

and with the relation $X_k \lrcorner \omega_0 = X_k \lrcorner \omega_a$ *computed above*

$$X_0 \lrcorner dV = -\sum_{k=1}^{m} \xi_k \cdot (X_k \lrcorner \omega_a) + X_0 \lrcorner \omega_a \tag{4.43}$$

$$= \left(X_0 - \sum_{k=1}^{m} \xi_k \cdot X_k \right) \lrcorner \omega_a < 0, \quad x \in \mathcal{O} \setminus \{0\}, \tag{4.44}$$

where $\xi_k = \kappa_k(X_k \lrcorner \omega_{0,e})$.

As mentioned earlier, restating the results from Section 4.1.2 using differential forms is practical in the sense that locally, one can obtained a Lyapunov function by application of the homotopy operator on the one-form constructed using the closed-loop system. An application of this construction is presented in the next Section to illustrate these observations.

4.1.4 Simple Mechanical System Example

The above construction is applied to a controlled pendulum example, taken from (Sontag, 1998). Consider the control affine system

$$\dot{x}_1 = x_2 \tag{4.45}$$

$$\dot{x}_2 = -\sin(x_1) + u. \tag{4.46}$$

Setting $\Omega = dx_1 \wedge dx_2$, a first dissipative potential is obtained using $\omega_0 = X_0 \lrcorner \Omega$, given as

$$\omega_0 = \sin(x_1)dx_1 + x_2 dx_2. \tag{4.47}$$

Applying the radial homotopy operator defined in Section 3.3, the potential

$$\psi = (\mathbb{H}\omega) = \frac{1}{2}x_2^2 + (1 - \cos(x_1)), \tag{4.48}$$

is obtained with the desired properties locally. In this particular case ω_0 is already exact, hence $\omega_a = \omega_0 - \omega_e = 0$. The obtained function ψ is positive definite, but it is not a control Lyapunov function (it is in fact a first integral, *i.e.*, $\mathcal{L}_{X_0}\psi \equiv 0$). The damping feedback controller $u(x)$ is given as $-\kappa X_1 \lrcorner d\psi = -\kappa X_1 \lrcorner \omega_e = -\kappa x_2$. Applying the radial homotopy operator on the new one-form, one has

$$\omega = -(\kappa x_2 + \sin(x_1))dx_1 + x_2 dx_2 \tag{4.49}$$

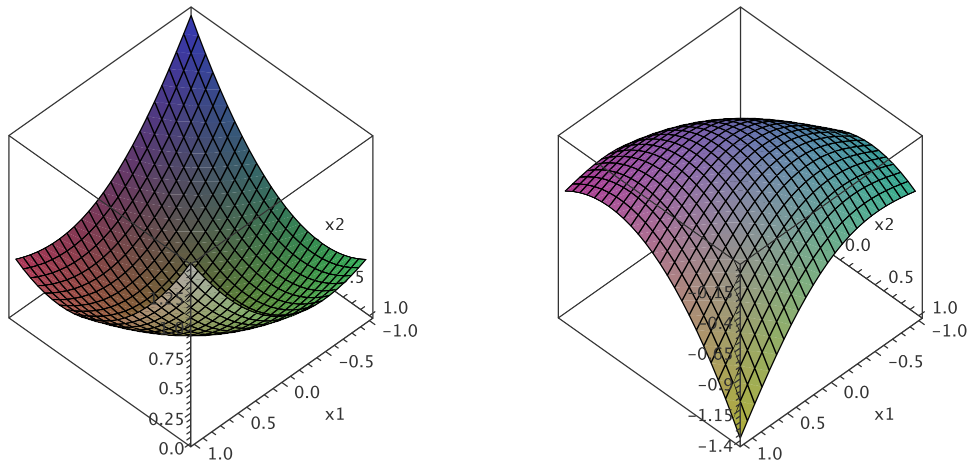
which results after integration, to the local Lyapunov function

$$V = 1 - \cos(x_1) + \frac{1}{2}(\kappa x_1 x_2 + x_2^2). \tag{4.50}$$

Computing $\nabla^T V \cdot X(x, u(x))$, one has

$$\dot{V} = -\frac{\kappa}{2}\left(x_2^2 + \kappa x_1 x_2 + x_1 \sin(x_1)\right) < 0, \tag{4.51}$$

which, for small values of x_1 such that $\sin(x_1) \approx x_1$, is true for $\|x\| < \kappa$. Figures 4.1a and 4.1b present the locally obtained $V(x)$ and $\dot{V}(x)$ in a neighborhood of the origin for $\kappa = 1$. One important aspect to note here is that the one-form ω is not exact. In fact, after the



(a) $V(x)$ (b) $\dot{V}(x)$

Figure 4.1: Simple mechanical example

second application of the homotopy operator, the anti-exact part is given as

$$\omega_a = \frac{\kappa}{2}\left(x_2 dx_1 - x_1 dx_2\right). \tag{4.52}$$

This non-exact part vanishes only at the origin, hence the largest invariant set for the dynamics, as outlined by the non-exact one-form is the origin $\{0\}$.

4.1.5 Stabilization of a Lotka–Volterra System

The stabilization of Lotka–Volterra dynamics, introduced in Chapter 3, is now considered, using the potential derived above. From a stabilization perspective, accessibility properties of this class of systems was studied in (De Leenheer and Aeyels, 2000), stabilization by positive control was presented in (Grognard and Gouzé, 2005) and recently by (Mazenc and Malisoff, 2009). Lyapunov stabilization for a larger class of population dynamics was presented in (Fall et al., 2007).

Consider the following controlled Lotka–Volterra ecology:

$$\begin{aligned}\dot{x}_i &= x_i\left(k_i + \sum_{j\neq i} a_{ij}x_j\right), \quad i = 1,\ldots,n-1\\ \dot{x}_n &= x_n\left(k_n + \sum_{j\neq n} a_{nj}x_j\right) + u\end{aligned}$$

with k_i, the net birth/mortality rate coefficients, $a_{ij} = -a_{ji}$, $\forall i \neq j$, the predation coefficients, and u, a "feeding rate" of specie x_n, $u(t) > 0$, $\forall t$ and $x \in \mathbb{R}^n_+$. Stabilization of this system was studied using Port-Controlled Hamiltonian techniques in (Ortega et al., 1999, 2000).

Consider the 2-dimensional Lotka–Volterra system presented in Section 3.5.2:

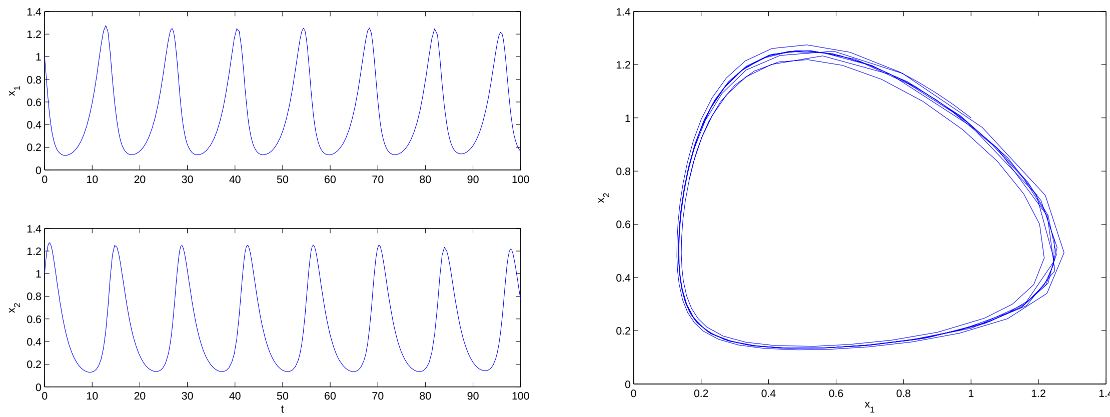
$$\begin{aligned}\dot{x}_1 &= ax_1 - bx_1x_2 + u\\ \dot{x}_2 &= -cx_2 + bx_1x_2\end{aligned}$$

with $a, b, c > 0$. The linearization of the uncontrolled part is given by

$$Df = \begin{bmatrix} a - bx_2 & -bx_1 \\ bx_2 & bx_1 - c \end{bmatrix}$$

Two equilibria exist for the uncontrolled case: A saddle point at $[0,\ 0]^T$ and a center equilibrium at $[\frac{c}{b}, \frac{a}{b}]^T$, surrounded by stable orbits, for example as presented in Figures 4.2a and 4.2b where $a = c = 0.5$, $b = 1$, hence $x^* = [0.5,\ 0.5]^T$, and initial conditions $x(0) = [1,\ 1]^T$.

The objective here is to stabilize the center equilibrium using damping feedback. Setting



(a) State trajectories (b) Phase diagram

Figure 4.2: Lotka–Volterra system in open loop

$\Omega = dx_1 \wedge dx_2$, the following dissipative potential is obtained:

$$\begin{aligned} \psi(x) &= b\frac{x_1x_1^*(x_2^*-x_2)+x_2x_2^*(x_1^*-x_1)-x_1(x_2^*)^2-(x_1^*)^2x_2+x_1^2x_2^*}{6} \\ &+ b\frac{(x_1+x_1^*)x_2^2-x_1^*(x_2^*)^2+x_1^2x_2-(x_1^*)^2x_2^*}{3} \\ &\quad + \frac{(c-a)(x_1^*x_2-x_1x_2^*)+(a+c)(x_1^*x_2^*-x_1x_2)}{2}. \end{aligned} \tag{4.53}$$

The plot of this function for the values given above is depicted in Figure 4.3.

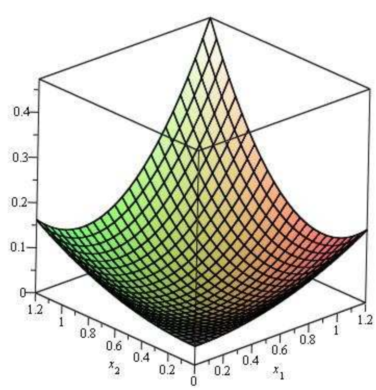
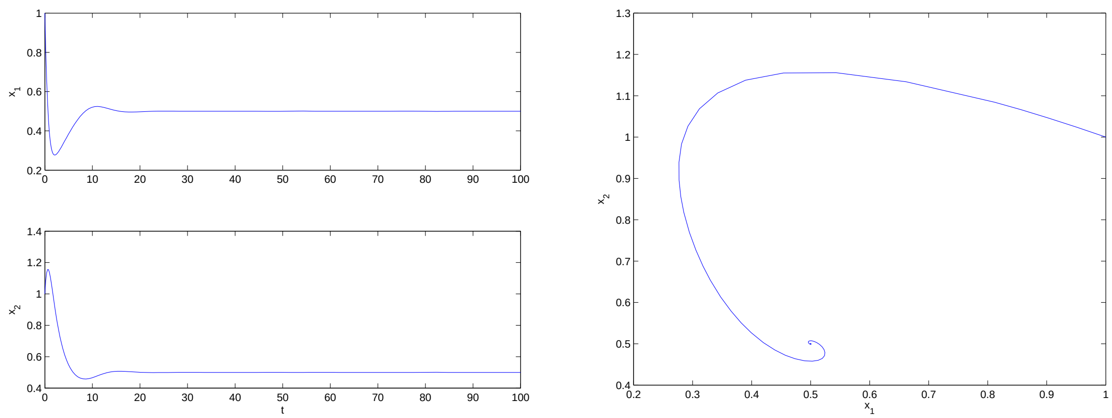


Figure 4.3: Potential function for Lotka–Volterra center stabilization

One can check that ω_a is zero only at the desired equilibrium x^*. The damping controller is given as

$$u(x) = \kappa \left(b\frac{x_1^*(x_2 - x_2^*) + x_2 x_2^* + (x_2^*)^2}{6} + \frac{-a(x_2 - x_2^*) + c(x_2 + x_2^*)}{2} - b\frac{x_1(x_2^* + x_2) + x_2^2}{3} \right). \quad (4.54)$$

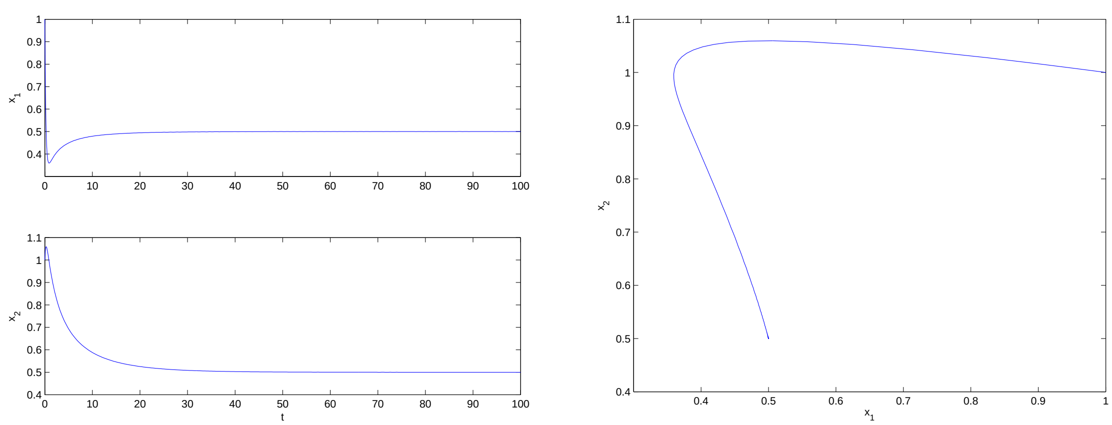
Numerical simulations are given in Figures 4.4a and 4.4b.



(a) State trajectories (b) Phase diagram

Figure 4.4: Lotka–Volterra system in closed loop

By increasing the gain κ, one can accelerate the convergence to the desired equilibrium, as depicted in Figures 4.5a and 4.5b.



(a) State trajectories (b) Phase diagram

Figure 4.5: Lotka–Volterra system in closed loop — higher gain

4.2 Dynamic Feedback Stabilization

4.2.1 Introduction

In this section, the problem of dynamic feedback stabilization is considered. Departing from the usual approaches to the design of dynamic regulator, the exogeneous reference dynamic is generated by a dissipative Hamiltonian system with an isolated equilibrium located at a desired isolated equilibrium of the system to be stabilized. The objective is to design damping feedback such that a control affine system tracks a reference trajectory generated by a desired dissipative Hamiltonian system and stabilizes the system at the origin.[3]

A similar idea was recently presented in Acosta and Astolfi (2009), where the PDEs arising from the IDA-PBC construction were replaced by algebraic inequalities by using an integral approximation construction and a dynamic extension of the control affine system. Stabilization to the origin of an extended system (x, e), where $e = \hat{x} - x$, was studied using a map $H(x, \hat{x}) : \mathbb{R}^n \to \mathbb{R}^n$, computed as

$$H(x, \hat{x}) = \sum_{i=1}^{n} \int_0^{x_i} h_i(\hat{x})|_{\hat{x}_i = s} ds \tag{4.55}$$

where $h(x) = \nabla_x^T H(x, e)|_{e=0}$. Moreover, it was shown that stabilization can be achieved even when the mapping $H(x, \hat{x})|_{\hat{x}=x}$ is not a solution of the matching PDEs. An alternative approach to the dynamic regulator problem is given in (Astolfi et al., 2008) for application in adaptive controller design. In the present section, a different but related approach to stabilize a control affine system using generalized Hamiltonian systems is proposed. In particular, the matching problem will be considered from a feedback regulator design perspective, where the reference signal to track is given by an admissible dissipative Hamiltonian realization. The problem of dynamic feedback stabilization is usually presented as follows

[3]Following the ideas of Chapter 3, the reference dynamic was chosen to be generated by a known Hamiltonian dissipative system. For general exogeneous systems, the reader is referred to (Marino and Tomei, 1995).

(Bacciotti, 1992; Marino and Tomei, 1995). Consider the system

$$\dot{x} = f(x) + g(x)u(x, w) \tag{4.56}$$

where w is generated by the exogeneous system

$$\dot{w} = S(w). \tag{4.57}$$

Solutions to this problem are reviewed extensively in (Byrnes et al., 1997). The approach considered here departs from the usual point of view, $i.e.$, instead of designing the dynamics of w as neutral, the reference system is given as a dissipative Hamiltonian system, of the form defined in Chapter 3.

4.2.2 Problem Formulation

Consider the control affine system (4.1) with $x \in \mathcal{O} \subset \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $X_0(x)$ and $X_i(x)$, $i = 1, \ldots, m$, of class $\mathcal{C}^\infty$ and where $\mathcal{O}$ is a neighborhood of a desired equilibrium. Full state feedback $u(x)$ is considered, and the error is given as $e = x - w$ where $w \in \mathbb{R}^n$ is generated by the dissipative Hamiltonian realization given by

$$\dot{w} = (J(w) - R(w))\nabla_w H(w) \tag{4.58}$$

with $J(w)$ skew-symmetric and $R(w)$ symmetric positive semi-definite. The function $H(w)$ is chosen such that the target system is asymptotically stable at the isolated desired equilibrium w^*. The objective is to compute a potential of the form $\psi(x, w)$ such that system (4.1) in closed-loop with the the damping feedback

$$u_p(x, w) = -\kappa \nabla_x^T \psi(x, w) \cdot X_p(x) \tag{4.59}$$

tracks the desired dissipative dynamics and asymptotically stabilizes a desired equilibrium $(x, e)^T = (x^*, 0)^T$ in the extended space. In other words, the objective is to stabilize the extended system (x, w) to a local isolated minimum (x^*, w^*), where w^* coincides with x^*. In the present section, the reference dynamics is fixed as a n-dimensional dissipative Hamiltonian system. The only restriction is that the isolated asymptotically stable equilibrium of the w-subsystem must coincide with the desired x-subsystem equilibrium. In the example considered in Section 4.2.4, the origin is not the desired admissible equilibrium.
The next section presents the construction of a damping feedback regulator based on a potential derived from the extended dynamics $(w, e) \in \mathbb{R}^{2n}$.

4.2.3 Construction of a Feedback Regulator

A dynamic feedback regulator is now constructed for the problem presented in Section 4.2.2. As outlined above, the reference system is of the form

$$\dot{w} = F(w)\nabla H(w) \tag{4.60}$$

with known structure $F(w)$ and Hamiltonian function $H(w)$, locally asymptotically stable at an isolated desired equilibrium $w^* = x^*$. Discussion on stability properties of these systems can be found, for example in Ortega et al. (2002). By definition, $x = w - e$, hence (4.1) is re-expressed in terms of w and e to obtain the extended drift system as

$$\dot{w} = F(w)\nabla_w H(w) \tag{4.61}$$

$$\dot{e} = F(w)\nabla_w H(w) - f(w, e). \tag{4.62}$$

The vector field of the augmented system is given by

$$X_0(w, e) = \left(F(w)\nabla_w H(w)\right)\partial_w + \left(F(w)\nabla_w H(w) - f(w, e)\right)\partial_e. \tag{4.63}$$

We define a non-vanishing closed two-form $\Omega(w,e)$ on $\mathbb{R}^{2n}$ as

$$\Omega = \sum_{1\leq i<j\leq n} \left(dw_i \wedge dw_j + de_i \wedge de_j\right). \tag{4.64}$$

The orientation of the two-form will be fixed, if necessary, by checking the sign of the obtained dissipative function, $\psi(w,e)$, in a neighborhood of the origin.

As in the previous section, a one-form associated to the system is obtained by contracting the above two-form with respect to the extended vector fields:

$$\omega_0 = \left(F(w)\nabla_w H(w)\right)\partial_w \lrcorner \Omega + \left(F(w)\nabla_w H(w) - f(w,e)\right)\partial_e \lrcorner \Omega. \tag{4.65}$$

Then, applying a homotopy centered at $e=0$ and $w=w^*=x^*$, such that

$$\mathfrak{X}(w,e) = \sum_{i=1}^{n} \lambda e_i \partial_{e_i} + \left(x^* + \lambda(w-x^*)\right)\partial_{w_i}, \tag{4.66}$$

one obtains

$$\psi(w,e) = \mathbb{H}\omega_0 = \int_0^1 \omega_{0,w}(\lambda(w-x^*))\lambda(w-x^*)d\lambda + \int_0^1 \omega_{0,e}(\lambda(w-x^*),\lambda e_i)\lambda e_i d\lambda \tag{4.67}$$

where

$$\omega_{0,w} = \left(F(w)\nabla_w H(w)\right)\partial_w \lrcorner dw_i \wedge dw_j \tag{4.68}$$

$$\omega_{0,e} = \left(F(w)\nabla_w H(w) - f(w,e)\right)\partial_e \lrcorner de_i \wedge de_j \tag{4.69}$$

evaluated on $\mathfrak{X}$.

The dissipative potential $\psi(w,e)$ centered at $e=0$ and $w=x^*$ will be the basis of the stabilization design that is presented in the following section. The main result of this section is now given.

Theorem 4.2.1. *The extended system (w, e) is locally asymptotically stable in a neighborhood of $(w^*, 0)$ provided that*

(i) The dissipative Hamiltonian reference dynamic is chosen such that w^* is an isolated asymptotically stable equilibrium of

$$\dot{w} = F(w)\nabla_w H(w),$$

(ii) The dissipative potential $\psi(w, e)$ is such that

$$\sum_{i=1}^{n} \frac{\partial^2 \psi}{\partial e_i^2} \neq 0 \tag{4.70}$$

for fixed w in a neighborhood of the origin $\mathcal{O} \subset \mathbb{R}^n \setminus \{0\}$ of the error dynamics.

Proof: Both conditions are consequences of the construction presented in Chapter 3. Since the choice of the reference dynamics is arbitrary, it can be shown, for a suitable choice of dynamics, that

$$\psi_w(w) = \int_0^1 \omega_{0,w}(\lambda(w - x^*))\lambda(w - x^*)d\lambda \tag{4.71}$$

is a dissipative potential centered at w^*. Asymptotic stability in a neighborhood of $(w^*, \cdot)$ follows from Ortega et al. (2002). For part (ii), note that the cross-terms in e and w from

$$\psi_e(w, e) = +\int_0^1 \omega_{0,e}(\lambda(w - x^*), \lambda e_i)\lambda e_i d\lambda \tag{4.72}$$

are convex in e_i. Hence, for fixed w, if $\frac{\partial^2 \psi}{\partial e_i^2} \neq 0$, it is possible to apply, following Chapter 3, a coordinate change defined by

$$z_i = -\frac{1}{2}\left(\frac{\partial \psi}{\partial e_i}\right)\left(\sum_{i=1}^{n} \frac{\partial^2 \psi}{\partial e_i^2}\right)^{-1} \tag{4.73}$$

that transforms the drift system to a dissipative Hamiltonian realization defined by

$$\dot{z} = (J(z) - R(z))\nabla H(z), \tag{4.74}$$

with $H(z)$ defined by $H(z) = \frac{1}{2}\sum_{i=1}^{n} z_i^2$. The transformation maps the origin of $(e_1, \ldots, e_n)$ to the origin of $(z_1, \ldots, z_n)$. In the new coordinates,

$$\bar{\omega}_e = -\frac{1}{2}\sum_{i=1}^{n} z_i dz_i, \tag{4.75}$$

from which a Lyapunov function can be computed, using $\bar{X}(z) = (J(z) - R(z))\nabla H(z)\partial_z$, as

$$V = \bar{X} \lrcorner \bar{\omega}_e \tag{4.76}$$

$$= \frac{1}{2}\sum_{i=1}^{n} z_i^2. \tag{4.77}$$

Taking the derivative with respect to time,

$$\dot{V} = -\sum_{i=1}^{n} z_i^2. \tag{4.78}$$

The origin of the error sub-system is therefore locally asymptotically stable for any fixed w. ■

The next section presents the construction of the dynamic feedback regulator using ψ in a Jurdevic–Quinn controller of the form $u(x, w) = -\kappa\nabla_x\psi(x, w) \cdot g(x)$.

The locally-defined dissipative potential function computed in the last section to stabilize the system in the extended space. The development follows the standard Jurdjevic–Quinn arguments (Malisoff and Mazenc, 2009, Chapter 4), summarized in Chapter 2. In the extended space, the Jurdjevic–Quinn controller design approach can be summarized as follows. Re-writing ψ using the definition $e = x - w$, it is assumed that $\psi(x, w)$ is locally a weak Jurdjevic–Quinn function , $i.e.$, such that $\psi(x,w) > 0$ and $(\nabla^T\psi \cdot X_0)(x,w) < 0$ for all (x,w) in a neighborhood $\mathcal{O} \subset \mathbb{R}^{2n} \setminus \{(x^*,w^*)\}$, $\psi(0) = 0$ and $(\nabla^T\psi \cdot X_k)(x^*) = 0$. In practice, one may use an integrating factor $\gamma(x,w)$ to guarantee that

$$\psi(x,w) = -\left(\mathbb{H}(\gamma\omega_0)\right)(x,w) \tag{4.79}$$

has the desired properties. The anti-exact part $\omega_a = \omega_0 - d\psi$ does not contribute locally to the dissipative dynamics and as a result it is not taken into account for the design presented in the present section. In practice, a feedback gain κ could be used to dominate the tangential dynamics, $i.e.$, it is possible to construct the damping feedback controller as

$$u_k(x) = -\kappa(\nabla^T\psi \cdot g)(x). \tag{4.80}$$

The approach for the regulator construction can be summarized as follows. Assume that for every $x \in \mathbb{R}^n \setminus \{x^*\}$,

$$\mathrm{span}\{X_0(x), \mathrm{ad}^k_{X_0}X_p(x),\ k \in \mathbb{N}\} = \mathbb{R}^n.$$

Let w be generated by the dissipative Hamiltonian reference system,

$$\dot{w} = F(w)\nabla_w H(w)$$

with isolated asymptotically stable equilibrium at $w^* = x^*$. Then, the nonlinear system

$$\dot{x} = X_0(x) + \sum_{i=1}^{p} X_i(x)u_i(x,w)$$

is locally stabilized to a desired isolated equilibrium x^* by damping feedback $u = -\kappa\nabla^T_x\psi(x,w)\cdot g(x)$, where $\psi(x,w)$ is a weak Jurdjevic–Quinn function computed above.

The next section illustrates the application of this construction for predator-prey systems.

4.2.4 Application to Predator-Prey Systems

The application of the dynamic feedback regulator is applied to the Lotka–Volterra system presented in Chapter 3 and in Section 4.1. First, consider a two-dimensional Lotka–Volterra system:

$$\dot{x}_1 = ax_1 - bx_1x_2 + u \tag{4.81}$$

$$\dot{x}_2 = -cx_2 + bx_1x_2, \tag{4.82}$$

with $x_1 \geq 0$ and $x_2 \geq 0$ and where a, b, c are known positive constants. It is desired to stabilize the system to the center equilibrium $x^* := [x_1^*,\ x_2^*]^T = [\frac{c}{b}, \frac{a}{b}]^T$ by tracking the dissipative Hamiltonian reference system centered at the desired equilibrium

$$\dot{w}_1 = -(w_1 - x_1^*) - (w_2 - x_2^*) \tag{4.83}$$

$$\dot{w}_2 = (w_1 - x_1^*) - (w_2 - x_2^*). \tag{4.84}$$

Letting $z_i = (w_i - x_i^*)$, the reference system is of the form

$$z = \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \nabla \bar{H}. \tag{4.85}$$

with $\bar{H} = \frac{1}{2}(z_1^2 + z_2^2)$. Stability of the origin of this reference signal can be shown using $\bar{H}$ as a Lyapunov function, as discussed previously in Chapter 3.

Set $e_i = x_i - w_i$, $i = 1, 2$, and re-express the drift dynamic of (4.81)-(4.82) in terms of e

and w, and one obtains

$$f(w,e) = \begin{bmatrix} a(e_1+w_1) - b(e_1+w_1)(e_2+w_2) \\ -c(e_2+w_2) + b(e_1+w_1) \end{bmatrix}. \tag{4.86}$$

With $\Omega = dw_1 \wedge dw_2 + de_1 \wedge de_2$ and the reference dynamic (4.83)-(4.84), one can compute $\omega_0(w,e)$ following (4.65) and a potential, $\psi(w,e)$, by application of the homotopy operator. Here, we omit the expression of $\psi(w,e)$, however it is possible to show that the obtained potential is convex with respect to the error (e_1, e_2) for $x_1 > 0$ and $x_2 > 0$, since

$$\frac{\partial^2 \psi}{\partial e_1^2} = b(e_2+w_2) = bx_2 \tag{4.87}$$

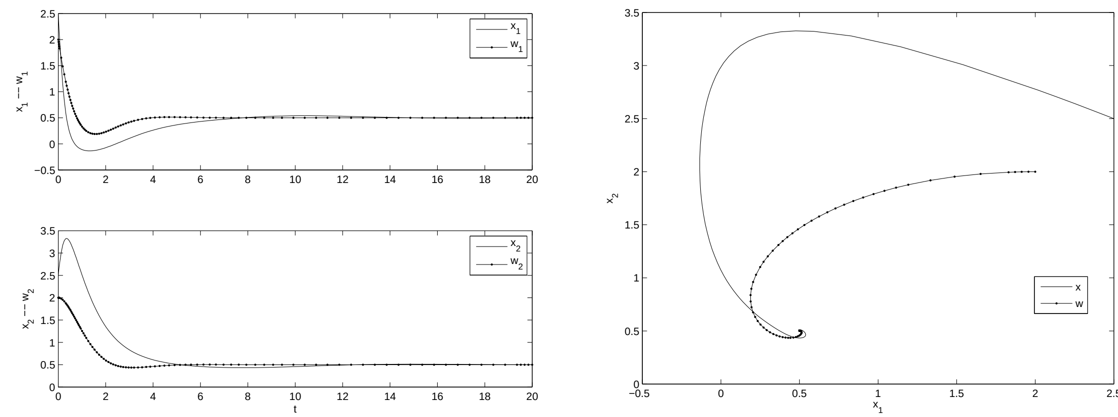
$$\frac{\partial^2 \psi}{\partial e_2^2} = b(e_1+w_1) = bx_1, \tag{4.88}$$

and hence following the argument from Section 4.2.3, $(e,w) = (0, w^*) \in \mathbb{R}^{2n}$ is an asymptotically stable point for the extended dynamics. Hence, as long as the initial conditions are different than zero, *i.e.*, $x_1(0) \neq 0$ and $x_2(0) \neq 0$, the dynamic regulator steers the system to the desired equilibrium x^*.

Re-expressing the obtained potential in terms of x and w using the definition of the error, one has that $\psi(x,w) > 0$ in the positive orthant and satisfies the Jurdjevic–Quinn conditions. The damping regulator $u(x,w) = -\nabla_x \psi^T(x,w) \cdot g(x)$ is given by

$$u(x,w) = \frac{bx_2(w_1+w_2)}{2} - \frac{w_2 - bx_2^2}{2} + \frac{cx_2}{2} + \frac{a(x_2-w_2)}{2} + \frac{(w_1 - x_1^* + x_2^*)}{2} - bx_1x_2. \tag{4.89}$$

For numerical simulations, the parameters are set to $a = c = 0.5$, $b = 1$, leading to the desired equilibrium at $x^* = [0.5,\ 0.5]$. Closed-loop simulations using the above regulator with initial conditions $x(0) = [2.5,\ 2.5]^T$ and $w(0) = [2,\ 2]^T$ are presented in Figures 4.6a, 4.6b and 4.6c.



(a) State trajectories (b) Phase diagram

Figure 4.6: Stabilization of Lotka–Volterra system by dynamic feedback

The potential of the present extension is now illustrated by considering a predator-prey model with a higher order coupling function, keeping the same dissipative Hamiltonian reference system as before. Consider

$$\dot{x}_1 = ax_1 - bx_1^5x_2^3 + u \tag{4.90}$$

$$\dot{x}_2 = -cx_2 + bx_1^5x_2^3. \tag{4.91}$$

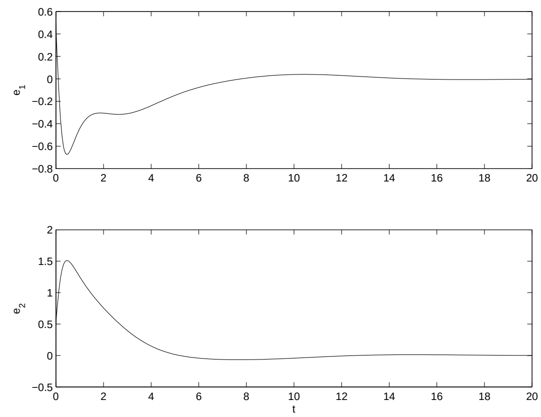
The new desired equilibrium is located at $x^* = [0.9,\ 0.9]^T$.

The performance of the regulator obtained from the application of the construction from Section 4.2.3, with initial conditions $x(0) = w(0) = [2,\ 2]^T$, is illustrated in Figures 4.7a and 4.7b.

4.3 Stabilization to a Periodic Orbit by Damping

4.3.1 Introduction

This section briefly illustrates the stabilization of admissible periodic orbits of the drift dynamics using state feedback damping control. The approach follows the extension of the



(c) Error dynamics

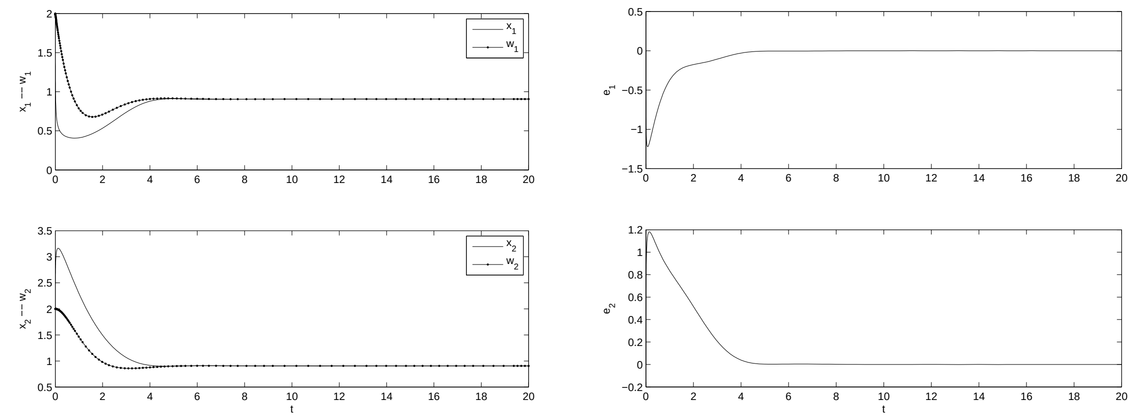
Figure 4.6: Stabilization of Lotka–Volterra system by dynamic feedback

Jurdjevic–Quinn approach suggested originally in (Bacciotti and Mazzi, 1995) and used in (Bombrun, 2007, Chapter 3) in the context of smooth orbital stabilization of satellite trajectories. A related approach, using a speed-gradient algorithm, was exploited in (Shiriaev and Fradkov, 2001), where periodic orbits of mechanical systems were stabilized using the Chetaev's method for Lyapunov function construction based on first integrals. Application of stabilization using first integrals of mass-action systems (and more generally positive systems) was presented in (De Leenheer and Aeyels, 2002). Stabilization of periodic orbits was also considered in (Aracil et al., 2005) using backstepping.

The result in (Bacciotti and Mazzi, 1995) simplifies the classical construction of Chetaev used in (Shiriaev and Fradkov, 2001) by showing that if a first integral I_0 is known for the system, then a suitable potential to apply the Jurdjevic–Quinn approach to stabilize existing periodic orbits Γ of the drift dynamics indexed by their "energy" level K is given by

$$\psi(x) = \frac{1}{2}(I_0 - K)^2. \tag{4.92}$$

The approach favored here is to compute a first integral for the system by finding suitable



(a) State trajectories

(b) Error dynamics

Figure 4.7: Stabilization of predator-prey model with higher coupling by dynamic feedback

integrating factors such that the one-form obtained by taking the interior product of a two-form with respect to a scaled vector field is closed (and hence, exact).

The section is divided as follows. The result from (Bacciotti and Mazzi, 1995) is summarized in Section 4.3.2. The construction of a damping feedback controller to a periodic orbit using this result is illustrated through an example in Section 4.3.3.

4.3.2 Construction of a Damping Feedback

A constructive result extending the Jurdjevic–Quinn approach to stabilize periodic orbits is given in (Bacciotti and Mazzi, 1995) and is presented here. The reader is referred to Section 2.2.4 for the background on orbital stability.

Theorem 4.3.1. *Let* (4.1) *be as above and assume it satisfies the following conditions:*

(i) There exists a cycle Γ *for the drift system* $X_0(x)$*,*

(ii) there exists a neighborhood U_0 *of* Γ *and a function* $V(x) \in \mathcal{C}^2$ *such that* $V|_\Gamma \equiv 0$*,* $V(x) > 0, \forall x \notin \Gamma$*,* $\nabla V \neq 0, \forall x \notin \Gamma$*, and* $\dot{V}(x) = (\nabla V \cdot X_0)(x) \leq 0$*,*

(iii) there exists a neighborhood of Γ, $U_1 \subset U_0$, *such that, for each* $x \in U_1 \setminus \Gamma$,

$$\mathrm{span}\{X_0(x), \mathrm{ad}_{X_0}^k X_i(x); i = 1, \ldots, m; k \geq 0\} = \mathbb{R}^n. \tag{4.93}$$

Then there exists m *functions* $u_1(x), \ldots, u_m(x)$ *defined on* U_1, *of class* $\mathcal{C}^1$ *such that* Γ *is an orbitally asymptotically stable limit cycle for the system* (4.1).

To construct a function V to stabilize the desired periodic orbit, (Bacciotti and Mazzi, 1995) proposed an approach using a first integral of the drift system. Given the uncontrolled part of the dynamics $\dot{x} = X_0(x)$, a scalar function $I_0(x)$ is a first integral for the system if I_0 is constant on an orbit, *i.e.*, $\nabla I_0 \cdot X_0(x) \equiv 0$ on an orbit.

An admissible potential function to stabilize an admissible energy level, K, is

$$\psi(x) = \frac{1}{2}(I_0 - K)^2.$$

The proposed construction for this problem, using the construction from above, is presented using an example in the following.

4.3.3 Application to a Lotka–Volterra System

The stabilization of the 2-dimensional Lotka–Volterra system presented above to a periodic orbit with "energy" level K surrounding the center equilibrium $x^* = [0.5,\ 0.5]^T$ is given. In this particular case, it is known that the center equilibrium is surrounded by stable periodic orbits. The control task is therefore to transfer the system from one level of periodic evolution to another. The proposed approach consists in finding integrating factors $\alpha(x_1, x_2)$ and $\beta(x_1, x_2)$ such that the one-form

$$\omega = \alpha(x_1, x_2) f_2 dx_1 - \beta(x_1, x_2) f_1 dx_2 \tag{4.94}$$

is closed. A first integral for the system can be computed solving the condition $d\omega \equiv 0$ for $\alpha(x_1, x_2)$ and $\beta(x_1, x_2)$, *i.e.*, computing integrating factors such that the one-form is closed. The problem results in the solution of the following partial differential equations :

$$\begin{aligned} \frac{\partial \alpha}{\partial x_2}(x_1, x_2) f_2(x_1, x_2) + \alpha(x_1, x_2) \frac{\partial f_2}{\partial x_2}(x_1, x_2) &= 0 \\ \frac{\partial \beta}{\partial x_1}(x_1, x_2) f_1(x_1, x_2) + \beta(x_1, x_2) \frac{\partial f_1}{\partial x_1}(x_1, x_2) &= 0. \end{aligned}$$

This set of partial differential equations can be related to the ones used in the context of port-controlled Hamiltonian passivity-based control given for Lotka–Volterra systems in (Ortega et al., 2000, 1999). A particular solution of the system of nonlinear PDEs is

$$\alpha(x_1, x_2) = \frac{1}{x_2} \exp(-x_1) \tag{4.95}$$

$$\beta(x_1, x_2) = \frac{1}{x_1} \exp(-x_2) \tag{4.96}$$

and the one-form to be used for the computation of the desired potential is

$$\omega = (-c + bx_1) \exp(-x_1) dx_1 + (-a + bx_2) \exp(-x_2) dx_2. \tag{4.97}$$

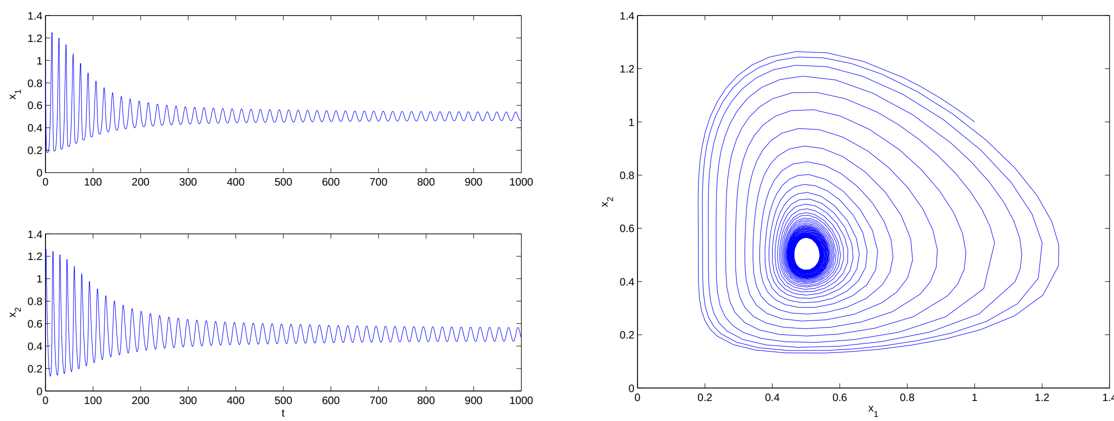
Using $I_0 = (\mathbb{H}\omega)$ as the first integral in a neighborhood of the center equilibrium x^*, one obtains

$$\begin{aligned} I_0 = {} & (b(1 + x_1^*) - c) \left(\exp(-x_1^*) - \exp(-x_1)\right) \\ & + (b(1 + x_2^*) - a) \left(\exp(-x_2^*) - \exp(-x_2)\right). \end{aligned} \tag{4.98}$$

Following (Bacciotti and Mazzi, 1995), the desired damping controller is given as $u = -\kappa \cdot \nabla(I_0 - K)^2$, for a given K:

$$u = \kappa\phi(x_2)\Big((b(1+x_1^*) - c)\,(\exp(-x_1^*) - \exp(-x_1)) + (b(1+x_2^*) - a)\,(\exp(-x_2^*) - \exp(-x_2)) - K\Big), \tag{4.99}$$

with $\phi(x_2) = (bx_2)(bx_2 - a)\exp(-x_2)$. Numerical simulations for $a = c = 0.5$ and $b = 1$ are presented in Figures 4.8a and 4.8b.



(a) State trajectories (b) Phase diagram

Figure 4.8: Lotka–Volterra cycle stabilization

As noted above, the solution presented here for the computation of the integrating factors using the set of partial differential equations is not systematic. The objective here was to show that the proposed approach could be applied to recover the original stabilization result presented in (Ortega et al., 1999). The same idea is exploited in Section 5.3.2. It is clear that further research on this problem requires a careful study of the partial differential equations involved.

4.4 Summary and Extensions

In this chapter, the problem of Lyapunov function construction was first considered for the stabilization of nonlinear control affine systems satisfying Jurdjevic–Quinn conditions, following the original contributions from (Faubourg and Pomet, 2000) and (Mazenc and Malisoff, 2006). Provided that a positive definite function (a dissipative potential) is obtained by taking the interior product of a non-vanishing two-form with respect to the drift vector field, as presented in Chapter 3, it was shown that a Lyapunov function can be computed for the closed-loop vector field subject to damping feedback control under some conditions on the anti-exact part of a one-form obtained for the system. The choice of the two-form must be made such that the potential function fulfills locally the Jurdjevic–Quinn weak conditions. The proposed method is local since the approach relies on a homotopy operator centered at a desired equilibrium point. The construction was extended to the construction of locally defined static feedback regulators for control affine systems to track trajectories generated by dissipative Hamiltonian reference systems, by applying the method in an extended space $[e,\ w]^T$. Under a convexity condition of the obtained potential with respect to the error and for a suitable choice of dissipative Hamiltonian reference dynamics, a damping feedback stabilizing controller was designed using the Jurdjevic–Quinn approach. Finally, following the ideas of Bacciotti and Mazzi (1995), the technique was used to construct damping feedback controllers stabilizing desired existing periodic orbits, by computing a first integral for the drift system using integrating factors.

Further research on time-independent systems will focus on systematic computation of the domain of attraction, and to develop an approach to extend this domain of attraction, by rendering the obtained dissipative potential convex, following the ideas presented in (Rantzer and Parrilo, 2000; Rantzer, 2001). Another possible investigation is the characterization of the non-exact part, ω_a, when it is not vanishing only at the origin. Cancelation techniques, such as the one discussed in Section 3.6 could be of interest in that case.

Some problems related to the Jurdjevic–Quinn technique not addressed in the present thesis are discussed in the following.

4.4.1 Multiple Time Scale Systems Stabilization

The stabilization problem and the construction of Lyapunov functions on multiple time scales was considered extensively in the literature, for example in (Chow and Kokotović, 1978), (Saberi and Khalil, 1984), (Saberi and Khalil, 1985), (Sharkey and O'Reilly, 1988), Sharkey (1988) (see the classical reference (Kokotović et al., 1999, Chapter 7) for an extensive review of the problem). A Lyapunov-based sequential design procedure for the design of fast and slow controls is given in (Saberi and Khalil, 1985). In (Marino and Kokotović, 1988), geometric properties of singularly perturbed systems are investigated, leading to a coordinate-free characterization of time-scales around an invariant manifold. This geometric approach was employed for composite control design in (Sharkey and O'Reilly, 1988) and (Sharkey, 1988). The problem of strict Lyapunov function construction was addressed recently in Malisoff and Mazenc (2009) (see especially Chapters 10 and 11 and references therein) in a different context than the one considered above.

The interest for singularly perturbed systems was revived in the last few years in the context of drug delivery applications. Many biological systems evolve at two or more time scales, especially when drug infusion dynamics is considered. For example the analysis of a simplified model of HIV controlled dynamics given in (Brandt and Chen, 2001; Barão and Lemos, 2007; Ge et al., 2005). The model is given as

$$\begin{aligned}
\dot{x}_1 &= (s - dx_1 - \beta x_1 x_3) + (\beta x_1 x_3)u_1 \\
\dot{x}_2 &= (\beta x_1 x_3 - \mu x_2) - (\beta x_1 x_3)u_1 \\
\dot{x}_3 &= (kx_2 - cx_3) - (kx_2)u_2
\end{aligned}$$

where x_1 denotes the concentration of healthy cells, x_2 is the concentration of infected cells and x_3 is the concentration of virions. Following (Barão and Lemos, 2007), and with the parameters provided therein, one can see that the equation for x_3 converges quickly to the equilibrium $x_3^* = (1-u_2)\frac{k}{c}x_2$. Assuming only one control ($i.e.$, setting $u_2 \equiv 0$), the study in (Barão and Lemos, 2007) lead to the stabilization of the slow dynamics

$$\dot{\tilde{x}}_1 = \left(s - d\tilde{x}_1 - \frac{\beta k}{c}\tilde{x}_1\tilde{x}_2\right) + \left(\frac{\beta k}{c}\tilde{x}_1\tilde{x}_2\right)u_1 \tag{4.100}$$

$$\dot{\tilde{x}}_2 = \left(\frac{\beta k}{c}\tilde{x}_1\tilde{x}_2 - \mu\tilde{x}_2\right) - \left(\frac{\beta k}{c}\tilde{x}_1\tilde{x}_2\right)u_1 \tag{4.101}$$

The idea that could be considered in future research is as follows. One could seek to construct two local functions $\psi_{\text{slow}}(x,y)$ and $\psi_{\text{fast}}(x,y)$ using a locally defined homotopy operator for $f_{0,\text{slow}}(x,y)$ and $f_{0,\text{fast}}(x,y)$. The stabilization problem would be, on two-time scale singularly perturbed control affine systems, to study the construction of a Lyapunov function for closed-loop dynamics of the form

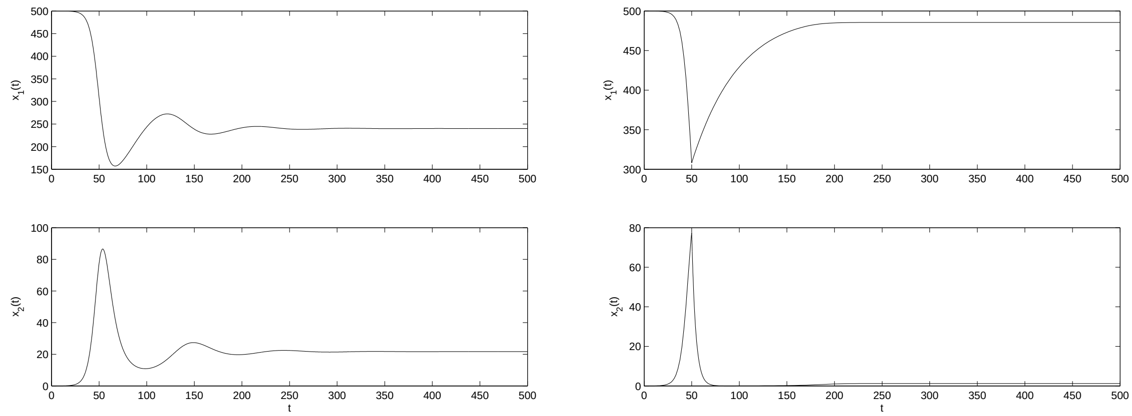
$$\dot{x} = f_{0,\text{slow}}(x,y,\epsilon) + \sum_{i=1}^{p_1} g_{i,\text{slow}}(x,y,\epsilon)u_s \tag{4.102}$$

$$\epsilon\dot{y} = f_{0,\text{fast}}(x,y,\epsilon) + \sum_{j=1}^{p_2} g_{i,\text{fast}}(x,y,\epsilon)u_f, \tag{4.103}$$

with $x \in \mathbb{R}^n$ is the slow part of the state dynamics , $y \in \mathbb{R}^m$ is the fast part of the dynamics, $u_s \in \mathbb{R}^{p_1}$ is the slow control, $u_f \in \mathbb{R}^{p_2}$ is the fast control, and ϵ is a scalar constant parameter. Assuming that f_i, $i = 0, 1, \ldots, p_1$, and g_i, $j = 0, 1, \ldots, p_2$ are smooth, one would seek to obtain a potential

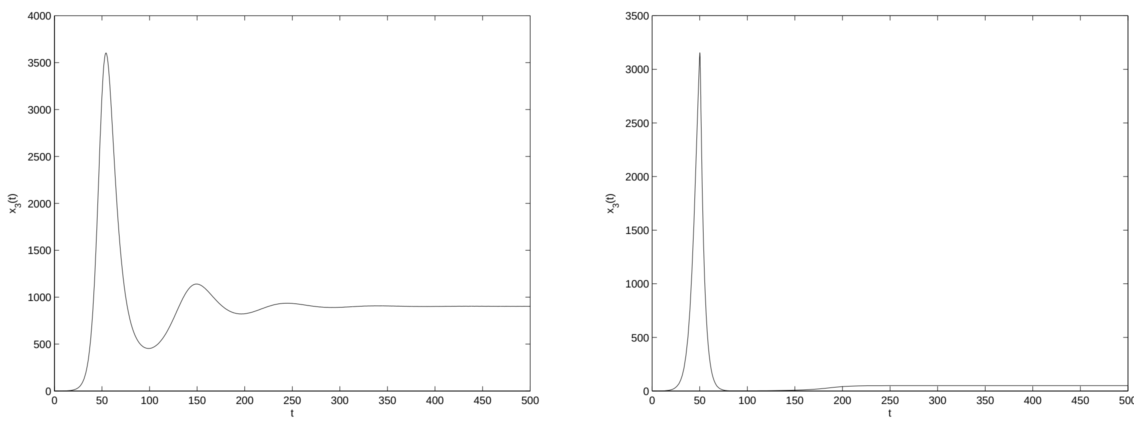
$$\psi(x,y,\epsilon) = \psi_{\text{slow}}(x,y,\epsilon) + \epsilon\psi_{\text{fast}}(x,y). \tag{4.104}$$

The problem of designing stabilizing control $u(\cdot)$ would then follow the approach of Saberi and Khalil (1985). Application of this approach to the stabilization of the reduced HIV model, without Lyapunov function construction, are given in Figures 4.9a and 4.9b for the slow dynamics and 4.10a and 4.10b for the fast dynamics.



(a) State trajectories — open-loop (b) State trajectories — closed-loop

Figure 4.9: Reduced HIV model control on two-time scales — slow dynamics



(a) State trajectories — open-loop (b) State trajectories — closed-loop

Figure 4.10: Reduced HIV model control on two-time scales — fast dynamics

4.4.2 Nonaffine Systems Stabilization

The problem of stabilizing nonlinear systems of the form

$$\dot{x} = F(x, u) \tag{4.105}$$

with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ and with F assumed to be smooth and $F(0,0) = 0$, was not considered in the present thesis. However, the construction of smooth damping feedback controllers for this general case was considered in (Lin, 1994, 1995a,b) and in (Malisoff and Mazenc, 2009, Section 4.3) using Jurdjevic–Quinn feedback design. Precisely, in (Malisoff and Mazenc, 2009), the system is first re-written as

$$F(x,u) = f(x) + g(x)u + h(x,u)u, \tag{4.106}$$

where $f(x) = F(x,0)$, $g(x) = \frac{\partial F}{\partial u}(x,0)$ and

$$h(x,u) = \int_0^1 \left[\frac{\partial F}{\partial u}(x,\lambda u) - \frac{\partial F}{\partial u}(x,0)\right] d\lambda. \tag{4.107}$$

The following assumption is used in (Malisoff and Mazenc, 2009).

Assumption 4.4.1. *There is a storage function* $V : \mathbb{R}^n \to [0,\infty)$ *such that* $\mathcal{L}_f V(x) \leq 0$ *everywhere. Moreover, there is a smooth scalar function* ψ *such that if* $x \neq 0$ *is such that* $\mathcal{L}_f V(x) = 0$ *and* $\mathcal{L}_g V(x) = 0$ *both hold, then* $\mathcal{L}_f \psi(x) < 0$.

Then, Malisoff and Mazenc (2009, Theorem 4.2) showed how to construct a Lyapunov function for this class of systems using a deformation of the form

$$\mathcal{V}(x) = \lambda(V(x))\psi(x) + \Gamma(V(x)), \tag{4.108}$$

and a corresponding Jurdjevic–Quinn controller

$$u(x) = -\xi(x)\mathcal{L}_g V(x)^T \tag{4.109}$$

that globally asymptotically stabilizes the equilibrium (origin) of (4.105).

The control of nonlinear systems of that form is of importance in practice, for example for the control of tumors, using for example the model suggested in (de Pillis and Radunskaya,

2003) for the tumor control chemotherapy. In that case, the killing rate differs for each cells and is expressed as a response curve of the form

$$F(u) = a(1 - \exp(-ku)). \tag{4.110}$$

The model to consider (de Pillis and Radunskaya, 2003) is, in that particular case, given by

$$\begin{aligned}
\dot{N} &= r_2 N(1 - b_2 N) - c_4 TN - a_3(1 - \exp(-u))N \\
\dot{T} &= r_1 T(1 - b_1 T) - c_2 IT - a_2(1 - \exp(-u))T \\
\dot{I} &= s + \frac{\rho IT}{\alpha + T} - c_1 IT - -d_1 I - a_1(1 - \exp(-u))I,
\end{aligned}$$

with N denoting the number of hosts, T the number of tumor cells and I the number of immune cells.

The problem of damping stabilization of a desired equilibrium point for this problem could be considered in the future.

4.4.3 Bilinear Systems Stabilization

As mentioned usually in the literature, the Jurdjevic–Quinn approach is often related, from a historical point of view, to a construction of Lyapunov-based stabilizers presented originally by (Jacobson, 1977) for the bilinear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} u_1 + \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} u_2. \tag{4.111}$$

Stabilization of bilinear systems was also considered in (Bacciotti and Boieri, 1991), (Chabour et al., 1993), (Gauthier and Kuptka, 1992), (Gutman, 1981), (Quinn, 1980) and (Ryan and Buckingham, 1983). As noted in (Bacciotti, 1992), the Jurdjevic–Quinn approach for the construction of damping feedback controllers is suited for systems where dissipation is already present. This observation follows, to some extent, the original definitions given in (Willems, 1972). In practice, this "natural dissipation" is found by inspecting the driftless term. However, in the bilinear case from (Jacobson, 1977) there are no drift dynamics, hence one could ask if the construction presented in this thesis can be modified. For the particular example from (Jacobson, 1977), a function $V(x)$ is to be designed such that

$$\mathcal{L}_{X_1} V \cdot X_1 + \mathcal{L}_{X_2} V \cdot X_2 < 0 \tag{4.112}$$

on $\mathbb{R}^2 \setminus \{0\}$. This can be re-expressed as

$$\begin{aligned}(X_1 \lrcorner dV) \cdot X_1 + (X_1 \lrcorner dV) \cdot X_2 &< 0 && (4.113)\\ (X_1 \lrcorner \omega_e) \cdot X_1 + (X_1 \lrcorner \omega_e) \cdot X_2 &< 0. && (4.114)\end{aligned}$$

Let $\omega = [X_1, X_2] \lrcorner \Omega$. By relation (A.11), one obtains

$$\begin{aligned}\omega &= [X_1, X_2] \lrcorner \Omega && (4.115)\\ &= X_1 \lrcorner d(X_2 \lrcorner \Omega) - X_2 \lrcorner d(X_1 \lrcorner \Omega) - X_2 \lrcorner X_1 \lrcorner d\Omega + d(X_1 \lrcorner X_2 \lrcorner \Omega). && (4.116)\end{aligned}$$

Using the fact that $X_k \lrcorner X_k \lrcorner \Omega = 0$, the above stabilization condition is re-written using $\omega_e = \omega - \omega_a$ as

$$\begin{aligned}&\Big(- X_1 \lrcorner X_2 \lrcorner d(X_1 \lrcorner \Omega) + X_1 \lrcorner d(X_1 \lrcorner X_2 \lrcorner \Omega) - X_1 \lrcorner \omega_a \Big) \cdot X_1 \\ &\quad + \Big(X_2 \lrcorner X_1 \lrcorner d(X_2 \lrcorner \Omega) + X_2 \lrcorner d(X_1 \lrcorner X_2 \lrcorner \Omega) - X_2 \lrcorner \omega_a \Big) \cdot X_2 < 0.\end{aligned} \tag{4.117}$$

By changing the ordering and applying Cartan's identity formula (A.12), one obtains

$$-X_1 \lrcorner X_2 \lrcorner (\mathcal{L}_{X_1} \Omega \cdot X_1 - \mathcal{L}_{X_2} \Omega \cdot X_2) - (X_1 \lrcorner \omega_a \cdot X_1 + X_2 \lrcorner \omega_a \cdot X_2) < 0. \tag{4.118}$$

Depending on the orientation of the two-form, one can show that the condition for stabilization is indeed satisfied, in this particular case using the fact that $\omega_a = 0$. Applying the homotopy operator to the one-form ω, one obtains $V = -(\mathbb{H}\omega) = x_1^2 + x_2^2$, leading to the damping controller (up to a scaling factor of 1/2)

$$u_1 = -x_1 x_2 \tag{4.119}$$

$$u_2 = x_1^2 - x_2^2, \tag{4.120}$$

as obtained originally in Jacobson (1977).

Obviously, this result in not general in any sense, and the choice of the particular vector field $[X_1, X_2](x)$ is *ad hoc*. However, further studies could be considered in that particular directions, *i.e.*, when the Brockett's conditions are satisfied, or, using a regulator controller where $n = m$, following the result from (Pomet, 1992), given in Section 2.3.2.

In cases where these conditions are not satisfied, it was shown, for example in (Pomet, 1992) and (Moreau and Aeyels, 1999a) how to construct time-dependent feedback controllers. This point is discussed as a future area for research in Section 5.3.2, after presenting results on the stabilization of control-affine time dependent systems using the Jurdjevic–Quinn approach throughout Chapter 5.

Chapter 5

Stabilization of Time-Dependent Systems

This chapter extends the Jurdjevic–Quinn approach to stabilization presented in Chapter 4 to time-varying control affine systems. Comments on the application of this particular stabilization approach to time-varying systems were given in (Outbib and Vivalda, 1999). An extension to time-dependent control affine systems was originally presented in (Aeyels and Sepulchre, 1995). More recently, Mazenc and Malisoff (2009) (see also Malisoff and Mazenc (2009, Chapter 8)) obtained results in this direction. Stabilization of time-varying control affine nonlinear systems is presented in Section 5.1. Provided that a time-varying Jurdjevic–Quinn potential is obtained using the approach presented in Chapters 3 and 4, the proposed approach consists in computing a time-varying function that cancels the time dependence of the potential to compute a semi-definite Lyapunov function and ensure stability of the closed-loop dynamics. The stability argument follows a contribution by Aeyels (1995). In Section 5.2, the result is used for asymptotic stabilization of periodic orbits using a tracking controller, *i.e.*, by stabilizing the origin of the time-varying error dynamics. Section 5.3 discusses potential extensions to synchronization and driftless systems stabilization using time-varying feedback following the approach outlined here.

5.1 Time-Dependent Systems Stabilization

5.1.1 Introduction

This section considers the stabilization of time-dependent affine nonlinear control systems

$$\dot{x}(t) = X_0(t,x) + \sum_{i=1}^{m} X_i(t,x)u_i(t,x), \tag{5.1}$$

where $X_i(t,x)$, $i = 0, 1, \ldots, m$ are assumed smooth in x, continuous and bounded over bounded intervals I in $\mathbb{R}_+$. Furthermore, it is assumed that $X_0(t,0) = 0$. The objective is to construct a function $\psi(t,x)$ such that the time-dependent feedback controls

$$u_p(t,x) = -\frac{\partial \psi}{\partial x_p}(t,x) \cdot X_p(t,x), \quad p = 1, \ldots, m \tag{5.2}$$

stabilize the origin of the closed-loop system over $t \in I \subset \mathbb{R}_+$.

A summary of Lyapunov stability results for time-dependent systems are presented in (Khalil, 2002). Early results on the topic can also be found in (Rouche et al., 1977). Stability results using semi-definite Lyapunov functions were given in (Aeyels, 1995) and (Iggidr and Sallet, 2003). Construction of strict Lyapunov functions for time-varying systems appeared in (Mazenc, 2003) and (Malisoff and Mazenc, 2005) (see (Malisoff and Mazenc, 2009, Part III) for a complete review of strict time-varying Lyapunov function methods).

General results on stabilization of time-dependent nonlinear control systems also appeared in the literature. For example, an existence result for time-varying CLF was given in (Albertini and Sontag, 1999). Building on that result, a generalization of Sontag's formula was presented in (Moulay and Perruquetti, 2005). Here, following the approach presented in (Aeyels, 1995), a method to construct semi-definite (non-strict) Lyapunov functions for time-varying closed-loop systems using the damping control feedback approach presented in Chapter 4 is given. The extension of the Jurdjevic–Quinn stabilization approach to time-varying control systems was presented originally in (Aeyels and Sepulchre, 1995). The challenge in applying the construction from (Aeyels and Sepulchre, 1995) is to find a suitable auxiliary function $\psi(\cdot)$ to construct the damping feedback controller (5.2) and a suitable Lyapunov function $V(\cdot)$ to show the stability of the closed-loop system. Finally, an extension of the Jurdjevic–Quinn method for time-varying systems using strict time-dependent Lyapunov functions is given in (Malisoff and Mazenc, 2009, Section 8.6).

In this section, elements related to stability and stabilization of time-dependent nonlinear systems are recalled, following Aeyels (1995) and Aeyels and Sepulchre (1995).

In the following, the objective is to construct a potential $\psi(t,x)$ such that the feedback $u_k(t,x) = -(\nabla^T \psi \cdot X_k)(t,x)$ stabilizes the system (5.1). In Aeyels and Sepulchre (1995), it was shown that this construction extends to time-varying affine systems provided that

$$\frac{\partial \psi}{\partial t}(t,x) + \nabla_x^T \psi(t,x) \cdot X_0(t,x) \tag{5.3}$$

is negative semi-definite. In Aeyels and Sepulchre (1995), the existence of a time-varying Lyapunov function $V(t,x)$ was assumed to prove asymptotic stability. In the present Section, it is shown that the existence of a potential function $\psi(t,x)$ and the knowledge of a function $\gamma(t,x)$ such that $\frac{d}{dt}(\gamma(t,x)\psi(t,x)) = 0$ is enough to show the stability of the affine time-varying system in closed-loop with controls of the form $u_k(t,x) = -\nabla^T \psi(t,x) \cdot X_k(t,x)$, using $V(x) = (\gamma(t,x)\psi(t,x))(x)$ as a Lyapunov function. In this particular case, the following result from Aeyels (1995) will be used.

Theorem 5.1.1. *Consider*

$$\dot{x} = f(t,x) \tag{5.4}$$

and assume existence and uniqueness of solutions over bounded interval $I \subset \mathbb{R}_+$. *Let* $V : \mathbb{R}^n \supset \mathcal{X} \to \mathbb{R}$ *be a continuously differentiable map, for which there is some neighborhood* $\mathcal{O}$ *of the origin such that the following hold:*

(i) $V(0) = 0$ *and* $V(x) > 0$ *for* $x \in \mathcal{O}$, $x \neq 0$*;*

(ii) $\dot{V}(t,x) := \nabla V(x) \cdot f(t,x) \leq 0$, *for all* $x \in \mathcal{O}$, $t \in I$*;*

(iii) For all $p \in \mathcal{O}$, $p \neq 0$, *there is a finite* $r(p) > 0$ *such that*

$$\limsup_{t\to\infty} V(x(t+r(p);p,t)) < V(p),$$

where $x(t+r(p);p,t)$ *denotes the state at time* $t+r(p)$ *corresponding to the solution of* (5.4) *with initial condition* p *at time* t*.*

Then (5.4) *is asymptotically stable at the origin.*

5.1.2 Construction of a Damping Feedback

Following the approach from Chapter 4, define a non-vanishing closed two-form Ω on $\mathbb{R}^n$ as

$$\Omega = \sum_{1\leq i<j\leq n} dx_i \wedge dx_j. \tag{5.5}$$

As above, the choice of this two-form is arbitrary, and it is set such that the potential satisfies Jurdjevic–Quinn conditions are fulfilled. A one-form associated to the system is obtained by contracting this two-form with respect to the drift vector field,

$$\omega_0(t,x) = X_0(t,x) \lrcorner \Omega. \tag{5.6}$$

From Section 3.2, a homotopy operator can be constructed locally on $\mathbb{R}^n$ such that $\omega_0(t,x) = \omega_{0,e}(t,x) + \omega_{0,a}(t,x)$. Since $\omega_{0,e}(t,x)$ is exact on $\mathbb{R}^n$, it is given as the exterior derivative of a potential function and it is denoted in the following way for the sequel

$$\omega_0(t,x) = -d\psi(t,x) + \omega_{0,a}(t,x). \tag{5.7}$$

It is assumed that the dissipative potential $\psi(t,x)$, obtained after application of the homotopy operator ($i.e.$, $\psi(t,x) = (\mathbb{H}\omega_{0,e})(t,x)$), is such that $\mathcal{L}_{X_0(t,x)}\psi(t,x) < 0$ for $x \in \mathcal{O} \subset \mathbb{R}^n \setminus \{0\}$ for all $t \in I$. As given in Chapters 4, a damping feedback controller is constructed as

$$u_k(t,x) = -\nabla^T\psi(t,x) \cdot X_k(t,x). \tag{5.8}$$

In the next section, it is showed that by computing a function such that $\omega_0(t,x)$ is invariant with respect to time, it is possible to transform $\psi(t,x)$ to obtain a Lyapunov function $V(x)$ and prove asymptotic stability of the origin for the closed-loop system.

5.1.3 Construction of a Lyapunov Function

A Lyapunov function $V(x)$ for the closed-loop system based on the dissipative potential computed by the homotopy above is now constructed. In order to do so, a function $\gamma(t,x)$ such that

$$\frac{\partial}{\partial t}(\gamma(t,x)\psi(t,x)) = 0 \tag{5.9}$$

is computed. From the last expression, a particular solution is

$$\gamma(t,x) = \exp\left(\int_0^t \frac{\psi'(\tau,x)}{\psi(\tau,x)}d\tau\right) > 0 \tag{5.10}$$

where $F'(t,x)$ denotes the time derivative $\frac{\partial}{\partial t}F(t,x)$. This expression will be used in Section 5.1.4 for explicit computations. This particular choice of time-varying canceling factor can be related to time-scale transformation used in feedback linearization (see for example Guay (2002) and references therein). It is also similar to the construction given in (Mazenc, 2003) for construction of strict Lyapunov functions. The main advantage of this particular construction guarantees in the present context is that $\gamma(t,x)$ is positive.

In the sequel, it will be shown that $V(x) = (\gamma\psi)(x)$ is a Lyapunov function for the closed-loop system under damping feedback $\psi(t,x)$ provided, as noted above, that $\mathcal{L}_{X_0(t,x)}\psi(t,x) < 0$ for $x \in \mathcal{O} \subset \mathbb{R}^n \setminus \{0\}$ for all $t \in I$.

Consider the time-dependent affine system under the damping feedback law constructed above, *i.e.*, the system

$$X_0 - \sum_{k=1}^{m} \mathcal{L}_{X_k}\psi \cdot X_k. \tag{5.11}$$

The main result of this section is now stated and proven.

Theorem 5.1.2. *Assume that for all $x \in \mathcal{O} \subset \mathbb{R}^n \setminus \{0\}$ and all $t \in I \subset \mathbb{R}_+$, one can construct $\psi(t,x)$ such that ω_0 is exact, i.e., $\omega_0(t,x) = -d\psi(t,x) \neq 0$ and that $d\psi$ vanishes at the origin, an isolated equilibrium for the system. Then, the damping feedback $u_k(t,x) = -(\mathcal{L}_{X_k}\psi)(t,x)$ locally asymptotically stabilizes the system at the origin. Moreover, a Lyapunov function for the closed-loop system can be computed using $V(x) = (\gamma\psi)(x)$ where $\gamma(t,x)$ is given by* (5.10) *and renders $\omega(t,x)$ invariant with respect to time, i.e., $\omega(x) = \gamma(t,x)\omega_0(t,x)$, for all $t \in I$.*

Proof: Consider the closed loop system under damping feedback control with $\psi(t,x)$ as required. By definition, the closed-loop system is written as

$$X_0 - \sum_{k=1}^{m} \mathcal{L}_{X_k}\psi \cdot X_k = X_0 - \sum_{k=1}^{m} (X_k \lrcorner d\psi) \cdot X_k. \tag{5.12}$$

Let $V(x)$ be given by $(\gamma\psi)(x)$. Then

$$\dot{V} = \frac{\partial}{\partial t}(\gamma\psi) + X_0 \cdot \nabla(\gamma\psi) - \sum_{k=1}^{m}(X_k \lrcorner d\psi) \cdot X_k \cdot \nabla(\gamma\psi). \tag{5.13}$$

By construction of the integrating factor $\gamma(t,x)$, the term $\frac{\partial}{\partial t}(\gamma\psi) = 0$, and by the product rule, $\nabla(\gamma\psi)(x)$ is given as

$$\nabla(\gamma\psi)(x) = \psi\nabla\gamma + \gamma\nabla\psi. \tag{5.14}$$

Since it was assumed that $\psi(t,x)$ is a first integral of $X_0(t,x)$ in x ($\omega_{0,a} \equiv 0$). It can be shown using the closed one-form ω_0 that $X_0\nabla(\gamma\psi) = 0$. By definition of the interior product,

$$X_0 \cdot \nabla V = X_0 \lrcorner dV. \tag{5.15}$$

Moreover, the exterior derivative of V can be expressed as

$$dV = \gamma d\psi + \psi d\gamma \tag{5.16}$$

$$= \gamma\omega_0 + \psi d\gamma. \tag{5.17}$$

The exterior derivative of $\gamma(t,x)$ on $\mathbb{R}^n$ is given by

$$d\gamma = d\left(\exp(\int_0^t \frac{\psi\prime}{\psi}d\tau)\right) \tag{5.18}$$

$$= \frac{\partial}{\partial t}\gamma(t,x)d\psi(t,x) \tag{5.19}$$

$$= \frac{\partial}{\partial t}\gamma(t,x)\omega_0. \tag{5.20}$$

From these relations, and from the construction of $\omega_0 = X_0 \lrcorner \Omega$, (5.15) becomes

$$
\begin{aligned}
X_0 \lrcorner dV &= \left(\gamma + \psi \frac{\partial}{\partial t}\gamma\right) X_0 \lrcorner \omega_0 && (5.21)\\
&= \left(\gamma + \psi \frac{\partial}{\partial t}\gamma\right) X_0 \lrcorner X_0 \lrcorner \Omega && (5.22)\\
&= 0 && (5.23)
\end{aligned}
$$

from the fact that the interior product of a vector field with itself $X_i \lrcorner X_i \equiv 0$ (see Edelen (2005)). Hence, (5.13) can be re-written as

$$\dot{V} = -\sum_{k=1}^{m} (X_k \lrcorner d\psi) \cdot X_k \cdot \nabla(\gamma\psi). \tag{5.24}$$

As above, $X_k \cdot \nabla(\gamma\psi) = X_k \lrcorner (\psi d\gamma + \gamma d\psi)$, and

$$
\begin{aligned}
\dot{V} &= -\sum_{k=1}^{m} (X_k \lrcorner d\psi) \cdot \left(X_k \lrcorner (\gamma + \psi \frac{\partial}{\partial t}\gamma) d\psi\right) && (5.25)\\
&= -\left(\gamma + \psi \frac{\partial}{\partial t}\gamma\right) \sum_{k=1}^{m} (X_k \lrcorner d\psi)^2 && (5.26)
\end{aligned}
$$

which is negative definite.

Following Aeyels (1995), we now have to show that the origin is the largest invariant set of the dynamics, *i.e.*, that

$$W = \{x : dV = 0\} \tag{5.27}$$

is $\{0\}$, at least in a neighborhood $\mathcal{O}$ of the origin. Since $\omega_0 = -d\psi$ and ψ vanish only at the origin by assumption, and that by construction, $d\gamma = \frac{\partial}{\partial t}\gamma d\psi$, we have that W is the largest invariant set if the integral over $[0, t]$ of $\frac{\psi\prime}{\psi}$ is finite for some $t \in I$, which is ensured by the regularity assumption on $X_0(t, x)$ with respect to t. ■

Remark 5.1.3. *The domain of attraction for which a stabilizing control can be constructed* *or for which stability can be proven relies on the properness of* $\psi(\cdot, x)$ *obtained by application of the radial homotopy operator. The reader is referred to* (Faubourg, 2001, Chapter 2) *for a discussion on deformation of weak Jurdjevic–Quinn functions to obtain Lyapunov function in the case of time-varying homogeneous vector fields.*

5.1.4 Example

In this section, an application of the above construction is presented to illustrate the construction from the last section. The following example was originally given in (Malisoff and Mazenc, 2009, Chapter 8). Consider the system

$$\dot{x}_1 = \cos^2(t)x_2 \tag{5.28}$$

$$\dot{x}_2 = -\cos^2(t)x_1 + \cos^4(t)u(t,x). \tag{5.29}$$

Denote $X_{0,1}(x,t) = \cos^2(t)x_2$, $X_{0,2}(x,t) = -\cos^2(t)x_1$, $X_{1,1}(x,t) = 0$, and $X_{1,2}(x,t) = \cos^4(t)$. Applying the approach depicted above with $\Omega = dx_1 \wedge dx_2$, the obtained one-form is given as

$$\omega_0 = -X_{0,2}(t,x)dx_1 + X_{0,1}(t,x)dx_2,\ \forall t \in \mathbb{R}_+, \tag{5.30}$$

which is a closed one-form since

$$\frac{\partial}{\partial x_2}(-X_{0,2}) = \frac{\partial}{\partial x_1}(X_{0,1}) = 0. \tag{5.31}$$

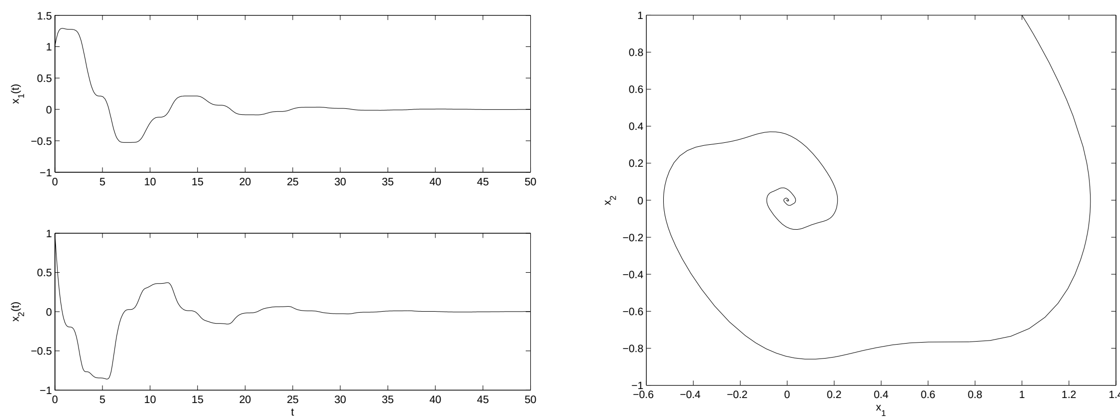
Applying the homotopy operator

$$(\mathbb{H}\omega) = \int_0^1 \left(-X_{0,2}(t,\lambda x)\cdot \lambda x_1 + X_{0,1}(t,\lambda x)\cdot \lambda x_2\right) d\lambda \tag{5.32}$$

results in the time-varying Jurdjevic–Quinn potential $\psi(t,x) = \cos^2(t)(x_1^2 + x_2^2)$. The control $u(t,x)$ is thus given by

$$u(t,x) = -\nabla^T \psi(t,x) \cdot X_{1,2}(t,x) = -\cos^6(t) x_2. \tag{5.33}$$

Figures 5.1a and 5.1b present the simulation of the closed-loop system with initial state $(x_1(0), x_2(0)) = (1,1)$. The controller value trajectory is given in Figure 5.2.



(a) State trajectories (b) Phase diagram

Figure 5.1: Time-varying stabilization example

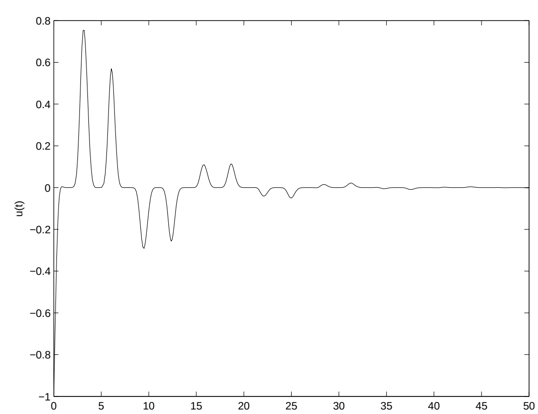


Figure 5.2: Time-dependent controller value with respect to time

Stability of the origin of the closed-loop system is now analyzed using the expression of $\psi(t,x)$ given above. Note that $\omega_0(t,x) = \cos^2(t)(-x_1dx_1 + x_2dx_2)$. The integrating factor $\gamma(t,x)$ is obtained by taking the time derivative of

$$\frac{\partial}{\partial t}(\gamma(t,x)\psi(t,x)) \quad = \quad 0 \tag{5.34}$$

which in this case leads to the condition (independent of the state x)

$$\gamma\prime(t)\cos^2(t) - 2\gamma(t)\sin(t)\cos(t) \quad = \quad 0. \tag{5.35}$$

The integrating factor is hence $\gamma(t) = \exp(2\int_0^t \tan(\tau)d\tau) = \exp(-2\ln|\cos(t)|) = \sec^2(t)$. The desired Lyapunov function is thus given by $V(x) = x_1^2 + x_2^2$. The time derivative with respect to the system under damping feedback is

$$\dot{V}(t,x) = -\cos^{10}(t)x_2^2 < 0, \quad \forall x \setminus \{0\}, \forall t \in I. \tag{5.36}$$

In this case, the closed-loop one-form is exact, and trivially $\omega_a \equiv 0$. The largest invariant set for the closed-loop is hence the origin. By the arguments from Aeyels (1995) (Barbashin–Krasovskĭ), the origin of the system under damping feedback is therefore asymptotically stable. Moreover, since $V(x)$ is proper, the origin is globally asymptotically stable.

5.2 Stabilization of a Trajectory

This section considers the problem of stabilizing the time-independent control affine nonlinear systems

$$\dot{x} = X(x) + \sum_{i=1}^{m} X_i(x)u_i \tag{5.37}$$

to periodic reference trajectories $\eta_r(t)$ using time-varying damping feedback controls of the form

$$u_p(t,x) = -\frac{\partial \psi}{\partial x_p}(t,x) \cdot X_p(t,x), \quad p = 1, \ldots, m. \tag{5.38}$$

It is assumed that $X_i(x)$, $i = 0, \ldots, m$ are locally smooth in x. Stability and stabilization of periodic trajectories was considered recently in (Mazenc et al., 2006) and (Mazenc et al., 2008) for bioreactors where two species coexist. In this section, stabilization of periodic orbits of predator-prey dynamics is considered.

The problem of periodic stabilization is here considered from an asymptotic tracking point of view, see for example the general discussion in (Marino and Tomei, 1995). More precisely, a reference trajectory $\eta_r(t)$, solution of the drift dynamics

$$\dot{\eta}_r(t) = X_0(\eta_r(t)) \tag{5.39}$$

is assumed to be known. Stabilization of the origin of the error dynamics defined by

$$\dot{e} = \dot{x} - \dot{\eta}_r(t) \tag{5.40}$$

is then considered to ensure convergence of x to η_r. Following the approach used in this thesis, the design of static state feedback tracking controllers $u = u(x, \eta_r, t)$ are considered using the Jurdjevic–Quinn approach for time-varying affine systems. The reader is referred to (Andrieu et al., 2007) for the related problem of output feedback tracking of periodic bounded solutions.

A potential function $\psi(t,e)$ is constructed such that the time-dependent feedback controls (5.38) stabilize the origin of the closed-loop system over $t \in I \subset \mathbb{R}_+$. The challenge in applying the method is to find a suitable auxiliary function $\psi(\cdot)$ to construct the damping feedback controller (5.38) and a suitable Lyapunov function $V(\cdot)$ to show the stability of the closed-loop system. The proposed approach relies on a locally defined homotopy-based decomposition technique proposed above to construct a dissipative potential $\psi(t,e)$ for the time-dependent error dynamics. In the present section, it is shown how a locally-defined potential function for the time-dependent error dynamics, $\psi(t,e)$, and the knowledge of a time-varying function $\gamma(t,e)$ such that $\frac{d}{dt}(\gamma(t,e)\psi(t,e)) = 0$ is enough to show the stability of the affine error time-varying system in closed-loop with controls of the form $u_k(t,e) = -\nabla^T\psi(t,e)\cdot g_k(t,e)$, using $V(e) = (\gamma(t,e)\psi(t,e))$ as a Lyapunov function. In particular, the construction from Section 5.1 is used.

5.2.1 Construction of a Damping Feedback

Following the discussion from above, the objective is to design a state feedback damping controller $u(x,\eta_r,t)$ such that the dynamics of (5.49)-(5.50) stabilizes asymptotically to the periodic orbit (5.55)-(5.56), *i.e.*, such that

$$\lim_{t\to\infty}(x(t)-\eta_r(t)) = 0. \tag{5.41}$$

In order to do so, consider the error, defined as $e_i(t,x) = x_i - \eta_{r,i}$ for $i = 1,2$. The error dynamics are given by

$$\dot{e} = \dot{x} - \dot{\eta}_r(t). \tag{5.42}$$

In the sequel, the resulting time-varying vector field is denoted as $X_0(t,e) = \dot{x} - \dot{\eta}_r(t)$ obtained after replacing $x = e + \eta_r(t)$ in the drift vector field of (5.37). The idea exploited here is to design a damping feedback tracking controller such that the origin of the error dynamics is asymptotically stable for initial conditions in a neighborhood of the origin (since the approach is local). The asymptotic tracking problem is therefore expressed as a time-varying stabilization problem.

First define a non-vanishing closed two-form Ω on $\mathbb{R}^n$ as

$$\Omega = \sum_{1\leq i<j\leq n} de_i \wedge de_j. \tag{5.43}$$

A one-form associated to the system is obtained by contracting this two-form with respect to the drift vector field,

$$\omega_0(t,e) = X_0(t,e) \lrcorner \Omega, \tag{5.44}$$

with $X_0(t,e)$ defined above as $X_0(t,e) = \dot{x} - \dot{\eta}_r(t)$. Since $\omega_{0,e}(t,e)$ is exact on $\mathbb{R}^n$, it is given as the exterior derivative of a potential function and it is denoted as

$$\omega_0(t,e) = -d\psi(t,e) + \omega_{0,a}(t,e). \tag{5.45}$$

Assume that the dissipative potential $\psi(t,e)$, obtained after application of the homotopy operator ($i.e.$, $\psi(t,e) = (\mathbb{H}\omega_{0,e})(t,e)$), is such that $\mathcal{L}_{X(t,e)}\psi(t,e) < 0$ for $x \in \mathcal{O} \subset \mathbb{R}^n \setminus \{0\}$ for all $t \in I$. The damping feedback controller is constructed as

$$u_k(t,e) = -\nabla^T\psi(t,e) \cdot g_k(t,e). \tag{5.46}$$

In the particular case of the Lotka–Volterra considered in the sequel, $g(t,e)$ is simply $[1,\ 0]^T$. In the next section, the result from Section 5.1 is specialized to obtain a Lyapunov function $V(e)$ and to prove asymptotic stability of the closed-loop using a time-varying function to render $\omega_0(t,e)$ independent of time.

5.2.2 Stability of the Time-Varying Error Dynamics

A Lyapunov function $V(e)$ is constructed for the closed-loop error dynamics system based on the dissipative potential computed by the homotopy above. As in Section 5.1, a time-varying function $\gamma(t, e)$ is computed such that

$$\frac{\partial}{\partial t}(\gamma(t, e)\psi(t, e)) = 0. \tag{5.47}$$

It is now shown that $V(e) = (\gamma\psi)(e)$ is a Lyapunov function for the closed-loop system under damping feedback $\psi(t, e)$ provided, as noted above, that $\mathcal{L}_{X_{0(t,e)}}\psi(t, e) < 0$ for $x \in \mathcal{O} \subset \mathbb{R}^n \setminus \{0\}$ for all $t \in I$.

Consider the time-dependent affine error dynamic system under the damping feedback law constructed above, $i.e.$, the system

$$X_0(t, e) - \sum_{k=1}^{m}(\mathcal{L}_{X_k}\psi \cdot X_k)(t, e). \tag{5.48}$$

The following result is given without proof, as a specialization of the result from Section 5.1.

Theorem 5.2.1. *Assume that for all $e \in \mathcal{O} \subset \mathbb{R}^n \setminus \{0\}$ and all $t \in I \subset \mathbb{R}_+$, one can construct $\psi(t, e)$ such that ω_0 is exact, i.e., $\omega_0(t, e) = -d\psi(t, e) \neq 0$ and that $d\psi$ vanishes at the origin. Then, the damping feedback $u_k(t, e) = -(\mathcal{L}_{X_k}\psi)(t, e)$ locally asymptotically stabilizes the origin of the error system. Moreover, a Lyapunov function for the closed-loop system can be computed using $V(e) = (\gamma\psi)(e)$ where $\gamma(t, e)$ is given by (5.10) and renders $\omega(t, e)$ invariant with respect to time, i.e., $\omega(e) = \gamma(t, e)\omega_0(t, e)$, for all $t \in I$.*

The next section illustrates the application of this result to a Lotka–Volterra system.

5.2.3 Asymptotic Tracking of a Lokta–Volterra Periodic Solution

The construction presented in Section 5.2.2 is now applied to the Lotka–Volterra system introduced in the previous chapters. Consider

$$\dot{x}_1 = ax_1 - bx_1x_2 + u \quad (5.49)$$

$$\dot{x}_2 = -cx_2 + bx_1x_2, \quad (5.50)$$

with $x_1 \geq 0$ and $x_2 \geq 0$ and where a, b, c are known positive constants. The first step is to give explicit periodic solutions for the drift system.

The periodic invariant solution to be tracked is given following the original construction given in (Evans and Findley, 1999). Define the invariant for the system as

$$\Lambda = bx_1 + bx_2 - c\ln x_1 - a\ln x_2. \quad (5.51)$$

Then, defining the auxiliary signal

$$w = \frac{\Lambda}{2(a+c)}(1 - \cos 2\phi), \quad (5.52)$$

it was shown that the periodic invariant of the uncontrolled part of the Lotka–Volterra system can be expressed as

$$x_1(t) = \frac{1}{b}(cw + \dot{w}) \quad (5.53)$$

$$x_2(t) = \frac{1}{b}(aw - \dot{w}). \quad (5.54)$$

In the sequel, those expressions will be used as the periodic reference trajectories, *i.e.*,

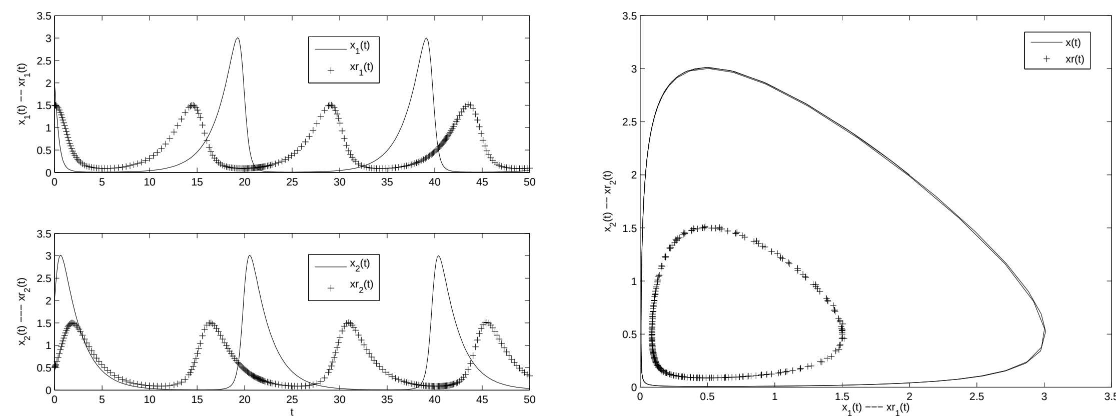
$$\eta_{r,1}(t) = \frac{1}{b}(cw + \dot{w}) \quad (5.55)$$

$$\eta_{r,2}(t) = \frac{1}{b}(aw - \dot{w}), \quad (5.56)$$

where w and $\dot{w}$ are generated by the expressions above or, as solutions of the second order differential equation

$$\ddot{w} - \dot{w}^2 - (c-a)(w-1)\dot{w} + acw(w-1) = 0. \tag{5.57}$$

The open-loop dynamics and reference trajectories are depicted in Figures 5.3a and 5.3b.



(a) State trajectories (b) Phase diagram

Figure 5.3: Uncontrolled dynamics and reference trajectories

The error dynamics is given as

$$\dot{e}_i = \dot{x}_i - \dot{\eta}_{r,i}(t)$$

with $i = 1, 2$. The time-varying control affine system to be stabilized is obtained by replacing $x_i = e_i + \eta_i(t)$ in the drift dynamics of (5.49)-(5.50):

$$\dot{e}_1 = -b(e_1 + \eta_{r,1})(e_2 + \eta_{r,2}) + a(e_1 + \eta_{r,1}) - \dot{\eta}_{r,1} \tag{5.58}$$

$$\dot{e}_2 = b(e_1 + \eta_{r,1})(e_2 + \eta_{r,2}) - c(e_2 + \eta_{r,2}) - \dot{\eta}_{r,2}. \tag{5.59}$$

Denoting this dynamics as the vector field

$$X_0 = \tilde{f}_1(t,e)\partial_{e_1} + \tilde{f}_2(t,e)\partial_{e_2} \tag{5.60}$$

and setting $\Omega = de_1 \wedge de_2$, the desired one-form is obtained by taking the interior product $X_0 \lrcorner \Omega$:

$$\omega_0 = -\tilde{f}_2 de_1 + \tilde{f}_1 de_2. \tag{5.61}$$

Applying the homotopy operator centered at the origin on $\mathfrak{X} = \lambda e_i \partial_{e_i}$, the dissipative potential is given by

$$\psi(t,e) \;\; = \;\; \tilde{f}_2 e_1 - \tilde{f}_1 e_2. \tag{5.62}$$

For the choice of parameters $a = c = 0.5$ and $b = 1$, and using the definition of $\eta_{r,i}$ in terms of w, one can show that for $x_i = e_i + \eta_{r,i} \geq 0$, $\psi(t,e) \geq 0$ in the neighborhood of the center equilibrium $(0.5,\ 0.5)^T$. Moreover, one can show that in this particular case, the anti exact part of the dynamics vanishes only for $\tilde{f}_i = 0$, and hence ω_0 is exact. The function

$$\gamma(t,e) = \exp\left(\int_0^t \frac{\psi\prime(\tau,e)}{\psi(\tau,e)} d\tau\right) \tag{5.63}$$

is not explicitly written. However, the regularity assumptions required for such a function to exist are met for the bounded reference trajectories (5.55)-(5.56). Hence, following (Aeyels, 1995) and Section 5.1, there exists a Lyapunov function $V(e) = (\gamma\psi)(e)$ for the closed-loop system.

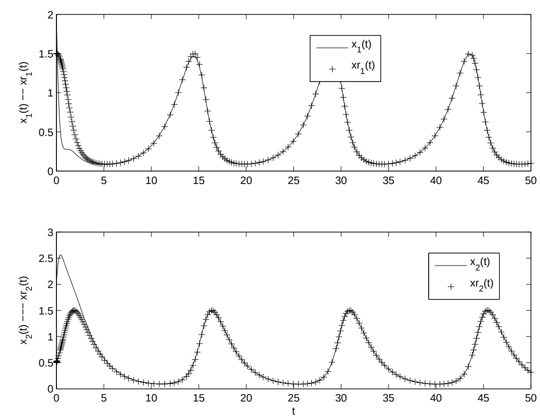
Replacing $e = x - \eta_r(t)$ in (5.62), the asymptotic tracking controller $u(t,x)$ is computed as

$$u(t,x) = -\nabla_x^T \psi(t,x) \cdot g(x). \tag{5.64}$$

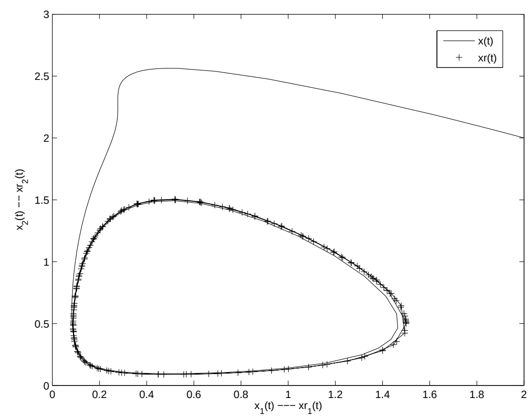
With $g(x) = [1,\ 0]^T$, it is given as

$$u(t,x) \quad = \quad a\frac{x_2 - \eta_{r,2}}{2} + b\left(\frac{x_2(\eta_{r,1}+\eta_{r,2}) - x_2^2}{2} - x_1 x_2\right) + c\frac{x_2}{2} + \frac{\eta_{r,2}}{2}. \qquad (5.65)$$

Closed-loop simulations using the above damping controller with initial conditions $x(0) = \eta_r(0) = [2,2]^T$ are presented in Figures 5.4a and 5.4b. The time evolution of the tracking error $[e_1(t),\ e_2(t)]^T$ is shown in Figure 5.4c.

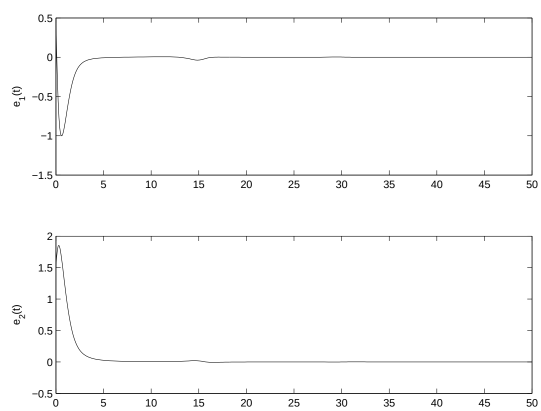


(a) State trajectories



(b) Phase diagram

Figure 5.4: Asymptotic tracking of periodic solutions for a Lotka–Volterra system



(c) Error dynamics

Figure 5.4: Asymptotic tracking of periodic solutions for a Lotka–Volterra system

5.3 Summary and Extensions

This chapter considered the problem of Lyapunov functions construction for the stabilization of time-dependent affine nonlinear control systems satisfying the weak Jurdjevic–Quinn conditions. Provided that a positive definite function (a dissipative potential) can be obtained by taking the interior product of a non-vanishing two-form with respect to the drift vector field, a damping feedback controlled was constructed. As discussed above, there is a certain freedom in the choice of the two-form to ensure that the potential has the desired properties. It was shown how a function cancelling the time-dependence of the dissipative potential can be used to deform the time-varying dissipative potential to obtain a non-strict Lyapunov function for the closed-loop vector field subject to time-varying damping feedback control. The proposed method is local since the approach relies on a homotopy operator centered at the origin. One problem under investigation is the extension for systems where the non-exact part is not trivial. In that case, the proposed construction could lead to stabilization of time-varying complex patterns, for example limit cycles.

The problem of asymptotic tracking of periodic orbits using smooth damping feedback tracking controllers was then considered. The proposed approach consists in stabilizing the origin of the time-varying error dynamics system, building on the result from Section 5.1. Application of the approach was illustrated on a Lotka–Volterra periodic stabilization example.

Future research of this approach to synchronization is discussed next.

5.3.1 Synchronization

In (Hudon and Guay, 2010b), a problem inspired by (Stan et al., 2007) was considered. The approach presented in Chapter 4 was applied to cyclic interconnection structure (Arcak and

Sontag, 2006) of the form

$$\dot{x}_1 = -a_1(x_1) + b_n(x_n) \quad (5.66)$$

$$\dot{x}_2 = -a_1(x_2) - b_1(x_1) \quad (5.67)$$

$$\vdots \quad (5.68)$$

$$\dot{x}_n = -a_n(x_n) + b_{n-1}(x_{n-1}), \quad (5.69)$$

where $a_i(\cdot)$ and $b_i(\cdot)$, $i = 1, \ldots, n$, are continuous functions satisfying $x_i a_i(x_i) > 0$ and $x_i b_i(x_i) > 0$ for $x_i \neq 0$. The situation considered here is the case where $a_i(\cdot)$, $i = 1, \ldots, n$ and $b_i(\cdot), i = 1, \ldots, n-1$ are increasing functions and $b_n(\cdot)$ is a decreasing function. This is a special case of the metabolic network with feedback inhibition considered in (Grognard et al., 2004).

An interesting example of such a system was given by Arcak and Sontag (2008),

$$\dot{x}_1 = -a_1 x_1 + \phi(x_3) \quad (5.70)$$

$$\dot{x}_2 = -a_2 x_2 + b_1 x_1 \quad (5.71)$$

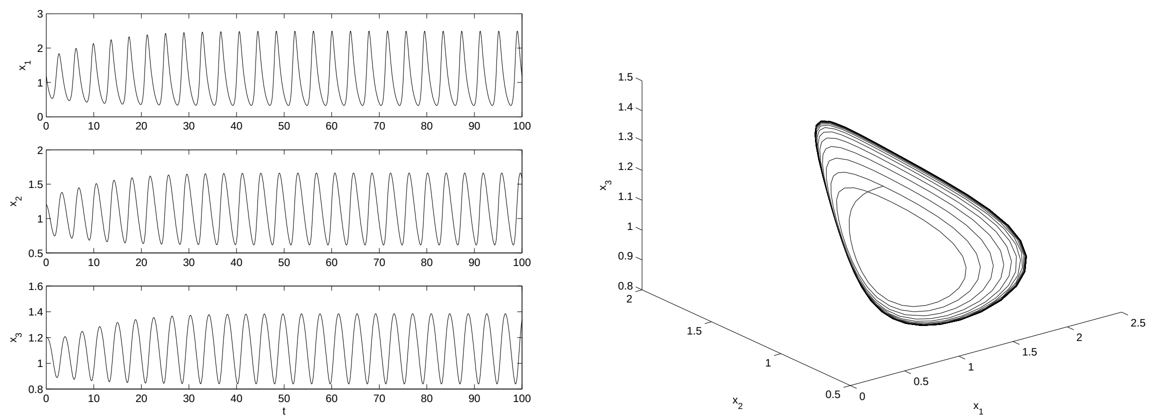
$$\dot{x}_3 = -a_3 x_3 + b_2 x_2. \quad (5.72)$$

In the case where $\phi(x_3)$ is given as a function of the form $\frac{1}{1+x_3^p}$, the above cyclic interconnection is known as the Goodwin oscillator for which, depending on the value of p, a stable periodic orbit exists (see for example Stan et al. (2007) and references therein).

This section considers the special case from (Arcak and Sontag, 2008) where $a_1 = a_2 = a_3 = 1$ and $b_1 = b_2 = 1$. In particular, Arcak and Sontag (2008) showed, for $\phi(x_3) = \exp(-10(x_3 - 1)) + 0.1\ \mathrm{sat}(25(x_3 - 1))$, that the equilibrium $x^* = [1,\ 1,\ 1]^T$ coexists with a periodic orbit. As a first approximation, $\phi(x_3)$ is considered to be

$$\phi(x_3) = \exp(-10(x_3 - 1)). \quad (5.73)$$

The open-loop cycle surrounding the desired equilibrium $x^* = [1,\ 1,\ 1]^T$, with initial conditions $x_0 = [1.2,\ 1.2,\ 1.2]^T$ is presented in Figures 5.5a and 5.5b.



(a) State trajectories (b) Phase diagram

Figure 5.5: Oscillator from (Arcak and Sontag, 2008) — open-loop

The synchronization of two cyclic systems originally on different time scales was considered in (Hudon and Guay, 2010e):

$$\dot{x}_{1,1} = -a_1(x_{1,1}) + b_n(x_{n,1}) + u_1 \tag{5.74}$$

$$\dot{x}_{2,1} = -a_1(x_{2,1}) - b_1(x_{1,1}) + u_2 \tag{5.75}$$

$$\vdots \tag{5.76}$$

$$\dot{x}_{n,1} = -a_n(x_{n,1}) + b_{n-1,1}(x_{n-1,1}) + u_n, \tag{5.77}$$

and

$$\begin{aligned}
\tau\dot{x}_{1,2} &= -a_1(x_{1,2}) + b_n(x_{n,2}) && (5.78)\\
\tau\dot{x}_{2,2} &= -a_1(x_{2,2}) - b_1(x_{1,2}) && (5.79)\\
&\vdots && (5.80)\\
\tau\dot{x}_{n,2} &= -a_n(x_{n,2}) + b_{n-1,2}(x_{n-1,2}), && (5.81)
\end{aligned}$$

with $a_{i,j}$ and $b_{i,j}$ as above. Consider two differential one-forms obtained using $f_1(x_{i,1})$ and $f_2(x_{i,2})$, the drift vector fields from the two systems:

$$\begin{aligned}
\eta_1 &= f_1(x_{i,1}) \lrcorner \Omega_1 && (5.82)\\
\eta_2 &= f_2(x_{i,2}) \lrcorner \Omega_2. && (5.83)
\end{aligned}$$

Since both systems have the same drift structure, $\Omega_2 = \delta\Omega_1$, where $\delta = \tau^2$. Define a closed one-form for the error as

$$\omega_0 = (f_1(x_{i,1}) - \delta f_2(x_{i,2})) \lrcorner \Omega_1. \tag{5.84}$$

Then, applying a homotopy centered at the origin for this dynamics, the dissipative potential

$$\psi(x_{i,1}, x_{i,2}) = \mathbb{H}\omega_0 = \int_0^1 \omega_0(\lambda(x_{i,1} - x_{i,2}))\lambda(x_{i,1} - x_{i,2})d\lambda \tag{5.85}$$

is obtained.

Assuming that all the reference trajectories $x_{i,2}(t)$ are available, the damping controllers for the system are given by

$$u_k = -k_k \nabla^T_{x_{i,1}} \psi \cdot g_k. \tag{5.86}$$

By the usual arguments of the damping method, stabilization of the slave system to the reference trajectories follows. The simulations with $\sigma = 2$ are presented in Figure 5.6, the control $u_i(x) = -k_i \frac{\partial \psi}{\partial x_i}$ are set to zero for $t < 20$. Both systems are initialized at $x_0 = [1.2,\ 1.2,\ 1.2]^T$.

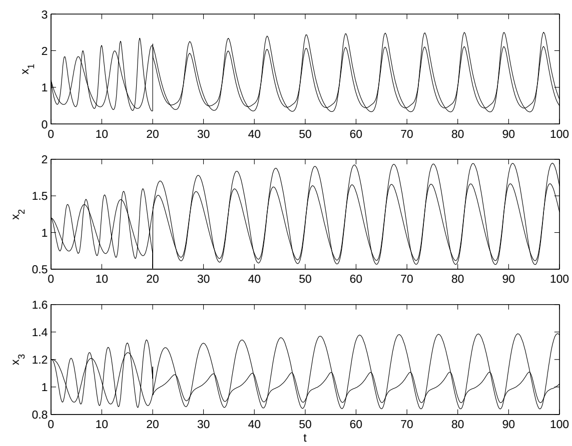


Figure 5.6: Synchronization simulations

The results show that the original system synchronizes with the reference dynamics, following essentially the discussion of Section 4.2. Current studies around that approach to synchronization seek to extend the procedure to the (output) feedback tracking problem. One promising approach was given in (Nijmeijer and Mareels, 1997). However, it should be noted that for larger problems, *i.e.*, when the number of subsystems to synchronize or coordinate is large, a time-dependent reference trajectory can be used. This is the case, for example, for the synchronization of network of circadian oscillators, presented for example in (Doyle III et al., 2006) and (Bagheri et al., 2007). From that perspective, synchronization problems are potential extensions of the stabilization problems presented in this chapter.

5.3.2 Driftless Systems

Following the discussion of Sections 4.4.3 on bilinear systems stabilization, the control of driftless system

$$\dot{x} = \sum_{j=1}^{m} u_k X_k \tag{5.87}$$

where $X = (X_1, \ldots, X_m)$ be m vector fields of class $\mathcal{C}^\infty$ are now discussed. It is known from the classical result of Brockett (1983), that local controllability on $\mathbb{R}^n \setminus \{0\}$ does not imply asymptotic stabilizability by means of a continuous feedback law $u = u(x)$. However, some systems of the form (5.87) can be stabilized by periodic time-varying feedback laws (Coron, 1990). In particular, constructive methods were presented in (Pomet, 1992) and (Moreau and Aeyels, 1999b). Jurdjevic–Quinn stabilization approach for this class of systems was briefly discussed in (Aeyels and Sepulchre, 1995), assuming a Jurdjevic–Quinn function constructed as proposed in (Pomet, 1992). For the special case where the system (5.87) is such that $x \in \mathbb{R}^3$ and $u \in \mathbb{R}^2$, under the assumptions that $X_1(x)$, $X_2(x)$ and $[X_1,\ X_2](x)$ span $\mathbb{R}^3$ and that $X_1(x)$ and $X_2(x)$ are homogeneous, Moreau and Aeyels (1999b) proposed a feedback of the form

$$u_1(t,x) = l_1(x) + \cos(t) l_3(x) \tag{5.88}$$

$$u_2(t,x) = l_2(x) + \sin(t) l_4(x) \tag{5.89}$$

with $l_i(x)$ smooth on $\mathbb{R}^3 \setminus \{0\}$ and homogeneous, and showed local asymptotic stability of the origin under that feedback.

An approach to consider following the construction proposed in this chapter and the preliminary result in Section 4.4.3 would be to compute time-varying functions such that

$$\omega = (\alpha_1(t,x) X_1 + \alpha_2(t,x) X_2 + \alpha_3(t,x)[X_1,\ X_2](x)) \lrcorner \Omega \tag{5.90}$$

is exact and use the associated dissipative potential for time-dependent feedback design. The main difficulty for that approach would be to solve the partial differential equations associated with the condition $\omega_a \equiv 0$.

Consider for example the example presented in (Pomet, 1992):

$$\dot{x}_1 = u_1 \tag{5.91}$$

$$\dot{x}_2 = x_1 u_2 \tag{5.92}$$

$$\dot{x}_3 = u_2. \tag{5.93}$$

This system does not satisfy Brockett's necessary conditions, hence it is not stabilizable at the origin by smooth state feedback. The problem of stabilizing the origin of this system using a time-dependent feedback law, as presented in (Pomet, 1992), is now discussed. Following the idea in (Moreau and Aeyels, 1999b) and as in Section 4.3, it is proposed to use the one-form obtained from (5.90). First note that $[X_1,\ X_2] = \partial_{x_2}$, and $\text{span}\{X_1, X_2, [X_1,\ X_2]\} = \mathbb{R}^3$. The scaled vector field is as follows

$$X = \alpha_1 \partial_{x_1} + \alpha_2(x_1 + 1)\partial_{x_2} + \alpha_3 \partial_{x_3}. \tag{5.94}$$

Letting

$$\Omega = dx_1 \wedge dx_2 + dx_2 \wedge dx_3 + dx_3 \wedge dx_1, \tag{5.95}$$

and taking the interior product $X \lrcorner \Omega$, the one-form computed from (5.90) is

$$\omega = (-\alpha_2(x_1 + 1)) + \alpha_3)\, dx_1 + (\alpha_1 - \alpha_3) dx_2 + (\alpha_2(x_1 + 1) - \alpha_1)\, dx_3. \tag{5.96}$$

Following (Pomet, 1992), it is assumed that the desired time-varying solutions are of the form

$$\alpha_1(t,x_1,x_2,x_3) = x_1 + \phi(t,x_2,x_3) \tag{5.97}$$
$$\alpha_2(t,x_1,x_3) = \phi(t,x_1,x_3) \tag{5.98}$$
$$\alpha_3(t,x_1,x_2) = \phi(t,x_1,x_2). \tag{5.99}$$

With this particular choice, the condition $\omega_a \equiv 0$ leads to the following set of partial differential equations

$$\frac{\partial}{\partial x_1}\phi_3(t,x_1,x_2) + \frac{\partial}{\partial x_2}\phi_3(t,x_1,x_2) - 1 = 0 \tag{5.100}$$
$$(x_1+1)\left(\frac{\partial}{\partial x_1}\phi_2(t,x_1,x_3) - \frac{\partial}{\partial x_3}\phi_2(t,x_1,x_3)\right) - \phi_2(t,x_1,x_3) + 1 = 0 \tag{5.101}$$
$$\frac{\partial}{\partial x_2}\phi_1(t,x_2,x_3) + \frac{\partial}{\partial x_3}\phi_1(t,x_2,x_3) = 0. \tag{5.102}$$

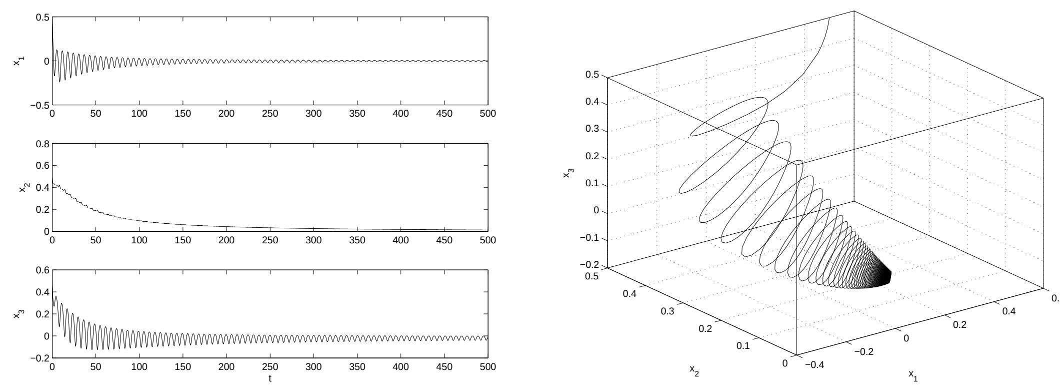
A particular solution for this system is given as

$$\phi_1(t,x_2,x_3) = F_1(t,-x_2+x_3) \tag{5.103}$$
$$\phi_2(t,x_1,x_3) = \frac{x_1 + F_3(t,-x_1+x_3)}{x_1+1} \tag{5.104}$$
$$\phi_3(t,x_1,x_2) = x_1 + F_2(t,-x_1+x_2). \tag{5.105}$$

Following the arguments given in (Pomet, 1992), the unknown functions F_i are fixed as

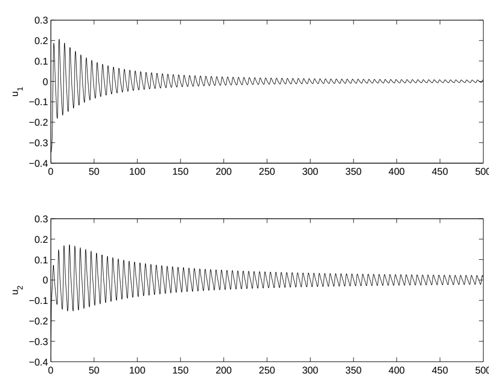
$$F_1(t,-x_2+x_3) = \cos(t)(-x_2-x_3) \tag{5.106}$$
$$F_2(t,-x_1+x_2) = \sin(t)(-x_1+x_2) \tag{5.107}$$
$$F_3(t,-x_1+x_3) = \exp(-t)(-x_1+x_3). \tag{5.108}$$

Using the modified one-form ω to construct a dissipative potential and applying the Jurdevic–Quinn control approach, the system is stabilized to a neighborhood of the origin. Simulations for the system are given in Figures 5.7a and 5.7b and the periodic smooth control in Figure 5.7c, using the initial conditions $x(0) = [0.5,\ 0.5,\ 0.5]^T$.



(a) State trajectories (b) Phase diagram

Figure 5.7: Time-dependent stabilization of a driftless system



(c) Time-dependent controller values

Figure 5.7: Time-dependent stabilization of a driftless system

The approach taken here is obviously not systematic. However, an extensive study of this approach could be considered in the future. The first element to be considered would be to study carefully the existence and uniqueness of solutions for the partial differential equations obtained by imposing that ω is closed.

Chapter 6

Conclusions

As discussed in the previous chapters, the results presented in this thesis suggest some potential extensions in the field of feedback stabilization design. Some of those potential extensions are discussed in the following. First, the contributions given in this thesis are summarized in Section 6.1. Some application problems to be addressed and possible areas for research are discussed in Section 6.2.

6.1 Summary

The objective of the thesis was to study how the decomposition of a one-form associated with a given nonlinear drift vector field can be used to design smooth feedback damping controllers. Originally motivated by the problem of dissipative Hamiltonian realization, following the decomposition approach given in (Cheng et al., 2000) and (Cheng et al., 2002), a homotopy-based approach dating back to (Edelen, 1973) and used in (Edelen, 2005) in the context of non-equilibrium thermodynamics was applied to compute a locally defined dissipative potential. A similar construction, using the Poincaré lemma, was recently presented in (Yap, 2009). This last approach to compute a potential using an exact one-form is related to the result presented in (Hudon and Guay, 2009a), summarized briefly in

Section 3.6, where feedback controller design was achieved by canceling the anti-exact part of the dynamics using feedback.

Since stabilization problems encountered in chemical engineering that can be solved by the existing IDA-PBC design method are limited, the aforementioned dissipative potential was exploited to construct smooth damping feedback controllers of the Jurdjevic–Quinn type, as described in (Malisoff and Mazenc, 2009), specializing here to control affine systems. Building on this result, the design of feedback regulator by computing a potential and use it for stabilization of the origin in an extended space was presented. To complete the discussion, it was shown that the same potential computation procedure can be used to obtain a first integral of the uncontrolled part of the dynamics and to stabilize closed orbits using the result from Bacciotti and Mazzi (1995) and (Bombrun, 2007). As noted in (Bacciotti, 1992), the Jurdjevic–Quinn approach works well when the drift system possesses some dissipative property. By extension, most control applications using Jurdjevic–Quinn controllers are built using the drift dynamics as a starting point. However, in some particular cases, such as the bilinear example presented originally by Jacobson (1977), it is possible to recover the idea of a potential using the proposed technique, as noted briefly in Section 4.4. This observation paved the way for further extensions of the techniques presented here.

Following the original result from Aeyels and Sepulchre (1995) and the recent construction by Malisoff and Mazenc (2009, Chapter 8), an extension of the Jurdjevic–Quinn approach to the stabilization of time-dependent control affine systems was considered in Chapter 5. It was shown that by computing a time-varying factor that makes the potential time-invariant, it is possible to find a semi-definite Lyapunov function proving stability of the system in closed-loop with a time-dependent Jurdjevic–Quinn control law. This result was then used to construct a controller for the asymptotic tracking of periodic orbits, following the problem studied by Mazenc et al. (2006). The construction of time-dependent feedback laws for the stabilization of driftless systems was then explored in Section 5.3.2 as a potential area for future research.

Some applications, not presented in the present thesis, should be mentioned. Damping feedback stabilization of a wastewater plant model (dimension $n = 4$) with mixed Monod and Haldane kinetics was given in (Hudon and Guay, 2010e). Application of the stabilization approach on cyclic interconnections, as treated by Arcak and Sontag (2006) and Arcak and Sontag (2008) and mentioned in Section 5.3.1 is presented in (Hudon and Guay, 2010b). This last class of extensions suggests many opportunities for theoretical as well as application-motivated problems that are discussed in the next section.

6.2 Future Research Problems

Potential research extensions based on the results presented in this thesis are now discussed. First, potential extensions of the damping approach to output feedback nonlinear control design and robust design are discussed. Then, motivated by application to drug delivery systems, long-term research areas are described.

6.2.1 Theoretical Extensions

The results of Sections 4.2 and 5.2 assumed full state feedback stabilization. As noted in Section 4.4, a natural extension of this research would be to consider output feedback stabilization. One approach to re-derive the results of Chapter 4 would be to use the construction of output feedback control Lyapunov functions, as proposed in (Tsinias and Kalouptsidis, 1990) and (Tsinias, 1991). An interesting starting point for the application presented in Section 4.2 would be to consider the design of Lyapunov-based observers as presented in (Nijmeijer and Berghuis, 1995). Geometric elements mirroring the contribution of Banaszuk and Hauser (1996) for observer design was presented in (Lynch and Bortoff, 1997). From that point of view, the research presented in this thesis could be related to approximate (output) feedback linearization. Moreover, synchronization problems, such as the problem discussed in Section 5.3.1 could be considered from an output feedback point of view, as originally proposed in (Stan et al., 2007).

Following the discussion from (Sepulchre et al., 1997) and (Malisoff and Mazenc, 2009), and more recently by Karafyllis and Tsinias (2009), the robustness issue of the proposed design could also be addressed in the future. Building on the idea from (Haddad and Chellaboina, 2009, Chapter 12), where Lyapunov functions of the form $V(x) = V_0 + V_\Delta$ were proposed to achieve robust stabilization with respect to parametric uncertainty, one could consider the idea of domination design based on the damping feedback stabilization technique proposed here, as briefly mentioned in the context of singular perturbation in Section 4.4.

In the present thesis, the structure of the system was rarely considered to simplify the proposed design approach. The work given in (De Leenheer and Aeyels, 2002) on positive systems and in (Angeli and Sontag, 2003) on monotone systems could be considered as potential case studies. The preliminary analysis carried on cyclic system proposed originally in (Arcak and Sontag, 2006; Stan et al., 2007) were given in 5.3.1. Representation results for this class of systems could lead to extensions of the technique presented here to general network stability and stabilization problems (Arcak and Sontag, 2008). From that perspective, considering network stabilization using connections of elements with known dissipative potential could lead to some practical results, following some results obtained using PCH-PBC.

6.2.2 Potential Applications

If one considers drug infusion stabilization problems, such as the example from (Chang and Astolfi, 2008) presented in Chapter 1, many theoretical extensions of the present work need to be addressed before application. First, since parameter uncertainty is inherent to drug infusion dynamics, elements on the time-dependent stabilization approach presented in Section 5.2 could be considered as a basis for adaptive control design, for example following the work presented in (Astolfi et al., 2008). The wide variety of drug infusion models also suggests some potential areas of future research. For example, age-structured models can be considered using state-delayed systems. An application of the Jurdjevic–Quinn method was recently proposed for discrete systems with delays by Aggoune (2007). Continuous models with delays of infection dynamics are emerging in the literature, for example the model of HIV dynamics proposed in (Nelson and Perelson, 2002). From a control point of view, work on stabilization of nonlinear delayed systems were extensively covered by Jankovic, using the "$\mathcal{L}_g V$" technique, for example in (Jankovic, 2003). Extensions of the potential construction technique contained in the present thesis from the point of view considered in (Jankovic, 2003) is possible, if one considers the homotopy approach on more general spaces than $\mathbb{R}^n$ where all the results in the present thesis are built.

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Appendix A

Elements of Exterior Calculus in $\mathbb{R}^n$

Basic elements of exterior calculus on $\mathbb{R}^n$ are reviewed. Of interest are the properties of the exterior derivative, the interior product, Lie derivative of exterior forms, and Cartan's formula. A complete account of exterior calculus can be found in (Edelen, 2005), (Flanders, 1963), and (Lee, 2006, Chapter 12).

We denote a smooth vector field $X \in \Gamma^\infty(\mathbb{R}^n)$ as a smooth map

$$X : \mathbb{R}^n \to T\mathbb{R}^n, \qquad X(x) = \sum_{i=1}^{n} v^i(x)\partial_{x_i}, \tag{A.1}$$

i.e., a map taking a point $x \in \mathbb{R}^n$ and assigning a tangent vector $X(x) \in T_x\mathbb{R}^n$. The cotangent (dual) space $T_x^*\mathbb{R}^n$ is the set of all linear functionals on $T_x\mathbb{R}^n$,

$$T_x^*\mathbb{R}^n = \{\omega(x) : T_x\mathbb{R}^n \to \mathbb{R}\} \tag{A.2}$$

where each $\omega(x)$ is linear, *i.e.*,

$$(a\omega_1(x) + b\omega_2(x))(X(x)) = a\omega_1|_x(X(x)) + b\omega_2|_x(X(x)). \tag{A.3}$$

The standard basis of $T_x^*\mathbb{R}^n$ is given by $\{dx_1, \ldots, dx_n\}$, where $dx_i(\partial_{x_j}) = \delta_j^i$, δ_j^i being the Kronecker delta. An element $\omega(x)$ in the cotangent space $T_x^*\mathbb{R}^n$ can be written as

$$\omega(x) = \sum_{i=1}^{n} \omega_i(x) dx_i, \quad \omega_i \in \mathbb{R}. \tag{A.4}$$

Differential one-forms are generated the following way. A differential one-form ω on $\mathbb{R}^n$ is a smooth map taking a point $x \in \mathbb{R}^n$ and assigning an element of its dual space $T_x^*\mathbb{R}^n$. We write

$$\omega(x) = \sum_{i=1}^{n} \omega_i(x) dx_i, \tag{A.5}$$

where $\omega_i(x)$ are smooth functions on $\mathbb{R}^n$. The exterior (wedge) product $\wedge$ is defined on $\Lambda^1(\mathbb{R}^n) \times \Lambda^1(\mathbb{R}^n)$ by the requirements

$$\begin{aligned} dx_i \wedge dx_j &= -dx_j \wedge dx_i \\ dx_i \wedge f(x) dx_j &= f(x) dx_i \wedge dx_j \end{aligned}$$

for all smooth functions $f(x)$ and by

$$\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma, \tag{A.6}$$

for all $\alpha, \beta, \gamma \in T^*\mathbb{R}^n$. If $\alpha \in \Lambda^k(\mathbb{R}^n)$, then we write $\deg \alpha = k$. We note that $\Lambda^1(\mathbb{R}^n) = T^*\mathbb{R}^n$ and $\Lambda^0(\mathbb{R}^n) = \mathcal{C}^\infty(\mathbb{R}^n)$.

The **exterior derivative** d is the unique operator on $\Lambda(\mathbb{R}^n) = \bigoplus_{k=0}^n \Lambda^k(\mathbb{R}^n)$,

$$d : \Lambda^k(\mathbb{R}^n) \to \Lambda^{k+1}(\mathbb{R}^n), \quad 0 \leq k \leq n-1, \tag{A.7}$$

with the following properties:

1. $d(\alpha + \beta) = d\alpha + d\beta$.
2. $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha}\alpha \wedge d\beta$.
3. $df = \sum_{i=1}^n (\frac{\partial f}{\partial x_i})dx_i, \ \forall f(x) \in \Lambda^0(\mathbb{R}^n)$.
4. $d \circ d\alpha = 0$.

A k-form α is said to be closed if $d\alpha = 0$. It is said to be exact if there exists a $(k-1)$-form β such that $d\beta = \alpha$.

The **interior product** $\lrcorner$ is a map

$$\lrcorner : \Gamma^\infty(\mathbb{R}^n) \times \Lambda^k(\mathbb{R}^n) \to \Lambda^{k-1}(\mathbb{R}^n), \quad 0 \leq k \leq n, \tag{A.8}$$

with the following properties $\forall \alpha, \beta \in \Lambda^k(\mathbb{R}^n)$, $\forall X, X_1, X_2 \in \Gamma^\infty(\mathbb{R}^n)$ and $\forall f, g \in \Lambda^0(\mathbb{R}^n)$:

1. $X \lrcorner f = 0$.
2. $X \lrcorner \omega = \omega(X), \forall \omega \in \Lambda^1(\mathbb{R}^n)$.
3. $X \lrcorner (\alpha + \beta) = X \lrcorner \alpha + X \lrcorner \beta$.
4. $(fX_1 + gX_2) \lrcorner \alpha = f \cdot (X_1 \lrcorner \alpha) + g \cdot (X_2 \lrcorner \alpha)$.
5. $X \lrcorner (\alpha \wedge \beta) = (X \lrcorner \alpha) \wedge \beta + (-1)^{\deg(\alpha)}\alpha \wedge (X \lrcorner \beta)$.
6. $X_1 \lrcorner (X_2 \lrcorner \alpha) = -X_2 \lrcorner (X_1 \lrcorner \alpha)$.

The last property leads to the following composition rule for interior product: $X \lrcorner (X \lrcorner \alpha) = 0$. The **Lie derivative** $\mathcal{L}$ is a map

$$\mathcal{L} : \Lambda^k(\mathbb{R}^n) \to \Lambda^k(\mathbb{R}^n), \quad 0 \leq k \leq n, \tag{A.9}$$

with the following properties:

1. $\mathcal{L}_X f = X \lrcorner df$.
2. $\mathcal{L}_X(\alpha + \beta) = \mathcal{L}_X \alpha + \mathcal{L}_X \beta$.
3. $\mathcal{L}_X(\alpha \wedge \beta) = (\mathcal{L}_X \alpha) \wedge \beta + \alpha \wedge (\mathcal{L}_X \beta)$.
4. $\mathcal{L}_X d\alpha = d(\mathcal{L}_X \alpha)$.
5. $\mathcal{L}_{f \cdot X} \alpha = f \cdot \mathcal{L}_X \alpha + df \wedge (X \lrcorner \alpha)$.
6. $\mathcal{L}_{X_1 + X_2} \alpha = \mathcal{L}_{X_1} \alpha + \mathcal{L}_{X_2} \alpha$.
7. $\mathcal{L}_{X_1}(X_2 \lrcorner \alpha) = \mathcal{L}_{X_1}(X_2 \lrcorner \alpha) - X_2 \lrcorner (\mathcal{L}_{X_1} \alpha)$.

The action of the Lie derivative for $X_1, X_2 \in \Gamma^\infty(\mathbb{R}^n)$ is given by the Lie bracket $\mathcal{L}_{X_1} X_2 = [X_1, X_2]$ and satisfies

$$\mathcal{L}_{[X_1, X_2]} \alpha = (\mathcal{L}_{X_1} \mathcal{L}_{X_2} - \mathcal{L}_{X_2} \mathcal{L}_{X_1}) \alpha. \tag{A.10}$$

We also define $\mathrm{ad}^0_{X_1} X_2 = X_2$ and the k iterates for $k = 1, 2, \ldots$ by $\mathrm{ad}^{k+1}_{X_1} X_2 = [X_1, \mathrm{ad}^k_{X_1} X_2]$. The definitions above yield to the following relation that is used in Section 4.4:

$$[X_1, X_2] \lrcorner \alpha \;=\; X_1 \lrcorner d(X_2 \lrcorner \alpha) - X_2 \lrcorner d(X_1 \lrcorner \alpha) - X_2 \lrcorner X_1 \lrcorner d\alpha + d(X_1 \lrcorner X_2 \lrcorner \alpha). \tag{A.11}$$

We complete this review with Cartan's identity, which relate the Lie derivative of a differential form to exterior differentiation:

$$\mathcal{L}_X \omega = X \lrcorner d\omega + d(X \lrcorner \omega). \tag{A.12}$$