ATOMICITY IN RINGS WITH ZERO DIVISORS

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ABSTRACT

Trentham, Stacy Michelle, Ph D , Department of Mathematics, College of Science and Mathematics, North Dakota State University, March 2011 Atomicity In Rings With Zero Divisors Major Professor Dr James Barker Coykendall IV

In this dissertation, we examine atomicity in rings with zero divisors We begin by examining the relationship between a ring's level of atomicity and the highest level of irreducibility shared by the ring's irreducible elements Later, we choose one of the higher forms of atomicity and identify ways of building large classes of examples of rings that rise to this level of atomicity but no higher Characteristics of the various types of irreducible elements will also be examined Next, we extend our view to include polynomial extensions of rings with zero divisors In particular, we focus on properties of the three forms of maximal common divisors and how a ring's classification as an MCD, SMCD, or VSMCD ring affects its atomicity To conclude, we identify some unsolved problems relating to the topics discussed in this dissertation

DEDICATION

I want to take this opportunity to thank my parents, Terry and Lois Iszler They have always placed great value in education and hard work I remember a time in middle school when I failed an English assignment I had done all of the problems correctly, but I had misread the assignment and had done problems 1-5 instead of problems 1-15 My parents, much to my surprise, were not angry about the low grade Instead, they had me complete the assignment as the teacher had intended and had him check it for accuracy The grade was not important to them, instead, the learning and hard work required to complete the assignment were their priority

As I got older, I began to adopt their values regarding education I was the first member of my family to attend a four year college Upon graduation, I struggled to find a job My parents suggested that I take the GRE and consider going to graduate school I did not know anything about graduate school and thought that it was not an option for me So I took the exam to prove to them that I would not score high enough to get into graduate school When I got my scores, I found that I had done pretty well and shortly thereafter scheduled a visit to the NDSU math department

Without my parent's encouragement, enthusiasm, and high value for education, I would have never entered graduate school and as a result, this dissertation would never have been written This dissertation is as much a product of their hard work and dedication as it is mine!

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CHAPTER 1. INTRODUCTION

The ınterest ın factorızatıon ıs not, by any means, a modern fascınatıon We know that ıdeas of factorızatıon have been floatıng around sınce 300 B C durıng the tıme when Euclıd composed *The Elements* At the begınnıng of Book VII of *The Elements*, a lıst of defınıtıons can be found ıncludıng the defınıtıons for even, odd, prıme, and composıte numbers We also find Euclıd's Algorıthm for findıng the greatest common ıntegral dıvısor of two posıtıve ıntegers ın thıs book Thıs algorıthm and ıts applıcatıons are stıll taught ın contemporary Abstract Algebra courses One of the earlıest results ın factorızatıon ıs the *The Fundamental Theorem of Arıthmetıc*, an equıvalent form ıs found ın Book IX of Euclıd's *The Elements* Thıs theorem states that any ınteger greater than one can be wrıtten unıquely as the product of prıme numbers, up to orderıng [3], [5]

Factorızatıon theory ıs a branch of commutatıve algebra where varıous types of commutatıve rıngs and theır propertıes are studıed These rıngs and theır ıdeals are studıed much lıke a chemıst studıes the molecular structure of a substance We look at the "smallest" components of the rıng (ıf such a thıng exısts) and examıne how these buıld "larger" components We not only look at the structure of these components but also how they ınteract wıth one another vıa addıtıon and multıplıcatıon Much of the research done ın factorızatıon today ıs focused on ıntegral domaıns The defınıtıons and theorems ın thıs chapter can be found ın a varıety of texts such as [6] and [4]

Defınıtıon 1 1 A *rıng* R ıs a nonempty set wıth two bınary operatıons denoted $+$ and $*$ wıth the followıng three propertıes

1 $(R, +)$ ıs an abelıan group

2 $(R, *)$ ıs assocıatıve

3 $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for every $a, b, c \in R$

A ring R is called *commutative* if for each $a, b \in R$ we have that $ab = ba$ If R contains an element 1_R such that $a1_R = 1_Ra$ for each $a \in R$, then R is said to be a ring with identity

In this dissertation, we always will assume that rings are commutative with identity

Definition 1 2 Let R be a ring An element $r \in R$ is called *regular* if $rs = 0$ only when $s = 0$ An element $r \in R$ is called a *zero divisor* if $rs = 0$ for some nonzero $s \in R$

A ring may possess both regular elements and zero divisors For example, in the ring $\mathbb{Z}_6[x]$ the element x is regular and the element 2 is a zero divisor with $(2)(3) = 0$ Particular focus has been put on those commutative rings whose nonzero elements are all regular Such a ring is called an *(integral) domain* We encounter domains on a daily basis The ring consisting of the integers $\mathbb{Z}$, the ring consisting of the rational numbers $\mathbb{Q}$, the ring consisting of the real numbers $\mathbb{R}$, and the ring consisting of the complex numbers $\mathbb{C}$ are all examples of domains We also examine the structure and behavior of a ring's ideals This can give us valuable insight into the factorization properties of the ring We can also use the various types of ideals to generate examples of rings with specific factorization properties

Definition 1 3 Let R be a commutative ring A subset $I \subseteq R$ is an *ideal* of R if I is itself a ring and if for each $x \in I$ and each $r \in R$, the element rx is an element of I

Definition 1 4 An ideal $I \subseteq R$ is called a *principal ideal* if it generated by a single element of R

Definition 1 5 If every ideal of a commutative ring R is a principal ideal, then R

is called a *principal ideal ring (PIR)* Moreover, if R is a domain, then it is called a *principal ideal domain (PID)*

The familiar domain $\mathbb{Z}$ is an example of a PID In this domain, the ideal $I = (6)$ which consists of all integers divisible by 6 is a principal ideal If we generate an ideal with more than one element, say $I = (8, 12)$, then this ideal is the same as the ideal generated by the greatest common divisor of 8 and 12, i e $I = (8, 12) = (4)$ More generally, if an ideal $J \subseteq \mathbb{Z}$ is generated by a finite set S, then $J = (d)$ where d is the greatest common divisor of S That is, any finitely generated ideal in $\mathbb{Z}$ is principal As it turns out, every ideal in $\mathbb{Z}$ is finitely generated

Definition 1 6 A ring is called *Noetherian* if every ideal in the ring is finitely generated

PIR's are special cases of Noetherian rings However, a Noetherian ring need not be a PIR For example, the ring $R = \mathbb{Z}[x, y]$ is a Noetherian domain The ideal $I = (x, y)$ cannot be generated by only one element so R is not a PIR Equivalent definitions of a Noetherian ring exist One such definition is that R is a Noetherian ring if given an ascending chain of ideals $I_1 \subseteq I_2 \subseteq$ there exists an $N \in \mathbb{N}$ such that for every $j, k > N$ we have $I_j = I_k$

Definition 1 7 Consider an ascending chain of principal ideals $I_1 \subseteq I_2 \subseteq$ in R If there exists an $N \in \mathbb{N}$ such that for every $j, k > N$ we have $I_j = I_k$ then we say that R satisfies the *ascending chain condition on principal ideals (ACCP)*

Definition 1 8 Let M be an ideal in a commutative ring R If $M \subseteq I$ for some nontrivial ideal $I \subseteq R$ only when $M = I$, then M is called a *maximal ideal* of R

Definition 1 9 Let $P \subseteq R$ be an ideal Then P is called a *prime ideal* of R if whenever $IJ \subseteq P$ for some ideals $I, J \in R$ we have that either $I \subseteq P$ or $J \subseteq P$

Proposition 1 10 *If M is a maximal ideal in R, then M is a prime ideal in R*

Proof Let $ab \in M$ with $a \notin M$ Then the ideal (M, a) must be R This tells us that $1 - ra \in M$ for some $r \in R$ Now we look at the element $b(1 - ra) = b - rab$ This element is in M so we can say that $b - rab = m$ for some $m \in M$ Thus, $b = m + rab \in M$ and we have that M is prime □

The ideal $I = (2)$ in $\mathbb{Z}$ is a maximal ideal The previous theorem leads us to conclude that $I = (2)$ is also prime While the ideal $J = (3)$ is a prime ideal in the domain $R = \mathbb{Z}[x]$ However, J is not maximal as $J \subsetneq (3, x)$

Definition 1 11 An ideal $I \subseteq R$ is called a *radical ideal* if whenever $x^n \in I$ then $x \in I$ If $J \subseteq R$ is an ideal of R, then the radical of J, written $rad(J)$ is the set $\{x \in R | x^n \in J \text{ for some } n \in \mathbb{N}\}$

Definition 1 12 An ideal $I \subseteq R$ is *primary* if given $ab \in I$, then either $a \in I$ or $b^n \in I$ for some $n \in \mathbb{N}$

Proposition 1 13 *I is a prime ideal in R if and only if I is both radical and primary*

Proof First we will assume that I is both radical and primary Let $ab \in I$ If $a \notin I$, then we know that $b^n \in I$ for some $n \in \mathbb{N}$ Since I is radical, we also have that $b \in I$ Thus, I is prime

Now assume that I is prime and let $ab \in I$ This means that if $a \notin I$, then $b^1 \in I$ so I is primary If $a^n \in I$, then $a \in I$ since I is prime and we have that I is radical □

Let $R = \mathbb{Z}[x]$ The ideal $I = (2x)$ is a radical ideal in R The element $2x$ is in I but neither 2 nor x^n is in I for any $n \in \mathbb{N}$ Thus, I is not primary This tells us that the ideal I is not prime The ideal $J = (8)$ in $\mathbb{Z}$ is a primary ideal However, the element 2^3 is in J but 2 is not in J and we have that J is not radical Thus, J is also not prime

Proposition 1 14 *Let I be a primary ideal in R Then $rad(I)$ is a prime ideal in R*

Proof Let $ab \in rad(I)$ This means that there is a positive integer n such that $(ab)^n = a^n b^n \in I$ So we have that either $a^n \in I$ or $b^k \in I$ where $k = mn$ for some $m \in \mathbb{N}$ That is, either $a \in rad(I)$ or $b \in rad(I)$ So $rad(I)$ is prime □

Definition 1 15 Let R be a commutative ring We say that $a \in R$ is a *nilpotent* element if $a^n = 0$ for some $n \in \mathbb{N}$ We say that the ideal $I \subseteq R$ is *nilpotent* if $I^n = 0$ for some $n \in \mathbb{N}$

If R is a domain, then the only nilpotent element is 0 and the only nilpotent ideal is (0) However, if we look at rings with zero divisors, we find many examples of nilpotent elements and ideals Considering the ring $R = \mathbb{Z}_{64}$ we find that the element 2 is nilpotent since $2^6 = 0$ and $I = (4)$ is a nilpotent ideal since $I^3 = 0$

Theorem 1 16 *Let R be a commutative ring and let I be an ideal in R*

1. *R/I is a field if and only if I is a maximal ideal*

2. *R/I is a domain if and only if I is a prime ideal*

3. *R/I has no nonzero nilpotent elements if and only if I is a radical ideal*

4. *All zero divisors in R/I are nilpotent if and only if I is a primary ideal*

Proof

1. We will begin by assuming that R/I is a field and let J be an ideal such that $I \subsetneq J$ Then there exists an element $a \in J - I$ This means that for some $b \in R$, we have that $ab + I = 1 + I$ or $ab - 1 \in I \subsetneq J$ So there is some element $j \in J$ such that $ab - 1 = j$ but this means that $1 = ab - j \in J$ Thus, $J = R$ and I is maximal

Now assume that I is a maximal ideal of R and choose some nonzero element $a+I \in R/I$ Since $a \notin I$, we know that $(I,a) = R$ So for some $r \in R - I$ and some $i \in I$, we have $i + ra = 1$ Now if we look at $(i+ra)+I = 1+I$, we will see that $ra+I = (r+I)(a+I) = 1+I$ Thus, $a+I$ is a unit and R/I is a field

2 Here we will assume that R/I is a domain and assume that $ab \in I$ This means that $ab + I = 0 + I$ or $(a+I)(b+I) = 0+I$ Since R/I is a domain, we have that $a+I = 0+I$ or $b+I = 0+I$, i e $a \in I$ or $b \in I$ and we have that I is a prime ideal

Next we will begin with I as a prime ideal Let $ab+I = 0+I$ This means that $ab \in I$ Since I is prime, we have that $a \in I$ or $b \in I$ That is, $a+I = 0+I$ or $b+I = 0+I$ and we have that R/I is a domain

3 Let $a^n \in I$ for some $n \in \mathbb{N}$ Here, we are assuming that R/I has no nonzero nilpotent elements so this means that $a^n + I = 0+I$ means that $a+I = 0+I$ Thus, $a \in I$ and I is radical

Let $a^n + I = 0 + I$ where I is a radical ideal Since $a^n \in I$ and I is radical, we have $a \in I$ or $a+I = 0+I$ So R/I has no nonzero nilpotent elements

4 Here we will assume that all zero divisors of R/I are nilpotent Let $ab \in I$ such that $a \notin I$ This means that $ab+I = 0+I$ in R/I with $a+I \neq 0+I$ So $b+I$ is a zero divisor in R/I and must therefore be nilpotent, say $b^n + I = 0+I$ where $n \in \mathbb{N}$ This means that if $a \notin I$, then $b^n \in I$ for some natural number n and we have that I is primary

Lastly, we will assume that I is primary Let $b+I$ be a zero divisor in R/I This means that there is a nonzero element $a+I \in R/I$ such that $ab+I = 0+I$ giving us that $ab \in I$ Since I is primary and $a \notin I$, we have that $b^n \in I$, i e $(b+I)^n = 0+I$ So the zero divisors in R/I are nilpotent $\square$

Our goal is to generalize concepts used to describe domains so that we may use these generalizations to describe rings in general To this end, our focus will be on rings with zero divisors or nondomains We must first agree on definitions for the fundamental ideas commonly used in factorization For example, there are several equivalent definitions for associate elements when working with domains However, before we begin we must first examine these definitions as applied to nondomains to see if they remain equivalent If not, we must fine tune our lexicon to allow us to properly describe rings regardless of the presence of zero divisors This will be the focus of our next section

CHAPTER 2. DEFINITIONS

A domain is *atomic* if every nonzero, nonunit can be written as a finite product of irreducibles To generalize this definition, we begin by replacing the word "domain" with "ring" However, this raises a new question, "What is an irreducible in a nondomain?" An *irreducible* in a domain is an element x such that whenever $x = yz$ then x is associate to either y or z To properly generalize this definition, we must first revisit the definition for associate elements We continue to assume that rings are commutative with identity $1_R \neq 0_R$

Theorem 2 1 *Let D be an integral domain with nonzero elements a and b The following statements are equivalent*

1 $a \mid b$ and $b \mid a$

2 There exists a unit $u \in D$ such that $a = ub$

3 If we have $a \mid b$, $b \mid a$, and $a = bc$, then c must be a unit in D

Proof Clearly, $3 \Rightarrow 2 \Rightarrow 1$ So it suffices to show that $1 \Rightarrow 3$ If $a \mid b$, $b \mid a$, and $a = bc$, then there exists a nonzero element $d \in D$ such that $ad = b$ This means that $a = adc$ or $a(1 - dc) = 0$ Because a is nonzero, we know that $1 - dc = 0$ or $dc = 1$ Thus, both c and d are units in D □

If two elements $a, b \in D$ satisfy one, hence all of these properties, then we say that a and b are associates in D If we remove the domain restriction, then the three statements are no longer equivalent If two elements in a ring R satisfy the first statement, then we say that these elements are *associates* ($\sim$) Two elements that satisfy the second condition are called *strong associates* ($\approx$) Lastly, elements that satisfy the third statement are called *very strong associates* ($\cong$) We also define 0 to be very strongly associate to itself It is easily verified that very strong associates $\Rightarrow$

strong associates $\Rightarrow$ associates It is worth noting that none of these implications can be reversed [1]

Example 2 2 Let $R = \mathbb{Z}_6 \times \mathbb{Z}_9$ Notice that $(2,2) = (5,8)(4,7)$ where $(5,8)$ is a unit in R So $(2,2) \approx (4,7)$ Also, $(2,2) = (2,8)(4,7)$ where $(2,8)$ is not a unit in R Thus, $(2,2) \not\cong (4,7)$

Example 2 3 Let $R = \dfrac{\mathbb{Q}[x,y]}{(x - xy^2)}$ In R, $x = xy^2$ so $x \sim xy$ so there exists z such that $xz = xy$ Assume that z is a unit in R Then $xz - xy = rx - rxy^2 \in \mathbb{Q}[x,y]$ for some $r \in \mathbb{Q}[x,y]$ Since x is prime, we have $z - y = r - ry^2$ and $z = y + r - ry^2$ If z is a unit in R, then $(z, x - xy^2) = \mathbb{Q}[x,y]$, i e $1 = az + b(x - xy^2) = ay + ar - ary^2 + bx - bxy^2$ Note that $\mathbb{Q}[x,y]$ is a domain, so we must have $ar = 1$ and $ay - y^2 + bx - bxy^2 = 0$ This means that both a and r are units in $\mathbb{Q}[x,y]$ so they are elements of $\mathbb{Q}$ So $b \in (y)$ and $y(a - y^2) \in (x)$ We know that $y \notin (x)$ so $a - y^2 \in (x)$ and $ar - ry^2 = 1 - ry^2 \in (x)$ Since $ry^2 \in (y)$, this gives us that $(x,y) = \mathbb{Q}[x,y]$, a contradiction So there is no unit u in R such that $x = uxy$ which means $x \not\approx xy$ A similar example can be found in [1]

In domains, we have two equivalent definitions for irreducible elements We know that a is irreducible in a domain D if given $a = bc$, then b is a unit or c is a unit in D The three levels of associate elements along with this definition give us three types of irreducible elements Equivalently, a is irreducible in a domain D if and only if the ideal $I = (a)$ is maximal among all principal ideals of D Using this definition for an irreducible element, we find that there is also a fourth type of irreducible that exists in rings with zero divisors

Definition 2 4 [1] Let $a \in R$ be a nonunit We say that a is *irreducible* if $a = bc$ implies that $a \sim b$ or $a \sim c$ Equivalently, a is *irreducible* if $(a) = (b)$

Definition 2 5 [1] Let $a \in R$ be a nonunit We say that a is *strongly irreducible* if $a = bc$ implies that $a \approx b$ or $a \approx c$

Definition 2 6 [1] Let $a \in R$ be a nonunit We say that a is *very strongly irreducible* if $a = bc$ implies that $a \cong b$ or $a \cong c$

Definition 2 7 [1] Let $a \in R$ be a nonunit We say that a is *m-irreducible* if (a) is maximal among proper principal ideals

In domains, these four definitions are equivalent We must now show that when we generalize to include nondomains, these are four unique levels of irreducibles Note that for nonzero elements of R, very strongly irreducible $\Rightarrow$ m-irreducible $\Rightarrow$ strongly irreducible $\Rightarrow$ irreducible [1]

Definition 2 8 Let $a \in R$ We say that a is *prime* if the ideal (a) is prime ideal Equivalently, we say that a is *prime* if $a \mid xy$ implies that $a \mid x$ or $a \mid y$

Proposition 2 9 *If $a \in R$ is prime, then a is irreducible*

Proof Let a be prime in R and assume that $a = xy$ for some $x, y \in R$ This means that either $x \in (a)$ or $y \in (a)$ That is, either $a \sim x$ or $a \sim y$ Thus, a is irreducible □

When defining new classifications of elements, we must verify that each class is nonempty and unique We know that prime elements are irreducible but we have yet to determine whether or not irreducible elements are prime Let $R = \mathbb{Z}[\sqrt{-3}]$ Then $1 + \sqrt{-3}$ is irreducible but not prime Thus, the class of prime elements and the class of irreducible elements are distinct Similarly, we can show that the remaining classes of irreducibles are unique by providing examples to show that the implications above cannot be reversed First, let $R = \dfrac{\mathbb{Q}[x, y]}{(x - xy^2)}$ Notice that x is prime so it is irreducible However, considering x and xy we know that $x = xy^2$ and $x \sim xy$

but there is no unit $u \in R$ such that $x = u(xy)$ So $x \not\approx xy$ Clearly, $x \nmid y$ so $x \nsim y$ That is, x is irreducible but not strongly irreducible Now, let $R = \mathbb{Z} \times \mathbb{Q}$ If $(0,5) = (a,b)(c,d)$, then either $a = 0$ or $c = 0$ in $\mathbb{Z}$ with both b and d being units in $\mathbb{Q}$ That is, either (a,b) or (c,d) is a unit multiple of $(0,5)$ So $(0,5)$ is strongly irreducible However, if we let $I =< (0,5) >$ and $J =< (2,5) >$, then $I \subsetneq J$ So $(0,5)$ is strongly irreducible but not m-irreducible Lastly, let $R = \mathbb{Z}_6$ Clearly, (3) is maximal among principal ideals so 3 is m-irreducible However, $3 = (3)(3)$ but 3 is not a unit in R So 3 is m-irreducible but not very strongly irreducible

Using these four levels of irreducible elements along with primes, we find that nondomains may come in five different flavors of atomic

Definition 2 10 [1] R is *atomic* if every nonzero, nonunit can be written as a finite product of irreducibles

Definition 2 11 [1] R is *strongly atomic* if every nonzero, nonunit can be written as a finite product of strong irreducibles

Definition 2 12 [1] R is *m-atomic* if every nonzero, nonunit can be written as a finite product of m-irreducibles

Definition 2 13 [1] R is *very strongly atomic* if every nonzero, nonunit can be written as a finite product of very strong irreducibles

Definition 2 14 [1] R is *p-atomic* if every nonzero, nonunit can be written as a finite product of primes

It is easily shown that very strongly atomic $\Rightarrow$ m-atomic $\Rightarrow$ strongly atomic $\Rightarrow$ atomic In [1], the following theorems were introduced Using Theorem 2 16, we are able to use familiar rings to construct examples to show that the various levels of atomicity are indeed unique We credit this theorem with many of the examples given in this dissertation

Theorem 2 15 *[1] Let* $\{R_\alpha\}_{\alpha\in\Lambda}$ *be a famıly of commutatıve rıngs and* $R = \prod_{\alpha\in\Lambda} R_\alpha$ *Consıder the elements* $a = (a_\alpha), b = (b_\alpha) \in R$

1 $a \sim b \Leftrightarrow a_\alpha \sim b_\alpha$ *for each* $\alpha \in \Lambda$, $a \approx b \Leftrightarrow a_\alpha \approx b_\alpha$ *for each* $\alpha \in \Lambda$ *and ıf some* $a_\beta = 0$, *then* $a = 0$

2 *a ıs ırreducıble (respectıvely, strongly ırreducıble, m-ırreducıble, prıme)* $\Leftrightarrow$ *each* $a_\alpha \in U(R_\alpha)$ *except for one* $\beta \in \Lambda$ *where* a_β *ıs ırreducıble (respectıvely, strongly ırreducıble, m-ırreducıble, prıme) ın* R_β

3 *a ıs very strongly ırreducıble* $\Leftrightarrow$ *each* $a_\alpha \in U(R_\alpha)$ *except for one* $\beta \in \Lambda$ *where* a_β *ıs very strongly ırreducıble ın* R_β *but ıs not* 0 *unless* $\mid \Lambda \mid= 1$ *and* R_β *ıs a domaın*

Proof 1 Fırst we wıll assume that $a \sim b$ Thıs means that $ac = b$ for some $c = (c_\alpha)$ and $a = bd$ for some $d = (d_\alpha)$ So for each $\alpha \in \Lambda$, we have $a_\alpha c_\alpha = b_\alpha$ and $a_\alpha = b_\alpha d_\alpha$ for $c_\alpha, d_\alpha \in R_\alpha$ Thus, $a_\alpha \sim b_\alpha$ for all $\alpha \in \Lambda$

Now assume that $a_\alpha \sim b_\alpha$ for each $\alpha \in \Lambda$ So there exısts $c_\alpha, d_\alpha \in R_\alpha$ such that $a_\alpha c_\alpha = b_\alpha$ and $a_\alpha = b_\alpha d_\alpha$ That ıs, $ac = b$ and $a = bd$ where $c = (c_\alpha)$ and $d = (d_\alpha)$ and we have $a \sim b$

Let $a \approx b$ So there exısts some unıt $u = (u_\alpha) \in R$ such that $a = ub$ Thıs means that $a_\alpha = u_\alpha b_\alpha$ where u_α ıs a unıt ın R_α and thus, $a_\alpha \approx b_\alpha$ for all $\alpha \in \Lambda$

If $a_\alpha \approx b_\alpha$ for all $\alpha \in \Lambda$, then there exısts some unıt $u_\alpha \in R_\alpha$ such that $a_\alpha = u_\alpha b_\alpha$ That ıs, $a = ub$ where $u = (u_\alpha)$ and $a \approx b$

Here we wıll begın by assumıng that $a \cong b$ So eıther $a = b = 0$ or ıf $a = bc$ then

$c \in U(R)$ This gives us that either $a_\alpha = b_\alpha = 0$ for all $\alpha \in \Lambda$ or if $a_\alpha = b_\alpha d_\alpha$ then $a = bd$ where $d = (d_\alpha)$ so $d \in U(R)$ That is, each $d_\alpha \in U(R_\alpha)$ and we have $a_\alpha \cong b_\alpha$ If $a_\beta = 0$ for some $\beta \in \Lambda$, then we have $0 = b_\beta d_\beta$ If $a \neq 0$, then we must have that $d_\beta \in U(R_\beta)$ which means that $b_\beta = 0$ However, if $a_\beta = b_\beta = 0$, then $a_\beta = b_\beta x$ for any $x \in R_\beta$ This gives us nonunit elements $c \in R$ such that $a = bc$ where $a \neq 0$, a contradiction So if $a_\beta = 0$, then we must have $a = 0$

Next we assume that $a_\alpha \cong b_\alpha$ and if $a_\alpha = 0$ for some $\alpha \in \Lambda$, then $a = 0$ This means that if $a_\alpha = 0$ for some α, then $a = b = 0$ and we have $a \cong b$ If $a_\alpha \neq 0$ for all α and $a = bc$ for some $c \in R$, then $a_\alpha = b_\alpha c_\alpha$ for all α So each c_α is a unit in R_α and thus, c is a unit in R and we have that $a \cong b$

2 Let $a = (a_\alpha) \in R$ be irreducible This means that $a_\imath$ is a nonunit in $R_\imath$ for some $\imath \in \Lambda$ If $a_\imath = b_\imath c_\imath$, then we can say that $a = bc$ where $b = (\widehat{b}_\alpha)$ and $c = (\widehat{c}_\alpha)$ with $\widehat{b}_\alpha = 1$ if $\alpha \neq \imath$, $\widehat{b}_\alpha = b_\imath$ if $\alpha = \imath$, $\widehat{c}_\alpha = a_\alpha$ if $\alpha \neq \imath$, $\widehat{c}_\alpha = c_\imath$ if $\alpha = \imath$ Since a is irreducible, we know that either $a \sim b$ or $a \sim c$ and we have that either $a_\alpha \sim \widehat{b}_\alpha$ or $a_\alpha \sim \widehat{c}_\alpha$ for all $\alpha \in \Lambda$ More specifically, $a_\imath \sim \widehat{b}_\imath = b_\imath$ or $a_\imath \sim \widehat{c}_\imath = c_\imath$ and we have that $a_\imath$ is irreducible

Now consider $a_\jmath$ where $\jmath \neq \imath \in \Lambda$ Let $\bar{b} = (\bar{b}_k)$ and $\bar{c} = (\bar{c}_k)$ where $\bar{b}_k = b_\alpha$ and $\bar{c}_k = c_\alpha$ if $k \neq \imath, \jmath$, $\bar{b}_k = a_\jmath$ and $\bar{c}_k = 1$ if $k = \jmath$, and finally, $\bar{b}_k = 1$ and $\bar{c}_k = a_\imath$ if $k = \imath$ Recall that we are assuming a is irreducible and we now have $a = \bar{b}\bar{c}$ So either $a \sim \bar{b}$ or $a \sim \bar{c}$, i e $a_\imath \sim \bar{b}_\imath = 1$ or $a_\jmath \sim \bar{c}_\jmath = 1$ That is, either $a_\imath$ or $a_\jmath$ is a unit Since $a_\imath$ is irreducible, it cannot be a unit and therefore, $a_\jmath$ must be a unit for all $\jmath \neq \imath$

Let $\imath \in \Lambda$ and $a = (a_\alpha)$ where $a_\alpha = 1$ for $\alpha \neq \imath$ and $a_\imath$ irreducible in $R_\imath$ Now assume that $a = bc$ with $b = (b_\alpha)$ and $c = (c_\alpha)$ This gives us that $a_\alpha = b_\alpha c_\alpha$ So we have that b_α and c_α are units for $\alpha \neq i$ and $a_i \sim b_i$ or $a_i \sim c_i$ That is, $a \sim b$ or $a \sim c$ so a is irreducible

The proofs for the strongly irreducible, m-irreducible, and prime cases are very similar and are left to the reader

3 Let $a = (a_\alpha)$ be very strongly irreducible in R This gives us that a is irreducible and hence, $a_\alpha = 1$ for all α except one, call it a_β, which is irreducible in R_β If we assume $a = bc$, then we know that either $a \cong b$ or $a \cong c$ That is, either b or c is a unit in R So $a_\beta = b_\beta c_\beta$ where either b_β or c_β is a unit Now we have that either $a_\beta \cong b_\beta$ or $a_\beta \cong c_\beta$ and a_β is very strongly irreducible in R_β

Assume $a_\beta = 0$ and recall that a_β is very strongly irreducible so it is also m-irreducible This means that $(a_\beta) = (0)$ is maximal among principal ideals Now let I be an ideal in R_β such that $(a_\beta) \subsetneq I$ This means that for any nonzero element $x \in I$ we have $(a_\beta) \subseteq (x) \subsetneq I$ However, (a_β) is maximal among principal ideals so $(a_\beta) = (x)$, a contradiction Hence, $(a_\beta) = (0)$ is a maximal ideal in R_β and R_β is a domain

Let a be very strongly irreducible where $a_\beta = 0$ We know that $a \sim a$ and if $a = ak$ for some $k \in R$, then either a or k is a unit Since a is very strongly irreducible, we know that a is not a unit This means that k is a unit and $a \cong a$ Also, if $a_\beta = 0$, then we know from 1 that each $a_\alpha = 0$ Now if $\mid \Lambda \mid > 1$, then we can write $a = bc$ where $b = (b_\alpha)$ and $c = (c_\alpha)$ with $b_\alpha = 0$ for all $\alpha \in \Lambda$ and c_α is a nonunit for some α Notice that neither b nor c is a unit However, since

a is very strongly irreducible, we must have that either b or c is a unit in R and we have reached a contradiction Thus, $\mid \Lambda \mid = 1$

□

Theorem 2 16 *[1] Let $\{R_\alpha\}_{\alpha \in \Lambda}$ be a family of commutative rings, and let $R = \Pi R_\alpha$ If R satisfies ACCP or any of the forms of atomicity, then Λ is finite*

Let $R_1, R_2, \quad R_n$ be commutative rings and $R = R_1 \times R_2 \times \quad \times R_n$

1 *R satisfies ACCP (respectively, is atomic, strongly atomic, p-atomic) if and only if each R_i satisfies ACCP (respectively, is atomic, strongly atomic, p-atomic)*

2 *R is m-atomic if and only if each R_i is m-atomic and if $n > 1$ and some R_i is a domain, then R_i must be a field*

3 *R is very strongly atomic if and only if each R_i is very strongly atomic and if some R_i is a domain we must have $n = 1$*

Proof Note that if R is an atomic ring with zero divisors, then $0 = ab$ for some $a, b \in R$ So we can write 0 as a finite product of irreducible elements Now if R is a domain and $0 = ab$, then we have that $0 \sim a$ or $0 \sim b$ so 0 is irreducible So if R is ACCP or any form of atomic, then we must have that 0 can be written as a finite product of irreducible elements in R From the previous theorem, we can see that if Λ is infinite, then any finite product of irreducible elements must be nonzero So we must have that Λ is finite

For the remainder of this proof, we will assume that $\Lambda = n$ and $R = R_1 \times R_2 \times \quad \times R_n$

1 We know that the principal ideals of R are all ideals of the form $I_1 \times I_2 \times \quad \times I_n$ where each I_α is principal in R_α Now assume that $J_1 \subseteq J_2 \subseteq$ is an ascending chain of principal ideals in R_1 We will call this Chain 1 This gives us an ascending chain of principal ideals $J_1 \times R_2 \times \quad \times R_n \subseteq J_2 \times R_2 \times \quad \times R_n \subseteq$

in R which we will call Chain 2 Since R is ACCP, we know that Chain 2 must stabilize This means that Chain 1 must also stabilize and thus, we have that R_1 is also ACCP Similarly, each R_i must also be ACCP

Now we will assume that each R_i is ACCP Let $I_{1,1} \times I_{2,1} \times \quad I_{n,1} \subseteq I_{1,2} \times I_{2,2} \times \quad I_{n,2} \subseteq$ be an ascending chain of principal ideals in R Since each R_i is ACCP, we know that each of the chains $I_{i,j} \subseteq I_{i,j+1} \subseteq$ must stabilize Thus, our original chain of principal ideals must also stabilize and we have that R is ACCP

We know that $r = (r_i) \in R$ is irreducible (respectively, strongly irreducbile, prime) if and only if each r_i is a unit in R_i except one, say r_j, which must be irreducible (respectively, strongly irreducible, prime) in R_j From this we can conclude that R is atomic (respectively, strongly atomic, p-atomic) if and only if each R_i is atomic (respectively, strongly atomic, p-atomic)

2 First we will assume that R is m-atomic Notice that the element (a_i) where $a_i = 1$ for $i \neq j$ and a_j is a nonzero, nonunit in R_j can be written as a finite product of m-irreducibles in R This gives us a factorization of a_j into a finite product of m-irreducibles in R_j Thus, every nonzero, nonunit in R_j can be written as a finite product of m-irreducibles in R_j and R_j is m-atomic Also, if R_j is a domain, then 0 must be m-irreducible and hence, R_j is a field

If each R_i is m-atomic, then each element of the form (a_i) where $a_i = 1$ for $i \neq j$ and a_j is a nonzero, nonunit in R_j can be written as a finite product of m-irreducibles in R Also, every nonzero, nonunit in R can be written as a finite product of elements of this form Thus, R must also be m-atomic

3 Assume that R is very strongly atomic Recall that $a = (a_\alpha)$ is very strongly irreducible if and only if each a_α is a unit in R_α except one, call it a_i which must be very strongly irreducible in R_i and cannot be zero unless $n = 1$ and R_i is a domain By this we see that each R_i must be very strongly atomic Now assume that R_j is a domain for some j If $n > 1$, then we see that the element (x_i) where $x_i = 1$ if $i \neq j$ and $x_j = 0$ is irreducible but not very strongly irreducible, a contradiction since 0 is very strongly irreducible in R_j This means that if R_j is a domain for any $1 \leq j \leq n$, then $n = 1$

Now assume that each R_i is very strongly atomic If $n = 1$ and $R = R_1$ is a domain, then R is very strongly atomic So we will assume that $n > 1$ and each R_i is not a domain Then if $a \in R_i$ is nonzero and very strongly irreducible, we have that $(1, \quad , 1, a, 1, \quad , 1)$ is very strongly irreducible in R Notice that every element of R can be written as a finite product of these types of elements Thus, R is very strongly atomic □

Example 2 17 Let $R = \dfrac{\mathbb{Q}[x, y]}{(x - xy^2)}$ Then R is atomic but not strongly atomic R is Noetherian so it is atomic However, as we will see in the next chapter, because $x \in R$ is irreducible but not strongly irreducible, we know that R cannot be strongly atomic Now if we let $R = \mathbb{Z} \times \mathbb{Q}$, then R is strongly atomic but not m-atomic by Theorem 2 16 Using this same theorem, if we let $R = \mathbb{Z}_6$, then R is m-atomic but not very strongly atomic

The following theorems provide us with the tools we need to show that if R is p-atomic, then R is both strongly atomic and ACCP Recall that in domains, if R is ACCP, then R is atomic This implication remains true when the domain condition is removed

Definition 2 18 [1] A principal ideal ring (PIR) is called a *special principal ideal ring (SPIR)* if it has only one proper prime ideal P and $P^2 = 0$

Theorem 2 19 *[1] For a commutative ring R, the following statements are equivalent*

1 *R is p-atomic*

2 *R is a finite direct product of SPIRs and UFDs*

3 *Every (nonzero) proper principal ideal of R is a product of principal prime ideals*

Proposition 2 20 *If R is a SPIR, then R is very strongly atomic*

Proof We know that if R is a SPIR, then R is ACCP and hence, atomic Let $M = (m)$ be the unique maximal ideal We wish to show that m is irreducible We know that $M^2 = 0$ Now let $a \in R$ be a nonzero irreducible element This means that $a \in M$ so we have $a = rm$ for some $r \in R$ Since a is irreducible, we have that either $a \sim r$ or $a \sim m$ If $a \sim r$, then $r \in M$ and $ab = r$ for some $b \in R$ We now have $a = rm = abm = rmbm = ab^2m^2 = 0$ However, we know that a is nonzero so we must have $a \sim m$ This means that $ad = m$ for some $d \in R$ Now we will again look at our original factorization of a So we have $a = rm = rad = rrmd = r^2ad^2$ Now $r^2ad^2 = 0$ if either r or d is a nonunit Since a is nonzero, we know that r and d must both be units giving us that $a \cong m$ So m is also irreducible

Now we wish to show that m is very strongly irreducible so we assume that $m = st$ This means that either $m \sim s$ or $m \sim t$ Without loss of generality, we will assume that $m \sim t$ So for some $x \in R$, we have $mx = t$ This gives us $m = st = smx = sstx = s^2mx^2$ Now $s^2mx^2 = 0$ if either s or x is a nonunit Thus, s must be a unit in R and we have that m is very strongly irreducible Since $a \cong m$, this means that a is also very strongly irreducible □

Proposition 2 21 *[1] If R is p-atomic, then R is strongly atomic*

Proof Since R is a finite direct product of SPIRs and UFDs, we know that it is a finite direct product of very strongly atomic rings We will say that $R = R_1 \times R_2 \times \quad \times R_n$ If each R_j is not a domain or if $n = 1$, then R is very strongly atomic If $n > 1$ and each R_j is either a field or a nondomain SPIR, then R is m-atomic If any one of the rings R_j is a domain but not a field, then R is strongly atomic □

The following diagram shows the relationships between the various forms of atomicity

$$\begin{array}{ccccccc} & & & & \textbf{P-Atomic} & & \\ & & & & \Downarrow & & \\ \textbf{Very Strongly Atomic} & \Longrightarrow & \textbf{M-Atomic} & \Longrightarrow & \textbf{Strongly Atomic} & \Longrightarrow & \textbf{Atomic} \end{array}$$

We would like to show that the class of p-atomic rings does not coincide with another class of atomic ring If we let $R = \mathbb{Z}_4 \times \mathbb{Z}$, then R is p-atomic but not m-atomic Next we let $R = \mathbb{Z}[\sqrt{-3}] \times \mathbb{Z}$ Then R is strongly atomic but not p-atomic In our next chapter, we will dig a little deeper to uncover additional properties of the rings and elements identified in this chapter

CHAPTER 3. THEOREMS

Now that we have identified these five types of atomicity and have verified that they are unique, we want to know, "Given a ring, how do we identify its level of atomicity?" The atomicity of some rings can be identified using Theorem 2 16 However, this theorem may always not be useful We strive to identify additional methods for determining a ring's atomicity Also, we will examine some of the behavior of rings with various levels of atomicity

As we have seen in the previous chapter, when working with nondomains we cannot make any assumptions, no matter how logical they may seem We will begin by verifying whether or not a unit multiple of an element will retain the irreducibility/prime status of the original element

Proposition 3 1 *Let a be irreducible (respectively strongly irreducible, m-irreducible, very strongly irreducible, prime) in R and u a unit in R Then ua is irreducible (respectively strongly irreducible, m-irreducible, very strongly irreducible, prime) in R*

Proof Let a be irreducible in R and $\alpha = ua$ where u is a unit in R Assume that $\alpha = xy$ for some x and y in R Then $a = (u^{-1}x)y$ So either $a \sim u^{-1}x$ or $a \sim y$ If $a \sim u^{-1}x$, then $ab = u^{-1}x$ for some b in R That is, $uab = \alpha b = x$ and $\alpha \sim x$ If $a \sim y$, then $ab = y$ for some b in R That is, $ua(u^{-1}b) = \alpha u^{-1}b = y$ and $\alpha \sim y$ Thus, α is irreducible

Let a be strongly irreducible in R and $\alpha = ua$ where u is a unit in R Assume that $\alpha = xy$ for some x and y in R Then $a = (u^{-1}x)y$ So either $a \approx u^{-1}x$ or $a \approx y$ If $a \approx u^{-1}x$, then $ab = u^{-1}x$ for some unit b in R That is, $uab = \alpha b = x$ and $\alpha \approx x$ If $a \approx y$, then $ab = y$ for some unit b in R That is, $ua(u^{-1}b) = \alpha u^{-1}b = y$ and $\alpha \approx y$ Thus, α is strongly irreducible

Let a be m-irreducible in R and $\alpha = ua$ where u is a unit in R Then $(a) = (\alpha)$ which is maximal among principal ideals Thus, α is m-irreducible

Let a be very strongly irreducible in R and $\alpha = ua$ where u is a unit in R Assume that $\alpha = xy$ for some $x, y \in R$ Then $a = (u^{-1}x)y$ So either $a \cong u^{-1}x$ or $a \cong y$ If $a \cong u^{-1}x$, then y is a unit in R and $\alpha \cong x$ If $a \cong y$, then $u^{-1}x$ is a unit in R so x is a unit in R and $\alpha \cong y$ Thus, α is very strongly irreducible

Let a be prime in R and $\alpha = ua$ where u is a unit in R Then $(a) = (\alpha)$ which is a prime ideal Thus, α is prime □

Another matter of great interest is whether or not a ring's atomicity status has any relationship with the level of irreducibility reached by its irreducible elements Must the ring's atomicity status agree with the highest level of irreducibility shared by all irreducible elements? For example, can a very strongly atomic ring contain an irreducible element that attains no higher level of atomicity?

Theorem 3 2 *If R is very strongly atomic, then a is irreducible if and only if a is very strongly irreducible*

Proof Clearly, if a is very strongly irreducible, then a is irreducible So it suffices to show that if R is very strongly atomic, then each irreducible is very strongly irreducible

Let a be irreducible in R Since R is very strongly atomic, we can write a as a finite product of very strong irreducibles, say $a = \alpha_1\alpha_2 \quad \alpha_n$ where each α_i is very strongly irreducible Now a is irreducible, so without loss of generality $a \sim \alpha_1$ That is, $ab = \alpha_1$ for some b in R but α_1 is very strongly irreducible so b must be a unit Thus, a is very strongly irreducible □

Theorem 3 3 *If R is strongly atomic, then a is irreducible if and only if a is strongly irreducible*

Proof If a is strongly irreducible then a is irreducible so we will assume that a in R is irreducible and we can write $a = \alpha_1\alpha_2 \quad \alpha_n$ where each α_i is strongly irreducible Since a is irreducible, we know that $a \sim \alpha_j$ for some j Without loss of generality, we will say that $a \sim \alpha_1$ This means that $\alpha_1 = ak$ for some k in R Since α_1 is strongly irreducible, we have that either $\alpha_1 = ua$ or $\alpha_1 = vk$ for some units u and v in R If $\alpha_1 = ua$, then a is strongly irreducible and we are done So we assume that $\alpha_1 = vk$ This means that $(a) = (\alpha_1) = (k) = (a)(k) = (\alpha_1)^2 = (a)^2 = (k)^2$ More specifically, $(k) = (k)^2$ and we have $k = rk^2$ for some $r \in R$ Also, rk is idempotent since $(rk)^2 = rk^2r = rk$ Now we let $I = (rk) = (\alpha_1) = (a)$ and $J = (1 - rk)$ be ideals in R Notice that I and J are comaximal

Let $f \quad R \to R/I \times R/J$ be given by $a \mapsto (\overline{a}, \widehat{a})$ where $\overline{a}$ represents the coset $a + I$ and $\widehat{a}$ represents the coset $a + J$ The map f is a well-defined homomorphism Let $x \in R$ be such that $f(x) = (0,0)$ This means that $x \in I \bigcap J$ So $x = m(rk) = n(1 - rk)$ for some $m, n \in R$ and we have that $(m + n)rk = n$ which gives us $n \in I$ We will say $n = trk$ for some $t \in R$ Now we have $x = trk(1 - rk) = trk - t(rk)^2 = trk - trk = 0$ Thus, f is injective Now let $(\overline{m}, \widehat{n}) \in R/I \times R/J$ Notice that $f(n + (m - n)(1 - rk)) = (\overline{m}, \widehat{n})$ So f is bijective Thus, $R \cong R/I \times R/J$

We know that $a \sim rk$ so for some $b \in R$ we have $rk = ab$ This gives us that $f(ab) = f(a)f(b) = f(rk) = (0,1)$ That is, $(0, \widehat{a})(\overline{b}, \widehat{b}) = (0,1)$ and we have that $\widehat{a}$ is a unit in R/J Similarly, $f(\alpha_1) = (0, \widehat{\alpha_1})$ where $\widehat{\alpha_1}$ is a unit in R/J Now we have $f(a) = (0, \widehat{a}) = (1, \widehat{a})(0,1) = (1, \widehat{a})f(rk)$ Let $f^{-1}((1, \widehat{a})) = y$ in R We wish to show that y is a unit Since $f(y) = (\overline{y}, \widehat{y}) = (1, \widehat{a})$, we have $yz + I = 1 + I$ and $yw + J = 1 + J$ for some $w, z \in R$ This means that there exists $s, t \in R$ such that $yz = 1 + srk$ and $yw = 1 + t(1 - rk)$ So $yw(srk) = (1 + t - trk)(srk) = srk$ and $yz = 1 + ywrsk$, i e $y(z - wrsk) = 1$ and $y \in U(R)$ We now have $a = yrk$ where $y \in U(R)$ Similarly, $\alpha_1 = zrk$ for some $z \in U(R)$ Thus, $a = yz^{-1}(zrk) = (yz^{-1})\alpha_1$

So a is strongly irreducible □

Theorem 3 4 *If R is m-atomic, then a is irreducible if and only if a is m-irreducible*

Proof Clearly if a is m-irreducible, then a is irreducible We need to show that if a irreducible, then a is m-irreducible

Let a be irreducible in R Since R is m-atomic, a can be written as a finite product of m-irreducibles, say $a = \alpha_1\alpha_2 \quad \alpha_n$ where each α_i is m-irreducible Now a is irreducible, so without loss of generality, $a \sim \alpha_1$ That is, $(a) = (\alpha_1)$ Since α_1 is m-irreducible, $(\alpha_1) = (a)$ is maximal among principal ideals Thus, a is m-irreducible □

Theorem 3 5 *If R is p-atomic, then a is irreducible if and only if a is prime*

Proof It suffices to show that an irreducible a is also prime

Let a be irreducible in R Since R is p-atomic, a can be written as a finite product of primes, say $a = p_1p_2 \quad p_n$ where each p_i is prime Now a is irreducible, so without loss of generality, $a \sim p_1$ That is, $(a) = (p_1)$ Since p_1 is prime, $(p_1) = (a)$ is a prime ideal Thus, a is prime □

It is important to point out that the irreducibles of a ring with a particular form of atomicity will always fall into the corresponding class of irreducible However, this does not mean that the ring may not contain irreducibles from a "higher" class For example, if we let $R = \mathbb{Z} \times \mathbb{Z}$, then R is strongly atomic and has no higher form of atomicity However, all elements of the form $(p, 1)$ and $(1, p)$ where p is prime in $\mathbb{Z}$ are both very strongly irreducible and prime The elements $(1, 0)$ and $(0, 1)$ are only strongly irreducible but also prime Now let $R = \mathbb{Z}_4 \times \mathbb{Z}_6$ Notice that R is m-atomic but has no higher form of atomicity but the element $(2, 1)$ is very strongly irreducible in R

To assure ourselves that the classes of atomic rings we are studying are nonempty, we look for methods of generating examples One such method has been shown in Theorem 2 16 Another possible way of generating examples is by looking at classes of domains R along with specific types of ideals I and examining the atomic structure of R/I

Theorem 3 6 *Let R be a Noetherian domain and $I \subseteq R$ a primary ideal Then R/I is very strongly atomic*

Proof R is Noetherian and hence ACCP Thus, R/I is also ACCP and hence atomic Let $a + I$ be irreducible in R/I and assume that $a + I = bc + I$ Without loss of generality, we have that $ad + I = b + I$ for some $d + I \in R/I$ This gives us $a + I = acd + I$ or, equivalently, $a(1 - cd) + I = 0 + I$ Since $a \notin I$ and I is primary, we have that $(1 - cd)^n \in I$ for some n This means that for some $x \in R$, the element $1 - cx \in I$ So c is a unit in R/I and $a + I$ is very strongly irreducible Therefore, R is very strongly atomic □

This theorem remains true if we let R be any ring such that R/I is atomic It is also important to note that the converse does not hold true A ring R may be Noetherian and R/I may be very strongly atomic for some ideal I in R However, I need not be primary For example, let $R = \mathbb{Z}$ and $I = (900) = (4)(9)(25)$ with (4), (9), and (25) pairwise comaximal and primary Then $R/I \cong R/(4) \times R/(9) \times R/(25)$ is very strongly atomic but I is not primary

What happens if R is Noetherian domain and I is a product of primary ideals? We have seen that R/I may be very strongly atomic but we wish to know if this will always be the case Is there a Noetherian domain R with an ideal I that is a product of primary ideals such that R/I is no longer very strongly atomic?

Theorem 3 7 *Let R be a Noetherian domain and $I = I_1 I_2 \quad I_n$ where each I_j is a* *nonprime primary ideal and $I_1, I_2, \ldots, I_n$ are pairwise comaximal Then R/I is very strongly atomic*

Proof $R/I \cong R/I_1 \times R/I_2 \times \cdots \times R/I_n$ where each R/I_j is very strongly atomic If each I_j is not prime, then each R/I_j is a nondomain and we have that R/I is very strongly atomic □

Notice that, if any of the I_j's in the previous theorem is maximal, then R/I is m-atomic If one of the I_j's is a non-maximal prime ideal, then R/I is strongly atomic As in the previous theorem, we only need R to be a ring where R/I is atomic

Corollary 3 8 *Let R be a PID and $I = (\alpha)$ with $\alpha = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$ where each p_i is prime and each $a_j > 1$ Then R/I is very strongly atomic*

The converse does not hold true If R is a PID, then R/I need not be very strongly atomic For example, let $R = \mathbb{Z}$ and $I = (6)$ Then $3 + I$ is irreducible but not very strongly irreducible So R is a PID but R/I is not very strongly atomic

We now turn our attention from Noetherian domains to Dedekind domains Recall that every ideal in a Dedekind domain can be written as a finite product of prime ideals Since both prime ideals and powers of prime ideals are primary in a Dedekind domain, we wonder if we can use Theorem 3 6 to deduce the atomic status of the rings R/I where R is a Dedekind domain and I is any ideal in R

Lemma 3 9 *If R is a one-dimensional domain with nonzero primary ideals Q_1 and Q_2 such that $rad(Q_1) \neq rad(Q_2)$, then Q_1 and Q_2 are comaximal*

Proof Recall that the radical of a primary ideal is prime and since R is one-dimensional, every nonzero prime ideal is maximal Let $P_1 = rad(Q_1)$ and $P_2 = rad(Q_2)$ Assume that $Q_1 + Q_2$ is contained in some maximal ideal $M \subsetneq R$ Then $Q_1 \subseteq Q_1 + Q_2 \subseteq M$ So $rad(Q_1) \subseteq M$ However, $rad(Q_1) = P_1$ so $P_1 = M$ Similarly, $P_2 = M$ This gives us that $P_1 = P_2$, a contradiction So $Q_1 + Q_2 = R$ □

Theorem 3.10 *Let R be a Dedekind domain and $I = P_1^{a_1}P_2^{a_2} \cdots P_n^{a_n}$ be an ideal in R where each P_i is a prime ideal in R and each $a_i \geq 1$. Then R/I is m-atomic. If $a_i > 1$ for each i, then R/I is very strongly atomic.*

Proof. If we have a factorization of I into the product of primary ideals where $P_i = P_j$ for some i and j, then we can adjust the exponents and rewrite the factorization so that $P_i \neq P_j$ for all $i \neq j$. For our purposes, we will assume that $P_i \neq P_j$ for all $i \neq j$.

We know that prime ideals are maximal and powers of prime ideals are primary because R is a Dedekind domain. Note that $rad(P_i^{a_i}) = P_i$ so by the previous lemma, we have that $P_i^{a_i}$ and $P_j^{a_j}$ are comaximal for each $i \neq j$. Thus, $R/I \cong R/P_1^{a_1} \times R/P_2^{a_2} \times \cdots \times R/P_n^{a_n}$ by the Chinese Remainder Theorem. Each $R/P_i^{a_i}$ is very strongly atomic and if $a_i = 1$, we have that $R/P_i^{a_i}$ is a field. Giving us that R/I is m-atomic. If each $a_i > 1$, then each $R/P_i^{a_i}$ is a very strongly atomic nondomain so R/I is very strongly atomic. □

We know that a domain R is Dedekind if and only if it is Noetherian, one-dimensional, and integrally closed. What happens to the atomicity of R/I if we weaken the conditions of R. That is, what happens to the atomicity of R/I if we require R to be both Noetherian and one-dimensional but not necessarily integrally closed?

Theorem 3.11 *Let R be a one dimensional Noetherian domain and I be an ideal in R. Then R/I is m-atomic. If I can be written as the product of primary ideals that are not prime, then R/I is very strongly atomic.*

Proof. R is Noetherian and one dimensional so each ideal I in R has a primary decomposition. Say $I = Q_1 \bigcap Q_2 \bigcap \cdots \bigcap Q_n$ is a primary decomposition of I. Let $T = \{P_i = rad(Q_i) | 1 \leq i \leq n\}$. Then T is a set of prime ideals with $Q_i \subseteq P_i$. Now let $S_j = \{Q_i | Q_i \subseteq P_j\}$. The set $\{S_j | 1 \leq j \leq n\}$ forms a partition of $\{Q_i | 1 \leq i \leq n\}$

since all nonzero prime ideals in R are maximal Now define $I_j = \bigcap_{Q_i \in S_j} Q_i$ and Y to be the set of all distinct ideals I_j Now considering only the ideals I_j in Y, we have that I_j is primary, $I_j \subseteq P_j$, and $I_k \subseteq P_k$ with $P_j \neq P_k$, $I_j \nsubseteq P_k$, and $I_k \nsubseteq P_j$ Then $I = \bigcap_{I_j \in Y} I_j$ is a reduced primary decomposition Also, $I_j \in Y$ and $I_k \in Y$ are pairwise comaximal for all $j \neq k$ as needed to apply Lemma 3 9 Thus, we can write $R/I \cong R/I_1 \times R/I_2 \times \quad \times R/I_m$ using the elements I_j from Y So each R/I_i is very strongly atomic and if I_j is prime, then R/I_j is a field This gives us that R/I is m-atomic If I_j is not prime for all j, then R/I is very strongly atomic □

Can we generalize Theorem 3 10 any further? What happens if we now remove the requirement that R be one-dimensional? Let $R = \mathbb{Q}[x, y]$ and $I = (x - xy^2)$ Then R is a 2-dimensional Noetherian domain However, R/I is not m-atomic In fact, R/I is not even strongly atomic

We will now switch gears and look a little closer at the elements of a ring Our hope is that a better understanding of these elements will give us insight into the ring's factorization

Proposition 3 12 *If $m \in R$ is m-irreducible but not very strongly irreducible, then* $(m) = (m)^2$

Proof Let m in R be m-irreducible and say $m = ab$ for some $a, b \in R$ Since we assume that m is not very strongly irreducible, we can assume that neither a nor b are units So we have that $(m) \subseteq (a)$ and $(m) \subseteq (b)$ Since a and b are nonunits, $(a) \neq R$ and $(b) \neq R$ Thus, $(a) = (b) = (m) = (a)(b) = (m)^2$ □

While this theorem shows us an interesting property of m-irreducibles, it does not provide us with a method for identifying m-irreducibles A principal ideal may be idempotent and its generator not be m-irreducible If we let $R = \mathbb{Z} \times \mathbb{Z}$ and let $m = (1, 0)$ and $I = (m)$, then $I = I^2$ but m is not m-irreducible since $I \subsetneq <(1, 2)>$

Theorem 3 13 *If $r \in R$ ιs regular and ιrreducιble, then r ιs very strongly ιrreducιble but not necessarιly prιme*

Proof Let $r \in R$ be regular and ırreducıble Assume that $r = ab$ Sınce r ıs ırreducıble, eıther $r \sim a$ or $r \sim b$ Wıthout loss of generalıty, we wıll say $r \sim a$ So $rk = a$ for some k ın R Thıs means that $r = rkb$ or $r(1 - kb) = 0$ We know that r ıs regular so thıs must mean that $1 - kb = 0$ That ıs, k and b are unıts ın R Thus, r ıs very strongly ırreducıble

Let $R = \mathbb{Z}_4[x]$ Then x ıs regular and $(x+2)^2 \in (x)$ but $x + 2 \notin (x)$ so x ıs not prıme □

Now that we are more famılıar wıth some of the ıntrıcacıes of atomıcıty ın nondomaıns, we wısh to take the next step and look at polynomıal extensıons of our nondomaıns wıth varyıng levels of atomıcıty Before we do thıs we wıll look ınto a concept that can be used to verıfy the atomıcıty of a polynomıal extensıon of a domaın called a maxımal common dıvısor

CHAPTER 4. MAXIMAL COMMON DIVISORS IN DOMAINS

In 1993, Moshe Roitman published *Polynomial Extensions of Atomic Domains* [7] Here he constructs an example of an atomic commutative domain R such that $R[x]$ is not atomic One of the key ingredients in this construction is the notion of *maximal common divisor (MCD)* Given a finite, nonempty set S in R, we say that $m \in R$ is an MCD of S if m divides each element of S and if n is another common divisor of S such that $m \mid n$, then m and n are associates [7] A domain in which every finite set has an MCD is called an *MCD domain* [7] It is worth noting that if R is a GCD domain, then R is an MCD domain If we let $R = \mathbb{F}_2[x^2, x^3]$, then we know that R is not a GCD domain because the set $S = \{x^5, x^6\}$ does not have a GCD However, it does have an MCD In fact, both x^2 and x^3 are MCD's of S We wish to show that this ring is an MCD domain To do this, we must first establish that R is atomic Notice that R is a Noetherian domain This gives us that $R[y, z]$ is also a Noetherian domain Hence, both R and $R[y, z]$ are atomic domains

In the first section of his paper, Roitman explores the connection between the MCD property and the atomicity of polynomial extensions of the domain The following theorem was first introduced and proven in [2] but is restated in Roitman's paper adjusting the language to include the MCD property It is this theorem that verifies that $\mathbb{F}_2[x^2, x^3]$ is an MCD domain We will later provide an alternate proof of this theorem using maximal common divisors

Theorem 4 1 *[7] Let R be an commutative domain with identity The following are equivalent*

1 *$R[x, y]$ is atomic*

2 *Given any indexing set I, the polynomial extension $R[\{x_i\}]_{i \in I}$ is atomic*

3. *R is an atomic MCD domain*

This theorem shows how the MCD status of a domain can influence the atomicity of its polynomial extensions. However, we wish to know more of the finer details of this property. For example, do we need every finite set in R to have an MCD in order for $R[x]$ to be atomic or is it necessary only for some sets? Before we attempt to answer that question, we need to identify a special class of polynomials in $R[x]$. A polynomial $f \in R[x]$ is called *indecomposable* if it cannot be written as the product of two polynomials with positive degree [7]. In the ring $\mathbb{Z}[x]$, the polynomial $2x + 2$ is indecomposable. Notice that we can write $6x - 3 = 3(2x - 1)$ but we are unable to write $6x - 3$ as the product of two polynomials of positive degree. In general, if R is a domain, then any linear polynomial in $R[x]$ is indecomposable.

Theorem 4.2 *[7] Let R be a domain. The following conditions are equivalent.*

1. *R is atomic and the set of coefficients of any indecomposable polynomial in $R[x]$ has an MCD in R.*
2. *$R[x]$ is atomic.*

Proof. (1 $\Rightarrow$ 2) Since any polynomial in $R[x]$ can be written as a finite product of indecomposable polynomials, it suffices to show that any indecomposable polynomial can be written as a finite product of irreducibles.

Let $f = \sum_{i=0}^{n} f_i x^i$ be an indecomposable polynomial and m be the MCD of the coefficients of f. If the degree of f is 0, then we have that $f \in R$ so f can be written as a finite product of irreducibles. So we will assume that $deg(f) > 0$. Let $g = \sum_{i=0}^{n} \frac{f_i}{m} x^i$. We claim that g is irreducible. Assume that $g = hk$ for some $h, k \in R[x]$. Since f is indecomposable, we know that g must also be indecomposable so without loss of generality we say that $h \in R$. This means that $mh \mid f_i$ and $m \mid mh$ so we now have that m and mh are associates. Thus, h is a unit in R and g is irreducible.

(2 $\Rightarrow$ 1) Let f be an indecomposable polynomial in $R[x]$ Now look at an irreducible factorization of f say $f = f_1 f_2 \quad f_k$ Since f is indecomposable, we know that $k-1$ of these irreducible factors must be elements of R Without loss of generality say $f_1, f_2, \quad , f_{k-1}$ are elements of R and let $m = f_1 f_2 \quad f_{k-1}$ Now assume that $c \in R$ is a common divisor of the coefficients of f where $m \mid c$ That is, $md = c$ for some $d \in R$ and $f = md(\frac{f_k}{d})$ but f_k is irreducible so d is a unit Therefore, m and c are associates and m is an MCD of the coefficients of f □

If we tighten the conditions on R slightly, we see that if R is an atomic MCD domain, then $R[x]$ is atomic On our quest to provide an alternative proof of Theorem 4 1, we need to know if $R[x]$ inherits the MCD property from R More generally, we want to know if any polynomial extension of R is an MCD domain if R is an MCD domain

Theorem 4 3 *[7] Let R be a commutative domain The following are equivalent*

1 R is an MCD domain

2 R[x] is an MCD domain

3 R[x] is a weak GCD domain (every set of two distinct elements in R has an MCD)

4 Any polynomial extension of R is an MCD domain

5 Any polynomial extension of R is a weak GCD domain

Proof It suffices to show that $3 \Rightarrow 1 \Rightarrow 4$

(3 $\Rightarrow$ 1) Consider the set $S_1 = \{r_1, r_2, \quad , r_n\}$ in R and assume that $n > 2$ Let $f(x) = r_1 + r_2 x + \quad + r_{n-1} x^{n-2}$ be a polynomial in $R[x]$ We know that the set $S_2 = \{f, r_n\}$ has an MCD in $R[x]$ call it m This means that $m \mid S_1$ Now assume that $c \in R$ such that $c \mid S_1$ and $m \mid c$ Then $c \mid S_2$ so c and m are associates Thus, m is an MCD for S_1

$(1 \Rightarrow 4)$ Let X be a family of indeterminants and let $S_1 = \{f_1, f_2, \ldots, f_n\}$ be a set of polynomials in $R[X]$. If CD_{S_1} is the set of all common divisors of S_1, then there exists at least one polynomial in CD_{S_1} that has the highest combined degree. Choose one such polynomial and call it g. Now let $m \in R$ be the MCD of all of the coefficients of the polynomials in the set $S_2 = \{\frac{f_1}{g}, \frac{f_2}{g}, \ldots, \frac{f_n}{g}\}$. We will show that mg is an MCD of S_1. If h is a common divisor of S_1 such that $mg \mid h$, then $mgk = h$ for some $k \in R[X]$. Since g has the highest combined degree, we know that k must be an element in R. Thus, mk is a common divisor of the coefficients of S_2 and $m \mid mk$ so m and mk are associates. Thus, k is a unit and mg is an MCD of S_1. □

We now have the tools we need to provide an alternate proof of Theorem 4.1.

Proof. $(3 \Rightarrow 2)$ Let X be a set of indeterminates and choose $f \in R[X]$. Since $R[X]$ is a domain, we know that if $a = bc$ then $deg_x(a) = deg_x(b) + deg_x(c)$ for all $x \in X$. Thus, we can write $f = f_1 f_2 \cdots f_n$ where each f_i is indecomposable. Now let S_i be the set of coefficients of f_i. Since R is an MCD domain, each S_i has an MCD call it m_i. So we have $f = m_1 m_2 \cdots m_n \frac{f_1}{m_1} \frac{f_2}{m_2} \cdots \frac{f_n}{m_n}$. Now R is atomic so $m_1 m_2 \cdots m_n$ can be written as a product of irreducible elements in $R[X]$. We claim that each $\frac{f_i}{m_i}$ is irreducible. Assume that $\frac{f_i}{m_i} = gh$. Then $f_i = (m_i g)h$ so either $deg(m_i g) = 0$ which means that $deg(g) = 0$ or $deg(h) = 0$. Without loss of generality, we will assume that $deg(g) = 0$. This means that $m_i g$ divides each element in S_i and $m_i \mid m_i g$. We know that m_i is the MCD of S_i so we must have that m_i and $m_i g$ are associates. That is, g is a unit in R. So $\frac{f_i}{m_i}$ is irreducible and $R[X]$ is atomic.

Since we can easily see that $(2 \Rightarrow 1)$, we will conclude by proving that $(1 \Rightarrow 3)$. R inherits its atomicity from $R[x, y]$ so we need only show that R is an MCD domain. Let $S = \{s_1, s_2, \ldots, s_n\}$ be a finite set in R. Then $f = s_1 + s_2 x + \cdots + s_{n-1} x^{n-2} + s_n y$ is an indecomposable polynomial in $R[x, y]$. Since $R[x, y]$ is atomic, we know that the set of coefficients of any indecomposable polynomial in $R[x][y] = R[x, y]$ has an MCD in $R[x]$ So if we rewrite f as $f = g_1 + g_2 y$ where $g_1 = s_1 + s_2 x + \quad + s_{n-1}x^{n-2}$ and $g_2 = s_n$, then we know that the set $\{g_1, g_2\}$ has an MCD in $R[x]$ call it m However, $g_2 \in R$ so $deg(m) = 0$, i e $m \in R$ This means that m also divides each coefficient of g_1 So m divides each element of S Now assume that k also divides each element of S and $m \mid k$ This means that k also divides both g_1 and g_2 Since m is the MCD of $\{g_1, g_2\}$ we must have that m and k are associates Thus, $m \in R$ is an MCD of S and R is an MCD domain $\square$

Our goal in the next chapter is to generalize some of these theorems by removing the domain condition However, as we will see, rings with zero divisors can display behavior that can make this challenging To accommodate this behavior we will need to specify additional properties that the ring must possess in order for the result to hold true We will also provide examples of rings with some of this troublesome behavior

CHAPTER 5. MAXIMAL COMMON DIVISORS IN RINGS WITH ZERO DIVISORS

We begin by defining three different types of maximal common divisors using the definition Roitman used when working with domains and incorporating the three levels of associate elements

Definition 5 1 Given a set S in R, m is a *maximal common divisor (MCD)* of S if m has the following two properties

1 m divides every element in S and

2 if n is another common divisor of the elements of S such that $m \mid n$, then $m \sim n$

Definition 5 2 Given a set S in R, m is a *strong maximal common divisor (SMCD)* of S if m has the following two properties

1 if m divides every element in S and

2 if n is another common divisor of the elements of S such that $m \mid n$, then $m \approx n$

Definition 5 3 Given a set S in R, m is a *very strong maximal common divisor (VSMCD)* of S if m has the following two properties

1 if m divides every element in S and

2 if n is another common divisor of the elements of S such that $m \mid n$, then $m \cong n$

We can see that when generalizing an MCD result in domains, we will have three corresponding results to verify in nondomains We begin by first defining three new types of rings

Definition 5 4 R is an *MCD ring* if every finite set in R has an MCD

Definition 5 5 R is an *SMCD ring* if every finite set in R has an SMCD

Definition 5 6 R is a *VSMCD ring* if every finite set in R has a VSMCD

Before we go any further we need to verify that these are three distinct, nonempty classes of rings

The ring $R = \mathbb{Z} \times \mathbb{Z}$ is a VSMCD ring Because $\mathbb{Z}$ is a UFD, it is also a GCD domain so it is a VSMCD ring We will see later that the product of VSMCD rings is also a VSMCD ring

Now if we let $R = \mathbb{Z}_6$ Then R is an SMCD ring but not a VSMCD ring If the set contains 1 or 5, then 1 is an SMCD of the set If the set contains 2 and 3, then 1 is an SMCD of the set If the set contains 3 and 4, then 1 is an SMCD of the set If the set is $S = \{3\}$, then the only common divisors of S are $1, 3, 5$ Since 1 and 5 divide 3 but are not associate to 3, we know that they are not MCD's of S Notice here that 3 is a common divisor of S such that $3 \mid 3$ However, 3 is strongly associate but not very strongly associate to itself So 3 is an SMCD of S but S does not have a VSMCD If the set is $\{2\}$, $\{4\}$, or $\{2, 4\}$, then 2 is an SMCD of the set For any of these three sets, 4 is a common divisor such that $2 \mid 4$ Also, we know that $2 \approx 4$ but $2 \not\cong 4$ This means that S has an SMCD but not a VSMCD

At this point in time, an MCD ring that is not an SMCD ring has not been identified As we will see later, if R is an atomic SMCD ring, then R is strongly atomic Since we know that $\dfrac{\mathbb{Q}[x, y]}{(x - xy^2)}$ is atomic but not strongly atomic, then this is the logical ring to begin with when looking for an example of a ring that is an MCD ring but not an SMCD ring

We will begin, as Roitman did, by examining how the various MCD properties affect the polynomial extension of a ring

Recall that in a domain R, every polynomial in $R[x]$ can be written as a finite product of indecomposable polynomials This useful fact does not necessarily hold if R is contains zero divisors For example, if we let $R = \mathbb{Z}_4$, then we see that $1 + 2x^n$ is a unit in $R[x]$ for all $n \in \mathbb{N}$ So given any polynomial $f \in R[x]$ such that $2 \nmid f$

and $deg(f) \geq 1$, we can write $f = (1+2x^n)((1+2x^n)f)$ where both $1+2x^n$ and $(1+2x^n)f$ have positive degree for any $n \in \mathbb{N}$ This means that any polynomial that is not divisible by 2 can be written as the product of two polynomials of positive degree If $2|f$ and $deg(f) \geq 1$, then $f = 2g$ for some g in $R[x]$ Notice here that $2 \nmid g$ since $f \neq 0$ This means that $f = (1+2x)[(1+2x)(2g)] = (1+2x)[2(1+2x)]g = (1+2x)2g = (1+2x)f$ Here we have that both $1+2x$ and f have positive degree Thus, no polynomial in $R[x]$ of positive degree is indecomposable and consequently no nonconstant polynomial can be written as a finite product of indecomposable polynomials This behavior is often problematic causing the need for an additional condition when generalizing theorems from domains to rings

It is important to point out that polynomial rings exist outside the realm of domains where each polynomial can be written as a finite product of indecomposable polynomials One such ring is $R = \mathbb{Z}_6[x]$ Notice that in $\mathbb{Z}_6[x]$, the ideals $I = (2)$ and $J = (3)$ are comaximal So we have that $R \cong R/I \times R/J$ Now since both R/I and R/J are both domains, we know that any polynomial in R/I, for example, can be written as a finite product of polynomials in R/I Thus, if we have a polynomial in R call it f, then we can rewrite it as $f = (g, h)$ If the degree of g is n and the degree of h is m, then f can be factored into at most $n+m$ polynomials in R with positive degree This means that we can find a factorization of f into polynomials of positive degree that has maximum length, say it is $f = f_1 f_2 \quad f_k$ where each f_i is of positive degree Now assume that $f_i = ab$ If $a \notin R$ and $b \notin R$, then we have a factorization of f into nonconstant polynomials of length $k+1$ This contradicts the maximality of the length of the original factorization of f So we have that every polynomial in $\mathbb{Z}_6[x]$ can be written as a finite product of indecomposable polynomials

Conjecture 5 7 Let R be an atomic ring and let f be a polynomial in $R[x]$ If S is the set of coefficients of f, then there exists an MCD of S, call it m, and a polynomial g such that $f = mg$ where an MCD of the set of coefficients of g is 1

If we are working with a domain, then this conjecture is easily proven to be true However, if $R = \mathbb{Z}_6$, for example, then we can have factorizations like $f(x) = 2x + 4 = 2(4x + 2)$ where the MCD of $\{2, 4\} \neq 1$ In this case, we can choose to factor $f(x) = 2x + 4$ as $f(x) = 2(x + 2)$ and here the MCD of $\{1, 2\}$ is 1 We use this conjecture to prove the following two theorems

Theorem 5 8 *Let R be a ring such that all polynomials in $R[x]$ can be written as a finite product of indecomposable polynomials If R is atomic and the set of coefficients of any indecomposable polynomial in $R[x]$ has an MCD, then $R[x]$ is atomic*

Proof Let f be a polynomial in $R[x]$ Since all polynomials in $R[x]$ can be written as a finite product of indecomposable polynomials, we may assume that f is indecomposable That is, if $f = gh$, then without loss of generality $h \in R$

Let S_f be the set of coefficients of f and let m be an MCD of S_f Also, let g be a polynomial such that $f = mg$ and the MCD of S_g, the set of coefficients of g, is 1 We now need to show that g is irreducible in $R[x]$

Assume that $g = kt$ for some $k, t \in R[x]$ Then without loss of generality, we may assume that $t \in R$ since f is indecomposable This means that t is a common divisor of S_g and $1 \mid t$ which gives us $1 \sim t$ and t is a unit Thus, g is irreducible In [1], we find that an element $a \in R$ is irreducible in R if and only if it is irreducible in $R[x]$ and we now have that $R[x]$ is atomic □

Theorem 5 9 *Let R be a ring such that all polynomials in $R[x]$ can be written as a finite product of indecomposable polynomials If R is strongly atomic and the set of coefficients of any indecomposable polynomial in $R[x]$ has an SMCD in R, then $R[x]$ is strongly atomic*

Proof Let f be a polynomial in $R[x]$ Since all polynomials in $R[x]$ can be written as a finite product of indecomposable polynomials, we may assume that f is indecomposable That is, if $f = gh$, then without loss of generality $h \in R$

Let S_f be the set of coefficients of f and let m be an SMCD of S_f Also, let g be a polynomial such that $f = mg$ and the SMCD of S_g, the set of coefficients of g, is 1 We now need to show that g is strongly irreducible in $R[x]$

Assume that $g = kt$ for some $k, t \in R[x]$ Then without loss of generality, we may assume that $t \in R$ since f is indecomposable This means that t is a common divisor of S_g and $1 \mid t$ which gives us $1 \approx t$ and t is a unit Thus, g is strongly irreducible In [1], we find that an element $a \in R$ is strongly irreducible in R if and only if it is strongly irreducible in $R[x]$ and we now have that $R[x]$ is strongly atomic □

Theorem 5 10 *Let R be a ring such that all polynomials in $R[x]$ can be written as a finite product of indecomposable polynomials If R is very strongly atomic and the set of coefficients of any indecomposable polynomial in $R[x]$ has a VSMCD, then $R[x]$ is very strongly atomic*

Proof Let $f(x) = a_0 + a_1x + a_2x^2 + \quad + a_nx^n$ be a polynomial in $R[x]$ Since all polynomials in $R[x]$ can be written as a finite product of indecomposable polynomials, we may assume that f is indecomposable That is, if $f = gh$, then without loss of generality $h \in R$ Let $S = \{a_0, a_1, a_2, \quad , a_n\}$ be the set of all coefficients of f and m be a VSMCD of S So $f(x) = m(\frac{a_0}{m} + \frac{a_1}{m}x + \frac{a_2}{m}x^2 + \quad + \frac{a_n}{m}x^n)$ Let $g(x) = \frac{a_0}{m} + \frac{a_1}{m}x + \frac{a_2}{m}x^2 + \quad + \frac{a_n}{m}x^n$ We now need to show that g is very strongly irreducible in $R[x]$

Assume that $g(x) = k(x)t(x)$ for some $k, t \in R[x]$ Then without loss of generality, we may assume that $t(x) \in R$ since f is indecomposable This means that mt is a common divisor of S and $m \mid mt$ So $m \cong mt$ which means there exists a unit $u \in R$ such that $m = umt$ and if $mt = rm$ for some $r \in R$, then r must be a unit. Since $mt = umt^2 = (ut^2)m$ we can conclude that $t(x)$ is a unit in R and in $R[x]$. Thus, g is very strongly irreducible. □

We have seen that a direct product of rings with a particular type of atomicity does not necessarily possess the same form of atomicity. In fact, this direct product may not have any form of atomicity. What happens when we take a direct product of rings with a particular MCD property? Is it still an MCD ring? Do we have to bound the indexing set to retain any level of the MCD property?

Theorem 5.11 *Let $\{R_\alpha\}_{\alpha\in\Lambda}$ be a family of rings and let $R = \prod_{\alpha\in\Lambda} R_\alpha$. Then R is an MCD ring if and only if each R_α is an MCD ring. R is an SMCD ring if and only if each R_α is an SMCD ring. R is a VSMCD ring if and only if each R_α is a VSMCD ring.*

Proof. The proofs for each of the three statements are nearly identical so we will prove only the first statement.

Let $R = \prod_{\alpha\in\Lambda} R_\alpha$ be an MCD ring and let $S_j = \{s_1, s_2, \ldots, s_n\}$ be a finite set in R_j for some $j \in \Lambda$. Let $\widehat{s_i} = (x_\alpha)_{\alpha\in\Lambda}$ where $x_\alpha = s_i$ if $\alpha = j$ and $x_\alpha = 0$ if $\alpha \neq j$. Consider the set $\widehat{S_j} = \{\widehat{s_1}, \widehat{s_2}, \ldots, \widehat{s_n}\}$ in R. Note that $\widehat{S_j}$ has an MCD in R, call it $m = (m_\alpha)_{\alpha\in\Lambda}$. We now have that $m_j \mid S_j$ so we assume that $c \mid S_j$ and $m_j \mid c$ for some $c \in R_j$. If $m_c = (y_\alpha)_{\alpha\in\Lambda}$ where $y_\alpha = m_\alpha$ if $\alpha \neq j$ and $y_\alpha = c$ if $\alpha = j$, then m_c is a common divisor of $\widehat{S_j}$ with $m \mid m_c$. Thus, $m \sim m_c$ so $m_j \sim c$ and we have that m_j is an MCD of S_j giving us that R_j is an MCD ring.

Let $R = \prod_{\alpha\in\Lambda} R_\alpha$ where each R_α is an MCD ring. Consider the set $S = \{s_1, s_2, \ldots, s_n\}$ in R where each $s_i = \{x_{i,\alpha}\}_{\alpha\in\Lambda}$. Now look at the set $S_\alpha = (x_{1,\alpha}, x_{2,\alpha}, \ldots, x_{n,\alpha})$ in R_α. This set has an MCD in R_α call it m_α. We wish to show that $m = (m_\alpha)_{\alpha\in\Lambda}$ is an MCD of S. Clearly, $m \mid S$ so we now assume that $c \mid S$ and $m \mid c$ for some

$c = c_\alpha)_{\alpha \in \Lambda}$ in R This means that $c_\alpha \mid S_\alpha$ and $m_\alpha \mid c_\alpha$ Thus, $m_\alpha \sim c_\alpha$ and $m \sim c$ □

In Chapter 3, we found nice ways of generating large classes of rings with the various forms of atomicity We would like to also generate large classes of rings that possess the various levels of the MCD property

Theorem 5 12 *If R is a PIR, then R is an MCD ring If R is a SPIR, then R is a VSMCD ring*

Proof Let R be a PIR and let $S \subseteq R$ be a finite set Since R is a PIR, we know that the ideal (S) is principally generated We will say $(S) = (d)$ This means that d is a common divisor of S Now assume that x is another common divisor of S such that $d \mid x$ We now have $S \subseteq (x)$ and $x \in (d)$ That is, $S = (d) \subseteq (x)$ and we have $(x) = (d)$, i e $x \sim d$ and d is an MCD of S

Now we will let R be a SPIR with maximal ideal M and $S \subseteq R$ be a finite set We know that if $(S) = (d)$, then d is an MCD of S Now let c be a common divisor of S such that $d \mid c$ This means that $d \sim c$ If $d = 0$, then $(d) = (0) = (c)$ so we have that $c = 0$ and $d \cong c$ If d is a unit, then c is a unit and we have that $d \cong c$ So we will assume that c and d are nonzero, nonunits where $d = cx$ for some $x \in R$ Since R is a SPIR, if $x \in M$ then $cx = 0 = d$, a contradiction So $x \notin M$ which means that x is a unit Thus, $d \cong c$ and d is a VSMCD of S □

Theorem 5 13 *If R is p-atomic, then R is a VSMCD ring*

Proof If R is p-atomic, then $R = \prod_{i=1}^{n} R_i$ where each R_i is either a UFD or a SPIR This means that each R_i is a VSMCD ring which gives us that R is a VSMCD ring □

A ring R is called *présimplifiable* if $x = xy$ implies that either $x = 0$ or y is a unit [1] Notice that any domain is présimplifiable For rings with zero divisors, the ring $R = \mathbb{Z}_9$ is présimplifiable Since R is very strongly atomic, we know that $x \cong x$ for all $x \in R$ so if we have $x = xy$, either $x = 0$ or y is a unit The ring $\mathbb{Z}_6$ is not preesimplifiable We know that $2 = 2 \cdot 4$ where $2 \neq 0$ and 4 is not a unit

Theorem 5 14 *If R is a présimplifiable MCD ring, then R is a VSMCD ring*

Proof Let S be a finite set in R and let m be an MCD of S Now assume that c is a common divisor of S where $m \mid c$ This means that $m \sim c$ That is, $m = cd$ and $c = mk$ for some $d, k \in R$ So we now have that $m = m(kd)$ If $m = 0$, then $c = 0$ and we have that $m \cong c$ If $m \neq 0$, then we have that kd is a unit in R or, more importantly, k and d are each units in R Thus, $m \cong c$ and every finite set in R has a VSMCD □

We have different levels of atomicity and different levels of MCD rings all influenced by the three forms of associate elements If a ring is some form of atomic and has some level of the MCD property, then how does its MCD level relate to its level of atomicity, if at all?

Theorem 5 15 *Let R be an atomic ring*

1 *If R is an VSMCD ring, then R is very strongly atomic*

2 *If R be an SMCD ring, then R is strongly atomic*

Proof 1 Let α be irreducible in R and consider the set $S = \{\alpha\}$ in R It suffices to show that α is very strongly irreducible Since R is a VSMCD ring, S has a VSMCD call it m Notice that α is a common divisor of S and $m \mid \alpha$ This means that $m \cong \alpha$ and α is also a VSMCD of S

Now assume that $\alpha = rt$ for some $r, t \in R$ Without loss of generality, we have that $\alpha \sim r$ That is, $r \mid S$ and $\alpha \mid r$ Thus, we have $\alpha \cong r$ and α is very strongly irreducible

2 Let α be irreducible in R and consider the set $S = \{\alpha\}$ in R It suffices to show that α is strongly irreducible Since R is a SMCD ring, S has an SMCD call it m Notice that α is a common divisor of S and $m \mid \alpha$ This means that $m\alpha$ and α is also an SMCD of S

Now assume that $\alpha = rt$ for some $r, t \in R$ Without loss of generality, we have that $\alpha \sim r$ That is, $r \mid S$ and $\alpha \mid r$ Thus, we have αr and α is strongly irreducible

□

We would also like to generalize Theorem 4 3 However, the proof for this theorem relies heavily on degree arguments, a luxury we do not have when dealing with nondomains

The research of factorization properties in rings with zero divisors is limited and there are several cases where we find many different theories surrounding a single topic The idea of factoring an element has taken on two different flavors We may factor an element in a nondomain just as we would factor an element in a domain Alternatively, we may use an idea called u-factorization which separates an element's factors into relevant and irrelevant factors When using the u-factorizations, it is only the relevant factors that are examined The research on MCD domains/rings has spawned very little published works Our final chapter will provide a sampling of interesting unsolved questions

CHAPTER 6. FUTURE RESEARCH IDEAS

In 1993, Roitman states a conjecture that is a variation of Theorem 4 1

Conjecture 5 1 [7] Let R be a domain The following are equivalent

1 $R[x]$ is atomic

2 $R[x, y]$ is atomic

3 R is an atomic MCD domain

The proof of this conjecture comes down to verifying that given a set of elements in R, there exists an indecomposable polynomial in $R[x]$ whose coefficients are exactly the elements of the set It is important to point out that the some of the coefficients of the polynomial may be zero For example, if the set is $S = \{2, 3, 4\}$, then a polynomial of the form $f(x) = 2x^6 + 3x^2 + 4$ would be acceptable

Rings with zero divisors do not always behave in predictable ways For example, we can use degree arguments when working with polynomial extensions of domains However, as we have seen this technique cannot necessarily be used for polynomial extensions of rings with zero divisors A ring is indecomposable if it contains no nontrivial idempotent elements In an indecomposable ring R, can every polynomial in $R[x]$ be written as a product of indecomposable polynomials? What characteristics must R have in order for each polynomial in $R[x]$ to be written as a product of indecomposable polynomials? We also know that if R is a domain, then if $R[x]$ is atomic we know that R must also be atomic What happens if R is not a domain, is it possible to find a ring R that is not atomic but its polynomial extension $R[x]$ is atomic?

Various aspects of MCD domains and the different flavors of MCD rings are also of great interest We wish to generalize more of Roitman's theorems or at least portions of them We also wish to know if there is any relation between a ring being indecomposable and having some level of the MCD property

There is a wealth of research to be done involving MCD domains/rings and their various levels of atomicity For domains, we often look beyond atomicity and examine rings with properties such as unique factorization, bounded factorization, and finite factorization We wish to follow a similar path for rings with zero divisors To this end, some additional areas of interest are unique factorization in rings with zero divisors, bounded factorization in rings with zero divisors, and u-factorizations

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APPENDIX A. GLOSSARY

ACCP Consider an ascending chain of principal ideals $I_1 \subseteq I_2 \subseteq$ in R If there exists an $N \in \mathbb{N}$ such that for every $j, k > N$ we have $I_j = I_k$ then we say that R satisfies the ascending chain condition on principal ideals (ACCP)

Associate Elements Let a and b be elements of a ring We say that a and b are associates if $a \mid b$ and $b \mid a$

Associate Elements (Domain) Let D be an integral domain and $a, b \in D$ The following statements are equivalent

1 a and b are associates

2 $a|b$ and $b|a$

3 There exists a unit $u \in D$ such that $a = ub$

4 If $a|b$, $b|a$ and whenever $a = bc$ with $a \neq 0$, then c must be a unit in D

Atomic Domain A domain is atomic if every nonzero, nonunit can be written as a finite product of irreducibles

Atomic Ring A ring R is atomic if every nonzero, nonunit can be written as a finite product of irreducibles

Commutative Ring A ring R is called commutative if for each $a, b \in R$ we have that $ab = ba$ If R contains an element 1_R such that $a1_R = 1_R a$ for each $a \in R$, then R is said to be a ring with identity

Ideal: Let R be a commutative ring A subset $I \subseteq R$ is an ideal of R if I is itself a ring and if for each $x \in I$ and each $r \in R$, the element rx is an element of I

Indecomposable Polynomial Let R be a ring A polynomial $f \in R[x]$ is said to be indecomposable if whenever $f = gh$ for some $g, h \in R[x]$, we have that either $g \in R$ or $h \in R$

Indecomposable Ring A ring is indecomposable if it contains no nontrivial idempotent elements

Irreducible Let a be a nonunit element of a ring We say that a is irreducible if $a = bc$ implies that $a \sim b$ or $a \sim c$

Irreducible (Domain) An irreducible in a domain is an element x such that whenever $x = yz$ then x is associate to either y or z

M-Atomic Ring A ring R is m-atomic if every nonzero, nonunit can be written as a finite product of m-irreducibles

M-Irreducible Let a be a nonunit element of a ring We say that a is m-irreducible if (a) is maximal among proper principal ideals

Maximal Common Divisor Given a set S in a ring R, we say m is a maximal common divisor (MCD) of S if m has the following two properties

1 m divides every element in S and

2 if n is another common divisor of the elements of S such that $m \mid n$, then $m \sim n$

Maximal Common Divisor (Domain) Given a finite, nonempty set S in a domain D, we say $m \in R$ is a maximal common divisor (MCD) of S if m divides each element of S and if n is another element in R that divides each element of S with $m \mid n$, then m and n are associates

Maximal Ideal Let M be an ideal in a commutative ring R If $M \subseteq I$ for some nontrivial ideal $I \subseteq R$ only when $M = I$, then M is called a maximal ideal of R

MCD See Maximal Common Divisor

MCD Domain A domain in which every finite set has an MCD is called an MCD domain

MCD Ring R is an MCD ring if every finite set in R has an MCD

Nilpotent Let R be a commutative ring We say that $a \in R$ is a nilpotent element if $a^n = 0$ for some $n \in \mathbb{N}$ We say that the ideal $I \subseteq R$ is nilpotent if $I^n = 0$ for some

$n \in \mathbb{N}$

Noetherian ring A ring is called Noetherian if every ideal in the ring is finitely generated

P-Atomic Ring A ring R is p-atomic if every nonzero, nonunit can be written as a finite product of primes

PID See Principal Ideal Domain

PIR See Principal Ideal Ring

Présimplifiable A ring R is called présimplifiable if $x = xy$ implies that either $x = 0$ or y is a unit in R

Primary Ideal An ideal I of a ring R is primary if given $ab \in I$, then either $a \in I$ or $b^n \in I$ for some $n \in \mathbb{N}$

Prime Element Let a be an element of a ring We say that a is prime if (a) is prime ideal

Prime Ideal An ideal P is called a prime ideal of a ring R if whenever $IJ \subseteq P$ for some ideals $I, J \in R$ we have that either $I \subseteq P$ or $J \subseteq P$

Principal Ideal An ideal I of a ring R is called a principal ideal if it generated by a single element of R

Principal Ideal Domain If every ideal of a commutative domain D is a principal ideal, then D is called a principal ideal domain (PID)

Principal Ideal Ring If every ideal of a commutative ring R is a principal ideal, then R is called a principal ideal ring (PIR)

Radical Ideal An ideal I of a ring R is called a radical ideal if whenever $x^n \in I$ then $x \in I$ If $J \subseteq R$ is an ideal of R, then the radical of J, written $rad(J)$ is the set $\{x \in R | x^n \in J \text{ for some } n \in \mathbb{N}\}$

Regular Let R be a commutative ring An element $r \in R$ is called regular if $rs = 0$ only when $s = 0$

Ring A ring R is a nonempty set with two binary operations denoted $+$ and $*$ with the following three properties

1. $(R, +)$ is an abelian group
2. $(R, *)$ is associative
3. $a(b+c) = ab + ac$ and $(a+b)c = ac + bc$ for every $a, b, c \in R$

SMCD See Strong Maximal Common Divisor

SMCD Ring A ring R is an SMCD ring if every finite set in R has an SMCD

Special Principal Ideal Ring A principal ideal ring (PIR) is called a special principal ideal ring (SPIR) if it has only one proper prime ideal P and $P^2 = 0$

SPIR See Special Principal Ideal Ring

Strong Associate Elements Let a and b be elements of a ring Then a and b are strong associates if there exists a unit u in the ring such that $a = ub$

Strong Irreducible Let a be a nonunit element of a ring We say that a is strongly irreducible if $a = bc$ implies that $a \approx b$ or $a \approx c$

Strong Maximal Common Divisor Given a set S in a ring R, m is a strong maximal common divisor (SMCD) of S if m has the following two properties

1. if m divides every element in S and
2. if n is another common divisor of the elements of S such that $m \mid n$, then $m \approx n$

Strongly Atomic Ring A ring R is strongly atomic if every nonzero, nonunit can be written as a finite product of strong irreducibles

Very Strong Associate Elements Let a and b be elements of a ring Then a and b are very strong associates if either $a = b = 0$ or whenever $a = bc$ we have that c must be a unit in the ring

Very Strong Irreducible Let a be a nonunit element of a ring We say that a is very strongly irreducible if $a = bc$ implies that $a \cong b$ or $a \cong c$

Very Strong Maximal Common Divisor Given a set S in a ring R, m is a very strong maximal common divisor (VSMCD) of S if m has the following two properties

1 if m divides every element in S and

2 if n is another common divisor of the elements of S such that $m \mid n$, then $m \cong n$

Very Strongly Atomic Ring A ring R is very strongly atomic if every nonzero, nonunit can be written as a finite product of very strong irreducibles

VSMCD See Very Strong Maximal Common Divisor

VSMCD Ring A ring R is a VSMCD ring if every finite set in R has a VSMCD

Zero Divisor An element a of a ring R is called a zero divisor if $ab = 0$ for some nonzero $b \in R$