DG HOMOLOGICAL ALGEBRA, PROPERTIES OF RING HOMOMORPHISMS, AND THE GENERALIZED AUSLANDER-REITEN CONJECTURE

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ABSTRACT

This dissertation contains three aspects of my research that are listed as three joint papers with my advisor and a solo paper in the bibliography [56]–[59]. Each paper will be discussed in a different chapter.

Chapter 1 contains the introduction to this dissertation. In this chapter we give the statements of the most important results discussed in Chapters 3–6.

Chapter 2 contains notation and background material for use in the subsequent chapters.

In Chapter 3, we prove lifting results for DG modules that are akin to Auslander, Ding, and Solberg's famous lifting results for modules.

Chapter 4 contains the complete answer to a question of Vasconcelos from 1974. We show that a local ring has only finitely many shift-isomorphism classes of semidualizing complexes. Our proof relies on certain aspects of deformation theory for DG modules over a finite dimensional DG algebra, which we develop.

In Chapter 5, we investigate Cohen factorizations of local ring homomorphisms from three perspectives. First, we prove a "weak functoriality" result for Cohen factorizations: certain morphisms of local ring homomorphisms induce morphisms of Cohen factorizations. Second, we use Cohen factorizations to study the properties of local ring homomorphisms in certain commutative diagrams. Third, we use Cohen factorizations to investigate the structure of quasi-deformations of local rings, with an eye on the question of the behavior of CI-dimension in short exact sequences.

In Chapter 6, we show under some conditions that a Gorenstein ring R satisfies the Generalized Auslander-Reiten Conjecture if and only if so does $R[x]$. When R is a local ring we prove the same result for some localizations of $R[x]$.

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CHAPTER 1. INTRODUCTION

Convention. *In this dissertation, all rings are commutative and noetherian with identity, R is a ring, and all modules are unital. A local ring is "complete" when it is complete with respect to its maximal ideal.*

1.1. Main Ideas

This dissertation contains three aspects of my research that are listed as three joint papers with my advisor and a solo paper in the bibliography [56]–[59]. Each paper will be discussed in a different chapter.

The first aspect of my research that will be discussed in Chapters 3 and 4 (see also [57, 59]) is concerned with *differential graded algebras* (*DG algebras* for short) and *differential graded modules* (*DG modules* for short), and their applications. These notions that come originally from algebraic topology provide powerful tools to investigate problems in homological algebra. One of these problems is a conjecture of Vasconcelos from 1974 that says there are only finitely many isomorphism classes of semidualizing modules over a Cohen-Macaulay local ring. In Chapters 3 and 4 we use these concepts and give a complete solution to this conjecture.

The second aspect of my research that will be discussed in Chapter 5 (see also [58]) is concerned with the properties of ring homomorphisms. More precisely, we work on *Cohen factorizations* of a local homomorphism and their applications. Cohen factorizations are important tools in commutative algebra and homological algebra because they enable us to transfer many properties along ring homomorphisms. The main point is that they allow one to study a local ring homomorphism by replacing it with a surjective one; thus, one can assume that the target is finitely generated over the source, so one can apply finite homological algebra techniques.

In the process of investigating properties of rings using properties of ring homomorphisms we work with some invariants called *homological dimensions*. One of these dimensions that is called *complete intersection dimension* (CI dimension for short) and is defined in terms of certain diagrams of local ring homomorphisms called *quasi-deformations* is quite nice in many respects. However, at this time we do not know how it behaves with respect to short exact sequences: If two of the modules in an exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ have finite CI-dimension, must the third module also have finite CI-dimension? The difficulty lies in attempting to combine the two given quasi-deformations into a single one that works for both M_1 and M_2. In Chapter 5, using Cohen factorizations, we deal with half of this problem under certain assumptions.

Finally, the third aspect of my research that is discussed in Chapter 6 (see also [56]) is related to the generalized version of a conjecture of Auslander and Reiten about vanishing of cohomology modules from 1975. It says that over a ring R, vanishing of $\operatorname{Ext}^i_R(M, M \oplus R)$ for a finitely generated R-module M and for all $i > 0$, implies that M is projective. This conjecture is solved in many cases, and there is not a counter-example to it in the commutative case. This conjecture is closely related to many other important conjectures in commutative algebra about vanishing of homology and cohomology such as Tachikawa's conjecture [69] and the finitistic dimension conjecture [43].

1.2. Summary of Chapter 2

This chapter contains notation and background material for use in the subsequent chapters.

1.3. Summary of Chapter 3

Hochster famously wrote that "life is really worth living" in a Cohen-Macaulay ring [47].[1] For instance, if R is Cohen-Macaulay and local with maximal regular sequence $\underline{t}$, then $R/(\underline{t})$ is artinian and the natural epimorphism $R \to R/(\underline{t})$ is nice

[1] We know of this quote from [19].

enough to allow for transfer of properties between the two rings. Thus, if one can prove a result for artinian local rings, then one can (often) prove a similar result for Cohen-Macaulay local rings by showing that the desired conclusion descends from $R/(\underline{t})$ to R. When R is complete, then this is aided sometimes by the lifting result of Auslander, Ding, and Solberg.

Theorem 1.1 ([6, Propositions 1.7 and 2.6]). *Let $\underline{t} \in R$ be an R-regular sequence, and let M be a finitely generated $R/(\underline{t})$-module. Assume that R is local and $(\underline{t})$-adically complete.*

(a) *If $\operatorname{Ext}^2_{R/(\underline{t})}(M, M) = 0$, then M is "liftable" to R, that is, there is a finitely generated R-module N such that $R/(\underline{t}) \otimes_R N \cong M$ and $\operatorname{Tor}_i^R(R/(\underline{t}), N) = 0$ for all $i \geqslant 1$.*

(b) *If $\operatorname{Ext}^1_{R/(\underline{t})}(M, M) = 0$, then M has at most one lift to R.*

In Chapter 3, we are concerned with what happens when the sequence $\underline{t}$ is not R-regular. One would like a similar mechanism for reducing questions about arbitrary local rings to the artinian case.

It is well known that the map $R \to R/(\underline{t})$ is not nice enough in general to guarantee good descent/lifting behavior. Our perspective[2] in this matter is that this is not the right map to consider in general: the correct one is the natural map from R to the Koszul complex $K = K^R(\underline{t})$. This perspective requires one to make some adjustments. For instance, K is a differential graded R-algebra, so not a commutative ring in the traditional sense. This may cause some consternation, but the payoff can be handsome. For instance, in Chapter 4 we use this perspective to answer a question of Vasconcelos [70]. One of the tools for the proof of this result is the following version of Auslander, Ding, and Solberg's lifting result. Note that we do not assume that R is local in part (a) of this result.

[2]This perspective is not original to our work. We learned of it from Avramov and Iyengar.

Theorem 1.2. *Let $\underline{t} = t_1, \cdots, t_n$ be a sequence of elements of R, and assume that R is $\underline{t}R$-adically complete. Let D be a DG $K^R(\underline{t})$-module that is homologically bounded below and homologically degreewise finite.*

(a) *If $\operatorname{Ext}^2_{K^R(\underline{t})}(D, D) = 0$, then D is quasi-liftable to R, that is, there is a semi-free R-complex D' such that $D \simeq K^R(\underline{t}) \otimes_R D'$.*

(b) *Assume that R is local. If D is quasi-liftable to R and $\operatorname{Ext}^1_{K^R(\underline{t})}(D, D) = 0$, then any two homologically degreewise finite quasi-liftings of D to R are quasiisomorphic over R.*

This result is proved in Corollaries 3.19 and 3.22, which follow from more general results on liftings along morphisms of DG algebras. Note that it is similar to, but quite different from, some results of Yoshino [80].

1.4. Summary of Chapter 4

Chapter 4 is concerned with the solution to a question of Vasconcelos from 1974 about *semidulizing R-modules*, that is, the finitely generated R-modules C such that $\operatorname{Hom}_R(C, C) \cong R$ and $\operatorname{Ext}^i_R(C, C) = 0$ for $i \geqslant 1$. These modules were introduced, as best we know, by Foxby [28]. They were rediscovered independently by several authors including Vasconcelos [70], Golod [37], and Wakamatsu [74], who all used different terminology for them. Special cases of these modules include Grothendieck's canonical modules over Cohen-Macaulay rings, and duality with respect to a semidualizing module extends Auslander and Bridger's G-dimension [4, 5].

Vasconcelos posed the following in [70, p. 97].

Question 1.3. If R is local and Cohen-Macaulay, must the set of isomorphism classes of semidualizing R-modules be finite?

Christensen and Sather-Wagstaff [22] answer this question affirmatively in the case when R contains a field. Their proof motivates our own techniques, so we describe some aspects of it here. Using standard ideas, they reduce to the case where R is complete with algebraically closed residue field F. Given a maximal R-sequence $\mathbf{x}$, they replace R with the quotient $R/(\mathbf{x})$, which is a finite dimensional algebra over F. The desired result then follows from a deformation-theoretic theorem of Happel [44] which states that, in this context, there are only finitely many R-modules C of a given length r such that $\operatorname{Ext}^1_R(C, C) = 0$.

To prove Happel's result, one parametrizes all such modules by an algebraic variety Mod^R_r that is acted on by the general linear group GL^F_r so that the isomorphism class of C is precisely the orbit $\operatorname{GL}^F_r \cdot C$. A theorem of Voigt [73] (see also Gabriel [34]) provides an isomorphism between $\operatorname{Ext}^1_R(C, C)$ and the quotient of tangent spaces $\mathsf{T}^{\operatorname{Mod}^R_r}_C / \mathsf{T}^{\operatorname{GL}^F_r \cdot C}_C$. Thus, the vanishing $\operatorname{Ext}^1_R(C, C) = 0$ implies that the orbit $\operatorname{GL}^F_r \cdot C$ is open in Mod^R_r. Since Mod^R_r is quasi-compact, it can only have finitely many open orbits, so R can only have finitely many such modules up to isomorphism.

The main result of Chapter 4, stated next, provides a complete answer to Vasconcelos' question. Note that it does not assume that R is Cohen-Macaulay. Section 4.3 is devoted to its proof (see 4.27) and some consequences.

Theorem 1.4. *Let R be a local ring. Then the set of isomorphism classes of semidualizing R-modules is finite.*

The idea behind our proof is the same as in Christensen and Sather-Wagstaff's proof, with one important difference, pioneered by Avramov: instead of replacing R with $R/(\mathbf{x})$, we use the Koszul complex K on a minimal generating sequence for the maximal ideal of R. More specifically, we replace R with a finite dimensional *DG F-algebra U* that is quasiisomorphic to K.

In order to prove versions of the results of Happel and Voigt, we develop certain aspects of deformation theory for DG modules over a finite dimensional DG F-algebra U. This is the subject of Section 4.2. In short, we parametrize all finite dimensional

DG U-modules M with fixed underlying graded F-vector space W by an algebraic variety $\mathrm{Mod}^U(W)$. This variety is acted on by a product $\mathrm{GL}(W)_0$ of general linear groups so that the isomorphism class of M is precisely the orbit $\mathrm{GL}(W)_0 \cdot M$. Following Gabriel, we focus on the associated functors of points $\underline{\mathrm{Mod}}^U(W)$, $\underline{\mathrm{GL}}(W)_0$, and $\underline{\mathrm{GL}}(W)_0 \cdot M$. (Our notational conventions are spelled out explicitly in Notations 4.13 and 4.17.) Our version of Voigt's result for this context is the following, which we prove in 4.23.

Theorem 1.5. *We work in the setting of Notations* 4.13 *and* 4.17*. Given an element $M = (\partial, \mu) \in \mathrm{Mod}^U(W)$, there is an isomorphism of abelian groups*

$$\mathcal{T}_M^{\underline{\mathrm{Mod}}^U(W)} \, / \, \mathcal{T}_M^{\underline{\mathrm{GL}}(W)_0 \cdot M} \cong \mathrm{YExt}_U^1(M, M).$$

(See the paragraph before Theorem 1.6 *for the definition of YExt.)*

As a consequence, we deduce that if $\mathrm{Ext}_U^1(M, M) = 0$, then the orbit $\underline{\mathrm{GL}}(W)_0 \cdot M$ is open in $\underline{\mathrm{Mod}}^U(W)$; see Corollary 4.24. The proof of Theorem 1.4 concludes like that of Christensen and Sather-Wagstaff, with a few technical differences.

One technical difference is the following: given DG U-modules M and N, there are (at least) two different modules that one might write as $\mathrm{Ext}_U^1(M, N)$. This is the topic of Section 4.1. First, there is the derived category version: this is the module $\mathrm{Ext}_U^1(M, N) = \mathrm{H}^1(\mathrm{Hom}_U(F, N))$ where F is a "semi-free resolution" of M. Second, there is the abelian category version: this is the module $\mathrm{YExt}_U^1(M, N)$ that is the set of equivalence classes of exact sequences $0 \to N \to X \to M \to 0$. In general, one has $\mathrm{Ext}_U^1(M, N) \ncong \mathrm{YExt}_U^1(M, N)$. This is problematic as the passage from R to U uses $\mathrm{Ext}_U^1(M, N)$, but Theorem 1.5 uses $\mathrm{YExt}_U^1(M, N)$. These are reconciled in the next result which we prove in 4.5.

Theorem 1.6. *Let A be a DG R-algebra, and let P, Q be DG A-modules such that Q is graded-projective (e.g., Q is semi-free). Then there is an isomorphism $\mathrm{YExt}_A^1(Q, P) \xrightarrow{\cong} \mathrm{Ext}_A^1(Q, P)$ of abelian groups.*

We actually prove a version of Theorem 1.4 for semidualizing complexes over a local ring. We do this in Theorem 4.26. Moreover, we prove versions of these results for certain non-local rings, including all semilocal rings in Theorem 4.35.

1.5. Summary of Chapter 5

In Chapter 5, we are concerned with the notion of *Cohen factorizations* and their applications. Cohen factorizations were introduced in [16] as tools to study local ring homomorphisms. The utility of these factorizations can be seen in their many applications; see, e.g., [9, 10, 11, 13, 14, 31, 50, 64]. The main point of this construction is that it allows one to study a local ring homomorphism by replacing it with a surjective one; thus, one can assume that the target is finitely generated over the source, so one can apply finite homological algebra techniques.

In Section 5.1, we investigate functorial properties of Cohen factorizations. The main result of this section is the following; its proof is in 5.4. Example 5.5 shows that the separability assumptions (2) are necessary.

Theorem 1.7. *Consider a commutative diagram of local ring homomorphisms*

$$\begin{array}{ccc} (R,\mathfrak{m}) & \xrightarrow{\varphi} & (S,\mathfrak{n}) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ (\widetilde{R},\widetilde{\mathfrak{m}}) & \xrightarrow{\widetilde{\varphi}} & (\widetilde{S},\widetilde{\mathfrak{n}}) \end{array} \tag{1.7.1}$$

with the following properties:

(1) *α and φ have regular factorizations $R \xrightarrow{\dot{\alpha}} R'' \xrightarrow{\alpha'} \widetilde{R}$ and $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} S$, e.g., $\widetilde{R}$ and S are complete,*

(2) *$\widetilde{S}$ is complete, and the field extensions $R/\mathfrak{m} \to \widetilde{R}/\widetilde{\mathfrak{m}} \to \widetilde{S}/\widetilde{\mathfrak{n}}$ are separable.*

Let $S \xrightarrow{\dot{\beta}} S' \xrightarrow{\beta'} \widetilde{S}$ and $\widetilde{R} \xrightarrow{\dot{\widetilde{\varphi}}} \widetilde{R}' \xrightarrow{\widetilde{\varphi}'} \widetilde{S}$ be Cohen factorizations of β and $\widetilde{\varphi}$. Then there is a commutative diagram of local ring homomorphisms

$$
\begin{array}{ccccc}
R & \xrightarrow{\dot{\varphi}} & R' & \xrightarrow{\varphi'} & S \\
{\scriptstyle\dot{\alpha}}\downarrow & & \downarrow{\scriptstyle\dot{\gamma}} & & \downarrow{\scriptstyle\dot{\beta}} \\
R'' & \xrightarrow{\dot{\sigma}} & T & \xrightarrow{\sigma'} & S' \\
{\scriptstyle\alpha'}\downarrow & & \downarrow{\scriptstyle\gamma'} & & \downarrow{\scriptstyle\beta'} \\
\widetilde{R} & \xrightarrow{\dot{\widetilde{\varphi}}} & \widetilde{R}' & \xrightarrow{\widetilde{\varphi}'} & \widetilde{S}
\end{array} \qquad (1.7.2)
$$

such that the diagrams $R' \xrightarrow{\dot{\gamma}} T \xrightarrow{\gamma'} \widetilde{R}'$ *and* $R'' \xrightarrow{\dot{\sigma}} T \xrightarrow{\sigma'} S'$ *are minimal Cohen factorizations.*

We think of this as a result about functoriality of regular (e.g., Cohen) factorizations as follows. The diagram (1.7.1) is a morphism in the category of local ring homomorphisms. Our result provides the following commutative diagram of local ring homomorphisms where $\gamma = \gamma'\dot{\gamma}$

$$
\begin{array}{ccccc}
R & \xrightarrow{\dot{\varphi}} & R' & \xrightarrow{\varphi'} & S \\
{\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} \\
\widetilde{R} & \xrightarrow{\dot{\widetilde{\varphi}}} & \widetilde{R}' & \xrightarrow{\widetilde{\varphi}'} & \widetilde{S}
\end{array}
$$

which is a morphism in the category of regular factorizations. Of course, the operation that maps a local ring homomorphism to a regular (or Cohen) factorization is not well-defined; hence our terminology "weak functoriality".

Given a diagram (1.7.1) where α and β are "nice", the maps φ and $\widetilde{\varphi}$ are intimately related. This maxim is the subject of Section 5.2, which culminates in the proof of the next result. It is one of the applications of Cohen factorizations mentioned in the title of Chapter 5; see 5.15 for the proof.

Theorem 1.8. *Consider a commutative diagram of local ring homomorphisms*

$$
\begin{array}{ccc}
(R,\mathfrak{m}) & \xrightarrow{\varphi} & (S,\mathfrak{n}) \\
{\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\
(\widetilde{R},\widetilde{\mathfrak{m}}) & \xrightarrow{\widetilde{\varphi}} & (\widetilde{S},\widetilde{\mathfrak{n}})
\end{array}
$$

such that α is weakly Cohen, β is weakly regular, and the induced map $\widetilde{R}/\widetilde{\mathfrak{m}} \to \widetilde{S}/\widetilde{\mathfrak{n}}$ is separable. Let P be one of the following conditions: Gorenstein, quasi-Gorenstein, complete intersection, Cohen-Macaulay, quasi-Cohen-Macaulay. Then φ is P if and only if $\widetilde{\varphi}$ is P.

Section 5.2 also contains, among other things, a base change result for CI-dimension and CM-dimension (Proposition 5.13) which may be of independent interest.

Chapter 5 concludes with Section 5.3, which is devoted almost entirely to the proof of the next result; see 5.18. It is another application of Cohen factorizations. Also, it is related to the short exact sequence question for CI-dimension, as we describe in Remark 5.17.

Theorem 1.9. *For $i = 1, 2$ let $(R, \mathfrak{m}, k) \xrightarrow{\varphi_i} (R_i, \mathfrak{m}_i, k_i) \xleftarrow{\tau_i} (Q_i, \mathfrak{n}_i, k_i)$ be a quasi-deformation such that the field extension $k \to k_i$ is separable.*

(a) *Then there exists a commutative diagram of local ring homomorphisms*

$$\begin{array}{ccccccccc}
(Q_1,\mathfrak{n}_1,k_1) & \xrightarrow{\tau_1} & (R_1,\mathfrak{m}_1,k_1) & \xleftarrow{\varphi_1} & (R,\mathfrak{m},k) & \xrightarrow{\varphi_2} & (R_2,\mathfrak{m}_2,k_2) & \xleftarrow{\tau_2} & (Q_2,\mathfrak{n}_2,k_2) \\
 & \searrow^{\overline{\delta}_1\delta_1} & & \searrow^{\sigma_1\beta_1} & \downarrow{\alpha'} & \swarrow^{\sigma_2\beta_2} & & \swarrow^{\overline{\delta}_2\delta_2} & \\
 & & (\overline{Q}_1,\overline{\mathfrak{n}}_1,k') & \xrightarrow[\overline{\gamma}_1]{} & (R',\mathfrak{m}',k') & \xleftarrow[\overline{\gamma}_2]{} & (\overline{Q}_2,\overline{\mathfrak{n}}_1,k') & &
\end{array} \qquad \text{(C.1)}$$

such that α' is flat, each map $\overline{\delta}_i\delta_i$ is weakly Cohen, and each map $\overline{\gamma}_i$ is surjective.

(b) *Assume that each map φ_i is weakly Cohen. Then the maps in diagram* (C.1) *satisfy the following properties for $i = 1, 2$:*

(b1) *The maps $R_i \to R'$ are weakly Cohen.*

(b2) *The diagram $R \to R' \twoheadleftarrow \overline{Q}_i$ is a quasi-deformation and each parallelogram diagram is a pushout.*

(b3) *Given an R-module M, if $\mathrm{pd}_{Q_i}(M \otimes_R R_i) < \infty$, then $\mathrm{pd}_{\overline{Q}_i}(M \otimes_R R') < \infty$.*

1.6. Summary of Chapter 6

The original version of a conjecture of Auslander and Reiten [7] is a generalized version of a conjecture of Nakayama. It says that over an artin algebra Λ, vanishing of $\operatorname{Ext}^i_\Lambda(M, M \oplus \Lambda)$ for a finitely generated Λ-module M and for all $i > 0$, implies that M is projective. Auslander and Reiten proved this conjecture for several classes of rings. Later Auslander, Ding, and Solberg [6] studied this conjecture for arbitrary commutative noetherian rings. This version is known as the *Auslander-Reiten conjecture* and is proved affirmatively in some special cases; see for instance [24, 48, 49, 51, 60, 66]. A generalized version of the Auslander-Reiten conjecture, studied by several authors including [26, 75, 76], is the following.

Conjecture 1.10. *Let n be a positive integer, and let M be a finitely generated R-module. If $\operatorname{Ext}^i_R(M, M \oplus R) = 0$ for all $i > n$, then $\operatorname{pd}_R(M) \leqslant n$.*

Related to this, the *finitistic extension degree* of the ring R (defined below), denoted $\operatorname{fed}(R)$, has been recently introduced by Diveris in [26]. This invariant is tightly connected to Conjecture 1.10 over Gorenstein rings, i.e.

Theorem 1.11 ([26, Corollary 3.2])**.** *If R is Gorenstein, then $\operatorname{fed}(R)$ is finite if and only if Conjecture 1.10 holds for R.*

By definition,

$$\operatorname{fed}(R) = \sup\{n \mid \operatorname{Ext}^i_R(M, M) = 0 \text{ for all } i > n \text{ and } \operatorname{Ext}^n_R(M, M) \neq 0\}$$

where the supremum is taken over all finitely generated R-modules M such that $\operatorname{Ext}^i_R(M, M) = 0$ for $i \gg 0$. Diveris also studies the behavior of this dimension under certain changes of rings. Among other results, he proves that when $(R, \mathfrak{m})$ is Gorenstein and local, finiteness of any of $\operatorname{fed}(R)$, $\operatorname{fed}(\widehat{R})$, $\operatorname{fed}(R[[x]])$, or $\operatorname{fed}(R[x]_{(\mathfrak{m},x)})$ implies finiteness of the others; see [26, Theorem 4.4]. In other words, when $(R, \mathfrak{m})$

is a Gorenstein local ring, if one of the rings R, $\widehat{R}$, $R[[x]]$, or $R[x]_{(\mathfrak{m},x)}$ satisfies Conjecture 1.10, then they all do.

In Chapter 6 we examine how finiteness of the finitistic extension degree of a Gorenstein ring, hence the condition of Conjecture 1.10, is preserved under certain faithfully flat ring extensions. In this chapter, we prove the following theorems that are related to the base ring extensions $R \to R[x]$ and $R \to R[x]_{\mathfrak{m}R[x]}$ when $(R, \mathfrak{m})$ is a local ring.

Theorem 1.12. *Let k be an uncountable algebraically closed field, and let R be a finite dimensional k-algebra that is Gorenstein. Then* $\operatorname{fed}(R) < \infty$ *if and only if* $\operatorname{fed}(R[x]) < \infty$. *Therefore, by Theorem* 1.11, *R satisfies Conjecture* 1.10 *if and only if so does $R[x]$.*

Theorem 1.13. *Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring with an uncountable coefficient field k. Then* $\operatorname{fed}(R) < \infty$ *if and only if* $\operatorname{fed}(R[x]_{\mathfrak{m}R[x]}) < \infty$. *Therefore, by Theorem* 1.11, *R satisfies Conjecture* 1.10 *if and only if so does $R[x]_{\mathfrak{m}R[x]}$.*

CHAPTER 2. BACKGROUND MATERIAL

This chapter, which consists of four sections, is devoted to definitions and background material on the notions that will be used in the entire dissertation. Useful references for each topic will be introduced in each section.

2.1. DG Modules

In this section, we list fundamental material for use in Chapters 3 and 4. The central objects that we use in these chapters are DG algebras and DG modules which essentially come from algebraic topology. We start with the definition of R-complexes.

Definition 2.1. A complex of R-modules ("R-complex" for short) is a chain of R-modules and R-module homomorphisms:

$$M = \cdots \xrightarrow{\partial^M_{n+2}} M_{n+1} \xrightarrow{\partial^M_{n+1}} M_n \xrightarrow{\partial^M_n} M_{n-1} \xrightarrow{\partial^M_{n-1}} \cdots$$

such that $\partial^M_i \partial^M_{i+1} = 0$ for all i. Sometimes we write (M, ∂^M) to specify the differential on M. The degree of an element $m \in M$ is denoted $|m|$. The *infimum*, *supremum*, and *amplitude* of M are

$$\begin{aligned} \inf(M) &:= \inf\{n \in \mathbb{Z} \mid \mathrm{H}_n(M) \neq 0\} \\ \sup(M) &:= \sup\{n \in \mathbb{Z} \mid \mathrm{H}_n(M) \neq 0\} \\ \mathrm{amp}(M) &:= \sup(M) - \inf(M) \end{aligned}$$

where $\mathrm{H}_i(M) := \mathrm{Ker}(\partial^M_i)/\,\mathrm{Im}(\partial^M_{i+1})$ for all i.

Definition 2.2. The *tensor product* of two R-complexes M, N, denoted $M \otimes_R N$, is defined as follows:

$$(M \otimes_R N)_l := \coprod_{p \in \mathbb{Z}} M_p \otimes_R N_{l-p}$$

with the differential $\partial_l^{M\otimes_R N}(m_p \otimes n_{l-p}) := \partial_p^M(m_p)\otimes n_{l-p} + (-1)^p m_p \otimes \partial_{l-p}^N(n_{l-p})$. Also the *Hom complex*, denoted $\operatorname{Hom}_R(M,N)$, is defined as follows:

$$\operatorname{Hom}_R(M,N)_l := \prod_{p\in\mathbb{Z}} \operatorname{Hom}_R(M_p, N_{l+p})$$

with the differential $\partial_l^{\operatorname{Hom}_R(M,N)}(\psi)_p := \partial_{p+l}^N \psi_p - (-1)^l \psi_{p-1}\partial_p^M$. A *chain map* $f\colon M \to N$ is a cycle of degree 0 in $\operatorname{Hom}_R(M,N)$, i.e., a commutative diagram of R-module homomorphisms

$$\begin{array}{ccccccc} \cdots \longrightarrow & M_{i+1} & \xrightarrow{\partial_{i+1}^M} & M_i & \longrightarrow \cdots \\ & \downarrow{\scriptstyle f_{i+1}} & & \downarrow{\scriptstyle f_i} & \\ \cdots \longrightarrow & N_{i+1} & \xrightarrow{\partial_{i+1}^N} & N_i & \longrightarrow \cdots . \end{array}$$

Isomorphisms in the category of R-complexes are identified by the symbol $\cong$. A *quasiisomorphism* is a morphism $M \to N$ of R-complexes such that each induced map $\mathrm{H}_i(M) \to \mathrm{H}_i(N)$ is an isomorphism; these are identified by the symbol $\simeq$. Two R-complexes M and N are *quasiisomorphic* if there is a chain of quasiisomorphisms (in alternating directions) from M to N; this equivalence relation is denoted by the symbol $\simeq$.

Definition 2.3. Let i be an integer. The ith *suspension* of an R-complex M is the R-complex $\Sigma^i M$ defined by $(\Sigma^i M)_n := M_{n-i}$ and $\partial_n^{\Sigma^i M} := (-1)^i \partial_{n-i}^M$. Two R-complexes M and N are *shift-quasiisomorphic* if there is an integer m such that $M \simeq \Sigma^m N$; this equivalence relation is denoted by the symbol $\sim$.

Definition 2.4. An R-complex M is *bounded below* if $M_n = 0$ for all $n \ll 0$; it is *bounded* if $M_n = 0$ when $|n| \gg 0$; it is *degree-wise finite* if M_i is finitely generated over R for each i; it is *homologically bounded below* if the total homology module $\mathrm{H}(M) := \bigoplus_{i\in\mathbb{Z}} \mathrm{H}_i(M)$ is bounded below; it is *homologically bounded* if $\mathrm{H}(M)$ is bounded; it is *homologically degree-wise finite* if each R-module $\mathrm{H}_n(M)$ is finitely generated; and it is *homologically finite* if it is homologically both bounded and degree-wise finite.

Definition 2.5. A *free* (resp. *projective*) *resolution* of an R-complex M is a quasiisomorphism $F \xrightarrow{\simeq} M$ of R-complexes such that F is a bounded below complex of free (resp. projective) R-modules. For a homologically bounded R-complex M the *projective dimension*, denoted $\mathrm{pd}_R(M)$, is defined to be

$$\mathrm{pd}_R(M) := \inf \{ \sup \{ l \in \mathbb{Z} \mid P_l \neq 0\} \mid P \simeq M \text{ is a projective resolution}\}.$$

Fact 2.6. An R-complex M has a free resolution if and only if it has a projective resolution if and only if it is homologically bounded below [20].

Definition 2.7. The derived category $\mathcal{D}(R)$ is formed from the category of R-complexes by formally inverting the quasiisomorphisms. Isomorphisms in $\mathcal{D}(R)$ are identified by $\simeq$, and isomorphisms up to shift in $\mathcal{D}(R)$ are identified by $\sim$.

Given a free resolution $F \xrightarrow{\simeq} M$ and an R-complex N, set $M \otimes^{\mathbf{L}}_R N := F \otimes_R N$, $\mathrm{Tor}^R_i(M,N) := \mathrm{H}_i(M \otimes^{\mathbf{L}}_R N)$, $\mathbf{R}\mathrm{Hom}_A(M,N) := \mathrm{Hom}_A(F,N)$, and $\mathrm{Ext}^i_A(M,N) := \mathrm{H}_{-i}(\mathbf{R}\mathrm{Hom}_A(M,N))$ for each integer i. The functors $\mathbf{R}\mathrm{Hom}_R(-,-)$ and $- \otimes^{\mathbf{L}}_R -$ are called the *derived functors* of Hom and tensor product.

Standard references for the category of R-complexes and the derived category $\mathcal{D}(R)$ are [8, 15, 20, 29, 35, 45, 71, 72].

Next we discuss DG algebras, which are treated in, e.g., [1, 2, 8, 15, 53, 54].

Definition 2.8. A *commutative differential graded algebra over* R (*DG* R-*algebra* for short) is an R-complex A equipped with a chain map $\mu^A \colon A \otimes_R A \to A$ with $ab := \mu^A(a \otimes b)$ that is:

associative for all $a, b, c \in A$ we have $(ab)c = a(bc)$;

unital there is an element $1 \in A_0$ such that for all $a \in A$ we have $1a = a$;

graded commutative for all $a, b \in A$ we have $ab = (-1)^{|a||b|}ba$ and $a^2 = 0$ when $|a|$ is odd; and

positively graded $A_i = 0$ for $i < 0$.

The map μ^A is the *product* on A. Given a DG R-algebra A, the *underlying algebra* is the graded commutative R-algebra $A^\natural = \oplus_{i=0}^{\infty} A_i$. When R is a field and $\operatorname{rank}_R(\oplus_{i \geqslant 0} A_i) < \infty$, we say that A is *finite-dimensional* over R.

A *morphism* of DG R-algebras is a chain map $f\colon A \to B$ between DG R-algebras respecting products and multiplicative identities: $f(aa') = f(a)f(a')$ and $f(1) = 1$.

Fact 2.9. Let A be a DG R-algebra. The fact that the product on A is a chain map says that ∂^A satisfies the *Leibniz rule*: $\partial^A_{|a|+|b|}(ab) = \partial^A_{|a|}(a)b + (-1)^{|a|}a\partial^A_{|b|}(b)$.

The ring R, considered as a complex concentrated in degree 0, is a DG R-algebra. The map $R \to A$ given by $r \mapsto r \cdot 1$ is a morphism of DG R-algebras. It is straightforward to show that the R-module A_0 is an R-algebra. Moreover, the natural map $A_0 \to A$ is a morphism of DG R-algebras. The condition $A_{-1} = 0$ implies that A_0 surjects onto $\mathrm{H}_0(A)$ and that $\mathrm{H}_0(A)$ is an A_0-algebra. Furthermore, the R-module A_i is an A_0-module, and $\mathrm{H}_i(A)$ is an $\mathrm{H}_0(A)$-module for each i.

Given a second DG R-algebra K, the tensor product $K \otimes_R A$ is also a DG R-algebra with multiplication $(x \otimes a)(x' \otimes a') := (-1)^{|a||x'|}(xx') \otimes (aa')$.

Definition 2.10. Let A be a DG R-algebra. We say that A is *noetherian* if $\mathrm{H}_0(A)$ is noetherian and the $\mathrm{H}_0(A)$-module $\mathrm{H}_i(A)$ is finitely generated for all $i \geqslant 0$. When R is local, we say that A is *local* if it is noetherian and the ring $\mathrm{H}_0(A)$ is a local R-algebra, that is, $\mathrm{H}_0(A)$ is a local ring whose maximal ideal contains the extension of the maximal ideal of R.

Fact 2.11. Assume that R is local with maximal ideal $\mathfrak{m}$. Let A be a local DG R-algebra, and let $\mathfrak{m}_{\mathrm{H}_0(A)}$ be the maximal ideal of $\mathrm{H}_0(A)$. The composition $A \to \mathrm{H}_0(A) \to \mathrm{H}_0(A)/\mathfrak{m}_{\mathrm{H}_0(A)}$ is a surjective morphism of DG R-algebras with kernel of the form $\mathfrak{m}_A = \cdots \xrightarrow{\partial_2^A} A_1 \xrightarrow{\partial_1^A} \mathfrak{m}_0 \to 0$ for some maximal ideal $\mathfrak{m}_0 \subsetneq A_0$. The quotient $A/\mathfrak{m}_A$ is isomorphic to $\mathrm{H}_0(A)/\mathfrak{m}_{\mathrm{H}_0(A)}$. Since $\mathrm{H}_0(A)$ is a local R-algebra, we have $\mathfrak{m}A_0 \subseteq \mathfrak{m}_0$.

Definition 2.12. Assume that R is local. Given a local DG R-algebra A, the subcomplex $\mathfrak{m}_A$ from Fact 2.11 is the *augmentation ideal* of A.

For Chapters 3 and 4 of this dissertation, an important example is the next one.

Example 2.13. Given a sequence $\underline{t} = t_1, \cdots, t_n \in R$, the Koszul complex $K = K^R(\underline{t})$ is a DG R-algebra with product given by the wedge product. If R is a local ring with the maximal ideal $\mathfrak{m}$ and if $\underline{t} \in \mathfrak{m}$, then K is a local DG R-algebra with augmentation ideal $\mathfrak{m}_K = (0 \to R \to \cdots \to R^n \to \mathfrak{m} \to 0)$.

In the passage to DG algebras, we must focus on DG modules, described next.

Definition 2.14. Let A be a DG R-algebra. A *differential graded module over A* (*DG A-module* for short) is an R-complex M with a chain map $\mu^M \colon A \otimes_R M \to M$ such that the rule $am := \mu^M(a \otimes m)$ is associative and unital. The map μ^M is the *scalar multiplication* on M. The *underlying $A^\natural$-module* associated to M is the $A^\natural$-module $M^\natural = \oplus_{i=-\infty}^{\infty} M_i$.

Example 2.15. Consider the ring R as a DG R-algebra. A DG R-module is just an R-complex, and a morphism of DG R-modules is simply a chain map.

Fact 2.16. Let A be a DG R-algebra, and let M be a DG A-module. The fact that the scalar multiplication on M is a chain map says that ∂^M satisfies the *Leibniz rule*: $\partial^A_{|a|+|m|}(am) = \partial^A_{|a|}(a)m + (-1)^{|a|}a\partial^M_{|m|}(m)$. The R-module M_i is an A_0-module, and $\mathrm{H}_i(M)$ is an $\mathrm{H}_0(A)$-module for each i.

Definition 2.17. Let A be a DG R-algebra, and let i be an integer. The ith *suspension* of a DG A-module M is the DG A-module $\Sigma^i M$ defined by $(\Sigma^i M)_n := M_{n-i}$ and $\partial_n^{\Sigma^i M} := (-1)^i \partial_{n-i}^M$. The scalar multiplication on $\Sigma^i M$ is defined by the formula $\mu^{\Sigma^i M}(a \otimes m) := (-1)^{i|a|}\mu^M(a \otimes m)$.

A *morphism* of DG A-modules is a chain map $f\colon M \to N$ between DG A-modules that respects scalar multiplication: $f(am) = af(m)$. Isomorphisms in the category of DG A-modules are identified by the symbol $\cong$. A *quasiisomorphism* is a morphism $M \to N$ such that each induced map $\mathrm{H}_i(M) \to \mathrm{H}_i(N)$ is an isomorphism; these are identified by the symbol $\simeq$. Two DG A-modules M and N are *quasiisomorphic* if there is a chain of quasiisomorphisms (in alternating directions) from M to N; this equivalence relation is denoted by the symbol $\simeq$. Two DG A-modules M and N are *shift-quasiisomorphic* if there is an integer m such that $M \simeq \Sigma^m N$; this equivalence relation is denoted by the symbol $\sim$.

Notation 2.18. The derived category $\mathcal{D}(A)$ is formed from the category of DG A-modules by formally inverting the quasiisomorphisms; see [53]. Isomorphisms in $\mathcal{D}(A)$ are identified by $\simeq$, and isomorphisms up to shift in $\mathcal{D}(A)$ are identified by $\sim$.

Definition 2.19. Let A be a DG R-algebra, and let M, N be DG A-modules. The *tensor product* $M \otimes_A N$ is the quotient $(M \otimes_R N)/U$ where U is the subcomplex generated by all elements of the form $(am) \otimes n - (-1)^{|a||m|} m \otimes (an)$. Given an element $m \otimes n \in M \otimes_R N$, we denote the image in $M \otimes_A N$ as $m \otimes n$.

Fact 2.20. Let A be a DG R-algebra, and let M, N be DG A-modules. The tensor product $M \otimes_A N$ is a DG A-module via the scalar multiplication

$$a(m \otimes n) := (am) \otimes n = (-1)^{|a||m|} m \otimes (an).$$

Definition 2.21. Let A be a DG R-algebra. A DG A-module M is *bounded below* if $M_n = 0$ for all $n \ll 0$; it is *bounded* if $M_n = 0$ when $|n| \gg 0$; it is *degree-wise finite* if M_i is finitely generated over A_0 for each i; it is *homologically bounded below* if the total homology module $\mathrm{H}(M) := \bigoplus_{i\in\mathbb{Z}} \mathrm{H}_i(M)$ is bounded below; it is *homologically bounded* if $\mathrm{H}(M)$ is bounded; it is *homologically degree-wise finite* if each $\mathrm{H}_0(A)$-module $\mathrm{H}_n(M)$ is finitely generated; and it is *homologically finite* if it is homologically both bounded and degree-wise finite. The full subcategory of $\mathcal{D}(A)$ whose objects are the homologically bounded below DG A-modules is denoted $\mathcal{D}_+(A)$.

Here we discuss one type of resolution for DG modules. See Section 4.1 for a discussion of other kinds of resolutions.

Definition 2.22. Let A be a DG R-algebra, and let L be a DG A-module. A subset E of L is called a *semibasis* if it is a basis of the underlying $A^\natural$-module $L^\natural$. If L is bounded below, then L is called *semi-free* if it has a semibasis.[3] A *semi-free resolution* of a DG A-module M is a quasiisomorphism $F \xrightarrow{\simeq} M$ of DG A-modules such that F is semi-free. Given a semi-free resolution $F \xrightarrow{\simeq} M$ and a DG A-module N, set $M \otimes^{\mathbf{L}}_A N := F \otimes_A N$ and $\mathrm{Tor}^A_i(M,N) := \mathrm{H}_i(M \otimes^{\mathbf{L}}_A N)$.

Assume that R and A are local. A *minimal semi-free resolution* of M is a semi-free resolution $F \xrightarrow{\simeq} M$ such that F is *minimal*, i.e., each (equivalently, some) semibasis of F is finite in each degree and the differential on $(A/\mathfrak{m}_A) \otimes_A F$ is 0.[4]

Fact 2.23. Let A be a DG R-algebra, and let M be a DG A-module that is

[3] As is noted in [15], when L is not bounded below, the definition of "semi-free" is significantly more technical. Using the more general notion, one can define the up-coming derived functors $M \otimes^{\mathbf{L}}_A N$, $\mathrm{Tor}^A_i(M,N)$, $\mathbf{R}\mathrm{Hom}_A(M,N)$, and $\mathrm{Ext}^i_A(M,N)$ for any pair of DG A-modules, with no boundedness assumptions. However, our results do not require this level of generality, so we focus only on this case. Furthermore, for $M \otimes^{\mathbf{L}}_A N$ and $\mathrm{Tor}^A_i(M,N)$, one only needs semi-flat resolutions, and for $\mathbf{R}\mathrm{Hom}_A(M,N)$ and $\mathrm{Ext}^i_A(M,N)$, one only needs semi-projective resolutions. Consult [15, Sections 2.8 and 2.10] for a discussion of these notions and the relations between them.

[4] Note that our definition of minimality differs from the definition found in [15, (2.12.1)]. However, the definitions are often equivalent, and we do not use any technical aspects of the definition from [15, (2.12.1)] in this dissertation.

homologically bounded below. Then M has a semi-free resolution over A by [15, Theorem 2.7.4.2]. For each DG A-module N, the complex $M \otimes^{\mathbf{L}}_A N$ is well-defined (up to isomorphism) in $\mathcal{D}(A)$; hence the modules $\operatorname{Tor}^A_i(M,N)$ are well-defined over $\mathrm{H}_0(A)$ and over R. Given a semi-free resolution $G \xrightarrow{\simeq} N$, one has $M \otimes^{\mathbf{L}}_A N \simeq M \otimes_R G$.

Assume that A is noetherian, and let j be an integer. Assume that M is homologically degree-wise finite and $\mathrm{H}_i(M) = 0$ for $i < j$. Then M has a semi-free resolution $F \xrightarrow{\simeq} M$ such that $F^{\natural} \cong \oplus_{i=j}^{\infty} \Sigma^i (A^{\natural})^{\beta_i}$ for some integers β_i, and so $F_i = 0$ for all $i < j$; see [2, Proposition 1]. In particular, homologically finite DG A-modules admit such "degree-wise finite, bounded below" semi-free resolutions.

Assume that R and A are local with $k = A/\mathfrak{m}_A$. Then M has a minimal semi-free resolution $F \xrightarrow{\simeq} M$ such that $F_i = 0$ for all $i < j$; see [2, Proposition 2]. In particular, homologically finite DG A-modules admit minimal semi-free resolutions. Moreover, the condition $\partial^{k \otimes_A F} = 0$ shows that $\beta_i = \operatorname{rank}_k(\operatorname{Tor}^A_i(M,k))$ for all i.

Definition 2.24. Let A be a DG R-algebra, and let M, N be DG A-modules. Given an integer i, a *DG A-module homomorphism of degree i* is an element $f \in \operatorname{Hom}_R(M,N)_i$ such that $f(am) = (-1)^{i|a|} a f(m)$ for all $a \in A$ and $m \in M$. The graded submodule of $\operatorname{Hom}_R(M,N)$ consisting of all DG A-module homomorphisms $M \to N$ is denoted $\operatorname{Hom}_A(M,N)$. A homomorphism $f \in \operatorname{Hom}_A(M,N)_i$ is *null-homotopic* if it is in $\operatorname{Im}(\partial^{\operatorname{Hom}_A(M,N)}_{i+1})$. Two homomorphisms $M \to N$ are *homotopic* if their difference is null-homotopic.

Given a semi-free resolution $F \xrightarrow{\simeq} M$, set $\mathbf{R}\operatorname{Hom}_A(M,N) := \operatorname{Hom}_A(F,N)$ and $\operatorname{Ext}^i_A(M,N) := \mathrm{H}_{-i}(\mathbf{R}\operatorname{Hom}_A(M,N))$ for each integer i. Equivalently, the elements of $\operatorname{Ext}^i_A(M,N)$ are the homotopy equivalence classes of morphisms of DG A-modules $F \to \Sigma^{-i} N$.

Fact 2.25. Let A be a DG R-algebra, and let M, N be DG A-modules. The complex $\operatorname{Hom}_A(M,N)$ is a DG A-module via the action

$$(af)(m) := a(f(m)) = (-1)^{|a||f|} f(am).$$

For each $a \in A$ the multiplication map $\mu^{M,a} \colon M \to M$ given by $m \mapsto am$ is a homomorphism of degree $|a|$.

Assume that M is homologically bounded below. The complex $\mathbf{R}\mathrm{Hom}_A(M, N)$ is independent of the choice of semi-free resolution of M, and we have an isomorphism $\mathbf{R}\mathrm{Hom}_A(M, N) \simeq \mathbf{R}\mathrm{Hom}_A(M', N')$ in $\mathcal{D}(A)$ whenever $M \simeq M'$ and $N \simeq N'$; see [8, Propositions 1.3.1–1.3.3].

In our proof of Theorem 1.4, we use Christensen and Sather-Wagstaff's notion of semidualizing DG U-modules from [23], defined next.

Definition 2.26. Let A be a DG R-algebra, and let M be a DG A-module. The *homothety morphism* $X^A_M \colon A \to \mathrm{Hom}_A(M, M)$ is given by $X^A_M(a) := \mu^{M,a}$, i.e., $X^A_M(a)(m) = am$. When M is homologically bounded below, this induces a *homothety morphism* $\chi^A_M \colon A \to \mathbf{R}\mathrm{Hom}_A(M, M)$.

Assume that A is noetherian. Then M is a *semidualizing* DG A-module if M is homologically finite and the homothety morphism $\chi^A_M \colon A \to \mathbf{R}\mathrm{Hom}_A(M, M)$ is an isomorphism in $\mathcal{D}(A)$. Let $\mathfrak{S}(A)$ denote the set of shift-isomorphism classes in $\mathcal{D}(A)$ of semidualizing DG A-modules, that is, the set of equivalence classes of semidualizing DG A-modules under the relation $\sim$ from Notation 2.18.

The following base-change results are used in our proofs of Theorems 1.2 and 1.4.

Remark 2.27. Let $A \to B$ be a morphism of DG R-algebras, and let M and N be DG A-modules. The "base changed" complex $B \otimes_A M$ has the structure of a DG B-module by the action $b(b' \otimes m) := (bb') \otimes m$. This structure is compatible with the DG A-module structure on $B \otimes_A M$ via restriction of scalars. Furthermore, this induces a well-defined operation $\mathcal{D}_+(A) \to \mathcal{D}_+(B)$ given by $M \mapsto B \otimes^{\mathbf{L}}_A M$.

Given $f \in \mathrm{Hom}_A(M,N)_i$, define $B \otimes_A f \in \mathrm{Hom}_B(B \otimes_A M, B \otimes_A N)_i$ by the formula $(B \otimes_A f)(b \otimes m) := (-1)^{i|b|} b \otimes f(m)$. This yields a morphism of DG A-modules $\mathrm{Hom}_A(M,N) \to \mathrm{Hom}_B(B \otimes_A M, B \otimes_A N)$ given by $f \mapsto B \otimes_A f$. When M is homologically bounded below, this provides a well-defined morphism $\mathbf{R}\mathrm{Hom}_A(M,N) \to \mathbf{R}\mathrm{Hom}_B(B \otimes^{\mathbf{L}}_A M, B \otimes^{\mathbf{L}}_A N)$ in $\mathcal{D}(A)$.

The next lemma is essentially from [54] and [61].

Lemma 2.28. *Let $\varphi \colon A \to B$ be a quasiisomorphism of noetherian DG R-algebras, that is, a morphism of DG R-algebras that is also a quasiisomorphism.*

(a) *The base change functor $B \otimes^{\mathbf{L}}_A -$ induces an equivalence of derived categories $\mathcal{D}_+(A) \to \mathcal{D}_+(B)$ whose quasi-inverse is given by restriction of scalars.*

(b) *For each DG A-module $X \in \mathcal{D}_+(A)$, one has $X \simeq B \otimes^{\mathbf{L}}_A X$ in $\mathcal{D}(A)$, and thus*

$$\inf(B \otimes^{\mathbf{L}}_A X) = \inf(X)$$
$$\sup(B \otimes^{\mathbf{L}}_A X) = \sup(X)$$
$$\mathrm{amp}(B \otimes^{\mathbf{L}}_A X) = \mathrm{amp}(X).$$

(c) *The equivalence from part* (a) *induces a bijection from $\mathfrak{S}(A)$ to $\mathfrak{S}(B)$.*

Proof. (a) See, e.g., [54, 7.6 Example].

(b) The equivalence from part (a) implies that the map $X \to B \otimes^{\mathbf{L}}_A X$ is a quasiisomorphism, and the displayed equalities follow directly.

(c) Let X be a homologically bounded below DG A-module. We show that X is a semidualizing DG A-module if and only if $B \otimes^{\mathbf{L}}_A X$ is a semidualizing DG B-module. Since the maps $X \to B \otimes^{\mathbf{L}}_A X$ and $A \to B$ are quasiisomorphisms, it follows that X is homologially finite over A if and only if $B \otimes^{\mathbf{L}}_A X$ is homologially finite over B. It remains to show that the homothety morphism $\chi^A_X \colon A \to \mathbf{R}\mathrm{Hom}_A(X,X)$ is an isomorphism in $\mathcal{D}(A)$ if and only if $\chi^B_{B\otimes^{\mathbf{L}}_A X}\colon B \to \mathbf{R}\mathrm{Hom}_B(B\otimes^{\mathbf{L}}_A X, B\otimes^{\mathbf{L}}_A X)$ is an isomorphism in $\mathcal{D}(B)$. It is routine to show that the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\chi^A_X} & \mathbf{R}\mathrm{Hom}_A(X,X) \\ {\scriptstyle\varphi}\downarrow{\scriptstyle\simeq} & & {\scriptstyle\simeq}\downarrow{\scriptstyle\omega} \\ B & \xrightarrow[\chi^B_{B\otimes^{\mathbf{L}}_A X}]{} & \mathbf{R}\mathrm{Hom}_B(B\otimes^{\mathbf{L}}_A X, B\otimes^{\mathbf{L}}_A X) \end{array}$$

where ω is the morphism from Remark 2.27. As ω is an isomorphism by [61, Proposition 2.1], the desired equivalence follows. $\square$

Definition 2.29. Let A be a DG R-algebra, and let M be a DG A-module. Given an integer n, the *nth soft left truncation of M* is the complex

$$\tau(M)_{(\leqslant n)} := \cdots \to 0 \to M_n/\operatorname{Im}(\partial^M_{n+1}) \to M_{n-1} \to M_{n-2} \to \cdots$$

with differential induced by ∂^M and the induced scalar multiplication.

Remark 2.30. Let A be a DG R-algebra, and let M be a DG A-module. Fix an integer n. Then the truncation $\tau(M)_{(\leqslant n)}$ is a DG A-module, and the natural chain map $M \to \tau(M)_{(\leqslant n)}$ is a morphism of DG A-modules. This morphism is a quasiisomorphism if and only if $n \geqslant \sup(M)$. See [15, (4.1)].

Definition 2.31. Let A be a local DG R-algebra, and let M be a homologically finite DG A-module. For each integer i, the ith *Betti and Bass numbers* are

$$\beta^A_i(M) := \operatorname{rank}_k(\operatorname{Tor}^A_i(k,M)) \qquad \mu^i_A(M) := \operatorname{rank}_k(\operatorname{Ext}^i_A(k,M))$$

respectively, where $k = A/\mathfrak{m}_A$. The *Poincaré and Bass series* of M are the formal Laurent series

$$P^M_A(t) := \sum_{i\in\mathbb{Z}} \beta^A_i(M)t^i \qquad I^A_M(t) := \sum_{i\in\mathbb{Z}} \mu^i_A(M)t^i.$$

2.2. Factorizations

In this section we introduce the notion of factorization of a local ring homomorphism, especially Cohen factorizations from the work of Avramov, Foxby, and B. Herzog [16] that are central objects of study for Chapter 5.

Definition 2.32. Let $\varphi\colon (R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a local ring homomorphism.

(a) The *embedding dimension* of φ is $\operatorname{edim}(\varphi) = \operatorname{edim}(S/\mathfrak{m}S)$.

(b) The map φ is *weakly regular* if it is flat and the closed fiber $S/\mathfrak{m}S$ is regular.

(c) The map φ is a *weakly Cohen* if it is weakly regular, and the induced field extension $R/\mathfrak{m} \to S/\mathfrak{n}$ is separable, e.g., if $\operatorname{char}(R/\mathfrak{m}) = 0$ or $R/\mathfrak{m}$ is perfect of positive characteristic.

(d) The map φ is *Cohen* if it is weakly Cohen such that $\mathfrak{n} = \mathfrak{m}S$, that is, it is weakly Cohen such that the closed fiber is a field.

Fact 2.33. Let $R \xrightarrow{\varphi} S \xrightarrow{\psi} T$ be weakly regular local ring homomorphisms. Then the composition $\psi\varphi$ is weakly regular such that $\operatorname{edim}(\psi\varphi) = \operatorname{edim}(\psi)+\operatorname{edim}(\varphi)$; see [50, 5.9].

Definition 2.34. Let $\varphi\colon (R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a local ring homomorphism.

(a) A *regular factorization* of φ is a diagram of local homomorphisms $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} S$ such that $\varphi = \varphi'\dot{\varphi}$, the map $\dot{\varphi}$ is weakly regular, and φ' is surjective.

(b) A *Cohen factorization* of φ is a regular factorization $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} S$ of φ such that R' is complete.

(c) A *comparison* of one Cohen factorization $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} S$ of φ to another one $R \xrightarrow{\ddot{\varphi}} R'' \xrightarrow{\varphi''} S$ is a local homomorphism $\upsilon\colon R' \to R''$ making the following diagram commute:

$$\begin{array}{ccccc} & & R' & & \\ & \overset{\dot\varphi}{\nearrow} & \big\downarrow{\scriptstyle \upsilon} & \overset{\varphi'}{\searrow} & \\ R & \xrightarrow{\ddot\varphi} & R'' & \xrightarrow{\varphi''} & S. \end{array}$$

(d) The *semicompletion* of φ, denoted $\grave{\varphi}\colon R \to \widehat{S}$, is the composition of φ with the natural map $S \to \widehat{S}$.

Remark 2.35. Let $R \xrightarrow{\dot\varphi} R' \xrightarrow{\varphi'} S$ be a regular factorization of a local ring homomorphism $\varphi\colon R \to S$. The surjective homomorphism $\varphi'\colon R' \to S$ induces a surjective homomorphism $R'/\mathfrak{m}R' \to S/\mathfrak{m}S$ which implies that $\operatorname{edim}(\dot\varphi) \geqslant \operatorname{edim}(\varphi)$. Since $\dot\varphi$ is weakly regular, we conclude that

$$\dim(R') - \dim(R) = \operatorname{edim}(\dot\varphi) \geqslant \operatorname{edim}(\varphi).$$

Definition 2.36. A regular factorization $R \xrightarrow{\dot\varphi} R' \xrightarrow{\varphi'} S$ of a local ring homomorphism $\varphi\colon R \to S$ is *minimal* if $\dim(R') - \dim(R) = \operatorname{edim}(\varphi)$.

Fact 2.37. Let $\varphi\colon (R, \mathfrak{m}) \to (S, \mathfrak{n})$ and $\psi\colon S \to T$ be local ring homomorphisms.

(a) By [16, (1.1) Theorem and (1.5) Proposition], if S is complete, then φ has a minimal Cohen factorization. Since $\widehat{S}$ is complete, it follows that the semicompletion $\grave{\varphi}\colon R \to \widehat{S}$ has a minimal Cohen factorization.

(b) If φ is surjective, ψ is weakly regular, and T is complete, then in any minimal Cohen factorization $R \to R' \to T$ of the composition $\psi\varphi$, we have $T \cong R' \otimes_R S$, that is, the diagram

$$\begin{array}{ccc} R & \longrightarrow & R' \\ \big\downarrow & & \big\downarrow \\ S & \longrightarrow & T \end{array}$$

is a pushout; see the proof of [16, (1.6) Theorem].

(c) Assume that S is complete, and consider two Cohen factorizations $R \to R' \to S$ and $R \to R'' \to S$ of φ. If the extension $R/\mathfrak{m} \to S/\mathfrak{n}$ is separable, then there is a comparison $\upsilon\colon R' \to R''$ of the first factorization to the second one; moreover, if both Cohen factorizations are minimal, then any comparison between them is an isomorphism. See [16, (1.7) Proposition].

2.3. Homological Notions

In this section we introduce some homological dimensions for complexes. Some of these dimensions are defined in terms of certain diagrams of local ring homomorphisms.

Definition 2.38. The flat dimension of a local ring homomorphism $\varphi\colon R \to S$ is $\mathrm{fd}(\varphi) = \mathrm{fd}_R(S)$.

The next definition originates with work of Auslander and Bridger [4, 5].

Definition 2.39. Let X be a homologically finite R-complex. Then X is *derived reflexive* if $\mathbf{R}\mathrm{Hom}_R(X,R)$ is homologically finite (that is, homologically bounded) and the natural biduality map $X \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X,R),R)$ is an isomorphism in $\mathcal{D}(R)$. The *G-dimension* of X is

$$\mathrm{G\text{-}dim}_R(X) := \begin{cases} -\inf(\mathbf{R}\mathrm{Hom}_R(X,R)) & \text{if } X \text{ is derived reflexive over } R \\ \infty & \text{otherwise.} \end{cases}$$

Remark 2.40. For a finitely generated R-module, the G-dimension defined above is the same as the one defined by Auslander and Bridger; see [78, 2.7. Theorem]. More generally, the G-dimension of a homologically finite R-complex has a similar interpretation by Christensen [20, (2.3.8) GD Corollary].

Given a finitely generated R-module M, the fact that $\operatorname{Ext}^i_R(M,R)=0$ for all $i>\operatorname{G-dim}_R(M)$ implies that $\operatorname{grade}_R(M)\leqslant \operatorname{G-dim}_R(M)$, where $\operatorname{grade}_R(M)$ is

$$\operatorname{grade}_R(M) := \min\{\, n\geqslant 0 \mid \operatorname{Ext}^n_R(M,R)\neq 0 \,\}.$$

This motivates the next definition.

Definition 2.41. A finitely generated R-module M is *G-perfect* if $\operatorname{grade}_R(M) = \operatorname{G-dim}_R(M)$.

We proceed with CI-dimension of Avramov, Gasharov, and Peeva [17, 65, 64] and Gerko's CM-dimension [36].

Definition 2.42. Consider a diagram $R \xrightarrow{\varphi} R' \xleftarrow{\tau} Q$ of local ring homomorphisms such that φ is flat, and τ is surjective. Such a diagram is a *G-quasi-deformation* if R' is G-perfect as a Q-module. Such a diagram is a *quasi-deformation* if $\operatorname{Ker}(\tau)$ is generated by a Q-regular sequence.

Definition 2.43. Let X be a homologically finite complex over a local ring R. The *CM-dimension* and *CI-dimension* of X are

$$\operatorname{CM-dim}_R(X) := \inf\left\{ \operatorname{G-dim}_Q(R'\otimes^{\mathbf{L}}_R X) - \operatorname{G-dim}_Q(R') \;\middle|\; \begin{array}{c} R\to R'\leftarrow Q \text{ is a} \\ \text{G-quasi-deformation}\end{array}\right\}$$

$$\operatorname{CI-dim}_R(X) := \inf\left\{ \operatorname{pd}_Q(R'\otimes^{\mathbf{L}}_R X) - \operatorname{pd}_Q(R') \;\middle|\; \begin{array}{c} R\to R'\leftarrow Q \text{ is a} \\ \text{quasi-deformation}\end{array}\right\}.$$

Fact 2.44. Let $\alpha'\colon (R',\mathfrak{m}',k')\to(\widetilde{R}',\widetilde{\mathfrak{m}}',\widetilde{k}')$ be a flat local ring homomorphism, and let M be a homologically finite R'-complex. The proof of [17, (1.11) Proposition] shows that $\operatorname{CI-dim}_{\widetilde{R}'}(\widetilde{R}'\otimes^{\mathbf{L}}_{R'} M)\geqslant \operatorname{CI-dim}_{R'}(M)$ with equality holding when

$\mathrm{CI\text{-}dim}_{\widetilde{R}'}(\widetilde{R}' \otimes^{\mathbf{L}}_{R'} M) < \infty$. A similar argument shows that $\mathrm{CM\text{-}dim}_{\widetilde{R}'}(\widetilde{R}' \otimes^{\mathbf{L}}_{R'} M) \geqslant \mathrm{CM\text{-}dim}_{R'}(M)$ with equality holding when $\mathrm{CM\text{-}dim}_{\widetilde{R}'}(\widetilde{R}' \otimes^{\mathbf{L}}_{R'} M) < \infty$.

Definition 2.45. A *dualizing complex* for R is a semidualizing complex D of finite injective dimension, i.e., such that D is isomorphic in $\mathcal{D}(R)$ to a bounded complex of injective R-modules.

Remark 2.46. Assume that R is local. Then R has a dualizing complex if and only if it is a homomorphic image of a Gorenstein local ring; one implication is from Grothendieck and Hartshorne [45], and the other is by Kawasaki [52]. In particular, a complete local ring has a dualizing complex by the Cohen Structure Theorem.

2.4. Properties of Ring Homomorphisms

In this section we introduce different homological dimensions for local ring homomorphisms. We will use these homological dimensions in Chapter 5 where we study properties of local ring homomorphisms in certain commutative diagrams. The first notion in this subsection is from Avramov [9].

Definition 2.47. Let $\varphi\colon R \to S$ be a local ring homomorphism. Given a Cohen factorization $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} \widehat{S}$ of the semicompletion $\grave{\varphi}\colon R \to \widehat{S}$, we say that φ is *complete intersection* if $\mathrm{Ker}(\varphi')$ is generated by an R'-regular sequence.

Remark 2.48. Let $\varphi\colon R \to S$ be a local ring homomorphism. The complete intersection property for φ is independent of the choice of Cohen factorization by [9, (3.3) Remark]. Also R and φ are complete intersection if and only if S is complete intersection and $\mathrm{fd}(\varphi)$ is finite; see [9, (5.9), (5.10), and (5.12)].

The next notion is mostly due to Avramov and Foxby [14], with some contributions from Iyengar and Sather-Wagstaff [50].

Definition 2.49. Let $\varphi\colon R \to S$ be a local ring homomorphism. Given a Cohen factorization $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} \widehat{S}$ of the semicompletion $\grave{\varphi}\colon R \to \widehat{S}$, we set

$$\text{G-dim}(\varphi) := \text{G-dim}_{R'}(\widehat{S}) - \text{edim}(\dot{\varphi}).$$

Remark 2.50. The G-dimension of a local homomorphism is independent of the choice of Cohen factorization by [50, 3.2. Theorem].

Definition 2.51. Let $\varphi\colon R \to S$ be a local ring homomorphism, and let $D^{\widehat{R}}$ be a dualizing complex for $\widehat{R}$. A *dualizing complex* for φ is a semidualizing S-complex D^{φ} such that $D^{\widehat{R}} \otimes^{\mathbf{L}}_{\widehat{R}} (\widehat{S} \otimes^{\mathbf{L}}_{S} D^{\varphi})$ is a dualizing complex for $\widehat{S}$.

Fact 2.52. Let $\varphi\colon R \to S$ be a local ring homomorphism of finite G-dimension, e.g., finite flat dimension; see [14, (4.4.2)]. If S is complete, then φ has a dualizing complex by [14, (6.7) Lemma]; specifically, given a Cohen factorization $R \to R' \to S$ of φ, the S-complex $\mathbf{R}\text{Hom}_{R'}(S, R')$ is dualizing for φ. In particular, the semicompletion $\grave{\varphi}\colon R \to \widehat{S}$ has a dualizing complex.

Next are some notions of Avramov and Foxby [10, 14] and Frankild [31].

Definition 2.53. Let $\varphi\colon R \to S$ be a local ring homomorphism of finite G-dimension and let $D^{\grave{\varphi}}$ be a dualizing complex for the semicompletion $\grave{\varphi}\colon R \to \widehat{S}$. The *quasi-Cohen-Macaulay defect* of φ is $\mathbf{q}\text{cmd}(\varphi) := \text{amp}(D^{\grave{\varphi}})$, and φ is *quasi-Cohen-Macaulay* if $\mathbf{q}\text{cmd}(\varphi) = 0$, that is, if $D^{\grave{\varphi}}$ is isomorphic in $\mathcal{D}(\widehat{S})$ to a module.

When $\text{fd}(\varphi) < \infty$, the *Cohen-Macaulay defect* of φ is $\text{cmd}(\varphi) = \mathbf{q}\text{cmd}(\varphi)$; see [14, (5.5)]. The map φ is *Cohen-Macaulay* if it is quasi-Cohen-Macaulay and has finite flat dimension.

Remark 2.54. Let $\varphi\colon R \to S$ be a local ring homomorphism of finite G-dimension. The quasi-Cohen-Macaulay defect of φ is independent of the choice of dualizing complex, as the dualizing complex is unique up to shift-isomorphism; see [14, (5.4)] and [31, (6.5)]. If R is Cohen-Macaulay and φ is quasi-Cohen-Macaulay, then S is Cohen-Macaulay; the converse holds when φ has finite flat dimension or the induced map $\operatorname{Spec}(\widehat{S}) \to \operatorname{Spec}(\widehat{R})$ is surjective by [16, (3.10) Theorem] and [31, (7.7)].

Definition 2.55. Let $\varphi\colon R \to S$ be a local ring homomorphism of finite G-dimension, and let $D^{\dot{\varphi}}$ be a dualizing complex for the semicompletion $\dot{\varphi}\colon R \to \widehat{S}$ such that $\inf(D^{\dot{\varphi}}) = \operatorname{depth}(S) - \operatorname{depth}(R)$. The *Bass series* for φ is the Poincaré series $I_\varphi(t) := P^{\widehat{S}}_{D^{\dot{\varphi}}}(t)$.

Remark 2.56. Let $\varphi\colon R \to S$ be a local ring homomorphism of finite G-dimension. The Bass series for φ is a formal Laurent series $I_\varphi(t)$ with non-negative integer coefficients satisfying the formal relation $I^S_S(t) = I^R_R(t) I_\varphi(t)$; see [14, (7.1) Theorem].

Definition 2.57. Let $\varphi\colon R \to S$ be a local ring homomorphism of finite G-dimension. Then φ is *quasi-Gorenstein* if $I_\varphi(t) = t^{\operatorname{depth}(S)-\operatorname{depth}(R)}$, that is, if $\widehat{S}$ is dualizing for $\dot{\varphi}\colon R \to \widehat{S}$. The map φ is *Gorenstein* if it is quasi-Gorenstein and has finite flat dimension.

Remark 2.58. If $\varphi\colon R \to S$ is a local homomorphism with $\operatorname{G-dim}(\varphi) < \infty$, then S is Gorenstein if and only if R is Gorenstein and φ is quasi-Gorenstein by [14, (7.7.2)].

Definition 2.59. Let $\varphi\colon R \to S$ be a local ring homomorphism. The *CM-dimension of* φ and *CI-dimension of* φ are

$$\operatorname{CM-dim}(\varphi) := \inf \left\{ \operatorname{CM-dim}_{R'}(\widehat{S}) - \operatorname{edim}(\dot{\varphi}) \;\middle|\; \begin{array}{c} R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} \widehat{S} \text{ is a Cohen} \\ \text{factorization of } \dot{\varphi} \end{array} \right\}$$
$$\operatorname{CI-dim}(\varphi) := \inf \left\{ \operatorname{CI-dim}_{R'}(\widehat{S}) - \operatorname{edim}(\dot{\varphi}) \;\middle|\; \begin{array}{c} R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} \widehat{S} \text{ is a Cohen} \\ \text{factorization of } \dot{\varphi} \end{array} \right\}.$$

Remark 2.60. We do not know whether the finiteness of CM-dimension and/or CI-dimension of a local homomorphism is independent of the choice of Cohen factorization.

CHAPTER 3. LIFTINGS AND QUASI-LIFTINGS OF DG MODULES

In this chapter, we prove lifting results for DG modules including Theorem 1.2 from the introduction that are akin to Auslander, Ding, and Solberg's famous lifting results for modules [6]. These results will be used in Chapter 4 to solve a question of Vasconcelos from 1974.

3.1. Structure of Semi-free DG Modules and DG Homomorphisms

The proof of Theorem 1.2 involves the manipulation of the differentials on certain DG modules to construct isomorphisms that are amenable to lifting. For this, we need a concrete understanding of these differentials and the homomorphisms between these DG modules. This concrete understanding is the goal of this section. We begin by establishing some notation to be used for much of this chapter.

Notation 3.1. Let A be a DG R-algebra such that each A_i is free over R of finite rank. Given an element $t \in R$, let $K = K^R(t)$ denote the Koszul complex $0 \to K_1 \xrightarrow{t} K_0 \to 0$ with $K_1 \cong R \cong K_0$ and basis elements $1 \in K_0$ and $e \in K_1$. We fix a basis $\{\gamma_{i,1}, \dots, \gamma_{i,r_i}\}$ for A_i. Let B denote the DG R-algebra $K^R(t) \otimes_R A$, which has the following form

$$B \cong \cdots \xrightarrow{\partial^B_{i+1}} A_{i-1} \oplus A_i \xrightarrow{\partial^B_i} A_{i-2} \oplus A_{i-1} \xrightarrow{\partial^B_{i-1}} \cdots \xrightarrow{\partial^B_2} A_0 \oplus A_1 \xrightarrow{\partial^B_1} 0 \oplus A_0 \to 0.$$

This uses the isomoprhism $B_i = (K_1 \otimes_R A_{i-1}) \oplus (K_0 \otimes_R A_i) \cong A_{i-1} \oplus A_i$. We identify B_i with $A_{i-1} \oplus A_i$ for the remainder of this dissertation. Under this identification, the sum $e \otimes a_{i-1} + 1 \otimes a_i \in B_i$ corresponds to the column vector $\left[\begin{smallmatrix} a_{i-1} \\ a_i \end{smallmatrix}\right] \in A_{i-1} \oplus A_i$. The use of column vectors allows us to identify the differential of B as the matrix

$$\partial^B_i = \begin{bmatrix} -\partial^A_{i-1} & 0 \\ t & \partial^A_i \end{bmatrix}.$$

Remark 3.2. In Notation 3.1, the algebra structure on B translates to the formula

$$\begin{bmatrix} a_{i-1} \\ a_i \end{bmatrix} \begin{bmatrix} c_{j-1} \\ c_j \end{bmatrix} = \begin{bmatrix} a_{i-1}c_j + (-1)^i a_i c_{j-1} \\ a_i c_j \end{bmatrix}$$

where $\left[\begin{smallmatrix} a_{i-1} \\ a_i \end{smallmatrix}\right] \in B_i$ and $\left[\begin{smallmatrix} c_{j-1} \\ c_j \end{smallmatrix}\right] \in B_j$. This uses the fact that $e^2 = 0$ in K.

Note that a basis of B_i is

$$\left\{ \begin{bmatrix} \gamma_{i-1,1} \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} \gamma_{i-1,r_{i-1}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \gamma_{i,1} \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ \gamma_{i,r_i} \end{bmatrix} \right\}.$$

Also, note that the assumptions on A imply that A and B are noetherian. From the explicit description of ∂^B, it follows that $\mathrm{H}_0(B) \cong \mathrm{H}_0(A)/t\,\mathrm{H}_0(A)$.

Assume that R and A are local. Then B is also local. Moreover, given the augmentation ideal $\mathfrak{m}_A = \cdots \xrightarrow{\partial_2^A} A_1 \xrightarrow{\partial_1^A} \mathfrak{m}_0 \to 0$ it is straightforward to show that the augmentation ideal of B is

$$\mathfrak{m}_B = \cdots \xrightarrow{\partial_{i+1}^B} A_{i-1} \oplus A_i \xrightarrow{\partial_i^B} A_{i-2} \oplus A_{i-1} \xrightarrow{\partial_{i-1}^B} \cdots \xrightarrow{\partial_2^B} A_0 \oplus A_1 \xrightarrow{\partial_1^B} 0 \oplus \mathfrak{m}_0 \to 0$$

and we have $B/\mathfrak{m}_B \cong A/\mathfrak{m}_A$.

Notation 3.3. We work in the setting of Notation 3.1. Let $\{\beta_i\}_{i=-\infty}^{\infty}$ be a set of cardinal numbers such that $\beta_i = 0$ for $i \ll 0$. For each integer i, set

$$M_i = \bigoplus_{j=0}^{\infty} A_j^{(\beta_{i-j})}$$

where $A_j^{(\beta_{i-j})}$ is a direct sum of copies of A_j indexed by β_{i-j}. Identify each β_i with a basis of $A_0^{(\beta_i)}$ over A_0, and set $\beta = \bigcup_i \beta_i$ considered as a subset of the disjoint union $\bigsqcup_i M_i$. Define scalar multiplication on M over A using the scalar multiplication on each $A^{(\beta_i)}$.

Consider R-module homomorphisms

$$\xi_i \colon M_i \to M_{i-1}, \quad \tau_i \colon M_i \to M_i, \quad \delta_i \colon M_i \to M_{i-2}, \quad \text{and} \quad \alpha_i \colon M_i \to M_{i-1}.$$

For each i, set

$$N_i = M_{i-1} \oplus M_i \qquad \text{and} \qquad \partial_i^N = \begin{bmatrix} \xi_{i-1} & \delta_i \\ \tau_{i-1} & \alpha_i \end{bmatrix} : N_i \to N_{i-1}.$$

We consider the sequences

$$M = \cdots \xrightarrow{\alpha_{i+1}} M_i \xrightarrow{\alpha_i} M_{i-1} \xrightarrow{\alpha_{i-1}} \cdots$$

and

$$N = \cdots \xrightarrow{\partial_{i+1}^N} N_i \xrightarrow{\partial_i^N} N_{i-1} \xrightarrow{\partial_{i-1}^N} \cdots .$$

Given elements $\left[\begin{smallmatrix} a_{i-1} \\ a_i \end{smallmatrix}\right] \in B_i$ and $\left[\begin{smallmatrix} m_{j-1} \\ m_j \end{smallmatrix}\right] \in N_j$, we define

$$\begin{bmatrix} a_{i-1} \\ a_i \end{bmatrix} \begin{bmatrix} m_{j-1} \\ m_j \end{bmatrix} = \begin{bmatrix} a_{i-1}m_j + (-1)^i a_i m_{j-1} \\ a_i m_j \end{bmatrix}.$$

For each $\beta_{i,j} \in \beta_i$ we set $e_{i,j} = \left[\begin{smallmatrix} 0 \\ \beta_{i,j} \end{smallmatrix}\right] \in N_i$. For each i, set $E_i = \{e_{i,j}\}_j$. Let $E = \bigcup_i E_i$ considered as a subset of the disjoint union $\bigsqcup_i N_i$.

Remark 3.4. In Notation 3.3, the sequences M and N may not be complexes. Note that the scalar multiplications defined on M and N make $\oplus_i M_i$ and $\oplus_i N_i$ into graded free modules over $A^\natural$ and $B^\natural$, respectively.

The next result is a straightforward consequence of the definitions in 3.3.

Lemma 3.5. *We work in the setting of Notations* 3.1 *and* 3.3. *The following conditions are equivalent.*

(i) *The sequence M is a semi-free DG A-module;*

(ii) *The sequence M is a DG A-module; and*

(iii) *For all integers i and j we have*

$$\alpha_{i-1}\alpha_i = 0 \qquad \alpha_{i+j}(\gamma_{i,s}m_j) = \partial_i^A(\gamma_{i,s})m_j + (-1)^i\gamma_{i,s}\alpha_j(m_j) \tag{3.5.1}$$

for $s = 1, \dots, r_i$ and for all $m_j \in M_j$.

Next, we give a similar result for the sequence N.

Lemma 3.6. *We work in the setting of Notations* 3.1 *and* 3.3*. The following conditions are equivalent.*

(i) *The sequence N is a semi-free DG B-module;*

(ii) *The sequence N is a DG B-module; and*

(iii) *For all integers i and j we have*

$$\xi_i = -\alpha_i \qquad \tau_i = t \tag{3.6.1}$$

$$\alpha_{i-1}\alpha_i = -t\delta_i \qquad \delta_i\alpha_{i+1} = \alpha_{i-1}\delta_{i+1} \tag{3.6.2}$$

$$\delta_{i+j}(\gamma_{i,s}m_j) = \gamma_{i,s}\delta_j(m_j) \tag{3.6.3}$$

$$\alpha_{i+j}(\gamma_{i,s}m_j) = \partial_i^A(\gamma_{i,s})m_j + (-1)^i\gamma_{i,s}\alpha_j(m_j) \tag{3.6.4}$$

for $s = 1, \dots, r_i$ and for all $m_j \in M_j$.

In particular, if N is a DG B-module, then

$$\partial_i^N = \begin{bmatrix} -\alpha_{i-1} & \delta_i \\ t & \alpha_i \end{bmatrix}. \tag{3.6.5}$$

Proof. (ii) $\Longrightarrow$ (iii) Assume that N is a DG B-module. Then the scalar multiplication defined in Notation 3.3 must satisfy the Leibniz rule. The Leibniz rule for products of the form $\left[\begin{smallmatrix} 0 \\ \gamma_{i,s} \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 \\ m_j \end{smallmatrix}\right]$, where $1 \leqslant s \leqslant r_i$ and $m_j \in M_j$, is equivalent to the following relations:

$$\delta_{i+j}(\gamma_{i,s}m_j) = \gamma_{i,s}\delta_j(m_j) \tag{3.6.6}$$

$$\alpha_{i+j}(\gamma_{i,s}m_j) = \partial_i^A(\gamma_{i,s})m_j + (-1)^i\gamma_{i,s}\alpha_j(m_j). \tag{3.6.7}$$

The Leibniz rule for products of the form $\left[\begin{smallmatrix} \gamma_{i,s} \\ 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 \\ m_j \end{smallmatrix}\right]$ is equivalent to the following:

$$\tau_{i+j}(\gamma_{i,s}m_j) = t\gamma_{i,s}m_j \tag{3.6.8}$$

$$\xi_{i+j}(\gamma_{i,s}m_j) = -(\partial_i^A(\gamma_{i,s})m_j + (-1)^i\gamma_{i,s}\alpha_j(m_j)). \tag{3.6.9}$$

The Leibniz rule for products of the form $\left[\begin{smallmatrix} 0 \\ \gamma_{i,s} \end{smallmatrix}\right]\left[\begin{smallmatrix} m_j \\ 0 \end{smallmatrix}\right]$ is equivalent to the following:

$$\tau_{i+j}(\gamma_{i,s}m_j) = \gamma_{i,s}\tau_j(m_j) \tag{3.6.10}$$

$$\xi_{i+j}(\gamma_{i,s}m_j) = -\partial_i^A(\gamma_{i,s})m_j + (-1)^i\gamma_{i,s}\xi_j(m_j). \tag{3.6.11}$$

The Leibniz rule for $\left[\begin{smallmatrix} \gamma_{i,s} \\ 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} m_j \\ 0 \end{smallmatrix}\right] = 0$ is equivalent to the following:

$$(-1)^i t\gamma_{i,s}m_j + (-1)^{i+1}\gamma_{i,s}\tau_j(m_j) = 0. \tag{3.6.12}$$

Equation (3.6.3) is the same as (3.6.6), and equation (3.6.4) is the same as (3.6.7). Comparing equations (3.6.7) and (3.6.9) with $\gamma_{0,1} = 1$, we find $\xi_i = -\alpha_i$. Using equation (3.6.8) also with $\gamma_{0,1} = 1$, we see that $\tau_i = t$. This explains (3.6.2) and (3.6.5). It also shows that (3.6.12) is trivial. Since N is an R-complex, we have $\partial_i^N\partial_{i+1}^N = 0$ which gives the equations in (3.6.2). This completes the proof of the implication.

The implication (iii) $\Longrightarrow$ (ii) is handled similarly, and the equivalence (i) $\Longleftrightarrow$ (ii) is straightforward. □

Our next two results characterize semi-free DG modules over A and B. The first one is straightforward.

Lemma 3.7. *We work in the setting of Notations* 3.1 *and* 3.3. *If F is a bounded below semi-free DG A-module with semi-basis G, then $F \cong M$ for some appropriate choice of α_i satisfying* (3.5.1) *for all i and j where $\beta_i = |G \cap F_i|$.*

Lemma 3.8. *We work in the setting of Notations* 3.1 *and* 3.3. *If F is a bounded below semi-free DG B-module with semi-basis G, then $F \cong N$ for some appropriate choices of ξ_i, τ_i, α_i, and δ_i satisfying* (3.6.1)–(3.6.4) *for all i and j where $\beta_i = |G \cap F_i|$.*

Proof. Let F be a bounded below semi-free DG B-module with semi-basis G. For each i, set $\beta_i = |G \bigcap F_i|$. Since F is semi-free, it is straightforward to show that $F_i \cong \bigoplus_{j=0}^{\infty} B_j^{(\beta_{i-j})}$. Decomposing B_j as $A_{j-1} \bigoplus A_j$, we see that $F_j \cong N_j$ for each j. Since the R-module homomorphisms $N_j \to N_{j-1}$ are necessarily of the form $\left[\begin{smallmatrix} \xi_{i-1} & \delta_i \\ \tau_{i-1} & \alpha_i \end{smallmatrix}\right]$, it follows that there are appropriate choices of ξ_i, τ_i, α_i, and δ_i such that $F \cong N$. Finally, the fact that F is a DG B-module implies that the maps ξ_i, τ_i, α_i, and δ_i satisfy (3.6.1)–(3.6.4), by Lemma 3.6. □

The next result indicates how a semi-free DG B-module should look in order to be liftable to A. See Section 3.2 for more about this.

Lemma 3.9. *We work in the setting of Notations* 3.1 *and* 3.3. *If M is a semi-free DG A-module, then $B \otimes_A M$ is a semi-free DG B-module, identified with a DG B-module N with*

$$\partial_i^N = \begin{bmatrix} -\alpha_{i-1} & 0 \\ t & \alpha_i \end{bmatrix}.$$

Proof. Using the isomorphisms

$$B \otimes_A M \cong (K^R(t) \otimes_R A) \otimes_A M \cong K^R(t) \otimes_R M$$

the result follows directly from the definitions in 3.3 with Lemmas 3.5 and 3.6. □

Next, we describe DG module homomorphisms over A and B. Again, the proof of the first of these results is straightforward.

Lemma 3.10. *We work in the setting of Notations* 3.1 *and* 3.3. *Assume that M is a semi-free DG A-module, and let M' be a second semi-free DG A-module. Fix an integer p. A sequence of R-module homomorphisms $\{u_i \colon M_i \to M'_{i+p}\}$ is a DG A-module homomorphism $M \to M'$ of degree p if and only if it is a degree-p homomorphism $M \to M'$ of the underlying R-complexes and*

$$u_{i+j}(\gamma_{i,s} m_j) = (-1)^{pi} \gamma_{i,s} u_j(m_j)$$

for $s = 1, \ldots, r_i$ and for all $m_j \in M_j$ for each integer j.

Lemma 3.11. *We work in the setting of Notations* 3.1 *and* 3.3. *Assume that N is a semi-free DG B-module. Let N' be a second semi-free DG B-module built from modules M'_i and maps ξ'_i, τ'_i, δ'_i, and α'_i as in* 3.3. *Fix an integer p. A sequence of R-module homomorphisms $\left\{S_i \colon N_i \to N'_{i+p}\right\}$ is a DG B-module homomorphism $N \to N'$ of degree p if and only if it is a degree-p homomorphism $N \to N'$ of the underlying R-complexes such that for all integers i we have $S_i = \left[\begin{smallmatrix} (-1)^p z_{i-1} & v_i \\ 0 & z_i \end{smallmatrix}\right]$ for some $z_i \colon M_i \to M'_{i+p}$ and $v_i \colon M_i \to M'_{i+p-1}$ and*

$$v_{i+j}(\gamma_{i,s} m_j) = (-1)^{i(p+1)} \gamma_{i,s} v_j(m_j) \tag{3.11.1}$$

$$z_{i+j}(\gamma_{i,s} m_j) = (-1)^{ip} \gamma_{i,s} z_j(m_j) \tag{3.11.2}$$

for $s = 1, \ldots, r_i$ and for all $m_j \in M_j$ for each integer j.

Proof. Fix a sequence of R-module homomorphisms $S = \left\{S_i \colon N_i \to N'_{i+p}\right\}$. By assumption, we have $N_i = M_{i-1} \oplus M_i$ and $N'_i = M'_{i-1} \oplus M'_i$, so the maps S_i have the form

$$S_i = \begin{bmatrix} u_{i-1} & v_i \\ y_{i-1} & z_i \end{bmatrix} : M_{i-1} \oplus M_i \to M'_{i+p-1} \oplus M'_{i+p}$$

where $u_{i-1}\colon M_{i-1} \to M'_{i+p-1}$, $v_i\colon M_i \to M'_{i+p-1}$, $y_{i-1}\colon M_{i-1} \to M'_{i+p}$, and $z_i\colon M_i \to M'_{i+p}$.

Assume that S is a DG B-module homomorphism $N \to N'$ of degree p. For each $b_q \in B_q$ and $n_d \in N_d$, we have

$$S_{q+d}(b_q n_d) = (-1)^{pq} b_q S_d(n_d). \tag{3.11.3}$$

Therefore, given integers i and j, for $s = 1, \ldots, r_i$ and for all $m_j \in M_j$, by writing equation (3.11.3) for the elements $\left[\begin{smallmatrix} \gamma_{i,s} \\ 0 \end{smallmatrix}\right] \in B_{i+1}$ and $\left[\begin{smallmatrix} 0 \\ m_j \end{smallmatrix}\right] \in N_j$ we have

$$y_{i+j}(\gamma_{i,s} m_j) = 0 \tag{3.11.4}$$

$$u_{i+j}(\gamma_{i,s} m_j) = (-1)^{(i+1)p} \gamma_{i,s} z_j(m_j). \tag{3.11.5}$$

Using the elements $\left[\begin{smallmatrix} 0 \\ \gamma_{i,s} \end{smallmatrix}\right] \in B_i$ and $\left[\begin{smallmatrix} 0 \\ m_j \end{smallmatrix}\right] \in N_j$ we have

$$v_{i+j}(\gamma_{i,s} m_j) = (-1)^{i(p+1)} \gamma_{i,s} v_j(m_j) \tag{3.11.6}$$

$$z_{i+j}(\gamma_{i,s} m_j) = (-1)^{ip} \gamma_{i,s} z_j(m_j). \tag{3.11.7}$$

Similar equations arise using the elements $\left[\begin{smallmatrix} \gamma_{i,s} \\ 0 \end{smallmatrix}\right] \in B_{i+1}$ and $\left[\begin{smallmatrix} m_j \\ 0 \end{smallmatrix}\right] \in N_{j+1}$ and the elements $\left[\begin{smallmatrix} 0 \\ \gamma_{i,s} \end{smallmatrix}\right] \in B_i$ and $\left[\begin{smallmatrix} m_j \\ 0 \end{smallmatrix}\right] \in N_{j+1}$.

By comparing equations (3.11.5) and (3.11.7) we conclude that $z_i = (-1)^p u_i$. Equation (3.11.4) with $\gamma_{0,1} = 1$ implies that $y_i = 0$ for all i. Therefore, S_i has the desired form $S_i = \left[\begin{smallmatrix} (-1)^p z_{i-1} & v_i \\ 0 & z_i \end{smallmatrix}\right]$. Also, equations (3.11.6) and (3.11.7) are exactly (3.11.1) and (3.11.2). This completes the proof of the forward implication. The converse is established similarly. $\square$

The last result of this section describes some homomorphisms between semi-free DG B-modules that are liftable to A.

Lemma 3.12. *We work in the setting of Notations* 3.1 *and* 3.3. *Let M and M' be semi-free DG A-modules, and let $f \in \operatorname{Hom}_A(M, M')_j$. If $(B \otimes_A M)_i$ is identified with $M_{i-1} \oplus M_i$ and similarly for $(B \otimes_A M')_i$, then the map $(B \otimes_A f)_i \colon (B \otimes_A M)_i \to (B \otimes_A M')_{i+j}$ is identified with the matrix $\left[\begin{smallmatrix} (-1)^j f_{i-1} & 0 \\ 0 & f_i \end{smallmatrix}\right]$.*

Proof. This follows directly from Definition 2.24. □

3.2. Liftings and Quasi-liftings of DG Modules

In this section we prove Theorem 1.2, starting with the definitions of our notions of liftings in the DG arena.

Definition 3.13. Let $T \to S$ be a morphism of DG R-algebras, and let D be a DG S-module. Then D is *quasi-liftable* to T if there is a semi-free DG T-module D' such that $D \simeq S \otimes_T D'$; in this case D' is called a *quasi-lifting* of D to T. We say that D is *liftable* to T if there is a DG T-module D' such that $D \cong S \otimes_T D'$; in this case D' is called a *lifting* of D to T.

Remark 3.14. In the definition of "quasi-liftable" we require that D' is semi-free in order to avoid the need for derived categories. If one prefers, one can remove the semi-free assumption and require that $D \simeq S \otimes^{\mathbf{L}}_T D'$ instead.

Our next result is a technical lemma for use in the proof of Theorem 1.2.

Lemma 3.15. *We work in the setting of Notations* 3.1 *and* 3.3. *Let $n \geqslant 1$, and let $N^{(n-1)}$ be a semi-free DG B-module*

$$N^{(n-1)} = \cdots \to M_{i-1} \oplus M_i \xrightarrow{\left[\begin{smallmatrix} -\alpha^{(n-1)}_{i-1} & t^{n-1}\delta^{(n-1)}_i \\ t & \alpha^{(n-1)}_i \end{smallmatrix}\right]} M_{i-2} \oplus M_{i-1} \to \cdots$$

whose semi-basis over B is finite in each degree. In the case $n \geqslant 2$, assume that for each i there are R-module homomorphisms $v_i^{(n-2)}\colon M_i \to M_{i-2}$ and $z_i^{(n-2)}\colon M_i \to M_{i-1}$ such that

$$\alpha_i^{(n-1)} = \alpha_i^{(n-2)} + t^{n-1} z_i^{(n-2)} \tag{3.15.1}$$

$$\delta_i^{(n-1)} = v_i^{(n-2)} - t^{n-2} z_{i-1}^{(n-2)} z_i^{(n-2)} \tag{3.15.2}$$

$$v_{i+j}^{(n-2)}(\gamma_{i,s} m_j) = \gamma_{i,s} v_j^{(n-2)}(m_j) \tag{3.15.3}$$

$$z_{i+j}^{(n-2)}(\gamma_{i,s} m_j) = (-1)^i \gamma_{i,s} z_j^{(n-2)}(m_j). \tag{3.15.4}$$

$$\alpha_{i-2}^{(n-2)} z_{i-1}^{(n-2)} + z_{i-2}^{(n-2)} \alpha_{i-1}^{(n-2)} + t v_{i-1}^{(n-2)} = \delta_{i-1}^{(n-2)} \tag{3.15.5}$$

$$-\alpha_{i-2}^{(n-2)} v_i^{(n-2)} + t^{n-2} \delta_{i-1}^{(n-2)} z_i^{(n-2)} - t^{n-2} z_{i-2}^{(n-2)} \delta_i^{(n-2)} + v_{i-1}^{(n-2)} \alpha_i^{(n-2)} = 0. \tag{3.15.6}$$

for $s = 1, \cdots, r_i$, for all $m_j \in M_j$, and for each integer j.

If $\operatorname{Ext}_B^2(N^{(n-1)}, N^{(n-1)}) = 0$, then there is a semi-free DG B-module $N^{(n)}$ and an isomorphism of DG B-modules

$$\begin{array}{ccccccc}
N^{(n-1)} = \cdots & \longrightarrow & M_{i-1} \oplus M_i & \xrightarrow{\begin{bmatrix} -\alpha_{i-1}^{(n-1)} & t^{n-1}\delta_i^{(n-1)} \\ t & \alpha_i^{(n-1)} \end{bmatrix}} & M_{i-2} \oplus M_{i-1} & \longrightarrow & \cdots \\
 & & \Big\downarrow {\scriptstyle \begin{bmatrix} 1 & -t^{n-1} z_i^{(n-1)} \\ 0 & 1 \end{bmatrix}} & & \Big\downarrow {\scriptstyle \begin{bmatrix} 1 & -t^{n-1} z_{i-1}^{(n-1)} \\ 0 & 1 \end{bmatrix}} & & \\
N^{(n)} = \cdots & \longrightarrow & M_{i-1} \oplus M_i & \xrightarrow{\begin{bmatrix} -\alpha_{i-1}^{(n)} & t^{n}\delta_i^{(n)} \\ t & \alpha_i^{(n)} \end{bmatrix}} & M_{i-2} \oplus M_{i-1} & \longrightarrow & \cdots .
\end{array}$$

such that for each index i there are R-module homomorphisms $v_i^{(n-1)}\colon M_i \to M_{i-2}$ and $z_i^{(n-1)}\colon M_i \to M_{i-1}$ such that

$$\alpha_i^{(n)} = \alpha_i^{(n-1)} + t^{n} z_i^{(n-1)} \tag{3.15.7}$$

$$\delta_i^{(n)} = v_i^{(n-1)} - t^{n-1} z_{i-1}^{(n-1)} z_i^{(n-1)} \tag{3.15.8}$$

$$v_{i+j}^{(n-1)}(\gamma_{i,s} m_j) = \gamma_{i,s} v_j^{(n-1)}(m_j) \tag{3.15.9}$$

$$z_{i+j}^{(n-1)}(\gamma_{i,s} m_j) = (-1)^i \gamma_{i,s} z_j^{(n-1)}(m_j). \tag{3.15.10}$$

$$\alpha_{i-2}^{(n-1)} z_{i-1}^{(n-1)} + z_{i-2}^{(n-1)} \alpha_{i-1}^{(n-1)} + t v_{i-1}^{(n-1)} = \delta_{i-1}^{(n-1)} \tag{3.15.11}$$

$$-\alpha_{i-2}^{(n-1)} v_i^{(n-1)} + t^{n-1} \delta_{i-1}^{(n-1)} z_i^{(n-1)} - t^{n-1} z_{i-2}^{(n-1)} \delta_i^{(n-1)} + v_{i-1}^{(n-1)} \alpha_i^{(n-1)} = 0. \tag{3.15.12}$$

for $s = 1, \cdots, r_i$*, for all* $m_j \in M_j$*, and for each integer* j*.*

Proof. By Lemma 3.6, we conclude that for all integers i, j we have

$$\alpha_{i-1}^{(n-1)}\alpha_i^{(n-1)} = -t^n\delta_i^{(n-1)} \qquad t^{n-1}\delta_i^{(n-1)}\alpha_{i+1}^{(n-1)} = \alpha_{i-1}^{(n-1)}t^{n-1}\delta_{i+1}^{(n-1)} \tag{3.15.13}$$

$$t^{n-1}\delta_{i+j}^{(n-1)}(\gamma_{i,s}m_j) = \gamma_{i,s}t^{n-1}\delta_j^{(n-1)}(m_j) \tag{3.15.14}$$

$$\alpha_{i+j}^{(n-1)}(\gamma_{i,s}m_j) = \partial_i^A(\gamma_{i,s})m_j + (-1)^i\gamma_{i,s}\alpha_j^{(n-1)}(m_j) \tag{3.15.15}$$

for $s = 1, \ldots, r_i$ and for all $m_j \in M_j$.

Note that the sequence $\left\{ \begin{bmatrix} \delta_{i-1}^{(n-1)} & 0 \\ 0 & \delta_i^{(n-1)} \end{bmatrix} : M_{i-1} \oplus M_i \to M_{i-3} \oplus M_{i-2} \right\}$ is a cycle of degree -2 in the complex $\mathrm{Hom}_B(N^{(n-1)}, N^{(n-1)})$. Indeed, in the case $n = 1$, this follows from equations (3.15.13)–(3.15.14); in the case $n \geqslant 2$, this follows from equations (3.15.1)–(3.15.6). The assumption $\mathrm{Ext}_B^2(N^{(n-1)}, N^{(n-1)}) = 0$ implies that this cycle is null-homotopic, that is, there is a DG B-module homomorphism $S^{(n-1)} = \{S_i^{(n-1)}\}\colon N^{(n-1)} \to N^{(n-1)}$ of degree -1 such that

$$\begin{bmatrix} \delta_{i-1}^{(n-1)} & 0 \\ 0 & \delta_i^{(n-1)} \end{bmatrix} = \begin{bmatrix} -\alpha_{i-2}^{(n-1)} & t^{n-1}\delta_{i-1}^{(n-1)} \\ t & \alpha_{i-1}^{(n-1)} \end{bmatrix} S_i^{(n-1)} + S_{i-1}^{(n-1)} \begin{bmatrix} -\alpha_{i-1}^{(n-1)} & t^{n-1}\delta_i^{(n-1)} \\ t & \alpha_i^{(n-1)} \end{bmatrix}. \tag{3.15.16}$$

Lemma 3.11 implies that each $S_i^{(n-1)}$ is of the form

$$S_i^{(n-1)} = \begin{bmatrix} -z_{i-1}^{(n-1)} & v_i^{(n-1)} \\ 0 & z_i^{(n-1)} \end{bmatrix}$$

where $v_i^{(n-1)}\colon M_i \to M_{i-2}$ and $z_i^{(n-1)}\colon M_i \to M_{i-1}$, and for $s = 1, \cdots, r_i$, and for all $m_j \in M_j$ for each integer j we have

$$v_{i+j}^{(n-1)}(\gamma_{i,s}m_j) = \gamma_{i,s}v_j^{(n-1)}(m_j) \tag{3.15.17}$$

$$z_{i+j}^{(n-1)}(\gamma_{i,s}m_j) = (-1)^i\gamma_{i,s}z_j^{(n-1)}(m_j). \tag{3.15.18}$$

Hence for every i the equality (3.15.16) implies that we have

$$\alpha_{i-2}^{(n-1)} z_{i-1}^{(n-1)} + z_{i-2}^{(n-1)} \alpha_{i-1}^{(n-1)} + t v_{i-1}^{(n-1)} = \delta_{i-1}^{(n-1)} \tag{3.15.19}$$

$$-\alpha_{i-2}^{(n-1)} v_i^{(n-1)} + t^{n-1} \delta_{i-1}^{(n-1)} z_i^{(n-1)} - t^{n-1} z_{i-2}^{(n-1)} \delta_i^{(n-1)} + v_{i-1}^{(n-1)} \alpha_i^{(n-1)} = 0. \tag{3.15.20}$$

Now let

$$\alpha_i^{(n)} = \alpha_i^{(n-1)} + t^n z_i^{(n-1)} \qquad \delta_i^{(n)} = v_i^{(n-1)} - t^{n-1} z_{i-1}^{(n-1)} z_i^{(n-1)} \tag{3.15.21}$$

and

$$N^{(n)} = \cdots \to M_{i-1} \oplus M_i \xrightarrow{\begin{bmatrix} -\alpha_{i-1}^{(n)} & t^n \delta_i^{(n)} \\ t & \alpha_i^{(n)} \end{bmatrix}} M_{i-2} \oplus M_{i-1} \to \cdots .$$

Note that the conclusions (3.15.7)–(3.15.12) follow directly from (3.15.17)–(3.15.21).

Since $N^{(n-1)}$ is a DG B-module, equations (3.15.13), (3.15.19), and (3.15.21) give the following equation for all i:

$$\alpha_{i-1}^{(n)} \alpha_i^{(n)} + t^{n+1} \delta_i^{(n)} = 0. \tag{3.15.22}$$

By equations (3.15.13), (3.15.19), (3.15.20), and (3.15.21), for all i we have

$$-\alpha_{i-1}^{(n)} t^n \delta_{i+1}^{(n)} + t^n \delta_i^{(n)} \alpha_{i+1}^{(n)} = 0. \tag{3.15.23}$$

For $s = 1, \cdots, r_i$, and for all $m_j \in M_j$, equations (3.15.14), (3.15.17), (3.15.18), and (3.15.21) give the following equality:

$$t^n \delta_{i+j}^{(n)} (\gamma_{i,s} m_j) = \gamma_{i,s} t^n \delta_j^{(n)} (m_j). \tag{3.15.24}$$

Also, by equations (3.15.15), (3.15.18), and (3.15.21) we conclude that

$$\alpha_{i+j}^{(n)}(\gamma_{i,s}m_j) = \partial_i^A(\gamma_{i,s})m_j + (-1)^i\gamma_{i,s}\alpha_j^{(n)}(m_j). \tag{3.15.25}$$

Therefore, equations (3.15.22)–(3.15.25) and Lemma 3.6 imply that $N^{(n)}$ is a semi-free DG B-module. Equations (3.15.18)–(3.15.19) and (3.15.21) provide the next morphism of DG B-modules:

$$\begin{array}{ccccccc} N^{(n-1)} = \cdots & \longrightarrow & M_{i-1} \oplus M_i & \xrightarrow{\left[\begin{smallmatrix} -\alpha_{i-1}^{(n-1)} & t^{n-1}\delta_i^{(n-1)} \\ t & \alpha_i^{(n-1)} \end{smallmatrix}\right]} & M_{i-2} \oplus M_{i-1} & \longrightarrow & \cdots \\ & & \Big\downarrow {\left[\begin{smallmatrix} 1 & -t^{n-1}z_i^{(n-1)} \\ 0 & 1 \end{smallmatrix}\right]} & & \Big\downarrow {\left[\begin{smallmatrix} 1 & -t^{n-1}z_{i-1}^{(n-1)} \\ 0 & 1 \end{smallmatrix}\right]} & & \\ N^{(n)} = \cdots & \longrightarrow & M_{i-1} \oplus M_i & \xrightarrow{\left[\begin{smallmatrix} -\alpha_{i-1}^{(n)} & t^{n}\delta_i^{(n)} \\ t & \alpha_i^{(n)} \end{smallmatrix}\right]} & M_{i-2} \oplus M_{i-1} & \longrightarrow & \cdots . \end{array}$$

Similar reasoning shows that the sequence $\left\{\left[\begin{smallmatrix} 1 & t^{n-1}z_i^{(n-1)} \\ 0 & 1 \end{smallmatrix}\right]\right\}$ is a morphism of DG B-modules, and it is straightforward to show that these sequences are inverse isomorphisms. □

Part (a) of Theorem 1.2 is a consequence of the next result.

Theorem 3.16. *We work in the setting of Notations* 3.1 *and* 3.3*. Assume that R is tR-adically complete and that N is semi-free such that its semi-basis over B is finite in each degree. If* $\operatorname{Ext}_B^2(N,N) = 0$*, then N is liftable to A with semi-free lifting.*

Proof. Set $N^{(0)} = N$. Here, we use a natural variation of Notation 3.3; see equation (3.6.5):

$$N^{(0)} = \cdots \to M_{i-1} \oplus M_i \xrightarrow{\left[\begin{smallmatrix} -\alpha_{i-1}^{(0)} & \delta_i^{(0)} \\ t & \alpha_i^{(0)} \end{smallmatrix}\right]} M_{i-2} \oplus M_{i-1} \to \cdots .$$

Because of our assumptions, each M_i is a finitely generated free R-module.

Lemma 3.15 implies that for each $n \geqslant 1$ there is a semi-free DG B-module $N^{(n)}$ and an isomorphism of DG B-modules

$$\begin{array}{ccccccc}
N^{(n-1)} = \cdots & \longrightarrow & M_{i-1} \oplus M_i & \xrightarrow{\begin{bmatrix} -\alpha_{i-1}^{(n-1)} & t^{n-1}\delta_i^{(n-1)} \\ t & \alpha_i^{(n-1)} \end{bmatrix}} & M_{i-2} \oplus M_{i-1} & \longrightarrow & \cdots \\
 & & \Big\downarrow {\scriptstyle \begin{bmatrix} 1 & -t^{n-1}z_i^{(n-1)} \\ 0 & 1 \end{bmatrix}} & & \Big\downarrow {\scriptstyle \begin{bmatrix} 1 & -t^{n-1}z_{i-1}^{(n-1)} \\ 0 & 1 \end{bmatrix}} & & \\
N^{(n)} = \cdots & \longrightarrow & M_{i-1} \oplus M_i & \xrightarrow{\begin{bmatrix} -\alpha_{i-1}^{(n)} & t^{n}\delta_i^{(n)} \\ t & \alpha_i^{(n)} \end{bmatrix}} & M_{i-2} \oplus M_{i-1} & \longrightarrow & \cdots
\end{array}$$

such that for each index i there are R-module homomorphisms $v_i^{(n-1)} \colon M_i \to M_{i-2}$ and $z_i^{(n-1)} \colon M_i \to M_{i-1}$ such that

$$\alpha_i^{(n)} = \alpha_i^{(n-1)} + t^n z_i^{(n-1)} \tag{3.16.1}$$

$$\delta_i^{(n)} = v_i^{(n-1)} - t^{n-1} z_{i-1}^{(n-1)} z_i^{(n-1)} \tag{3.16.2}$$

$$v_{i+j}^{(n-1)}(\gamma_{i,s} m_j) = \gamma_{i,s} v_j^{(n-1)}(m_j) \tag{3.16.3}$$

$$z_{i+j}^{(n-1)}(\gamma_{i,s} m_j) = (-1)^i \gamma_{i,s} z_j^{(n-1)}(m_j). \tag{3.16.4}$$

$$\alpha_{i-2}^{(n-1)} z_{i-1}^{(n-1)} + z_{i-2}^{(n-1)} \alpha_{i-1}^{(n-1)} + t v_{i-1}^{(n-1)} = \delta_{i-1}^{(n-1)} \tag{3.16.5}$$

$$-\alpha_{i-2}^{(n-1)} v_i^{(n-1)} + t^{n-1} \delta_{i-1}^{(n-1)} z_i^{(n-1)} - t^{n-1} z_{i-2}^{(n-1)} \delta_i^{(n-1)} + v_{i-1}^{(n-1)} \alpha_i^{(n-1)} = 0. \tag{3.16.6}$$

for $s = 1, \cdots, r_i$, for all $m_j \in M_j$, and for each integer j. This follows by induction on n; note that this uses the isomorphism $N^{(n-1)} \cong N$ in the induction step to conclude that $\mathrm{Ext}_B^2(N^{(n-1)}, N^{(n-1)}) \cong \mathrm{Ext}_B^2(N, N) = 0$.

A straightforward calculation shows that

$$\prod_{j=0}^{n-1} \begin{bmatrix} 1 & -t^j z_i^{(j)} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\sum_{j=0}^{n-1} t^j z_i^{(j)} \\ 0 & 1 \end{bmatrix}.$$

Hence, the composite isomorphism $N^{(0)} \to N^{(n)}$ has the following form:

$$\begin{array}{ccccc} N^{(0)} = \cdots \longrightarrow & M_{i-1} \oplus M_i & \xrightarrow{\begin{bmatrix} -\alpha_{i-1}^{(0)} & \delta_i^{(0)} \\ t & \alpha_i^{(0)} \end{bmatrix}} & M_{i-2} \oplus M_{i-1} & \longrightarrow \cdots \\ & \Big\downarrow {\scriptstyle \begin{bmatrix} 1 & -\sum_{j=0}^{n-1} t^j z_i^{(j)} \\ 0 & 1 \end{bmatrix}} & & \Big\downarrow {\scriptstyle \begin{bmatrix} 1 & -\sum_{j=0}^{n-1} t^j z_i^{(j)} \\ 0 & 1 \end{bmatrix}} & \\ N^{(n)} = \cdots \longrightarrow & M_{i-1} \oplus M_i & \xrightarrow{\begin{bmatrix} -\alpha_{i-1}^{(n)} & t^n \delta_i^{(n)} \\ t & \alpha_i^{(n)} \end{bmatrix}} & M_{i-2} \oplus M_{i-1} & \longrightarrow \cdots . \end{array}$$

Furthermore, equation (3.16.1) shows that

$$\alpha_i^{(n)} = \alpha_i^{(0)} + \sum_{j=0}^{n-1} t^{j+1} z_i^{(j)}.$$

We now define $N^{(\infty)}$ as follows:

$$N^{(\infty)} = \cdots \to M_{i-1} \oplus M_i \xrightarrow{\begin{bmatrix} -\alpha_{i-1}^{(\infty)} & 0 \\ t & \alpha_i^{(\infty)} \end{bmatrix}} M_{i-2} \oplus M_{i-1} \to \cdots$$

where

$$\alpha_i^{(\infty)} = \alpha_i^{(0)} + \sum_{j=0}^{\infty} t^{j+1} z_i^{(j)}.$$

Note that $\alpha_i^{(\infty)}$ is well-defined because R is tR-adically complete and the modules M_i and M_{i-1} are finitely generated free R-modules .

We claim that $N^{(\infty)}$ is a semi-free DG B-module. For all indices i and n, set

$$\zeta_i^{(n)} = \sum_{j=0}^{\infty} t^j z_i^{(j+n)} \tag{3.16.7}$$

and notice that

$$\alpha_i^{(\infty)} = \alpha_i^{(n)} + t^{n+1} \zeta_i^{(n)}. \tag{3.16.8}$$

Using (3.15.22), it follows that

$$\alpha_i^{(\infty)}\alpha_{i+1}^{(\infty)} = t^{n+1}(-\delta_{i+1}^{(n)} + \alpha_i^{(n)}\zeta_{i+1}^{(n)} + \zeta_i^{(n)}\alpha_{i+1}^{(n)} + t^{n+1}\zeta_i^{(n)}\zeta_{i+1}^{(n)}). \tag{3.16.9}$$

It follows that $\alpha_i^{(\infty)}\alpha_{i+1}^{(\infty)} \in \bigcap_{n=1}^{\infty} t^{n+1}\operatorname{Hom}_R(M_i, M_{i-1}) = 0$, by Krull's Intersection Theorem, so we have

$$\alpha_i^{(\infty)}\alpha_{i+1}^{(\infty)} = 0. \tag{3.16.10}$$

Furthermore, for $s = 1, \cdots, r_i$ and for all $m_j \in M_j$ equations (3.15.25), (3.16.4), (3.16.7), and (3.16.8) imply that

$$\alpha_{i+j}^{(\infty)}(\gamma_{i,s}m_j) = \partial_i^A(\gamma_{i,s})m_j + (-1)^i\gamma_{i,s}\alpha_j^{(\infty)}(m_j). \tag{3.16.11}$$

Thus by equations (3.16.10), (3.16.11) and Lemma 3.6 we conclude that $N^{(\infty)}$ is a DG B-module.

Now consider the chain map $\varphi = \{\varphi_i\}\colon N^{(0)} \to N^{(\infty)}$ defined by

$$\varphi_i = \prod_{j=0}^{\infty} \left[\begin{smallmatrix} 1 & -t^j z_i^{(j)} \\ 0 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & -\sum_{j=0}^{\infty} t^j z_i^{(j)} \\ 0 & 1 \end{smallmatrix}\right]$$

This map is well-defined because R is tR-adically complete and N_i is finitely generated over R. Using these assumptions with equation (3.16.4) we conclude that

$$\left(-\textstyle\sum_{l=0}^{\infty} t^l z_{i+j}^{(l)}\right)(\gamma_{i,s}m_j) = (-1)^i\gamma_{i,s}\left(-\textstyle\sum_{l=0}^{\infty} t^l z_j^{(l)}\right)(m_j)$$

for $s = 1, \cdots, r_i$, and for all $m_j \in M_j$ for each integer j. Thus φ is B-linear and satisfies the assumptions of Lemma 3.11, so φ is a morphism of DG B-modules. Similar reasoning shows that the sequence $\left\{\left[\begin{smallmatrix} 1 & \sum_{j=0}^{\infty} t^j z_i^{(j)} \\ 0 & 1 \end{smallmatrix}\right]\right\}$ is a morphism of DG B-modules, and it is easy to show that these sequences are inverse isomorphisms.

On the other hand, by equations (3.16.10)–(3.16.11) and Lemma 3.5, the sequence

$$M^{(\infty)} = \cdots \to M_i \xrightarrow{\alpha_i^{(\infty)}} M_{i-1} \to \cdots$$

is a semi-free DG A-module. Now Lemma 3.9 implies that $M^{(\infty)}$ is a lifting of $N^{(\infty)}$ to A. Since $N \cong N^{(\infty)}$ by definition, we conclude that $M^{(\infty)}$ is a lifting of N to A, so N is liftable to A. □

Corollary 3.17. *We work in the setting of Notation* 3.1*. Assume that R is tR-adically complete. Let D be a DG B-module that is homologically both bounded below and degreewise finite. If* $\operatorname{Ext}_B^2(D,D)=0$*, then D is quasi-liftable to A.*

Proof. Fact 2.23 implies that D has a semi-free resolution $N \simeq D$ over B such that the semi-basis for N is finite in each degree. Since $\operatorname{Ext}_B^2(N,N) \cong \operatorname{Ext}_B^2(D,D) = 0$, Theorem 3.16 implies that N is liftable to A, with semi-free lifting M. Thus, we have $B \otimes_A M \cong N \simeq D$, so D is quasi-liftable to A. □

To prove Theorem 1.2 we need to use induction on n, the length of the sequence $\underline{t}$. The next result is useful for the induction step in the proof.

Proposition 3.18. *We work in the setting of Notation* 3.1*. Assume that R is tR-adically complete, and let D be a DG B-module that is homologically bounded below and homologically degreewise finite such that* $\operatorname{Ext}_B^d(D,D)=0$ *for some integer d. If M is a quasi-lifting of D to A, then* $\operatorname{Ext}_A^d(M,M)=0$*.*

Proof. Assume without loss of generality that M is degreewise finite and bounded below; see Fact 2.23. Lemma 3.7 shows that M has the shape dictated by Notation 3.3. Since M is a quasi-lifting of D to A, we see that $N = B \otimes_A M \cong K^R(t) \otimes_R M$ is a semi-free resolution of D over B; see Lemma 3.9. To show that $\operatorname{Ext}_A^d(M,M)=0$, let $f = \{f_i \colon M_i \to M_{i-d}\}$ be a cycle in $\operatorname{Hom}_A(M,M)_{-d}$; we need to show that f is null-homotopic. The fact that f is a cycle says that for every i we have $f_i \alpha_{i+1} = (-1)^d \alpha_{i+1-d} f_{i+1}$. For each i set $v_i^{(-1)} = f_i$

Claim: For all $n \geqslant 0$ and for all $i \in \mathbb{Z}$, there are maps $v_i^{(n)} \colon M_i \to M_{i-d}$ and $z_i^{(n)} \colon M_i \to M_{i+1-d}$ such that for $s = 1, \cdots, r_i$, and for all $m_j \in M_j$ for each j

$$v_{i+j}^{(n)}(\gamma_{i,s} m_j) = (-1)^{-id} \gamma_{i,s} v_j^{(n)}(m_j) \tag{3.18.1}$$

$$z_{i+j}^{(n)}(\gamma_{i,s} m_j) = (-1)^{i(1-d)} \gamma_{i,s} z_j^{(n)}(m_j) \tag{3.18.2}$$

$$(-1)^d z_{i-2}^{(n)} \alpha_{i-1} + t v_{i-1}^{(n)} + \alpha_{i-d} z_{i-1}^{(n)} = v_{i-1}^{(n-1)} \tag{3.18.3}$$

$$(-1)^d v_{i-1}^{(n)} \alpha_i - \alpha_{i-d} v_i^{(n)} = 0. \tag{3.18.4}$$

To prove the claim, we proceed by induction on n. We verify the base case and the inductive step simultaneously. Let $n \geqslant 0$ and assume that for each i there exists $v_i^{(n-1)} \colon M_i \to M_{i-d}$ such that for $s = 1, \cdots, r_i$, and for all $m_j \in M_j$ for each integer j, we have

$$v_{i+j}^{(n-1)}(\gamma_{i,s} m_j) = (-1)^{id} \gamma_{i,s} v_j^{(n-1)}(m_j) \tag{3.18.5}$$

$$v_{i-1}^{(n-1)} \alpha_i - (-1)^d \alpha_{i-d} v_i^{(n-1)} = 0. \tag{3.18.6}$$

Thus, the sequence $\left\{ \begin{bmatrix} (-1)^d v_{i-1}^{(n-1)} & 0 \\ 0 & v_i^{(n-1)} \end{bmatrix} : M_{i-1} \oplus M_i \to M_{i-d-1} \oplus M_{i-d} \right\}$ is a cycle in $\operatorname{Hom}_B(N, N)_{-d}$, by Lemma 3.11. As $\operatorname{Ext}_B^d(D, D) = 0$, this morphism is null-homotopic. Thus there exists a DG B-module homomorphism $S^{(n)} = \{S_i^{(n)}\} \in \operatorname{Hom}_B(N, N)_{1-d}$ such that for every i we have

$$\begin{bmatrix} -\alpha_{i-d} & 0 \\ t & \alpha_{i-d+1} \end{bmatrix} S_i^{(n)} - (-1)^{1-d} S_{i-1}^{(n)} \begin{bmatrix} -\alpha_{i-1} & 0 \\ t & \alpha_i \end{bmatrix} = \begin{bmatrix} (-1)^d v_{i-1}^{(n-1)} & 0 \\ 0 & v_i^{(n-1)} \end{bmatrix}. \tag{3.18.7}$$

Lemma 3.11 implies that each $S_i^{(n)}$ is of the form

$$S_i^{(n)} = \begin{bmatrix} (-1)^{1-d} z_{i-1}^{(n)} & v_i^{(n)} \\ 0 & z_i^{(n)} \end{bmatrix} : N_i \to N_{i-d+1}$$

where $v_i^{(n)} \colon M_i \to M_{i-d}$ and $z_i^{(n)} \colon M_i \to M_{i+1-d}$, and for $s = 1, \cdots, r_i$, and for all $m_j \in M_j$ for each integer j we have

$$v_{i+j}^{(n)}(\gamma_{i,s} m_j) = (-1)^{-id} \gamma_{i,s} v_j^{(n)}(m_j) \tag{3.18.8}$$

$$z_{i+j}^{(n)}(\gamma_{i,s} m_j) = (-1)^{i(1-d)} \gamma_{i,s} z_j^{(n)}(m_j). \tag{3.18.9}$$

Hence for each i, equation (3.18.7) implies that we have

$$(-1)^d z_{i-2}^{(n)} \alpha_{i-1} + t v_{i-1}^{(n)} + \alpha_{i-d} z_{i-1}^{(n)} = v_{i-1}^{(n-1)} \tag{3.18.10}$$

$$(-1)^d v_{i-1}^{(n)} \alpha_i - \alpha_{i-d} v_i^{(n)} = 0. \tag{3.18.11}$$

This completes the proof of the claim.

Equation (3.18.3) implies the following equality for each i:

$$f_i = (-1)^d \left[\sum_{j=0}^{n} t^j z_{i-1}^{(j)} \right] \alpha_i + \alpha_{i+1-d} \left[\sum_{j=0}^{n} t^j z_i^{(j)} \right] + t^{n+1} v_i^{(n)}.$$

Since R is tR-adically complete and each M_i is finitely generated over R, each series $\eta_i = \sum_{j=0}^{\infty} t^j z_i^{(j)}$ converges in $\operatorname{Hom}_R(M_i, M_{i+1-d})$, and for every i we have

$$f_i = (-1)^d \eta_{i-1} \alpha_i + \alpha_{i+1-d} \eta_i. \tag{3.18.12}$$

By equation (3.18.2), we conclude that $\eta_{i+j}(\gamma_{i,s} m_j) = (-1)^{i(1-d)} \gamma_{i,s} \eta_j(m_j)$ for all i, j, s. Thus, Lemma 3.10 implies that $\eta = \{\eta_i\} \in \operatorname{Hom}_A(M, M)$ is a DG A-module homomorphism of degree $1 - d$. Equation (3.18.12) implies that $f = \{f_i\}$ is null-homotopic, as desired. $\square$

Corollary 3.19. *Let $\underline{t} = t_1, \cdots, t_n$ be a sequence of elements of R, and assume that R is $\underline{t}R$-adically complete. Let D be a DG $K^R(\underline{t})$-module that is homologically bounded below and homologically degreewise finite. If $\mathrm{Ext}^2_{K^R(\underline{t})}(D, D) = 0$, then D is quasi-liftable to R.*

Proof. By induction on n, using Corollary 3.17 and Proposition 3.18. □

Here is Theorem 1.2 from the introduction and its proof.

Corollary 3.20. *Let $\underline{t} = t_1, \cdots, t_n$ be a sequence of elements of R, and assume that R is $\underline{t}R$-adically complete. If D is a semidualizing DG $K^R(\underline{t})$-module, then there exists a semidualizing R-complex C which is a quasi-lifting of D to R. Moreover, the base-change operation $C \mapsto K^R(\underline{t}) \otimes_R C$ induces a bijection $\mathfrak{S}(R) \xrightarrow{\cong} \mathfrak{S}(K^R(\underline{t}))$.*

Proof. Note that the fact that R is $\underline{t}R$-adically complete implies that $\underline{t}R$ is contained in the Jacobson radical of R. Using this, one checks readily that the conclusions of [23, Lemma A.3] hold in our setting. The existence of an R-complex C that is a quasi-lifting of D to R follows from Corollary 3.19; and C is semidualizing over R by [23, Lemma A.3(a)]. This says that the base-change map $\mathfrak{S}(R) \to \mathfrak{S}(K^R(\underline{t}))$ is surjective; it is injective by [23, Lemma A.3(b)]. □

The next theorem, is a result about uniqueness of quasi-liftings.

Theorem 3.21. *We work in the setting of Notation* 3.1. *Assume that R is tR-adically complete, and assume that R, A, and A_0 are local, and let D be a DG B-module that is homologically bounded below and homologically degreewise finite. If D is quasi-liftable to A and $\mathrm{Ext}^1_B(D, D) = 0$, then any two homologically degreewise finite quasi-liftings of D to A are quasiisomorphic over A.*

Proof. The assumption that R is tR-adically complete and local implies that t is in the maximal ideal $\mathfrak{m} \subset R$.

Let C and C' be two homologically degreewise finite semi-free DG A-modules such that $B \otimes_A C \simeq D \simeq B \otimes_A C'$. Let $M \xrightarrow{\simeq} C$ and $M' \xrightarrow{\simeq} C'$ be minimal semi-free resolutions of C and C' over A. Lemma 3.7 shows that M and M' have the shape dictated by Notation 3.3. Since C and C' are quasi-liftings of D to A, we see that $N := B \otimes_A M \cong K^R(t) \otimes_R M$ and $N' := B \otimes_A M' \cong K^R(t) \otimes_R M'$ are semi-free resolutions of D over B; see Lemma 3.9. Furthermore, from Remark 3.2, we have the isomorphism $B/\mathfrak{m}_B \cong A/\mathfrak{m}_A$, which implies that

$$B/\mathfrak{m}_B \otimes_B (B \otimes_A M) \cong A/\mathfrak{m}_A \otimes_A M.$$

Since M is minimal over A, the differential on this complex is 0, so $B \otimes_A M$ is minimal over B, and similarly for $B \otimes_A M'$.

From [15, Theorem 2.12.5.2 and Example 2.12.5.4] there exists an isomorphism $\Upsilon\colon N \xrightarrow{\cong} N'$. Lemma 3.11 implies that Υ has the following form

$$\begin{array}{ccccccc}
N = \cdots & \longrightarrow & M_{i-1} \oplus M_i & \xrightarrow{\left[\begin{smallmatrix} -\alpha_{i-1} & 0 \\ t & \alpha_i \end{smallmatrix}\right]} & M_{i-2} \oplus M_{i-1} & \longrightarrow & \cdots \\
 & & \Big\downarrow {\scriptstyle \left[\begin{smallmatrix} z_{i-1} & v_i \\ 0 & z_i \end{smallmatrix}\right]} & & \Big\downarrow {\scriptstyle \left[\begin{smallmatrix} z_{i-2} & v_{i-1} \\ 0 & z_{i-1} \end{smallmatrix}\right]} & & \\
N' = \cdots & \longrightarrow & M'_{i-1} \oplus M'_i & \xrightarrow{\left[\begin{smallmatrix} -\alpha'_{i-1} & 0 \\ t & \alpha'_i \end{smallmatrix}\right]} & M'_{i-2} \oplus M'_{i-1} & \longrightarrow & \cdots
\end{array} \tag{3.21.1}$$

and that we have

$$v_{i+j}(\gamma_{i,s} m_j) = (-1)^i \gamma_{i,s} v_j(m_j) \tag{3.21.2}$$

$$z_{i+j}(\gamma_{i,s} m_j) = \gamma_{i,s} z_j(m_j) \tag{3.21.3}$$

for all i, for $s = 1, \ldots, r_i$ and for all $m_j \in M_j$ for each integer j. As the diagram (3.21.1) commutes, we have

$$z_{i-1}\alpha_i = tv_i + \alpha'_i z_i \tag{3.21.4}$$

for all i. Since Υ is an isomorphism, it follows that the z_i's are isomorphisms.

The condition (3.21.2) implies that $v = \{v_i\} \in \mathrm{Hom}_A(M, M')_{-1}$; for instance see Lemma 3.10. As the diagram (3.21.1) commutes, we have $v_{i-1}\alpha_i = -\alpha'_{i-1}v_i$ for all i, so v is a cycle in $\mathrm{Hom}_A(M, M')_{-1}$. This yields a cycle $B \otimes_A v \in \mathrm{Hom}_B(N, N')_{-1}$ which has the form $\left\{\left[\begin{smallmatrix} -v_{i-1} & 0 \\ 0 & v_i \end{smallmatrix}\right] : M_{i-1} \oplus M_i \to M_{i-d-1} \oplus M_{i-d}\right\}$; see Fact 2.25 and Lemma 3.12. Since $B \otimes_A v$ is a cycle, our Ext-vanishing assumption implies that there is a DG B-module homomorphism

$$T^{(0)} = \left\{ \begin{bmatrix} u_{i-1}^{(0)} & p_i^{(0)} \\ 0 & u_i^{(0)} \end{bmatrix} \right\} \in \mathrm{Hom}_B(N, N')_0$$

such that for every i we have

$$\begin{bmatrix} -\alpha'_{i-1} & 0 \\ t & \alpha'_i \end{bmatrix} T_i^{(0)} - T_{i-1}^{(0)} \begin{bmatrix} -\alpha_{i-1} & 0 \\ t & \alpha_i \end{bmatrix} = \begin{bmatrix} -v_{i-1} & 0 \\ 0 & v_i \end{bmatrix}.$$

Therefore for all $i \in \mathbb{Z}$ we obtain the following equations:

$$-u_{i-1}^{(0)}\alpha_i + tp_i^{(0)} + \alpha'_i u_i^{(0)} = v_i$$

$$u_{i+j}^{(0)}(\gamma_{i,s}m_j) = \gamma_{i,s}u_j^{(0)}(m_j)$$

$$p_{i+j}^{(0)}(\gamma_{i,s}m_j) = (-1)^i\gamma_{i,s}p_i^{(0)}(m_j)$$

$$p_{i-1}^{(0)}\alpha_i = -\alpha'_{i-1}p_i^{(0)}.$$

The process repeats using $p^{(0)} = \{p_i^{(0)}\}$ in place of $p^{(-1)} = v$. Inductively, for each $n \geqslant 0$ one can construct a DG B-module homomorphism

$$T^{(n)} = \left\{ \begin{bmatrix} u_{i-1}^{(n)} & p_i^{(n)} \\ 0 & u_i^{(n)} \end{bmatrix} \right\} \in \mathrm{Hom}_B(N, N')_0$$

such that for every i we have

$$\left[\begin{smallmatrix} -\alpha'_{i-1} & 0 \\ t & \alpha'_i \end{smallmatrix}\right] T_i^{(n)} - T_{i-1}^{(n)} \left[\begin{smallmatrix} -\alpha_{i-1} & 0 \\ t & \alpha_i \end{smallmatrix}\right] = \left[\begin{smallmatrix} -p_{i-1}^{(n-1)} & 0 \\ 0 & p_i^{(n-1)} \end{smallmatrix}\right].$$

Therefore for all $i \in \mathbb{Z}$ and $n \geqslant 0$ we get the following equations:

$$\begin{aligned} -u_{i-1}^{(n)}\alpha_i + tp_i^{(n)} + \alpha'_i u_i^{(n)} &= p_i^{(n-1)} \\ u_{i+j}^{(n)}(\gamma_{i,s} m_j) &= \gamma_{i,s} u_j^{(n)}(m_j) \end{aligned} \tag{3.21.5}$$

and hence

$$v_i = p_i^{(-1)} = \alpha'_i \left[\sum_{j=0}^{n} t^j u_i^{(j)}\right] + t^{n+1} p_i^{(n)} - \left[\sum_{j=0}^{n} t^j u_{i-1}^{(j)}\right] \alpha_i. \tag{3.21.6}$$

Since R is tR-adically complete, the next series converges for each i

$$\xi_i = \sum_{j=0}^{\infty} t^j u_i^{(j)}$$

and equation (3.21.6) implies that

$$v_i = \alpha'_i \xi_i - \xi_{i-1}\alpha_i. \tag{3.21.7}$$

Combining equations (3.21.4) and (3.21.7), for each i we have

$$(z_{i-1} + t\xi_{i-1})\alpha_i = \alpha'_i(z_i + t\xi_i). \tag{3.21.8}$$

This shows that the sequence $z + t\xi\colon M \to M'$ is a degree-0 homomorphism of the underlying R-complexes. Combining equations (3.21.3) and (3.21.5), we see that

$$(z + t\xi)_{i+j}(\gamma_{i,s} m_j) = \gamma_{i,s}(z + t\xi)_j(m_j)$$

for all i, for $s = 1, \dots, r_i$ and for all $m_j \in M_j$ for each j. So, Lemma 3.10 implies that $z + t\xi$ is a cycle in $\mathrm{Hom}_A(M, M')_0$. Since each z_i is bijective and $t \in \mathfrak{m}$, Nakayama's Lemma implies that for every i, the map $z_i + t\xi_i$ is also bijective. Hence $z + t\xi$ is an isomorphism $M \xrightarrow{\cong} M'$, so $C \simeq M \cong M' \simeq C'$, as desired. □

Corollary 3.22. *Let $\underline{t} = t_1, \cdots, t_n$ be a sequence of elements of R, and assume that R is local and $\underline{t}R$-adically complete. Let D be a DG $K^R(\underline{t})$-module that is homologically bounded below and homologically degreewise finite. If D is quasi-liftable to R and $\mathrm{Ext}^1_{K^R(\underline{t})}(D, D) = 0$, then any two homologically degreewise finite quasi-liftings of D to R are quasiisomorphic over R.*

Proof. By induction on n, using Theorem 3.21 and Proposition 3.18. □

We conclude this chapter with an example showing that the quasi-liftings in the previous two results must be homologically degreewise finite.

Example 3.23. Let $(R, \mathfrak{m})$ be a local integral domain that is not a field. Let $Q(R)$ be the field of fractions of R, and let $0 \neq t \in \mathfrak{m}$. If F is an R-free resolution of $Q(R)$, then F and 0 are both quasi-liftings of 0 from $K^R(t)$ to R.

CHAPTER 4. SOLUTION TO VASCONCELOS' QUESTION

This chapter contains the complete answer to a question of Vasconcelos from 1974; see Question 1.3 and Theorem 1.4 from the introduction.

4.1. DG Ext and Yoneda Ext

Given a DG R-algebra A, and DG A-modules M and N such that M is homologically bounded below, we have the DG-Ext module $\mathrm{Ext}^1_A(M,N)$ from Definition 2.24. In general, this module does not parametrize the extensions $0 \to N \to L \to M \to 0$; see Example 4.2. To parametrize such extensions, we need "Yoneda Ext", which we describe next. The main point of this chapter is to prove Theorem 1.6 which is important for our solution of Vasconcelos' question.

Definition 4.1. Let A be a DG R-algebra. The category of DG A-modules described in Definition 2.17 is an abelian category; see, e.g., [53, Introduction]. So, given DG A-modules L, M, the *Yoneda Ext group* $\mathrm{YExt}^1_A(L,M)$, defined as the set of equivalence classes of exact sequences $0 \to M \to X \to L \to 0$ of DG A-modules, is a well-defined abelian group under the Baer sum; see, e.g., [77, (3.4.6)].

The next example shows that $\mathrm{YExt}^1_A(M,N)$ and $\mathrm{Ext}^1_A(M,N)$ are distinct.

Example 4.2. Let $R = k[\![X]\!]$, and consider the following exact sequence of DG R-modules, i.e., exact sequence of R-complexes:

$$0 \longrightarrow \underline{R} \longrightarrow \underline{R} \longrightarrow \underline{k} \longrightarrow 0$$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R & \xrightarrow{X} & R & \longrightarrow & k & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \\
0 & \longrightarrow & R & \xrightarrow{X} & R & \longrightarrow & k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

This sequence does not split over R (it is not even degree-wise split) so it gives a non-trivial class in $\mathrm{YExt}^1_R(\underline{k}, \underline{R})$, and we conclude that $\mathrm{YExt}^1_R(\underline{k}, \underline{R}) \neq 0$. On the other hand, $\underline{k}$ is homologically trivial, so we have $\mathrm{Ext}^1_R(\underline{k}, \underline{R}) = 0$ since 0 is a semi-free resolution of $\underline{k}$.

In preparation for the proof of Theorem 1.6, we require two more items.

Definition 4.3. Let A be a DG R-algebra. A DG A-module Q is *graded-projective* if $\mathrm{Hom}_A(Q, -)$ preserves surjective morphisms, that is, if $Q^\natural$ is a projective graded $R^\natural$-module; see [15, Theorem 2.8.3.1].

Remark 4.4. If Q is semi-free, then $Q^\natural \cong \oplus_i \Sigma^i (R^\natural)^{(\beta_i)}$ is a free (hence projective) graded $R^\natural$-module, so Q is graded-projective.

4.5 (Proof of Theorem 1.6)**.** Let $\zeta \in \mathrm{YExt}^1_A(Q, P)$ be represented by the sequence

$$0 \to P \to X \to Q \to 0. \tag{4.5.1}$$

Since Q is graded-projective, the sequence (4.5.1) graded-splits (see [15, (2.8.3.1)]), that is, this sequence is isomorphic to one of the form

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle \partial^P_{i+1}} & & \downarrow{\scriptstyle \partial^X_{i+1}} & & \downarrow{\scriptstyle \partial^Q_{i+1}} & & \\
0 & \longrightarrow & P_i & \xrightarrow{\epsilon_i} & P_i \oplus Q_i & \xrightarrow{\pi_i} & Q_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial^P_{i}} & & \downarrow{\scriptstyle \partial^X_{i}} & & \downarrow{\scriptstyle \partial^Q_{i}} & & \\
0 & \longrightarrow & P_{i-1} & \xrightarrow{\epsilon_{i-1}} & P_{i-1} \oplus Q_{i-1} & \xrightarrow{\pi_{i-1}} & Q_{i-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial^P_{i-1}} & & \downarrow{\scriptstyle \partial^X_{i-1}} & & \downarrow{\scriptstyle \partial^Q_{i-1}} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array} \tag{4.5.2}$$

where ϵ_j is the natural inclusion and π_j is the natural surjection for each j. Since this diagram comes from a graded-splitting of (4.5.1), the scalar multiplication on the middle column of (4.5.2) is the natural one $a\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right] = \left[\begin{smallmatrix} ap \\ aq \end{smallmatrix}\right]$.[5]

The fact that (4.5.2) commutes implies that ∂_i^X has a specific form:

$$\partial_i^X = \begin{bmatrix} \partial_i^P & \lambda_i \\ 0 & \partial_i^Q \end{bmatrix}. \tag{4.5.3}$$

Here, we have $\lambda_i \colon Q_i \to P_{i-1}$, that is, $\lambda \in \operatorname{Hom}_R(Q,P)_{-1}$. Since the maps in the sequence (4.5.2) are morphisms of DG A-modules, it follows that λ is a cycle in $\operatorname{Hom}_A(Q,P)_{-1}$. Thus, λ represents a homology class in $\operatorname{Ext}^1_A(Q,P)$, and we define $\Psi\colon \operatorname{YExt}^1_A(Q,P) \to \operatorname{Ext}^1_A(Q,P)$ by the formula $\Psi(\zeta) := \lambda$.

We show that Ψ is well-defined. Let ζ be represented by another exact sequence

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle \partial^P_{i+1}} & & \downarrow{\scriptstyle \partial^{X'}_{i+1}} & & \downarrow{\scriptstyle \partial^Q_{i+1}} & & \\
0 & \longrightarrow & P_i & \xrightarrow{\epsilon_i} & P_i \oplus Q_i & \xrightarrow{\pi_i} & Q_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial^P_{i}} & & \downarrow{\scriptstyle \partial^{X'}_{i}} & & \downarrow{\scriptstyle \partial^Q_{i}} & & \\
0 & \longrightarrow & P_{i-1} & \xrightarrow{\epsilon_{i-1}} & P_{i-1} \oplus Q_{i-1} & \xrightarrow{\pi_{i-1}} & Q_{i-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial^P_{i-1}} & & \downarrow{\scriptstyle \partial^{X'}_{i-1}} & & \downarrow{\scriptstyle \partial^Q_{i-1}} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array} \tag{4.5.4}$$

where

$$\partial_i^{X'} = \begin{bmatrix} \partial_i^P & \lambda'_i \\ 0 & \partial_i^Q \end{bmatrix}. \tag{4.5.5}$$

We need to show that $\lambda - \lambda' \in \operatorname{Im}(\partial_0^{\operatorname{Hom}_A(Q,P)})$. The sequences (4.5.2) and (4.5.4) are equivalent in $\operatorname{YExt}^1_R(Q,P)$, so for each i there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_i & \xrightarrow{\epsilon_i} & P_i \oplus Q_i & \xrightarrow{\pi_i} & Q_i & \longrightarrow & 0 \\
 & & {\scriptstyle =}\downarrow & & {\scriptstyle \left[\begin{smallmatrix} u_i & v_i \\ w_i & x_i \end{smallmatrix}\right]}\downarrow{\scriptstyle \cong} & & {\scriptstyle =}\downarrow & & \\
0 & \longrightarrow & P_i & \xrightarrow{\epsilon_i} & P_i \oplus Q_i & \xrightarrow{\pi_i} & Q_i & \longrightarrow & 0
\end{array} \tag{4.5.6}$$

[5]Given R-modules M and N, we write elements of $M \oplus N$ as column vectors $\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]$ with $m \in M$ and $n \in N$. This permits us to use matrix notation for homomorphisms between such modules.

where the middle vertical arrow is a DG A-module isomorphism, and such that the following diagram commutes

$$\begin{array}{ccc} P_i \oplus Q_i & \xrightarrow[\cong]{\left[\begin{smallmatrix} u_i & v_i \\ w_i & x_i \end{smallmatrix}\right]} & P_i \oplus Q_i \\ {\scriptstyle \left[\begin{smallmatrix} \partial_i^P & \lambda_i \\ 0 & \partial_i^Q \end{smallmatrix}\right]}\downarrow & & \downarrow{\scriptstyle \left[\begin{smallmatrix} \partial_i^P & \lambda'_i \\ 0 & \partial_i^Q \end{smallmatrix}\right]} \\ P_{i-1} \oplus Q_{i-1} & \xrightarrow[\cong]{\left[\begin{smallmatrix} u_{i-1} & v_{i-1} \\ w_{i-1} & x_{i-1} \end{smallmatrix}\right]} & P_{i-1} \oplus Q_{i-1}. \end{array} \tag{4.5.7}$$

The fact that diagram (4.5.6) commutes implies that $u_i = \mathrm{id}_{P_i}$, $x_i = \mathrm{id}_{Q_i}$, and $w_i = 0$. Also, the fact that the middle vertical arrow in diagram (4.5.6) describes a DG A-module morphism implies that the sequence $v_i \colon Q_i \to P_i$ respects scalar multiplication, i.e., we have $v \in \mathrm{Hom}_A(Q,P)_0$. The fact that diagram (4.5.7) commutes implies that $\lambda_i - \lambda'_i = \partial_i^P v_i - v_{i-1}\partial_i^Q$. We conclude that $\lambda - \lambda' = \partial_0^{\mathrm{Hom}_A(Q,P)}(v) \in \mathrm{Im}(\partial_0^{\mathrm{Hom}_A(Q,P)})$, so Ψ is well-defined.

Next we show that Ψ is additive. Let $\zeta, \zeta' \in \mathrm{YExt}_A^1(Q,P)$ be represented by exact sequences $0 \to P \xrightarrow{\epsilon} X \xrightarrow{\pi} Q \to 0$ and $0 \to P \xrightarrow{\epsilon'} X' \xrightarrow{\pi'} Q \to 0$ respectively, where $X_i = P_i \oplus Q_i = X'_i$ and the differentials ∂^X and $\partial^{X'}$ are described as in (4.5.3) and (4.5.5), respectively. We need to show that the Baer sum $\zeta + \zeta'$ is represented by an exact sequence $0 \to P \xrightarrow{\widetilde{\epsilon}} \widetilde{X} \xrightarrow{\widetilde{\pi}} Q \to 0$ respectively, where $\widetilde{X}_i = P_i \oplus Q_i$ and $\partial_i^{\widetilde{X}} = \left[\begin{smallmatrix} \partial_i^P & \lambda_i + \lambda'_i \\ 0 & \partial_i^Q \end{smallmatrix}\right]$, with scalar multiplication $a \left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right] = \left[\begin{smallmatrix} ap \\ aq \end{smallmatrix}\right]$. Note that it is straightforward to show that the sequence $\widetilde{X}$ defined in this way is a DG A-module, and the natural maps $P \xrightarrow{\widetilde{\epsilon}} \widetilde{X} \xrightarrow{\widetilde{\pi}} Q$ are DG-linear, using the analogous properties for X and X'.

We construct the Baer sum in two steps. The first step is to construct the pull-back diagram

$$\begin{array}{ccc} X'' & \xrightarrow{\pi''} & X' \\ {\scriptstyle \pi'''}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi'} \\ X & \xrightarrow{\pi} & Q. \end{array}$$

The DG module X'' is a submodule of the direct sum $X \oplus X'$, so each X''_i is the submodule of

$$X \oplus X' = X_i \oplus X'_i = P_i \oplus Q_i \oplus P_i \oplus Q_i$$

consisting of all vectors $\left[\begin{smallmatrix} x \\ x' \end{smallmatrix}\right]$ such that $\pi'_i(x') = \pi_i(x)$, that is, all vectors of the form $[p \quad q \quad p' \quad q']^T$ such that $q = q'$. In other words, we have

$$P_i \oplus Q_i \oplus P_i \xrightarrow{\cong} X''_i \tag{4.5.8}$$

where the isomorphism is given by $[p \quad q \quad p']^T \mapsto [p \quad q \quad p' \quad q]^T$. The differential on $X \oplus X'$ is the natural diagonal map. So, under the isomorphism (4.5.8), the differential on X'' has the form

$$X''_i \cong P_i \oplus Q_i \oplus P_i \xrightarrow{\partial_i^{X''} = \left[\begin{smallmatrix} \partial_i^P & \lambda_i & 0 \\ 0 & \partial_i^Q & 0 \\ 0 & \lambda'_i & \partial_i^P \end{smallmatrix}\right]} P_{i-1} \oplus Q_{i-1} \oplus P_{i-1} \cong X''_{i-1}.$$

The second step is to construct $\widetilde{X}$, which is the cokernel of the morphism $\gamma\colon P \to X''$ given by $p \mapsto \left[\begin{smallmatrix} -p \\ 0 \\ p \end{smallmatrix}\right]$. In other words, since γ is injective, the complex $\widetilde{X}$ is determined by the exact sequence $0 \to P \xrightarrow{\gamma} X'' \xrightarrow{\tau} \widetilde{X} \to 0$. It is straightforward to show that the following diagram describes such an exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_i & \xrightarrow{\left[\begin{smallmatrix} -1 \\ 0 \\ 1 \end{smallmatrix}\right]} & P_i \oplus Q_i \oplus P_i & \xrightarrow{\left[\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right]} & P_i \oplus Q_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_i^P} & & \downarrow{\scriptstyle \left[\begin{smallmatrix} \partial_i^P & \lambda_i & 0 \\ 0 & \partial_i^Q & 0 \\ 0 & \lambda'_i & \partial_i^P \end{smallmatrix}\right]} & & \downarrow{\scriptstyle \left[\begin{smallmatrix} \partial_i^P & \lambda_i + \lambda'_i \\ 0 & \partial_i^Q \end{smallmatrix}\right]} & & \\
0 & \longrightarrow & P_{i-1} & \xrightarrow{\left[\begin{smallmatrix} -1 \\ 0 \\ 1 \end{smallmatrix}\right]} & P_{i-1} \oplus Q_{i-1} \oplus P_{i-1} & \xrightarrow{\left[\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right]} & P_{i-1} \oplus Q_{i-1} & \longrightarrow & 0.
\end{array}$$

By inspecting the right-most column of this diagram, we see that $\widetilde{X}$ has the desired form. Furthermore, checking the module structures at each step of the construction, we see that the scalar multiplication on $\widetilde{X}$ is the natural one $a\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right] = \left[\begin{smallmatrix} ap \\ aq \end{smallmatrix}\right]$.

Next, we show that Ψ is injective. Suppose that $\zeta \in \operatorname{Ker}(\Psi)$ is represented by the displays (4.5.1)–(4.5.3). The condition $\Psi(\zeta) = 0$ says that $\lambda \in \operatorname{Im}(\partial_0^{\operatorname{Hom}_A(Q,P)})$, so there is an element $s \in \operatorname{Hom}_A(Q,P)_0$ such that $\zeta = \partial_0^{\operatorname{Hom}_A(Q,P)}(s)$. This says that for each i we have $\lambda_i = \partial_i^P s_i - s_{i-1}\partial_i^Q$. From this, it is straightforward to show that the following diagram commutes:

$$\begin{array}{ccc} P_i \oplus Q_i & \xrightarrow[\cong]{\left[\begin{smallmatrix} 1 & s_i \\ 0 & 1 \end{smallmatrix}\right]} & P_i \oplus Q_i \\ {\scriptstyle \left[\begin{smallmatrix} \partial_i^P & \lambda_i \\ 0 & \partial_i^Q \end{smallmatrix}\right]} \downarrow & & \downarrow {\scriptstyle \left[\begin{smallmatrix} \partial_i^P & 0 \\ 0 & \partial_i^Q \end{smallmatrix}\right]} \\ P_{i-1} \oplus Q_{i-1} & \xrightarrow[\cong]{\left[\begin{smallmatrix} 1 & s_{i-1} \\ 0 & 1 \end{smallmatrix}\right]} & P_{i-1} \oplus Q_{i-1}. \end{array}$$

From the fact that s is A-linear, it follows that the maps $\left[\begin{smallmatrix} 1 & s_i \\ 0 & 1 \end{smallmatrix}\right]$ describe an A-linear isomorphism $X \xrightarrow{\cong} P \oplus Q$ making the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P & \xrightarrow{\epsilon} & X & \xrightarrow{\pi} & Q & \longrightarrow & 0 \\ & & {\scriptstyle =}\downarrow & & {\scriptstyle \cong}\downarrow & & {\scriptstyle =}\downarrow & & \\ 0 & \longrightarrow & P & \xrightarrow{\epsilon} & P \oplus Q & \xrightarrow{\pi} & Q & \longrightarrow & 0. \end{array}$$

In other words, the sequence (4.5.1) splits, so we have $\zeta = 0$, and Ψ is injective.

Finally, we show that Ψ is surjective. For this, let $\xi \in \operatorname{Ext}_A^1(Q,P)$ be represented by $\lambda \in \operatorname{Ker}(\partial_{-1}^{\operatorname{Hom}_A(Q,P)})$. Using the fact that λ is A-linear such that $\partial_{-1}^{\operatorname{Hom}_A(Q,P)}(\lambda) = 0$, one checks directly that the displays (4.5.2)–(4.5.3) describe an exact sequence of DG A-module homomorphisms of the form (4.5.1) whose image under Ψ is ξ. $\square$

To describe higher Yoneda Ext groups, we need another variant of the notion of projectivity for DG modules.

Definition 4.6. Let A be a DG R-algebra. Projective objects in the category of DG A-modules are called *categorically projective* DG A-modules.

Remark 4.7. Let A be a DG R-algebra. Our definition of "categorically projective" is equivalent to the one given in [15, Section 2.8.1], because of [15, Theorem 2.8.7.1]. Furthermore, the category of DG A-modules has enough projectives by [15, Corollary 2.7.5.4 and Theorem 2.8.7.1(iv)]. Thus, given DG A-modules L and M, for each $i \geqslant 0$ we have a well-defined Yoneda Ext group $\mathrm{YExt}^i_A(L, M)$, defined in terms of a resolution of L by categorically projective DG A-modules:

$$\cdots \to Q_1 \to Q_0 \to L \to 0.$$

A standard result shows that when $i = 1$, this definition of Yoneda Ext is equivalent to the one given in Definition 4.1.

Corollary 4.8. *Let A be a DG R-algebra, and let P, Q be DG A-modules such that Q is graded-projective (e.g., Q is semi-free). Then there is an isomorphism $\mathrm{YExt}^i_A(Q, P) \cong \mathrm{Ext}^i_A(Q, P)$ of abelian groups for all $i \geqslant 1$.*

Proof. Using Theorem 1.6, we need only justify the isomorphism $\mathrm{YExt}^i_A(Q, P) \cong \mathrm{Ext}^i_A(Q, P)$ for $i \geqslant 2$. Let

$$L^+_\bullet = \cdots \xrightarrow{\partial^L_2} L_1 \xrightarrow{\partial^L_1} L_0 \xrightarrow{\pi} Q \to 0$$

be a resolution of Q by categorically projective DG A-modules. Since each L_j is categorically projective, we have $\mathrm{YExt}^i_A(L_j, -) = 0$ for all $i \geqslant 1$. From [15, Theorem 2.8.7.1] we conclude that $L_j \simeq 0$ for each j, so we have $\mathrm{Ext}^i_A(L_j, -) = 0$ for all i. Set $Q_i = \operatorname{Im} \partial^L_i$ for each $i \geqslant 1$. Each L_i is graded-projective by [15, Theorems 2.8.6.1 and 2.8.7.1], so the fact that Q is graded-projective implies that each Q_i is graded-projective.

Now, a straightforward dimension-shifting argument to explain the first and third isomorphisms in the following display for $i \geqslant 2$:

$$\mathrm{YExt}^i_A(Q, P) \cong \mathrm{YExt}^1_A(Q_{i-1}, P) \cong \mathrm{Ext}^1_A(Q_{i-1}, P) \cong \mathrm{Ext}^i_A(Q, P).$$

The second isomorphism is from Theorem 1.6 since each Q_i is graded-projective. $\square$

The next example shows that one can have $\mathrm{YExt}^0_A(Q, P) \ncong \mathrm{Ext}^0_A(Q, P)$, even when Q is semi-free.

Example 4.9. Continue with the assumptions and notation of Example 4.2, and set $Q = P = \underline{R}$. It is straightforward to show that the morphisms $\underline{R} \to \underline{R}$ are precisely given by multiplication by fixed elements of R, so we have the first step in the next display:

$$\mathrm{YExt}^0_A(\underline{R}, \underline{R}) \cong R \neq 0 = \mathrm{Ext}^0_A(\underline{R}, \underline{R}).$$

The third step follows from the condition $\underline{R} \simeq 0$.

In our proof of Theorem 1.4, we need to know when YExt respects truncations.

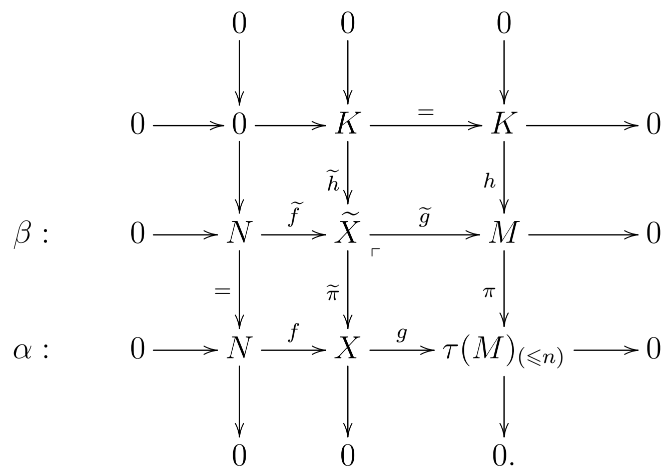
Proposition 4.10. *Let A be a DG R-algebra, and let M and N be DG A-modules. Assume that n is an integer such that $N_i = 0$ for all $i > n$. Then the natural map $\mathrm{YExt}^1_A(\tau(M)_{(\leqslant n)}, N) \to \mathrm{YExt}^1_A(M, N)$ induced by the morphism $\pi\colon M \to \tau(M)_{(\leqslant n)}$ is a monomorphism.*

Proof. Let Υ denote the map $\mathrm{YExt}^1_A(\tau(M)_{(\leqslant n)}, N) \to \mathrm{YExt}^1_A(M, N)$ induced by π. Let $\alpha \in \mathrm{Ker}(\Upsilon) \subseteq \mathrm{YExt}^1_A(\tau(M)_{(\leqslant n)}, N)$ be represented by the exact sequence

$$0 \to N \xrightarrow{f} X \xrightarrow{g} \tau(M)_{(\leqslant n)} \to 0. \tag{4.10.1}$$

Note that, since $N_i = 0 = (\tau(M)_{(\leqslant n)})_i$ for all $i > n$, we have $X_i = 0$ for all $i > n$.

Then $0 = \Upsilon([\alpha]) = [\beta]$ where β comes from the following pull-back diagram:



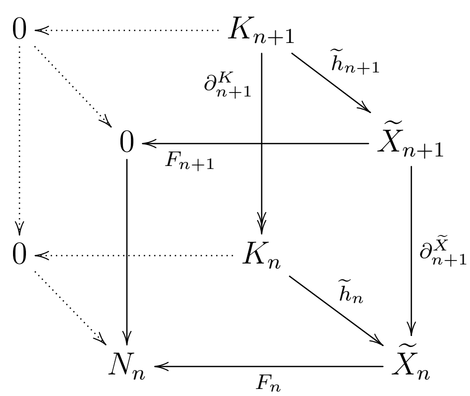
(4.10.2)

The middle row β of this diagram is split exact since $[\beta] = 0$, so there is a morphism $F\colon \widetilde{X} \to N$ of DG A-modules such that $F \circ \widetilde{f} = \mathrm{id}_N$. Note that K has the form

$$K = \cdots \xrightarrow{\partial^M_{n+2}} M_{n+1} \xrightarrow{\partial^M_{n+1}} \mathrm{Im}(\partial^M_{n+1}) \to 0 \qquad (4.10.3)$$

because of the right-most column of the diagram.

We claim that $F \circ \widetilde{h} = 0$. It suffices to check this degree-wise. When $i > n$, we have $N_i = 0$, so $F_i = 0$, and $F_i \circ \widetilde{h}_i = 0$. When $i < n$, the display (4.10.3) shows that $K_i = 0$, so $\widetilde{h}_i = 0$, and $F_i \circ \widetilde{h}_i = 0$. For $i = n$, we first note that the display (4.10.3) shows that ∂^K_{n+1} is surjective. In the following diagram, the faces with solid arrows commute because $\widetilde{h}$ and F are morphisms:



Since ∂^K_{n+1} is surjective, a simple diagram chase shows that $F_n \circ \widetilde{h}_n = 0$. This establishes the claim.

To conclude the proof, note that the previous claim shows that the map $K \to 0$ is a left-splitting of the top row of diagram (4.10.2) that is compatible with the left-splitting F of the middle row. It is then straightforward to show that F induces a morphism $\overline{F}\colon X \to N$ of DG A-modules that left-splits the bottom row of diagram (4.10.2). Since this row represents $\alpha \in \mathrm{YExt}^1_A(\tau(M)_{(\leqslant n)}, N)$, we conclude that $[\alpha] = 0$, so Υ is a monomorphism. □

The next example shows that the monomorphism from Proposition 4.10 may not be an isomorphism.

Example 4.11. Continue with the assumptions and notation of Example 4.2. The following diagram describes a non-zero element of $\mathrm{YExt}^1_R(M, N)$:

$$0 \longrightarrow N \longrightarrow \underline{R} \longrightarrow M \longrightarrow 0$$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & R & \xrightarrow{1} & R & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \pi} & & \\
0 & \longrightarrow & R & \xrightarrow{X} & R & \xrightarrow{\pi} & k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

It is straightforward to show that $\tau(M)_{(\leqslant 0)} = 0$, so we have

$$0 = \mathrm{YExt}^1_A(\tau(M)_{(\leqslant 0)}, N) \hookrightarrow \mathrm{YExt}^1_A(M, N) \neq 0$$

so this map is not an isomorphism.

Proposition 4.12. *Let A be a DG R-algebra, and let C be a graded-projective (e.g., semi-free) DG A-module such that* $\mathrm{Ext}^1_R(C,C) = 0$. *For $n \geqslant \sup(C)$, one has*

$$\mathrm{YExt}^1_A(C,C) = 0 = \mathrm{YExt}^1_A(\tau(C)_{(\leqslant n)}, \tau(C)_{(\leqslant n)}).$$

Proof. From Theorem 1.6, we have $\mathrm{YExt}^1_A(C,C) \cong \mathrm{Ext}^1_A(C,C) = 0$. For the remainder of the proof, assume without loss of generality that $\sup(C) < \infty$. Another application of Theorem 1.6 explains the first step in the next display:

$$\mathrm{YExt}^1_A(C, \tau(C)_{(\leqslant n)}) \cong \mathrm{Ext}^1_A(C, \tau(C)_{(\leqslant n)}) \cong \mathrm{Ext}^1_A(C,C) = 0.$$

The second step comes from the assumption $n \geqslant \sup(C)$ which guarantees that the natural map $C \to \tau(C)_{(\leqslant n)}$ is a quasiisomorphism. Proposition 4.10 implies that $\mathrm{YExt}^1_A(\tau(C)_{(\leqslant n)}, \tau(C)_{(\leqslant n)})$ is isomorphic to a subgroup of $\mathrm{YExt}^1_A(C, \tau(C)_{(\leqslant n)}) = 0$, so we have $\mathrm{YExt}^1_A(\tau(C)_{(\leqslant n)}, \tau(C)_{(\leqslant n)}) = 0$, as desired. □

4.2. Some Deformation Theory for DG Modules

The ideas for this section are from [3, 34, 44].

Notation 4.13. Let F be an algebraically closed field, and let

$$U := (0 \to U_q \xrightarrow{\partial^U_q} U_{q-1} \xrightarrow{\partial^U_{q-1}} \cdots \xrightarrow{\partial^U_1} U_0 \to 0)$$

be a finite-dimensional DG F-algebra. Let $\dim_F(U_i) = n_i$ for $i = 0, \ldots, q$. Let

$$W := \bigoplus_{i=0}^{s} W_i$$

be a graded F-vector space with $r_i := \dim_F(W_i)$ for $i = 0, \ldots, s$.

A DG U-module structure on W consists of two pieces of data. First, we need a differential ∂. Second, once the differential ∂ has been chosen, we need a scalar multiplication μ. Let $\mathrm{Mod}^U(W)$ denote the set of all ordered pairs (∂, μ) making W into a DG U-module. Let $\mathrm{End}_F(W)_0$ denote the set of F-linear endomorphisms of W that are homogeneous of degree 0. Let $\mathrm{GL}(W)_0$ denote the set of F-linear automorphisms of W that are homogeneous of degree 0, that is, the invertible elements of $\mathrm{End}_F(W)_0$.

Let $F[\epsilon] := F\epsilon \oplus F$ be the algebra of dual numbers, where $\epsilon^2 = 0$. For our convenience, we write elements of $F[\epsilon]$ as column vectors: $a\epsilon + b = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$. We identify $U[\epsilon] := F[\epsilon] \otimes_F U$ with $U\epsilon \oplus U \cong U \oplus U$, and $W[\epsilon] := F[\epsilon] \otimes_F W$ with $W\epsilon \oplus W \cong W \oplus W$. Using this protocol, we have $\partial_i^{U[\epsilon]} = \left[\begin{smallmatrix} \partial_i^U & 0 \\ 0 & \partial_i^U \end{smallmatrix}\right]$.

We next describe geometric structures on the sets $\mathrm{Mod}^U(W)$ and $\mathrm{GL}(W)_0$.

Remark 4.14. We work in the setting of Notation 4.13.

A differential ∂ on W is an element of the graded vector space $\mathrm{Hom}_F(W, W)_{-1}$ such that $\partial\partial = 0$. The vector space $\mathrm{Hom}_F(W_i, W_{i-1})$ has dimension $r_i r_{i-1}$, so the map ∂ corresponds to an element of the affine space $\mathbb{A}_F^d$ where $d := \sum_i r_i r_{i-1}$. The vanishing condition $\partial\partial = 0$ is equivalent to the entries of the matrices representing ∂ satisfying certain fixed homogeneous quadratic polynomial equations over F. Hence, the set of all differentials on W is a Zariski-closed subset of $\mathbb{A}_F^d$.

Once the differential ∂ has been chosen, a scalar multiplication μ is in particular a cycle in $\mathrm{Hom}_F(U \otimes_F W, W)_0$. For all i, j, the vector space $\mathrm{Hom}_F(U_i \otimes_F W_j, W_{i+j})$ has dimension $n_i r_j r_{i+j}$, so the map μ corresponds to an element of the affine space $\mathbb{A}_F^{d'}$ where $d' := \sum_c \sum_i n_i r_{c-i} r_c$. The condition that μ be an associative, unital cycle is equivalent to the entries of the matrices representing ∂ and μ satisfying certain fixed polynomials over F. Thus, the set $\mathrm{Mod}^U(W)$ is a Zariski-closed subset of $\mathbb{A}_F^d \times \mathbb{A}_F^{d'} \cong \mathbb{A}_F^{d+d'}$.

Remark 4.15. We work in the setting of Notation 4.13.

An element $\alpha \in \mathrm{GL}(W)_0$ is an element of the graded vector space $\mathrm{Hom}_F(W, W)_0$ with a multiplicative inverse. The vector space $\mathrm{Hom}_F(W_i, W_i)$ has dimension r_i^2, so the map α corresponds to an element of the affine space $\mathbb{A}_F^e$ where $e := \sum_i r_i^2$. The invertibility of α is equivalent to the invertibility of each "block" $\alpha_i \in \mathrm{Hom}_F(W_i, W_i)$, which is an open condition. Thus, the set $\mathrm{GL}(W)_0$ is a Zariski-open subset of $\mathbb{A}_F^e$, so it is smooth over F.

Alternately, one can view $\mathrm{GL}(W)_0$ as the product $\mathrm{GL}(W_0) \times \cdots \times \mathrm{GL}(W_s)$. Since each $\mathrm{GL}(W_i)$ is an algebraic group smooth over F, it follows that $\mathrm{GL}(W)_0$ is also an algebraic group that is smooth over F.

Next, we describe an action of $\mathrm{GL}(W)_0$ on $\mathrm{Mod}^U(W)$.

Remark 4.16. We work in the setting of Notation 4.13.

Let $\alpha \in \mathrm{GL}(W)_0$. For every $(\partial, \mu) \in \mathrm{Mod}^U(W)$, we define $\alpha \cdot (\partial, \mu) := (\widetilde{\partial}, \widetilde{\mu})$, where $\widetilde{\partial} := \alpha \circ \partial \circ \alpha^{-1}$ and $\widetilde{\mu} := \alpha \circ \mu \circ (U \otimes_F \alpha^{-1})$. It is straightforward to show that the ordered pair $(\widetilde{\partial}, \widetilde{\mu})$ describes a DG U-module structure for W, that is, we have $\alpha \cdot (\partial, \mu) := (\widetilde{\partial}, \widetilde{\mu}) \in \mathrm{Mod}^U(W)$. From the definition of $\alpha \cdot (\partial, \mu)$, it follows readily that this describes a $\mathrm{GL}(W)_0$-action on $\mathrm{Mod}^U(W)$.

It is straightforward to show that the map α gives a DG U-module isomorphism $(W, \partial, \mu) \xrightarrow{\cong} (W, \widetilde{\partial}, \widetilde{\mu})$. Conversely, given another element $(\partial', \mu') \in \mathrm{Mod}^U(W)$, if there is a DG U-module isomorphism $\beta\colon (W, \partial, \mu) \xrightarrow{\cong} (W, \partial', \mu')$, then $\beta \in \mathrm{GL}(W)_0$ and $(\partial', \mu') = \beta \cdot (\partial, \mu)$. In other words, the orbits in $\mathrm{Mod}^U(W)$ under the action of $\mathrm{GL}(W)_0$ are the isomorphism classes of DG U-module structures on W.

Note that the maps defining the action of $\mathrm{GL}(W)_0$ on $\mathrm{Mod}^U(W)$ are regular, that is, determined by polynomial functions. This is because the inversion map $\alpha \mapsto \alpha^{-1}$ on $\mathrm{GL}(W)_0$ is regular, as is the multiplication of matrices corresponding to the compositions defining $\widetilde{\partial}$ and $\widetilde{\mu}$.

Notation 4.17. We work in the setting of Notation 4.13.

The set $\mathrm{Mod}^U(W)$ is the set of F-rational points of a scheme $\underline{\mathrm{Mod}}^U(W)$ over F, which we describe using the functorial point of view, following [25, 27]: for each commutative F-algebra S, we have[6]

$$\underline{\mathrm{Mod}}^U(W)(S) := \{\text{DG } S \otimes_F U\text{-module structures on } S \otimes_F W\}.$$

Sometimes we write $\mathrm{Mod}^{S\otimes_F U}(S \otimes_F W)$ in place of $\underline{\mathrm{Mod}}^U(W)(S)$. Similarly, $\mathrm{GL}(W)_0$ is the set of F-rational points of a scheme $\underline{\mathrm{GL}}(W)_0$ over F: for each commutative F-algebra S, we have

$$\underline{\mathrm{GL}}(W)_0(S) := \{\text{homogeneous } S\text{-linear automorphisms of } S \otimes_F W \text{ of degree } 0\}.$$

The fact that $\mathrm{Mod}^U(W)$ and $\mathrm{GL}(W)_0$ are the sets of F-rational points of these schemes means that $\mathrm{Mod}^U(W) = \underline{\mathrm{Mod}}^U(W)(F)$ and $\mathrm{GL}(W)_0 = \underline{\mathrm{GL}}(W)_0(F)$.

Fix a commutative F-algebra S. As in Remark 4.16, the group $\underline{\mathrm{GL}}(W)_0(S)$ acts on $\underline{\mathrm{Mod}}^U(W)(S)$: for each $\alpha \in \underline{\mathrm{GL}}(W)_0(S)$ and $(\partial, \mu) \in \underline{\mathrm{Mod}}^U(W)(S)$, define $\alpha\cdot(\partial, \mu) := (\widetilde{\partial}, \widetilde{\mu})$, where $\widetilde{\partial} := \alpha\circ\partial\circ\alpha^{-1}$ and $\widetilde{\mu} := \alpha\circ\mu\circ((S\otimes_F U)\otimes_S \alpha^{-1})$. Again, the orbits in $\underline{\mathrm{Mod}}^U(W)(S)$ under the action of $\underline{\mathrm{GL}}(W)_0(S)$ are the isomorphism classes of DG $S \otimes_F U$-module structures on $S \otimes_F W$.

Let $M = (\partial, \mu) \in \mathrm{Mod}^U(W)$. The orbit of M under $\underline{\mathrm{GL}}(W)_0$ is the subscheme $\underline{\mathrm{GL}}(W)_0 \cdot M$ of $\underline{\mathrm{Mod}}^U(W)$ defined as

$$(\underline{\mathrm{GL}}(W)_0 \cdot M)(S) := \underline{\mathrm{GL}}(W)_0(S) \cdot (S \otimes_F M)$$

which is the DG isomorphism class of $S \otimes_F M$ over $S \otimes_F U$. Let $\varrho\colon \underline{\mathrm{GL}}(W)_0 \to$

[6] Technically, the inputs for this functor should be taken from the category of affine schemes over $\mathrm{Spec}(F)$, but the equivalence between this category and the category of commutative F-algebras makes this equivalent to our approach.

$\underline{\mathrm{GL}}(W)_0 \cdot M$ denote the following natural map: for each commutative F-algebra S and each $\alpha \in \underline{\mathrm{GL}}(W)_0(S)$ we have $\varrho(\alpha) := \alpha \cdot (S \otimes_F M)$.

Given a scheme $\underline{X}$ over F and a point $x \in \underline{X}(F)$, let $\mathsf{T}_x^{\underline{X}}$ denote the Zariski tangent space to $\underline{X}$ at x.

Remark 4.18. We work in the setting of Notations 4.13 and 4.17. Let $M \in \mathrm{Mod}^U(W)$. From [25, II, §5, 3], we know that the orbit $\underline{\mathrm{GL}}(W)_0 \cdot M$, equipped with its natural reduced subscheme structure, is locally closed in $\underline{\mathrm{Mod}}^U(W)$, and the map ϱ is regular and faithfully flat. Also, [25, II, §5, 2.6] tells us that $\underline{\mathrm{GL}}(W)_0$ is smooth.

Lemma 4.19. *We work in the setting of Notations* 4.13 *and* 4.17*. If* $M \in \mathrm{Mod}^U(W)$*, then the map* $\varrho\colon \underline{\mathrm{GL}}(W)_0 \to \underline{\mathrm{GL}}(W)_0 \cdot M$ *and the orbit* $\underline{\mathrm{GL}}(W)_0 \cdot M$ *are smooth.*

Proof. We begin by showing that the fiber $\mathrm{Stab}(M)$ of $\varrho(F)$ over M is smooth over F. Since F is algebraically closed, it suffices to show that $\mathrm{Stab}(M)$ is regular. Since $\mathrm{GL}(W)_0$ is regular, to show that $\mathrm{Stab}(M) \subseteq \mathrm{GL}(W)_0$ is regular it suffices to show that $\mathrm{Stab}(M)$ is defined by linear equations. To find these linear equations, note that the stabilizer condition $\alpha \cdot M = M$ is equivalent to the conditions $\partial = \alpha \circ \partial \circ \alpha^{-1}$ and $\mu = \alpha \circ \mu \circ (U \otimes_F \alpha^{-1})$, that is, $\partial \circ \alpha = \alpha \circ \partial$ and $\mu \circ (U \otimes_F \alpha) = \alpha \circ \mu$; since the matrices defining ∂ and μ are fixed, these equations are described by a system of linear equations in the variables describing α. Thus, the fiber $\mathrm{Stab}(M)$ is smooth.

Now, each closed fiber of $\varrho(F)$ is isomorphic to $\mathrm{Stab}(M)$ by translation, so it is smooth over F. Hilbert's Nullstellensatz implies that $\varrho(F)$ maps closed points to closed points, so it follows from [42, Théorème (17.5.1)] that $\varrho(F)$ is smooth at every closed point of $\underline{\mathrm{GL}}(W)_0(F)$. Since smoothness is an open condition on the source by [41, Corollaire (6.8.7)], it follows that $\varrho(F)$ is smooth at every point (closed or not) of $\underline{\mathrm{GL}}(W)_0(F)$. The fact that $\varrho(F)$ is smooth implies that ϱ is smooth, by [25, I.4.4.1].

Finally, because $\varrho(F)$ is faithfully flat, it is surjective. We know that $\mathrm{GL}(W)_0$ is smooth over F, and $\varrho(F)$ is smooth, so $\mathrm{GL}(W)_0 \cdot M$ is also smooth over F by [41, Proposition (6.8.3)(ii)]. It follows from [25, I.4.4.1] that $\underline{\mathrm{GL}}(W)_0 \cdot M$ is smooth. $\square$

Lemma 4.20. *We work in the setting of Notations 4.13 and 4.17. Given an element $M = (\partial, \mu) \in \mathrm{Mod}^U(W)$, the tangent space $T_M^{\underline{\mathrm{Mod}}^U(W)}$ is the set of all ordered pairs $(\overline{\partial}, \overline{\mu}) \in \underline{\mathrm{Mod}}^U(W)(F[\epsilon])$ that give rise to M modulo ϵ. Equivalently, $T_M^{\underline{\mathrm{Mod}}^U(W)}$ is the set of all ordered pairs $(\overline{\partial}, \overline{\mu}) = (\{\overline{\partial}_i\}, \{\overline{\mu}_i\})$ satisfying the following conditions:*

(1) *For each i, we have $\overline{\partial}_i = \left[\begin{smallmatrix} \partial_i & \gamma_i \\ 0 & \partial_i \end{smallmatrix}\right]$ where $\gamma_i \colon W_i \to W_{i-1}$ is an F-linear transformation such that $\partial_i \gamma_{i+1} + \gamma_i \partial_{i+1} = 0$.*

(2) *There is a degree-0 graded homomorphism $\theta \colon U \otimes_F W \to W$ of F-vector spaces such that the map $\overline{\mu} \colon U[\epsilon] \otimes_{F[\epsilon]} W[\epsilon] \to W[\epsilon]$ is given by the formula*

$$\overline{\mu}_{i+j}\left(\left[\begin{smallmatrix} a' \\ a \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} \theta_{i+j}(a \otimes w) + \mu_{i+j}(a \otimes w') + \mu_{i+j}(a' \otimes w) \\ \mu_{i+j}(a \otimes w) \end{smallmatrix}\right]$$

for all $\left[\begin{smallmatrix} a' \\ a \end{smallmatrix}\right] \in U_i \oplus U_i$ and $\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right] \in W_j \oplus W_j$, and $\overline{\mu}$ is a degree-0 graded homomorphism of $F[\epsilon]$-modules.

(3) *For each $a \in U_i$ and $w \in W_j$, we have*

$$\gamma_{i+j}(\mu_{i+j}(a \otimes w)) + \partial_{i+j}(\theta_{i+j}(a \otimes w)) \\ = \theta_{i-1+j}(\partial_i^U(a) \otimes w) + (-1)^i \theta_{i+j-1}(a \otimes \partial_j(w)) + (-1)^i \mu_{i-1+j}(a \otimes \gamma_j(w)).$$

(4) *For each $a \in U_i$, $b \in U_p$ and $w \in W_j$, we have*

$$\theta_{i+p+j}((ab) \otimes w) = \theta_{i+p+j}(a \otimes \mu_{p+j}(b \otimes w)) + \mu_{i+p+j}(a \otimes \theta_{p+j}(b \otimes w)).$$

Proof. The natural map $F[\epsilon] \to F$ induces a morphism

$$\mathrm{Mod}^{U[\epsilon]}(W[\epsilon]) = \underline{\mathrm{Mod}}^{U}(W)(F[\epsilon]) \to \underline{\mathrm{Mod}}^{U}(W)(F) = \mathrm{Mod}^{U}(W)$$

and the tangent space $\mathsf{T}_M^{\underline{\mathrm{Mod}}^U(W)}$ is the fiber of this morphism over M. Thus, an element of $\mathsf{T}_M^{\underline{\mathrm{Mod}}^U(W)}$ is precisely a DG $U[\epsilon]$-module structure on $W[\epsilon]$ that gives rise to M modulo ϵ.

Let $N = (\overline{\partial}, \overline{\mu}) \in \mathsf{T}_M^{\underline{\mathrm{Mod}}^U(W)}$; we show that conditions (1)–(4) are satisfied. The fact that $\overline{\partial}$ is $F[\epsilon]$-linear and gives rise to ∂ modulo ϵ, implies that $\overline{\partial}$ has the form $\overline{\partial}_i = \left[\begin{smallmatrix} \delta_i & \gamma_i \\ \beta_i & \partial_i \end{smallmatrix}\right]$ where $\beta_i, \gamma_i, \delta_i \colon W_i \to W_{i-1}$. Since the ordered pair $(\overline{\partial}, \overline{\mu})$ endows $W[\epsilon]$ with a DG $U[\epsilon]$-module structure, the Leibniz rule must be satisfied. In particular, for all $w \in W_j$, we have

$$\begin{aligned}
\overline{\partial}_j \left(\overline{\mu}_j \left(\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 0 \\ w \end{smallmatrix}\right]\right)\right) &= \overline{\mu}_{j-1}\left(\partial_j^{U[\epsilon]}\left(\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\right) \otimes \left[\begin{smallmatrix} 0 \\ w \end{smallmatrix}\right]\right) + \overline{\mu}_{j-1}\left(\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \otimes \overline{\partial}_j\left(\left[\begin{smallmatrix} 0 \\ w \end{smallmatrix}\right]\right)\right) \\
\begin{bmatrix} \delta_j & \gamma_j \\ \beta_j & \partial_j \end{bmatrix} \left[\begin{smallmatrix} w \\ 0 \end{smallmatrix}\right] &= 0 + \overline{\mu}_{j-1}\left(\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \otimes \left(\begin{bmatrix} \delta_j & \gamma_j \\ \beta_j & \partial_j \end{bmatrix} \left[\begin{smallmatrix} 0 \\ w \end{smallmatrix}\right]\right)\right) \\
\begin{bmatrix} \delta_j(w) \\ \beta_j(w) \end{bmatrix} &= \overline{\mu}_{j-1}\left(\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \otimes \begin{bmatrix} \gamma_j(w) \\ \partial_j(w) \end{bmatrix}\right) \\
\begin{bmatrix} \delta_j(w) \\ \beta_j(w) \end{bmatrix} &= \left[\begin{smallmatrix} \partial_j(w) \\ 0 \end{smallmatrix}\right].
\end{aligned}$$

It follows that $\beta_j = 0$ and $\partial_j = \delta_j$, so we have $\overline{\partial}_i = \left[\begin{smallmatrix} \partial_i & \gamma_i \\ 0 & \partial_i \end{smallmatrix}\right]$. Also for each i the condition $\overline{\partial}_i \overline{\partial}_{i+1} = 0$ implies that $\partial_i \gamma_{i+1} + \gamma_i \partial_{i+1} = 0$ for all i. This establishes (1).

The map $\overline{\mu}$ is a chain map over $F[\epsilon]$ from $U[\epsilon] \otimes_{F[\epsilon]} (W[\epsilon], \overline{\partial})$ to $(W[\epsilon], \overline{\partial})$. The fact that $\overline{\mu}$ is $F[\epsilon]$-linear and gives rise to μ modulo ϵ, implies that $\overline{\mu}$ satisfies the following conditions:

$$\begin{aligned}
\overline{\mu}_{i+j}\left(\left[\begin{smallmatrix} 0 \\ a \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 0 \\ w \end{smallmatrix}\right]\right) &= \begin{bmatrix} \theta_{i+j}(a \otimes w) \\ \mu_{i+j}(a \otimes w) \end{bmatrix} \\
\overline{\mu}_{i+j}\left(\left[\begin{smallmatrix} 0 \\ a \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} w \\ 0 \end{smallmatrix}\right]\right) &= \left[\begin{smallmatrix} \mu_{i+j}(a \otimes w) \\ 0 \end{smallmatrix}\right] \\
\overline{\mu}_{i+j}\left(\left[\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} 0 \\ w \end{smallmatrix}\right]\right) &= \left[\begin{smallmatrix} \mu_{i+j}(a \otimes w) \\ 0 \end{smallmatrix}\right] \\
\overline{\mu}_{i+j}\left(\left[\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} w \\ 0 \end{smallmatrix}\right]\right) &= \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right].
\end{aligned}$$

Here we have $a \in U_i$ and $w \in W_j$, and $\theta\colon U \otimes_F W \to W$ is a degree-0 graded homomorphism of F-vector spaces. Condition (2) follows by linearity. Condition (3) follows from the Leibniz rule for elements of the form $\left[\begin{smallmatrix} 0 \\ w \end{smallmatrix}\right] \in W[\epsilon]_j$ and $\left[\begin{smallmatrix} 0 \\ a \end{smallmatrix}\right] \in U[\epsilon]_i$, and condition (4) follows from the associativity of the scalar multiplication $\overline{\mu}$.

Similar reasoning shows that any ordered pair $(\overline{\partial}, \overline{\mu}) \in \underline{\mathrm{Mod}}^U(W)(F[\epsilon])$ satisfying conditions (1)–(4) is a DG $U[\epsilon]$-module structure on $W[\epsilon]$ that gives rise to M modulo ϵ, that is, an element of $\mathsf{T}_M^{\underline{\mathrm{Mod}}^U(W)}$. Note that condition (4) implies that $\theta_j(1 \otimes w) = 0$ for all $w \in W_j$ for all j, which is used in this implication. □

Lemma 4.21. *We work in the setting of Notations* 4.13 *and* 4.17. *Given an element $M = (\partial, \mu) \in \mathrm{Mod}^U(W)$, the tangent space $\mathsf{T}_M^{\underline{\mathrm{Mod}}^U(W)}$ is an F-vector space under the following operations: Let $N^{(1)}, N^{(2)} \in \mathsf{T}_M^{\underline{\mathrm{Mod}}^U(W)}$ where $N^{(n)} = (\overline{\partial}^{(n)}, \overline{\mu}^{(n)})$ such that $\overline{\partial}_i^{(n)} = \left[\begin{smallmatrix} \partial_i & \gamma_i^{(n)} \\ 0 & \partial_i \end{smallmatrix}\right]$ and*

$$\overline{\mu}_{i+j}^{(n)}\left(\left[\begin{smallmatrix} a' \\ a \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} \theta_{i+j}^{(n)}(a \otimes w) + \mu_{i+j}(a \otimes w') + \mu_{i+j}(a' \otimes w) \\ \mu_{i+j}(a \otimes w) \end{smallmatrix}\right]$$

for $n = 1, 2$ as in Lemma 4.20. *For $\alpha_1, \alpha_2 \in F$ the element $\alpha_1 N^{(1)} + \alpha_2 N^{(2)}$ in $\mathsf{T}_M^{\underline{\mathrm{Mod}}^U(W)}$ is given using the functions $\alpha_1\gamma^{(1)} + \alpha_2\gamma^{(2)}$ and $\alpha_1\theta^{(1)} + \alpha_2\theta^{(2)}$, that is, we have $\alpha_1 N^{(1)} + \alpha_2 N^{(2)} = (\overline{\partial}, \overline{\mu})$ where $\overline{\partial}_i = \left[\begin{smallmatrix} \partial_i & \alpha_1\gamma_i^{(1)} + \alpha_2\gamma_i^{(2)} \\ 0 & \partial_i \end{smallmatrix}\right]$ and*

$$\overline{\mu}_{i+j}\left(\left[\begin{smallmatrix} a' \\ a \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} \alpha_1\theta_{i+j}^{(1)}(a \otimes w) + \alpha_2\theta_{i+j}^{(2)}(a \otimes w) + \mu_{i+j}(a \otimes w') + \mu_{i+j}(a' \otimes w) \\ \mu_{i+j}(a \otimes w) \end{smallmatrix}\right].$$

Proof. It is straightforward to show that the ordered pair $(\overline{\partial}, \overline{\mu})$ satisfies conditions (1)–(4) from Lemma 4.20. That is, the tangent space $\mathsf{T}_M^{\underline{\mathrm{Mod}}^U(W)}$ is closed under linear combinations. The other vector space axioms follow readily. □

Lemma 4.22. *We work in the setting of Notations* 4.13 *and* 4.17. *The tangent space $\mathsf{T}_{\mathrm{id}_W}^{\underline{\mathrm{GL}}(W)_0}$ is the set of all elements of $\underline{\mathrm{GL}}(W)_0(F[\epsilon])$ that give rise to id_W modulo*

ϵ. *Equivalently,* $\mathsf{T}_{\mathrm{id}_W}^{\underline{\mathrm{GL}}(W)_0}$ *is the set of all matrices of the form* $\xi = \left[\begin{smallmatrix} \mathrm{id}_W & D \\ 0 & \mathrm{id}_W \end{smallmatrix}\right]$*, where* $D \in \mathrm{End}_F(W)_0$.

Proof. Arguing as in the proof of Lemma 4.20, one checks readily that $\mathsf{T}_{\mathrm{id}_W}^{\underline{\mathrm{GL}}(W)_0}$ is the set of all elements of $\underline{\mathrm{GL}}(W)_0(F[\epsilon])$ that give rise to id_W modulo ϵ. To describe the elements of $\mathsf{T}_{\mathrm{id}_W}^{\underline{\mathrm{GL}}(W)_0}$ explicitly, recall from Notation 4.13 that we write $W[\epsilon]$ as $W \oplus W$. Thus, the elements of $\mathsf{T}_{\mathrm{id}_W}^{\underline{\mathrm{GL}}(W)_0} \subseteq \underline{\mathrm{GL}}(W)_0(F[\epsilon])$ have the form $\xi = \left[\begin{smallmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{smallmatrix}\right]$ where each $\xi_{ij} \colon W \to W$. Since ξ gives rise to id_W modulo ϵ, we must have $\xi_{22} = \mathrm{id}_W$. Also, the condition $\epsilon\xi(\left[\begin{smallmatrix} 0 \\ w \end{smallmatrix}\right]) = \xi(\left[\begin{smallmatrix} w \\ 0 \end{smallmatrix}\right])$ for all $w \in W$ implies that $\xi_{21} = 0$ and $\xi_{11} = \mathrm{id}_W$, and hence ξ has the desired form. One checks similarly that every matrix of the form $\left[\begin{smallmatrix} \mathrm{id}_W & D \\ 0 & \mathrm{id}_W \end{smallmatrix}\right]$, where $D \in \mathrm{End}_F(W)_0$ is an element of $\underline{\mathrm{GL}}(W)_0(F[\epsilon])$ that gives rise to id_W modulo ϵ, that is, it is in $\mathsf{T}_{\mathrm{id}_W}^{\underline{\mathrm{GL}}(W)_0}$. $\square$

4.23 (Proof of Theorem 1.5)**.** Using the notation of Lemma 4.20, let $N = (\overline{\partial}, \overline{\mu})$ be an element of $\mathsf{T}_M^{\underline{\mathrm{Mod}}^U(W)}$. Since N is a DG $U[\epsilon]$-module, restriction of scalars along the natural inclusion $U \to U[\epsilon]$ makes N a DG U-module with scalar multiplication given by the following formula

$$a\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right] := \overline{\mu}_{i+j}\left(\left[\begin{smallmatrix} 0 \\ a \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} \mu_{i+j}(a \otimes w') + \theta_{i+j}(a \otimes w) \\ \mu_{i+j}(a \otimes w) \end{smallmatrix}\right] \tag{4.23.1}$$

for all $a \in U_i$ and $\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right] \in N_j = W_j \oplus W_j$.

Define $\rho \colon M \to N$ and $\pi \colon N \to M$ by the formulas $\rho(w) := \left[\begin{smallmatrix} w \\ 0 \end{smallmatrix}\right]$ and $\pi\left(\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right) := w$. Using the equation $\overline{\partial}_i = \left[\begin{smallmatrix} \partial_i & \gamma_i \\ 0 & \partial_i \end{smallmatrix}\right]$ from Lemma 4.20, it is straightforward to show that ρ and π are chain maps. From equation (4.23.1), we conclude that ρ and π are U-linear. In other words, we have an exact sequence

$$0 \to M \xrightarrow{\rho} N \xrightarrow{\pi} M \to 0$$

of DG U-module morphisms. So, we obtain a map $\tau \colon \mathsf{T}_M^{\underline{\mathrm{Mod}}^U(W)} \to \mathrm{YExt}_U^1(M, M)$

where $\tau(N)$ is the equivalence class of the displayed sequence in $\mathrm{YExt}^1_U(M,M)$. We show that τ is a surjective abelian group homomorphism with $\mathrm{Ker}(\tau) = \mathsf{T}^{\mathrm{GL}(W)_0 \cdot M}_{M}$.

To show that τ is additive, let $N^{(1)}, N^{(2)} \in \mathsf{T}^{\underline{\mathrm{Mod}}^U(W)}_{M}$ where $N^{(n)} = (\overline{\partial}^{(n)}, \overline{\mu}^{(n)})$ such that $\overline{\partial}^{(n)}_i = \left[\begin{smallmatrix} \partial_i & \gamma^{(n)}_i \\ 0 & \partial_i \end{smallmatrix}\right]$ and

$$\overline{\mu}^{(n)}_{i+j}\left(\left[\begin{smallmatrix} a' \\ a \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} \theta^{(n)}_{i+j}(a\otimes w)+\mu_{i+j}(a\otimes w')+\mu_{i+j}(a'\otimes w) \\ \mu_{i+j}(a\otimes w) \end{smallmatrix}\right]$$

for $n = 1, 2$ as in Lemma 4.20. Then $\tau(N^{(n)})$ is represented by the exact sequence

$$0 \to M \xrightarrow{\rho} N^{(n)} \xrightarrow{\pi} M \to 0$$

for $n = 1, 2$.[7] The Baer sum $\tau(N^{(1)}) + \tau(N^{(2)})$ is represented by the exact sequence

$$0 \to M \xrightarrow{\rho'} T \xrightarrow{\pi'} M \to 0 \tag{4.23.2}$$

which is constructed in the following four steps:

(1) Let L denote the pull-back of $\pi\colon N^{(1)} \to M$ and $\pi\colon N^{(2)} \to M$, which is a DG U-module with[8]

$$\begin{aligned}
L_i &= \left\{\left(\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right], \left[\begin{smallmatrix} v' \\ v \end{smallmatrix}\right]\right) \in N^{(1)}_i \oplus N^{(2)}_i \mid \pi\left(\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right) = \pi\left(\left[\begin{smallmatrix} v' \\ v \end{smallmatrix}\right]\right)\right\} \\
&= \left\{\left(\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right], \left[\begin{smallmatrix} v' \\ v \end{smallmatrix}\right]\right) \in N^{(1)}_i \oplus N^{(2)}_i \mid w = v\right\} \\
&= \left\{\left(\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right], \left[\begin{smallmatrix} v' \\ w \end{smallmatrix}\right]\right) \in N^{(1)}_i \oplus N^{(2)}_i\right\} \\
\partial^L_i\left(\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right], \left[\begin{smallmatrix} v' \\ w \end{smallmatrix}\right]\right) &= \left(\overline{\partial}^{(1)}_i\left(\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right), \overline{\partial}^{(2)}_i\left(\left[\begin{smallmatrix} v' \\ w \end{smallmatrix}\right]\right)\right) \\
&= \left(\left[\begin{smallmatrix} \partial_i(w')+\gamma^{(1)}_i(w) \\ \partial_i(w) \end{smallmatrix}\right], \left[\begin{smallmatrix} \partial_i(v')+\gamma^{(2)}_i(w) \\ \partial_i(w) \end{smallmatrix}\right]\right) \\
a\left(\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right], \left[\begin{smallmatrix} v' \\ w \end{smallmatrix}\right]\right) &= \left(a\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right], a\left[\begin{smallmatrix} v' \\ w \end{smallmatrix}\right]\right) \\
&= \left(\left[\begin{smallmatrix} \theta^{(1)}_{i+j}(a\otimes w)+\mu_{i+j}(a\otimes w') \\ \mu_{i+j}(a\otimes w) \end{smallmatrix}\right], \left[\begin{smallmatrix} \theta^{(2)}_{i+j}(a\otimes w)+\mu_{i+j}(a\otimes w') \\ \mu_{i+j}(a\otimes w) \end{smallmatrix}\right]\right)
\end{aligned}$$

for all $a \in U$.

[7] We abuse notation slightly here: for instance, the maps $M \to N^{(1)}$ and $M \to N^{(2)}$ have different domains, so they should not both be called ρ. However, the maps of underlying vector spaces are the same, so there should be no confusion.

[8] We conveniently abuse our notational protocol for direct sums here.

(2) The map $\sigma\colon M \to N^{(1)} \oplus N^{(2)}$ given by $\sigma(m) = \left(\left[\begin{smallmatrix} -m \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} m \\ 0 \end{smallmatrix}\right]\right)$ is a well-defined DG U-module morphism such that $\operatorname{Im}(\sigma) \subseteq L$. Let $\overline{\sigma}\colon M \to L$ denote the induced DG U-module morphism.

(3) Set $T = \operatorname{Coker}(\overline{\sigma})$. Let the element of T represented by the ordered pair $\left(\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right], \left[\begin{smallmatrix} v' \\ w \end{smallmatrix}\right]\right)$ be denoted $\left[\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right], \left[\begin{smallmatrix} v' \\ w \end{smallmatrix}\right]\right]$. Then T is a DG U-module with differential and scalar multiplication induced from L:

$$\begin{aligned} \partial_i^T \left(\left[\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right], \left[\begin{smallmatrix} v' \\ w \end{smallmatrix}\right]\right]\right) &= \left[\left[\begin{smallmatrix} \partial_i(w') + \gamma_i^{(1)}(w) \\ \partial_i(w) \end{smallmatrix}\right], \left[\begin{smallmatrix} \partial_i(v') + \gamma_i^{(2)}(w) \\ \partial_i(w) \end{smallmatrix}\right]\right] \\ a\left[\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right], \left[\begin{smallmatrix} v' \\ w \end{smallmatrix}\right]\right] &= \left[\left[\begin{smallmatrix} \theta_{i+j}^{(1)}(a\otimes w) + \mu_{i+j}(a\otimes w') \\ \mu_{i+j}(a\otimes w) \end{smallmatrix}\right], \left[\begin{smallmatrix} \theta_{i+j}^{(2)}(a\otimes w) + \mu_{i+j}(a\otimes w') \\ \mu_{i+j}(a\otimes w) \end{smallmatrix}\right]\right] \end{aligned}$$

(4) Let $\rho'\colon M \to T$ be given by $\rho'(m) = \left[\left[\begin{smallmatrix} m \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]\right] = \left[\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} m \\ 0 \end{smallmatrix}\right]\right]$. Let $\pi'\colon T \to M$ be given by $\pi'\left(\left[\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right], \left[\begin{smallmatrix} v' \\ w \end{smallmatrix}\right]\right]\right) = w$. It is straightforward to show that ρ' and τ' are well-defined DG U-module morphisms making (4.23.2) exact.

On the other hand, Lemma 4.21 implies that the element $N = N^{(1)} + N^{(2)}$ in $\mathsf{T}_M^{\mathrm{Mod}^U(W)}$ is given using the functions $\gamma^{(1)} + \gamma^{(2)}$ and $\theta^{(1)} + \theta^{(2)}$, that is, we have $N = (\overline{\partial}, \overline{\mu})$ where $\overline{\partial}_i = \left[\begin{smallmatrix} \partial_i & \gamma_i^{(1)} + \gamma_i^{(2)} \\ 0 & \partial_i \end{smallmatrix}\right]$ and

$$\overline{\mu}_{i+j}\left(\left[\begin{smallmatrix} a' \\ a \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} \theta_{i+j}^{(1)}(a\otimes w) + \theta_{i+j}^{(2)}(a\otimes w) + \mu_{i+j}(a\otimes w') + \mu_{i+j}(a'\otimes w) \\ \mu_{i+j}(a\otimes w) \end{smallmatrix}\right].$$

The element $\tau(N)$ is represented by the exact sequence

$$0 \to M \xrightarrow{\rho} N \xrightarrow{\pi} M \to 0$$

where ρ and π are the canonical inclusion and surjection. To prove that $\tau(N) = \tau(N^{(1)}) + \tau(N^{(2)})$, we need to construct a DG U-module morphism $\phi\colon T \to N$ making the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{\rho'} & T & \xrightarrow{\pi'} & M & \longrightarrow & 0 \\ & & \Big\downarrow{=} & & \Big\downarrow{\phi} & & \Big\downarrow{=} & & \\ 0 & \longrightarrow & M & \xrightarrow{\rho} & N & \xrightarrow{\pi} & M & \longrightarrow & 0. \end{array}$$

Define $\phi_i\left(\left[\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right],\left[\begin{smallmatrix} v' \\ w \end{smallmatrix}\right]\right]\right) = \left[\begin{smallmatrix} w'+v' \\ w \end{smallmatrix}\right]$ for each i. It is straightforward to show that ϕ is a well-defined DG U-module morphism making the above diagram commute. Thus, the map τ is additive.

Now we show that τ is onto. Fix an arbitrary element $\zeta \in \operatorname{YExt}^1_U(M,M)$, represented by the sequence $0 \to M \xrightarrow{f} Z \xrightarrow{g} M \to 0$. In particular, this is an exact sequence of F-complexes, so it is degree-wise split. This implies that we have a commutative diagram of graded vector spaces:

$$\begin{CD}
0 @>>> M @>{f}>> Z @>{g}>> M @>>> 0\\
@. @V{=}VV @VV{\vartheta}V @VV{=}V @.\\
0 @>>> M @>{\rho}>> W[\epsilon] @>{\pi}>> M @>>> 0
\end{CD}$$

where $\rho(w) = \left[\begin{smallmatrix} w \\ 0 \end{smallmatrix}\right]$, $\pi\left(\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right) = w$, and ϑ is an isomorphism of graded F-vector spaces. The map ϑ allows us to transfer a DG U-module structure to $W[\epsilon]$ as follows: let the differential on $W[\epsilon]$ be given by the formula $\overline{\partial}_i = \vartheta_{i-1}\partial^Z_i\vartheta^{-1}_i$, and define scalar multiplication μ' over U on $W[\epsilon]$ by the formula $\mu'_{i+j}\left(a \otimes \left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right) = \vartheta_{i+j}\left(\mu_{i+j}\left(a \otimes \vartheta^{-1}_j\left(\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right)\right)\right)$ for all $a \in U_i$ and $\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right] \in W[\epsilon]_j$. These definitions provide an exact sequence

$$0 \to M \xrightarrow{\rho} \left(W[\epsilon], \overline{\partial}, \mu'\right) \xrightarrow{\pi} M \to 0 \tag{4.23.3}$$

of DG U-modules equivalent to the original sequence $0 \to M \xrightarrow{f} Z \xrightarrow{g} M \to 0$. Next, define scalar multiplication $\overline{\mu}$ over $U[\epsilon]$ on $W[\epsilon]$ by the formulas

$$\begin{aligned}
\overline{\mu}_{i+j}\left(\left[\begin{smallmatrix} 0 \\ a \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right) &= \mu'_{i+j}\left(a \otimes \left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right) = \vartheta_{i+j}\left(\mu_{i+j}\left(a \otimes \vartheta^{-1}_j\left(\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right)\right)\right)\\
\overline{\mu}_{i+j}\left(\left[\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right) &= \left[\begin{smallmatrix} \mu_{i+j}(a \otimes w) \\ 0 \end{smallmatrix}\right]
\end{aligned}$$

for all $a \in U_i$ and $\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right] \in W[\epsilon]_j$. These definitions endow $W[\epsilon]$ with a DG $U[\epsilon]$-module structure $(\overline{\partial}, \overline{\mu})$ that gives rise to M modulo ϵ, so $N = (\overline{\partial}, \overline{\mu}) \in \mathsf{T}^{\mathrm{Mod}^U(W)}_M$.

Furthermore, since the sequence (4.23.3) is equivalent to the sequence representing ζ, we have $\tau(N) = \zeta$, so τ is surjective.

By Lemma 4.19, the map $\varrho : \underline{\mathrm{GL}}(W)_0 \to \underline{\mathrm{GL}}(W)_0 \cdot M$ defined by $g \mapsto g \cdot M$ is smooth. Thus, the induced map on tangent spaces $\mathsf{T}^{\underline{\mathrm{GL}}(W)}_{\mathrm{id}_W} \xrightarrow{\varpi} \mathsf{T}^{\underline{\mathrm{GL}}(W)_0 \cdot M}_{M}$ is surjective; see [25, I.4.4.15].

To describe $\mathsf{T}^{\underline{\mathrm{GL}}(W)_0 \cdot M}_{M}$ as a subset of $\mathsf{T}^{\underline{\mathrm{Mod}}^U(W)}_{M}$, consider the next commutative diagram of morphisms of schemes where $\underline{\Psi}_M$ is (induced by) multiplication by M

$$\begin{array}{ccc} \underline{\mathrm{GL}}(W)_0 & \xrightarrow{\underline{\Psi}_M} & \underline{\mathrm{Mod}}^U(W) \\ & {\scriptstyle \cdot M} \searrow \quad \nearrow & \\ & \underline{\mathrm{GL}}(W)_0 \cdot M & \end{array}$$

and the unspecified arrow is the natural inclusion. Note that for each F-algebra S, the map $\underline{\Psi}_M(S)\colon\ \underline{\mathrm{GL}}(W)_0(S) \to \underline{\mathrm{Mod}}^U(W)(S)$ is given by multiplication on $S \otimes_F M$. This diagram induces the following commutative diagram of maps of tangent spaces:

$$\begin{array}{ccc} \mathsf{T}^{\underline{\mathrm{GL}}(W)_0}_{\mathrm{id}_W} & \xrightarrow{\psi_M} & \mathsf{T}^{\underline{\mathrm{Mod}}^U(W)}_{M} \\ & {\scriptstyle \varpi} \searrow \quad \nearrow {\scriptstyle \upsilon} & \\ & \mathsf{T}^{\underline{\mathrm{GL}}(W)_0 \cdot M}_{M} . & \end{array}$$

Since the orbit $\mathrm{GL}(W)_0 \cdot M$ is a locally closed subset of $\mathrm{Mod}^U(W)$, the map υ is injective. Since ϖ is surjective, it follows that $\mathsf{T}^{\underline{\mathrm{GL}}(W)_0 \cdot M}_{M}$ is isomorphic to the image of ψ_M. Thus, we identify $\mathsf{T}^{\underline{\mathrm{GL}}(W)_0 \cdot M}_{M}$ with $\mathrm{Im}(\psi_M) \subseteq \mathsf{T}^{\underline{\mathrm{Mod}}^U(W)}_{M}$.

To continue our description of $\mathsf{T}^{\underline{\mathrm{GL}}(W)_0 \cdot M}_{M}$, we describe ψ_M explicitly. Again, $\mathsf{T}^{\underline{\mathrm{GL}}(W)_0}_{\mathrm{id}_W}$ is the fiber over id_W in the map $\underline{\mathrm{GL}}(W)_0(F[\epsilon]) \to \underline{\mathrm{GL}}(W)_0(F) = \mathrm{GL}(W)_0$ induced by the natural ring epimorphism $F[\epsilon] \to F$. And $\mathsf{T}^{\underline{\mathrm{Mod}}^U(W)}_{M}$ is the fiber over M in the induced map $\underline{\mathrm{Mod}}^U(W)(F[\epsilon]) \to \underline{\mathrm{Mod}}^U(W)(F) = \mathrm{Mod}^U(W)$. Thus, the map ψ_M is induced by $\underline{\Psi}_M(F[\epsilon])\colon\ \underline{\mathrm{GL}}(W)_0(F[\epsilon]) \to \underline{\mathrm{Mod}}^U(W)(F[\epsilon])$, so it is given by multiplication on $F[\epsilon] \otimes_F M$. In a variation of the notation of Lemma 4.20, we have $F[\epsilon] \otimes_F M = (\partial', \mu')$ where $\partial'_i = \left[\begin{smallmatrix} \partial_i & 0 \\ 0 & \partial_i \end{smallmatrix}\right]$ and

$$\mu'_{i+j}(\left[\begin{smallmatrix} a' \\ a \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]) = \left[\begin{smallmatrix} \mu_{i+j}(a \otimes w') + \mu_{i+j}(a' \otimes w) \\ \mu_{i+j}(a \otimes w) \end{smallmatrix}\right]$$

for all $\left[\begin{smallmatrix} a' \\ a \end{smallmatrix}\right] \in U[\epsilon]_i$ and all $\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right] \in W[\epsilon]_j$.

By definition, the map $\underline{\Psi}_M(F[\epsilon])\colon \underline{\mathrm{GL}}(W)(F[\epsilon]) \to \underline{\mathrm{Mod}}^U(F[\epsilon])$ is given by $\underline{\Psi}_M(F[\epsilon])(\xi) = (\partial'', \mu'')$ where $\partial'' = \xi \circ \partial' \circ \xi^{-1}$ and $\mu'' = \xi \circ \mu' \circ (U[\epsilon] \otimes \xi^{-1})$. Then ψ_M is the restriction of $\underline{\Psi}_M(F[\epsilon])$ to $\mathsf{T}^{\underline{\mathrm{GL}}(W)}_{\mathrm{id}_W}$.

Lemma 4.22 implies that each element $\xi \in \mathsf{T}^{\underline{\mathrm{GL}}(W)}_{\mathrm{id}_W}$ is of the form

$$\xi = \left[\begin{smallmatrix} \mathrm{id}_W & D \\ 0 & \mathrm{id}_W \end{smallmatrix}\right] \tag{4.23.4}$$

where $D \in \mathrm{End}_F(W)_0$. Note that $\xi^{-1} = \left[\begin{smallmatrix} \mathrm{id}_W & -D \\ 0 & \mathrm{id}_W \end{smallmatrix}\right]$. It follows that

$$\begin{aligned} \mu''_{i+j}(\left[\begin{smallmatrix} a' \\ a \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]) &= \xi_{i+j}\left(\mu'_{i+j}\left(\left[\begin{smallmatrix} a' \\ a \end{smallmatrix}\right] \otimes \xi_j^{-1}\left(\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right)\right)\right) \\ &= \left[\begin{smallmatrix} \mu_{i+j}(a' \otimes w) + \mu_{i+j}(a \otimes w') - \mu_{i+j}(a \otimes D_j(w)) + D_{i+j}(\mu_{i+j}(a \otimes w)) \\ \mu_{i+j}(a \otimes w) \end{smallmatrix}\right] \end{aligned}$$

and similarly $\partial''_i = \left[\begin{smallmatrix} \partial_i & D_{i-1}\partial_i - \partial_i D_i \\ 0 & \partial_i \end{smallmatrix}\right]$. So, we have $\psi_M(\xi) = (\partial'', \mu'')$ where ∂'' and μ'' are given by the above formulas.

We now show that $\mathsf{T}^{\underline{\mathrm{GL}}(W)_0 \cdot M}_M \subseteq \mathrm{Ker}(\tau)$. Let $N \in \mathsf{T}^{\underline{\mathrm{GL}}(W)_0 \cdot M}_M$, and write $N = \psi_M(\xi)$ where ξ is as in (4.23.4). Define $h\colon M \to N$ by the formula $h_j(w) = \left[\begin{smallmatrix} D_j(w) \\ w \end{smallmatrix}\right]$. By definition, the DG U-module structure on N gives

$$a \left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mu_{i+j}(a \otimes w') - \mu_{i+j}(a \otimes D_j(w)) + D_{i+j}(\mu_{i+j}(a \otimes w)) \\ \mu_{i+j}(a \otimes w) \end{smallmatrix}\right].$$

From this (using the explicit description of $\psi_M(\xi)$) it is straightforward to show that h is a morphism of DG U-modules such that $\pi \circ h = \mathrm{id}_M$. Therefore the exact sequence

$0 \to M \xrightarrow{\rho} N \xrightarrow{\pi} M \to 0$ representing $\tau(N)$ splits. This means that $N \in \operatorname{Ker}(\tau)$, as desired.

We conclude the proof by showing that $\mathsf{T}_M^{\underline{\mathrm{GL}}(W)_0 \cdot M} \supseteq \operatorname{Ker}(\tau)$. Given an element $N = (\overline{\partial}, \overline{\mu}) \in \operatorname{Ker}(\tau)$, the short exact sequence $0 \to M \xrightarrow{\rho} N \xrightarrow{\pi} M \to 0$ representing $\tau(N)$ splits over U. Therefore there exists a morphism of DG U-modules $h\colon M \to N$ such that $\pi \circ h = \mathrm{id}_M$. The condition $\pi \circ h = \mathrm{id}_M$ implies that $h_j(w) = \left[\begin{smallmatrix} D_j(w) \\ w \end{smallmatrix}\right]$ for some $D \in \operatorname{End}_F(W)_0$. The fact that h is a chain map implies that $\gamma_i = D_{i-1}\partial_i - \partial_i D_i$, in the notation of Lemma 4.20, so we have $\overline{\partial}_i = \left[\begin{smallmatrix} \partial_i & D_{i-1}\partial_i - \partial_i D_i \\ 0 & \partial_i \end{smallmatrix}\right]$. By checking the condition of h being a DG U-homomorphism, we get

$$\theta_{i,j}(a \otimes w) = D_{i+j}(\mu_{i+j}(a \otimes w)) - \mu_{i+j}(a \otimes D_j(w))$$

for $a \in U_i$ and $w \in W_j$. Thus

$$\overline{\mu}_{i+j}(\left[\begin{smallmatrix} a' \\ a \end{smallmatrix}\right] \otimes \left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]) = \left[\begin{smallmatrix} \mu_{i+j}(a' \otimes w) + \mu_{i+j}(a \otimes w') - \mu_{i+j}(a \otimes D_j(w)) + D_{i+j}(\mu_{i+j}(a \otimes w)) \\ \mu_{i+j}(a \otimes w) \end{smallmatrix}\right].$$

This means that $N = \psi_M(\xi) \in \operatorname{Im}(\psi_M) = \mathsf{T}_M^{\underline{\mathrm{GL}}(W)_0 \cdot M}$, where ξ is as in (4.23.4). $\square$

The next result follows the ideas of Gabriel [34, 1.2 Corollary].

Corollary 4.24. *We work in the setting of Notations* 4.13 *and* 4.17. *Let C be a degree-wise finite graded-projective (e.g., semi-free) semidualizing DG U-module, and let $s \geqslant \sup(C)$. Set $M = \tau(C)_{(\leqslant s)}$ and $W = M^\natural$. Then the orbit $\underline{\mathrm{GL}}(W)_0 \cdot M$ is open in $\underline{\mathrm{Mod}}^U(W)$.*

Proof. Proposition 4.12 implies that $\operatorname{YExt}^1_U(M, M) = 0$, so by Theorem 1.5 we have $\mathsf{T}_M^{\underline{\mathrm{Mod}}^U(W)} = \mathsf{T}_M^{\underline{\mathrm{GL}}(W)_0 \cdot M}$. Lemma 4.20 implies that the orbit $\underline{\mathrm{GL}}(W)_0 \cdot M$ is smooth.

This explains the first step in the next sequence

$$\begin{aligned}\dim(\mathcal{O}_{\underline{\mathrm{GL}}(W)_0\cdot M,M}) &= \mathrm{rank}_F(\mathsf{T}_M^{\underline{\mathrm{GL}}(W)_0\cdot M}) \\ &= \mathrm{rank}_F(\mathsf{T}_M^{\underline{\mathrm{Mod}}^U(W)}) \\ &\geqslant \dim(\mathcal{O}_{\underline{\mathrm{Mod}}^U(W),M}) \\ &\geqslant \dim(\mathcal{O}_{\underline{\mathrm{GL}}(W)_0\cdot M,M}).\end{aligned}$$

The second step follows from the equality $\mathsf{T}_M^{\underline{\mathrm{Mod}}^U(W)} = \mathsf{T}_M^{\underline{\mathrm{GL}}(W)_0\cdot M}$. The third step is standard, and the last step follows from the fact that $\underline{\mathrm{GL}}(W)_0 \cdot M$ is a locally closed subscheme of $\underline{\mathrm{Mod}}^U(W)$. It follows that $\underline{\mathrm{Mod}}^U(W)$ is smooth at M such that $\dim(\mathcal{O}_{\underline{\mathrm{Mod}}^U(W),M}) = \dim(\mathcal{O}_{\underline{\mathrm{GL}}(W)_0\cdot M,M})$. Since $\underline{\mathrm{GL}}(W)_0 \cdot M$ is a locally closed subscheme of $\underline{\mathrm{Mod}}^U(W)$, the ring $\mathcal{O}_{\underline{\mathrm{GL}}(W)_0\cdot M,M}$ is a localization of a quotient of $\mathcal{O}_{\underline{\mathrm{Mod}}^U(W),M}$. However, since $\mathcal{O}_{\underline{\mathrm{Mod}}^U(W),M}$ is a regular local ring, any proper quotient or localization has strictly smaller Krull dimension. It follows that $\mathcal{O}_{\underline{\mathrm{GL}}(W)_0\cdot M,M} = \mathcal{O}_{\underline{\mathrm{Mod}}^U(W),M}$, so $\mathrm{GL}(W)_0 \cdot M$ and $\mathrm{Mod}^U(W)$ are equal in an open neighborhood V of M in $\mathrm{Mod}^U(W)$.

Every closed point $M' \in \mathrm{GL}(W)_0 \cdot M$ is of the form $M' = \sigma \cdot M$ for some element $\sigma \in \mathrm{GL}(W)_0$. Translating by σ, we see that $\mathrm{GL}(W)_0 \cdot M$ and $\mathrm{Mod}^U(W)$ are equal in an open neighborhood $\sigma \cdot V$ of $\sigma \cdot M = M'$ in $\mathrm{Mod}^U(W)$. Since this is true for every closed point of $\mathrm{GL}(W)_0 \cdot M$, it is true for every point of $\mathrm{GL}(W)_0 \cdot M$. This uses the fact that $\mathrm{Mod}^U(W)$ is of finite type over a field. We conclude that $\mathrm{GL}(W)_0 \cdot M$ is open in $\mathrm{Mod}^U(W)$. Hence, $\underline{\mathrm{GL}}(W)_0 \cdot M$ is open in $\underline{\mathrm{Mod}}^U(W)$. □

4.3. Answering Vasconcelos' Question

The final steps of our proof of Theorem 1.4 begin with the next result which is motivated by Happel [44].

Lemma 4.25. *We work in the setting of Notations 4.13 and 4.17. Let $\mathfrak{S}_W(U)$ denote the set of quasiisomorphism classes of degree-wise finite semi-free semidualizing DG U-modules C such that $s \geqslant \sup(C)$, $C_i = 0$ for all $i < 0$, and $(\tau(C)_{(\leqslant s)})^\natural \cong W$. Then $\mathfrak{S}_W(U)$ is a finite set.*

Proof. Fix a representative C for each quasiisomorphism class in $\mathfrak{S}_W(U)$, and write $[C] \in \mathfrak{S}_W(U)$ and $M_C = \tau(C)_{(\leqslant s)}$.

Let $[C], [C'] \in \mathfrak{S}_W(U)$. If $\underline{\mathrm{GL}}(W)_0 \cdot M_C = \underline{\mathrm{GL}}(W)_0 \cdot M_{C'}$, then $[C] = [C']$: indeed, Remark 4.16 explains the second step in the next display

$$C \simeq M_C \cong M_{C'} \simeq C'$$

and the remaining steps follow from the assumptions $s \geqslant \sup(C)$ and $s \geqslant \sup(C')$.

Now, each orbit $\underline{\mathrm{GL}}(W)_0 \cdot M_C$ is open in $\underline{\mathrm{Mod}}^U(W)$ by Corollary 4.24. Since $\underline{\mathrm{Mod}}^U(W)$ is quasi-compact, it can only have finitely many open orbits. By the previous paragraph, this implies that there are only finitely many distinct elements $[C] \in \mathfrak{S}_W(U)$. □

Theorem 1.4 is a corollary of the following result whose proof uses techniques we learned from Avramov and Iyengar. Recall the notation $\mathfrak{S}(R)$ from Definition 2.26.

Theorem 4.26. *Let $(R, \mathfrak{m}, k)$ be a local ring. Then the set $\mathfrak{S}(R)$ is finite.*

Proof. A result of Grothendieck [39, Proposition (0.10.3.1)] provides a flat local ring homomorphism $R \to (R', \mathfrak{m}', k')$ such that k' is algebraically closed. Composing with the natural map from R' to its $\mathfrak{m}'$-adic completion, we assume that R' is complete. By [32, Theorem II(c)], the induced map $\mathfrak{S}(R) \to \mathfrak{S}(R')$ is a monomorphism. Thus it suffices to prove the result for R', so we assume that R is complete with algebraically closed residue field.

Let $\underline{t} = t_1, \cdots, t_n$ be a minimal generating sequence for $\mathfrak{m}$, and set $K = K^R(\underline{t})$, the Koszul complex. The map $\mathfrak{S}(R) \to \mathfrak{S}(K)$ induced by $C \mapsto K \otimes_R C$ is bijective by Corollary 3.20. Thus, it suffices to show that $\mathfrak{S}(K)$ is finite. Note that for each semidualizing R-complex C, we have $\operatorname{amp}(C) \leqslant \dim(R) - \operatorname{depth}(R)$ by [21, (3.4) Corollary]. A standard result about K (see, e.g., [30, 1.3]) implies that

$$\operatorname{amp}(K \otimes_R C) \leqslant \operatorname{amp}(C) + n \leqslant \dim(R) - \operatorname{depth}(R) + n. \tag{4.26.1}$$

Set $s = \dim(R) - \operatorname{depth}(R) + n$.

Since R is complete, the Cohen Structure Theorem provides a complete regular local ring $(A, \mathfrak{M}, k)$ and an epimorphism $A \to R$ such that $\mathfrak{M}$ is generated by a sequence $\underline{a} = a_1, \ldots, a_n \in \mathfrak{M}$ where a_i is a lifting of t_i to A. The Koszul complex $K^A(\underline{a})$ is a minimal A-free resolution of k. The lifting assumption for $\underline{a}$ implies that $K \cong K^A(\underline{a}) \otimes_A R$. The fact that A is regular and local implies that $q := \operatorname{pd}_A(R) < \infty$.

From [8, Proposition 2.2.8] we know that there is a "DG algebra resolution" B of R over A such that $B_i = 0$ for all $i > q$. This means that B is a DG A-algebra with a quasiisomorphism of DG A-algebras $B \xrightarrow{\simeq} R$ such that each B_i is finitely generated and free over A and $B_i = 0$ for all $i > q$. Since $K^A(\underline{a})$ and B are DG A-algebras that are bounded below and consist of flat A-modules, we have the following (quasi)isomorphisms of DG A-algebras:

$$K \cong K^A(\underline{a}) \otimes_A R \xleftarrow{\simeq} K^A(\underline{a}) \otimes_A B \xrightarrow{\simeq} k \otimes_A B =: U. \tag{4.26.2}$$

Note that the assumptions on B imply that U is a finite dimensional DG k-algebra, as in Notation 4.13 with $F = k$. With the second paragraph of this proof, Lemma 2.28(c) says that base change yields bijections

$$\mathfrak{S}(R) \xrightarrow{\cong} \mathfrak{S}(K) \cong \mathfrak{S}(K^A(\underline{a}) \otimes_A R) \xleftarrow{\cong} \mathfrak{S}(K^A(\underline{a}) \otimes_A B) \xrightarrow{\cong} \mathfrak{S}(U).$$

Thus, it suffices to show that $\mathfrak{S}(U)$ is finite. Note that each algebra in (4.26.2) is a local DG A-algebra, as is R.

Let C' be a semidualizing DG U-module, and let C be a semidualizing R-complex corresponding to C' under the bijections given above. Assume without loss of generality that C is not shift-isomorphic in $\mathcal{D}(R)$ to R. (Removing this from consideration only removes a single semidualizing DG U-module, so does not affect the discussion of the finiteness of $\mathfrak{S}(U)$.) Since R is local, it follows from [21, (8.1) Theorem] that $\mathrm{pd}_R(C) = \infty$.

From Lemma 2.28(b) and the display (4.26.1), we have

$$\mathrm{amp}(C') = \mathrm{amp}(K \otimes_R C) \leqslant s.$$

By applying an appropriate shift we assume without loss of generality that $\inf(C) = 0 = \inf(C')$, so we have $\sup(C') \leqslant s$. Let $L \xrightarrow{\simeq} C'$ be a minimal semi-free resolution of C' over U. The conditions $\sup(L) = \sup(C') \leqslant s$ imply that L (and hence C') is quasiisomorphic to the truncation $\widetilde{L} := \tau(L)_{\leqslant s}$. We set $W := \widetilde{L}^{\natural}$ and work in the setting of Notations 4.13 and 4.17.

We claim that $\beta_p^R(C) \leqslant \mu_R^{p+\operatorname{depth} R}(R)$ for all $p \geqslant 0$. To see this, first note that the isomorphism $\mathbf{R}\mathrm{Hom}_R(C, C) \simeq R$ implies the following equality of power series $I_R^R(t) = P_R^C(t) I_C^R(t)$. See [14, (1.5.3)]. We conclude that for each m we have

$$\mu_R^m(R) = \sum_{t=0}^{m} \beta_t^R(C) \mu_R^{m-t}(C).$$

In particular, for $m < \operatorname{depth}(R)$, we have

$$0 = \mu_R^m(R) = \sum_{t=0}^{m} \beta_t^R(C) \mu_R^{m-t}(C) \geqslant \beta_0^R(C) \mu_R^m(C).$$

The equality $\inf(C) = 0$ implies that $\beta_0^R(C) \neq 0$ by [21, (1.7.1)], so it follows that $\mu_R^m(C) = 0$. For $m = \operatorname{depth}(R)$, we conclude from this that

$$0 \neq \mu_R^{\operatorname{depth}(R)}(R) = \sum_{t=0}^{\operatorname{depth}(R)} \beta_t^R(C)\mu_R^{\operatorname{depth}(R)-t}(C) = \beta_0^R(C)\mu_R^{\operatorname{depth}(R)}(C)$$

and hence $\mu_R^{\operatorname{depth}(R)}(C) \neq 0$. Similarly, for $m = p + \operatorname{depth}(R)$, we have

$$\begin{aligned}\mu_R^{p+\operatorname{depth}(R)}(R) &= \sum_{t=0}^{p+\operatorname{depth}(R)} \beta_t^R(C)\mu_R^{p+\operatorname{depth}(R)-t}(C) \\ &\geqslant \beta_p^R(C)\mu_R^{\operatorname{depth}(R)}(C) \\ &\geqslant \beta_p^R(C)\end{aligned}$$

as claimed.

Next, we claim that there is an integer $\lambda \geqslant 0$, depending only on R and U, such that $\sum_{i=0}^{s} r_i \leqslant \lambda$. (Recall that r_i and other quantities are fixed in Notation 4.13.) To see this, first note that for $i = 1, \ldots, s$ we have $L_i = \bigoplus_{j=0}^{i} U_j^{\beta_{i-j}^U(C')}$; see Fact 2.23. From [1, p. 44, Proposition] and the previous claim, we conclude that

$$\beta_j^U(C') = \beta_j^R(C) \leqslant \mu_R^{j+\operatorname{depth}(R)}(R)$$

for all j. It follows that

$$r_i \leqslant \operatorname{rank}_F(L_i) = \sum_{j=0}^{i} n_{i-j}\beta_j^U(C') = \sum_{j=0}^{i} n_{i-j}\beta_j^R(C) \leqslant \sum_{j=0}^{i} n_{i-j}\mu_R^{j+\operatorname{depth}(R)}(R).$$

And we conclude that

$$\sum_{i=0}^{s} r_i \leqslant \sum_{i=0}^{s}\sum_{j=0}^{i} n_{i-j}\mu_R^{j+\operatorname{depth}(R)}(R).$$

Since the numbers in the right hand side of this inequality only depend on R and U, we have found the desired value for λ.

Because there are only finitely many $(r_0, \ldots, r_s) \in \mathbb{N}^{s+1}$ with $\sum_{i=0}^{s} r_i \leqslant \lambda$, there are only finitely many W that occur from this construction, say $W^{(1)}, \ldots, W^{(b)}$. Lemma 4.25 implies that $\mathfrak{S}(U) = \mathfrak{S}_{W^{(1)}}(U) \cup \cdots \cup \mathfrak{S}_{W^{(b)}}(U) \cup \{[U]\}$ is finite. $\square$

4.27 (Proof of Theorem 1.4)**.** The set $\mathfrak{S}_0(R) \subseteq \mathfrak{S}(R)$ is finite by Theorem 4.26. $\square$

Now, we prove versions of Theorems 1.4 and 4.26 for semilocal rings. We note that, over a non-local ring, the set $\mathfrak{S}(R)$ may not be finite. For instance, the *Picard group* $\mathrm{Pic}(R)$, consisting of finitely generated rank-1 projective R-modules, is contained in $\mathfrak{S}_0(R) \subseteq \mathfrak{S}(R)$, so $\mathfrak{S}(R)$ can even be infinite when R is a Dedekind domain. We use some notions from [33] to deal with this.

Definition 4.28. A *tilting R-complex* is a semidualizing R-complex of finite projective dimension. The *derived Picard group of R* is the set $\mathrm{DPic}(R)$ of isomoprhism classes in $\mathcal{D}(R)$ of tilting R-complexes. The isomorphism class of a tilting R-complex L is denoted $[L] \in \mathrm{DPic}(R)$.

Remark 4.29. A homologically finite R-complex L is tilting if and only if $L_\mathfrak{m} \sim R_\mathfrak{m}$ for all maximal (equivalently, for all prime) ideals $\mathfrak{m} \subset R$, by [33, Proposition 4.4 and Remark 4.7]. In [18] tilting complexes are called "invertible".

The derived Picard group $\mathrm{DPic}(R)$ is an abelian group under the operation $[L][L'] := [L \otimes^{\mathbf{L}}_R L']$. The identity in $\mathrm{DPic}(R)$ is $[R]$, and $[L]^{-1} = [\mathbf{R}\mathrm{Hom}_R(L, R)]$. The classical Picard group $\mathrm{Pic}(R)$ is naturally a subgroup of $\mathrm{DPic}(R)$. The group $\mathrm{DPic}(R)$ acts on $\mathfrak{S}(R)$ in a natural way: $[L][C] := [L \otimes^{\mathbf{L}}_R C]$. See [33, Properties 4.3 and Remark 4.9]. This action restricts to an action of $\mathrm{Pic}(R)$ on $\mathfrak{S}_0(R)$ given by $[L][C] := [L \otimes_R C]$.

Notation 4.30. The set of orbits in $\mathfrak{S}(R)$ under the action of $\mathrm{DPic}(R)$ is denoted $\overline{\mathfrak{S}}(R)$.[9] Given $[C] \in \mathfrak{S}(R)$, the orbit in $\overline{\mathfrak{S}}(R)$ is denoted $\langle C \rangle$. The set of

[9] Observe that the notations $\mathfrak{S}(R)$ and $\overline{\mathfrak{S}}(R)$ represent different sets in [33].

orbits in $\mathfrak{S}_0(R)$ under the action of $\operatorname{Pic}(R)$ is denoted $\overline{\mathfrak{S}_0}(R)$, and the orbit in $\overline{\mathfrak{S}_0}(R)$ of a given semidualizing R-module C is denoted $\langle C \rangle$.

Fact 4.31. Given semidualizing R-complexes A and B, the following conditions are equivalent by [33, Proposition 5.1]:

(i) there is an element $[P] \in \operatorname{DPic}(R)$ such that $B \simeq P \otimes^{\mathbf{L}}_R A$; and

(ii) $A_{\mathfrak{m}} \sim B_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m} \subset R$ and $\operatorname{Ext}^i_R(A, M) = 0$ for $i \gg 0$.

It is straightforward to show that the natural inclusion $\mathfrak{S}_0(R) \subseteq \mathfrak{S}(R)$ gives an inclusion $\overline{\mathfrak{S}_0}(R) \subseteq \overline{\mathfrak{S}}(R)$.

Lemma 4.32. *Assume that R is Cohen-Macaulay (not necessarily local), and let C be a semidualizing R-complex. There is an element $[L] \in \operatorname{DPic}(R)$ such that $L \otimes_R C$ is isomorphic in $\mathcal{D}(R)$ to a module.*

Proof. For each $\mathfrak{p} \in \operatorname{Spec}(R)$, since R is Cohen-Macaulay by [21, (3.4) Corollary] we have $\operatorname{amp}(C_{\mathfrak{p}}) = 0$, that is, $C_{\mathfrak{p}} \sim \operatorname{H}_i(C)_{\mathfrak{p}} \neq 0$ for some i. As $\operatorname{amp}(C) < \infty$, this implies that $\operatorname{Spec}(R)$ is the disjoint union

$$\operatorname{Spec}(R) = \bigcup_{i=\inf(C)}^{\sup(C)} \operatorname{Supp}_R(\operatorname{H}_i(C)).$$

It follows that each set $\operatorname{Supp}_R(\operatorname{H}_i(C))$ is both open and closed. So, if $\operatorname{Supp}_R(\operatorname{H}_i(C))$ is non-empty, then it is a union of connected components of $\operatorname{Spec}(R)$.

Let $e_1, \ldots, e_p$ be a "complete set of orthogonal primitive idempotents of R" as in [18, 4.8]. Then $R \cong R_{e_1} \times \cdots \times R_{e_p}$ and each $\operatorname{Spec}(R_{e_i})$ is naturally homeomorphic to a connected component of $\operatorname{Spec}(R)$. From the previous paragraph, for $i = 1, \ldots, p$ we have $C_{e_i} \simeq \mathsf{\Sigma}^{u_i} \operatorname{H}_{u_i}(C_{e_i})$, and $\operatorname{H}_{u_i}(C_{e_i})$ is a semidualizing R_{e_i}-module. Each R-module M has a natural decomposition $M \cong \oplus_{i=1}^p M_{e_i}$ that is compatible with the product decomposition of R, and it follows that $C \simeq \oplus_{i=1}^p \mathsf{\Sigma}^{u_i} \operatorname{H}_{u_i}(C_{e_i})$.

Let $L = \oplus_{i=1}^{p} \Sigma^{-u_i} R_{e_i}$. Then L is a tilting R-complex by Remark 4.29, and

$$
\begin{aligned}
L \otimes_R C &\simeq (\oplus_{i=1}^{p} \Sigma^{-u_i} R_{e_i}) \otimes_R (\oplus_{i=1}^{p} \Sigma^{u_i} \operatorname{H}_{u_i}(C_{e_i})) \\
&\simeq \oplus_{i=1}^{p} (\Sigma^{-u_i} R_{e_i}) \otimes_{R_{e_i}} (\Sigma^{u_i} \operatorname{H}_{u_i}(C_{e_i})) \\
&\simeq \oplus_{i=1}^{p} \operatorname{H}_{u_i}(C_{e_i}).
\end{aligned}
$$

Since $\oplus_{i=1}^{p} \operatorname{H}_{u_i}(C_{e_i})$ is an R-module, this establishes the lemma. $\square$

Definition 4.33. The *non-Gorenstein locus* of R is

$$
\operatorname{nGor}(R) := \{\text{maximal ideals } \mathfrak{m} \subset R \mid R_{\mathfrak{m}} \text{ is not Gorenstein}\} \subseteq \text{m-Spec}(R)
$$

where m-Spec(R) is the set of maximal ideals of R.

Remark 4.34. For "nice" rings, e.g. rings with a dualizing complex [68], the set $\operatorname{nGor}(R)$ is closed in m-Spec(R), so it is small in some sense.

Theorem 4.35. *Assume that R satisfies one of the following conditions:*

(1) *R is semilocal, or*

(2) *R is Cohen-Macaulay and* $\operatorname{nGor}(R)$ *is finite.*

Then the sets $\overline{\mathfrak{S}_0}(R)$ and $\overline{\mathfrak{S}}(R)$ are finite.

Proof. Because of the containment $\overline{\mathfrak{S}_0}(R) \subseteq \overline{\mathfrak{S}}(R)$, it suffices to show that $\overline{\mathfrak{S}}(R)$ is finite. Let $X = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\} \subseteq$ m-Spec(R), and let $f\colon \overline{\mathfrak{S}}(R) \to \prod_{i=1}^{n} \mathfrak{S}(R_{\mathfrak{m}_i})$ be given by the formula $f(\langle C \rangle) := ([C_{\mathfrak{m}_1}], \ldots, [C_{\mathfrak{m}_n}])$. This is well-defined because if $\langle B \rangle = \langle C \rangle$, then there is an element $[P] \in \operatorname{DPic}(R)$ such that $C \simeq P \otimes_R^{\mathbf{L}} B$, and the fact that $P_{\mathfrak{m}} \sim R_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m} \subset R$ implies that $B_{\mathfrak{m}_i} \sim C_{\mathfrak{m}_i}$ for $i = 1, \ldots, n$.

In each case (1)–(2) we show that there is a finite set X such that f is 1-1. Then Theorem 4.26 implies that the set $\prod_{i=1}^{n} \mathfrak{S}(R_{\mathfrak{m}_i})$ is finite, so $\overline{\mathfrak{S}}(R)$ is also finite.

(1) Assume that R is semilocal, and set $X := \text{m-Spec}(R)$. To show that f is 1-1, let $\langle B\rangle, \langle C\rangle \in \overline{\mathfrak{S}}(R)$ such that $B_{\mathfrak{m}_i} \sim C_{\mathfrak{m}_i}$ for $i = 1, \ldots, n$. We need to show that there is an element $[P] \in \text{DPic}(R)$ such that $C \simeq P \otimes^{\mathbf{L}}_R B$. According to Fact 4.31, it suffices to show that $\text{Ext}^j_R(B, C) = 0$ for $j \gg 0$. Since we know that $\text{Ext}^j_{R_{\mathfrak{m}_i}}(B_{\mathfrak{m}_i}, B_{\mathfrak{m}_i}) = 0$ for all $j \geqslant 1$, we conclude that there are integers $j_1, \ldots, j_n$ such that for $i = 1, \ldots, n$ we have $\text{Ext}^j_{R_{\mathfrak{m}_i}}(B_{\mathfrak{m}_i}, C_{\mathfrak{m}_i}) = 0$ for all $j \geqslant j_i$. Since B is homologically finite, we have

$$0 = \text{Ext}^j_{R_{\mathfrak{m}_i}}(B_{\mathfrak{m}_i}, C_{\mathfrak{m}_i}) \cong \text{Ext}^j_R(B, C)_{\mathfrak{m}_i}$$

for all $j \geqslant \max_i j_i$. Since vanishing is a local property, it follows that $\text{Ext}^j_R(B, C) = 0$ for $j \gg 0$, as desired.

(2) Now, assume that R is Cohen-Macaulay and $\text{nGor}(R)$ is finite, and set $X := \text{nGor}(R)$. To show that f is 1-1, let $\langle B\rangle, \langle C\rangle \in \overline{\mathfrak{S}}(R)$ such that $B_{\mathfrak{m}_i} \sim C_{\mathfrak{m}_i}$ for $i = 1, \ldots, n$. Lemma 4.32 provides tilting R-complexes L and M such that $L \otimes_R B$ and $M \otimes_R C$ are isomorphic in $\mathcal{D}(R)$ to modules B' and C', respectively. Thus, we have $\langle B\rangle = \langle L \otimes_R B\rangle = \langle B'\rangle$ and $\langle C\rangle = \langle M \otimes_R C\rangle = \langle C'\rangle$, so we may replace B and C with B' and C' to assume that B and C are modules.

We claim that $B_{\mathfrak{m}} \sim C_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m} \subset R$ and $\text{Ext}^i_R(B, C) = 0$ for all $i \geqslant 1$. (Then the desired conclusion follows from Fact 4.31.) Since B is a finitely generated R-module, it suffices to show that $B_{\mathfrak{m}} \sim C_{\mathfrak{m}}$ and $\text{Ext}^i_{R_{\mathfrak{m}}}(B_{\mathfrak{m}}, C_{\mathfrak{m}}) = 0$ for all $i \geqslant 1$ and for all maximal ideals $\mathfrak{m} \subset R$.

Case 1: $\mathfrak{m} \in \text{nGor}(R)$. In this case, we have $B_{\mathfrak{m}} \sim C_{\mathfrak{m}}$, by assumption. Since B and C are both modules, this implies that $B_{\mathfrak{m}} \cong C_{\mathfrak{m}}$, so the fact that $B_{\mathfrak{m}}$ is semidualizing over $R_{\mathfrak{m}}$ implies that

$$\text{Ext}^i_{R_{\mathfrak{m}}}(B_{\mathfrak{m}}, C_{\mathfrak{m}}) \cong \text{Ext}^i_{R_{\mathfrak{m}}}(B_{\mathfrak{m}}, B_{\mathfrak{m}}) = 0.$$

Case 2: $\mathfrak{m} \notin \text{nGor}(R)$. In this case, the ring $R_{\mathfrak{m}}$ is Gorenstein, so we have $B_{\mathfrak{m}} \cong R_{\mathfrak{m}} \cong C_{\mathfrak{m}}$ by [21, (8.6) Corollary], and the desired vanishing follows. $\square$

CHAPTER 5. COHEN FACTORIZATIONS: WEAK FUNCTORIALITY

In this chapter, we investigate Cohen factorizations of local ring homomorphisms from three perspectives. First, we prove a "weak functoriality" result for Cohen factorizations: certain morphisms of local ring homomorphisms induce morphisms of Cohen factorizations. Second, we use Cohen factorizations to study the properties of local ring homomorphisms (Gorenstein, Cohen-Macaulay, etc.) in certain commutative diagrams. Third, we use Cohen factorizations to investigate the structure of quasi-deformations of local rings, with an eye on the question of the behavior of CI-dimension in short exact sequences. In particular, we prove Theorems 1.7, 1.8, and 1.9 from the introduction.

5.1. Weak Functoriality of Regular/Cohen Factorizations

The main objective of this section is the proof of Theorem 1.7 from the introduction. We begin with a lemma.

Lemma 5.1. *Let* $\varphi\colon (R,\mathfrak{m}) \to (S,\mathfrak{n})$ *be a weakly regular local ring homomorphism. Then* $\operatorname{edim}(R) + \operatorname{edim}(S/\mathfrak{m}S) = \operatorname{edim}(S)$*.*

Proof. Set $e = \operatorname{edim}(R)$, and let $\mathbf{x} = x_1, \ldots, x_e \in \mathfrak{m}$ be a minimal generating sequence for $\mathfrak{m}$. Set $d = \operatorname{edim}(S/\mathfrak{m}S)$, and let $\mathbf{y} = y_1, \ldots, y_d \in \mathfrak{n}$ be such that the residue sequence $\overline{\mathbf{y}} \in \mathfrak{n}/\mathfrak{m}S$ generates $\mathfrak{n}/\mathfrak{m}S$ minimally over S and over $S/\mathfrak{m}S$. It is straightforward to show that the concatenated sequence $\varphi(\mathbf{x}), \mathbf{y} \in \mathfrak{n}$ generates $\mathfrak{n}$. It remains to show that this sequence generates $\mathfrak{n}$ minimally.

Suppose by way of contradiction that the sequence $\varphi(\mathbf{x}), \mathbf{y}$ does not generate $\mathfrak{n}$ minimally. The sequence $\varphi(\mathbf{x}), \mathbf{y}$ contains a minimal generating sequence for $\mathfrak{n}$. Thus, either one of the $\varphi(x_i)$ or one of the y_j is redundant as part of this generating sequence.

Case 1: y_j is redundant as part of this generating sequence. Assume without loss of generality that $j = 1$. In this case we have $\mathfrak{n} = (\varphi(x_1), \ldots, \varphi(x_e), y_2, \ldots, y_d)S$. Since $\mathfrak{m}S = (\varphi(\mathbf{x}))S$ it follows that the ideal $\mathfrak{n}/\mathfrak{m}S$ is generated by the residue sequence $\overline{y_2}, \ldots, \overline{y_d}$. This contradicts the minimality of the original sequence $\mathbf{y}$.

Case 2: $\varphi(x_i)$ is redundant as part of generating sequence for $\mathfrak{n}$. Assume without loss of generality that $i = 1$. It follows that the ideal $\mathfrak{n}/(\mathbf{y})S \subset S/(\mathbf{y})S$ is generated by the residue sequence $\overline{\varphi(x_2)}, \ldots, \overline{\varphi(x_e)} \in \mathfrak{n}/(\mathbf{y})S$.

The map φ is weakly regular, and the sequence $\overline{\mathbf{y}}$ minimally generates the maximal ideal of the regular local ring $S/\mathfrak{m}S$. In particular, this sequence is $S/\mathfrak{m}S$-regular, so we know that the sequence $\mathbf{y}$ is S-regular and the induced map $R \to S/(\mathbf{y})S$ is flat; see, e.g. [55, Corollary to Theorem 22.5]. Since this map is flat and local, it follows from [46, (2.3) Lemma] that the ideal $\mathfrak{m}(S/(\mathbf{y})S) = \mathfrak{n}/(\mathbf{y})S$ is minimally generated by the sequence $\overline{\varphi(x_1)}, \overline{\varphi(x_2)}, \ldots, \overline{\varphi(x_e)}$. This contradicts the conclusion of the previous paragraph. $\square$

Our next result will provide a vertical map in the proof of Theorem 1.7.

Proposition 5.2. *Consider a commutative diagram of local ring homomorphisms*

$$\begin{array}{ccc} (R,\mathfrak{m}) & \xrightarrow{\varphi} & (S,\mathfrak{n}) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ (\widetilde{R},\widetilde{\mathfrak{m}}) & \xrightarrow{\widetilde{\varphi}} & (\widetilde{S},\widetilde{\mathfrak{n}}) \end{array}$$

such that α is weakly regular, $\widetilde{S}$ is complete, β is weakly regular, and the induced map $R/\mathfrak{m} \to \widetilde{S}/\widetilde{\mathfrak{n}}$ is separable. Assume that φ has a minimal regular factorization $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} S$, and fix a minimal Cohen factorization $\widetilde{R} \xrightarrow{\dot{\widetilde{\varphi}}} S' \xrightarrow{\widetilde{\varphi}'} \widetilde{S}$ of $\widetilde{\varphi}$. Then there is a weakly regular local ring homomorphism $\alpha' \colon R' \to S'$ such that the next diagram *commutes*

$$\begin{CD} R @>{\dot{\varphi}}>> R' @>{\varphi'}>> S \\ @V{\alpha}VV @V{\alpha'}VV @VV{\beta}V \\ \widetilde{R} @>{\tilde{\dot{\varphi}}}>> S' @>{\widetilde{\varphi}'}>> \widetilde{S} \end{CD} \tag{5.2.1}$$

and such that the second square is a pushout.

Proof. Fact 2.37(b) provides a commutative diagram of local ring homomorphisms

$$\begin{CD} (R', \mathfrak{m}') @>{\varphi'}>> (S, \mathfrak{n}) \\ @V{\beta'}VV @VV{\beta}V \\ (S'', \mathfrak{n}'') @>{\rho''}>> (\widetilde{S}, \widetilde{\mathfrak{n}}) \end{CD} \tag{5.2.2}$$

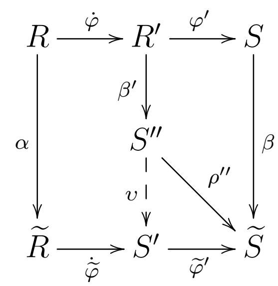
where $R' \xrightarrow{\beta'} S'' \xrightarrow{\rho''} \widetilde{S}$ is a minimal Cohen factorization for $\beta\varphi'$ and the diagram is a pushout. Fact 2.33 implies that the diagrams $R \xrightarrow{\tilde{\dot{\varphi}}\alpha} S' \xrightarrow{\widetilde{\varphi}'} \widetilde{S}$ and $R \xrightarrow{\beta'\dot{\varphi}} S'' \xrightarrow{\rho''} \widetilde{S}$ are Cohen factorizations of $\beta\varphi$. In particular, the next diagram commutes:

$$\begin{array}{ccc} R & \xrightarrow{\tilde{\dot{\varphi}}\alpha} & S' \\ {\scriptstyle \beta'\dot{\varphi}}\downarrow & \searrow^{\beta\varphi} & \downarrow{\scriptstyle \widetilde{\varphi}'} \\ S'' & \xrightarrow[\rho'']{} & \widetilde{S}. \end{array}$$

By assumption, the extension $R/\mathfrak{m} \to \widetilde{S}/\widetilde{\mathfrak{n}}$ is separable. Hence by Fact 2.37(c), there exists a comparison $\upsilon \colon S'' \to S'$ from $R \xrightarrow{\beta'\dot{\varphi}} S'' \xrightarrow{\rho''} \widetilde{S}$ to $R \xrightarrow{\tilde{\dot{\varphi}}\alpha} S' \xrightarrow{\widetilde{\varphi}'} \widetilde{S}$. In particular, the following diagram commutes:

$$\begin{array}{ccccc} & & S'' & & \\ & {\scriptstyle \beta'\dot{\varphi}}\nearrow & \downarrow{\scriptstyle \upsilon} & \searrow{\scriptstyle \rho''} & \\ R & \xrightarrow{\tilde{\dot{\varphi}}\alpha} & S' & \xrightarrow{\widetilde{\varphi}'} & \widetilde{S}. \end{array}$$

Combining this with (5.2.2), we obtain the next commutative diagram:



(5.2.3)

We claim that υ is an isomorphism. Once this is shown, it follows from the commutativity of (5.2.3) that the map $\alpha' = \upsilon\beta'$ makes the diagram (5.2.1) commute. The fact that the diagram (5.2.2) is a pushout then implies that the second square in the diagram (5.2.1) is also a pushout. Lastly, the fact that β' is weakly regular implies that α' is also weakly regular, so the desired conclusions hold.

To prove that υ is an isomorphism it is enough to show that both Cohen factorizations $R \xrightarrow{\beta'\dot{\varphi}} S'' \xrightarrow{\rho''} \widetilde{S}$ and $R \xrightarrow{\dot{\widetilde{\varphi}}\alpha} S' \xrightarrow{\widetilde{\varphi}'} \widetilde{S}$ are minimal; see Fact 2.37(c). Therefore by definition we need to show the following equalities:

$$\dim(S'') - \dim(R) \overset{(*)}{=} \operatorname{edim}(\widetilde{S}/\mathfrak{m}\widetilde{S}) \overset{(\dagger)}{=} \dim(S') - \dim(R).$$

We begin with equality $(*)$. The fact that $R' \xrightarrow{\beta'} S'' \xrightarrow{\rho''} \widetilde{S}$ is a minimal Cohen factorization explains the first step in the next sequence:

$$\dim(S'') - \dim(R') = \operatorname{edim}(\widetilde{S}/\mathfrak{m}'\widetilde{S}) = \operatorname{edim}(\widetilde{S}/\mathfrak{n}\widetilde{S}).$$

The second equality is from the fact that the map $R' \xrightarrow{\varphi'} S$ is surjective. Therefore, since $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} S$ is a minimal regular factorization, we conclude that

$$\begin{aligned}\dim(S'') - \dim(R) &= (\dim(S'') - \dim(R')) + (\dim(R') - \dim(R)) \\ &= \operatorname{edim}(\widetilde{S}/\mathfrak{n}\widetilde{S}) + \operatorname{edim}(S/\mathfrak{m}S).\end{aligned}$$

Now, the induced map $\overline{\beta}\colon S/\mathfrak{m}S \to \widetilde{S}/\mathfrak{m}\widetilde{S}$ is a flat local homomorphism. Furthermore, the closed fiber of this map is the same as the closed fiber for β, hence it is regular. Using Lemma 5.1, we conclude that

$$\operatorname{edim}(\widetilde{S}/\mathfrak{m}\widetilde{S}) = \operatorname{edim}(\widetilde{S}/\mathfrak{n}\widetilde{S}) + \operatorname{edim}(S/\mathfrak{m}S).$$

The equality $(*)$ follows from this with the previous display.

Next, we explain the equality (†). By assumption $\widetilde{R} \xrightarrow{\dot{\widetilde{\varphi}}} S' \xrightarrow{\widetilde{\varphi}'} \widetilde{S}$ is a minimal Cohen factorization and $\alpha\colon R \to \widetilde{R}$ is a flat local homomorphism. This explains the second step in the next display:

$$\begin{aligned}
\dim(S') - \dim(R) &= (\dim(S') - \dim(\widetilde{R})) + (\dim(\widetilde{R}) - \dim(R)) \\
&= \operatorname{edim}(\widetilde{S}/\widetilde{\mathfrak{m}}\widetilde{S}) + \dim(\widetilde{R}/\mathfrak{m}\widetilde{R}) \\
&= \operatorname{edim}(\widetilde{S}/\widetilde{\mathfrak{m}}\widetilde{S}) + \operatorname{edim}(\widetilde{R}/\mathfrak{m}\widetilde{R}) \\
&= \operatorname{edim}(\widetilde{S}/\mathfrak{m}\widetilde{S}).
\end{aligned}$$

The third step comes from the fact that α is weakly regular. The fourth step is from Fact 2.33 since the maps $R \xrightarrow{\alpha} \widetilde{R} \xrightarrow{\widetilde{\varphi}} \widetilde{S}$ are weakly regular. This display explains (†), and the proof is complete. □

Our next result complements the previous one by providing a horizontal map in the proof of Theorem 1.7.

Proposition 5.3. *Consider a commutative diagram of local ring homomorphisms*

$$\begin{array}{ccc}
(R,\mathfrak{m}) & \xrightarrow{\varphi} & (S,\mathfrak{n}) \\
{\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\beta} \\
(\widetilde{R},\widetilde{\mathfrak{m}}) & \xrightarrow{\widetilde{\varphi}} & (\widetilde{S},\widetilde{\mathfrak{n}})
\end{array} \tag{5.3.1}$$

such that α has a regular factorization $R \xrightarrow{\dot{\alpha}} R'' \xrightarrow{\alpha'} \widetilde{R}$, the ring $\widetilde{S}$ is complete, and the field extension $R/\mathfrak{m} \to \widetilde{S}/\widetilde{\mathfrak{n}}$ is separable. Let $S \xrightarrow{\dot{\beta}} S' \xrightarrow{\beta'} \widetilde{S}$ be a Cohen factorization of β. Then there is a commutative diagram of local ring homomorphisms

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ {\scriptstyle\dot{\alpha}}\downarrow & & \downarrow{\scriptstyle\dot{\beta}} \\ R'' & \xrightarrow{\sigma} & S' \\ {\scriptstyle\alpha'}\downarrow & & \downarrow{\scriptstyle\beta'} \\ \widetilde{R} & \xrightarrow{\dot{\varphi}} & \widetilde{S}. \end{array} \tag{5.3.2}$$

Proof. We prove the result in two cases.

Case 1: the ring S is complete. Let $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} S$ and $\widetilde{R} \xrightarrow{\dot{\widetilde{\varphi}}} \widetilde{R}' \xrightarrow{\widetilde{\varphi}'} \widetilde{S}$ be Cohen factorizations of φ and $\widetilde{\varphi}$, respectively. Since $\widetilde{R}'$ and S' are complete, the maps $R'' \xrightarrow{\dot{\widetilde{\varphi}}\alpha'} \widetilde{R}'$ and $R' \xrightarrow{\dot{\beta}\varphi'} S'$ have Cohen factorizations $R'' \xrightarrow{\dot{\pi}} T \xrightarrow{\pi'} \widetilde{R}'$ and $R' \xrightarrow{\dot{\tau}} T' \xrightarrow{\tau'} S'$, respectively. Since diagram (5.3.1) commutes, one sees readily that $\widetilde{\varphi}'\pi'\dot{\pi}\dot{\alpha} = \beta'\tau'\dot{\tau}\dot{\varphi}$. In other words, the diagrams

$$R \xrightarrow{\dot{\pi}\dot{\alpha}} T \xrightarrow{\widetilde{\varphi}'\pi'} \widetilde{S} \qquad\qquad R \xrightarrow{\dot{\tau}\dot{\varphi}} T' \xrightarrow{\beta'\tau'} \widetilde{S} \tag{5.3.3}$$

are Cohen factorizations of the same map; see Fact 2.33.

By assumption, the induced field extension $R/\mathfrak{m} \to \widetilde{S}/\widetilde{\mathfrak{n}}$ is separable. Thus, from Fact 2.37(c) we conclude that there is a comparison $\upsilon\colon T \to T'$ between the factorizations (5.3.3). It follows that the next diagram commutes

$$\begin{array}{ccccccc} R & & \xrightarrow{\dot{\varphi}} & & R' & \xrightarrow{\varphi'} & S \\ {\scriptstyle\dot{\alpha}}\downarrow & & & & \downarrow{\scriptstyle\dot{\tau}} & & \downarrow{\scriptstyle\dot{\beta}} \\ R'' & \xrightarrow{\dot{\pi}} & T & \xrightarrow{\upsilon} & T' & \xrightarrow{\tau'} & S' \\ {\scriptstyle\alpha'}\downarrow & & \downarrow{\scriptstyle\pi'} & & & & \downarrow{\scriptstyle\beta'} \\ \widetilde{R} & \xrightarrow{\dot{\widetilde{\varphi}}} & \widetilde{R}' & & \xrightarrow{\widetilde{\varphi}'} & & \widetilde{S}. \end{array}$$

So the map $\sigma = \tau' \upsilon \dot{\pi} \colon R'' \to S'$ makes diagram (5.3.2) commute.

Case 2: the general case. Let $\widehat{(-)}^{\mathfrak{n}}$ denote the $\mathfrak{n}$-adic completion functor, and consider the following commutative diagram of local ring homomorphisms

$$\begin{array}{ccc} S & \longrightarrow & \widehat{S}^{\mathfrak{n}} \\ {\scriptstyle \dot{\beta}}\downarrow & & \downarrow{\scriptstyle \widehat{\beta}^{\mathfrak{n}}} \\ S' & \xrightarrow{\cong} & \widehat{S'}^{\mathfrak{n}} \end{array}$$

where the horizontal maps are the natural ones. The map $S' \to \widehat{S'}^{\mathfrak{n}}$ is an isomorphism since S' is complete and $\dot{\beta}$ is local. Accordingly, we identify S' and $\widehat{S'}^{\mathfrak{n}}$, and similarly for $\widetilde{S}$ and $\widehat{\widetilde{S}}^{\mathfrak{n}}$. From [16, (1.9) Remark] we know that the diagram $\widehat{S}^{\mathfrak{n}} \xrightarrow{\widehat{\dot{\beta}}^{\mathfrak{n}}} S' \xrightarrow{\beta'} \widetilde{S}$ is a Cohen factorization of the map $\widehat{S}^{\mathfrak{n}} \xrightarrow{\widehat{\beta}^{\mathfrak{n}}} \widetilde{S}$. The previous diagram provides the second commutative triangle in the next diagram, and the other triangle commutes by definition of $\dot{\varphi}$:

$$\begin{array}{ccccc} & & & & S \\ & & {\scriptstyle \varphi}\nearrow & \swarrow & \\ R & \xrightarrow{\dot{\varphi}} & \widehat{S}^{\mathfrak{n}} & & \downarrow{\scriptstyle \dot{\beta}} \\ {\scriptstyle \dot{\alpha}}\downarrow & & \downarrow{\scriptstyle \widehat{\dot{\beta}}^{\mathfrak{n}}} & \swarrow & \\ R'' & \overset{\sigma}{\dashrightarrow} & S' & & \\ {\scriptstyle \alpha'}\downarrow & & \downarrow{\scriptstyle \beta'} & & \\ \widetilde{R} & \xrightarrow{\dot{\varphi}} & \widetilde{S}. & & \end{array}$$

Case 1 provides a local ring homomorphism σ making the two squares commute. It follows readily that σ also makes the diagram (5.3.2) commute. □

5.4 (Proof of Theorem 1.7)**.** By Proposition 5.3 there is a local ring homomorphism $\sigma \colon R'' \to S'$ making the next diagram commute:

$$\begin{array}{ccccc} R & \xrightarrow{\dot{\varphi}} & R' & \xrightarrow{\varphi'} & S \\ {\scriptstyle \dot{\alpha}}\downarrow & & & & \downarrow{\scriptstyle \dot{\beta}} \\ R'' & & \xrightarrow{\qquad\sigma\qquad} & & S' \\ {\scriptstyle \alpha'}\downarrow & & & & \downarrow{\scriptstyle \beta'} \\ \widetilde{R} & \xrightarrow{\dot{\widetilde{\varphi}}} & \widetilde{R}' & \xrightarrow{\widetilde{\varphi}'} & \widetilde{S}. \end{array} \qquad (5.4.1)$$

Since S' is complete, the map σ has a Cohen factorization $R'' \xrightarrow{\dot{\sigma}} T' \xrightarrow{\sigma'} S'$. Proposition 5.2 provides a weakly regular local ring homomorphism $\delta\colon R' \to T'$ making the next diagram commute, where $R'' \xrightarrow{\dot{\pi}} T'' \xrightarrow{\pi'} \widetilde{R}'$ is a Cohen factorization of $R'' \xrightarrow{\dot{\widetilde{\varphi}}\alpha'} \widetilde{R}'$:

$$
\begin{array}{ccccc}
R & \xrightarrow{\quad\dot{\varphi}\quad} & R' & \xrightarrow{\varphi'} & S \\
{\scriptstyle\dot{\alpha}}\downarrow & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\dot{\beta}} \\
R'' & \xrightarrow{\quad\dot{\sigma}\quad} & T' & \xrightarrow{\sigma'} & S' \\
 & {\scriptstyle\dot{\pi}}\searrow & & & \\
{\scriptstyle\alpha'}\downarrow & & T'' & & \downarrow{\scriptstyle\beta'} \\
 & & & \searrow{\scriptstyle\pi'} & \\
\widetilde{R} & \xrightarrow{\quad\dot{\widetilde{\varphi}}\quad} & \widetilde{R}' & \xrightarrow{\widetilde{\varphi}'} & \widetilde{S}.
\end{array}
$$

Fact 2.33 implies that the diagrams $R'' \xrightarrow{\dot{\pi}} T'' \xrightarrow{\widetilde{\varphi}'\pi'} \widetilde{S}$ and $R'' \xrightarrow{\dot{\sigma}} T' \xrightarrow{\beta'\sigma'} \widetilde{S}$ are two Cohen factorizations of the map $\widetilde{\varphi}\alpha' = \beta'\sigma$. By assumption (2), the induced field extension $R/\mathfrak{m} \to \widetilde{S}/\widetilde{\mathfrak{n}}$ is separable, so there is a comparison $v\colon T' \to T''$ of these Cohen factorizations, by Fact 2.37(c). This is a local ring homomorphism making the next diagram commute:

$$
\begin{array}{ccccc}
R & \xrightarrow{\dot{\varphi}} & R' & \xrightarrow{\varphi'} & S \\
{\scriptstyle\dot{\alpha}}\downarrow & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\dot{\beta}} \\
R'' & \xrightarrow{\dot{\sigma}} & T' & \xrightarrow{\sigma'} & S' \\
{\scriptstyle\alpha'}\downarrow & & \downarrow{\scriptstyle\pi' v} & & \downarrow{\scriptstyle\beta'} \\
\widetilde{R} & \xrightarrow{\dot{\widetilde{\varphi}}} & \widetilde{R}' & \xrightarrow{\widetilde{\varphi}'} & \widetilde{S}.
\end{array}
\tag{5.4.2}
$$

At this point, the only problem is that $\pi' v$ may not be surjective. We use a technique from [16, 42] to remedy this.

Since $\widetilde{R}'$ and S' are local and complete and the maps $\widetilde{\varphi}'$ and β' are surjective, in the pull-back square

$$
\begin{array}{ccc}
\widetilde{T} & \xrightarrow{f} & S' \\
{\scriptstyle g}\downarrow & \lrcorner & \downarrow{\scriptstyle\beta'} \\
\widetilde{R}' & \xrightarrow{\widetilde{\varphi}'} & \widetilde{S}
\end{array}
$$

the ring $\widetilde{T}$ is noetherian, local, and complete by [42, (19.3.2.1)]. The commutativity of the diagram (5.4.2) implies that there is a local ring homomorphism $w\colon T' \to \widetilde{T}$ making the next diagram commute:

$$\xymatrix{
R \ar[r]^{\dot\varphi} \ar[d]_{\dot\alpha} & R' \ar[rr]^{\varphi'} \ar[d]^{\delta} & & S \ar[d]^{\dot\beta} \\
R'' \ar[r]^{\dot\sigma} \ar[dd]_{\alpha'} & T' \ar[rr]^{\sigma'} \ar[dr]^{w} \ar[dd]_{\pi' v} & & S' \ar[dd]^{\beta'} \\
& & \widetilde{T} \ar[ur]_{f} \ar[dl]^{g} & \\
\widetilde{R} \ar[r]^{\dot{\widetilde{\varphi}}} & \widetilde{R}' \ar[rr]^{\widetilde{\varphi}'} & & \widetilde{S}.
} \tag{5.4.3}$$

The ring $\widetilde{T}$ is complete, so the map w has a Cohen factorization $T' \xrightarrow{\dot{w}} T \xrightarrow{w'} \widetilde{T}$. Consider the following diagram

$$\xymatrix{
R \ar[r]^{\dot\varphi} \ar[d]_{\dot\alpha} & R' \ar[r]^{\varphi'} \ar[d]^{\dot{w}\delta} & S \ar[d]^{\dot\beta} \\
R'' \ar[r]^{\dot{w}\dot\sigma} \ar[d]_{\alpha'} & T \ar[r]^{fw'} \ar[d]^{gw'} & S' \ar[d]^{\beta'} \\
\widetilde{R} \ar[r]^{\dot{\widetilde{\varphi}}} & \widetilde{R}' \ar[r]^{\widetilde{\varphi}'} & \widetilde{S}.
}$$

This diagram has the desired properties for (1.7.2). Indeed, commutativity follows from the commutativity of (5.4.3), using the equation $w = w'\dot{w}$, and the middle column and middle row are Cohen factorizations by Fact 2.33. $\square$

The following example, from [16, (1.8) Example] shows that the separability assumptions (2) in Theorem 1.7 are necessary.

Example 5.5. In [16, (1.8) Example], the authors construct a local ring homomorphism $\varphi\colon R \to S$ with a pair of Cohen factorizations

$$
\begin{array}{ccccc}
 & & R_1 & & \\
 & \overset{\dot\varphi_1}{\nearrow} & \vdots\ \nexists & \overset{\varphi_1'}{\searrow} & \\
R & & \xrightarrow{\qquad\varphi\qquad} & & S \\
 & \underset{\dot\varphi_2}{\searrow} & \vdots & \underset{\varphi_2'}{\nearrow} & \\
 & & R_2 & &
\end{array}
\tag{5.5.1}
$$

such that there is not a comparison from the top one to the bottom one. (Note that, by Fact 2.37(c), the map on residue fields induced by φ cannot be separable.) We begin with the following diagram, as in (1.7.1)

$$
\begin{array}{ccc}
R & \xrightarrow{\varphi} & S \\
{\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
R & \xrightarrow{\varphi} & S
\end{array}
$$

with the given Cohen factorizations represented by the horizontal solid arrows in the next diagram.

$$
\begin{array}{ccccc}
R & \xrightarrow{\dot\varphi_1} & R_1 & \xrightarrow{\varphi_1'} & S \\
{\scriptstyle \mathrm{id}}\downarrow & & \vdots\,{\scriptstyle \dot\gamma} & & \downarrow{\scriptstyle \mathrm{id}} \\
R & \overset{\dot\sigma}{\cdots\cdots>} & T & \overset{\sigma'}{\cdots\cdots>} & S \\
{\scriptstyle \mathrm{id}}\downarrow & & \vdots\,{\scriptstyle \gamma'} & & \downarrow{\scriptstyle \mathrm{id}} \\
R & \xrightarrow{\dot\varphi_2} & R_2 & \xrightarrow{\varphi_2'} & S
\end{array}
$$

If there were Cohen factorizations (as represented by the dotted arrows in the diagram) then the map $\gamma'\dot\gamma$ would provide a comparison of the Cohen factorizations from (5.5.1), contradicting the conclusion of [16, (1.8) Example]. Thus, in the notation of Theorem 1.7, the field extension $\widetilde{R}/\widetilde{\mathfrak{m}} \to \widetilde{S}/\widetilde{\mathfrak{n}}$ must be separable (even when the extension $R/\mathfrak{m} \to \widetilde{R}/\widetilde{\mathfrak{m}}$ is trivial). Similarly, the reflected diagrams

$$
\begin{array}{ccc}
R & \xrightarrow{\text{id}} & R \\
{\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\
S & \xrightarrow{\text{id}} & S
\end{array}
\qquad\qquad
\begin{array}{ccccc}
R & \xrightarrow{\text{id}} & R & \xrightarrow{\text{id}} & R \\
{\scriptstyle\dot\varphi_1}\downarrow & & \vdots{\scriptstyle\dot\gamma} & & \downarrow{\scriptstyle\dot\varphi_2} \\
R_1 & \xrightarrow{\dot\sigma} & T & \xrightarrow{\sigma'} & R_2 \\
{\scriptstyle\varphi_1'}\downarrow & & \vdots{\scriptstyle\gamma'} & & \downarrow{\scriptstyle\varphi_2'} \\
S & \xrightarrow{\text{id}} & S & \xrightarrow{\text{id}} & S
\end{array}
$$

show that the field extension $R/\mathfrak{m} \to \widetilde{R}/\widetilde{\mathfrak{m}}$ must be separable (even when the extension $\widetilde{R}/\widetilde{\mathfrak{m}} \to \widetilde{S}/\widetilde{\mathfrak{n}}$ is trivial).

The next example shows that, even when the factorizations $R \xrightarrow{\dot\varphi} R' \xrightarrow{\varphi'} S$ and $\widetilde{R} \xrightarrow{\dot{\widetilde{\varphi}}} \widetilde{R}' \xrightarrow{\widetilde{\varphi}'} \widetilde{S}$ are minimal, the map $\gamma = \gamma'\dot\gamma$ in Theorem 1.7 is not uniquely determined.

Example 5.6. Let k be a field, and consider the following diagram where the unspecified maps are the natural ones:

$$
\begin{array}{ccccc}
k & \longrightarrow & k[\![X]\!] & \longrightarrow & k[\![X]\!] \\
\downarrow & & \vdots{\scriptstyle\gamma} & & \downarrow \\
k & \longrightarrow & k[\![X]\!] & \longrightarrow & k[\![X]\!]/(X^2).
\end{array}
$$

One can define γ by mapping X to any power series of the form $X + X^2 f$ to make the diagram commute.

5.2. Commutative Diagrams of Local Ring Homomorphisms

In this section, we prove Theorem 1.8 from the introduction.

Lemma 5.7. *Fix a commutative diagram of field extensions*

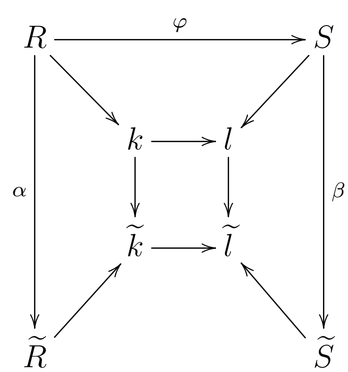
$$
\begin{array}{ccc}
k & \xrightarrow{\varphi_0} & l \\
{\scriptstyle\alpha_0}\downarrow & & \downarrow{\scriptstyle\beta_0} \\
\widetilde{k} & \xrightarrow{\widetilde{\varphi}_0} & \widetilde{l}
\end{array}
\tag{5.7.1}
$$

Let $\varphi\colon (R,\mathfrak{m},k) \to (S,\mathfrak{n},l)$ and $\alpha\colon R \to (\widetilde{R},\widetilde{\mathfrak{m}},\widetilde{k})$ and $\beta\colon S \to (\widetilde{S},\widetilde{\mathfrak{n}},\widetilde{l})$ be local ring homomorphisms that induce the maps φ_0 and α_0 and β_0 on residue fields. If α is weakly Cohen and $\widetilde{S}$ is complete, then there is a commutative diagram of local ring homomorphisms

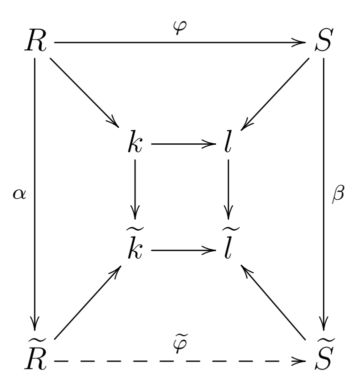
$$\begin{array}{ccc}
(R,\mathfrak{m},k) & \xrightarrow{\varphi} & (S,\mathfrak{n},l) \\
{\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\
(\widetilde{R},\widetilde{\mathfrak{m}},\widetilde{k}) & \overset{\widetilde{\varphi}}{\dashrightarrow} & (\widetilde{S},\widetilde{\mathfrak{n}},\widetilde{l})
\end{array}$$

such that $\widetilde{\varphi}$ induces the map $\widetilde{\varphi}_0$ on residue fields.

Proof. By assumption, the maps α and β make the following diagram commute



Since the extension $k \hookrightarrow \widetilde{k}$ is separable, the map α is formally smooth by [40, Théorème (19.8.2.i)], so [40, Corollaire (19.3.11)] provides a local homomorphism $\widetilde{\varphi}\colon \widetilde{R} \to \widetilde{S}$ making the next diagram commute:



It follows that the outer square in this diagram has the desired properties. $\square$

Remark 5.8. Given a diagram (5.7.1), there exist Cohen maps α and β, as in Lemma 5.7, by [40, Théorèm 19.8.2(ii)].

We use the next result to relate the flat dimensions of the maps in Theorem 1.8.

Proposition 5.9. *Consider a commutative diagram of local ring homomorphisms*

$$\begin{array}{ccc} (R,\mathfrak{m},k) & \xrightarrow{\varphi} & (S,\mathfrak{n},l) \\ \alpha\downarrow & & \downarrow\beta \\ (\widetilde{R},\widetilde{\mathfrak{m}},\widetilde{k}) & \xrightarrow{\widetilde{\varphi}} & (\widetilde{S},\widetilde{\mathfrak{n}},\widetilde{l}) \end{array} \tag{5.9.1}$$

such that $\mathrm{fd}(\alpha)$ *and* $\mathrm{fd}(\beta)$ *are both finite.*

(a) *One has* $\mathrm{fd}(\varphi) \leqslant \mathrm{fd}(\widetilde{\varphi}) + \mathrm{fd}(\alpha)$*; in particular, if* $\mathrm{fd}(\widetilde{\varphi})$ *is finite, then so is* $\mathrm{fd}(\varphi)$*.*

(b) *If* α *is weakly regular, then* $\mathrm{fd}(\widetilde{\varphi}) \leqslant \mathrm{edim}(\alpha) + \mathrm{fd}(\beta) + \mathrm{fd}(\varphi)$*; in particular,* $\mathrm{fd}(\varphi)$ *and* $\mathrm{fd}(\widetilde{\varphi})$ *are simultaneously finite.*

(c) *If* α *is flat with closed fiber a field and* β *is flat, then* $\mathrm{fd}(\widetilde{\varphi}) = \mathrm{fd}(\varphi)$*.*

Proof. (a) This is proved like [50, Theorem 5.7].

(b) Assume without loss of generality that $\mathrm{fd}(\varphi) < \infty$. Let $\mathbf{x} = x_1, \ldots, x_e \in \widetilde{\mathfrak{m}}$ be a sequence whose residue sequence in $\widetilde{R}/\mathfrak{m}\widetilde{R}$ is a regular system of parameters. This explains the first isomorphism in the derived category $\mathcal{D}(\widetilde{S})$ in the next sequence:

$$\begin{aligned} \widetilde{k} \otimes^{\mathbf{L}}_{\widetilde{R}} \widetilde{S} &\simeq [(\widetilde{R}/\mathbf{x}\widetilde{R}) \otimes^{\mathbf{L}}_{\widetilde{R}} (\widetilde{R}/\mathfrak{m}\widetilde{R})] \otimes^{\mathbf{L}}_{\widetilde{R}} \widetilde{S} \\ &\simeq (\widetilde{R}/\mathbf{x}\widetilde{R}) \otimes^{\mathbf{L}}_{\widetilde{R}} [(\widetilde{R}/\mathfrak{m}\widetilde{R}) \otimes^{\mathbf{L}}_{\widetilde{R}} \widetilde{S}] \\ &\simeq (\widetilde{R}/\mathbf{x}\widetilde{R}) \otimes^{\mathbf{L}}_{\widetilde{R}} [(k \otimes^{\mathbf{L}}_{R} \widetilde{R}) \otimes^{\mathbf{L}}_{\widetilde{R}} \widetilde{S}] \\ &\simeq (\widetilde{R}/\mathbf{x}\widetilde{R}) \otimes^{\mathbf{L}}_{\widetilde{R}} [(k \otimes^{\mathbf{L}}_{R} S) \otimes^{\mathbf{L}}_{S} \widetilde{S}]. \end{aligned}$$

The second isomorphism is associativity. The third isomorphism is from the flatness of α. The fourth isomorphism is from the commutativity of (5.9.1). This explains the second step in the next display:

$$
\begin{aligned}
\operatorname{fd}(\widetilde{\varphi}) &= \operatorname{amp}(\widetilde{k} \otimes^{\mathbf{L}}_{\widetilde{R}} \widetilde{S}) \\
&= \operatorname{amp}((\widetilde{R}/\mathbf{x}\widetilde{R}) \otimes^{\mathbf{L}}_{\widetilde{R}} [(k \otimes^{\mathbf{L}}_{R} S) \otimes^{\mathbf{L}}_{S} \widetilde{S}]) \\
&\leqslant \operatorname{pd}_{\widetilde{R}}(\widetilde{R}/\mathbf{x}\widetilde{R}) + \operatorname{amp}((k \otimes^{\mathbf{L}}_{R} S) \otimes^{\mathbf{L}}_{S} \widetilde{S}) \\
&= e + \operatorname{amp}((k \otimes^{\mathbf{L}}_{R} S) \otimes^{\mathbf{L}}_{S} \widetilde{S}) \\
&= e + \operatorname{fd}(\beta\varphi) \\
&\leqslant e + \operatorname{fd}(\beta) + \operatorname{fd}(\varphi) \\
&= \operatorname{edim}(\widetilde{R}/\mathfrak{m}\widetilde{R}) + \operatorname{fd}(\beta) + \operatorname{fd}(\varphi).
\end{aligned}
$$

The first and fifth steps are from [12, Proposition 5.5]. The third step is standard; see [29, (7.28) and (8.17)]. The fourth step is from [55, Corollary to Theorem 22.5], which tells us that $\mathbf{x}$ is $\widetilde{R}$-regular. The sixth step is by [13, (1.8(a))]. The seventh step is by definition of e as $\operatorname{edim}(\widetilde{R}/\mathfrak{m}\widetilde{R})$.

(c) Part (a) implies that $\operatorname{fd}(\widetilde{\varphi}) \geqslant \operatorname{fd}(\varphi)$, and (b) implies that $\operatorname{fd}(\widetilde{\varphi}) \leqslant \operatorname{fd}(\varphi)$. $\square$

Here are some examples that show the limitations of the previous result.

Example 5.10. Let k be a field, and consider the following commutative diagram of natural local ring homomorphisms:

$$
\begin{array}{ccc}
k & \xrightarrow{\varphi} & k[\![X]\!] \\
{\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\
k[\![X]\!] & \xrightarrow{\widetilde{\varphi}} & k[\![X]\!]/(X^2).
\end{array}
$$

Note that α is weakly regular, and we have

$$\mathrm{fd}(\widetilde{\varphi}) = 1 < 2 = 1 + 1 + 0 = \mathrm{edim}(\alpha) + \mathrm{fd}(\beta) + \mathrm{fd}(\varphi)$$

so we can have strict inequality in Proposition 5.9(b).

Example 5.11. Let k be a field, and let S be a complete local ring with coefficient field k, and assume that S is not a field. Consider the following commutative diagram of natural local ring homomorphisms:

$$\begin{array}{ccc} k & \xrightarrow{\varphi} & S \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ k & \xrightarrow{\widetilde{\varphi}} & k. \end{array}$$

Then we have $\mathrm{fd}(\widetilde{\varphi}) = 0 < 0 + \mathrm{fd}(\beta) = \mathrm{fd}(\varphi) + \mathrm{fd}(\beta)$. so we can have strict inequality in Proposition 5.9(c) if β is not flat.

Lemma 5.12. *Let* $(\widetilde{R}', \widetilde{\mathfrak{m}}', \widetilde{k}') \xleftarrow{\alpha'} (R', \mathfrak{m}', k') \xrightarrow{\rho} (R_1, \mathfrak{m}_1, k_1) \xleftarrow{\tau} (Q, \mathfrak{r}, k_1)$ *be local ring homomorphisms such that* α' *is Cohen,* ρ *is flat, and* τ *is surjective. Fix a commutative diagram*

$$\begin{array}{ccc} k' & \xrightarrow{\rho_0} & k_1 \\ {\scriptstyle\alpha'_0}\downarrow & & \downarrow{\scriptstyle\alpha_0} \\ \widetilde{k}' & \xrightarrow{\widetilde{\rho}_0} & \widetilde{k}_1 \end{array}$$

of field extensions such that ρ_0 *and* α'_0 *are the maps induced on residue fields by* ρ *and* α'*. Fix a weakly regular homomorphism* $\alpha_1 \colon R_1 \to (\widetilde{R}_1, \widetilde{\mathfrak{m}}_1, \widetilde{k}_1)$ *that induces the map* α_0 *on residue fields. If* $\widetilde{R}_1$ *is complete, then there is a commutative diagram of local ring homomorphisms*

$$\begin{array}{ccccc} R' & \xrightarrow{\rho} & R_1 & \xleftarrow{\tau} & Q \\ {\scriptstyle\alpha'}\downarrow & & \downarrow{\scriptstyle\alpha_1} & & \downarrow{\scriptstyle\delta} \\ \widetilde{R}' & \xrightarrow{\widetilde{\rho}} & \widetilde{R}_1 & \xleftarrow{\widetilde{\tau}} & \widetilde{Q} \end{array}$$

such that $\widetilde{\rho}$ is flat, δ is weakly regular, the right-hand square is a pushout, and $\widetilde{\rho}$ induces the map $\widetilde{\rho}_0$ on residue fields.

Proof. Lemma 5.7 provides a local ring homomorphism $\widetilde{\rho}\colon \widetilde{R}' \to \widetilde{R}_1$ making the left-hand square commute. Proposition 5.9(c) implies that $\widetilde{\rho}$ is flat. We get the right-hand square from Fact 2.37(b). □

The next result augments [17, (1.13) Proposition].

Proposition 5.13. *Let $\alpha'\colon (R', \mathfrak{m}', k') \to (\widetilde{R}', \widetilde{\mathfrak{m}}', \widetilde{k}')$ be a Cohen homomorphism, and let M be a homologically finite R'-complex. Then we have*

(a) $\text{CI-dim}_{\widetilde{R}'}(\widetilde{R}' \otimes^{\mathbf{L}}_{R'} M) = \text{CI-dim}_{R'}(M)$*; thus, the quantities* $\text{CI-dim}_{\widetilde{R}'}(\widetilde{R}' \otimes^{\mathbf{L}}_{R'} M)$ *and* $\text{CI-dim}_{R'}(M)$ *are simultaneously finite.*

(b) $\text{CM-dim}_{\widetilde{R}'}(\widetilde{R}' \otimes^{\mathbf{L}}_{R'} M) = \text{CM-dim}_{R'}(M)$*; in particular,* $\text{CM-dim}_{\widetilde{R}'}(\widetilde{R}' \otimes^{\mathbf{L}}_{R'} M)$ *and* $\text{CM-dim}_{R'}(M)$ *are simultaneously finite.*

Proof. (a) It suffices by Fact 2.44 to assume that $\text{CI-dim}_{R'}(M)$ is finite and prove that $\text{CI-dim}_{\widetilde{R}'}(\widetilde{R}' \otimes^{\mathbf{L}}_{\widetilde{R}'} M)$ is finite as well. Let $R' \xrightarrow{\rho} (R_1, \mathfrak{m}_1, k_1) \xleftarrow{\tau} (Q, \mathfrak{r}, k_1)$ be a quasi-deformation such that $\text{pd}_Q(R_1 \otimes^{\mathbf{L}}_{R'} M) < \infty$. Let $\widetilde{k}_1$ be a join of k_1 and $\widetilde{k}'$ over k', so there is a commutative diagram

$$\begin{array}{ccc} k' & \longrightarrow & k_1 \\ \downarrow & & \downarrow \\ \widetilde{k}' & \longrightarrow & \widetilde{k}_1 \end{array}$$

of field extensions. There is a weakly regular homomorphism $\alpha_1\colon R_1 \to (\widetilde{R}_1, \widetilde{\mathfrak{m}}_1, \widetilde{k}_1)$ that induces the map $k_1 \to \widetilde{k}_1$ on residue fields by [39, Proposition (0.10.3.1)]. Compose with the natural map from $\widetilde{R}_1$ to its completion if necessary to assume that $\widetilde{R}_1$ is complete. Thus, Lemma 5.12 provides a commutative diagram of local ring homomorphisms

$$\begin{array}{ccccc} R' & \xrightarrow{\rho} & R_1 & \xleftarrow{\tau} & Q \\ \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \delta} \\ \widetilde{R}' & \xrightarrow{\widetilde{\rho}} & \widetilde{R}_1 & \xleftarrow{\widetilde{\tau}} & \widetilde{Q} \end{array}$$

such that $\widetilde{\rho}$ is flat, δ is weakly regular, and the right-hand square is a pushout. In particular, the bottom row of this diagram is a quasi-deformation. Also, in the following sequence, the second and fourth equalities are from the flatness of δ:

$$\begin{aligned} \mathrm{pd}_{\widetilde{Q}}(\widetilde{R}_1 \otimes^{\mathbf{L}}_{\widetilde{R}'} (\widetilde{R}' \otimes^{\mathbf{L}}_{R'} M)) &= \mathrm{pd}_{\widetilde{Q}}(\widetilde{R}_1 \otimes^{\mathbf{L}}_{R'} M) \\ &= \mathrm{pd}_{\widetilde{Q}}((\widetilde{Q} \otimes^{\mathbf{L}}_{Q} R_1) \otimes^{\mathbf{L}}_{R'} M) \\ &= \mathrm{pd}_{\widetilde{Q}}(\widetilde{Q} \otimes^{\mathbf{L}}_{Q} (R_1 \otimes^{\mathbf{L}}_{R'} M)) \\ &= \mathrm{pd}_{Q}(R_1 \otimes^{\mathbf{L}}_{R'} M). \end{aligned}$$

By definition, it follows that $\text{CI-dim}_{\widetilde{R}'}(\widetilde{R}' \otimes^{\mathbf{L}}_{\widetilde{R}'} M)$ is finite, as desired.

(b) This is proved as above, starting with a G-quasi-deformation $(R', \mathfrak{m}', k') \xrightarrow{\rho} (R_1, \mathfrak{m}_1, k_1) \xleftarrow{\tau} (Q, \mathfrak{r}, k_1)$, and observing that the diagram $\widetilde{R}' \xrightarrow{\widetilde{\rho}} \widetilde{R}_1 \xleftarrow{\widetilde{\tau}} \widetilde{Q}$ is also a a G-quasi-deformation. This follows from the fact that the flat base change of a G-perfect ideal is G-perfect, which is readily checked. □

The next result compares to Proposition 5.9. It is important for our proof of Theorem 1.8 from the introduction.

Proposition 5.14. *Consider a commutative diagram of local ring homomorphisms*

$$\begin{array}{ccc} (R, \mathfrak{m}) & \xrightarrow{\varphi} & (S, \mathfrak{n}) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ (\widetilde{R}, \widetilde{\mathfrak{m}}) & \xrightarrow{\widetilde{\varphi}} & (\widetilde{S}, \widetilde{\mathfrak{n}}) \end{array}$$

such that α is weakly Cohen, β is weakly regular, and the induced map $\widetilde{R}/\widetilde{\mathfrak{m}} \to \widetilde{S}/\widetilde{\mathfrak{n}}$

is separable. Then we have

(a) $\operatorname{G-dim}(\widetilde{\varphi}) = \operatorname{G-dim}(\varphi) + \operatorname{edim}(\alpha) - \operatorname{edim}(\beta)$. *Hence, the quantities* $\operatorname{G-dim}(\varphi)$ *and* $\operatorname{G-dim}(\widetilde{\varphi})$ *are simultaneously finite.*

(b) *If β is Cohen, then* $\operatorname{CI-dim}(\widetilde{\varphi}) = \operatorname{CI-dim}(\varphi) + \operatorname{edim}(\alpha)$, *so the* $\operatorname{CI-dim}(\varphi)$ *and* $\operatorname{CI-dim}(\widetilde{\varphi})$ *are simultaneously finite.*

(c) *If β is Cohen, then* $\operatorname{CM-dim}(\widetilde{\varphi}) = \operatorname{CM-dim}(\varphi) + \operatorname{edim}(\alpha)$, *so the quantities* $\operatorname{CM-dim}(\varphi)$ *and* $\operatorname{CM-dim}(\widetilde{\varphi})$ *are simultaneously finite.*

Proof. (b) Assume that β is Cohen.

Case 1: S and $\widetilde{S}$ are complete. Let $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} S$ be a minimal Cohen factorization of φ, and let $\widetilde{R} \xrightarrow{\dot{\widetilde{\varphi}}} S' \xrightarrow{\widetilde{\varphi}'} \widetilde{S}$ be a minimal Cohen factorization of $\widetilde{\varphi}$. Proposition 5.2 provides a weakly regular local ring homomorphism $\alpha' \colon R' \to S'$ such that the next diagram commutes

$$\begin{array}{ccccc} R & \xrightarrow{\dot{\varphi}} & R' & \xrightarrow{\varphi'} & S \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha'} & & \downarrow{\scriptstyle\beta} \\ \widetilde{R} & \xrightarrow{\dot{\widetilde{\varphi}}} & S' & \xrightarrow{\widetilde{\varphi}'} & \widetilde{S} \end{array} \tag{5.14.1}$$

and such that the second square is a pushout. The residue field extension induced by α' is the same as the one induced by β, since the maps φ' and $\widetilde{\varphi}'$ are surjective. Thus, the fact that α' is weakly regular and β is Cohen implies that α' is weakly Cohen. Furthermore, α' and β have isomorphic closed fibers, so α' is Cohen.

The pushout condition and the flatness of α' explain the first equality in the next display

$$\operatorname{CI-dim}_{S'}(\widetilde{S}) = \operatorname{CI-dim}_{S'}(S' \otimes^{\mathbf{L}}_{R'} S) = \operatorname{CI-dim}_{R'}(S)$$

and the second equality is from Proposition 5.13(a). With Fact 2.33, this explains the second equality in the next display:

$$\begin{aligned}
\text{CI-dim}(\widetilde{\varphi}) &= \text{CI-dim}_{S'}(\widetilde{S}) - \text{edim}(\dot{\widetilde{\varphi}}) \\
&= \text{CI-dim}_{R'}(S) - [\text{edim}(\dot{\varphi}) + \text{edim}(\alpha') - \text{edim}(\alpha)] \\
&= [\text{CI-dim}_{R'}(S) - \text{edim}(\dot{\varphi})] - \text{edim}(\beta) + \text{edim}(\alpha) \\
&= \text{CI-dim}(\varphi) + \text{edim}(\alpha).
\end{aligned}$$

The first and last equalities are by definition, since β is Cohen; the third equality follows from the fact that α' and β have isomorphic closed fibers.

Case 2: the general case. There is a natural ring homomorphism $\widehat{\beta}\colon \widehat{S} \to \widehat{\widetilde{S}}$ making the following diagram commute

$$\begin{array}{ccc}
S & \longrightarrow & \widehat{S} \\
{\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \widehat{\beta}} \\
\widetilde{S} & \longrightarrow & \widehat{\widetilde{S}}
\end{array}$$

where the horizontal maps are the natural ones. Proposition 5.9(c) implies that $\widehat{\beta}$ is flat, and it is straightforward to show that its closed fiber is isomorphic to the completion of the closed fiber of β. Thus, $\widehat{\beta}$ is Cohen. With [64, 2.14.2] this explains the first and third equalities in the next display:

$$\text{CI-dim}(\widetilde{\varphi}) = \text{CI-dim}(\grave{\widetilde{\varphi}}) = \text{CI-dim}(\dot{\varphi}) + \text{edim}(\alpha) = \text{CI-dim}(\varphi) + \text{edim}(\alpha).$$

The second equality is from Case 1 applied to the diagram

$$\begin{array}{ccc}
R & \xrightarrow{\dot{\varphi}} & \widehat{S} \\
{\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \widehat{\beta}} \\
\widetilde{R} & \xrightarrow{\grave{\widetilde{\varphi}}} & \widehat{\widetilde{S}}.
\end{array}$$

This completes the proof of part (b).

For parts (a) and (c), argue as above, using [14, (4.1.4)] and Proposition 5.13(b) in place of Proposition 5.13(a). $\square$

5.15 (Proof of Theorem 1.8)**.** Case 1: P = Gorenstein. Since α and β are both flat with Gorenstein closed fibers, they are both Gorenstein. It follows that there are integers a and b such that $I_{\widetilde{R}}^{\widetilde{R}}(t) = t^a I_R^R(t)$ and $I_{\widetilde{S}}^{\widetilde{S}}(t) = t^b I_S^S(t)$. Thus, there is an integer c such that $I_S^S(t) = t^c I_R^R(t)$ if and only if there is an integer d such that $I_{\widetilde{S}}^{\widetilde{S}}(t) = t^d I_{\widetilde{R}}^{\widetilde{R}}(t)$. Proposition 5.9(b) says that $\operatorname{fd}(\widetilde{\varphi})$ is finite if and only if $\operatorname{fd}(\varphi)$ is finite, so φ is Gorenstein if and only if $\widetilde{\varphi}$ is Gorenstein, by definition.

Case 2: P = quasi-Gorenstein. This follows like Case 1, by using Proposition 5.14(a).

For the remaining cases, let $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} S$ be a minimal Cohen factorization of $\dot{\varphi}$, and let $\widetilde{R} \xrightarrow{\dot{\widetilde{\varphi}}} S' \xrightarrow{\widetilde{\varphi}'} \widetilde{S}$ be a minimal Cohen factorization of $\grave{\widetilde{\varphi}}$. Proposition 5.2 provides a weakly regular local ring homomorphism $\alpha'\colon R' \to S'$ such that the next diagram commutes

$$\begin{array}{ccccc} R & \xrightarrow{\dot{\varphi}} & R' & \xrightarrow{\varphi'} & \widehat{S} \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\alpha'} & & \big\downarrow{\scriptstyle\beta} \\ \widetilde{R} & \xrightarrow{\dot{\widetilde{\varphi}}} & \widetilde{R}' & \xrightarrow{\widetilde{\varphi}'} & \widehat{\widetilde{S}} \end{array} \tag{5.15.1}$$

and such that the second square is a pushout.

Case 3: P = complete intersection. Since the second square of diagram (5.15.1) is a pushout, we have $\operatorname{Ker}(\widetilde{\varphi}') = \operatorname{Ker}(\varphi')\widetilde{R}'$. The fact that α' is flat and local implies that a minimal generating sequence for $\operatorname{Ker}(\varphi')$ extends to a minimal generating sequence for $\operatorname{Ker}(\varphi')\widetilde{R}'$, and that this sequence is R'-regular if and only if it is $\widetilde{R}'$-regular. That is, the ideal $\operatorname{Ker}(\widetilde{\varphi}') = \operatorname{Ker}(\varphi')\widetilde{R}'$ in $\widetilde{R}'$ is a complete intersection if and only if $\operatorname{Ker}(\varphi') \subseteq R'$ is a complete intersection. Thus, φ is complete intersection if and only if $\widetilde{\varphi}$ is complete intersection, by definition.

Case 4: P = (quasi-)Cohen-Macaulay. Assume for the moment that G-dim$(\varphi) < \infty$. Then Proposition 5.14(a) implies that G-dim$(\widetilde{\varphi}) < \infty$. From [14, (6.7) Lemma], a relative dualizing complex for $\grave{\varphi}$ is $D^{\grave{\varphi}} = \mathbf{R}\mathrm{Hom}_{R'}(\widehat{S}, R')$, and it follows that

$$\begin{aligned}
D^{\grave{\widetilde{\varphi}}} &= \mathbf{R}\mathrm{Hom}_{\widetilde{R}'}(\widehat{\widetilde{S}}, \widetilde{R}') \\
&\simeq \mathbf{R}\mathrm{Hom}_{\widetilde{R}'}(\widetilde{R}' \otimes^{\mathbf{L}}_{R'} \widehat{S}, \widetilde{R}' \otimes^{\mathbf{L}}_{R'} R') \\
&\simeq \widetilde{R}' \otimes^{\mathbf{L}}_{R'} \mathbf{R}\mathrm{Hom}_{R'}(\widehat{S}, R') \\
&= \widetilde{R}' \otimes^{\mathbf{L}}_{R'} D^{\grave{\varphi}}.
\end{aligned}$$

The second step is from the fact that the second square of diagram (5.15.1) is a pushout and α' is flat. It follows that

$$\begin{aligned}
\mathbf{q}\mathrm{cmd}(\widetilde{\varphi}) = \mathbf{q}\mathrm{cmd}(\grave{\widetilde{\varphi}}) = \mathrm{amp}(D^{\grave{\widetilde{\varphi}}}) &= \mathrm{amp}(\widetilde{R}' \otimes^{\mathbf{L}}_{R'} D^{\grave{\varphi}}) \\
= \mathrm{amp}(D^{\grave{\varphi}}) = \mathbf{q}\mathrm{cmd}(\grave{\varphi}) &= \mathbf{q}\mathrm{cmd}(\varphi).
\end{aligned}$$

A similar argument shows that if $\mathrm{fd}(\varphi) < \infty$, then $\mathrm{cmd}(\widetilde{\varphi}) = \mathrm{cmd}(\varphi)$.

In view of the simultaneous finiteness given in Propositions 5.9 parts (a),(b) and 5.14 part (a), we see that φ is quasi-Cohen-Macaulay if and only if $\widetilde{\varphi}$ is quasi-Cohen-Macaulay, and φ is Cohen-Macaulay if and only if $\widetilde{\varphi}$ is Cohen-Macaulay. □

Remark 5.16. At this time, we do not know if the assumptions of weak Cohenness and separability are necessary in Theorem 1.8.

5.3. Proof of Theorem 1.9

Remark 5.17. Complete intersection dimension is quite nice in many respects. However, at this time we do not know how it behaves with respect to short exact sequences: If two of the modules in an exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ have finite CI-dimension, must the third module also have finite CI-dimension?

The difficulty with this question is the following. Assume, for instance, that $\text{CI-dim}_R(M_1)$ and $\text{CI-dim}_R(M_2)$ are finite. Then for $i = 1, 2$ there is a quasi-deformation $R \to R_i \leftarrow Q_i$ such that $\text{pd}_{Q_i}(R_i \otimes_R M_i) < \infty$. If there were a single quasi-deformation $R \to R' \leftarrow Q$ such that $\text{pd}_Q(R' \otimes_R M_i) < \infty$ for $i = 1, 2$ then we could conclude easily that $\text{pd}_Q(R' \otimes_R M_3) < \infty$, so $\text{CI-dim}_R(M_3) < \infty$. The difficulty lies in attempting to combine the two given quasi-deformations into a single one that works for both M_1 and M_2. Theorem 1.9 deals with half of this problem when the maps $R \to R_i$ are weakly Cohen by showing how to combine the flat maps in the given quasi-deformations. However, we do not see how to use Theorem 1.9 to answer the above question in any special cases, e.g., when R contains a field of characteristic 0.

5.18 (Proof of Theorem 1.9)**.** (a) Consider the induced maps $k \to k_i$ for $i = 1, 2$. Since these are separable by assumption, there is a commutative diagram of separable field extensions

$$\begin{array}{ccc} k & \longrightarrow & k_1 \\ \downarrow & & \downarrow \\ k_2 & \longrightarrow & k'. \end{array}$$

Let $\alpha\colon (R, \mathfrak{m}, k) \to (\widetilde{R}, \widetilde{\mathfrak{m}}, k')$ and $\beta_i\colon (R_i, \mathfrak{m}_i, k_i) \to (\widetilde{R}_i, \widetilde{\mathfrak{m}}_i, k')$ be Cohen extensions corresponding to the separable field extensions $k \to k'$ and $k_i \to k'$ such that the rings $\widetilde{R}$ and $\widetilde{R}_i$ are complete; see [40, Théorèm 19.8.2(ii)].

By Lemma 5.7 and Proposition 5.9(c), each φ_i $(i = 1, 2)$ can be extended to a flat local homomorphism $\widetilde{\varphi}_i$ making the following diagram commute:

$$\begin{array}{ccc} (R, \mathfrak{m}, k) & \xrightarrow{\varphi_i} & (R_i, \mathfrak{m}_i, k_i) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta_i} \\ (\widetilde{R}, \widetilde{\mathfrak{m}}, k') & \xrightarrow{\widetilde{\varphi}_i} & (\widetilde{R}_i, \widetilde{\mathfrak{m}}_i, k'). \end{array}$$

For $i = 1, 2$, Fact 2.37(b) provides a commutative diagram of local ring homomorphisms

$$\begin{array}{ccc} (Q_i, \mathfrak{n}_i, k_i) & \xrightarrow{\delta_i} & (\widetilde{Q}_i, \widetilde{\mathfrak{n}}_i, k') \\ {\scriptstyle\tau_i}\downarrow & & \downarrow{\scriptstyle\gamma_i} \\ (R_i, \mathfrak{m}_i, k_i) & \xrightarrow{\beta_i} & (\widetilde{R}_i, \widetilde{\mathfrak{m}}_i, k') \end{array} \tag{5.18.1}$$

such that $\widetilde{Q}_i$ is complete, δ_i is weakly regular, γ_i is surjective, and the induced map $R_i \otimes_{Q_i} \widetilde{Q}_i \to \widetilde{R}_i$ is an isomorphism. Because of this isomorphism and the flatness of δ_i, the fact that τ_i is surjective with kernel generated by a Q_i-regular sequence implies that γ_i is surjective with kernel generated by a $\widetilde{Q}_i$-regular sequence. Also, since the maps $R \xrightarrow{\varphi_i} R_i \xrightarrow{\beta_i} \widetilde{R}_i$ are flat and local, the same is true of the composition. Thus, for $i = 1, 2$, the diagram $R \xrightarrow{\beta_i\varphi_i} \widetilde{R}_i \xleftarrow{\gamma_i} \widetilde{Q}_i$ is a quasi-deformation.

Now consider the complete tensor product $R' = \widetilde{R}_1 \widehat{\otimes}_{\widetilde{R}} \widetilde{R}_2$; see, e.g., [38, Section 0.7.7] and [67, Section V.B.2] for background on this. This is a complete semi-local noetherian ring that is flat over each $\widetilde{R}_i$ by [40, Lemme 19.7.1.2]. Moreover, the proof of [40, Lemme 19.7.1.2] shows that the maximal ideals of R' are in bijection with the maximal ideals of $(\widetilde{R}_1/\widetilde{\mathfrak{m}}_1) \otimes_{\widetilde{R}} (\widetilde{R}_2/\widetilde{\mathfrak{m}}_2) \cong k' \otimes_{k'} k' \cong k'$, so R' is local. For $i = 1, 2$ let $\sigma_i \colon (\widetilde{R}_i, \widetilde{\mathfrak{m}}_i, k') \to (R', \mathfrak{m}', k')$ be the natural (flat local) map.

Hence, each σ_i is flat with complete target, so by Fact 2.37(2.37) there exists a commutative diagram of local homomorphisms

$$\begin{array}{ccc} (\widetilde{Q}_i, \widetilde{\mathfrak{n}}_i, k') & \overset{\overline{\delta}_i}{\dashrightarrow} & (\overline{Q}_i, \overline{\mathfrak{n}}_i, k') \\ {\scriptstyle\gamma_i}\big\downarrow & & \big\downarrow{\scriptstyle\overline{\gamma}_i} \\ (\widetilde{R}_i, \widetilde{\mathfrak{m}}_i, k') & \xrightarrow{\sigma_i} & (R', \mathfrak{m}', k') \end{array} \tag{5.18.2}$$

such that $\overline{Q}_i$ is complete, $\overline{\delta}_i$ is weakly regular, and $\overline{\gamma}_i$ is surjective.

Thus, the following diagram

$$\begin{array}{ccccccccc} (Q_1, \mathfrak{n}_1, k_1) & \xrightarrow{\tau_1} & (R_1, \mathfrak{m}_1, k_1) & \xleftarrow{\varphi_1} & (R, \mathfrak{m}, k) & \xrightarrow{\varphi_2} & (R_2, \mathfrak{m}_2, k_2) & \xleftarrow{\tau_2} & (Q_2, \mathfrak{n}_2, k_2) \\ & {\scriptstyle\overline{\delta}_1\delta_1}\searrow & & {\scriptstyle\sigma_1\beta_1}\searrow & \big\downarrow{\scriptstyle\alpha'} & \swarrow{\scriptstyle\sigma_2\beta_2} & & \swarrow{\scriptstyle\overline{\delta}_2\delta_2} & \\ & & (\overline{Q}_1, \overline{\mathfrak{n}}_1, k') & \xrightarrow[\overline{\gamma}_1]{} & (R', \mathfrak{m}', k') & \xleftarrow[\overline{\gamma}_2]{} & (\overline{Q}_2, \overline{\mathfrak{n}}_1, k') & & \end{array} \tag{C.1}$$

commutes where $\alpha' = \sigma_1\beta_1\varphi_1 = \sigma_2\beta_2\varphi_2$. By assumption, each $\overline{\gamma}_i$ is surjective. Since σ_1, β_1, and φ_1 are flat, so is their composition α'. The maps $\overline{\delta}_i$ and δ_i are weakly regular, hence Fact 2.33 implies that their composition is weakly regular. Moreover, since the field extension $k_i \to k'$ is separable, the composition $\overline{\delta}_i\delta_i$ is weakly Cohen.

(b) Assume that each map φ_i is weakly Cohen. Then the closed fiber $R_i/\mathfrak{m}R_i$ is regular. Since $\beta_i \colon R_i \to \widetilde{R}_i$ is Cohen, the same is true of the induced map $R_i/\mathfrak{m}R_i \to \widetilde{R}_i/\mathfrak{m}\widetilde{R}_i$. Thus, the fact that $R_i/\mathfrak{m}R_i$ is regular implies that $\widetilde{R}_i/\mathfrak{m}\widetilde{R}_i$ is also regular. Since $\alpha \colon R \to \widetilde{R}$ is Cohen, we have $\mathfrak{m}\widetilde{R} = \widetilde{\mathfrak{m}}$, and it follows that $\widetilde{R}_i/\mathfrak{m}\widetilde{R}_i = \widetilde{R}_i/\widetilde{\mathfrak{m}}\widetilde{R}_i$ is regular. From [40, Lemme 19.7.1.2] we know that the closed fiber of the map $\sigma_1 \colon (\widetilde{R}_1, \widetilde{\mathfrak{m}}_1, k') \to (R', \mathfrak{m}', k')$ is

$$R'/\widetilde{\mathfrak{m}}_1 R' \cong [\widetilde{R}_1/\widetilde{\mathfrak{m}}_1] \otimes_{\widetilde{R}} \widetilde{R}_2 \cong [\widetilde{R}_1/\widetilde{\mathfrak{m}}_1] \otimes_{k'} [\widetilde{R}_2/\widetilde{\mathfrak{m}}\widetilde{R}_2] \cong k' \otimes_{k'} [\widetilde{R}_2/\widetilde{\mathfrak{m}}\widetilde{R}_2] \cong \widetilde{R}_2/\widetilde{\mathfrak{m}}\widetilde{R}_2.$$

Since this ring is regular and the field extension $k' \to k'$ is trivially separable, the map σ_1 is weakly Cohen, as is σ_2 by similar argument. Thus, Fact 2.37(b) implies that diagram (5.18.2) is a pushout. As in an earlier part of this proof, for each $i = 1, 2$ the diagram $R \xrightarrow{\sigma_i\beta_i\varphi_i} R' \xleftarrow{\overline{\gamma}_i} \overline{Q}_i$ is a quasi-deformation.

By construction, the maps $R_i \xrightarrow{\sigma_i\beta_i} R'$ are compositions of weakly Cohen maps, so they are weakly Cohen; hence conclusion (b1) from the statement of Theorem 1.9 is satisfied. Since the diagrams (5.18.1) and (5.18.2) are pushouts, the same is true of the parallelograms in the diagram above; hence conclusion (b2) from the statement of Theorem 1.9 is satisfied. Lastly, given an R-module M, the flatness of the map $Q_i \xrightarrow{\overline{\delta}_i\delta_i} \overline{Q}_i$ provides the first equality in the next display:

$$\begin{aligned}
\operatorname{pd}_{Q_i}(M \otimes_R R_i) &= \operatorname{pd}_{\overline{Q}_i}((M \otimes_R R_i) \otimes_{Q_i} \overline{Q}_i) \\
&= \operatorname{pd}_{\overline{Q}_i}(M \otimes_R (R_i \otimes_{Q_i} \overline{Q}_i)) \\
&= \operatorname{pd}_{\overline{Q}_i}(M \otimes_R R').
\end{aligned}$$

(See [62, Theorem 9.6] and [63].) The last equality is from the pushout conclusion on each parallelogram. This shows that conclusion (b3) from the statement of Theorem 1.9 is satisfied. □

Remark 5.19. At this time, we do not know if the weakly Cohen assumption is necessary in Theorem 1.9.

CHAPTER 6. THE GENERALIZED AUSLANDER-REITEN CONJECTURE UNDER CERTAIN RING EXTENSIONS

In this chapter, we show under some conditions that a Gorenstein ring R satisfies the Generalized Auslander-Reiten Conjecture if and only if so does $R[x]$. In this direction we give the proofs of Theorems 1.12 and 1.13 from the introduction.

6.1. Preliminaries and Notation

We begin by establishing the notation that will be used in this chapter.

Notation 6.1. Let R be an algebra over a field k, and let M be a module over the polynomial ring $R[x]$. For an element $\alpha \in k$ we set

$$M_\alpha := M \otimes_{k[x]} (k[x]/(x-\alpha)k[x]).$$

Remark 6.2. We work in the setting of Notation 6.1. For each element $\alpha \in k$, the module M_α is simply a residue module of M by the submodule $(x-\alpha)M$. If M is a finitely generated $R[x]$-module then it has a presentation of the form

$$R[x]^{\beta_1} \xrightarrow{m(x)} R[x]^{\beta_0} \to M \to 0$$

where β_0, β_1 are non-negative integers and $m(x)$ is a $\beta_0 \times \beta_1$-matrix consisting of polynomials in $R[x]$. Therefore, by using the isomorphism

$$R = R[\alpha] \cong R[x]/(x-\alpha)R[x] \tag{6.2.1}$$

the module M_α has a presentation of the form

$$R^{\beta_1} \xrightarrow{m(\alpha)} R^{\beta_0} \to M_\alpha \to 0$$

where $m(\alpha)$ is a $\beta_0 \times \beta_1$-matrix consisting of elements in R. In particular, if M is a finitely generated $R[x]$-module, then M_α is a finitely generated R-module.

For more information about the meaning of M_α see [79].

The following lemma will be used several times in this chapter.

Lemma 6.3. *Let R be an algebra over a field k, and let $\alpha \in k$. If $x-\alpha$ is a non-zero-divisor on an $R[x]$-module M, then for each integer i we have* $\mathrm{Ext}^i_{R[x]}(M, M_\alpha) \cong \mathrm{Ext}^i_R(M_\alpha, M_\alpha)$.

Proof. This isomorphism follows from (6.2.1) and [55, Lemma 2(ii), p. 140]. □

The following two lemmas are important tools that we use for the proofs of our main results in the next section.

Lemma 6.4. *Let R be an algebra over a field k, and let M be an $R[x]$-module. Then for each ideal I of $R[x]$ and each $\alpha \in k$ we have $(IM)_\alpha = IM_\alpha$ and $(M/IM)_\alpha \cong M_\alpha/IM_\alpha$.*

Proof. Without loss of generality we can assume that I is non-zero. For the first equality note that M_α has an $R[x]$-module structure that uses the natural ring homomorphism $R[x] \to R[x]_\alpha$. This gives the second equality in the next display:

$$(IM)_\alpha = (IM) \otimes_{k[x]} k[x]/(x-\alpha)k[x] = I(M \otimes_{k[x]} k[x]/(x-\alpha)k[x]) = IM_\alpha.$$

For the second equality in the statement of the lemma, let $I = (t_1, ..., t_l)$ for some positive integer l. Then we have an exact sequence

$$\bigoplus_{j=1}^{l} M \xrightarrow{f} M \to M/IM \to 0$$

where f is the matrix multiplication by $(t_1, ..., t_l)$. By applying the right exact functor $- \otimes_{k[x]} k[x]/(x-\alpha)k[x]$ we get an exact sequence

$$\bigoplus_{j=1}^{l} M_\alpha \xrightarrow{f \otimes \mathrm{id}} M_\alpha \to (M/IM)_\alpha \to 0$$

where id is the identity map on $k[x]/(x-\alpha)k[x]$. Now we have

$$(M/IM)_\alpha \cong M_\alpha / \operatorname{Im}(f \otimes \mathrm{id}) = M_\alpha/(\operatorname{Im}(f))_\alpha = M_\alpha/(IM)_\alpha = M_\alpha/IM_\alpha$$

as desired. □

Lemma 6.5. *Let R be an algebra over a field k, and let $\alpha \in k$. If $x - \alpha$ is a non-zero-divisor on an $R[x]$-module M, then for each $i \geqslant 0$ there exists an exact sequence*

$$\begin{aligned} 0 \to \operatorname{Ext}^i_{R[x]}(M, M)_\alpha \to \operatorname{Ext}^i_R(M_\alpha, M_\alpha) \\ \to \operatorname{Tor}_1^{k[x]}(\operatorname{Ext}^{i+1}_{R[x]}(M, M), k[x]/(x-\alpha)k[x]) \to 0. \end{aligned}$$

Proof. Using the exact sequence

$$0 \to M \xrightarrow{x-\alpha} M \to M_\alpha \to 0$$

we get the long exact sequence

$$\begin{aligned} \operatorname{Ext}^i_{R[x]}(M, M) \xrightarrow{x-\alpha} \operatorname{Ext}^i_{R[x]}(M, M) \\ \to \operatorname{Ext}^i_{R[x]}(M, M_\alpha) \to \operatorname{Ext}^{i+1}_{R[x]}(M, M) \xrightarrow{x-\alpha} \cdots \end{aligned}$$

where $x - \alpha$ represents the multiplication map $\operatorname{Ext}^i_{R[x]}(M, M) \xrightarrow{x-\alpha} \operatorname{Ext}^i_{R[x]}(M, M)$.

By Lemma 6.3 we have an isomorphism $\operatorname{Ext}^i_{R[x]}(M, M_\alpha) \cong \operatorname{Ext}^i_R(M_\alpha, M_\alpha)$ for each integer i. Thus, for each integer i we get the following short exact sequence

$$0 \to \operatorname{Coker}(x-\alpha) \to \operatorname{Ext}^i_R(M_\alpha, M_\alpha) \to \operatorname{Ker}(x-\alpha) \to 0.$$

By the definition we have $\operatorname{Coker}(x-\alpha) = \operatorname{Ext}^i_{R[x]}(M, M)_\alpha$. Now to compute $\operatorname{Ker}(x-\alpha)$, the short exact sequence

$$0 \to k[x] \xrightarrow{x-\alpha} k[x] \to k[x]/(x-\alpha)k[x] \to 0$$

gives us the following long exact sequence

$$0 \to \operatorname{Tor}_1^{k[x]}(\operatorname{Ext}^{i+1}_{R[x]}(M, M), k[x]/(x-\alpha)k[x]) \to \operatorname{Ext}^{i+1}_{R[x]}(M, M) \otimes_{k[x]} k[x] \xrightarrow{x-\alpha} \operatorname{Ext}^{i+1}_{R[x]}(M, M) \otimes_{k[x]} k[x]$$

So we have $\operatorname{Ker}(x-\alpha) \cong \operatorname{Tor}_1^{k[x]}(\operatorname{Ext}^{i+1}_{R[x]}(M, M), k[x]/(x-\alpha)k[x])$. This completes the proof of the lemma. □

We recall a straightforward fact that will be used later.

Fact 6.6. Let R be a Gorenstein ring, and let $0 \to M_1 \to F \to M \to 0$ be an exact sequence of finitely generated R-modules where F is projective. It follows from the long exact sequence

$$\operatorname{Ext}^i_R(M, F) \to \operatorname{Ext}^i_R(M, M) \to \operatorname{Ext}^{i+1}_R(M, M_1) \to \operatorname{Ext}^{i+1}_R(M, F)$$

that $\operatorname{Ext}^i_R(M, M) \cong \operatorname{Ext}^{i+1}_R(M, M_1)$ for all $i > \operatorname{id}_R(F)$. Also by using the long exact sequence

$$\operatorname{Ext}_R^i(F, M_1) \to \operatorname{Ext}_R^i(M_1, M_1) \to \operatorname{Ext}_R^{i+1}(M, M_1) \to \operatorname{Ext}_R^{i+1}(F, M_1)$$

we get the isomorphism $\operatorname{Ext}_R^i(M_1, M_1) \cong \operatorname{Ext}_R^{i+1}(M, M_1)$ for all $i > 0$. Therefore, for the finitely generated module M over the Gorenstein ring R, we have $\operatorname{Ext}_R^i(M, M) = 0$ for all $i \gg 0$ if and only if $\operatorname{Ext}_R^i(M_1, M_1) = 0$ for all $i \gg 0$.

6.2. Proofs of Theorems 1.12 and 1.13

In this section, we prove our main theorems about Conjecture 1.10. The method that we use for the proofs of both theorems uses the well-known decomposition of finitely generated modules over principal ideal domains. We begin this section with the following lemma. Recall the definition of fed from Section 1.6 of the Introduction.

Lemma 6.7. *Let k be an algebraically closed field, and let R be a Gorenstein k-algebra. Let $k(x)$ be a transcendental extension of k, and assume that both* $\operatorname{fed}(R)$ *and* $\operatorname{fed}(R \otimes_k k(x))$ *are finite. Then* $\operatorname{fed}(R[x]) < \infty$.

Proof. Set $f := \operatorname{fed}(R) < \infty$ and $g := \operatorname{fed}(R \otimes_k k(x)) < \infty$. Let M be a finitely generated $R[x]$-module such that $\operatorname{Ext}_{R[x]}^i(M, M) = 0$ for all $i \gg 0$. Since $R[x]$ is a Gorenstein ring, by Fact 6.6 we can replace M with its first syzygy in an arbitrary free resolution and assume that M is submodule of a free $R[x]$-module. In particular, M is assumed to be a torsion-free $R[x]$-module.

Now for each $\alpha \in k$, by using the short exact sequence

$$0 \to M \xrightarrow{x-\alpha} M \to M_\alpha \to 0 \tag{6.7.1}$$

our vanishing assumption is equivalent to $\operatorname{Ext}_{R[x]}^i(M, M_\alpha) = 0$ for all $i \gg 0$, and by Lemma 6.3, this is equivalent to $\operatorname{Ext}_R^i(M_\alpha, M_\alpha) = 0$ for all $i \gg 0$. Since by Remark 6.2

the R-module M_α is finitely generated, our assumption implies that $\operatorname{Ext}^i_R(M_\alpha, M_\alpha) = 0$ for all $i > f$. Therefore, by using the long exact sequence of Ext obtained from the short exact sequence (6.7.1) and the isomorphism $\operatorname{Ext}^i_{R[x]}(M, M_\alpha) \cong \operatorname{Ext}^i_R(M_\alpha, M_\alpha)$ from Lemma 6.3, we observe that $x - \alpha$ acts bijectively on $\operatorname{Ext}^i_{R[x]}(M, M)$ for all $i > f + 1$ and all $\alpha \in k$.

Now consider the following set

$$S := \{(x - \alpha_1)^{n_1} ... (x - \alpha_l)^{n_l} \mid n_j \in \mathbb{N} \cup \{0\} \text{ and } \alpha_j \in k \text{ for } 1 \leqslant j \leqslant l\}.$$

Note that S is a multiplicatively closed subset of $k[x]$ and for each $i > f + 1$ the cohomology module $\operatorname{Ext}^i_{R[x]}(M, M)$ is an $S^{-1}k[x]$-module.

On the other hand, since k is algebraically closed we have the equality $S^{-1}k[x] = k(x)$ and therefore,

$$S^{-1}R[x] \cong S^{-1}(R \otimes_k k[x]) = R \otimes_k (S^{-1}k[x]) = R \otimes_k k(x).$$

This isomorphism gives the last isomorphism in the next display for each $i > f + 1$:

$$\begin{aligned} \operatorname{Ext}^i_{R[x]}(M, M) &\cong S^{-1} \operatorname{Ext}^i_{R[x]}(M, M) \\ &\cong \operatorname{Ext}^i_{S^{-1}R[x]}(S^{-1}M, S^{-1}M) \\ &\cong \operatorname{Ext}^i_{R \otimes_k k(x)}(S^{-1}M, S^{-1}M). \end{aligned}$$

Hence, $\operatorname{Ext}^i_{R \otimes_k k(x)}(S^{-1}M, S^{-1}M) = 0$ for all $i \gg 0$. Our assumption implies that $\operatorname{Ext}^i_{R \otimes_k k(x)}(S^{-1}M, S^{-1}M) = 0$ for all $i > g$. Therefore, $\operatorname{Ext}^i_{R[x]}(M, M) = 0$ for all $i > h := \max\{f + 1, g\}$. Note that h is independent of the choice of the finitely generated $R[x]$-module M and we conclude that $\operatorname{fed}(R[x]) \leqslant h < \infty$. □

Remark 6.8. It is straightforward to check that if $R \to S$ is a faithfully flat ring homomorphism with $\mathrm{fed}(S) < \infty$, then $\mathrm{fed}(R) \leqslant \mathrm{fed}(S) < \infty$. This follows from the fact that for every finitely generated R module M and for each integer i we have $\mathrm{Ext}^i_R(M, M) = 0$ if and only if $\mathrm{Ext}^i_S(M \otimes_R S, M \otimes_R S) = 0$. Therefore, it follows from Theorem 1.11 that if R and S are Gorenstein rings such that S satisfies Conjecture 1.10, then so does R.

Proposition 6.9. *Let k be an uncountable field, and let R be a Gorenstein finite dimensional k-algebra. Let $k(x)$ be a transcendental extension of k, and assume that* $\mathrm{fed}(R) < \infty$. *Then* $\mathrm{fed}(R \otimes_k k(x)) < \infty$. *More precisely,* $\mathrm{fed}(R) = \mathrm{fed}(R \otimes_k k(x))$.

Proof. Set $f := \mathrm{fed}(R) < \infty$. Let N be a finitely generated $R \otimes_k k(x)$-module such that $\mathrm{Ext}^i_{R\otimes_k k(x)}(N, N) = 0$ for $i \gg 0$. We know that $R \otimes_k k(x) \cong S^{-1}R[x]$ where S is the multiplicatively closed subset $k[x]\backslash\{0\}$. Thus, there exists a finitely generated $R[x]$-submodule M of N such that

$$\begin{aligned} N \cong M \otimes_{R[x]} S^{-1}R[x] &\cong M \otimes_{R[x]} (R[x] \otimes_{k[x]} S^{-1}k[x]) \\ &\cong M \otimes_{k[x]} S^{-1}k[x] \\ &= M \otimes_{k[x]} k(x). \end{aligned}$$

Note that N is a $k(x)$-module by using the natural ring homomorphism $k(x) \to R \otimes_k k(x)$. More precisely, for each $a(x) \in k(x)$ and each $n \in N$ the scalar multiplication is defined by $a(x).n := (1 \otimes a(x))n$. Now, let $f(x) \in k[x] \subseteq k(x)$ be a non-zero polynomial such that $f(x).n = 0$ for some $n \in N$. It follows that $n = (1/f(x))f(x).n = 0$. This implies that N is a torsion-free $k[x]$-module. Hence, M is a torsion-free $k[x]$-module as a submodule of N. This implies that $x - \alpha$ is a non-zero-divisor on M for each $\alpha \in k$.

Now by the isomorphism $\mathrm{Ext}^i_{R\otimes_k k(x)}(N, N) \cong \mathrm{Ext}^i_{R[x]}(M, M) \otimes_{k[x]} k(x)$, we see that $\mathrm{Ext}^i_{R[x]}(M, M) \otimes_{k[x]} k(x) = 0$ for $i \gg 0$.

On the other hand, since by assumption R is a finite dimensional k-algebra, for each $i \geqslant 0$ the $R[x]$-module $\operatorname{Ext}^i_{R[x]}(M,M)$ is a finitely generated $k[x]$-module, i.e. $\operatorname{Ext}^i_{R[x]}(M,M)$ is finitely generated over a principal ideal domain. Therefore, $\operatorname{Ext}^i_{R[x]}(M,M)$ has a $k[x]$-module decomposition as

$$\operatorname{Ext}^i_{R[x]}(M,M) \cong \bigoplus_{j=1}^{c_i} k[x]/(w_{ij}(x))k[x] \oplus k[x]^{v_i},$$

where $0 \neq w_{ij}(x) \in k[x]$ and each v_i is a non-negative integer.

Since $\operatorname{Ext}^i_{R[x]}(M,M) \otimes_{k[x]} k(x) = 0$ for all $i \gg 0$, we get $v_i = 0$ for $i \gg 0$. Also there are only countably many polynomials $w_{ij}(x)$. Since by assumption k is uncountable, there exists $\alpha \in k$ such that $w_{ij}(\alpha) \neq 0$ for all i,j. Therefore, $x-\alpha$ acts bijectively on $k[x]/(w_{ij}(x))k[x]$ for all i,j. Since $x-\alpha$ is a non-zero-divisor on each $k[x]^{v_i}$, we conclude that $\operatorname{Tor}_1^{k[x]}(\operatorname{Ext}^{i+1}_{R[x]}(M,M), k[x]/(x-\alpha)k[x]) = 0$ for all i. Since $v_i = 0$ for $i \gg 0$, we obtain that $\operatorname{Ext}^i_{R[x]}(M,M)_\alpha = 0$ for $i \gg 0$. Therefore, Lemma 6.5 implies that $\operatorname{Ext}^i_R(M_\alpha, M_\alpha) = 0$ for $i \gg 0$. Note that by Remark 6.2 the R-module M_α is finitely generated. Thus, by assumption we have $\operatorname{Ext}^i_R(M_\alpha, M_\alpha) = 0$ for all $i > f$. Since $\operatorname{Ext}^i_{R[x]}(M,M)_\alpha$ is a submodule of $\operatorname{Ext}^i_R(M_\alpha, M_\alpha)$ by Lemma 6.5, we have $\operatorname{Ext}^i_{R[x]}(M,M)_\alpha = 0$ for all $i > f$. This implies that $v_i = 0$ for $i > f$, which is equivalent to $\operatorname{Ext}^i_{R[x]}(M,M) \otimes_{k[x]} k(x) = 0$ for $i > f$. This is also equivalent to $\operatorname{Ext}^i_{R \otimes_k k(x)}(N,N) = 0$ for $i > f$, and this shows that $\operatorname{fed}(R \otimes_k k(x)) \leqslant f < \infty$. Equality now follows from Remark 6.8. $\square$

With the previous results in hand, we are ready to prove Theorem 1.12.

6.10 (Proof of Theorem 1.12)**.** It follows from Proposition 6.9 that $\operatorname{fed}(R \otimes_k k(x)) < \infty$. Therefore, by Lemma 6.7 we conclude that $\operatorname{fed}(R[x]) < \infty$. This shows that if R satisfies Conjecture 1.10, then so does $R[x]$. The reverse implication follows directly from Remark 6.8.

Remark 6.11. By [26, Corollary 3.5], for a Gorenstein ring R, finiteness of $\operatorname{fed}(R)$ implies that $\operatorname{fed}(R) = \operatorname{id}(R)$. Therefore, under assumptions of Theorem 1.12, we have $\operatorname{fed}(R) = \operatorname{fed}(R \otimes_k k(x)) = 0$, and $\operatorname{fed}(R[x]) = 1$.

Next is the proof of Theorem 1.13.

6.12 (Proof of Theorem 1.13)**.** Set $f := \operatorname{fed}(R) < \infty$, and let N be a finitely generated $R[x]_{\mathfrak{m}R[x]}$-module such that $\operatorname{Ext}^i_{R[x]_{\mathfrak{m}R[x]}}(N, N) = 0$ for $i \gg 0$. Since $R[x]_{\mathfrak{m}R[x]}$ is a Gorenstein ring, by Fact 6.6 we can replace N by its first syzygy in an arbitrary free resolution and assume that N is torsion-free.

Notice that since we are working with localization, we can find a finitely generated $R[x]$-submodule M of N such that $\operatorname{Ext}^i_{R[x]_{\mathfrak{m}R[x]}}(N, N) \cong \operatorname{Ext}^i_{R[x]}(M, M)_{\mathfrak{m}R[x]}$ for all i. Therefore, by our assumption we get $\operatorname{Ext}^i_{R[x]}(M, M)_{\mathfrak{m}R[x]} = 0$ for all $i \gg 0$.

On the other hand, for each i, Nakayama's lemma implies the first equivalence in the next display

$$\begin{aligned}\operatorname{Ext}^i_{R[x]}(M, M)_{\mathfrak{m}R[x]} = 0 &\iff \left(\operatorname{Ext}^i_{R[x]}(M, M)/\mathfrak{m}\operatorname{Ext}^i_{R[x]}(M, M)\right)_{\mathfrak{m}R[x]} = 0 \\ &\iff \left(\operatorname{Ext}^i_{R[x]}(M, M)/\mathfrak{m}\operatorname{Ext}^i_{R[x]}(M, M)\right) \otimes_{k[x]} k(x) = 0.\end{aligned}$$

The second equivalence follows from the fact that $\mathfrak{m}R[x]$ is the zero ideal of $k[x]$. Hence, $\left(\operatorname{Ext}^i_{R[x]}(M, M)/\mathfrak{m}\operatorname{Ext}^i_{R[x]}(M, M)\right) \otimes_{k[x]} k(x) = 0$ for all $i \gg 0$.

Since for each i the $R[x]$-module $\operatorname{Ext}^i_{R[x]}(M, M)$ is finitely generated, the $k[x]$-module $\operatorname{Ext}^i_{R[x]}(M, M)/\mathfrak{m}\operatorname{Ext}^i_{R[x]}(M, M)$ is finitely generated as well. Hence, as we explained in the proof of Proposition 6.9, each $\operatorname{Ext}^i_{R[x]}(M, M)/\mathfrak{m}\operatorname{Ext}^i_{R[x]}(M, M)$ has a $k[x]$-modules decomposition as

$$\operatorname{Ext}^i_{R[x]}(M, M)/\mathfrak{m}\operatorname{Ext}^i_{R[x]}(M, M) \cong \bigoplus_{j=1}^{c_i} k[x]/(w_{ij}(x))k[x] \oplus k[x]^{v_i},$$

where $0 \neq w_{ij}(x) \in k[x]$ and each v_i is a non-negative integer.

Since $\left(\operatorname{Ext}^i_{R[x]}(M,M)/\mathfrak{m}\operatorname{Ext}^i_{R[x]}(M,M)\right) \otimes_{k[x]} k(x) = 0$ for all $i \gg 0$, we get $v_i = 0$ for $i \gg 0$. As in the proof of Proposition 6.9, there exists $\alpha \in k$ such that $w_{ij}(\alpha) \neq 0$ for all i, j. Hence, $x - \alpha$ acts bijectively on $k[x]/(w_{ij}(x))k[x]$ for all i, j. This implies that

$$\left(\operatorname{Ext}^i_{R[x]}(M,M)/\mathfrak{m}\operatorname{Ext}^i_{R[x]}(M,M)\right)_\alpha \cong k^{v_i} \tag{6.12.1}$$

for all i. Since $v_i = 0$ for all $i \gg 0$, it follows for all $i \gg 0$ that

$$\left(\operatorname{Ext}^i_{R[x]}(M,M)/\mathfrak{m}\operatorname{Ext}^i_{R[x]}(M,M)\right)_\alpha = 0.$$

On the other hand, for all i we have the following isomorphism by Lemma 6.4:

$$\left(\operatorname{Ext}^i_{R[x]}(M,M)/\mathfrak{m}\operatorname{Ext}^i_{R[x]}(M,M)\right)_\alpha \cong \operatorname{Ext}^i_{R[x]}(M,M)_\alpha/\mathfrak{m}\operatorname{Ext}^i_{R[x]}(M,M)_\alpha. \tag{6.12.2}$$

Therefore, Nakayama's lemma implies that $\operatorname{Ext}^i_{R[x]}(M,M)_\alpha = 0$ for all $i \gg 0$. Equivalently, $\operatorname{Ext}^i_{R[x]}(M,M) \xrightarrow{x-\alpha} \operatorname{Ext}^i_{R[x]}(M,M)$ is a bijective map for all $i \gg 0$. Thus, $\operatorname{Tor}_1^{k[x]}(\operatorname{Ext}^{i+1}_{R[x]}(M,M), k[x]/(x-\alpha)k[x]) = 0$ for all $i \gg 0$.

It follows from Lemma 6.5 that $\operatorname{Ext}^i_R(M_\alpha, M_\alpha) = 0$ for all $i \gg 0$. Since M_α is a finitely generated R-module by Remark 6.2, our assumption implies that $\operatorname{Ext}^i_R(M_\alpha, M_\alpha) = 0$ for all $i > f$. Again from Lemma 6.5 we get $\operatorname{Ext}^i_{R[x]}(M,M)_\alpha = 0$ for all $i > f$. In particular, $(\operatorname{Ext}^i_{R[x]}(M,M)/\mathfrak{m}\operatorname{Ext}^i_{R[x]}(M,M))_\alpha = 0$ for all $i > f$ by (6.12.2). It follows from equation (6.12.1) that $v_i = 0$ for $i > f$. We conclude now that $\operatorname{Ext}^i_{R[x]}(M,M)/\mathfrak{m}\operatorname{Ext}^i_{R[x]}(M,M) \otimes_{k[x]} k(x) = 0$ for all $i > f$. As we explained above this is equivalent to saying $\operatorname{Ext}^i_{R[x]}(M,M)_{\mathfrak{m}R[x]} = 0$ for all $i > f$. Hence, $\operatorname{Ext}^i_{R[x]_{\mathfrak{m}R[x]}}(N,N) = 0$ for all $i > f$. Thus, $\operatorname{fed}(R[x]_{\mathfrak{m}R[x]}) \leqslant f < \infty$. This shows that if R satisfies Conjecture 1.10, then so does $R[x]_{\mathfrak{m}R[x]}$. The reverse implication follows from Remark 6.8.

6.3. Conjecture 1.10 Under Another Type of Ring Extension

Let R be a local ring. In this section, we investigate the condition of Conjecture 1.10 under the base ring extension $R \to R[x]_{\mathfrak{M}}$ where $\mathfrak{M}$ is a maximal ideal of the polynomial ring that contracts to the maximal ideal of R. The next proposition is our main result in this section.

Proposition 6.13. *Let* $(R, \mathfrak{m})$ *be a Gorenstein local ring with algebraically closed residue field, and let* $\mathfrak{M}$ *be a maximal ideal of* $R[x]$ *such that* $\mathfrak{m} = \mathfrak{M} \cap R$*. Then* $\operatorname{fed}(R) < \infty$ *if and only if* $\operatorname{fed}(R[x]_{\mathfrak{M}}) < \infty$*. Therefore, R satisfies Conjecture* 1.10 *if and only if so does* $R[x]_{\mathfrak{M}}$*.*

Proof. Suppose that $\operatorname{fed}(R) < \infty$. By Hilbert's nullstellensatz, there exists an element $r \in R$ such that $\mathfrak{M} = (\mathfrak{m}, x - r)R[x]$. Since, $R \cong R[x]/(x - r)R[x]$ and $x - r$ is a non-zero-divisor on $R[x]$, we conclude that $R[x]/(x-r)R[x]$ is a local Gorenstein ring with maximal ideal $\mathfrak{M}/(x - r)R[x]$ such that $\operatorname{fed}(R[x]/(x - r)R[x]) < \infty$; see [26, Proposition 4.3]. On the other hand we have a ring isomorphism

$$R[x]/(x - r)R[x] \cong R[x]_{\mathfrak{M}}/(x - r)R[x]_{\mathfrak{M}}$$

which implies that $\operatorname{fed}(R[x]_{\mathfrak{M}}/(x - r)R[x]_{\mathfrak{M}}) < \infty$. Since $x - r$ is a non-zero-divisor on $R[x]_{\mathfrak{M}}$, again [26, Proposition 4.3] implies that $\operatorname{fed}(R[x]_{\mathfrak{M}}) < \infty$. This shows that if R satisfies Conjecture 1.10, then so does $R[x]_{\mathfrak{M}}$. The reverse implication follows from Remark 6.8. □

We conclude this chapter by proving a corollary of Proposition 6.13 that improves Theorem 1.12 when R is a local ring. Before stating the corollary, we need to prove the following lemma.

Lemma 6.14. *Let R be a Gorenstein ring of finite Krull dimension. If* $\operatorname{fed}(R_{\mathfrak{m}}) < \infty$ *for each maximal ideal* $\mathfrak{m}$ *of* R*, then* $\operatorname{fed}(R) < \infty$.

Proof. By [26, Theorem 3.1], for each maximal ideal $\mathfrak{m}$ we have $\operatorname{fed}(R_{\mathfrak{m}}) \leqslant \operatorname{id}(R_{\mathfrak{m}}) = \dim(R_{\mathfrak{m}}) \leqslant \dim(R)$. Therefore,

$$c := \sup\{\operatorname{fed}(R_{\mathfrak{m}}) \mid \ \mathfrak{m} \text{ is a maximal ideal of } R\} \leqslant \dim(R) < \infty. \tag{6.14.1}$$

Let M be a finitely generated R-module, and assume that $\operatorname{Ext}^i_R(M, M) = 0$ for all $i \gg 0$. Thus for every maximal ideal $\mathfrak{m}$ of R we have $\operatorname{Ext}^i_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}, M_{\mathfrak{m}}) \cong \operatorname{Ext}^i_R(M, M)_{\mathfrak{m}} = 0$ for all $i \gg 0$. Therefore, by (6.14.1), for every maximal ideal $\mathfrak{m}$ of R we get $\operatorname{Ext}^i_R(M, M)_{\mathfrak{m}} = 0$ for all $i > c$. This implies that $\operatorname{Ext}^i_R(M, M) = 0$ for all $i > c$. Thus, $\operatorname{fed}(R) \leqslant c < \infty$. □

Corollary 6.15. *Let* $(R, \mathfrak{m})$ *be an artinian Gorenstein local ring with algebraically closed residue field. Then* $\operatorname{fed}(R) < \infty$ *if and only if* $\operatorname{fed}(R[x]) < \infty$. *Therefore, R satisfies Conjecture* 1.10 *if and only if so does* $R[x]$.

Proof. Assume that $\operatorname{fed}(R) < \infty$. Since R is artinian, for each maximal ideal $\mathfrak{M}$ of $R[x]$ we have $\mathfrak{M} \cap R = \mathfrak{m}$. It follows from Proposition 6.13 that for each maximal ideal $\mathfrak{M}$ of $R[x]$ we have $\operatorname{fed}(R[x]_{\mathfrak{M}}) < \infty$. Therefore, Lemma 6.14 implies that $\operatorname{fed}(R[x]) < \infty$. This shows that if R satisfies Conjecture 1.10, then so does $R[x]$. The reverse implication follows from Remark 6.8. □

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