KNOT GROUPS AND BI-ORDERABLE HNN EXTENSIONS OF FREE GROUPS

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The supervisory committee certifies that this dissertation complies with North Dakota State University's regulations and meets the accepted standards for the degree of

DOCTOR OF PHILOSOPHY

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ABSTRACT

Suppose K is a fibered knot with bi-orderable knot group. We perform a topological winding operation to half-twist bands in a free incompressible Seifert surface Σ of K. This results in a Seifert surface Σ' with boundary that is a non-fibered knot K'. We call K a *fibered base* of K'. A fibered base exists for a large class of non-fibered knots.

We prove K' has a bi-orderable knot group if Σ' is obtained from applying the winding operation to only one half-twist band of Σ. Utilizing a Seifert surface gluing technique, we obtain HNN extension group presentations for both knot groups that differ by only one relation. To show the knot group of K' is bi-orderable, we apply the following:

Let G be a bi-ordered free group with order preserving automorphism α. It is well known that the semidirect product $\mathbb{Z} \ltimes_\alpha G$ is a bi-orderable group. If X is a basis of G, a presentation of $\mathbb{Z} \ltimes_\alpha G$ is $\langle\, t, X \mid R \,\rangle$, where the relations are $R = \{txt^{-1}\alpha(x)^{-1} : x \in X\}$. If R' is any subset of R, we prove that the group $H = \langle\, t, X \mid R' \,\rangle$ is bi-orderable. H is a special case of an HNN extension of G. Finally, we add new relations to the group presentation of H such that bi-orderability is preserved.

ACKNOWLEDGEMENTS

I would like to thank my advisor Azer Akhmedov for his guidance on many of the essential arguments found below. His insight has been invaluable.

DEDICATION

This thesis is dedicated to my father Michael Martin. Solely due to him, my interest in mathematics was fueled at a very young age.

PREFACE

In the beginning, I had different goals for this thesis. Shifts in focus and life delays made for a long, bumpy journey, setting its completion back much longer than I intended.

Version one of my thesis was modest. The original plan was to apply results from my first paper [2] to the remaining knots with crossing number 8 with unknown bi-orderability. The techniques in [2] require finding group presentations using Seifert surfaces of the knots, and computing such presentations calls for a tedious, hands-on approach. I had spent hours making Seifert surfaces out of paper and working with string and yarn to determine the groups. While working on the crossing-8 knots, I noticed a nice relationship in the group presentations if one surface is obtained from another by adding an even number of half-twists to some of the connecting bands between disks in the Seifert surface, and this relationship is not obvious in the classic Wirtinger presentation. Most importantly, for known examples the bi-orderability of both knot groups is the same. I observed that all non-fibered knots up to eight crossings can be obtained by adding such half-twists to Seifert surfaces of fibered knots. This was the push towards a new direction.

Version two of my thesis was too ambitious. Now my goal was to relate every knot to a fibered knot, or fibered base, through similar alterations to Seifert surfaces, carefully measuring the effect on the groups, and to obtain a necessary and sufficient condition for bi-orderability. I also wanted a combinatorial method to deduce bi-orderability from the structure of the Seifert surfaces. At nine crossings and above, knots that are both non-alternating and non-fibered became a big problem. There are many non-fibered non-alternating knots that cannot be obtained from adding half-twists to fibered knots. I explored cutting and gluing operations on the surfaces, but none of these had a manageable effect on the group presentations.

Version three of my thesis is what you see now. Work towards the original purpose of the thesis (version one above) was left unfinished, but some related data was collected into a last chapter.

TABLE OF CONTENTS

LIST OF FIGURES

1. INTRODUCTION

Given a knot K in $\mathbb{S}^3$, the knot group is $\pi_1(\mathbb{S}^3 \backslash K)$. Knot groups are left-orderable [15]; however, not every knot group is bi-orderable. For the class of fibered knots, a sufficient condition for bi-orderability can be found in [22]. A partial converse for this result is in [6]. Dropping the fibered assumption, there are results for two-bridge knots [5].

The knot groups of 6_2 and 7_6 are shown to be non-bi-orderable in [2]. The methods do not assume the knots are fibered. Instead, a Seifert surface gluing technique from [12, 17] is applied that results in HNN extension presentations for the knot groups. The Seifert surface gluing technique and resulting knot group presentations are still largely unexplored. We exploit this group presentation to show the following.

Theorem. *Let K be a fibered knot with $\pi_1(\mathbb{S}^3 \backslash K)$ bi-orderable such that Σ_K is a non-flat, free incompressible Seifert surface. Let $\Sigma_{K'}$ and K' be the respective Seifert surface and knot obtained after l-times winding one band in Σ_K. Then $\pi_1(\mathbb{S}^3 \backslash K')$ is bi-orderable.*

Winding a band is adding half-twists to a connecting band in a Seifert surface. If two knots are related by this winding operation, we will show that there exists HNN extension presentations for the knot groups that only differ by a single relation. To prove the above theorem, we also prove the following for HNN extensions of free groups.

Theorem. *Let G be a free group with primitive subgroups A and B, and let $\alpha_\circ \in \mathrm{Aut}(G)$ such that $\alpha_\circ(A) = B$. Define $\alpha = \alpha_\circ|_A$. If $\mathbb{Z} \ltimes_{\alpha_\circ} G$ is bi-orderable then the HNN extension $G*_\alpha$ is bi-orderable.*

Theorem. *Let G be a free group with primitive subgroups A and B, and let $\alpha_\circ \in \mathrm{Aut}(G)$ such that $\alpha_\circ(A) = B$. Define $\alpha = \alpha_\circ|_A$ and let C be the algebraic closure of $\langle A, B\rangle$. Suppose for some indexing set I, $S_1 := \{x_i\}_{i\in I}$ and $S_2 := \{y_i\}_{i\in I}$ are subsets of G such that $D := \langle C, S_1, S_2\rangle \cong C*(*_{i\in I}\langle x_i\rangle)*(*_{i\in I}\langle y_i\rangle)$ is algebraically closed. Let $A' = \langle A, S_1\rangle$, $B' = \langle B, S_2\rangle$, and the isomorphism $\gamma : A' \to B'$ be the extension of α such that $\gamma(x_i) = y_i$ for all $i \in I$. If $\mathbb{Z} \ltimes_{\alpha_\circ} G$ is bi-orderable then the HNN extension $G*_\gamma$ is bi-orderable.*

1.1. Orderable Groups

Left-orderable groups were first explored by Hölder in the early 20th century in a dynamical context [14]. A Group G is called *left-orderable* if there exists a total order $<$ on G such that for all $g, h, k \in G$, $g < h \Rightarrow kg < kh$. $<$ is called a *left-order* on G. Right-orderable groups and right-orders are defined analogously: If there exists a total order $<$ on G such that for all $g, h, k \in G$, $g < h \Rightarrow gk < hk$, we call G *right-orderable*, and $<$ is called a *right-order* on G. If $<$ is simultaneously both a left and right-order on G, we call $<$ a *bi-order*, and G is said to be *bi-orderable.*

Left-orderability and right-orderability are equivalent. Given a left-order $<$ on G, we can define a right-order $\prec$ via $g \prec h \Leftrightarrow h^{-1} < g^{-1}$. We take the left-order convention in this document. Left-orderability is not equivalent to bi-orderability, as we will see below.

1.1.1. Properties of Orderable Groups and Examples

$\mathbb{Z}$ and $\mathbb{R}$ are bi-orderable groups. The standard total order on both of these groups respects addition. More generally, torsion-free Abelian groups are bi-orderable, and non-Abelian examples of bi-orderable groups include free groups, torsion-free nilpotent groups, and pure braid groups. Examples of left-orderable groups that are not bi-orderable are the fundamental group of the Klein bottle, braid groups (in general), and $\mathrm{Homeo}_+(\mathbb{R})$. All surface groups except the fundamental groups of the real projective plane and the Klein bottle are bi-orderable. All of the above examples can be found in [7] and some also in [10].

A left-order on $\mathrm{Homeo}_+(\mathbb{R})$, the group of orientation preserving homeomorphisms of the real line, can be defined as follows: Fix a sequence of real numbers $(x_1, x_2, \ldots)$ that is dense in $\mathbb{R}$. Define $f < g$ if and only if $f(x_i) < g(x_i)$, where x_i is the first element of the sequence where f and g differ. For $h \in \mathrm{Homeo}_+(\mathbb{R})$, $f(x_i) < g(x_i)$ implies $h(f(x_i)) < h(g(x_i))$ since h is increasing. Thus $f < g \Rightarrow h \circ f < h \circ g$, and $<$ is a left-order on $\mathrm{Homeo}_+(\mathbb{R})$. For countable left-orderable groups, $\mathrm{Homeo}_+(\mathbb{R})$ is universal. A countable group G is left-orderable if and only if G embeds in $\mathrm{Homeo}_+(\mathbb{R})$ [7, 10].

For a fixed order $<$ on a group G, we say that $g \in G$ is positive if $g > 1$ and negative if $g < 1$. If g is positive (respectively negative), then g^{-1} is negative (respectively positive). Products of positive elements are positive. Subgroups of an orderable group G are orderable, since one can simply restrict the ordering of G to the subgroup. A more nontrivial fact is that orderability is a local property [7, 10]: G is left-orderable (respectively bi-orderable) if and only if every finitely generated subgroup of G is left-orderable (respectively bi-orderable). Finally, orderable groups are torsion-free. If $g \neq 1$ is a torsion element, then $g^n = 1$ for some $n \geq 2$. Without loss of generality suppose $g > 1$, else consider g^{-1}. Then multiplying both sides by g yields $g^2 > g$. Transitivity implies $g^2 > 1$. Repeating this $n-2$ more times we have $1 = g^n > 1$. Contradiction.

Proposition 1.1.1. *For a left-order $<$ on a group G, the following are equivalent:*

(a) $<$ is a bi-order on G.

(b) Conjugation preserves the positive cone; that is $1 < g$ implies $1 < hgh^{-1}$ for all $h \in G$.

(c) For every $g, h \in G$, if $g < h$ then $h^{-1} < g^{-1}$.

(d) For every $f, g, h, k \in G$, if $f < g$ and $h < k$ then $fh < gk$.

Proof. *(a) ⇒ (d)* Suppose $<$ is a bi-order, $f < g$, and $h < k$. Then $fh < gh$ and $gh < gk$. Transitivity implies $fh < gk$.
(d) ⇒ (c) Suppose $g < h$, and assume for contradiction that $g^{-1} < h^{-1}$. Apply *(d)* to obtain $1 < 1$. Contradiction.
(c) ⇒ (b) Suppose $1 < g$ and $h \in G$. Then $h < hg$. Apply *(c)* to obtain $g^{-1}h^{-1} < h^{-1}$. Left-multiply by hg to obtain $1 < hgh^{-1}$.
(b) ⇒ (a) We need to show the left-order $<$ is also a right-order. Suppose $g < h$ and $k \in G$. Then $1 < g^{-1}h$. Conjugate by k^{-1} to obtain $1 < k^{-1}g^{-1}hk$. Now left-multiply by gk to obtain $gk < hk$. Thus $<$ is also a right-order.
□

Three consequences of Proposition 1.1.1 are the following [10]: Bi-orderable groups do not have generalized torsion; if g is not the identity then products of conjugates of g are not the identity. Second, bi-orderable groups also possess the unique root property; if $f^n = g^n$ and $n \neq 0$, then $f = g$. Finally, in a bi-orderable group if a nonzero power of g commutes with f, then g commutes with f.

A specific left or bi-order on a group G is completely determined by its positive cone P. Conversely if there exists a subsemigroup P of a group G such that $G = P \sqcup \{1\} \sqcup P^{-1}$, then a left-order $<$ on G can be defined by $g < h$ if and only if $g^{-1}h \in P$; furthermore, $<$ is a bi-order if and only if P is preserved under conjugation by elements of G. It is sometimes convenient to define a left or bi-order on a group by first defining positive elements.

A direct product (and therefore a direct sum) of orderable groups is orderable. Given a direct product of orderable groups $G = \prod_{i \in I} G_i$ first choose a well-order on I, and then use lexicographical ordering. Given $g \in G$, define $g = (g_i)_{i \in I}$ to be positive if and only if $g_{i_\circ}$ is positive where $i_\circ$ is the first element of I such that $g_{i_\circ}$ is not the identity in its respective factor $G_{i_\circ}$. An extension of a left-orderable group by a left-orderable group is left-orderable, so in particular semidirect products of left-orderable groups are left-orderable. Consider the following short exact sequence:

$$1 \to N \to G \to G/N \to 1$$

If N and G/N have left-orders $<_N$ and $\prec$ respectively, we can define a left-order $<$ on G as follows: Define $g < h$ if and only if

- $\bar{g} \prec \bar{h}$, or
- $\bar{g} = \bar{h}$ and $1 <_N g^{-1}h$.

The order $<$ is invariant under left-multiplication: Suppose $g < h$ and $k \in G$. If $\bar{g} \prec \bar{h}$ then $\bar{k}\bar{g} \prec \bar{k}\bar{h}$, so $kg < kh$ by definition. Suppose that $\bar{g} = \bar{h}$. Then $1 <_N g^{-1}h = g^{-1}k^{-1}kh = (kg)^{-1}(kh) \Rightarrow kg < kh$. It is worth noting that this argument does not use the fact that $<_N$ is left-multiplication invariant; however, this is needed to prove that $<$ is transitive.

If $<_N$ and $\prec$ are bi-orders, $<$ is not necessarily a bi-order. One also needs that given $g < h$ with $\bar{g} = \bar{h}$, we have $1 <_N k^{-1}g^{-1}hk$ for all $k \in G$; or in other words, the conjugation action of G on N must preserve positive elements of N. This fails in general. Consider the fundamental group of the Klein bottle $\mathbb{K}$. A presentation for this group is

$$\pi_1(\mathbb{K}) = \langle\ x,\ y \mid xyx^{-1} = y^{-1}\ \rangle.$$

Conjugating y by x results in y^{-1}, so this group cannot be bi-orderable by Proposition 1.1.1 since y and y^{-1} are necessarily opposite in sign. However, this group is left-orderable since it is a semidirect product of left-orderable groups $\langle x \rangle \ltimes \langle y \rangle \cong \mathbb{Z} \ltimes \mathbb{Z}$. In general, a semidirect product

$G \ltimes H$ of bi-orderable groups G and H is bi-orderable if and only if there exists a particular bi-order on H such that the action of G on H preserves positive elements of H.

We conclude this subsection with some sketches concerning the orderability of free groups and free products of orderable groups. First consider the free group on n generators $F_n = \langle x_1, \ldots, x_n \rangle$. We bi-order F_n using a representation, due to Magnus [19], into a ring. Let $R = \mathbb{Z}[[X_1, \ldots, X_n]]$, the ring of formal power series of the non-commuting variables $X_1, \ldots, X_n$ with integer coefficients. The *Magnus expansion* map $\mu : F_n \to R$ is given via

- $\mu(x_i) = 1 + X_i$
- $\mu(x_i^{-1}) = 1 - X_i + X_i^2 - X_i^3 + X_i^4 - X_i^5 + \ldots$

A few computations show that μ is an injective homomorphism into the group of units of R that have constant term 1; call this group G. We define an order $<$ on G. For a fixed power series $r \in G \backslash \{1\}$ write the terms in the following form: Order the terms first by degree, and then within a fixed degree use lexicographical ordering. For example in degree five, the term $X_1^2 X_4 X_1 X_2$ would precede $X_1 X_2^2 X_3^2$. Not including the degree zero term of r, which is simply the coefficient 1, let $c(r)$ be the first nonzero coefficient of r respecting the form described. We define $r > 1$ if and only if $c(r) > 0$. A quick argument shows $<$ is a bi-order on G, so F_n is bi-orderable since it is isomorphic to a subgroup of G.

Free groups of arbitrary rank are bi-orderable, since orderability is a local property and every subgroup of a free group is free. More generally, a free product of arbitrarily many left-orderable (respectively bi-orderable) groups is left-orderable (respectively bi-orderable) ; in fact, a left-order (respectively bi-order) exists on the free product such that the restriction of the left-order (respectively bi-order) to any of the free factors is the original order on the free factor. The statement was first proved for two free factors by Vinogradov [27]. Such an order is also described in §2.1.2 of [10].

1.1.2. History of Orderable Groups and Some Applications

An orderable group is said to be *Archimedean* if any two non-identity elements are comparable; that is if for all $g, h \in G \backslash \{1\}$, there exists an integer n such that $g^n > h$. One of the first major results considering orderable groups is due to Hölder. In 1901 Hölder proved that every

Archimedean group embeds into $\mathbb{R}$, hence every Archimedean group is Abelian [14]. The motivation of this theorem was dynamical in nature. An implication of Hölder's theorem is that a freely acting subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ is Abelian.

Orderable groups caught the attention of algebraists in the mid 20th century. A conjecture, often credited to Kaplansky, is the following.

Conjecture 1.1.2. *Suppose R is a ring with identity and without zero divisors, and suppose G is a torsion-free group. Then the group ring RG has no zero divisors.*

The *group ring RG* is the free left R-module generated by elements of G, so an element of RG is a formal linear combination $\sum_{i=1}^{n} a_i g_i$ where each $a_i \in R$ and each $g_i \in G$. Multiplication of elements in RG is defined in the natural way, utilizing the group multiplication in G: To multiply two formal linear combinations, foil to obtain a new linear combination where each term is in the form $(ag)(bh)$ with $a, b \in R$ and $g, h \in G$. Now define $(ag)(bh)$ to be $(ab)(gh)$.

The truth of the conjecture is unknown even when $R = \mathbb{Z}$; however, if G is left-orderable, then the conjecture holds. Assume the hypothesis of the conjecture, and fix a left-order $<$ on G. Consider two nonzero elements $\sum_{i=1}^{n} a_i g_i$ and $\sum_{j=1}^{m} b_j h_j$ in RG. By possibly first combining or relabeling suppose the a_i and b_j are all nonzero, the g_i are all distinct, the h_j are all distinct, and $h_1 < h_2 < \ldots < h_m$. Multiplying these two elements, we obtain

$$\sum_{i=1}^{n} \sum_{j=1}^{m} (a_i b_j)(g_i h_j).$$

Now since $h_1 < h_j$ for all $j \in \{2, \ldots, m\}$, we have $g_i h_1 < g_i h_j$ for every $i \in \{1, \ldots, n\}$ since $<$ is a left-order on G. In other words, $g_i h_1 \neq g_i h_j$ for every $i \in \{1, \ldots, n\}$ and $j \neq 1$. Since the g_i are all distinct, $g_{i_1} h_1 \neq g_{i_2} h_1$ for all distinct $i_1, i_2 \in \{1, \ldots, n\}$. Thus there is exactly one term in the above product that is minimal, so this term will survive cancellation. Hence the product is nonzero, and RG has no zero divisors.

More recent applications of orderable groups in 3-manifold topology are the following: If M is a closed, orientable, irreducible 3-manifold with $\pi_1(M)$ non-left-orderable, and if N is a closed orientable 3-manifold with $\pi_1(N)$ left-orderable, then every map $f : M \to N$ has degree zero [23]. If surgery on a knot K results in a 3-manifold with finite fundamental group, then the knot group of K is not bi-orderable [6]. It is well known that every closed, orientable, connected 3-manifold is obtained as a Dehn surgery on a link. This famous result has been proved independently by Wallace [28] and Lickorish [16]. When does a Dehn filling result in a 3-manifold with left-orderable fundamental group? This question is particularly important and has been extensively studied for Dehn fillings on a knot. For example, it has been shown by Culler and Dunfield in [9] that for a knot K in $\mathbb{S}^3$, and more generally in a homology 3-sphere with a lean complement, if the Alexander polynomial has a simple root on a unit circle, then there exists a positive real number $a > 0$ such that for all rational $r \in (-a, a)$ the Dehn filling $D(r)$ on K produces a 3-manifold with left-orderable fundamental group. In general, the spectrum of r where the Dehn filling $D(r)$ on a given knot K has a left-orderable fundamental group is an important and well studied question.

1.2. Knots and Links

A *knot* K is an embedding of $\mathbb{S}^1$ in $\mathbb{S}^3$. Two knots K and J are said to be *equivalent* if there exists an ambient isotopy of $\mathbb{S}^3$ which takes K to J. We call the group $\pi_1(\mathbb{S}^3 \backslash K)$ the *knot group* of K. This is an invariant of the knot K; that is, if two knots are equivalent then their respective knot groups are isomorphic. A *link* is one or more disjointly embedded knots in $\mathbb{S}^3$. Each knot in the link is called a *component* of the link. Analogous definitions hold for link equivalence and link groups.

As customary in knot theory, we will assume our links (and therefore knots) are tame. A link L is *tame* if L is equivalent to a link L' such that there exists an extension of the embedding of L' in $\mathbb{S}^3$ to a tubular neighborhood of L'. There are many equivalent definitions of tame: Smooth links, polygonal links, and tame links all define the same objects [8]. We also assume that our links have finitely many components, so that this along with the tame condition above guarantees our links and knots have finitely many crossings in reduced diagrams.

A knot equivalent to a knot that has a diagram with no crossings is called the *unknot*, or the trivial knot, and it is denoted 0_1. The unknot is ambiently isotopic to a circle in the plane. An *unlink* is a link that is ambiently isotopic to finitely many disjoint circles in the plane. A separate, less trivial case that occurs with links is if there exists a topological 2-sphere in the complement of the link that separates some of the components of the link. We call such links *split links.* Every unlink with more than one component is a split link. Split links are more general, since a component in a split link may be a nontrivial knot.

The simplest nontrivial knot has three crossings and is called the trefoil; the trefoil is denoted 3_1 and is the only knot with three crossings. The next knot is the figure-8 knot, which has four crossings and is denoted 4_1. The simplest non-split link that has more than one component is called the Hopf link and has two crossings.

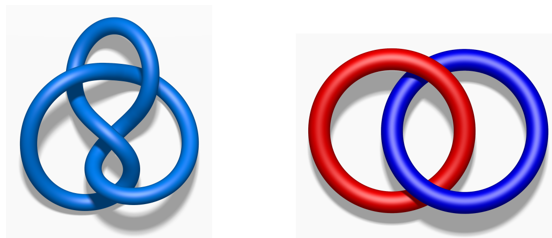

Figure 1.1. The figure-8 knot (left) and the Hopf link (right).

A link is called *alternating* if there exists an oriented diagram of the link such that all crossings alternate between over and under when following along the orientation of each component of the link. Note that both the figure-8 knot and Hopf link are alternating. The first non-alternating knot occurs at eight crossings.

1.2.1. Presentations of Link Groups and Seifert Surfaces

Historically, a Wirtinger presentation has been a commonly used presentation for knot and link groups. A *Wirtinger presentation* is a finite presentation in the form

$$\langle g_1, \ldots, g_n \mid w_1 g_{i_1} w_1^{-1} = g_{j_1}, \ldots, w_r g_{i_r} w_r^{-1} = g_{j_r} \rangle.$$

Here, each w_k is a word in the generators $\{g_1, \ldots, g_n\}$, and $i_l, j_l \in \{1, \ldots, n\}$ for each $l \in \{1, \ldots, r\}$. The above presentation is abstracted from Wirtinger's original method of computing knot groups around 1904 during his lectures in Vienna. Wirtinger's argument was first published in a paper of Tietze in 1908 [26]. We describe Wirtinger's approach. Fix an oriented diagram of a link L, and suppose L has n arcs. An arc is a segment of a link that begins at an under-crossing and ends at the next under-crossing. Label the arcs $g_1, \ldots, g_n$. These are the generators of the link group.

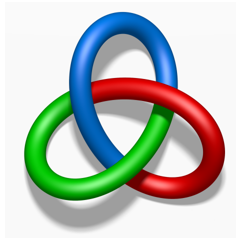

Figure 1.2. The trefoil knot with colored arcs.

A relation is acquired at each crossing. Note that there are three arcs at each crossing. Suppose the over-crossing arc is g_k, and at the under-crossing there are arcs g_i and g_j, with g_i occurring first with respect to the orientation of the link. The relation obtained is that either g_k or g_k^{-1} conjugates g_i to g_j, depending on the orientation of the crossing. In the abstract definition of Wirtinger presentation above, observe that in the case of link groups the words w_k are simply just one of the generators.

The unknot has one arc and no crossings, so its knot group is $\mathbb{Z}$. In general the link group for an n-component unlink is F_n. Suppose in a split link L, we have $L = L_1 \cup L_2$ where L_1 is separable from L_2 by an embedded 2-sphere in $\mathbb{S}^3$. Note that each L_i may itself be a link which is not a knot. Then when computing a Wirtinger presentation, none of the arcs in L_1 appear in crossings with arcs in L_2. Hence we obtain $\pi_1(\mathbb{S}^3 \backslash L) \cong \pi_1(\mathbb{S}^3 \backslash L_1) * \pi_1(\mathbb{S}^3 \backslash L_2)$.

For nontrivial examples, first consider the Hopf link pictured above. There are only two arcs. One arc conjugates the other arc to itself. The resulting group is therefore $\mathbb{Z}^2$. Next consider the trefoil 3_1. If the three arcs are x, y, and z, one obtains the following presentation applying Wirtinger's algorithm.

$$\begin{aligned} \pi_1(\mathbb{S}^3 \backslash 3_1) &= \langle\ x,\ y,\ z \mid x = yzy^{-1},\ y = zxz^{-1},\ z = xyx^{-1}\ \rangle \\ &= \langle\ x,\ y \mid xyx = yxy\ \rangle. \end{aligned}$$

Observe that the Abelianization of this groups is $\mathbb{Z}$; the Abelianization of a link group causes all arcs to be equal to one another in a component of the link. Therefore the first homology group of a link complement in $\mathbb{S}^3$ is $\mathbb{Z}^r$, where r is the number of components of the link. Thus the first homology group of a link complement is a poor invariant, since it only distinguishes the number of components. The above presentation of the trefoil is isomorphic to the following presentation.

$$\pi_1(\mathbb{S}^3 \backslash 3_1) \;\cong\; \langle\ a,\ b \mid a^2 = b^3\ \rangle$$

This group is not bi-orderable since b does not commute with a, but a power of b commutes with a; however, in the next subsection we will argue that this group is left-orderable. Applying the above for the figure-8 knot 4_1, we obtain the following presentation.

$$\pi_1(\mathbb{S}^3 \backslash 4_1) = \langle x, y, z, w | x = z^{-1}wz, y = wxw^{-1}, z = x^{-1}yx, w = yzy^{-1}\rangle = \langle x, y | xyx^{-1}yx = yxy^{-1}xy\rangle$$

Determining the bi-orderability of this knot is not immediately obvious. Wirtinger presentations for knot groups are not necessarily that useful for determining orderability. Borrowing a Seifert surface technique from [12, 17], we obtain a presentation for knot groups expressed in the form of an HNN extension of a free group. First we define Seifert surfaces and HNN extensions.

Definition 1.2.1. A *Seifert surface* Σ_L of a link L is a connected, compact, orientable surface whose boundary is L.

Frankl and Pontrjagin first proved the existence of Seifert surfaces in 1930 [11]. In 1934, Seifert constructed an algorithm for constructing these surfaces from a given link diagram [25]. We roughly describe Seifert's algorithm. Start on a point on a strand in an oriented link diagram; a strand is like an arc except it ends at the very next crossing, whether the crossing is over or under-handed. We follow the strand, respecting the orientation, until a crossing is reached. At the crossing there are two incoming and two outgoing strands, and we are arriving through one of the incoming strands. We now switch to the adjacent outgoing strand. We continue to travel along the new strand and repeat this at each crossing until we arrive at our starting point. This procedure creates a disk called a *Seifert disk*. Start on a new strand of the diagram that has not yet been exhausted from creating the first Seifert disk and repeat. We obtain a second Seifert disk, and we continue to create more Seifert disks until every possible strand is exhausted. Finally we attach half-twist bands between the Seifert disks at locations that correspond to the crossings in the original link diagram. The orientations of the half-twist bands respect the orientations of the original crossings.

Seifert surfaces are not unique, since one can attach a handle to a given Seifert surface of a link L to obtain a new Seifert surface of L. The *genus* of a link L is the minimal genus of all Seifert surfaces of L, and is an invariant of L. Seifert surfaces may themselves be knotted. A Seifert surface is called *free* if the complement of the surface in $\mathbb{S}^3$ is a handlebody. The *free genus* of a link L is the minimal genus of all free Seifert surfaces of L. It is worth noting that every Seifert surface obtained from applying Seifert's algorithm is free. This follows immediately from the algorithm. In general the genus of a link is difficult to compute; Seifert's algorithm fails to produce a minimal genus Seifert surface for many non-alternating knots, even when considering all reduced diagrams of such knots. This follows since the gap between free genus and genus can be arbitrarily large [20].

Let L be a link and Σ be an embedded Seifert surface of L in $\mathbb{S}^3$. We say that Σ is *incompressible* if $\Sigma \backslash L$ is π_1-injective in $\mathbb{S}^3 \backslash L$. An application of the Loop Theorem shows that every minimal genus Seifert surface of a knot is incompressible. It is also worth noting that there are knots that admit only non-free incompressible Seifert surfaces [18]. A nice special case is alternating knots. Applying Seifert's algorithm to a reduced diagram of an alternating knot results in a minimal genus Seifert surface [21], so every alternating knot admits a Seifert surface that is both free and incompressible.

The last adjective for Seifert surfaces we define is flat. A free Seifert surface Σ embedded in $\mathbb{S}^3$ is called *flat* if Σ is ambiently isotopic to a Seifert surface $\Sigma_\circ$, where $\Sigma_\circ$ can be constructed from disjoint disks in a plane and attaching half-twist bands between the disks such that no half-twist band overlaps with any of the disks or any of the other half-twist bands. The Seifert surface of the Hopf link above is flat. We can start with two disjoint disks in a plane and attach two half-twists bands between the disks. A flat incompressible Seifert surface Σ admits a particularly simple HNN extension group presentation of the link bounding Σ. Next we define HNN extensions, which were first introduced by Higman, Neumann, and Neumann in 1949 [13].

Definition 1.2.2. Let G be a group with presentation $\langle\, S \mid R \,\rangle$, and let $A, B \leq G$ with isomorphism $\alpha : A \to B$. The *HNN extension* $G*_\alpha$ of G is defined $G*_\alpha = \langle\, S, t \mid R, tat^{-1} = \alpha(a)\ \forall a \in A \,\rangle$. G is called the *base group*, and t is called the *stable letter*.

If $A = B = G$ then $G*_\alpha = \mathbb{Z} \ltimes_\alpha G$. In the other extreme, if A and B are trivial then $G*_\alpha = \mathbb{Z} * G$, so morally a proper HNN extension of G is somewhat between a free product and a semidirect product of $\mathbb{Z}$ with G. If G is left-orderable, then by the above both $\mathbb{Z} \ltimes G$ and $\mathbb{Z} * G$ are left-orderable; however, if G is left-orderable it is not necessarily the case that every HNN extension of G is left-orderable [1]. This even fails when G is a free group; it also fails when G is left-orderable solvable. If G is torsion-free nilpotent, then every HNN extension of G is left-orderable [1]. In particular every HNN extension of every torsion-free Abelian group is left-orderable.

Although HNN extensions are defined for an abstract group, they occur naturally when computing the fundamental group of the space W obtained by gluing together two homeomorphic subsets Y and Z of a path-connected topological space X. The stable letter t in the above definition is the loop created from gluing Y to Z. We utilize this idea to yield HNN extension presentations of link groups.

Applying the construction from [12] and also described in [2], let L be a link with incompressible Seifert surface Σ and $\nu(L)$ be a tubular neighborhood of L. Now let $W = \mathbb{S}^3 \backslash \nu(L)$, and consider $X = W \backslash (\Sigma \times (-\epsilon, \epsilon))$; we can glue the positive parallel copy of the Seifert surface $Y := \Sigma \times \{\epsilon\} \subseteq X$ to the negative $Z := \Sigma \times \{-\epsilon\} \subseteq X$ to recover W, so that a loop in the positive copy of Σ has a parallel translate on the negative copy. Let $x_+ \in \pi_1(Y)$, and let $p \subseteq Y$ be a representative of x_+. Then $p = p_0 \times \{\epsilon\}$ for some loop $p_0 \subseteq \Sigma$. Define $q = p_0 \times \{-\epsilon\}$, and finally define $x_- = [q] \in \pi_1(Z)$, assuming here that if $x_0 \times \{\epsilon\}$ is the base point for Y, then $x_0 \times \{-\epsilon\}$ is the base point for Z. Letting $G = \pi_1(X)$, $A = \pi_1(Y)$, and $B = \pi_1(Z)$, and applying the Seifert-van Kampen theorem, we obtain an HNN extension presentation of the link group:

$$\pi_1(\mathbb{S}^3 \backslash L) \;\cong\; \pi_1(W) \;=\; \langle\, G, t \mid t x_+ t^{-1} = x_- \;\forall x_+ \in A \,\rangle.$$

Note that the incompressibility of Σ in the above construction guarantees that the inclusions $i : Y \to X$ and $j : Z \to X$ are π_1-injective, so A and B make sense as subgroups of G. Note that G is a free group if Σ is also a free.

As a first example consider the unknot 0_1. The set X described above looks like the complement of a red blood cell. Y and Z are the top and bottom disk of the inner webbing of the red blood cell. Gluing Y to Z yields W, which is the complement of a solid torus in this case. The groups G, A, and B are all trivial so the presentation obtained is simply $\langle t \rangle \cong \mathbb{Z}$.

A Seifert surface Σ_{3_1} for the trefoil can be obtained by starting with two disks D_1 and D_2, and then attaching three single-half-twist bands with the same orientation between them. Since 3_1 (the boundary of the surface in this particular diagram) is reduced alternating, Σ_{3_1} is free and incompressible. In this case Σ_{3_1} is also flat. Both the surface and its complement in $\mathbb{S}^3$ are homotopy equivalent to a wedge of two circles, so we need two generators for each. Pictured is Σ_{3_1} with choice of generators $G = \langle a, b \rangle$ and $\pi_1(\Sigma_{3_1}) = \langle \alpha, \beta \rangle$.

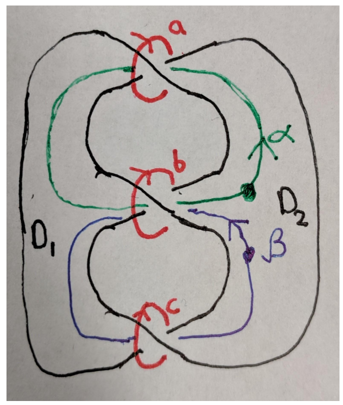

Figure 1.3. A flat incompressible Seifert surface of 3_1 with labeled generators.

The group A is obtained by lifting α and β off of one side of the surface and writing these as words in a, b, and c. We obtain $\alpha \mapsto a$ and $\beta \mapsto b$. From the other side of the surface we obtain $\alpha \mapsto b^{-1}$ and $\beta \mapsto c^{-1}$. G is generated by a and b. In the above figure $cba = 1$ so that $c^{-1} = ba$. The HNN extension presentation obtained is

$$\pi_1(\mathbb{S}^3 \backslash 3_1) \ = \ \langle \ t, a, b \mid tat^{-1} = b^{-1}, tbt^{-1} = ba \ \rangle.$$

Applying the construction to the figure-8, we have the following:

$$\pi_1(\mathbb{S}^3 \backslash 4_1) \ = \ \langle \ t, a, b \mid tat^{-1} = ba, tbt^{-1} = bab \ \rangle.$$

For both the trefoil and figure-8, the groups A and B are the entire group $G \cong F_2$, so the knot groups are both a semidirect product $\mathbb{Z} \ltimes F_2$. The HNN extension presentation results in a semidirect product if the link is fibered [12]. Both the trefoil and figure-8 knots are fibered knots.

Definition 1.2.3. A link L is called *fibered* if there exists a fibration $p : \mathbb{S}^3 \backslash L \to \mathbb{S}^1$ such that each fiber is a Seifert surface of L.

1.2.2. Orderability of Knot and Link Groups

We argue link groups are left-orderable by sketching the argument that link groups are *locally indicable*. Locally indicable groups lie strictly between left-orderable and bi-orderable groups [7], so we have the following: Bi-orderable $\Rightarrow$ locally indicable $\Rightarrow$ left-orderable $\Rightarrow$ torsion-free.

Definition 1.2.4. A group G is called *locally indicable* if every nontrivial finitely generated subgroup H of G surjects onto $\mathbb{Z}$.

Here is the sketch of the argument found in [7], following [15], showing that link groups are locally indicable: Suppose L is a link. Since the free product of locally indicable groups is locally indicable, we can assume that L is a non-split link; that is, there exists no 2-sphere in $\mathbb{S}^3 \backslash L$ that separates L. The complement of a non-split link L is an irreducible 3-manifold. Let $X = \mathbb{S}^3 \backslash L$, $G = \pi_1(X)$, and $H \leq G$ be nontrivial and finitely generated. $H_1(X) \cong \mathbb{Z}^r$ where r is the number of components in the link L, so if $[G : H] < \infty$ then the Abelianization of H is isomorphic to a nontrivial subgroup of $\mathbb{Z}^r$. Hence there exists a nontrivial homomorphism from H to $\mathbb{Z}$. Now suppose $[G : H] = \infty$. Let $p : Y \to X$ be a covering space of X such that $p_*(\pi_1(Y)) = H$. Y is a non-compact space with finitely generated fundamental group. The Scott core theorem [24] implies there exists a compact connected submanifold Z, or compact core, of Y such that the induced map $\pi_1(Z) \to \pi_1(Y)$ from the inclusion is an isomorphism. Since X is irreducible, and therefore Y as well, one can argue that Z has nonempty boundary consisting of no 2-sphere components. Consider the closed manifold Z' obtained by gluing two copies of Z along ∂Z via the identity. $\chi(Z') = 0$ and $\chi(\partial Z) \leq 0$, since ∂Z has no 2-sphere components. An Euler characteristic argument shows $\chi(Z) \leq 0$. From here, since $\chi(Z)$ is an alternating sum of ranks from the rational homology of Z, a computation shows that the rank of the first rational homology group of Z is nonzero. Thus the Abelianization of H, which is isomorphic to $H_1(Z)$, is infinite. Hence there exists a nontrivial homomorphism from H to $\mathbb{Z}$.

Although every link group is left-orderable, not every link group is bi-orderable. Two results that concern the bi-orderability of fibered knot groups are the following. The first result is from

Perron and Rolfsen in [22], and the second result, which is a partial converse, is due to Clay and Rolfsen in [6].

Theorem 1.2.5. *Let K be a fibered knot such that all of the roots of the Alexander polynomial $\Delta_K(t)$ are positive and real. Then $\pi_1(\mathbb{S}^3 \backslash K)$ is bi-orderable.*

Theorem 1.2.6. *Let K be a nontrivial fibered knot such that $\pi_1(\mathbb{S}^3 \backslash K)$ is bi-orderable. Then the Alexander polynomial $\Delta_K(t)$ has at least one positive real root.*

Recall that 3_1 and 4_1 are fibered. The Alexander polynomial for the trefoil knot is $\Delta_{3_1}(t) = t^2 - t + 1$. The roots of this polynomial are both non-real, so the second theorem implies $\pi_1(\mathbb{S}^3 \backslash 3_1)$ is not bi-orderable. The Alexander polynomial for the figure-eight knot is $\Delta_{4_1}(t) = t^2 - 3t + 1$, and the roots are $\frac{3 \pm \sqrt{5}}{2}$. Since both roots are positive, $\pi_1(\mathbb{S}^3 \backslash 4_1)$ is bi-orderable by the first theorem.

Another class of knots with results concerning bi-orderability are two-bridge knots. If K is a two-bridge knot there exists a presentation of K with two generators and one relation [5]. Arranging some conditions and applying results found in [4], the following theorem is obtained in [5].

Theorem 1.2.7. *Let K be a two-bridge knot such that $\pi_1(\mathbb{S}^3 \backslash K)$ is bi-orderable. Then the Alexander polynomial $\Delta_K(t)$ has at least one positive real root.*

There are many non-fibered knots that are two-bridge. For example, Theorem 1.2.7 implies the non-fibered knots 5_2, 7_2, 7_3, 7_4, 7_5, 8_8, 8_{13}, and 9_2 have non-bi-orderable knot groups. Twist knots are a simple case of two-bridge knots. Twist knots with an even (respectively odd) number of crossings have a bi-orderable (respectively non-bi-orderable) knot group [5]. Examples of non-fibered twist knots with bi-orderable knot group include 6_1, 8_1, 10_1, and $12a_{0803}$. A knot that is neither two-bridge nor fibered that is known to have a bi-orderable knot group is 10_{13} [7].

For the case of links, less is known concerning bi-orderability. The unlink and Hopf link have bi-orderable link groups, since F_n and $\mathbb{Z}^2$ are bi-orderable. A nice case is split links: Suppose $L = L_1 \cup L_2$ where L_1 is separable from L_2 by an embedded 2-sphere in $\mathbb{S}^3$. Recall from above that $\pi_1(\mathbb{S}^3 \backslash L) \cong \pi_1(\mathbb{S}^3 \backslash L_1) * \pi_1(\mathbb{S}^3 \backslash L_2)$. Since every subgroup of an orderable group is orderable, and since a free product of orderable groups is orderable, we have that L has a bi-orderable link group if and only if both L_1 and L_2 have bi-orderable link groups.

A broad class of examples of non-bi-orderable links can be obtained by the following *necklace operation*. By this construction we obtain non-bi-orderable link groups where every component has a bi-orderable knot group. Let K be a knot. A link L is called a *necklace* of K, if L is ambiently isotopic to a link L' such that we can order the components of L' cyclically as $K_1, K_2, \ldots, K_n, K_{n+1} = K_1$ such that the following conditions hold:

- There exist balls $B_1, \ldots, B_n$ such that $K_i \subset B_i$ for all $i \in \{1, \ldots, n\}$.
- For all distinct $i, j \in \{1, \ldots, n\}$, $B_i \cap B_j$ is nonempty if and only if $|i - j| = 1$ in $\mathbb{Z}/n\mathbb{Z}$.
- The union $B_1 \cup \cdots \cup B_n$ lies in a tubular neighborhood of K.
- For all $i \in \{1, \ldots, n\}$, the linking number of K_i and K_{i+1} equals $+1$ or -1.

Now notice that $\pi_1(\mathbb{S}^3 \backslash K)$ is embedded in $\pi_1(\mathbb{S}^3 \backslash L)$ as a subgroup. Thus if $\pi_1(\mathbb{S}^3 \backslash K)$ is not bi-orderable, then the link group $\pi_1(\mathbb{S}^3 \backslash L)$ is not bi-orderable.

2. SOME BI-ORDERABLE HNN EXTENSIONS OF FREE GROUPS

Recall in the previous chapter that if G is a bi-orderable group, then $\mathbb{Z} * G$ is bi-orderable and $\mathbb{Z} \ltimes_\alpha G$ is bi-orderable as long as there exists a bi-order on G that is invariant under the automorphism α. The two groups $\mathbb{Z} * G$ and $\mathbb{Z} \ltimes_\alpha G$ are the extreme cases of HNN extensions of G. There are two main results in this chapter: The first result shows that we can start with a bi-orderable semidirect product $\mathbb{Z} \ltimes_\alpha G$ with G a free group, delete relations, and then obtain a new group that is bi-orderable. The second result shows that after deleting, we can glue in a certain choice of new relations and obtain a group that is still bi-orderable. These will be referred to as the deleting and gluing theorems respectively.

We use the following notational conventions: If S is a subset of a group G and $g \in G$, we use $\langle g, S\rangle$ to denote the subgroup of G that is generated by the subsets $\{g\}$ and $\{S\}$ of G, and gSg^{-1} will denote the set $\{gsg^{-1} : s \in S\}$. In group presentations the symbol ":" is only used as the phrase "such that," and the symbol "|" is only used to separate generators and relations.

2.1. Preliminaries

The following two results, Britton's Lemma [3] and Vinogradov's Theorem [27], are major tools used in the proofs of the deleting and gluing theorems. Here is Britton's Lemma.

Lemma 2.1.1. *Let A and B be subgroups of a group G such that $\alpha : A \to B$ is an isomorphism, and let $G*_\alpha$ be the corresponding HNN extension with stable letter t. Let $w = g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} g_2 \ldots g_{n-1} t^{\epsilon_n} g_n$ such that $g_0, g_i \in G$ and $\epsilon_i \in \{-1, 1\}$ for all $i \in \{1, \ldots n\}$.*

- *If $n = 0$ and $g_0 \neq 1$ in G, then $w \neq 1$ in $G*_\alpha$.*
- *If $n > 0$ and w does not contain substrings of the form tat^{-1} or $t^{-1}bt$ where $a \in A$, $b \in B$, then $w \neq 1$ in $G*_\alpha$.*

A consequence of Britton's Lemma is G embeds into $G*_\alpha$, so we can view G as a subgroup of $G*_\alpha$. Following is Vinogradov's Theorem.

Theorem 2.1.2. *Suppose G and H are groups with bi-orders $<_G$ and $<_H$ respectively. Then there exists a bi-order on the free product $G * H$ whose restrictions to each factor G and H are the original respective bi-orders $<_G$ and $<_H$.*

From the proof of Vinogradov's Theorem, we have a stronger result. If the bi-orders on G and H are both invariant under some respective maps $\alpha \in \mathrm{Aut}(G)$ and $\beta \in \mathrm{Aut}(H)$, then a bi-order $<$ can be constructed on $G * H$, as stated above, such that we additionally have $<$ is invariant under $\alpha * \beta \in \mathrm{Aut}(G * H)$. As stated in the first chapter Theorem 2.1.2 holds for free products of arbitrarily many bi-orderable free factors [10]. A consequence of this theorem is the following.

Proposition 2.1.3. *Let $F = F(S)$ be the free group on the set $S = \{s_n : n \in \mathbb{Z}\}$, and let $\Gamma = \mathbb{Z} \ltimes_\alpha (H * F)$ where $\alpha|_H \in \mathrm{Aut}(H)$ and $\alpha(s_n) = s_{n+1}$ for all $n \in \mathbb{Z}$. If there exists a bi-order on H invariant under α, then Γ is bi-orderable.*

Proof. Let $<_1$ be a bi-order on H that is invariant under α. Let $<_2$ be a Magnus bi-order on F where terms in the formal power series ring are ordered lexicographically so that $s_n <_2 s_{n+1}$ for all $n \in \mathbb{Z}$. Then $<_2$ is preserved by α. Then by the proof of Vinogradov's Theorem there exists a bi-order $<$ on $H * F$ such that the restrictions of $<$ on H and F are $<_1$ and $<_2$ respectively, and $<$ is invariant under α. Hence the semidirect product Γ is bi-orderable.

$\square$

In particular if H is trivial in Proposition 2.1.3, we can place a bi-order on a free group with basis indexed by $\mathbb{Z}$ that is invariant under the index shift map. We close this preliminary section with some definitions.

Definition 2.1.4. Let Γ be a group such that $\Gamma_1 \leq \Gamma$. Γ_1 is a *root closed subgroup* of Γ if for all $u \in \Gamma \backslash \Gamma_1$ and $n \in \mathbb{Z} \backslash \{0\}$, $u^n \notin \Gamma_1$. For any subgroup $\Gamma_2 \leq \Gamma$, the intersection of all root closed subgroups of Γ containing Γ_2 is called the *root closure* of Γ_2 in Γ.

Definition 2.1.5. Let Γ be a group such that $\Gamma_1 \leq \Gamma$. Γ_1 is an *algebraically closed subgroup* of Γ if for all $u \in \Gamma \backslash \Gamma_1$, $\{w_1, \ldots, w_k\} \subseteq \Gamma_1 \backslash \{1\}$, and $\{n_1, \ldots, n_k\} \subseteq \mathbb{Z} \backslash \{0\}$, $u^{n_1} w_1 \cdots u^{n_k} w_k \notin \Gamma_1$. For any subgroup $\Gamma_2 \leq \Gamma$, the intersection of all algebraically closed subgroups of Γ containing Γ_2 is called the *algebraic closure* of Γ_2 in Γ.

Note that algebraically closed implies root closed. For a free group G with an HNN extension $G*_\alpha$ and map $\alpha : A \to B$, we need to be careful with elements in G outside of A or B when applying Britton's Lemma to obtain free subgroups of $G*_\alpha$. This will be an important tool in the next sections.

Remark 2.1.6. If A and B are algebraically closed subgroups of a group G such that $C := \langle A, B\rangle \cong A * B$, then C is not necessarily an algebraically closed subgroup of G. If $G = \langle a, b\rangle$, $A = \langle aba\rangle$, and $B = \langle b\rangle$ then both A and B are algebraically closed subgroups of G; however, $(ab)^2 \in C$ with $ab \notin C$, so C is not even root closed.

2.2. The Deleting Theorem

Let G be a free group with basis X. Let $\alpha \in \mathrm{Aut}(G)$ such that $\mathbb{Z} \ltimes_\alpha G$ is bi-orderable. If $\mathbb{Z}$ is generated by t, then we have the presentation $\mathbb{Z} \ltimes_\alpha G = \langle t, X | R\rangle$ where the exhaustive, non-redundant relations are precisely $R = \{txt^{-1}\alpha(x)^{-1} : x \in X\}$. Let R' be any subset of R. In this section we prove that $H = \langle t, X|R'\rangle$ is bi-orderable. H is an HNN extension of G. The associated subgroups A and B of G in the HNN extension H are generated by subsets of the bases X and $\alpha(X)$ of G. We define such subgroups to be primitive subgroups of G. Note that this is a different notion of a primitive subgroup than found elsewhere in the literature.

Definition 2.2.1. Let S be a set and $F(S)$ be the free group on S. An element $g \in F(S)$ is called *primitive* if there exists $s \in S$ and $\alpha \in \mathrm{Aut}(F(S))$ such that $\alpha(s) = g$. A subgroup $G \leq F(S)$ is called *primitive* if there exists a subset $T \subseteq S$ and $\alpha \in \mathrm{Aut}(F(S))$ such that $G = \alpha(F(T)) = \langle \alpha(t) : t \in T\rangle$.

For example if $a, b \in F(S)$ the commutator $[a, b]$ and a proper power a^n, $n \notin \{-1, 0, 1\}$, are not primitive. On the other hand, for distinct $a, b, c \in S$ the elements $a^p b$ and $(a^p b)^q c$ are primitive. If S is a singleton, then $F(S)$ is isomorphic to the integers. In this case $F(S)$ has no proper nontrivial primitive subgroups. If $S = \{a, b, c\}$ then $\langle a^2, b\rangle$ is not primitive but $\langle a^p b, (a^p b)^q c\rangle$ is primitive.

In the next two paragraphs assume that G is a primitive subgroup of the free group $F := F(S)$ on a set S. The next statements follow immediately from the definition: Automorphic images of primitive elements (respectively subgroups) are primitive elements (respectively subgroups), so there is no danger in simply saying G is a primitive subgroup of a free group F without specifying a basis; furthermore, we can find a basis of F so that α in the definition becomes trivial: Let $Y = \alpha^{-1}(T)$ and let $X = \alpha^{-1}(S)$ so that we have $F \cong F(X)$ with $G = \langle Y \rangle$ and $Y \subseteq X$. Since G is generated by a subset of a basis of F, G is generated by primitive elements and $\operatorname{rank}(G) \leq \operatorname{rank}(F)$; this inequality is strict if G is a proper subgroup and F is finitely generated. Letting $H = \langle X \backslash Y \rangle$, H is a primitive subgroup of F such that $F = \langle G, H \rangle \cong G * H$. A well-known special case of this is if F has rank n, then an element $g \in F$ is primitive if and only if there exists $g_1, g_2, \ldots, g_{n-1} \in F$ such that $F = \langle g_1, g_2, \ldots, g_{n-1}, g \rangle$. Since primitive subgroups are generated by a subset of a basis, primitive subgroups are algebraically closed.

If $x \in F \backslash G$ then $\langle G, x \rangle \cong G * \langle x \rangle$; however, $\langle G, x \rangle$ is not necessarily a primitive subgroup of F if x is primitive. For example $\langle ab, ba \rangle$ is not a primitive subgroup of $\langle a, b \rangle$, even though ab and ba are primitive elements. Note that in this case we still have that $\langle ab, ba \rangle \cong \langle ab \rangle * \langle ba \rangle$. More generally, if A and B are primitive subgroups of F then $\langle A, B \rangle$ is not necessarily primitive, even if $A \cap B$ is trivial; moreover, $\langle A, B \rangle$ may not even be root closed. The next proposition shows that we do not have to be careful with specification when considering nested primitive subgroups.

Proposition 2.2.2. *Suppose G is a primitive subgroup of a free group F such that $H \leq G$. Then H is a primitive subgroup of F if and only if H is a primitive subgroup of G.*

Proof. Choose a basis S of F such that $G = F(T) = \langle t : t \in T \rangle$ for some $T \subseteq S$. Assume that H is a primitive subgroup of F. Then there exist $U \subseteq S$ and $\alpha \in \operatorname{Aut}(F)$ where $H = \langle \alpha(u) : u \in U \rangle$, so that $B_1 = \{\alpha(u) : u \in U\}$ is a subset of a basis of F that is contained in $F(T) = G$. Then there exists a basis B of G such that $B_1 \subseteq B$. Choose $\beta \in \operatorname{Aut}(G)$ such that $\beta(T) = B$. There exists $T_1 \subseteq T$ such that $\beta(T_1) = B_1$. Thus $H = \beta(F(T_1))$ so H is a primitive subgroup of G. The converse follows immediately since an automorphism of $F(T)$ can be extended trivially to an automorphism of $F(S)$ by fixing all elements in $S \backslash T$.

□

Before proving the deleting theorem, we have one short and two technical lemmas. The first lemma lets us define an extended bi-order given two bi-orderable primitive subgroups with bi-orders that overlap nicely.

Definition 2.2.3. Let G be a group with $A, B \leq G$. Suppose $<_1$ and $<_2$ are bi-orders on A and B respectively. We say these bi-orders *agree* if for all $x, y \in A \cap B$, $x <_1 y$ if and only if $x <_2 y$.

Lemma 2.2.4. *Let G be a free group with primitive subgroups A and B having respective bi-orders $<_1$ and $<_2$ that agree. Also suppose that $A \cap B$ is a primitive subgroup of G. Then there exists a bi-order $<$ on $\langle A, B \rangle$ that agrees with both $<_1$ and $<_2$. If $\langle A, B \rangle$ is a primitive subgroup of G then $<$ can be extended to G.*

Proof. Applying Proposition 2.2.2 $A \cap B$ is a primitive subgroup of both A and B, so there exist subgroups $A_1 \leq A$ and $B_1 \leq B$ such that $A \cong A_1 * (A \cap B)$ and $B \cong (A \cap B) * B_1$. Then $A_1 \cap B$ and $A \cap B_1$ are trivial, and $\langle A, B \rangle \cong A_1 * (A \cap B) * B_1$. We apply Vinogradov's Theorem to construct a bi-order $<$ on the free product. Since $<_1$ and $<_2$ agree, we can use them in the construction for the free factors A_1, $A \cap B$, and B_1. Then the restrictions of $<$ to A and B are $<_1$ and $<_2$ respectively. Since $\langle A, B \rangle \cap A = A$ and $\langle A, B \rangle \cap B = B$, by definition $<$ agrees with both $<_1$ and $<_2$.

If $\langle A, B \rangle$ is a primitive subgroup of G there exists $G_1 \leq G$ such that $G \cong G_1 * \langle A, B \rangle$. Place an arbitrary bi-order on the free group G_1 and then apply Vinogradov's Theorem again to extend $<$ to G.

□

The key observation in proving the deleting theorem is that we can write an HNN extension as a semidirect product. Given an HNN extension $G*_\alpha$ with stable letter t, if we let F be the normal subgroup that consists of all elements of $G*_\alpha$ whose words are such that the sum of the powers of t in each the word is zero, or in other terminology $F := \langle \bigcup_{n \in \mathbb{Z}} t^n G t^{-n} \rangle$, then $G*_\alpha$ is the semidirect product $\mathbb{Z} \ltimes F$. If the associated subgroups A and B are primitive and come from removing relations in $\mathbb{Z} \ltimes G$ as described at the beginning of this section, we show that F is a free group and carefully find a basis of F in the second lemma. In the third and final lemma we argue that there is a bi-order on F invariant under t-conjugation if $\mathbb{Z} \ltimes G$ is bi-orderable.

Lemma 2.2.5. *Let G be a free group with primitive subgroups A and B, and let $\alpha_\circ \in \mathrm{Aut}(G)$ such that $\alpha_\circ(A) = B$. Let $\alpha = \alpha_\circ|_A$ and t be the stable letter of the HNN extension $G*_\alpha$. Then the normal subgroup $F := \langle \bigcup_{n \in \mathbb{Z}} t^n G t^{-n} \rangle$ is a free subgroup of $G*_\alpha$ and G is a primitive subgroup of F; furthermore, there exist bases X and Z of the free groups G and F respectively with $X \subseteq Z \subseteq W := (\bigcup_{n \geq 0} t^n X t^{-n}) \cup (\bigcup_{n < 0} t^n \alpha_\circ(X) t^{-n})$ such that for all $x \in X$*

- *if $t^n x t^{-n} \in Z$ for some $n > 0$ then $t^m x t^{-m} \in Z$ for all $m > 0$, and*
- *if $t^n \alpha_\circ(x) t^{-n} \in Z$ for some $n < 0$ then $t^m \alpha_\circ(x) t^{-m} \in Z$ for all $m < 0$.*

Proof. Since A is a primitive subgroup of G there exists a basis X of G such that $A = F(Y)$ for some $Y \subseteq X$. Let $Y' = X \backslash Y$, $V = \alpha(Y)$, and $V' = \alpha_\circ(Y')$. Then V is a basis of B, $V \cap V'$ is empty, and $S := V \cup V'$ is a basis of G. The set W generates F. If A is trivial then by Britton's Lemma F is a free group with basis W. If A is not trivial then $t^{n+1} y t^{-n-1} \in \langle t^n X t^{-n} \rangle$ for all $y \in Y$ and $n > 0$; we also have similar relations for $v \in V$ and negative n. Britton's Lemma implies that $Z := (\bigcup\limits_{n>0} t^n Y' t^{-n}) \cup X \cup (\bigcup\limits_{n<0} t^n V' t^{-n}) \subseteq W$ generates a free subgroup of F on itself, or in other words $\langle Z \rangle$ is a free subgroup of F with basis Z; furthermore, Z satisfies the two bulleted conditions above. We show $\langle Z \rangle = F$ to finish the proof. We have that $t^0 G t^{-0} \cup t^{-0} G t^0 = G = \langle X \rangle \subseteq \langle Z \rangle$. Assume for some $k \geq 1$ that $t^{k-1} G t^{-k+1} \cup t^{-k+1} G t^{k-1} \subseteq \langle Z \rangle$. Then,

$$\begin{aligned}
t^k X t^{-k} \cup t^{-k} S t^k &= \left(t^k Y t^{-k} \cup t^k Y' t^{-k}\right) \cup \left(t^{-k} V t^k \cup t^{-k} V' t^k\right) \\
&= \left(t^{k-1} V t^{-k+1} \cup t^k Y' t^{-k}\right) \cup \left(t^{-k+1} Y t^{k-1} \cup t^{-k} V' t^k\right) \\
&= t^{k-1} V t^{-k+1} \cup t^{-k+1} Y t^{k-1} \cup t^k Y' t^{-k} \cup t^{-k} V' t^k \\
&\subseteq t^{k-1} G t^{-k+1} \cup t^{-k+1} G t^{k-1} \cup t^k Y' t^{-k} \cup t^{-k} V' t^k \subseteq \langle Z \rangle.
\end{aligned}$$

Since $\langle Z \rangle$ contains generating sets for $t^k G t^{-k}$ and $t^{-k} G t^k$ we have $t^k G t^{-k} \cup t^{-k} G t^k \subseteq \langle Z \rangle$, so that by induction $F = \langle \bigcup\limits_{n \in \mathbb{Z}} t^n G t^{-n} \rangle \subseteq \langle Z \rangle$. Finally, since X is a subset of a basis of F, G is a primitive subgroup of F.

□

Remark 2.2.6. F as defined in Lemma 2.2.5 fails to be a free group for arbitrary HNN extensions of a free group G; more generally, F may fail to be bi-orderable group. Consider $G = \langle a \rangle$ with subgroups $A = B = \langle a^2 \rangle$. Let $\alpha \in \mathrm{Aut}(A)$ be the identity. Note that A and B fail to be root closed. By Britton's Lemma $tat^{-1} \neq a$. However $(tat^{-1})^2 = ta^2t^{-1} = a^2$. The element a^2 does not have a unique square root, so F cannot be bi-orderable; hence F is not free.

Lemma 2.2.7. *Let G be a free group with primitive subgroups A and B, and let $\alpha_\circ \in \mathrm{Aut}(G)$ such that $\alpha_\circ(A) = B$. Let $\alpha = \alpha_\circ|_A$ and t be the stable letter of the HNN extension $G*_\alpha$. If $\mathbb{Z} \ltimes_{\alpha_\circ} G$*

is bi-orderable then there exists a bi-order on the free subgroup $F := \langle \bigcup_{n\in\mathbb{Z}} t^n G t^{-n} \rangle \leq G*_\alpha$ *that is preserved under conjugation by* t.

Proof. Let Y, V, X, and Z be bases of A, B, G, and F respectively as in Lemma 2.2.5 and its proof, so that $Z = (\bigcup_{n>0} t^n Y' t^{-n}) \cup X \cup (\bigcup_{n<0} t^n V' t^{-n})$. Abusing notation we will also use α on the whole group F to denote conjugation by t. Since $\mathbb{Z} \ltimes_{\alpha_\circ} G$ is bi-orderable there exists a bi-order $<$ on G such that $<$ is preserved by $\alpha_\circ$. Then for all $a \in A$ such that $1 < a$, we have that $1 < \alpha(a)$, so if $g \in G$ and $1 < g$ such that $\alpha(g) = tgt^{-1} \in G$, then $1 < \alpha(g)$. For a subset $U \subseteq Z$ and $C := \langle U \rangle \leq F$ with bi-order $\prec$ we refer to the following properties as P, P', Q, and R respectively:

- If $c \in C$ and $1 \prec c$ such that $\alpha(c) = tct^{-1} \in C$, then $1 \prec \alpha(c)$.
- If $c \in C$ and $1 \prec c$ such that $\alpha^{-1}(c) = t^{-1}ct \in C$, then $1 \prec \alpha^{-1}(c)$.
- If $u \in Y'$ such that $\alpha^n(u) \notin U$ for some $n > 0$, then $\alpha^m(u) \notin U$ for all $m \geq n$.
- If $u \in V'$ such that $\alpha^n(u) \notin U$ for some $n < 0$, then $\alpha^m(u) \notin U$ for all $m \leq n$.

Properties P and P' are equivalent. Suppose $C \leq F$ with bi-order $\prec$ has property P. Also suppose $c, \alpha^{-1}(c) \in C$ with $1 \prec c$. Assume for contradiction that $\alpha^{-1}(c) \prec 1$. Then $1 \prec \alpha^{-1}(c^{-1}) \in C$ and $\alpha(\alpha^{-1}(c^{-1})) = c^{-1} \in C$, so apply property P to obtain $1 \prec c^{-1}$. Contradiction. The argument for P' implies P is analogous. Z and X have properties Q and R, and the group $G = \langle X \rangle$ has property P (and therefore P' as well) with the bi-order $<$. We claim that there exists a maximal set X' with $X \subseteq X' \subseteq Z$ satisfying the following:

- The bi-order $<$ can be extended to the primitive subgroup $G' = \langle X' \rangle$ of F.
- G' has property P with the extended bi-order $<$.
- X' has properties Q and R.

The proof of the claim is a usual Zorn's Lemma argument. Let $\{X_i\}_{i\in I}$ be a chain such that $X \subseteq X_i \subseteq Z$ for each $i \in I$. For convenience put a total order on I such that for all $i, j \in I$, $i < j \Leftrightarrow X_i \subsetneq X_j$. Suppose for each $i \in I$, X_i satisfies the conditions listed above for X' with bi-order $<_i$ on $G_i = \langle X_i \rangle$ such that for all $j < i$, $<_i$ is an extension of $<_j$. Let $X_\circ = \bigcup_{i\in I} X_i$ and $G_\circ = \langle X_\circ \rangle$. Then $X \subseteq X_\circ \subseteq Z$ and $G_\circ$ is primitive. If $g \in G_\circ$ then g is a word in some finitely many letters that belong to $\bigcup_{j=1}^{k} X_{i_j}$ for some $i_j \in I$ with $1 \leq j \leq k$. If i is the maximum of $\{i_1, i_2, \ldots, i_k\}$ then $g \in G_i$. Define an order $<_\circ$ on $G_\circ$ by $1 <_\circ g \Leftrightarrow 1 <_i g$. This is well-defined since for distinct $<_i$ and $<_j$, one of the orders is an extension of the other. For $g, \alpha(g), h, k \in G_\circ$ with g and h positive, we can again choose a maximum index like above so that $g, \alpha(g), h, k \in G_i$ for some $i \in I$. Then from the order $<_i$ we have $1 <_\circ gh$, $1 <_\circ kgk^{-1}$, and $1 <_\circ \alpha(g)$. Then $<_\circ$ is bi-order on $G_\circ$ with property P and is an extension of $<_i$ for all $i \in I$. Since each $<_i$ is an extension of $<$, $<_\circ$ is an extension of $<$. Suppose for some $n > 0$ and $u \in Y'$ that $\alpha^n(u) \notin X_\circ$. Assume for contradiction that there exists $m > n$ such that $\alpha^m(u) \in X_\circ$. There exists $i \in I$ such that $\alpha^m(u) \in X_i$. Since $X_i \subseteq X_\circ$, $\alpha^n(u) \notin X_i$. This contradicts property Q for X_i. Thus $X_\circ$ has property Q, and checking property R is analogous. Since $X_\circ$ is an upper bound of the chain satisfying all of the conditions, a maximal subset X' exists. We also use $<$ for the extended bi-order on the primitive subgroup $G' = \langle X' \rangle$.

If $X' = Z$ then $G' = F$, and we are done. Assume not. Since $X \subseteq X'$ at least one of the following holds: There exists $x_\circ \in (Z \backslash X') \cap (\bigcup_{n>0} \alpha^n(Y'))$ such that $\alpha^{-1}(x_\circ) \in G'$, or there exists $x_\circ \in (Z \backslash X') \cap (\bigcup_{n<0} \alpha^n(V'))$ such that $\alpha(x_\circ) \in G'$. Since G' has both properties P and P', and since X' has both properties Q and R, without loss of generality we can assume the latter else exchange α with α^{-1} for the remainder of the argument. Let $X'' = X' \cup \{x_\circ\} \supsetneq X'$. Then X'' has properties Q and R, and $G'' := \langle X'' \rangle \cong G' * \langle x_\circ \rangle$ is a primitive subgroup of F.

We construct a bi-order on G'' with property P that is an extension of $<$ to contradict the maximality of X' using primitive subgroups of G'' and applying the above lemma on agreeing bi-orders. Let $Y_1 = \{x \in X' : \alpha(x) \in G'\}$ and let $A' = \langle Y_1 \rangle$, which is primitive since $Y_1 \subseteq X' \subseteq Z$. Let $A'' = \langle Y_1, x_\circ \rangle \cong A' * \langle x_\circ \rangle$ which is a primitive subgroup; furthermore, $\alpha(A'')$ is a primitive subgroup of G' since $\langle \alpha(Y_1), \alpha(x_\circ) \rangle \leq G'$. Construct an order $<'$ on A'' using the order on $\alpha(A'') \leq G'$. For $a \in A''$ define $1 <' a \Leftrightarrow 1 < \alpha(a)$. Since $<$ is a bi-order and α is an isomorphism, $<'$ is a bi-order on A''. Now we have that $A'', G' \leq G''$ are primitive, $A'' \cap G' = A'$ is primitive, and $\langle A'', G' \rangle = G''$. Also for all $a \in A'$, $1 <' a \Leftrightarrow 1 < \alpha(a) \Leftrightarrow 1 < a$, so that $<'$ and $<$ agree. The last double implication follows since $a, \alpha(a) \in G'$ and G' has both properties P and P'. Apply Lemma 2.2.4: We have a bi-order $<''$ on G'' that agrees with both $<'$ and $<$, so that $<''$ is an extension of both $<'$ and $<$.

Finally, we check property P for G'' with $<''$. Suppose $g, \alpha(g) \in G''$ and $1 <'' g$. Then $\alpha(g) \in \alpha\big(\langle G', x_\circ\rangle\big) = \langle \alpha(G'), \alpha(x_\circ)\rangle$ and $\alpha(g) \in \langle G', x_\circ\rangle$. Since X' has property R, $\alpha^{-1}(x_\circ) \in Z\backslash X'$. This implies $x_\circ \notin \alpha(G')$, so that by Britton's Lemma $x_\circ \notin \langle \alpha(G'), \alpha(x_\circ)\rangle$ since $x_\circ$ is a basis element. Then $\alpha(g) \in G'$, implying $g \in \alpha^{-1}(G')$. Since $g \in G''$, write g as a reduced word $w = x_\circ^{n_1} g_1 x_\circ^{n_2} g_2 \cdots x_\circ^{n_k} g_k \in \alpha^{-1}(G')$ where $g_i \in G'$ for each $i \in \{1, \ldots, k\}$, n_1 is possibly 0, and g_k is possibly 1. Assume for contradiction that there exists $i_\circ \in \{1, \ldots, k\}$ such that $g_{i_\circ} \notin \alpha^{-1}(G')$. Since $g_{i_\circ} \in G'$ we can write $g_{i_\circ}$ as a reduced word $x_1^{m_1} x_2^{m_2} \ldots x_l^{m_l}$ where $x_j \in X'$ for all $j \in \{1, \ldots, l\}$. Then there exists $j_\circ \in \{1, \ldots, l\}$ such that $x_{j_\circ} \notin \alpha^{-1}(G')$. We can similarly write the other g_i as words in X', so that w above is a reduced word in $X'' \subseteq Z$ containing the basis element $x_{j_\circ} \notin \alpha^{-1}(G')$. Since $\alpha^{-1}(G') = \langle \alpha^{-1}(X')\rangle$ is a primitive subgroup, Britton's Lemma contradicts g being an element of $\alpha^{-1}(G')$, so $g_i \in \alpha^{-1}(G')$ for all $i \in \{1, \ldots, k\}$. Now,

$$g_i, \alpha(g_i) \in G' \ \forall i \in \{1, \ldots, k\} \ \Rightarrow \ g_i \in A' \ \forall i \in \{1, \ldots, k\} \ \Rightarrow \ g \in A''.$$

Since $<''$ is an extension of $<'$, $1 <' g$. Then $1 < \alpha(g)$ and since $<''$ is also an extension of $<$, $1 <'' \alpha(g)$. Thus G'' has property P with the bi-order $<''$. This contradicts the maximality of X', so $X' = Z$ and the bi-order $<$ extends to F such that is it preserved by α.

□

Theorem 2.2.8. *Let G be a free group with primitive subgroups A and B, and let $\alpha_\circ \in \mathrm{Aut}(G)$ such that $\alpha_\circ(A) = B$. Define $\alpha = \alpha_\circ|_A$. If $\mathbb{Z} \ltimes_{\alpha_\circ} G$ is bi-orderable then the HNN extension $G*_\alpha$ is bi-orderable.*

Proof. Let t be the stable letter of the HNN extension. Also use α to denote conjugation by t in F, where F is defined as in Lemma 2.2.5. Then $\alpha \in \mathrm{Aut}(F)$ and by Lemma 2.2.7 there exists a bi-order on F that is preserved by α. Thus the semidirect product $\mathbb{Z} \ltimes_\alpha F \cong G*_\alpha$ is bi-orderable.

□

2.3. The Gluing Theorem

Assume the hypothesis of the deleting theorem. If there is sufficient room in the base group, or more precisely if the algebraic closure of $\langle A, B\rangle$ is properly contained in G, we wish to carefully select (not necessarily primitive) elements $x, y \in G\backslash\langle A, B\rangle$ and add the relation $txt^{-1} = y$ so that the resulting new HNN extension is still bi-orderable; furthermore, we add more such relations as long as there is still sufficient room. Before the gluing theorem, we first prove a lesser result.

Lemma 2.3.1. *Let G be a free group with primitive subgroups A and B, and let $\alpha_\circ \in \mathrm{Aut}(G)$ such that $\alpha_\circ(A) = B$. Define $\alpha = \alpha_\circ|_A$ and let C be the algebraic closure of $\langle A, B\rangle$ in G. Suppose for some indexing set I, $S_1 := \{x_i\}_{i\in I}$ and $S_2 := \{y_i\}_{i\in I}$ are subsets of G such that $D := \langle C, S_1, S_2\rangle \cong C * (*_{i\in I}\langle x_i\rangle) * (*_{i\in I}\langle y_i\rangle)$ is an algebraically closed subgroup of G. Let $A' = \langle A, S_1\rangle$, $B' = \langle B, S_2\rangle$, and the isomorphism $\gamma : A' \to B'$ be the extension of α such that $\gamma(x_i) = y_i$ for all $i \in I$. If $\mathbb{Z} \ltimes_{\alpha_\circ} G$ is bi-orderable then the HNN extension $D*_\gamma$ is bi-orderable.*

Proof. Let t be the stable letter in HNN extension. We also use γ on the whole group $D*_\gamma$ to denote conjugation by t. Let $x_{i,n} = t^n x_i t^{-n}$ for all $n \in \mathbb{Z}$, so $x_{i,0} = x_i$ and $x_{i,1} = y_i$ for all $i \in I$. Let $H_i = \langle x_{i,n} : n \in \mathbb{Z}\rangle$, which by Britton's Lemma is the free group on the set $\{x_{i,n} : n \in \mathbb{Z}\}$ for each $i \in I$, since C is algebraically closed. Since C and D are algebraically closed subgroups of G, and since D splits as a free product as in the hypothesis, again by Britton's Lemma we have that $L := \langle \bigcup_{i\in I} H_i, C\rangle \cong (*_{i\in I}H_i) * C$, and this free product structure is preserved under t-conjugation.

By Theorem 2.2.8 $G*_\alpha$ is bi-orderable, so that the free subgroup $F_\circ := \langle \bigcup_{n\in\mathbb{Z}} t^n C t^{-n}\rangle \leq C*_\alpha \leq G*_\alpha$ has a bi-order $<_\circ$ that is invariant under conjugation by t. By Proposition 2.1.3 each H_i admits a bi-order $<_i$ which is preserved under conjugation by t. Then the bi-order constructed from $<_\circ$ and each $<_i$ on the free product $(*_{i\in I}H_i) * F_\circ$ as in the proof of Vinogradov's Theorem is invariant under conjugation by t. Now,

$$F := \left\langle \bigcup_{n\in\mathbb{Z}} t^n L t^{-n} \right\rangle \cong \left\langle \bigcup_{n\in\mathbb{Z}} t^n((*_{i\in I}H_i) * C)t^{-n} \right\rangle = \left\langle \bigcup_{n\in\mathbb{Z}} ((*_{i\in I}H_i) * t^n C t^{-n}) \right\rangle = (*_{i\in I}H_i) * F_\circ.$$

Then $\gamma \in \mathrm{Aut}(F)$ and F has a bi-order which is invariant under γ. Hence the semidirect product $\mathbb{Z} \ltimes_\gamma F \cong D*_\gamma$ is bi-orderable.

□

Theorem 2.3.2. *Let G be a free group with primitive subgroups A and B, and let $\alpha_\circ \in \mathrm{Aut}(G)$ such that $\alpha_\circ(A) = B$. Define $\alpha = \alpha_\circ|_A$ and let C be the algebraic closure of $\langle A, B\rangle$. Suppose for some indexing set I, $S_1 := \{x_i\}_{i\in I}$ and $S_2 := \{y_i\}_{i\in I}$ are subsets of G such that $D := \langle C, S_1, S_2\rangle \cong C*(*_{i\in I}\langle x_i\rangle)*(*_{i\in I}\langle y_i\rangle)$ is algebraically closed. Let $A' = \langle A, S_1\rangle$, $B' = \langle B, S_2\rangle$, and the isomorphism*

$\gamma : A' \to B'$ be the extension of α such that $\gamma(x_i) = y_i$ for all $i \in I$. If $\mathbb{Z} \ltimes_{\alpha_\circ} G$ is bi-orderable then the HNN extension $G*_\gamma$ is bi-orderable.

Proof. Let t be the stable letter in HNN extension. We also use γ on the whole group $G*_\gamma$ to denote conjugation by t. By Lemma 2.3.1 the subgroup $D*_\gamma$ of $G*_\gamma$ has some bi-order $<_\circ$. We show that this bi-order can be extended to $G*_\gamma$. Again using a Zorn's Lemma argument we show that there exists a maximal subgroup D' such that the following hold:

- $D \leq D' \leq G$
- The bi-order $<_\circ$ on $D*_\gamma$ can be extended to $D'*_\gamma$.
- D' is an algebraically closed subgroup of G.

Let $\{D_i\}_{i\in I}$ be a chain of subgroups with a respective chain of bi-orders $\{<_i\}_{i\in I}$ of $\{D_i*_\gamma\}_{i\in I}$ that satisfies the above bulleted conditions, indexed by a set I with a total order that respects the chain. Also suppose that $<_j$ is an extension of $<_i$ for all $i < j$. Let $D_\circ = \bigcup_{i\in I} D_i$. Then $D \leq D_\circ \leq G$. We define a bi-order on $D_\circ*_\gamma$ analogous to the bi-order defined in the Zorn's Lemma argument in the proof of Lemma 2.2.7. Then this bi-order is an extension of each $<_i$, and hence is an extension of $<_\circ$. We show that $D_\circ$ is an algebraically closed subgroup of G. Assume not. Then there exist $u \in G\backslash D_\circ$, $\{w_1, \ldots, w_k\} \subseteq G\backslash\{1\}$, and $\{n_1, \ldots, n_k\} \subseteq \mathbb{Z}\backslash\{0\}$ such that $w := u^{n_1}w_1 \cdots u^{n_k}w_k \in D_\circ$. Then w is an element of D_i for some $i \in I$. Since $D_i \leq D_\circ$, $u \in G\backslash D_i$. This contradicts D_i being an algebraically closed subgroup of G. Hence $D_\circ$ is an upper bound to the chain that satisfies the above bulleted conditions, so a maximal subgroup D' exists. We also use $<_\circ$ for the extended bi-order on $D'*_\gamma$. If $D' = G$ we are done. Assume not. Let $z \in G\backslash D'$ be such that $D'' := \langle D', z\rangle \cong D' * \langle z\rangle$ is an algebraically closed subgroup of G.

Now we proceed similarly to the proof of Lemma 2.3.1. Let $z_n = t^n z t^{-n}$ for all $n \in \mathbb{Z}$, and let $H = \langle z_n : n \in \mathbb{Z}\rangle$, which by Britton's Lemma is the free group on the set $\{z_n : n \in \mathbb{Z}\}$ since D' is algebraically closed. Then $L := \langle H, D'\rangle \cong H * D'$, and the free product structure is preserved by t-conjugation, again by Britton's Lemma since D' and D'' are algebraically closed subgroups of G and $D'' \cong D' * \langle z\rangle$. Since $<_\circ$ is a bi-order on $D'*_\gamma$, we have that $<_\circ$ is a bi-order on the subgroup $F_\circ := \langle \bigcup_{n\in\mathbb{Z}} t^n D' t^{-n}\rangle \leq D'*_\gamma$ that is preserved under conjugation by t. By Proposition 2.1.3 H admits a bi-order $<_1$ which is invariant under conjugation by t. From the proof of Vinogradov's

Theorem we have a bi-order on $H * F_\circ$ that is invariant by t-conjugation and is an extension of the bi-orders of both factors. Then,

$$F \;:=\; \Big\langle \bigcup_{n\in\mathbb{Z}} t^n L t^{-n} \Big\rangle \;\cong\; \Big\langle \bigcup_{n\in\mathbb{Z}} t^n (H * D') t^{-n} \Big\rangle \;=\; \Big\langle \bigcup_{n\in\mathbb{Z}} (H * t^n D' t^{-n}) \Big\rangle \;=\; H * F_\circ.$$

Then $\gamma \in \mathrm{Aut}(F)$ and F has a bi-order which is invariant under γ, so the semidirect product $\mathbb{Z} \ltimes_\gamma F \cong D'' *_\gamma$ has a bi-order that is an extension of $<_\circ$. This contradicts the maximality of D'. Hence $D' = G$.

$\square$

3. THE WINDING OPERATION

In this chapter we investigate the HNN extension group presentation for a knot K that admits a free incompressible Seifert surface and look to apply Theorems 2.2.8 and 2.3.2. A nice special case of such a knot K is an alternating knot. Recall that applying Seifert's algorithm to a reduced alternating diagram of an alternating knot yields a minimal-genus Seifert surface [21], so alternating knots admit a Seifert surface that is both free and incompressible. Incompressibility is required to apply the HNN extension Seifert surface gluing technique. Freeness guarantees the base group of the HNN extension $G*_\alpha$ is a free group. Finally, recall that for a fibered knot the HNN extension is simply the semidirect product $\mathbb{Z} \ltimes_\alpha G$. Theorems 2.2.8 and 2.3.2 require starting with such a semidirect product that is bi-orderable, and G must be free. If K is not fibered we try to relate K to a fibered knot $K_\circ$, such that HNN extension group presentations of K and $K_\circ$ are comparable. $K_\circ$ will be called a *fibered base* of K. Such a fibered base may not exist.

We will consistently use G to denote the free base group in the HNN extension, and $\alpha : A \to B$ to denote the isomorphism of the subgroups $A, B \leq G$ that correspond to the positive and negative parallel copies of the Seifert surface respectively. Due to the symmetry of the positive and negative copies of the Seifert surface in the Seifert surface gluing technique, A and B are not only isomorphic, but they lie in G analogously in the following sense: We have that $|G_1 : A_1| = |G_1 : B_1|$, where G_1 is the Abelianization of G, and A_1 and B_1 are the respective images of A and B in G_1.

Note: In the following two examples, and frequently in the rest of this chapter, the letter α is used to denote two distinct objects. α is used both for the map $\alpha : A \to B$ in the HNN extension $G*_\alpha$ and sometimes as a generator of $\pi_1(\Sigma)$. The use should be clear from context.

3.1. Some Examples

Example 3.1.1. Recall the Seifert surface Σ_{3_1} of the trefoil in the first chapter. The knot 5_2 is the first non-fibered knot. We can obtain a Seifert surface of 5_2 by adding a full-twist to any one of the half-twist bands in Σ_{3_1}, respecting the half-twist band's orientation. Up to homotopy the surface and its complement remain the same when adding the full-twist; furthermore, the boundary of the resulting surface is alternating, hence reduced, so that the resulting surface is free incompressible.

Abusing notation, we can use the same generators when computing HNN extension presentations for both groups.

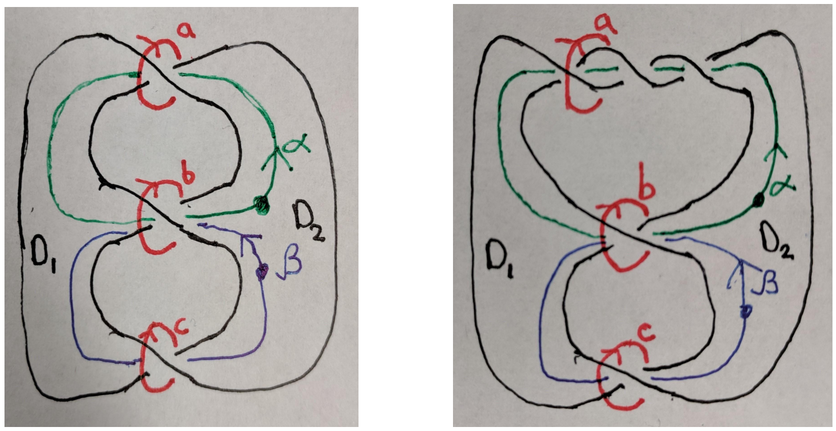

Figure 3.1. Seifert surfaces of 3_1 and 5_1 with labeled generators.

In other words the groups G and $\pi_1(\Sigma)$ are the same for both knots; however, lifting both the positive and negative copies of α off Σ_{5_2} results in an additional loop around the band a in the complement. For comparison here are the resulting presentations:

$$\pi_1(\mathbb{S}^3 \backslash 3_1) = \langle\, t, a, b \mid tat^{-1} = b^{-1},\ tbt^{-1} = ba \,\rangle$$

$$\pi_1(\mathbb{S}^3 \backslash 5_2) = \langle\, t, a, b \mid ta^2t^{-1} = ab^{-1},\ tbt^{-1} = ba \,\rangle$$

For 5_2, $A = \langle a^2, b\rangle$ and $|G_1 : A_1| = 2$. Again, G_1 is the Abelianization of G and A_1 is the image of A in G_1. The triple-half-twist band in the Seifert surface of 5_2 is precisely what causes 5_2 to be not fibered and for the HNN extension to be proper. In general full-twisting any one band in the trefoil n times results in a twist knot K whose group has the following presentation below. The first five knots in this form (i.e. $n = 0, 1, 2, 3, 4$) are $3_1, 5_2, 7_2, 9_2$, and $11a_{247}$. Since they are twist knots with an odd number of crossings, all of these knots have non-bi-orderable knot groups [5].

$$\pi_1(\mathbb{S}^3 \backslash K) = \langle\, t, a, b \mid ta^{n+1}t^{-1} = a^n b^{-1},\ tbt^{-1} = ba \,\rangle.$$

Example 3.1.2. Now consider the figure-eight knot. A free incompressible Seifert surface Σ_{4_1} of the figure eight knot consists of disks D, S_1, and S_2. There are two half-twist bands between D and S_i for $i \in \{1, 2\}$.

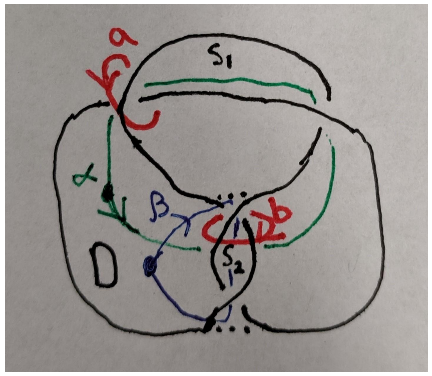

Figure 3.2. A Seifert surface of 4_1 with labeled generators.

For one side $\alpha \mapsto ba$ and $\beta \mapsto b$. For the other $\alpha \mapsto a$ and $\beta \mapsto a^{-1}b$, so we obtain the relations $tat^{-1} = ba$ and $ta^{-1}bt^{-1} = b \Rightarrow tbt^{-1} = bab$, resulting in the following presentation.

$$\pi_1(\mathbb{S}^3 \backslash 4_1) \;=\; \langle\, t, a, b \mid tat^{-1} = ba,\; tbt^{-1} = bab \,\rangle$$

The non-fibered twist knots 6_1 and 8_1 can be obtained by respectively adding a full-twist and a double-full-twist to any one of the half-twist bands in Σ_{4_1}. Recall from the introduction that these knot groups are bi-orderable. Below are presentations for the knot groups of 6_1 and 8_1 if we add the full-twist(s) to the band a. We have $|G_1 : A_1| = 2$ and $|G_1 : A_1| = 3$ for 6_1 and 8_1 respectively.

$$\pi_1(\mathbb{S}^3 \backslash 6_1) \;=\; \langle\, t, a, b \mid ta^2t^{-1} = ba^2,\; tbt^{-1} = bab \,\rangle$$

$$\pi_1(\mathbb{S}^3 \backslash 8_1) \;=\; \langle\, t, a, b \mid ta^3t^{-1} = ba^3,\; tbt^{-1} = bab \,\rangle$$

Note that Σ_{4_1} is not flat. The significance of this is that we obtain the letter b in one of the words from α even though α does not pass through the disk S_2 on the surface. In other words when lifting α off of one side of the surface, α is trapped around the band b. This also occurs with β and obtaining an a in one of the words. There is no choice of representatives on the surface that avoids this.

3.2. Winding, Unwinding, and the HNN Extension Presentation

We formalize adding and removing half-twists to a band. Let Σ_K be a Seifert surface of a knot K obtained by applying Seifert's algorithm to a reduced diagram of K. Recall that applying this algorithm results in disks, called Seifert disks, and attaching half-twist bands between these disks. These half-twist bands correspond to the crossings in the knot diagram. Two Seifert disks are said to be *adjacent* if they are connected by a half-twist band. We define a Seifert disk S to be *simple* if there are precisely two half-twist bands attached to S. For $1 \leq i \leq n-1$ suppose in Σ_K there exist Seifert disks D_1 and D_2, and distinct simple Seifert disks $S_i \notin \{D_1, D_2\}$ such that D_1 and S_1 are adjacent, S_i and S_{i+1} are adjacent for all $1 \leq i \leq n-2$, and S_{n-1} and D_2 are adjacent. We define $(S_1, \ldots, S_{n-1})$ to be an *n-half-twist band* between D_1 and D_2 in Σ_K, and we define D_1 and D_2 to be the *ends* of the n-half-twist band.

Since the knot diagram of K is reduced, every individual half-twist band in the n-half-twist band has the same orientation. Finally, if two distinct Seifert disks D_1 and D_2 are the ends of an n-half-twist band, we define D_1 and D_2 to be *pseudo-adjacent*. If $n = 1$, we have a 1-half-twist band between D_1 and D_2, or more simply a half-twist band in the usual sense, so D_1 and D_2 are adjacent. If $n = 2$, we have a single simple Seifert disk S between two Seifert disks D_1 and D_2. In the interest of performing the operation of removing a double-half-twist from an n-half-twist band in a Seifert surface, such as obtaining Σ_{3_1} from Σ_{5_2} above, we need to treat the case $n = 2$ with care. We have two cases: D_1 and D_2 are either distinct or they are the same disk. If $D_1 = D_2$, removing the double-half-twist from the band results in an untwisted band from D_1 to itself. The resulting surface is not incompressible. In this case, we define the double-half-twist band to be *essential*. The two double-half-twist bands in Σ_{4_1} above are essential.

If D_1 and D_2 are pseudo-adjacent, i.e. distinct, removing the double-half-twist band results in an untwisted band between D_1 and D_2. This is isotopic to merging the two Seifert disks into one disk. We say the double-half-twist band in this case is *non-essential*. Removing such a non-essential band in a prime knot may result in a composite knot, even if the knot is alternating. An example of a knot where this occurs is the non-fibered alternating knot 8_{15}. There is a Seifert surface $\Sigma_{8_{15}}$ of 8_{15} obtained from Seifert's algorithm such that replacing a non-essential double-half-twist band in $\Sigma_{8_{15}}$ with an untwisted band results in a Seifert surface whose boundary is the composite knot $3_1 \# 3_1^*$, the granny knot.

Note that the above is definable for a general Seifert surface Σ; it is not necessary that Σ comes from applying Seifert's algorithm.

Definition 3.2.1. Suppose Σ_K is Seifert surface of a knot K. For $n \geq 1$ we define replacing an n-half-twist band in Σ_K with a same-oriented $(n+2)$-half-twist band to be *winding* the band. For $n \geq 3$, or $n = 2$ and the band is non-essential, we define replacing an n-half-twist band with a same-oriented $(n-2)$-half-twist band to be *unwinding* the band.

Remark 3.2.2. Winding a band in Σ_K preserves both freeness and incompressibility, and we also have the following:

- If Σ_K is free incompressible and $\Sigma_{K'}$ is obtained from winding a band of Σ_K, then the HNN extension presentation Seifert surface gluing technique applies to both knots, and when computing the presentations for the knot groups of K and K', abusing notation we can choose the same generators for $\pi_1(\mathbb{S}^3\backslash\Sigma_K)$ and $\pi_1(\mathbb{S}^3\backslash\Sigma_{K'})$, and we can choose the same generators for $\pi_1(\Sigma_K)$ and $\pi_1(\Sigma_{K'})$.

- If the boundary of Σ_K is a (reduced) alternating knot K then winding is defined for $n = 0$, i.e. an untwisted band, since only one orientation choice keeps the knot alternating. Unwinding a band of Σ_K results in a new surface $\Sigma_{K_\circ}$ whose boundary is a reduced diagram of an alternating knot $K_\circ$, so $\Sigma_{K_\circ}$ is also free and incompressible, and $\mathrm{cr}(K_\circ) = \mathrm{cr}(K) - 2$.

Definition 3.2.3. If K' can be obtained from winding bands in a Seifert surface of a fibered knot K, K is called a *fibered base* of K'.

The crossing number statement above is false for unwinding a band in general. For example, using Seifert's algorithm on a reduced diagram of 9_{46} results in a Seifert surface with a triple-half-twist band. Once unwinding this band results in a diagram with seven crossings; however, this diagram is not reduced and is actually the knot 6_1.

Before investigating how winding and unwinding affect the knot group presentation, we carefully choose the generators for G and make some observations about the words obtained when lifting loops off of the surface. Let Σ be a free incompressible Seifert surface. First identity each distinct maximal band in Σ; suppose there are k of them. By a maximal band M, we mean M is an n-half-twist band where $n \neq 0$ and M is not part of an m-half-twist band where $m > n$. An exception to this is that in the context of unwinding a non-essential double-half-twist band (that is also maximal) and comparing fundamental groups, we will allow the resulting untwisted band to be maximal. Fix a base point p in $\mathbb{S}^3\backslash\Sigma$. For each maximal band $M_i \in \{M_1, \ldots, M_k\}$ choose a point $p_i \in \mathbb{S}^3\backslash\Sigma$ near M_i. Let γ_i be a path from p to p_i and let δ_i be a single loop around M_i starting and ending with p_i. Finally let $a_i = [\gamma_i\delta_i\gamma_i^{-1}] \in G$. Then each a_i is not the identity element and $G = \langle a_1, \ldots, a_k\rangle$. This follows since the complement of a free Seifert surface is homotopy equivalent to an bouquet of circles. We call a_i a simple loop. If we change the path γ_i, and obtain a new simple loop a_i', we have that $a_i = ga_i'g^{-1}$ for some $g \in G$, so that $S = \{a_1, \ldots, a_k\}$ is unique up to conjugation and sign of its elements. S is not a minimal generating set of G in general. For example Σ_{3_1} and Σ_{5_2} above have three maximal bands, and a and b are simple loops that generate G. When we specify generators of the base group G, we assume the generators are simple loops unless otherwise stated. For convenience and brevity we will also simply use a_i to refer to the maximal band M_i.

For a given generator $\alpha \in \pi_1(\Sigma)$ how does the word look in G with letters in S? We can assume that there is a representative r of α on Σ such that r passes through any given band at most once. Let us first assume that Σ is flat, and that r takes a *short path*; assume the path r injects into Σ, then since Σ is flat the loop r partitions the plane into an inner open disk and the complement of the closed disk. By a short path, we mean that there are no half-twist bands in the inner disk that r misses. Suppose we obtain the words $u, v \in G$ from α so that $tut^{-1} = v$ is a relation in the presentation. We walk along r to describe u and v. Suppose $a_{i_1}, a_{i_2} \ldots, a_{i_l} \in S$ are the distinct l bands that r passes through in order, and suppose each a_{i_j} is an n_{i_j}-half-twist band. As r approaches the band a_{i_j} we have separate cases depending on n_{i_j}:

Case 1: $n_{i_j} = 0$. This is an untwisted band, so r can be lifted of the band without picking up the generator a_{i_j}.

Case 2: $n_{i_j} = 1$. On one side of Σ, r can be slid free of the band, so precisely one of u or v does not pick up the generator a_{i_j}, and in the other (of u and v) we obtain an $a_{i_j}^{\pm 1}$.

Case 3: $n_{i_j} > 1$ is even. Every full-twist corresponds to an $a_{i_j}^{\pm 1}$, and the sign of a_{i_j} is the same in both of the words. Both u and v have an $a_{i_j}^{\pm n_{i_j}/2}$.

Case 4: $n_{i_j} > 1$ is odd. This case combines the previous two. One of u and v has $a_{i_j}^{\pm(n_{i_j}-1)/2}$ and the other has $a_{i_j}^{\pm(n_{i_j}+1)/2}$.

Recall that the the simple loops S are unique up to sign and conjugation. There is no guarantee that one can avoid conjugations of each band in the words u and v, and the conjugating element g_j for a_{i_j} may be different in u and v in general. If Σ is not flat there are two subtleties that may occur: First r may be trapped behind a band $b \in S$, even though r does not pass through b. This will introduce (a conjugation of) $b^{\pm 1}$ in the middle of the word u or the word v. An example of this is α and the band b in the figure-eight diagram above. The number of half-twists in the band b has no influence. Second if $a \in S$ is a band that r passes through, such that a has ends D_1 and D_2 with D_1 on a different level than D_2 in the diagram, then the power of a in the words u and v may be off by ± 1 from the four cases stated above. Placing the above together we have justified the following word lemma.

Lemma 3.2.4. *Let Σ be a free incompressible Seifert surface. Suppose $S = \{a_1, \dots, a_k\}$ are the maximal bands of Σ, each a_i is an n_i-half-twist band, α is a generator of $\pi_1(\Sigma)$ with representative r that passes through each band in Σ at most once, and the distinct l bands that r passes through in order are $a_{i_1}, a_{i_2} \dots, a_{i_l} \in S$. If u and v are the words in the group G obtained from α in the positive and negative copies of Σ respectively, then we have the following*

(a) If Σ is flat and r is a short path, then
$u = (g_1 a_{i_1}^{\epsilon_1} g_1^{-1})(g_2 a_{i_2}^{\epsilon_2} g_2^{-1}) \dots (g_l a_{i_l}^{\epsilon_l} g_l^{-1})$ and $v = (h_1 a_{i_1}^{\zeta_1} h_1^{-1})(h_2 a_{i_2}^{\zeta_2} h_2^{-1}) \dots (h_l a_{i_l}^{\zeta_l} h_l^{-1})$, where for each $1 \le j \le l$

- *$g_j, h_j \in G$.*
- *ϵ_j and ζ_j have the same sign.*
- *If n_{i_j} is even then $\epsilon_j = \zeta_j = \pm n_{i_j}/2$.*
- *If n_{i_j} is odd then $\epsilon_j = \pm(n_{i_j}+1)/2$ and $\zeta_j = \pm(n_{i_j}-1)/2$ or $\epsilon_j = \pm(n_{i_j}-1)/2$ and $\zeta_j = \pm(n_{i_j}+1)/2$.*

(b) If Σ is not flat or r is not a short path, then the words u and v are as in (a) with the following modifications. For each a_{i_j} with ends on different levels, the powers of a_{i_j} in u and v are $\epsilon_j + c$ and $\zeta_j + d$ where $c, d \in \{-1, 0, 1\}$. Furthermore the words u and v may have additional strings of letters in the form $g a^{\pm 1} g^{-1}$ where $a \in S$ and $g \in G$.

Now we formally describe the effect of winding or unwinding a band in the HNN extension presentation of a knot group. The above word lemma does not assume alternating, though note that the lemma can be applied to all alternating knots. Recall that for alternating knots we can make sense of winding a zero-half-twist band. The next theorem is written from the unwinding perspective for alternating knots to account for zero-half-twist bands.

Theorem 3.2.5. *Let K be an alternating knot with free incompressible Seifert surface Σ_K that has a non-essential n-half-twist band a_1 for some $n \geq 2$. Let $\Sigma_{K_\circ}$ and $K_\circ$ be the Seifert surface and respective knot obtained after once unwinding a_1. Then there exist the following group presentations:*

$$\pi_1(\mathbb{S}^3 \backslash K_\circ) \;=\; \langle\, t, a_1, \ldots, a_m \mid tu_1t^{-1} = v_1,\ tu_2t^{-1} = v_2, \ldots,\ tu_mt^{-1} = v_m \,\rangle \text{ and}$$

$$\pi_1(\mathbb{S}^3 \backslash K) \;=\; \langle\, t, a_1, \ldots, a_m \mid ta_1u_1t^{-1} = a_1v_1,\ tu_2t^{-1} = v_2, \ldots,\ tu_mt^{-1} = v_m \,\rangle, \text{ where}$$

- *$\{a_1, \ldots a_m\}$ is a minimal generating set of $G = \pi_1(\mathbb{S}^3 \backslash \Sigma_K)$ and t is the stable letter in the HNN extension.*
- *For $i \in \{2, \ldots, m\}$, u_i and v_i are words in the form described in Lemma 3.2.4.*
- *The words u_1 and v_1 are also in the form from Lemma 3.2.4 with the following: a_1 is a_{i_1}, $\epsilon_1, \zeta_1 \geq 0$, $h_1 = 1$, and $g_i = 1$ for all $1 \leq i \leq l$.*

Proof. Choose a minimal generating set of $\pi_1(\Sigma_K)$ with fixed representatives on Σ such that any representative passes through any particular band in Σ_K at most once and only one generator α has its representative passing through the band a_1; this is possible since the Seifert surface is homotopy equivalent to a bouquet of circles, and one can choose generators of the fundamental group of the bouquet so that only one generator passes through a particular circle. With possibly first adjusting the base point in the HNN extension construction we can also assume that the representative of α begins immediately before the band a_1. Choose $a_2, \ldots, a_m$ so that $\{a_1, \ldots, a_m\}$ is a minimal generating set of simple loops of G.

For both K and $K_\circ$ apply Lemma 3.2.4 to α. Since the base point of the HNN construction is near a_1, for α we have that a_{i_1} is a_1 (i.e the first band that the representative of α traverses is a_1), and $g_1 = h_1 = 1$. Recall that simple loops are unique up to sign and conjugation, so for the one generator α we can choose conjugates and then relabel the simple loops to ensure that $g_i = 1$ for $2 \leq i \leq l$. Note that there is no guarantee that we can do this simultaneously for $h_2, \ldots, h_l$.

Also, we can relabel the sign of a_1 so that $\epsilon_1, \zeta_1 \geq 0$. Now let us compare the words u, v in the preceding Lemma for α in both knots K and $K_\circ$. For $K_\circ$ we will name the words u_1 and v_1, so that $tu_1t^{-1} = v_1$ is the corresponding relation in the group $\pi_1(\mathbb{S}^3 \backslash K_\circ)$. Since winding adds precisely ± 1 to the power of a_1 in the relation, with sign agreeing with the band before winding, we obtain the relation $ta_1u_1t^{-1} = a_1v_1$ from α with the knot K. Note that with $\epsilon_1, \zeta_1 \geq 0$, we have that the sign of the additional a_1 from the winding is $+1$. If $n = 2$ the band a_1 in the surface $\Sigma_{K_\circ}$ is an untwisted band, so it is possible that $\epsilon_1, \zeta_1 = 0$. If this is the case we *now* relabel a_1 if needed with opposite sign to ensure we get the relation $ta_1u_1t^{-1} = a_1v_1$. For the remaining $m-1$ generators of $\pi_1(\Sigma_K)$, none of the fixed representatives traverse the band a_1, so when applying Lemma 3.2.4 the relations obtained for both knots are the same.

$\square$

From this theorem, its proof, and the word lemma we have the following immediate results for not necessarily alternating knots.

Corollary 3.2.6. Let K be a knot with free incompressible Seifert surface Σ_K, and let a_1 be a maximal band in Σ_K. Let $\Sigma_{K'}$ and K' be the Seifert surface and respective knot obtained after n-times winding a_1, where $n \geq 0$. Then there exist the following group presentations:

$$\pi_1(\mathbb{S}^3 \backslash K) \;=\; \langle\; t, a_1, \ldots, a_m \mid tu_1t^{-1} = v_1,\; tu_2t^{-1} = v_2, \ldots,\; tu_mt^{-1} = v_m \;\rangle \text{ and}$$

$$\pi_1(\mathbb{S}^3 \backslash K') \;=\; \langle\; t, a_1, \ldots, a_m \mid ta_1^nu_1t^{-1} = a_1^nv_1,\; tu_2t^{-1} = v_2, \ldots,\; tu_mt^{-1} = v_m \;\rangle, \text{ where}$$

- $\{a_1, \ldots a_m\}$ is a minimal generating set of $G = \pi_1(\mathbb{S}^3 \backslash \Sigma_K)$ and t is the stable letter in the HNN extension.

- For $i \in \{2, \ldots, m\}$, u_i and v_i are words in the form described in Lemma 3.2.4.

- The words u_1 and v_1 are also in the form from Lemma 3.2.4 with the following: a_1 is a_{i_1}, $\epsilon_1, \zeta_1 \geq 0$, $h_1 = 1$, and $g_i = 1$ for all $1 \leq i \leq l$.

If we have a sequence of knots $\{K_i\}_{0 \leq i \leq k}$ with respective free incompressible Seifert surfaces $\{\Sigma_{K_i}\}_{0 \leq i \leq k}$ such that Σ_{K_i} is obtained by n_i-times winding a band in $\Sigma_{K_{i-1}}$ for each $i \in \{1 \ldots k\}$, then group presentations exists for each pair (K_{i-1}, K_i) as in Corollary 3.2.6; however, there is no guarantee that we have a nice group presentation relationship between K_0 and K_k. This depends on the choice of generators on the Seifert surface.

In Σ_{3_1} above there are three maximal bands a, b, and c. Note that there is no choice of generators α and β of $\pi_1(\Sigma_{3_1})$ and representatives such that only one of the representatives passes through each maximal band in $\{a, b, c\}$. We can only arrange this for up to two of the bands in $\{a, b, c\}$. Above in Σ_{4_1}, we have representatives such that only one representative passes through each maximal band.

Corollary 3.2.7. Let K be a knot with free incompressible Seifert surface Σ_K, and $M := \{a_1, \ldots, a_k\}$ be a subset of a minimal generating set of maximal bands in Σ_K such that there exists a generating set $\pi_1(\Sigma_K)$ with fixed representatives S with the condition that for all $a \in M$, only one element of S passes through the band a.

Let $\Sigma_{K'}$ and K' be the Seifert surface and respective knot obtained after, for each $i \in \{1, \ldots, k\}$, n_i-times winding the band a_i where $n_i \geq 0$. Then there exist the following group presentations:

$$\begin{aligned}
\pi_1(\mathbb{S}^3 \backslash K) \;&=\; \langle\, t, a_1, \ldots, a_m \mid tu_1t^{-1} = v_1, \ldots,\ tu_kt^{-1} = v_k, \\
&\qquad\qquad tu_{k+1}t^{-1} = v_{k+1}, \ldots,\ tu_mt^{-1} = v_m \,\rangle \text{ and} \\
\pi_1(\mathbb{S}^3 \backslash K') \;&=\; \langle\, t, a_1, \ldots, a_m \mid ta_1^{n_1}u_1t^{-1} = a_1^{n_1}v_1, \ldots,\ ta_k^{n_k}u_kt^{-1} = a_k^{n_k}v_k, \\
&\qquad\qquad tu_{k+1}t^{-1} = v_{k+1}, \ldots,\ tu_mt^{-1} = v_m \,\rangle, \text{ where}
\end{aligned}$$

- $\{a_1, \ldots a_m\}$ is a minimal generating set of $G = \pi_1(\mathbb{S}^3 \backslash \Sigma_K)$ and t is the stable letter in the HNN extension.
- For $i \in \{1, \ldots, m\}$, u_i and v_i are words in the form described in Lemma 3.2.4.

In the figure-8 knot example, recall that 6_1 is obtained from winding one of maximal bands, and 8_1 is obtained from twice winding one of the maximal bands. The resulting presentations are:

$$\pi_1(\mathbb{S}^3 \backslash 4_1) \;=\; \langle\, t, a, b \mid tat^{-1} = ba,\ tbt^{-1} = bab \,\rangle$$

$$\pi_1(\mathbb{S}^3\backslash 6_1) \;=\; \langle\, t, a, b \mid ta^2t^{-1} = ba^2,\ tbt^{-1} = bab \,\rangle$$

$$\pi_1(\mathbb{S}^3\backslash 8_1) \;=\; \langle\, t, a, b \mid ta^3t^{-1} = ba^3,\ tbt^{-1} = bab \,\rangle$$

For an example of the form in Corollary 3.2.7 where $k > 1$, the knot 8_3 is obtained from 4_1 from winding both maximal bands once. This results in the following presentations. Note that at the time this presentation was computed, a slightly different generating set of the Seifert surface of 4_1 was used than the generating set used above.

$$\pi_1(\mathbb{S}^3\backslash 4_1) \;=\; \langle\, t, a, b \mid tat^{-1} = ab,\ tba^{-1}t^{-1} = b \,\rangle$$

$$\pi_1(\mathbb{S}^3\backslash 8_3) \;=\; \langle\, t, a, b \mid ta^2t^{-1} = a^2b,\ tb^2a^{-1}t^{-1} = b^2 \,\rangle$$

3.3. Choosing Generators to Satisfy Algebraic Conditions

For a positive integer l, we prove that l-times winding a single band of a free incompressible Seifert surface Σ_K of a fibered knot K preserves bi-orderability. Genus one will be treated as a special case: The only genus one fibered knot with bi-orderable knot group is $K = 4_1$, and the knots obtained from winding a single band are all twist knots. Since twist knots with an even crossing number have bi-orderable knot groups [5], we have that l-times winding one band of Σ_K preserves bi-orderability.

If the genus of K is at least two, then the base group in the HNN extension presentation is of rank at least four. To apply the gluing theorem, we delete not only the relation of the winding, but one more relation to ensure the algebraic closure of $\langle A, B\rangle$ is not all of G.

Let $K \neq 4_1$ be a fibered knot with bi-orderable knot group such that Σ_K is a free incompressible Seifert surface. Then in the group $\pi_1(\mathbb{S}^3\backslash K) = G*_{\alpha_\circ} \cong \mathbb{Z} \ltimes_{\alpha_\circ} G$, we have $m := \operatorname{rank}(G) \geq 4$. Let $\Sigma_{K'}$ and K' be the respective Seifert surface and knot obtained after l-times winding a band a_m in Σ_K. For a loop r on Σ_K, the image on $\Sigma_{K'}$ will be denoted r'. Positive and negative parallel translates of a loop r will be denoted r^+ and r^- respectively. For simplicity we will sometimes use the same symbol for both a generator of $\pi_1(\Sigma_K)$ and its fixed representative.

Assuming that Σ is not flat, suppose the band a_m connects two disks that are not at the same height. Fix a base point on Σ and choose $a_1, \ldots, a_{m-1}$ so that

$\{a_1, \ldots, a_m\}$ is a minimal generating set of simple loops of G and $\alpha_1, \ldots, \alpha_m$ are generators of $\pi_1(\Sigma_K)$ with fixed representatives $r_1, \ldots, r_m$ in Σ_K such that we have the following conditions:

1. Only α_m passes through the band a_m.
2. When lifting off α_{m-1} from the surface Σ_K, both positive and negative translates contain a_m exactly once as reduced words in G.
3. When lifting off α'_m from the surface $\Sigma_{K'}$, both positive and negative translates are a word of the form $u(a_1, \ldots, a_{m-1})a_m^p v(a_1, \ldots, a_{m-1})$.
4. $\alpha_1, \ldots, \alpha_{m-2}$ do not involve the letter a_m when either side is lifted off of the surface and written as a reduced word in G.
5. If a word w in the alphabet $\alpha_1^{\pm}, \ldots, \alpha_m^{\pm}$ is equal to a_m in G, then it either contains both of α_{m-1}^+ and α_m^+ or both of α_{m-1}^- and α_m^-.

We also have the following HNN extension presentation where for each $i \in \{1, \ldots, m\}$, u_i and v_i are the words in G obtained from $\alpha_i^{\pm}$:

$$\pi_1(\mathbb{S}^3 \backslash K) \;=\; \langle\, t, a_1, \ldots, a_m \mid tu_1t^{-1} = v_1,\ \ldots,\ tu_mt^{-1} = v_m \,\rangle$$

We have that $u_1, v_1, \ldots, u_{m-2}, v_{m-2}$ do not have the letter a_m, but $u_{m-1}, v_{m-1}, u_m, v_m$ do contain the letter a_m; furthermore, the elements u_{m-1} and v_{m-1} involve the letter a_m only once. Let $A = \langle u_1, \ldots, u_{m-2}\rangle$ and $B = \langle v_1, \ldots, v_{m-2}\rangle$. Since K is fibered A and B are primitive subgroups of G. Let $\alpha = \alpha_\circ|_A$. By condition 4 above, we have $\langle A, B\rangle \leq \langle a_1, \ldots, a_{m-1}\rangle$. If C is the algebraic closure of $\langle A, B\rangle$ in G, then we also have $C \leq \langle a_1, \ldots, a_{m-1}\rangle$.

We let $x_{m-1}, x_m, y_{m-1}, y_m$ be elements in $\pi_1(\mathbb{S}^3 \backslash K')$ represented with $(\alpha'_{m-1})^+, (\alpha'_m)^+$, $(\alpha'_{m-1})^-, (\alpha'_m)^-$ respectively. Then by the arrangement we have that $\langle x_{m-1}, y_{m-1}, x_m, y_m\rangle \cong F_4$, and $D := \langle C, x_{m-1}, x_m, y_{m-1}, y_m\rangle \cong \langle C\rangle * \langle x_{m-1}, x_m, y_{m-1}, y_m\rangle$.

D is an algebraically closed subgroup of G: Condition 5 guarantees that $\langle x_{m-1}, x_m, y_{m-1}, y_m\rangle$ is algebraically closed. On the other hand, no element of C in its form of a reduced word contains a_m, whereas a_m occurs in the reduced word form of every non-identity element of $\langle x_{m-1}, x_m, y_{m-1}, y_m\rangle$. Thus we obtain the decomposition of D as a free product above; furthermore, by the same property this free product decomposition is algebraically closed. It follows from conditions 1 through 5 that for all $f \in D\backslash C$, a reduced word representing f can be written as $w = uv$ where we have the following:

- $S \subseteq \langle a_1, \ldots, a_{m-1} \rangle$ such that $S \cap C$ is empty.
- v is a positive word in the alphabet $\mathcal{A} = \{a_m^i b : i \in \{-p, -1, 1, p\}, b \in S\}$.
- $u \in \langle a_1, \ldots, a_{m-1} \rangle$ such that $u \notin S^{-1}$.

Thus for all $x \in G$, if x^n is a positive word in the alphabet $\mathcal{A}$ for some $n \geq 1$, then x is also a positive word in the alphabet $\mathcal{A}$. Consequently, if $w = x^{p_1} h_1 x^{p_2} h_2 \ldots x^{p_k} h_k \in D\backslash C$ for some $p_i \in \mathbb{Z}\backslash\{0\}$ and $h_i \in D$ where $i \in \{1, \ldots, k\}$, then necessarily $x \in D$. On the other hand, if $f \in C$ then the equation $w = x^{p_1} h_1 x^{p_2} h_2 \ldots x^{p_k} h_k$ implies that $h_i \in C$ for all $i \in \{1, \ldots, k\}$. Therefore since C is algebraically closed, we obtain that $x \in C$.

Then by Theorem 2.3.2 the HNN extension $G*_\gamma$ is bi-orderable where $A' = \langle A, x_{m-1}, x_m \rangle$, $B' = \langle B, y_{m-1}, y_m \rangle$, and $\gamma : A' \to B'$ is the extension of α such that $\gamma(x_{m-1}) = y_{m-1}$ and $\gamma(x_m) = y_m$.

Theorem 3.3.1. *Let K be a fibered knot with $\pi_1(\mathbb{S}^3\backslash K)$ bi-orderable such that Σ_K is a non-flat, free incompressible Seifert surface. Let $\Sigma_{K'}$ and K' be the respective Seifert surface and knot obtained after l-times winding one band in Σ_K. Then $\pi_1(\mathbb{S}^3\backslash K')$ is bi-orderable.*

3.4. Bi-Orderable Knot Groups

For up to twelve crossings, we list some of the knots that are obtained from performing the winding operation to free incompressible Seifert surfaces of fibered knots that are known to have bi-orderable knot groups. Since winding increases crossing number, for the list we only need to consider the winding operation for crossing number up to ten. The fibered knots known to have bi-orderable group with crossing number ten or less are 4_1, 8_{12}, and 10_{137} [22]. In the lists below there may be knots missing; see Remark 3.4.1. We also distinguish when winding a single band or multiple.

The following knots are obtained from winding one band in 4_1. All of their respective knot groups are known to be bi-orderable [5]: 6_1, 8_1, 10_1, $12a_{0803}$

The following knots are obtained from winding both bands in 4_1. The bi-orderability of their respective knot groups is unknown: 8_3, 10_3, $12a_{1166}$, $12a_{1287}$

The following knots are obtained from winding one band in 8_{12}. The knot 10_{13} is known to have a bi-orderable knot group [7]. The bi-orderability of the other two knot groups was unknown before this document: 10_{13}, 10_{35}, $12a_{0691}$

The following knots are obtained from winding two bands in 8_{12}. The bi-orderability of their respective knot groups is unknown: $12a_{0471}$, $12a_{0482}$, $12a_{0690}$, $12a_{1127}$

Remark 3.4.1. Due to insufficient time, these lists were compiled after only a glance through some knot diagrams. The knot $12a_{0691}$ is obtained from winding the already wound band in 10_{13}. The crossing-12 knot obtained from winding the already wound band in 10_{35} appears to be missing. There are likely other knots missing.

Remark 3.4.2. The knot 10_{137} is non-alternating. The knots $12n_{0011}$ and $12n_{0046}$ are obtained from once winding a band in a Seifert surface Σ constructed from a reduced diagram of 10_{137}; however, Σ is not minimal genus, so Σ might not be incompressible. Again due to insufficient time, details were not checked carefully. It is not known by the author if $12n_{0011}$ and $12n_{0046}$ can be obtained from winding a band in a free incompressible Seifert surface of 10_{137}.

Applying Theorem 3.3.1 we have that the following knot groups are bi-orderable. The non-flat Seifert surface condition is mild. All fibered knots with knot groups that are known to be bi-orderable satisfy this condition.

Corollary 3.4.3. $\pi_1(\mathbb{S}^3\backslash 10_{35})$ and $\pi_1(\mathbb{S}^3\backslash 12a_{0691})$ are bi-orderable.

3.5. Closing Conjectures, Questions, and Remarks

The fibered base concept is underdeveloped. The definition depends on a choice of Seifert surface of the knot. If a fibered base exists, is it unique? Existence of a fibered base for all alternating knots seems likely, since alternating knots admit a minimal-genus (hence incompressible) free Seifert surface when applying Seifert's algorithm to a reduced diagram, every alternating knot admits an HNN extension presentation of an even-rank free group G that is close to a semidirect product $\mathbb{Z} \ltimes G$, in the sense that the index $|G_1 : A_1|$ is small and seems to respect the number of windings. Recall that G_1 is the Abelianization of G and A_1 is the image of A in G_1. What class of knots admit a fibered base?

Conjecture 3.5.1. *Every alternating knot admits a fibered base.*

Winding a band seems to create more room in the resulting group, which seems to increase the chance of bi-orderability.

Conjecture 3.5.2. *Winding any number of bands preserves bi-orderability.*

Question 3.5.3. Does there exists a fibered knot K with non-bi-orderable knot group such that a winding of K results in a bi-orderable knot group?

Question 3.5.4. Under what conditions can relations be deleted from a presentation of a bi-orderable group such that the resulting group is still bi-orderable?

4. DATA ON SOME CROSSING-EIGHT KNOTS

The final chapter consists of HNN extension presentation data using the Seifert surface gluing technique for some of the crossing-8 knots in anticipation of applying the methods in [2]; however, due to time constraints these methods have not yet been applied. The knots are 8_2, 8_3, 8_6, 8_8, 8_{13}, and 8_{15}. These are all non-fibered with the exception of 8_2, though the methods in [2] do not assume the knots are fibered. While the bi-orderability of four of these knot groups is unknown at the time of this writing, the bi-orderability of $\pi_1(\mathbb{S}^3 \backslash 8_8)$ and $\pi_1(\mathbb{S}^3 \backslash 8_{13})$ is known to be negative [5, 7]; both knots are non-fibered two-bridge knots with Alexander polynomial lacking real roots. All of these knots are alternating, so applying Seifert's algorithm to an alternating diagram results in a free incompressible Seifert surface. Hence the resulting groups are HNN extensions of even rank-free groups.

In this chapter G is the base group in the HNN extension, and $\alpha : A \to B$ is the isomorphism of subgroups $A, B \leq G$ corresponding to the positive and negative parallel copies of the Seifert surface respectively. Utilizing the lower central series to apply techniques in [2], we also let $G_0 = G$, $G_n = [G_0, G_{n-1}]$, and $K_n = G_{n-1}/G_n$ for $n \geq 1$. Abusing notation, we will write elements in the quotient K_n the same as elements in G_{n-1}. For $n \geq 1$, define A_n and B_n to be the respective images of A and B in K_n, and define α_n to be the induced isomorphism from A_n to B_n. In the case $n = 1$, we simply have that K_1 is the Abelianization of the group G, and A_1 and B_1 are the images of A and B respectively in the Abelianization of G. In the first section, we simply copy the data from both 6_2 and 7_6 in [2] since the data for the crossing-8 knots follows the same process. This data was collected before heavily exploring the winding operation: Instead of having the group presentations nicely reflect a possible winding, generators of the surface and its complement were chosen and relabeled to make A as simple as possible. For example in the non-fibered knot 8_6, $G = \langle x, a, b, c \rangle$ and $A = \langle x, a, b, c^2 \rangle$.

4.1. Non-bi-orderability of $\pi_1(\mathbb{S}^3 \backslash 6_2)$ and $\pi_1(\mathbb{S}^3 \backslash 7_6)$

The knots 6_2 and 7_6 are alternating and fibered; however, both of their respective Alexander polynomials have some (but not all) positive real roots, so Theorems 1.2.5 and 1.2.6 do not apply. Applying the Seifert surface gluing technique, we obtain an HNN extension presentations of $\pi_1(\mathbb{S}^3\backslash 6_2)$ and $\pi_1(\mathbb{S}^3\backslash 7_6)$. Images of their respective Seifert surfaces with labeled generators can be found in [2].

$$\begin{aligned}
\pi_1(\mathbb{S}^3\backslash 6_2) &= \langle\, t, x, a, b, c \mid ta^{-1}t^{-1} = xb,\ txat^{-1} = x, \\
&\qquad\qquad\qquad tbt^{-1} = c^{-1},\ tct^{-1} = abc\,\rangle \\
\pi_1(\mathbb{S}^3\backslash 7_6) &= \langle t, a, b, c, d \mid tat^{-1} = ab,\ tb^{-1}ac^{-1}t^{-1} = b^{-1}, \\
&\qquad\qquad\qquad tct^{-1} = b^{-1}d^{-1},\ tdt^{-1} = cd\,\rangle
\end{aligned}$$

In both cases the base group G is F_4, and $A = B = G$. We obtain semidirect products $\mathbb{Z} \ltimes F_4$. The acting group $\mathbb{Z}$ is generated by the stable letter t. The quotients $\pi_1(\mathbb{S}^3\backslash 6_2)/[G,G]$ and $\pi_1(\mathbb{S}^3\backslash 7_6)/[G,G]$ are isomorphic to $\mathbb{Z} \ltimes_{M_1} K_1$ and $\mathbb{Z} \ltimes_{N_1} K_1$ respectively where $K_1 \cong \mathbb{Z}^4$, and

$$M_1 = \begin{bmatrix} 2 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \ N_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix}.$$

The characteristic polynomials of M_1 and N_1 are

$$p_{M_1}(\lambda) = \lambda^4 - 3\lambda^3 + 3\lambda^2 - 3\lambda + 1, \text{ and}$$
$$p_{N_1}(\lambda) = \lambda^4 - 5\lambda^3 + 7\lambda^2 - 5\lambda + 1.$$

Note that these are the respective Alexander polynomials of 6_2 and 7_6. Since each G_n is invariant under conjugation by t, note that t acts on K_n. Passing to K_2 we obtain the groups $\mathbb{Z} \ltimes_{M_2} K_2$ and $\mathbb{Z} \ltimes_{N_2} K_2$, where t acts on $K_2 \cong \mathbb{Z}^6$ via M_2 or N_2. Below are generators of K_2 and their respective images under the action of conjugation by t via M_2 and N_2.

$$x_1 = [x, a] \overset{t}{\mapsto} x_2^{-1} \qquad x_2 = [x, b] \overset{t}{\mapsto} x_3^{-2} b_1^{-1}$$

$$x_3 = [x, c] \overset{t}{\mapsto} x_1^2 x_2^2 x_3^2 a_1^{-1} b_1 \qquad a_1 = [a, b] \overset{t}{\mapsto} x_3 b_1$$

$$a_2 = [a, c] \overset{t}{\mapsto} x_1^{-1} x_2^{-1} x_3^{-1} a_1 b_1^{-1} \qquad b_1 = [b, c] \overset{t}{\mapsto} a_2 b_1$$

$$M_2 = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & -1 \\ 2 & 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ -1 & -1 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$a_1 = [a, b] \overset{t}{\mapsto} a_1^2 a_3 b_2 \qquad a_2 = [a, c] \overset{t}{\mapsto} a_1^{-1} a_3^{-1} b_2^{-1}$$

$$a_3 = [a, d] \overset{t}{\mapsto} a_2 a_3 b_1 b_2 \qquad b_1 = [b, c] \overset{t}{\mapsto} a_1^{-1} a_3^{-1} b_2^{-2}$$

$$b_2 = [b, d] \overset{t}{\mapsto} a_2 a_3 b_1^3 b_2^3 c_1^{-1} \qquad c_1 = [c, d] \overset{t}{\mapsto} b_1^{-1} b_2^{-1} c_1$$

$$N_2 = \begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ -1 & 0 & -1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 3 & 3 & -1 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix}$$

The matrices M_2 and N_2 are used to prove that $\pi_1(\mathbb{S}^3 \backslash 6_2)$ and $\pi_1(\mathbb{S}^3 \backslash 7_6)$ are not bi-orderable. Details can be found in [2].

4.2. The knot 8_2

8_2 is a fibered knot. An HNN extension presentation for $\pi_1(\mathbb{S}^3\backslash 8_2)$ is

$$\begin{aligned}\pi_1(\mathbb{S}^3\backslash 8_2) \;=\; \langle\, t, x, a, b, c, d, e \mid\, & txt^{-1} = bx^2,\ tat^{-1} = x^{-1}b^{-1},\ tbt^{-1} = c^{-1}, \\ & tct^{-1} = d^{-1},\ tdt^{-1} = e^{-1},\ tet^{-1} = abcde\,\rangle.\end{aligned}$$

This group is a semidirect product $\mathbb{Z} \ltimes G$, where $G \cong F_6$. Abelianizing G, we obtain the group $\mathbb{Z} \ltimes_{M_1} K_1$, where t acts on $K_1 \cong \mathbb{Z}^6$ via multiplication by M_1:

$$M_1 = \begin{bmatrix} 2 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

The characteristic polynomial of M_1 is $p_1(\lambda) = \lambda^6 - 3\lambda^5 + 3\lambda^4 - 3\lambda^3 + 3\lambda^2 - 3\lambda + 1$, which is the Alexander polynomial $\Delta_{8_2}(\lambda)$. The roots are

$$\begin{aligned}&\lambda \approx 0.489598, \\ &\lambda \approx 2.04249, \\ &\lambda \approx -0.439693 \pm 0.898148i, \\ &\lambda \approx 0.673648 \pm 0.739052i.\end{aligned}$$

Note that the two real roots of $p_1(\lambda)$ are positive and irrational. Passing to K_2 we obtain the group $\mathbb{Z} \ltimes_{M_2} K_2$, where t acts on $K_2 \cong \mathbb{Z}^{15}$ via M_2. Below are generators of K_2, their respective images under the action of conjugation by t, and M_2.

$$[x,a] \overset{t}{\mapsto} x_2^{-1} \qquad [x,b] \overset{t}{\mapsto} x_3^{-2}b_1^{-1} \qquad [x,c] \overset{t}{\mapsto} x_4^{-2}b_2^{-1}$$

$$[x,d] \overset{t}{\mapsto} x_5^{-2}b_3^{-1} \qquad [x,e] \overset{t}{\mapsto} x_1^2x_2^2x_3^2x_4^2x_5^2a_1^{-1}b_1b_2b_3 \qquad [a,b] \overset{t}{\mapsto} x_3b_1$$

$$[a,c] \overset{t}{\mapsto} x_4b_2 \qquad [a,d] \overset{t}{\mapsto} x_5b_3 \qquad [a,e] \overset{t}{\mapsto} x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1}x_5^{-1}a_1b_1^{-1}b_2^{-1}b_3^{-1}$$

$$[b,c] \overset{t}{\mapsto} c_1 \qquad [b,d] \overset{t}{\mapsto} c_2 \qquad [b,e] \overset{t}{\mapsto} a_2b_1c_1^{-1}c_2^{-1}$$

$$[c,d] \overset{t}{\mapsto} d_1 \qquad [c,e] \overset{t}{\mapsto} a_3b_2c_1d_1^{-1} \qquad [d,e] \overset{t}{\mapsto} a_4b_3c_2d_1$$

$$M_2 = \begin{bmatrix}
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
2 & 2 & 2 & 2 & 2 & -1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & -1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1
\end{bmatrix}$$

The characteristic polynomial $p_2(\lambda)$ of M_2 is

$$p_2(\lambda) = -\lambda^{15} + 3\lambda^{14} - 6\lambda^{13} + 10\lambda^{12} - 24\lambda^{11} + 30\lambda^{10} + 2\lambda^9 - 36\lambda^8 + \\ 36\lambda^7 - 2\lambda^6 - 30\lambda^5 + 24\lambda^4 - 10\lambda^3 + 6\lambda^2 - 3\lambda + 1.$$

The roots of $p_2(\lambda)$ are listed below. Note that 1 is the only real root.

$\lambda = 1$ (multiplicity 3), $\lambda \approx 0.959977 \pm 0.28008i$, $\lambda \approx -0.898068 \pm 1.83446i$,

$\lambda \approx -0.215273 \pm 0.439732i$, $\lambda \approx 0.329817 \pm 0.361839i$, $\lambda \approx 0.36758 \pm 0.92992i$,

$\lambda \approx 1.37592 \pm 0.1.50951i$.

4.3. The Knot 8_3

8_3 is not fibered. An HNN extension presentation for $\pi_1(\mathbb{S}^3\backslash 8_3)$ is

$$\pi_1(\mathbb{S}^3\backslash 8_3) \;=\; \langle\ t, a, b \mid ta^2t^{-1} = a^2b,\ tb^2a^{-1}t^{-1} = b^2\ \rangle.$$

This is a proper HNN extension. Here $G = \langle a, b\rangle$ and $A = \langle a^2,\ b^2a^{-1}\rangle$. Observe that $|K_1 : A_1| = |K_1 : B_1| = 4$. Abelianizing G in $G*_\alpha$, the resulting group $K_1*_{\alpha_1}$ is a finitely presented subgroup of $\mathbb{Z} \ltimes_{M_1} (\mathbb{Z}[\frac{1}{2}])^2$ where

$$M_1 = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{4} \end{bmatrix}.$$

The characteristic polynomial of M_1 is $p_1(\lambda) = \lambda^2 - \frac{9}{4}\lambda + 1$, and the Alexander polynomial is $\Delta_{8_3}(\lambda) = 4\lambda^2 - 9\lambda + 4$, so we have $\Delta_{8_3}(\lambda) = |K_1 : A_1|p_1(\lambda)$. The roots are $\lambda = \frac{1}{8}(9 \pm \sqrt{17})$, which are both positive and irrational. Passing to K_2, the only generator is $a_1 := [a, b]$, and the only generator of A_2 is $a_1^4 = [a^2,\ b^2a^{-1}] \overset{t}{\mapsto} [a^2b, b^2] = a_1^4$, so $K_2*_{\alpha_2} = \langle\ t, a_1 \mid ta_1^4t^{-1} = a_1^4\ \rangle$.

4.4. The Knot 8_6

8_6 is not fibered. An HNN extension presentation for $\pi_1(\mathbb{S}^3\backslash 8_6)$ is

$$\begin{aligned}\pi_1(\mathbb{S}^3\backslash 8_6) \;=\; \langle\ t, x, a, b, c \mid\ & txt^{-1} = xc^{-1},\ tat^{-1} = b^{-1}, \\ & tbt^{-1} = cab,\ tc^2t^{-1} = b^{-1}a^{-1}c^{-1}bc^3x^{-1}\ \rangle.\end{aligned}$$

This is a proper HNN extension. Here $G = \langle x, a, b, c\rangle$ and $A = \langle x, a, b, c^2\rangle$. Observe that $|K_1 : A_1| = |K_1 : B_1| = 2$. Abelianizing G in $G*_\alpha$, the resulting group $K_1*_{\alpha_1}$ is a finitely presented subgroup of $\mathbb{Z} \ltimes_{M_1} (\mathbb{Z}[\frac{1}{2}])^4$ where

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 1 \end{bmatrix}.$$

The characteristic polynomial of M_1 is $p_1(\lambda) = \lambda^4 - 3\lambda^3 + \frac{7}{2}\lambda^2 - 3\lambda + 1$, and the Alexander polynomial is $\Delta_{8_6}(\lambda) = 2\lambda^4 - 6\lambda^3 + 7\lambda^2 - 6\lambda + 2$, so we have $\Delta_{8_6}(\lambda) = |K_1 : A_1| p_1(\lambda)$.

$$\begin{aligned} p_1(\lambda) &= \lambda^4 - 3\lambda^3 + \frac{7}{2}\lambda^2 - 3\lambda + 1 \\ &= \frac{1}{4}(2\lambda^2 - (3+\sqrt{3})\lambda + 2)(2\lambda^2 - (3-\sqrt{3})\lambda + 2) \end{aligned}$$

The roots are $\lambda = \frac{1}{4}\big(3 + \sqrt{3} \pm \sqrt{6\sqrt{3} - 4}\big)$ and $\lambda = \frac{1}{4}\big(3 - \sqrt{3} \pm i\sqrt{6\sqrt{3}+4}\big)$. Note that both of the real roots are positive and irrational. Passing to $K_2 \cong \mathbb{Z}^6$, we have generators $x_1 := [x, a]$, $x_2 := [x, b]$, $x_3 := [x, c]$, $a_1 := [a, b]$, $a_2 := [a, c]$, and $b_1 := [b, c]$. Then $A_2 = \langle x_1, x_2, x_3^2, a_1, a_2^2, b_1^2 \rangle$, and $K_2 *_{\alpha_2}$ is a subgroup of $\mathbb{Z} \ltimes_{M_2} (\mathbb{Z}[\frac{1}{2}])^6$. Below is M_2 and the generators of A_2 with their respective images under t.

$$x_1 = [x, a] \overset{t}{\mapsto} x_2^{-1} b_1^{-1} \qquad x_2 = [x, b] \overset{t}{\mapsto} x_1 x_2 x_3 a_2 b_1$$

$$x_3^2 = [x, c^2] \overset{t}{\mapsto} x_1^{-1} x_3 a_2^{-1} \qquad a_1 = [a, b] \overset{t}{\mapsto} a_1 b_1^{-1}$$

$$a_2^2 = [a, c^2] \overset{t}{\mapsto} x_2^{-1} a_1^{-1} b_1^{-2} \qquad b_1^2 = [b, c^2] \overset{t}{\mapsto} x_1 x_2 x_3 a_1 a_2^3 b_1^2$$

$$M_2 = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & -1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & 1 \end{bmatrix}$$

The characteristic polynomial of M_2 is

$$
\begin{aligned}
p_2(\lambda) &= \lambda^6 - \frac{7}{2}\lambda^5 + 8\lambda^4 - 11\lambda^3 + 8\lambda^2 - \frac{7}{2}\lambda + 1 \\
&= \frac{1}{2}(\lambda - 1)^2(2\lambda^4 - 3\lambda^3 + 8\lambda^2 - 3\lambda + 2).
\end{aligned}
$$

The roots of $p_2(\lambda)$ are listed below. Note that 1 is the only real root.

$$
\begin{aligned}
\lambda &= 1 \text{ (multiplicity 2)}, \\
\lambda &= \frac{1}{8}\big(3 - i\sqrt{23} \pm \sqrt{-78 - 6i\sqrt{23}}\big), \\
\lambda &= \frac{1}{8}\big(3 + i\sqrt{23} \pm \sqrt{-78 + 6i\sqrt{23}}\big).
\end{aligned}
$$

4.5. The Knot 8_8

8_8 is not fibered. An HNN extension presentation for $\pi_1(\mathbb{S}^3 \backslash 8_8)$ is

$$
\begin{aligned}
\pi_1(\mathbb{S}^3 \backslash 8_8) \;=\; \langle\; t, a, b, c, d \mid\; & ta^2t^{-1} = a^{-1}ba^2d^{-1}c^{-1}a,\; tbt^{-1} = a^{-1}b, \\
& tct^{-1} = a^{-1}cd,\; tdt^{-1} = c^{-1}a \;\rangle.
\end{aligned}
$$

This is a proper HNN extension. Here $G = \langle a, b, c, d\rangle$ and $A = \langle a^2, b, c, d\rangle$. Observe that $|K_1 : A_1| = |K_1 : B_1| = 2$. Abelianizing G in $G*_\alpha$, the resulting group $K_1*_{\alpha_1}$ is a subgroup of $\mathbb{Z} \ltimes_{M_1} (\mathbb{Z}[\frac{1}{2}])^4$ where

$$
M_1 = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}.
$$

The characteristic polynomial of M_1 is $p_1(\lambda) = \lambda^4 - 3\lambda^3 + \frac{9}{2}\lambda^2 - 3\lambda + 1$, and the Alexander polynomial is $\Delta_{8_8}(\lambda) = 2\lambda^4 - 6\lambda^3 + 9\lambda^2 - 6\lambda + 2$, so we have $\Delta_{8_6}(\lambda) = |K_1 : A_1|p_1(\lambda)$.

$$
\begin{aligned}
p_1(\lambda) &= \lambda^4 - 3\lambda^3 + \frac{9}{2}\lambda^2 - 3\lambda + 1 \\
&= \frac{1}{2}(2\lambda^2 - 2\lambda + 1)(2\lambda^2 - 2\lambda + 2)
\end{aligned}
$$

The roots are $\lambda = \frac{1}{2} \pm \frac{1}{2}i$ and $\lambda = 1 \pm i$, so there are no real roots. Passing to $K_2 \cong \mathbb{Z}^6$, we have generators $a_1 := [a,b]$, $a_2 := [a,c]$, $a_3 := [a,d]$, $b_1 := [b,c]$, $b_2 := [b,d]$, and $c_1 := [c,d]$. Then $A_2 = \langle a_1^2, a_2^2, a_3^2, b_1, b_2, c_1\rangle$, and $K_2 *_{\alpha_2}$ is a subgroup of $\mathbb{Z} \ltimes_{M_2} (\mathbb{Z}[\frac{1}{2}])^6$. Below is M_2 and the generators of A_2 with their respective images under t.

$$a_1^2 = [a^2,b] \overset{t}{\mapsto} a_1^3 a_2^{-1} a_3^{-1} b_1 b_2 \qquad a_2^2 = [a^2,c] \overset{t}{\mapsto} a_1 a_2 a_3 b_1 b_2$$
$$a_3^2 = [a^2,d] \overset{t}{\mapsto} a_1^{-1} a_2^{-1} a_3 b_1^{-1} c_1^{-1} \qquad b_1 = [b,c] \overset{t}{\mapsto} a_1 a_2^{-1} a_3^{-1} b_1 b_2$$
$$b_2 = [b,d] \overset{t}{\mapsto} a_1^{-1} a_2 b_1^{-1} \qquad c_1 = [c,d] \overset{t}{\mapsto} a_3^{-1} c_1$$

$$M_2 = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 1 & -1 & -1 & 1 & 1 & 0 \\ -1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

The characteristic polynomial of M_2 is

$$\begin{aligned} p_2(\lambda) &= \lambda^6 - \frac{9}{2}\lambda^5 + 8\lambda^4 - 9\lambda^3 + 8\lambda^2 - \frac{9}{2}\lambda + 1 \\ &= \frac{1}{2}(2\lambda - 1)(\lambda - 1)^2(\lambda - 2)(\lambda^2 + 1). \end{aligned}$$

The roots of $p_2(\lambda)$ are listed below. Note that the real roots are all positive and rational.

$$\lambda = 1 \text{ (multiplicity 2)},\ \lambda = \frac{1}{2},\ \lambda = 2,\ \lambda = \pm i.$$

4.6. The Knot 8_{13}

8_{13} is not fibered. An HNN extension presentation for $\pi_1(\mathbb{S}^3 \backslash 8_{13})$ is

$$\begin{aligned} \pi_1(\mathbb{S}^3 \backslash 8_{13}) \;=\; \langle\, t, a, b, c, d \mid tat^{-1} = ab^{-1}d,\ tb^2t^{-1} = abd, \\ tct^{-1} = cd,\ tdt^{-1} = ab^{-1}dc^{-1} \,\rangle. \end{aligned}$$

This is a proper HNN extension. Here $G = \langle a, b, c, d\rangle$ and $A = \langle a, b^2, c, d\rangle$. Observe that $|K_1 : A_1| = |K_1 : B_1| = 2$. Abelianizing G in $G{*_\alpha}$, the resulting group $K_1{*_{\alpha_1}}$ is a finitely presented subgroup of $\mathbb{Z} \ltimes_{M_1} (\mathbb{Z}[\frac{1}{2}])^4$ where

$$M_1 = \begin{bmatrix} 1 & -1 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

The characteristic polynomial of M_1 is $p_1(\lambda) = \lambda^4 - \frac{7}{2}\lambda^3 + \frac{11}{2}\lambda^2 - \frac{7}{2}\lambda + 1$, and the Alexander polynomial is $\Delta_{8_{13}}(\lambda) = 2\lambda^4 - 7\lambda^3 + 11\lambda^2 - 7\lambda + 2$, so we have $\Delta_{8_{13}}(\lambda) = |K_1 : A_1| p_1(\lambda)$.

The roots of $p_1(\lambda)$ are $\lambda = \frac{1}{8}\big(7 - i\sqrt{7} \pm \sqrt{-22 - 14i\sqrt{7}}\big)$ and $\lambda = \frac{1}{8}\big(7 + i\sqrt{7} \pm \sqrt{-22 + 14i\sqrt{7}}\big)$, so there are no real roots. Passing to $K_2 \cong \mathbb{Z}^6$, we have generators $a_1 := [a, b]$, $a_2 := [a, c]$, $a_3 := [a, d]$, $b_1 := [b, c]$, $b_2 := [b, d]$, and $c_1 := [c, d]$. Then $A_2 = \langle a_1^2, a_2, a_3, b_1^2, b_2^2, c_1\rangle$, and $K_2{*_{\alpha_2}}$ is a subgroup of $\mathbb{Z} \ltimes_{M_2} (\mathbb{Z}[\frac{1}{2}])^6$. Below is M_2 and the generators of A_2 with their respective images under t.

$$a_1^2 = [a, b^2] \overset{t}{\mapsto} a_1^2 b_2^{-2} \qquad a_2 = [a, c] \overset{t}{\mapsto} a_2 a_3 b_1^{-1} b_2^{-1} c_1^{-1}$$

$$a_3 = [a, d] \overset{t}{\mapsto} a_2^{-1} b_1 c_1 \qquad b_1^2 = [b^2, c] \overset{t}{\mapsto} a_2 a_3 b_1 b_2 c_1^{-1}$$

$$b_2^2 = [b^2, d] \overset{t}{\mapsto} a_1^{-2} a_2^{-1} b_1^{-1} b_2^2 c_1 \qquad c_1 = [c, d] \overset{t}{\mapsto} a_2^{-1} a_3^{-1} b_1 b_2 c_1^2$$

$$M_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 & -1 & -1 \\ 0 & -1 & 0 & 1 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 & \frac{1}{2} \\ 0 & -1 & -1 & 1 & 1 & 2 \end{bmatrix}$$

The characteristic polynomial of M_2 is

$$p_2(\lambda) = \lambda^6 - \frac{11}{2}\lambda^5 + \frac{45}{4}\lambda^4 - \frac{27}{2}\lambda^3 + \frac{45}{4}\lambda^2 - \frac{11}{2}\lambda + 1$$
$$= \frac{1}{4}(\lambda-1)^2(4\lambda^2 - (7+\sqrt{29})\lambda + 4)(4\lambda^2 - (7-\sqrt{29})\lambda + 4).$$

The roots of $p_2(\lambda)$ are listed below. Note that the four real roots are all positive, two of which are irrational.

$$\lambda = 1 \text{ (multiplicity 2)},$$
$$\lambda = \frac{1}{8}\big(7+\sqrt{29} \pm \sqrt{14+14\sqrt{29}}\big),$$
$$\lambda = \frac{1}{8}\big(7-\sqrt{29} \pm i\sqrt{14\sqrt{29}-14}\big).$$

4.7. The Knot 8_{15}

8_{15} is not fibered. An HNN extension presentation for $\pi_1(\mathbb{S}^3 \backslash 8_{15})$ is

$$\pi_1(\mathbb{S}^3 \backslash 8_{15}) \;=\; \langle\, t, a, b, c, d \mid tabt^{-1} = a,\ tcb^2t^{-1} = bcba,$$
$$tcbcdt^{-1} = bc^2,\ tdt^{-1} = dc\,\rangle.$$

This is a proper HNN extension. Here $G = \langle a, b, c, d\rangle$ and $A = \langle ab, cb^2, cbcd, d\rangle$. Observe that $|K_1 : A_1| = |K_1 : B_1| = 3$. Abelianizing G in $G*_\alpha$, the resulting group $K_1*_{\alpha_1}$ is a finitely presented subgroup of $\mathbb{Z} \ltimes_{M_1} (\mathbb{Z}[\frac{1}{3}])^4$ where

$$M_1 = \begin{bmatrix} \frac{1}{3} & -1 & -\frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & 1 & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & 0 & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

The characteristic polynomial of M_1 is $p_1(\lambda) = \lambda^4 - \frac{8}{3}\lambda^3 + \frac{11}{3}\lambda^2 - \frac{8}{3}\lambda + 1$, and the Alexander polynomial is $\Delta_{8_{15}}(\lambda) = 3\lambda^4 - 8\lambda^3 + 11\lambda^2 - 8\lambda + 3$, so we have $\Delta_{8_{15}}(\lambda) = |K_1 : A_1| p_1(\lambda)$.

$$\begin{aligned} p_1(\lambda) &= \lambda^4 - \frac{8}{3}\lambda^3 + \frac{11}{3}\lambda^2 - \frac{8}{3}\lambda + 1 \\ &= \frac{1}{3}(\lambda^2 - \lambda + 1)(3\lambda^2 - 5\lambda + 3) \end{aligned}$$

The roots of $p_1(\lambda)$ are $\lambda = \frac{1}{2}(1 \pm i\sqrt{3})$ and $\lambda = \frac{1}{6}(5 \pm i\sqrt{11})$, so there are no real roots. Passing to $K_2 \cong \mathbb{Z}^6$, we have generators $a_1 := [a, b]$, $a_2 := [a, c]$, $a_3 := [a, d]$, $b_1 := [b, c]$, $b_2 := [b, d]$, and $c_1 := [c, d]$. Then $A_2 = \langle\ a_1^2 a_2 b_1,\ a_1 a_2^2 a_3 b_1^2 b_2,\ a_3 b_2,\ b_1^3 b_2^2 c_1,\ b_2^2 c_1,\ b_2 c_1^2\ \rangle$, and $K_2 *_{\alpha_2}$ is a subgroup of $\mathbb{Z} \ltimes_{M_2} (\mathbb{Z}[\frac{1}{3}])^6$. Below is M_2 and the generators of A_2 with their respective images under t.

$$\begin{aligned} &a_1^2 a_2 b_1 = [ab, cb^2] \overset{t}{\mapsto} a_1^2 a_2 && a_1 a_2^2 a_3 b_1^2 b_2 = [ab, cbcd] \overset{t}{\mapsto} a_1 a_2^2 \\ &a_3 b_2 = [ab, d] \overset{t}{\mapsto} a_2 a_3 && b_1^3 b_2^2 c_1 = [cb^2, cbcd] \overset{t}{\mapsto} a_1 a_2^2 b_1^3 \\ &b_2^2 c_1 = [cb^2, d] \overset{t}{\mapsto} a_2 a_3 b_1^2 b_2^2 c_1 && b_2 c_1^2 = [cbcd, d] \overset{t}{\mapsto} b_1 b_2 c_1^2 \end{aligned}$$

$$M_2 = \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & -1 & -1 & 0 \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \\ 0 & \frac{2}{3} & \frac{2}{3} & 1 & 1 & 0 \\ 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 1 \end{bmatrix}$$

The characteristic polynomial of M_2 is

$$\begin{aligned} p_2(\lambda) &= \lambda^6 - \frac{11}{3}\lambda^5 + \frac{55}{9}\lambda^4 - \frac{62}{9}\lambda^3 + \frac{55}{9}\lambda^2 - \frac{11}{3}\lambda + 1 \\ &= \frac{1}{9}(\lambda - 1)^2(9\lambda^4 - 15\lambda^3 + 16\lambda^2 - 15\lambda + 9). \end{aligned}$$

The roots of $p_2(\lambda)$ are listed below. Note that 1 is the only real root.

$$
\begin{aligned}
&\lambda = 1 \text{ (multiplicity 2)},\\
&\lambda = \frac{1}{12}\big(5 + \sqrt{33} \pm i\sqrt{86 - 10\sqrt{33}}\big),\\
&\lambda = \frac{1}{12}\big(5 - \sqrt{33} \pm i\sqrt{86 + 10\sqrt{33}}\big).
\end{aligned}
$$

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