OPTIMIZATION PROBLEMS ARISING IN STABILITY ANALYSIS OF DISCRETE TIME RECURRENT NEURAL NETWORKS

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The supervisory committee certifies that this dissertation complies with North Dakota State University's regulations and meets the accepted standards for the degree of

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ABSTRACT

We consider the method of Reduction of Dissipativity Domain to prove global Lyapunov stability of Discrete Time Recurrent Neural Networks. The standard and advanced criteria for Absolute Stability of these essentially nonlinear systems produce rather weak results. The method mentioned above is proved to be more powerful. It involves a multi-step procedure with maximization of special nonconvex functions over polytopes on every step. We derive conditions which guarantee an existence of at most one point of local maximum for such functions over every hyperplane. This nontrivial result is valid for wide range of neuron transfer functions.

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1. INTRODUCTION

1.1. Background

Over the past few decades, Recurrent Neural Networks (RNNs) have got widespread attention. This is due to their versatile applications [6]. The applications of RNN include, but are not limited to modeling of nonlinear systems, and pattern recognition. RNN have dynamics, hence stability is an issue. We consider the following RNN:

$$
\begin{aligned}
x_1^{k+1} &= \phi(W_1 x_1^k + V_n x_n^k + b_1), \\
x_2^{k+1} &= \phi(W_2 x_2^k + V_1 x_1^{k+1} + b_2), \\
&\dots \\
x_n^{k+1} &= \phi(W_n x_n^k + V_{n-1} x_{n-1}^{k+1} + b_n),
\end{aligned} \tag{1.1}
$$

where n is the number of layers, x_j^k is the state vector of the layer j at time step k, W_j and V_j are fixed weight matrices, b_j is a fixed vector representing bias, and $\phi(\cdot)$ is a neuron transfer function. In most of the cases, $\phi(\cdot)$ is a smooth bounded nonlinear function. It is easy to notice that system (1.1) has local as well as global feedback. The goal is to analyze the global asymptotic stability of the RNN described in system (1.1).

RNN may approximate (in Hausdorff metric) the right hand side of a nonlinear system over a compact set to an arbitrary desired accuracy, and it exhibits nice properties of nonlinear dynamical systems. Hence, the problem of finding a general stability criteria for RNN occurs to be equivalent to finding stability criteria for a nonlinear dynamical system.

1.2. Previous Approaches

One of the famous approaches to address the problem of stability of nonlinear systems is based on the second Lyapunov method (i.e. method of Lyapunov functions).

Theorem 1.2.1. *System $y^{k+1} = f(y^k)$ with $f(0) = 0$ is globally asymptotically stable iff there exists a continuous function $V : \mathbb{R}^n \to \mathbb{R}$ such that,*

(a) $V(0) = 0,$

(b) $V(y) > 0$ *for* $y \neq 0$,

(c) $\Delta V(y) < 0$ *for* $y \neq 0$, *and*

(d) $V(y) \to \infty$ *as* $\|y\| \to \infty$.

This function $V(\cdot, \cdot)$ is called a Lyapunov function. As a consequence, if we can find a Lyapunov function for a given system, then system is globally asymptotically stable. There is no known method to find a Lyapunov function for an arbitrary system. In the following sections, we will study few well known methods to check the existence of a Lyapunov function for system (1.1).

1.2.1. Theory of Absolute Stability

One of the most efficient methods to check the existence of a Lyapunov function is based on the theory of Absolute Stability. Before discussing the results from the theory of absolute stability, we introduce the concept of Local quadratic constraint.

Definition 1.2.2. Suppose that $F(\cdot, \cdot)$ is a quadratic form. Let $\phi(\cdot)$ be a function such that $F(y, \phi(y)) \geq 0$ for all y. Then, we say that $\phi(\cdot)$ satisfies local quadratic constraint with quadratic form F.

In order to be analyzed for stability using this theory, a system should be written in the automatic control form:

$$\begin{aligned} x^{k+1} &= Ax^k + B\psi^k, \\ \sigma^k &= Cx^k, \\ \psi_i^k &= \phi_i(\sigma_i^k), i = 1, \ldots, m, \end{aligned} \tag{1.2}$$

where, A, B, C are matrices of suitable size, $\psi^k = (\psi_1^k, \ldots, \psi_m^k)$ is the input vector at step k, $\sigma^k = (\sigma_1^k, \ldots, \sigma_m^k)$ is the output vector at step k, and $\{\phi_i(\cdot)\}_{i=1}^m$ are nonlinear functions satisfying some constraints. In addition, assume that F is a quadratic function such that $F(x^k, \psi^k) \geq 0$ for all k. (i.e. the system (1.2) satisfies a local quadratic constraint with quadratic form $F(\cdot, \cdot)$). The problem is to find the conditions for the global asymptotic stability of system (1.2)[1].

To solve this problem, the Lyapunov function approach has been used. Consider the function $V(x) = x^* Hx$, where H is a positive definite Hermitian matrix and $*$ denotes the conjugate transpose. We need to find necessary and sufficient conditions for the existence of a Hermitian positive definite matrix H, such that $V(\cdot)$ is a Lyapunov function.

Problem 1: Suppose that $F(\cdot,\cdot)$ is a quadratic function. What are the necessary and sufficient conditions for the existence of a positive definite and Hermitian matrix H such that,

$$(Ax + B\psi)^* H(A + B\psi) - (x)^* Hx < 0 \tag{1.3}$$

for all nonzero pair of vectors (x, ψ), such that $F(x, \psi) \geq 0$?

It is easy to see that the left hand side of inequality (1.3) is the increment of the function $V(\cdot)$. Hence if there exists a positive definite matrix H such that inequality (1.3) holds true, then $V(\cdot)$ is a Lyapunov function for system (1.2). As a consequence, system (1.2) is globally asymptotically stable. Next, we consider another problem.

Problem 2: Suppose that $F(\cdot,\cdot)$ is a quadratic function. What are the necessary and sufficient conditions for the existence of a positive definite and Hermitian matrix H such that,

$$(Ax + B\psi)^* H(Ax + B\psi) - (x)^* Hx + F(x, \psi) < 0 \tag{1.4}$$

for all nonzero pair of vectors (x, ψ)?

Using Dine's theorem ([11, 5]), problem 1 has a solution if and only if there exists solution to problem 2. Now we formulate necessary and sufficient conditions for the existence of matrix H, such that inequality (1.4) holds true.

Suppose that inequality (1.4) is true. We can extend the left hand side of inequality (1.4) to complex domain in such a way that the resulting form is Hermitian. In particular, every product $x\psi$ has to be replaced by $\mathbb{R}e(z^* w)$, where the complex variables z, and w replace the real variables x, and ψ. In this case, $G(z, w) = G(\mathbb{R}e(z), \mathbb{R}(w)) + G(\mathbb{I}m(z), \mathbb{I}m(w))$, where $G(\cdot,\cdot)$ denotes the quadratic function on the left hand side of inequality (1.4). Therefore, if $G(x, \psi)$ is negative definite for all $(x, \psi) \neq 0$, then $G(z, w)$ is negative definite for all $(z, w) \neq 0$. We get

$$\mathbb{R}e((Az + Bw)^* H(Az + Bw) - z^* Hz + F(z, w)) < 0 \tag{1.5}$$

where $(w, z) \in \mathbb{C}^{n+m}/\{0\}$.

Pick $\omega \in [0, \pi]$ such that $e^{i\omega}$ is not an eigenvalue of matrix A. Consider a pair (z, w) such that $Az + Bw = e^{iw} z$. Then $z = (e^{iw} I - A)^{-1} Bw$. Using (1.5) we obtain, $\mathbb{R}e((e^{i\omega} z)^* H(e^{i\omega} z) -$

$z^*Hz + F(z,w)) < 0$. This implies $\mathbb{R}e(z^*(e^{i\omega})^*e^{i\omega}Hz - z^*Hz + F(z,w)) < 0$. Using the fact that $(e^{i\omega})^*e^{i\omega} = 1$, we get $\mathbb{R}e(F(z,w)) < 0$. From $z = (e^{iw}I - A)^{-1}Bw$, it follows that

$$\mathbb{R}e\Big(F\Big((e^{i\omega}I - A)^{-1}Bw, w\Big)\Big) < 0 \tag{1.6}$$

for all $\omega \in [0,\pi]$, and non zero $w \in \mathbb{C}^m$.

The inequality (1.6) is known as the Frequency Domain Inequality ([2],[10]). It represents the necessary condition for existence of a matrix H such that inequality (1.4) is satisfied.

Next, we present the sufficient conditions for existence of a matrix H.

Definition 1.2.3. A matrix pair (A,B) is called stabilizable if there exists a constant matrix L such that all the eigenvalues of matrix $(A + LB)$ lie inside unit circle(i.e. the matrix $A + BL$ is stable).

Assume the pair (A, B) is stabilizable. Using the Kalman Szegö lemma [8], we obtain that inequality (1.6) is also sufficient for existence of a positive definite matrix H.

Definition 1.2.4. System (1.2) is called minimally stable in the set of nonlinearities satisfying a local quadratic constraint with quadratic form $F(\cdot,\cdot)$ if there exists a constant matrix L such that $A + BL$ is stable and $F(x, Lx) \geq 0$ for all x.

The following lemma provides sufficient condition for the existence of a positive definite solution, H to inequality (1.4).

Lemma 1.2.5. *Suppose that system* (1.2) *is minimally stable. Moreover assume that inequality* (1.6) *holds. Then there exists* $H = H^* > 0$ *such that*

$$(Ax + B\psi)^*H(Ax + B\psi) - x^*Hx + F(x,\psi) < 0 \tag{1.7}$$

for all $(x,\psi) \neq 0$.

In many cases, the condition of minimal stability can be checked easily. Therefore, the Frequency Domain Inequality is a necessary as well as a sufficient condition for the existence of a matrix $H = H^* > 0$ such that (1.7) is satisfied. The relation in (1.7) is equivalent to a Linear

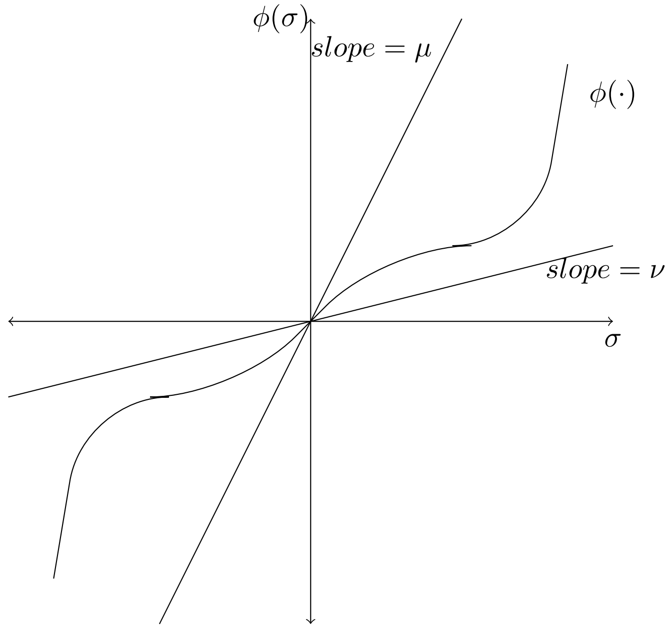

Figure 1.1. Sector Constraint

Matrix Inequality, which can be solved for matrix H using the standard LMI toolbox in MATLAB [9].

Next, we will introduce a particular type of frequency domain inequality. One of the most important forms of function $F(\cdot,\cdot)$ is the sector quadratic form. It utilizes the fact that the plots of function $\phi_i(\cdot,\cdot)$ lies in some sector $[\nu_i, \mu_i]$;

$$\nu_i \leq \frac{\phi_i(\sigma, t)}{\sigma} \leq \mu_i \tag{1.8}$$

for all (σ, t), and for all $i \in \{1 \dots m\}$. In this case, $\phi(0, t) = 0$ for all t. See Figure 1.1.

The inequality (1.8) may be rewritten as $F_i(x^k, \psi_i^k) = (\psi_i^k - \nu_i C x^k)(\mu_i C x^k - \psi_i^k) \geq 0$, where $\psi_i^k = \phi_i(\sigma_i^k), \sigma^k = Cx^k$. This implies that the function $\phi(\cdot,\cdot)$ satisfies a local quadratic constraint with quadratic form $F(\cdot,\cdot)$. For a multi input multi output system, the quadratic function $F(\cdot,\cdot)$ can be expressed as

$$F = \sum_{j=1}^{m} \tau_j F_j \tag{1.9}$$

where τ_j are positive numbers.

The frequency domain condition, that guarantees stability of system (1.2) with nonlinear functions $\phi_i(\cdot,\cdot)$ satisfying the local quadratic constraint with quadratic form $F_i(\cdot,\cdot)$ for all $i \in$

$\{1 \dots m\}$, is called the Circle criterion.

If $\nu_i = 0$, for all $i \in \{1 \dots m\}$, the Circle inequality is given by

$$\mathbb{R}e(\Gamma(W(e^{i\omega}) + M)) > 0 \tag{1.10}$$

for all $\omega \in [0, \pi]$. Here, $W(e^{i\omega}) = C(A - e^{i\omega} I)^{-1} B$ is the Transfer function matrix, $\Gamma = \text{diag}(\tau_j)_{j=1}^m$, and $M = \text{diag}(\mu_j^{-1})_{j=1}^m$.

In order to apply the result from the theory of Absolute stability, the discrete time dynamical system should be expressed in automatic control form, see (1.2). But in system (1.1), it is easy to notice that the right hand side depends on the value of state vector at time step $k+1$. Before we analyze the stability of system (1.1) using the theory of Absolute stability, it needs to be transformed to automatic control form.

1.2.1.1. System Transformation

In this section, we will show the State space extension method to transform system (1.1) to automatic control form [3]. We will illustrate this method using a simple example. Consider a two layer Recurrent Neural Network described by

$$\begin{aligned} x_1^{k+1} &= \tanh(W_1 x_1^k + V_2 x_2^k + b_1) \\ x_2^{k+1} &= \tanh(W_2 x_2^k + V_1 x_1^{k+1} + b_2). \end{aligned} \tag{1.11}$$

where W_j, V_j denote constant weight matrices of suitable size, and constant vectors b_j are called bias vector; $j \in \{1, 2\}$.

The above system can be transformed into the following form

$$\begin{aligned} x_{11}^{k+1} &= \tanh(W_1 x_{12}^k + V_2 x_{21}^k + b_1), \\ x_{12}^{k+1} &= x_{11}^k, \\ x_{21}^{k+1} &= \tanh(W_2 x_{22}^k + V_1 x_{11}^k + b_2), \\ x_{22}^{k+1} &= x_{21}^k. \end{aligned} \tag{1.12}$$

It is easy to notice that the right hand side of system (1.12) is independent of value of the state vector at step $k+1$. The system (1.12) is now in automatic control form described by

$$x^{k+1} = Ax^k + B\psi^k, \psi_j^k = \tanh(\sigma_j^k)$$
$$\sigma^k = \Theta x^k + b. \tag{1.13}$$

where $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}$, $B = \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}$, $x = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{bmatrix}$, $\Theta = \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, $\Theta_1 = [0, W_1, V_2, 0]$, $\Theta_2 = [V_1, 0, 0, W_2]$, and $j \in \{1, 2\}$.

It can be shown that system (1.12) can be transformed into two independent processes, and each process is a counterpart of original system (1.11). For the first counterpart, the set of equations defined by

$$x_{12}^{k+1} = x_{11}^k, x_{21}^{k+1} = \tanh(W_2 x_{22}^k + V_1 x_{11}^k + b_2). \tag{1.14}$$

can be rewritten as

$$x_{12}^{k+2} = \tanh(W_1 x_{12}^k + V_2 x_{21}^k + b_1),$$
$$x_{21}^{k+2} = \tanh(W_2 x_{21}^k + V_1 \tanh(W_1 x_{12}^k + V_2 x_{21}^k + b_1) + b_2). \tag{1.15}$$

Now, we will show that one time step of system (1.11) is identical to two time steps of the process defined by (1.14).

Re-indexing the above equations we get

$$x_{12}^{l+2} = \tanh(W_1 x_{12}^l + V_2 x_{21}^l + b_1),$$
$$x_{21}^{l+2} = \tanh(W_2 x_{21}^l + V_1(\tanh(W_1 x_{12}^l + V_2 x_{21}^l + b_1) + b_2). \tag{1.16}$$

Denoting $x_{12}^l := x_1^k$, and $x_{21}^l := x_2^k$, we obtain $x_{12}^{l+2} = \tanh(W_1 x_1^k + V_2 x_2^k + b_1) = x_1^{k+1}$, and $x_{21}^{l+2} = \tanh(W_2 x_2^k + V_1 x_1^{k+1} + b_2) = x_2^{k+1}$.

For second counterpart, consider the process defined by

$$x_{11}^{k+1} = \tanh(W_1 x_{12}^k + V_2 x_{21}^k + b_1), x_{22}^{k+1} = x_{21}^k. \tag{1.17}$$

Using (1.12), we obtain

$$\begin{aligned} x_{22}^{k+2} &= \tanh(W_2 x_{22}^k + V_1 x_{11}^k + b_2), \\ x_{11}^{k+2} &= \tanh(W_1 x_{11}^k + V_2 \tanh(W_2 x_{22}^k + V_1 x_{11}^k + b_2) + b_1) \end{aligned} \tag{1.18}$$

It can be checked that one time step of (1.11) corresponds to two time steps of process (1.18).

To recapitulate, system (1.11) can be represented in automatic Control form, (1.12). System (1.12) can be decomposed into two independent processes, which are counterparts of the original system (1.11). System (1.12) is stable, if and only if both the processes are stable. Since the processes are equivalent to the original system, system (1.11) is also stable.

In the next section, we will use more information about the nonlinear function $\phi(\cdot)$ to develop an improved stability criterion.

1.2.1.2. Monotonicity Approach

We showed earlier that system (1.2) is globally asymptotically stable if there exists $H = H^* > 0$ such that

$$(Ax^k + B\psi^k)^* H (Ax^k + B\psi^k) - (x^k)^* H (x^k) + \sum_{j=1}^{m} \tau_j F_j < 0 \tag{1.19}$$

for all $(x^k, \psi^k) \neq 0$, where $\tau_j > 0$, and $F_j(x^k, \psi^k) = (\psi_j^k - \nu_j C x^k)(\mu_j C x^k - \psi_j^k) \geq 0$ for all $j \in \{1 \dots m\}$. The inequality (1.19) is a Linear Matrix Inequality and can be solved using LMI toolbox in MATLAB for the matrix H and coefficients τ_j.

The circle criterion gives a sufficient condition for stability of nonlinear systems, with a nonlinear function $\phi(\cdot)$ satisfying the sector constraint. It only utilizes the fact that the nonlinear function $\phi(\cdot)$ satisfies a sector condition. It might happen that given a sector, defined by function

$\phi(\cdot)$, there exists a nonlinear function satisfying the sector condition, such that the corresponding system is unstable. Additional information about the nonlinear function $\phi(\cdot)$, can be used to check stability of nonlinear systems of particular kind, for example RNN. A modified stability criterion using additional information about the nonlinear function,(e.g. monotonicity) has been developed in [3]. One of the most common nonlinear functions used in RNN is $\tanh(\cdot)$. To improve the circle criterion, we use the fact that $\frac{d}{ds}(\tanh(s)) \in [0,1]$.

We illustrate this approach using an example of a single layer RNN. Consider the RNN defined by

$$x^{k+1} = \tanh(Wx^k) \tag{1.20}$$

where x^k defines the state vector of size m at time step k and W is the weight matrix of suitable size. We use the bounds on derivative of $\tanh(\cdot)$ to construct the quadratic function $F(\cdot,\cdot)$.

Denote $s := Wx$. Then $s_j = W_j x$, where W_j is the j th row of matrix W, and $j \in \{1,\ldots,m\}$. If $s_i < s_j$ then using mean value theorem, we get $0 \leq \frac{\tanh(s_j)-\tanh(s_i)}{s_j - s_i} \leq 1$. This implies $(W_j x - W_i x) - (\psi_j - \psi_i) \geq 0$, where $\psi := \tanh(s)$, and ψ_j is the j th row of vector $\psi(s)$.

Since $\tanh(\cdot)$ is a monotonically increasing function and $s_i < s_j$, we get $\psi_i < \psi_j$. Combining the above inequalities we get

$$(\psi_j - \psi_i)((W_j x - W_i x) - (\psi_j - \psi_i)) \geq 0 \tag{1.21}$$

for all $i, j \in \{1,\ldots,m\}$.

Since $\tanh(\cdot)$ satisfies the sector condition we obtain $\tanh(s_i)(s_i - \tanh(s_i)) \geq 0$ for all $i \in \{1,\ldots,m\}$. Therefore,

$$\psi_i(W_i x - \psi_i) \geq 0 \tag{1.22}$$

for all $i \in \{1,\ldots,m\}$.

We will use the equations (1.21) and (1.22) to construct the quadratic function $F(\cdot,\cdot)$. To this end, assume there exists a symmetric matrix $\Gamma = \{\gamma_{jk}\}_{j,k=1}^m$ such that $\gamma_{ij} < 0$ for all $i \neq j$ and $\sum_{k=1}^m \gamma_{jk} > 0$ for all $j \in \{1,\ldots,m\}$. This implies

$$\sum_{j=1}^m \psi_j(W_j x - \psi_j) \sum_{k=1}^m \gamma_{jk} \geq 0. \tag{1.23}$$

Combining the left hand side of inequality (1.21) with non negative weights, $-\gamma_{ij}, i \neq j$ and the left hand side of inequality (1.23) we get

$$\begin{aligned}
&\sum_{i=1}^{m}\Big(\sum_{j=1}^{m}(-\gamma_{ij})(\psi_j-\psi_i)((W_jx-W_ix)-(\psi_j-\psi_i))\Big)+\sum_{j=1}^{m}\psi_j(W_jx-\psi_j)\sum_{k=1}^{m}\gamma_{jk}\\
&=\sum_{i,j=1}^{m}\gamma_{ij}\Big((\psi_j-\psi_i)(W_ix-W_jx)+(\psi_j-\psi_i)^2\Big)+\sum_{j=1}^{m}\psi_j(W_jx-\psi_j)(\gamma_{j1}+\gamma_{j2}+\ldots+\gamma_{jm}).
\end{aligned} \tag{1.24}$$

Since Γ is a symmetric matrix, it can be checked that

$$\begin{aligned}
&\sum_{j=1}^{m}\psi_j(W_jx-\psi_j)(\gamma_{j1}+\gamma_{j2}+\ldots+\gamma_{jm})\\
&=\sum_{j=1}^{m}\psi_j(W_jx-\psi_j)\gamma_{jj}+\sum_{i=1}^{m}\Big(\sum_{j=1,j\neq i}^{m}\gamma_{ij}(\psi_i(W_ix-\psi_i)+\psi_j(W_jx-\psi_j))\Big).
\end{aligned} \tag{1.25}$$

At the same time,

$$\begin{aligned}
&\sum_{i=1}^{m}\Big(\sum_{j=1}^{m}\gamma_{ij}\Big((\psi_j-\psi_i)(W_ix-W_jx)+(\psi_j-\psi_i)^2\Big)\Big)\\
&=\sum_{i=1}^{m}\Big(\sum_{j=1,j\neq i}^{m}\gamma_{ij}\Big(\psi_j(\psi_j-W_jx)+\psi_i(\psi_i-W_ix)+(\psi_jW_ix+\psi_iW_jx)-2\psi_i\psi_j\Big)\Big).
\end{aligned} \tag{1.26}$$

Adding the right hand side of equations (1.25) and (1.26) we obtain

$$\begin{aligned}
&\sum_{j=1}^{m}\psi_j(W_jx-\psi_j)\gamma_{jj}+\sum_{i=1}^{m}\Big(\sum_{j=1,j\neq i}^{m}\Big(\gamma_{ij}(\psi_jW_ix+\psi_iW_jx)\Big)\Big)-2\sum_{i=1}^{m}\Big(\sum_{j=1,j\neq i}^{m}\Big(\gamma_{ij}\psi_j\psi_i\Big)\Big)\\
&=\sum_{j=1}^{m}\psi_j(W_jx-\psi_j)\gamma_{jj}+\sum_{i=1}^{m}\Big(\sum_{j=1,j\neq i}^{m}\Big(\gamma_{ij}\psi_j(W_ix-\psi_i)\Big)\Big)+\sum_{i=1}^{m}\Big(\sum_{j=1,j\neq i}^{m}\Big(\gamma_{ij}\psi_i(W_jx-\psi_j)\Big)\Big)\\
&=\sum_{j=1}^{m}\psi_j(W_jx-\psi_j)\gamma_{jj}+\sum_{i=1}^{m}\Big(\sum_{j=1,j\neq i}^{m}\Big(\gamma_{ij}(\psi_j(W_ix-\psi_i)+\psi_i(W_jx-\psi_j))\Big)\Big)\\
&=\psi^*\Gamma(Wx-\psi).
\end{aligned} \tag{1.27}$$

Hence, the quadratic function is given by $F(x,\psi)=\psi^*\Gamma(Wx-\psi)$.

The system (1.20) is globally asymptotically stable if there exists a positive definite Hermitian matrix H and a symmetric matrix Γ such that,

(i) $\psi^* H \psi - x^* H x + \psi^* \Gamma(Wx - \psi) < 0$ holds true for all $(x, \psi) \neq 0$, where $\psi(\cdot) = \tanh(\cdot)$, and

(ii) $\gamma_{jk} < 0$, when $j \neq k$, and $\sum_{k=1}^{m} \gamma_{jk} > 0$ for all $j \in \{1, \ldots, m\}$.

This condition is the same as saying that Γ is a diagonal row dominant matrix. The LMI in (i) can be solved for H and Γ using MATLAB toolbox.

In the next section, we will address the stability issue with non zero bias. It covers a broader group of RNN's.

1.2.1.3. Accounting for Nonzero Biases

It has been seen that solving stability problem with non zero bias covers a larger class of systems and it is more practical. Moreover, ignoring the bias limits the effectiveness of stability criteria. It has been shown in [3], that the stability criterion for case of RNN with non zero bias can be developed in an identical way as was done for the case of zero bias.

Consider a RNN with nonzero bias defined by

$$\begin{aligned} &x^{k+1} = Ax^k + B\psi^k, \sigma^k = \Theta x^k + b, \\ &\psi^k = \tanh(\sigma^k). \end{aligned} \tag{1.28}$$

Notice that if $b = 0$, then $x = 0$ is the equilibrium point. But for a non zero bias the equilibrium point is not at the origin. Let $z \neq 0$ be its equilibrium point. Since $z^{k+1} = z^k$, we get $z = Az + B\tanh(\Theta z + b)$. Denote $\Theta z + b := c$.

Next, we will transform system (1.28) into a system with zero bias. Consider the affine transformation, $y = x - z$. Then $y^{k+1} = x^{k+1} - z^{k+1} = A(x^k - z^{k+1}) + B(\psi^k - \tanh c)$. Denoting $\eta^k := \psi^k - \tanh c$, we get

$$y^{k+1} = Ay^k + B\eta^k. \tag{1.29}$$

Moreover, $\sigma_1^k = \Theta y^k = \Theta x^k - \Theta z$. Add c to both sides: $\sigma_1^k + c = \Theta x^k + b$. This implies $\tanh(\sigma_1^k + c) = \tanh(\sigma^k) = \psi^k$. Hence $\psi^k = \tanh(\sigma_1^k + c)$.

We obtain the new system defined by

$$y^{k+1} = Ay^k + B\eta^k, \quad \eta^k = \psi^k - \tanh c, \quad \sigma_1^k = \Theta y^k. \tag{1.30}$$

It can be easily checked that for system (1.30), $y = 0$ is the equilibrium point. Using the relation $\psi^k = \tanh(\sigma_1^k + c)$, we get $\eta^k = \tanh(\sigma_1^k + c) - \tanh c$. The function $\eta(\cdot)$ is still monotonic and satisfies a sector condition with new bounds. Next we find the new sector bounds for the function $\eta(\cdot)$.

First, we find the new upper bound for the sector, where the plot of the function $\eta(s) = \tanh(s + c) - \tanh(c)$ lies. It is equivalent to finding the upper bound for the function $\frac{\eta(s)}{s}$, where $s \neq 0$. To this end, we will compute the critical point for the function $\frac{\eta(s)}{s}$. Since (1.30) is a multi input multi output system, we need to compute $\mu_j := \max\{\frac{\eta_j(s)}{s}, s \neq 0\}$, where $j \in \{1 \dots m\}$.

Hence it is sufficient to find a root of the function $s\text{sech}^2(s + c_j) - (\tanh(s + c_j) - \tanh(c_j))$. It can be done using the MATLAB's fzero function. So, given c_j, we can find the upper bound, μ_j. It has been seen that due to the nonzero biases, the upper bound of the sector is less than unity.

Denote $M = \text{diag}\{\mu_j\}_{j=1}^m$ where $\mu_j < 1$. Then M defines the matrix of new upper bounds, and the quadratic function $F(\cdot, \cdot)$ is given by $F(y, \eta) = \eta^*\Gamma(M\Theta y - \eta)$, where Γ is any positive definite diagonal matrix. Hence system (1.30) is globally asymptotically stable if there exists a matrix $H = H^* > 0$ and a positive definite diagonal matrix Γ such that

$$(Ay^k + B\eta^k)^* H(Ay^k + B\eta^k) - (y^k)^* H(y^k) + (\eta^k)^*\Gamma(M\Theta y^k - \eta^k) < 0 \tag{1.31}$$

for all $(y^k, \eta^k) \neq 0$.

Next, we find the lower bound of the sector, where the function $\eta_j(s)$ lies. It has been seen that when the bias $b = 0$, the lower bound is zero. Due tothe boundedness of $\tanh(\cdot)$, the vector x^1 belongs to the interval $[-r_j - c_j, r_j - c_j]$, which can be calculated. The following lemma gives the lower bound for the sectors, where the plot of nonlinear function $\phi_j, j \in \{1 \dots m\}$ lies [3].

Lemma 1.2.6. *If* $|s + c_j| \leq r_j$, *then* $\frac{\eta_j(s)}{s} \geq \nu_j := \frac{\tanh(r_j) - \tanh(|c_j|)}{r_j - |c_j|}$.

Denote $N = \text{diag}\{\nu_j\}_{j=1}^m$. For an arbitrary positive definite diagonal matrix Γ, we obtain a local quadratic constraint with quadratic form $F(y^k, \eta^k) = (\eta^k - N\Theta y^k)^*\Gamma(M\Theta y^k - \eta^k)$. Then we will establish the stability criterion as before.

To recapitulate, the theory of Absolute Stability gives us necessary and sufficient conditions for the existence of quadratic Lyapunov functions in the set of nonlinear functions satisfying local quadratic constraint with quadratic form $F(\cdot,\cdot)$. This quadratic function $F(\cdot,\cdot)$ can be constructed using the properties of the nonlinear function $\phi(\cdot)$.

Next, we will discuss the stability criteria developed using NL_q approach [7].

1.2.2. NL_q Approach

A typical NL_q system is of the form,

$$p_{k+1} = P_1(Q_1 P_2(Q_2 \ldots P_q(Q_q p_k + B_q w_k) \ldots + B_2 w_k) + B_1 w_k) \tag{1.32}$$

where $p_k \in \mathbb{R}^n$, w_k is the input vector, P_i is a diagonal matrix, and Q_i is a constant matrix.

Assume $P_i = \text{diag}(\overline{p_j})_{j=1}^n$, where the values of $\overline{p_j}$ depend on p_k continuously. Assume $\|P_i\| \leq 1$, where the matrix norm is defined by $\|P_i\|_1 = \max_{1\leq j\leq n}(\overline{p_j}(p_k))$. It is easy to see that in system(1.32) there is an alternating sequence of linear and nonlinear operations. Hence it is called NL_q. Here the index q refers to number of alternating operations.

To analyze the stability of above system, set all the external inputs to zero. We consider

$$p_{k+1} = P_1 Q_1 P_2 Q_2 \ldots P_q Q_q p_k \tag{1.33}$$

which is written in automatic control form. The problem under consideration is to analyze the stability of system (1.33), with all matrices P_i such that $\|P_i\| \leq 1$.

First, the system (1.1) will be transformed into the form (1.33). For the case of $q = 3$, we obtain

$$x_{k+1} = P_3 \cdot Q_3 \cdot P_2 \cdot Q_2 \cdot P_1 \cdot Q_1 x_k, \tag{1.34}$$

where $P_1 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & f_1 \end{bmatrix}$, $Q_1 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & V_2 & W_3 \end{bmatrix}$, $P_2 = \begin{bmatrix} I & 0 & 0 \\ 0 & f_2 & 0 \\ 0 & 0 & I \end{bmatrix}$, $Q_2 = \begin{bmatrix} I & 0 & 0 \\ V_1 & W_2 & 0 \\ 0 & 0 & I \end{bmatrix}$,
$P_3 = \begin{bmatrix} f_3 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$, $Q_3 = \begin{bmatrix} W_1 & 0 & V_3 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$, $f(s) = \tanh(s)$, and $x^k = \begin{bmatrix} x_1^k \\ x_2^k \\ x_3^k \end{bmatrix}$.

Similarly, we can transform the general form to NL_q form.

The stability criterion [7] says that if there exist diagonal positive definite matrices D_j such that $\|D_j Q_j D_{j+1}^{-1}\| < 1$ for all $j = 1, \ldots, q (mod\ q)$, then system (1.34)(and hence (1.1)) is stable.

But we saw earlier that the theory of Absolute stability guarantees the existence of arbitrary positive definite matrix H, such that $x^* H x$ is a Lyapunov function. Hence it is more general as compared to NL_q theory, which requires the existence of a particular type of such matrix, namely a diagonal matrix.

In the next section, we go over an example of a globally asymptotically stable system, for which the circle inequality (1.10) does not hold true. It will help us deduce that the circle criterion gives essentially sufficient conditions for stability of a system.

1.3. Example

Consider a two dimensional system defined by

$$x_{k+1} = \tanh(W x_k) \tag{1.35}$$

where $W = \begin{bmatrix} 1.80 & 0.95 \\ -0.95 & 0.00 \end{bmatrix}$ is the weight matrix. Comparing this system with the standard automatic control form (1.2), we obtain $\varphi(\cdot) = \tanh(\cdot), A = 0, B = I$, and $C = W$.

We show that, for the system (1.35), the Circle inequality (1.10) does not hold true. To this end, we compute the transfer function matrix for the above system. Notice that, the transfer function matrix is given by $\tilde{W}(e^{i\omega}) = C(A - e^{i\omega} I)^{-1} B$. We get $\tilde{W}(e^{i\omega}) = e^{-i\omega} \begin{bmatrix} -1.80 & -0.95 \\ 0.95 & 0.00 \end{bmatrix}$.

Hence we get

$$\mathbb{R}e(\Gamma\tilde{W}(e^{i\omega}) + \Gamma I) = \frac{(\Gamma\tilde{W}(e^{i\omega}) + \Gamma) + (\Gamma\tilde{W}(e^{i\omega}) + \Gamma)^*}{2}$$

$$= \begin{bmatrix} \tau_1(1 - 1.80\cos\omega) & \frac{0.95}{2}(e^{i\omega}\tau_2 - e^{-i\omega}\tau_1) \\ \frac{0.95}{2}(e^{-i\omega}\tau_2 - e^{i\omega}\tau_1) & \tau_2 \end{bmatrix} \tag{1.36}$$

where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ defines matrix of upper bounds and $\Gamma = \begin{bmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{bmatrix}$.

We get $\det\Big(\mathbb{R}e(\Gamma\tilde{W}(e^{i\omega}) + \Gamma)\Big) = (0.95)^2\tau_1\tau_2\cos^2\omega - 1.80\tau_1\tau_2\cos\omega + \tau_1\tau_2 - \frac{(0.95)^2}{4}(\tau_1 + \tau_2)^2$, where I defines the matrix of upper bounds. It can be easily checked that, for $\omega = \arccos(\frac{1}{1.8})$, the matrix $\mathbb{R}e(\Gamma\tilde{W}(e^{i\omega}) + \Gamma)$ is not positive definite, for every positive τ_1, τ_2. Further, in chapter 2, we will show that this system is globally asymptotically stable. This implies that Frequency Domain Criterion gives an essentially sufficient condition for stability of nonlinear systems.

1.4. Summary

The problem of stability of Recurrent Neural Network in discrete time has been introduced. There exist stable systems for which the criterion developed using theory of Absolute stability does not have feasible solution. Therefore, it is necessary to develop a more general stability criterion. This is the subject of discussion in next chapter.

2. NEW STABILITY APPROACH

2.1. Overview

In this chapter, we will introduce an alternative stability criterion [4]. This criterion has proved to be more powerful than the well known stability criteria studied in the previous chapter. We will first describe the method of implementation. Then we will talk about the computational issues involved in this approach.

2.2. Method of Reduction of Dissipativity Domain

Consider the system

$$x^{k+1} = f(x^k) \tag{2.1}$$

where $x^k \in \mathbb{R}^n$ is a state vector at time step k, and $f(\cdot)$ is a bounded nonlinear function.

Let D_0 denote a set containing image of $f(\cdot)$. The set D_0 can be defined using the bounds of the function $f = \text{column}\{f_i\}_{i=1}^n$. Suppose there exists sets $\{D_k\}$ such that $D_{k+1} \subset D_k$,$f(D_k) \subset D_{k+1}$. Then, $x^k \in D_k$ if $x^0 \in D_0$. Thus, if $\{D_k\} \to 0$(in Hausdorff metric), then $\{x^k\} \to 0$, as $k \to \infty$, and system (2.1) is globally asymptotically stable. This approach is known as the method of reduction of dissipativity domain.

In order to implement this approach, the sets D_k need to be defined. Consider a set of functions $\{h_1, \ldots, h_{m_k}\}$, $h_j : \mathbb{R}^n \to \mathbb{R}$. Define $D_{k+1} := \{x \in D_k : h_j(x) \leq \alpha_{k+1,j}; j = 1, \ldots, m_{k+1}\}$, where m_k is number of constraints a time step k, and $\alpha_{k+1,j} = \max_{x \in D_k} h_j(f(x))$. Increase k and repeat the procedure.

First we will use induction to show that $f(D_k) \subset D_k$. For $k = 0$, $f(D_0) \subset D_0$ using definition of D_0. Assume that $f(D_k) \subset D_k$. We need to show that $f(D_{k+1}) \subset D_{k+1}$.

Suppose $y \in f(D_{k+1})$. We need to check that $y \in D_{k+1}$. Using the definition of the sets D_k it is easy to see that $D_{k+1} \subset D_k$. Therefore $f(D_{k+1}) \subset f(D_k)$. Using the inductive hypothesis we get $f(D_{k+1}) \subset D_k$. Therefore $y \in D_k$. Since $y \in f(D_{k+1})$ we get $y = f(z)$ for some $z \in D_{k+1} \subset D_k$. We get $h_j(y) = h_j(f(z)) \leq \max_{x \in D_k} h_j(f(x))$. This implies that $h_j(y) \leq \alpha_{k+1,j}$, which in turn gives that $y \in D_{k+1}$.

The process of constructing sets D_k is continued until the sequence $\{D_k\}$ tends to zero (in the Hausdorff metric) or the sequence stabilizes far from the origin. If the first possibility occurs, we can use stability by first approximation to check stability of the system. The second case takes place if the system is not globally asymptotically stable.

The sets D_k can be chosen as intersection of sets $\{x : h_j(x) \leq \gamma_{k,j}\}$ where $j \in \{1, \ldots, m_k\}$. Then the pairs $(h_j, \gamma_{k,j})_{j=1}^{m_k}$ define the set D_k. In order to define the set D_{k+1}, a new set of pairs $(h_j, \gamma_{k+1,j})$ need to be defined. It is beneficial to choose new function $h(\cdot)$ among the original set of functions $\{h_j\}_{j=1}^{m_k}$. This helps to prevent increase in the number of pairs (h, γ) used to define D_{k+1}. However if the sequence $\{D_k\}$ stabilizes far from the origin, then it is necessary to add new constraints to original set $\{h_j\}_{j=1}^{m_k}$. If we are unable to find new functions such that $D_{k+1} \subset D_k$ then we conclude that system (2.1) does not have a Lyapunov function, hence is unstable,[4]. Next we will discuss how to choose new functions h.

Recall that $D_k = \{x : h_j(x) \leq \gamma_{k,j}, j \in 1, \ldots, m_k\}$. Suppose that the sets $(h_j)_{j=1}^{m_k}$ and $(h_j)_{j=1}^{m_{k+1}}$ coincide, while the difference between $\gamma_{k,j}$ and $\gamma_{k+1,j}$ is sufficiently small for all $j \in \{1, \ldots, m_k\}$. Then the sets D_{k+1} and D_k are almost identical. Under these circumstances, we need to introduce new functions h. Since D_{k+1} and D_k coincide, for all $j \in \{1, \ldots, m_k\}$ there exists $a_j \in D_k$ such that difference between $h_j(a_j)$ and $\gamma_{k,j}$ is very small. Define $A = \{a_j : |h_j(a_j) - \gamma_{k,j}| < \varepsilon$ for all $j\}$ for sufficiently small $\varepsilon > 0$. Denote $B = f(A)$. If $B = A$ then we conclude that system (2.1) is unstable.

Assume that $B \neq A$. Take $x_0 \in A \setminus B$. Fix a function h such that $h(x_0) > \max_{y \in B}(h(y))$. Then $\gamma = \max_{z \in A}(h(f(z)) < h(x_0)$. Therefore $D_{k+1} = \{y \in D_k : h(y) < \gamma\}$ does not contain x_0 and hence $D_{k+1} \subset D_k$. In order to describe the set D_{k+1}, we add the function h to the set of functions $\{h_j\}_{j=1}^{m_k}$ to get the new set $\{h_j\}_{j=1}^{m_{k+1}}$, where $m_k + 1 = m_{k+1}$.

We can see that the main (and only) difficulty in implementation of RDD method is computing value of $\alpha_{k,j} = \max_{x \in D_k} h_j(f(x))$ for $j \in \{1, \ldots, m\}$. The complexity increases if the function $h_j(f)$ has several points of local maxima. It has been shown in [4] that computational complexity can be reduced if each $h(\cdot)$ can be chosen to be a linear function. This is the next subject of discussion.

2.3. Linear Constraints and Convex Lyapunov Functions

It has been seen that the vast majority of stability criteria check for existence of quadratic Lyapunov functions [4]. This new approach uses linear constraints h_j to test the existence of convex Lyapunov function.

Theorem 2.3.1. *Assume that system* (2.1) *has a convex Lyapunov function,* V*. Then for any time step* k*, there exists a linear function* h_k *such that* $D_{k+1} = \{x \in D_k : h_k(x) \leq \max\{h_k(f(y)) : y \in D_k\}\}$ *is a strict subset of* D_k*. In addition, for all* $\varepsilon > 0$ *there exists* $\delta > 0$ *and a linear function* h_k *such that, if the set* D_k *is not contained in the ball* $B(0, \varepsilon)$*, then there exists a point of set* D_k *such that distance of that point to set* D_{k+1} *is greater than* δ*.*

Proof. For any nonzero vector x, define $E(x) = \{y \in \mathbb{R}^n : V(y) \leq V(x)\}$. Since $V(\cdot)$ is a Lyapunov function we get $V(f(x)) < V(x)$. Pick $\varepsilon > 0$. Consider $x_1 \in D_0 \setminus B(0, \varepsilon)$, where D_0 is defined using the bounds of function $f(\cdot)$, and $B(0, \varepsilon) := \{x : \|x\| < \varepsilon\}$. Define $E(x_1) = \{y \in \mathbb{R}^n : V(y) \leq V(x_1)\}$. For the sake of brevity, we will denote $E(x_1)$ as E. We will show that there exists $\delta > 0$ and a unit vector z such that $z^T x_1 - \delta \geq \max\{z^T y : y \in \text{conv}(f(E))\}$, where $\text{conv}(f(E))$ denotes the convex hull of set $f(E)$.

To this end, we first show that $f(E) \subset E$. Let $y_1 = f(y) \in f(E)$. Then $y \in E$. We obtain $V(y_1) = V(f(y)) < V(y) \leq V(x_1)$. Therefore using the definition of E we obtain $f(y) \in E$ which in turn implies that $f(E) \subset E$. Next we will check that $f(E)$ has an empty intersection with ∂E.

Suppose $f(y) \in f(E) \cap \partial E$, where $y \in E$. At the same time there exists $\{y_n\}_{n=1}^{\infty} \nsubseteq E$ such that $\|y_n - f(y)\| \to 0$ as $n \to \infty$. Since $\{y_n\}_{n=1}^{\infty} \nsubseteq E$, we get $V(y_n) > V(x_1)$ for all n. Since $V(\cdot)$ is a continuous function we obtain $V(f(y)) \geq V(x_1)$. But we know that $V(f(y)) < V(y) \leq V(x_1)$. Hence contradiction.

Now we check that $x_1 \notin \text{conv}(f(E))$. Define $D := \{y : V(y) \leq \max_{y_1 \in E} V(f(y_1))\}$. Since $V(\cdot)$ is a convex function, it can be easily checked that D is a convex set. We need to show that $x_1 \notin \text{conv}(f(E))$. To this end, we will first show that $D \subset E$. Pick $z \in D$. Then we obtain $V(z) \leq \max_{y_1 \in E} V(f(y_1)) < \max_{y_1 \in E} V(y_1) \leq V(x_1)$. Therefore we get $V(z) < V(x_1)$. Using the definition of E we obtain that $D \subset E$.

Next we will show that $f(E) \subset D$. Pick $z \in f(E)$. This implies that $z = f(y)$, for some $y \in E$. It is easy to see that $V(z) = V(f(y)) \leq \max_{y_1 \in E} V(f(y_1))$. Therefore $z \in D$. Since D is convex, we get $\mathrm{conv}(f(E)) \subset D$. Now it is sufficient to show that $x_1 \notin D$. Suppose $x_1 \in D$. Then we obtain $V(x_1) \leq \max_{y_1 \in E} V(f(y_1)) < \max_{y_1 \in E} V(y_1) \leq V(x_1)$. Hence contradiction. Therefore $x_1 \notin \mathrm{conv}(f(E))$.

Since $V(\cdot)$ is a continuous function, E is a closed set. In addition, using an open covering argument we obtain that $f(E)$ is also a closed set. Since $f(E) \subset E$, we get that $f(E)$ is a closed and bounded set. Therefore, by Cartheodory theorem $\mathrm{conv}(f(E))$ is a closed and convex set. We have seen above that $x_1 \notin \mathrm{conv}(f(E))$. Then, using separation principle there exists $z \in \mathbb{R}^n \setminus \{0\}$ such that $z^T x_1 - \delta \geq \max\{z^T y : y \in \mathrm{conv}(f(E))\}$ for some $\delta > 0$.

Next, suppose $D_k \subsetneq B(0, \varepsilon)$ where ε is the same as above. Then we can choose $\varepsilon_1 > 0$ sufficiently small such that there exists $x_0 \in D_k \setminus B(0, \varepsilon_1)$ satisfying $V(x_0) = \max_{y \in D_k} V(y)$. Define $E(x_0) = \{x \in \mathbb{R}^n : V(x) \leq V(x_0)\}$. It is easy to see that $D_k \subset E$. Then using the above argument there exists $z_1 \in \mathbb{R}^n \setminus \{0\}$ such that $z_1^T x_0 - \delta_1 \geq \max\{z_1^T y : y \in \mathrm{conv}(f(E))\}$ for some $\delta_1 > 0$. It can be easily checked that $B(x_0, \delta_1) \cap \mathrm{conv}(f(E)) = \emptyset$.

We will define the linear function h_k as $h_k(x) = z_1^T x$. From the above discussion we obtain that $h_k(x_0) > \max_{y \in \mathrm{conv}(f(E))} h_k(y)$. Moreover, $f(D_k) \subset \mathrm{conv}(f(E))$ gives us $\max_{y \in f(D_k)} z_1^T y \leq \max_{y_1 \in \mathrm{conv}(f(E))} z_1^T y_1$. Therefore using the definition of D_{k+1} we conclude that $x_0 \in D_k \setminus D_{k+1}$. □

Using theorem 2.3.1, we obtain that if system (2.1) has a convex Lyapunov function then there exists a set of linear functions $\{h_j\}_{j=1}^{m_k}$ such that $\{D_k\} \to 0$ (in Hausdorff metric), as $k \to \infty$. Therefore if the system has a convex Lyapunov function, then it is sufficient to restrict our attention to set of linear functions in order to implement the method of reduction of dissipitavity domain.

2.4. Creating Linear Constraints for One Layer RNN

We consider the application of general procedure to single layer RNN, with zero bias. A single layer RNN with zero bias can be described by the following equation;

$$x^{k+1} = W\phi(x^k) \tag{2.2}$$

where W is a $n \times n$ matrix, x^k is the state vector at time step k, and $\phi(\cdot)$ is a neuron activation function.

For the case of zero bias and odd function $\phi(\cdot)$, the set D_k is symmetric with respect to the origin. When the constraints are linear functions, the sets D_k take the shape of polytopes defined by the matrix of constraints $L = \text{column}(l_1, \ldots, l_m)$. The set D_k is characterized by the set of pairs $(h_j, \alpha_{k,j})$ where $\alpha_{k+1,j} = \max_{x \in D_k} h_j(W\phi(x))$, $j \in \{1 \ldots m\}$. Here m is equal to the number of constraints.

A reasonable choice of linear functions is given by $h_j(x) = \langle l_j, W\phi(x) \rangle$, where l_j is a nonzero vector. If $\sup_x \phi(x) \leq 1$, and the vectors $\{l_1, \ldots, l_m\}$ are basis vectors, then the initial bounds can be taken as $\alpha_{0,j} = \sum_{i=1}^{n} |W_{ji}|$ for all $j = 1, \ldots, 2n$. For the next few steps, the set of functions $\{h_j\}_{j=1}^{m_k}$ need not be changed unless the sets $\{D_k\}$ stabilize far from the origin. If the sequence $\{D_k\}$ stabilizes then, new linear functions may be added to original set $\{h_j\}_{j=1}^{m_k}$. It helps to cut parts of D_k such that D_{k+1} is strictly contained in D_k.

A possible implementation of the above procedure is described below:

1. Define $D_0 = \{x : \mid l_j^T x \mid \leq \alpha_{0,j},\ j = 1, \ldots, m\}$. Notice that $m = 2n$.

2. Find $x^j = \arg\max_{x \in D_k}(l_j^T W\phi(x))$. Denote $\alpha_{k+1,j} = \left\langle l_j, W\phi(x^j) \right\rangle$ for all j. Then, $D_{k+1} := \{y : \langle l_j, y \rangle \leq \alpha_{k+1,j}\}$.

3. If $\max_j(\alpha_{k,j} - \alpha_{k+1,j}) > \varepsilon > 0$, increase k by 1 and go to step 2 and repeat. Here ε is some fixed threshold.

4. (a) If $\max_j(\alpha_{k,j} - \alpha_{k+1,j}) < \varepsilon$, then for all $j = 1, \ldots, m$ find

$$u_j = \arg\max(u_j^T x^j - \max_{x \in D_k}(u_j^T W\phi(x))). \tag{2.3}$$

   such that $\|u_j\| = 1$.

   (b) For all $j = 1, \ldots, m$, compute $\beta_j^1 = \max((u_j)^T x : L^T x \leq \alpha)$, $\beta_j^2 = \max((u_j)^T W\phi(x) : L^T x \leq \alpha)$. Here $\alpha = \text{row}(\alpha_1, \ldots, \alpha_m)$. Then compute $\beta_j = |\beta_j^1 - \beta_j^2|$.

   (c) Construct $\beta = \text{column}(\beta_1, \ldots, \beta_m)$. If $\max(\beta) < \tau$ for a given threshold $\tau > 0$, then we conclude that the system does not have a convex Lyapunov function and we stop this procedure. Otherwise we add the corresponding vector u_j to matrix L and go to step 2.

The choice of parameter ε affects the efficiency of this method. If ε is large, then we need to add more hyperplanes to define the new set D_k, thereby increasing the complexity of this process. Next we show that the stability criterion based on the method of reduction of dissipativity domain is more general as compared to the circle criterion of theory of absolute stability.

In chapter 1, we saw that the system

$$x_{k+1} = \tanh(Wx_k) \tag{2.4}$$

where $W = \begin{bmatrix} 1.80 & 0.95 \\ -0.95 & 0.00 \end{bmatrix}$, does not satisfy the circle criterion. But the algorithm described above established that, after 160 iterative steps the dissipativity set of this system is included in a cube $|x_1| \leq .02$, $|x_2| \leq .02$. For $|s| \leq .02$, the graph of the function $\tanh(s)$ lies in the sector $[.98, 1]$. For the new sector bounds, the quadratic form $F(\cdot, \cdot)$ is given by

$$F(x, \psi) = (\psi - NCx)^* \Gamma(MCx - \psi), \tag{2.5}$$

where $N = \begin{bmatrix} .98 & 0 \\ 0 & .98 \end{bmatrix}$, $M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\Gamma = \operatorname{diag}(\tau_j)_{j=1}^m$, τ_j are positive numbers. The circle inequality for system (2.4), satisfying local quadratic constraints, with quadratic form $F(\cdot, \cdot)$ defined in (2.5), is given by

$$\mathbb{R}e\Big((N\tilde{W}(e^{i\omega}) + I)^* \Gamma(I + M\tilde{W}(i\omega))\Big) > 0 \tag{2.6}$$

for all $\omega \in [0, \pi]$. Here, $\tilde{W}(e^{i\omega})$ is transfer function.

It has been shown in [4] that there exists a matrix Γ such that inequality (2.6) holds for all $\omega \in [0, \pi]$, and hence system (2.4) is globally asymptotically stable. Therefore the method of reduction of dissipativity domain (MRDD) proves to be more general as compared to the circle criterion given by theory of absolute stability.

The most challenging part in implementation of MRDD approach is to compute the points of global maxima for the function $f(x) := \langle l_j, W\phi(x)\rangle$ over the sets D_k. Since the function $\phi(\cdot)$ is nonconcave over the set D_k, it can have multiple points of local maxima. It has been seen that, in all the cases, the function $f(\cdot)$ has points of local maxima on the boundary of the polytope. We will first locate the points of local maxima for $f(\cdot)$ on an arbitrary hyperplane. The subject of this research is the solution to the following problem.

Problem Setting: Consider the hyperplane, $P = \{x : l^T x = b\}$ where l is a normal vector and $b \in \mathbb{R}$. Suppose the function $f(x) = \sum_{i=1}^{n} c_i \phi(x_i)$, $c_i \neq 0$ for all i, is defined on P. How many points of local maxima does $f(\cdot)$ have on P? Here $\phi(\cdot)$ is a standard neuron transfer function.

2.5. Summary

An alternative stability criterion, based on method of Reduction of dissipativity domain has been introduced. The computational difficulty encountered in this approach leads us to the main problem setting of this work. In the next chapter, we will present the solution to this problem.

3. OPTIMIZATION OF NONCONVEX FUNCTIONS OVER A HYPERPLANE

3.1. Overview

In this chapter, we will show that the function $f(x) = \sum_{j=1}^{n} c_j \phi(x_j)$ $c_j \neq 0$ for all j, has at most one point of local maximum on an arbitrary hyperplane. In section 2, we will develop necessary and sufficient conditions for the existence of points of local maxima. In section 3, some assumptions regarding function $\phi(\cdot)$ will be listed. Section 4 gives the possible location of points of local maxima. Then we will talk about number of points of local maxima in the main orthant, and side orthants with one negative coordinate. We will conclude with the main result of this chapter.

3.2. Identify the Points of Local Maxima

We will find the necessary and sufficient conditions for a critical point to be a point of local maximum for function $f(\cdot)$ on the hyperplane $P = \{x : l^T x = b\}$ where l is the normal vector and $b \in \mathbb{R}$. Let $K := (I - \frac{ll^T}{\|l\|^2})D(I - \frac{ll^T}{\|l\|^2})$ denote the projection matrix, where $D = \frac{\partial^2 f}{\partial x^2} |_{x=x_0}= \text{diag}(d_j)_{j=1}^{n}$ is the Hessian matrix. Here $d_j = c_j \phi''(x_j)$ for all $j \in \{1 \dots n\}$.

Theorem 3.2.1. *Suppose that x_0 is a critical point of $f(\cdot)$ over P (i.e. l is parallel to $\overrightarrow{\nabla f}(x_0)$). Then x_0 is a point of local maximum of $f(\cdot)$ over P only if $K \leq 0$. Moreover, if K has $n-1$ negative eigenvalues and one zero eigenvalue, then x_0 is a point of local maximum.*

Proof. Consider the Taylor expansion for $f(\cdot)$ in some neighborhood of x_0.

$$f(x) = f(x_0) + \left\langle \frac{\partial f}{\partial x}|_{x=x_0}, x - x_0 \right\rangle + \frac{1}{2} \left\langle x - x_0, \frac{\partial^2 f}{\partial x^2}|_{x=x_0}(x - x_0) \right\rangle + o(\|x - x_0\|^2) \tag{3.1}$$

Since $x \in P$, we have $l^T x = b$ and $l^T x_0 = b$, hence $x - x_0$ is orthogonal to l. Using the fact that $\overrightarrow{\nabla f}(x_0)$ is parallel to l, we get $\overrightarrow{\nabla f}(x_0)^T(x - x_0) = 0$. Moreover, since x_0 is a point of local maximum, we obtain $\left\langle x - x_0, \frac{\partial^2 f}{\partial x^2}|_{x=x_0}(x - x_0) \right\rangle \leq 0$ for all x such that $l^T(x - x_0) = 0$. Therefore,

$$y^T D y \leq 0 \tag{3.2}$$

for all y such that $l^T y = 0$.

Pick an arbitrary $z \in \mathbb{R}^n$. Define $y := (I - \frac{ll^T}{\|l\|^2})z$. Then, $l^T y = l^T(I - \frac{ll^T}{\|l\|^2})z$, and

$$\begin{aligned} z^T K z &= z^T\Big(I - \frac{ll^T}{\|l\|^2}\Big)D\Big(I - \frac{ll^T}{\|l\|^2}\Big)z \\ &= y^T D y \leq 0. \end{aligned} \tag{3.3}$$

Therefore, $K \leq 0$.

Now, suppose that K has $(n-1)$ negative eigenvalues and one zero eigenvalue. This implies that $z^T K z \leq 0$ for all $z \in \mathbb{R}^n$. We will show that if $z \in \{x : l^T x = 0\}$, then $z^T K z < 0$.

There exists an orthonormal basis $\{v_1, v_2, \ldots, v_{n-1}\}$ of the set $\{x : l^T x = 0\}$, consisting of eigenvectors of the matrix K, with eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_{n-1}\}$. Since $v_0 = l$ is the eigenvector with zero eigenvalue, we get $\lambda_i < 0$ for all $i \in \{1 \ldots n-1\}$.

Assume that $z \in \{x : l^T x = 0\}$. Then $z = \sum_{j=1}^{n-1} p_j v_j$, for some $p_j \in \mathbb{R}$ and,

$$\begin{aligned} z^T K z &= \Big\langle \sum_{i=1}^{n-1} p_i v_i, \sum_{i=1}^{n-1} p_i K v_i \Big\rangle = \Big\langle \sum_{i=1}^{n-1} p_i v_i, \sum_{i=1}^{n-1} p_i \lambda_i v_i \Big\rangle \\ &= \sum_{i=1}^{n-1} \lambda_i p_i^2 < \max(\lambda_i) \sum_{i=1}^{n-1} p_i^2 < 0. \end{aligned} \tag{3.4}$$

Pick x in a neighborhood of x_0 on P, where $x \neq x_0$. Then $(x_0 - x) \perp l$. Denote $z = x_0 - x$. Then, $(x_0 - x)^T K(x_0 - x) = (x_0 - x)^T D(x_0 - x) < 0$. Since x_0 is a critical point, we have $\Big\langle \frac{\partial f}{\partial x}|_{x=x_0}, x - x_0 \Big\rangle = 0$. Using the Taylor expansion for $f(\cdot)$, we get $f(x) < f(x_0)$ in some neighborhood of x_0 on P, and x_0 is a point of local maximum for $f(\cdot)$ on the hyperplane P.

□

In the following section, we will list some assumptions about the function $\phi(\cdot)$. These assumptions will be used to show the main result.

3.3. Assumptions About Cost Function

Notation: The following notation will be followed, unless specified.

$\psi'(s) := \frac{d\psi}{dx}|_{x=s}$, $\psi_\beta(\beta q) := \frac{d}{d\beta}(\psi(\beta q))$, $h_\beta(\beta, q_j, q_n) := \frac{\partial}{\partial \beta}(h(\beta, q_j, q_n))$, where $\psi(\cdot) = (\phi'(\cdot))^{-1}$ and, $h(\cdot)$ will be defined later.

Assumption 1: Function $\phi(\cdot)$ satisfies the following properties: $\phi(\cdot)$ is analytic, $\phi(-x) = -\phi(x)$, $\phi'(x) > 0$, $x\phi''(x) < 0$ for all $x \neq 0$, and $\lim_{x\to\infty} \phi(x) < \infty$.

Inequality $x\phi''(x) < 0$ implies that $\phi'(x)$ is decreasing for all $x > 0$. Hence $\phi'(\cdot)^{-1}$ exists. Denote $\phi'(\cdot)^{-1} = \psi(\cdot)$. Then $\psi : (0, \phi'(0)] \to [0, \infty)$. In addition, $\psi'(x) < 0$ for all $x \in (0, \phi'(0))$.

Assumption 2: The function $x(\ln|\psi'(x)|)'$ is a monotonically increasing function of x.

Assumption 3: Define $h(\beta, q_j, q_n) := \frac{\psi'(\beta q_j)}{\psi'(\beta q_n)}$. Then $\frac{\partial}{\partial\beta}\left[\frac{h_\beta(\beta,q_j,q_n)}{h_\beta(\beta,q_l,q_n)}\right]$ is sign definite, where $0 < q_j < q_n < q_l$.

Assumption 4 : For all $p > q > 0$, we have $\frac{d}{d\beta}\left(\frac{\psi(\beta p)}{\psi(\beta q)}\right) < 0$.

Assumption 5: For all $x > 0$, we have $\frac{d}{dx}\left(x\frac{d}{dx}\left(\frac{\psi(x)}{x\psi'(x)}\right)\right) \geq 0$.

Lemma 3.3.1. *Using assumption 2, we get* $\frac{d}{d\beta}\left(\frac{\psi'(\beta p)}{\psi'(\beta q)}\right) > 0$, *where* $p > q > 0$.

Proof. Since $\frac{d}{dx}\left(x(\ln|\psi'(x)|)'\right) > 0$, we obtain

$$\frac{x\psi''(x)\psi'(y) - y\psi''(y)\psi'(x)}{\psi'(x)\psi'(y)} > 0 \tag{3.5}$$

when $x > y$. Using the first assumption, we get $\psi'(x)\psi'(y) > 0$. Hence $x\psi''(x)\psi'(y) - y\psi''(y)\psi'(x) > 0$. If we put $x = \beta p$, and $y = \beta q$, where β is an arbitrary positive number, we get $\frac{d}{d\beta}(\frac{\psi'(\beta p)}{\psi'(\beta q)}) > 0$. □

3.4. Possible Locations of Points of Local Maxima

In the previous section, we listed some assumptions about the cost function. We will use these assumptions to locate the orthants on the hyperplane, where points of local maxima might lie. Recall that $f(x) = \sum_{i=1}^{n} c_i\phi(x_i)$, and $P = \{x : l^T x = b\}$ where l is the normal vector and $b \in \mathbb{R}$.

First, we change basis in order to get $c_j > 0$ for all $j \in \{1 \ldots n\}$. Suppose that $c_j = 0$ for some $j \in \{1 \ldots n\}$, then the corresponding term in the sum is zero, and we obtain $f(x) = \sum_{k=1}^{n-1} c_k x_k$. The problem is reduced to a similar problem of dimension $n - 1$. Without loss of generality, we assume that $c_j \neq 0$ for all $j \in \{1 \ldots n\}$. Next assume that $c_{j0} < 0$ for some $j0 \in \{1 \ldots n\}$. Using assumption 1, $\phi(\cdot)$ is odd function. This implies that $c_{j0}\phi(x_{j0}) = -c_{j0}\phi(-x_{j0})$. Hence, if $c_{j0} < 0$ then replacing c_{j0} by $-c_{j0}$, and x_{j0} by $-x_{j0}$, the function $f(x)$ remains unchanged, and coefficient $c_{j0} > 0$. Thus, without loss of generality, we assume that $c_j > 0$ for all $j \in \{1 \ldots n\}$.

Next, we consider the signs of the components of vector l. If $l_{j0} = 0$ for some $j0 \in \{1 \dots n\}$, then x_{j0} can be increased arbitrarily, still $\sum_{j=1}^{n} l_j x_j$ remains unchanged. Hence, the function $f(x)$ does not have a point of local maximum on P. Therefore we can assume that $l_j \neq 0$ for all $j \in \{1 \dots n\}$. Next, assume that there exists $j0$, $j1 \in \{1 \dots n\}$ where $j0 \neq j1$, such that $l_{j0} < 0 < l_{j1}$. Then we will increase x_{j0} and x_{j1} such that $\sum_{j=1}^{n} l_j x_j$ is unchanged. It is easy to see that the function $f(x)$ is increasing on P. The function $f(x)$ does not have a point of local maximum on P. Hence, we can assume that for every distinct values of $j0$,$j1 \in \{1 \dots n\}$, the product $l_{j0} l_{j1}$ is positive. Suppose that $l_j < 0$ for all $j \in \{1 \dots n\}$. Then we replace b by $-b$ and vector l by $-l$. Therefore, without loss of generality, we assume that $l_j > 0$ for all $j \in \{1 \dots n\}$.

Theorem 3.4.1. *Suppose that x_0 is a critical point for the function $f(\cdot)$. Then, x_0 is a point of local maximum only if the orthant has at most one negative coordinate(i.e. $x_j < 0$ for at most one j, where $j \in \{1 \dots n\}$).*

Proof. Recall $D = \text{diag}(d_j)_{j=1}^{n}$, and l denotes the normal vector. The characteristic polynomial of K is given by $\det(\lambda I - K) = 0$, where λ denotes an eigenvalue of matrix K. First, we will compute $\det(\lambda I - K)$.

$$\begin{aligned}
\det(\lambda I - K) &= \det(\lambda I - (I - \frac{ll^T}{\|l\|^2})D(I - \frac{ll^T}{\|l\|^2})) \\
&= \det(\lambda I - D(I - \frac{ll^T}{\|l\|^2})(I - \frac{ll^T}{\|l\|^2})) \\
&= \det(\lambda I - D(I - \frac{ll^T}{\|l\|^2})) \\
&= \det(\lambda I - D + D\frac{ll^T}{\|l\|^2}) \\
&= \det\Big((\lambda I - D)(I + (\lambda I - D)^{-1} D\frac{ll^T}{\|l\|^2})\Big) \\
&= \det\Big(\lambda I - D\Big) \det\Big(I + \frac{l^T(\lambda I - D)^{-1} Dl}{\|l\|^2}\Big). \qquad (3.6)
\end{aligned}$$

using Sylvester's identity.

Hence,

$$
\begin{aligned}
\det(\lambda I - K) &= \det\Big(\lambda I - D\Big)\Big(1 + \frac{l^T(\lambda I - D)^{-1}Dl}{\|l\|^2}\Big) \\
&= \det\Big(\lambda I - D\Big)\Big(1 + \sum_{i=1}^{n} \frac{l_i^2 d_i}{(\lambda - d_i)\|l\|^2}\Big) \\
&= \det\Big(\lambda I - D\Big)\Big(\sum_{i=1}^{n} \frac{l_i^2}{\|l\|^2} + \sum_{i=1}^{n} \frac{l_i^2 d_i}{(\lambda - d_i)\|l\|^2}\Big) \\
&= \Big(\Pi_{i=1}^{n}(\lambda - d_i)\Big) \cdot \sum_{i=1}^{n} \frac{\lambda l_i^2}{\|l\|^2(\lambda - d_i)}. \qquad (3.7)
\end{aligned}
$$

It follows from equation (3.7) that $\lambda = 0$ is an eigenvalue of K. Denote $g(\lambda) = \sum_{i=1}^{n} \frac{\lambda l_i^2}{\|l\|^2(\lambda - d_i)}$.

We can see that $g(\lambda)$ has vertical asymptotes at d_j. Suppose all d_j's are distinct. Then, we can arrange them as $d_1 < d_2 < \ldots < d_n$. It is easy to see that for all $d_j > 0$, $\lim_{\lambda \to d_j^+} g(\lambda) = \infty$ and $\lim_{\lambda \to d_j^-} g(\lambda) = -\infty$. If $d_j < 0$, then $\lim_{\lambda \to d_j^+} g(\lambda) = -\infty$ and $\lim_{\lambda \to d_j^-} g(\lambda) = \infty$. Since the function $g(\cdot)$ is continuous in (d_j, d_{j+1}), there exists a root, λ_j of the function $g(\cdot)$ in this open interval. The number λ_j is an eigenvalue of K for $j = 1, \ldots, n-1$. Thus, $\{0, \lambda_1, \ldots, \lambda_{n-1}\}$ are all eigenvalues of matrix K.

Now consider the general case. We order the values of d_j: $d_1 \le d_2 \le, \ldots, \le d_n$. If $d_j < d_{j+1}$ for all $j \in \{1, \ldots, n\}$, then the proof is same as above. Suppose that for some $j0$ we get $d_{j_0} = d_{j_0+1} = \ldots d_{j_0+k} \neq d_{j_0+k+1}$, then there are k eigenvalues $\lambda_j = \lambda_{j+1} = \ldots = \lambda_{j+k-1}$ of K at the point d_{j_0}. Together with the zero eigenvalue, the set of such numbers λ_j, $j = 1, 2, \ldots, n-1$ is the set of all eigenvalues of K.

Claim 1: $d_j > 0$ for at most one j where $j \in \{1 \cdots n\}$.

We will use the contrapositive approach to prove this claim. Suppose that there exist d_j and d_k such that $0 < d_j \le d_k$. If $d_j < d_k$, then $g(\lambda) = 0$ for some $\lambda \in (d_j, d_k)$. This implies that the matrix K has a positive eigenvalue, but $K \le 0$ (theorem 3.2.1). Next, if $d_j = d_k$ then K has a positive eigenvalue at d_j. Hence, we arrive to a contradiction. Therefore, $d_j > 0$ for at most one j, where $j \in \{1 \ldots n\}$. Claim 1 is proved.

Next, we will consider following two cases.

Case 1: Suppose that $d_j \le 0$ for all j. Using the definition of d_j, and assumption 1, we obtain that $x_j \ge 0$ for all j.

Case 2: Suppose that $d_j > 0$ for some j. It can be analyzed in a similar manner to Case 1. We obtain that $x_j < 0$ for some j.

Combining the results of Case 1 and Case 2, we obtain that a stationary point, x_0, is a point of local maximum only if the orthant has at most one negative coordinate. □

Using Theorem 3.4.1, we can deduce that the function $f(x)$ can have points of local maxima in the main orthant or side orthant with at most one negative coordinate. In sections 5 through 8, we will analyze the behavior of the function $f(x)$ in an open orthant (i.e. $x_j \neq 0$ for all j). The behavior at the boundary of orthant will be studied in section 9.

3.5. Points of Local Maxima in the Main Orthant

In this section, we will show that $f(x)$, defined on the hyperplane P, has at most one point of local maximum in the main orthant (i.e. $x_j > 0$ for all $j \in \{1 \ldots n\}$).

Proposition 3.5.1. *The function* $f(x) = \sum_{i=1}^{n} c_i \phi(x_i)$ *has at most one point of local maximum in the main orthant, over the hyperplane* P.

Proof. This is obvious, since $f(\cdot)$ is concave over the main orthant. □

Next, we will present the necessary and sufficient condition for existence of point of local maximum in the main orthant.

Proposition 3.5.2. *A point* x_0 *is a point of local maximum in the main orthant if and only if there exists* $\beta \in (0, \infty)$ *such that* $\sum_{j=1}^{n} l_j \psi(\beta \frac{l_j}{c_j}) = b$.

Proof. Necessity: Suppose that x_0 is a point of local maxima in the main orthant and $x_0 \in P$, the hyperplane. Then x_0 is a critical point. We obtain $\overrightarrow{\nabla f}(x_0) = \beta l$ for some $\beta \in \mathbb{R}$. Since $l_j > 0$ for all j, and $\frac{\partial f}{\partial x_j} = c_j \phi'(x_j) > 0$, we obtain that $\beta \in (0, \infty)$.

Hence, $\phi'(x_0^j) = \beta \frac{l_j}{c_j}$ for all $j \in \{1 \ldots n\}$, which in turn implies that $x_0^j = \psi(\beta q_j)$, where $q_j := \frac{l_j}{c_j}$, and $\psi := (\phi'(\cdot))^{-1}$. Since x_0 lies on hyperplane P, we get $b = l' x_0 = \sum_{j=1}^{n} l_j \psi(\beta q_j)$. for some $\beta \in (0, \infty)$.

Sufficiency: Assume that there exists $\beta \in (0, \infty)$ such that $b = \sum_{j=1}^{n} l_j \psi(\beta \frac{l_j}{c_j})$. Using the definition of $\psi(\cdot)$, we get $b = \sum_{j=1}^{n} l_j (\phi')^{-1}(\beta \frac{l_j}{c_j})$. Denote $x_0^j := (\phi')^{-1}(\beta \frac{l_j}{c_j})$. Then, $\phi'(x_0^j) = \beta \frac{l_j}{c_j}$ for all $j \in \{1 \ldots n\}$. Therefore, $\overrightarrow{\nabla} f(x_0) = \beta l$. Hence x_0 is a stationary point in the main orthant. Since $f(x)$ is concave over the main orthant, x_0 is a point of local maximum in the main orthant. □

In Theorem 3.4.1 we saw that a critical point x_0 for function $f(x) = \sum_{i=1}^{n} c_i \phi(x_i)$ can be a point of local maximum only if it is lying in the main orthant or in a side orthant, with at most one negative coordinate. In the previous section, we went over the necessary and sufficient conditions for the existence of local maximum in the main orthant. We also showed that $f(x)$ can have at most one point of local maximum in the main orthant. Next, we will show a similar result for the case of a side orthant, with exactly one negative coordinate.

3.6. Points of Local Maxima in a Side Orthant

In this section, we will show that the function $f(x)$, defined on the hyperplane P, has at most one point of local maximum in a side orthant, with exactly one negative coordinate. Here we have shown the result for the side orthant, which has last component negative. Other cases can be analyzed similarly.

First, we will develop the necessary and sufficient conditions for the existence of local maximum in the side orthant (i.e $x_j > 0$ for all $j \in \{1 \dots n-1\}$, $x_n < 0$). Suppose that x_0 is a critical point for the function $f(\cdot)$ in the side orthant (i.e $x_0^j > 0$ for all $j \in \{1 \dots n-1\}$, $x_0^n < 0$). Then, there exists $\beta \in (0, \infty)$ such that $c_j \phi'(x_0^j) = \beta l_j$ for all j. This implies $x_0^j = \mathrm{sign}(x_0^j)(\phi')^{-1}(\beta \frac{l_j}{c_j}) = \mathrm{sign}(x_0^j)(\phi')^{-1}(\beta q_j)$ for all $j \in \{1 \dots n\}$. Recall that $d_j = c_j \phi''(x_j)$. Using assumption 1, we obtain $d_1 \le d_2 \le \dots \le d_{n-1} < 0 < d_n$.

Denote $g_1(\beta) := l^T x_0 = \sum_{j=1}^{n-1} l_j (\phi')^{-1}(\beta q_j) - l_n (\phi')^{-1}(\beta q_n)$, where $q_j = \frac{l_j}{c_j}$. Using the definition of $\psi(\cdot)$, we represent $g_1(\beta)$ as follows

$$g_1(\beta) = \sum_{j=1}^{n-1} l_j \psi(\beta q_j) - l_n \psi(\beta q_n) \tag{3.8}$$

Theorem 3.6.1. *Let* $g(\lambda) = \sum_{i=1}^{n} \frac{\lambda l_i^2}{\|l\|^2(\lambda - d_i)}$. *If* $g'(0) < 0$ *and* $g_1(\beta) = b$, *for some* $\beta \in (0, \infty)$, *then* x_0 *is a point of local maximum for the function* $f(\cdot)$ *on the hyperplane* $P = \{x : l^T x = b\}$. *Moreover,* x_0 *is a point of local maximum only if* $g'(0) \le 0$ *and* $g_1(\beta) = b$, *for some* $\beta \in (0, \infty)$.

Proof. It is easy to see that $\lim_{\lambda \to d_n^-} g(\lambda) = -\infty$ and $\lim_{\lambda \to d_{n-1}^+} g(\lambda) = -\infty$. We saw earlier that there exists a unique eigenvalue of K, $\lambda \in (d_j, d_{j+1})$ for all $j \le n-2$. Thus, the matrix K has $n-2$ negative eigenvalues.

Therefore, we have two roots of $g(\lambda)$ in the interval (d_{n-1}, d_n). One of the roots is 0. Denote the other root by $\hat{\lambda}$. Notice that $d_n = c_n \phi''(x_0^n)$, where $x_0^n < 0$. Using the fact that $x\phi''(x) < 0$, and

$c_n > 0$, we get $d_n > 0$. This implies that $\hat{\lambda}$ can be negative or positive. But under the assumption that $g'(0) < 0$, we have $\hat{\lambda} < 0$. Hence we conclude that the matrix K has $n-1$ negative eigenvalues, and a zero eigenvalue. Using theorem 3.2.1, x_0 is a point of local maximum.

Suppose that x_0 is a point of local maximum. Then $K \leq 0$. This implies $g'(0) \leq 0$ and $b = \sum_{j=1}^{n-1} l_j \psi(\beta q_j) - l_n \psi(\beta q_n)$ for some $\beta \in (0, \infty)$. □

In theorem 3.6.1, we developed necessary and sufficient conditions for the existence of a point of local maximum in a side orthant. These conditions can be rewritten in terms of function $g_1(\beta)$. To this end, we need the following statement.

Lemma 3.6.2. *Consider the function $g(\lambda)$ as defined above. Then $g'(0) < 0$ if and only if $g_1'(\beta) > 0$.*

Proof. Necessity: We can rewrite $g'(0)$ as

$$g'(0) = \sum_{j=1}^{n} -\frac{l_j^2}{d_j \|l\|^2} = -\frac{1}{\|l\|^2} \sum_{j=1}^{n} \frac{l_j^2}{c_j \phi''(\text{sign}(x_0^j)\psi(\beta \frac{l_j}{c_j}))}. \tag{3.9}$$

where, $d_j = c_j \phi''(x_0^j) = c_j \phi''(\text{sign}(x_0^j)\psi(\beta \frac{l_j}{c_j}))$, and x_0 is critical point for $f(\cdot)$.

Therefore $g'(0)$ can be rewritten as

$$\begin{aligned} g'(0) &= -\frac{1}{\|l\|^2} \Big(\sum_{j=1}^{n-1} \frac{l_j q_j}{\phi''(\psi(\beta q_j))} + \frac{l_n q_n}{\phi''(-\psi(\beta q_n))} \Big) \\ &= \frac{1}{\|l\|^2} \Big(\frac{l_n q_n}{\phi''(\psi(\beta q_n))} - \sum_{j=1}^{n-1} \frac{l_j q_j}{\phi''(\psi(\beta q_j))} \Big). \end{aligned} \tag{3.10}$$

since $\phi''(\cdot)$ is odd.

Hence $g'(0) < 0$ implies

$$\frac{q_n l_n}{\phi''(\psi(\beta q_n))} < \sum_{j=1}^{n-1} \frac{q_j l_j}{\phi''(\psi(\beta q_j))}. \tag{3.11}$$

Next, we want to express $g_1'(\beta)$ in a form similar to $g'(0)$. To this end, we need an auxiliary result.

<u>Claim 2</u>: We have $\frac{1}{\phi''(\psi(\beta q))} = \psi'(\beta q)$. Using the definition of $\psi(\cdot)$, we obtain $\psi(\phi'(x)) = x$. Differentiating with respect to x, we get $\psi'(\phi'(x))\phi''(x) = 1$. This gives us $\phi''(x) = \frac{1}{\psi'(\phi'(x))}$. Let

$\phi'(x) = y$. Then $x = \psi(y)$. Evaluating, we get $\phi''(x) = \phi''(\psi(y))$. Hence $\frac{1}{\psi'(y)} = \phi''(\psi(y))$. Claim 2 is proved.

From equation (3.8), it is easy to see that

$$g_1'(\beta) = \sum_{j=1}^{n-1} l_j q_j \psi'(\beta q_j) - l_n q_n \psi'(\beta q_n) \tag{3.12}$$

where, $\frac{d\psi}{dx} \mid_{x=\beta q} := \psi'(s)$.

Using Claim 2, equation (3.12) can be rewritten as

$$g_1'(\beta) = \sum_{j=1}^{n-1} \frac{l_j q_j}{\phi''(\psi(\beta q_j))} - \frac{l_n q_n}{\phi''(\psi(\beta q_n))}. \tag{3.13}$$

Hence $g'(0) < 0$ implies $g_1'(\beta) > 0$.

Sufficiency: It easily follows from claim(2) and equation (3.12). □

As a consequence, we have the following result.

Corollary 3.6.3. *A critical point x_0 is a point of local maximum for function f in a side orthant if $g_1'(\beta) > 0, g_1(\beta) = b$, and only if $g_1'(\beta) \geq 0, g_1(\beta) = b$ for some $\beta \in (0, \infty)$.*

Proof. The proof easily follows from theorem 3.6.1 and lemma 3.6.2. □

Next, we will show that the function $f(x) = \sum_{i=1}^{n} c_i \phi(x_i)$ has at most one point of local maximum in the side orthant (i.e $x_j > 0$ for all $j \in \{1 \ldots n-1\}$, $x_n < 0$). Before going over the proof, we will prove some useful properties of function $g_1(\cdot)$, which will be used frequently in following sections. Notice that

$$g_1(\beta) = \sum_{j=1}^{n-1} l_j \psi(\beta q_j) - l_n \psi(\beta q_n) \tag{3.14}$$

and,

$$g_1'(\beta) = \sum_{j=1}^{n-1} l_j q_j \psi'(\beta q_j) - l_n q_n \psi'(\beta q_n) \tag{3.15}$$

Lemma 3.6.4. *Suppose that $q_n < q_{j0}$, where $j0 \neq n$. Then the following statements are true.*

(i) *$g_1'(\beta)$ has at most two roots on the interval $(0, \beta_{max}]$.*

(ii) $g_1'(\beta) \to -\infty,$ *as* $\beta \to \beta_{max}$.

Proof. Denote $h(\beta, q_j, q_n) := \frac{\psi'(\beta q_j)}{\psi'(\beta q_n)}$, and $h(\beta, q_l, q_n) := \frac{\psi'(\beta q_l)}{\psi'(\beta q_n)}$, where $j, l \in \{1 \dots n-1\}$. Recall that $\psi'(\beta q) = \frac{d\psi}{ds} |_{s=\beta q}$.

(i) Suppose that $q_n < q_{j0}$ for some $j0 \in \{1 \dots n-1\}$. Let $q_1 < q_2 < \dots < q_k < q_n < q_{k+1} < \dots < q_{n-1}$.

Using assumption 2, and definition of $h(\cdot,\cdot,\cdot)$, we get $h_\beta(\beta, q_j, q_n) \neq 0$ for all q_j,q_n such that $q_j \neq q_n$. In assumption 3, we saw that $\frac{\partial}{\partial \beta}\left[\frac{h_\beta(\beta,q_j,q_n)}{h_\beta(\beta,q_l,q_n)}\right]$ has the same sign for all (β, q_j, q_n, q_l) such that $\beta \in (0, \beta_{max})$, and $0 < q_j < q_n < q_l$.

Using assumptions 2 and 3, we obtain

$$\frac{\partial}{\partial \beta}\Big(\log | \frac{h_\beta(\beta, q_j, q_n)}{h_\beta(\beta, q_l, q_n)} | \Big) = \frac{h_\beta(\beta, q_l, q_n)}{h_\beta(\beta, q_j, q_n)} \cdot \frac{\partial}{\partial \beta}\Big(\frac{h_\beta(\beta, q_j, q_n)}{h_\beta(\beta, q_l, q_n)}\Big) \neq 0. \tag{3.16}$$

Notice that for all pairs (q_j, q_l) such that $q_j < q_l$, the expression

$$h_{\beta\beta}(\beta, q_j, q_n)h_\beta(\beta, q_l, q_n) - h_{\beta\beta}(\beta, q_l, q_n)h_\beta(\beta, q_j, q_n) \tag{3.17}$$

is sign definite, where $h_{\beta\beta}(\beta, q_j, q_n) = \frac{\partial}{\partial \beta}(h_\beta(\beta, q_j, q_n))$. This implies that

$$\sum_{j=1}^{k} \sum_{l=k+1}^{n-1} \underbrace{l_j q_j l_l q_l}_{\text{is positive}} \Big[h_{\beta\beta}(\beta, q_j, q_n)h_\beta(\beta, q_l, q_n) - h_\beta(\beta, q_j, q_n)h_{\beta\beta}(\beta, q_l, q_n)\Big] \neq 0, \tag{3.18}$$

which, in turn implies that

$$\sum_{j=1}^{k} \alpha_j h_{\beta\beta}(\beta, q_j, q_n) \sum_{l=k+1}^{n-1} \alpha_l h_\beta(\beta, q_l, q_n) - \sum_{j=1}^{k} \alpha_j h_\beta(\beta, q_j, q_n) \sum_{l=k+1}^{n-1} \alpha_l h_{\beta\beta}(\beta, q_l, q_n) \neq 0 \tag{3.19}$$

where, $\alpha_j := l_j q_j$, and $\alpha_l := l_l q_l$.

Denote $g(\beta) := \dfrac{\sum_{j=1}^{k} \alpha_j h_\beta(\beta, q_j, q_n)}{\sum_{l=k+1}^{n-1} \alpha_l h_\beta(\beta, q_l, q_n)}$. We can see that the left side of above equation is the same as the numerator of $g'(\beta)$. This implies that $g'(\beta)$ is sign definite. Therefore there exists at most one β, such that $g(\beta) = -1$. This implies

$$\sum_{j=1}^{k} \alpha_j h_\beta(\beta, q_j, q_n) + \sum_{l=k+1}^{n-1} \alpha_l h_\beta(\beta, q_l, q_n) = 0 \tag{3.20}$$

for at most one value of β.

Using the definition of $h_\beta(\beta, q, q_n)$, we obtain

$$\sum_{j=1}^{k} \alpha_j \frac{\partial}{\partial \beta}(h(\beta, q_j, q_n)) + \sum_{l=k+1}^{n-1} \alpha_l \frac{\partial}{\partial \beta}(h(\beta, q_l, q_n)) - l_n q_n \tag{3.21}$$

has at most one root. Recalling the definition of $h(\beta, q_j, q_n)$ we obtain that the function

$$\sum_{j=1}^{k} \alpha_j \frac{\partial}{\partial \beta}\Big(\frac{\psi'(\beta q_j)}{\psi'(\beta q_n)}\Big) + \sum_{l=k+1}^{n-1} \alpha_l \frac{\partial}{\partial \beta}\Big(\frac{\psi'(\beta q_l)}{\psi'(\beta q_n)}\Big) - l_n q_n \tag{3.22}$$

has at most one root. It is easy to see that the above function is equal to $\frac{d}{d\beta}\Big(\frac{g_1'(\beta)}{\psi'(\beta q_n)}\Big)$. Since $\psi'(\beta q) < 0$, we obtain that $g_1'(\beta)$ has at most two roots in $(0, \beta_{max})$. This completes the proof of part(i).

(ii) Using assumption 1, $\phi'(x)$ is a decreasing function for all $x > 0$, and $\lim_{x\to\infty} \phi'(x) = 0$ (since $\lim_{y\to\infty} \phi(y) < \infty$). Since $\psi(\cdot) := \phi'(\cdot)^{-1}$, we get $\psi : (0, \phi'(0)] \to [0, \infty)$. Moreover, $\psi'(x) < 0$ for all $x > 0$.

Using the fact that $\psi'(y) = \frac{1}{\phi''((\phi')^{-1}(y))}$(by Claim 2 in lemma 3.6.2) and $\phi''(0) = 0$, we obtain

$$\psi'(y) \to -\infty, \text{ as } y \to \phi'(0). \tag{3.23}$$

Recall that $g_1'(\beta) = \sum_{j=1}^{n-1} l_j q_j \psi'(\beta q_j) - l_n q_n \psi'(\beta q_n)$. Since, $q_n < q_{j0} = \max(q_j)_{j=1}^{n}$, we get $q_{j0}\beta_{max} = \phi'(0)$. Hence $g_1'(\beta) \to -\infty$, as $\beta \to \beta_{max}$. Proof of part (ii) is completed.

$\square$

Remark 3.6.5. Similarly we can show that if $q_n = \max(q_j)_{j=1}^{n}$, then $g_1'(\beta) \to \infty$, as $\beta \to \beta_{max}$.

Lemma 3.6.6. *Suppose that $q_n > q_j$ for all $j \in \{1 \ldots n-1\}$. Then, $g_1'(\beta)$ has at most one root. Moreover, if $\lim_{\beta\to 0} g_1'(\beta) \geq 0$, then $g_1'(\beta) \geq 0$ for all $\beta \in (0, \beta_{max})$.*

Proof. Assume that $q_n > q_j$ for all $j \in \{1 \dots n-1\}$. We will use the contrapositive approach to prove this claim. Suppose that there exist $\beta_1 \neq \beta_2$ such that $g_1'(\beta_1) = g_1'(\beta_2) = 0$. Since $g_1'(\beta_1) = 0$, we get $\dfrac{\sum_{j=1}^{n-1} l_j q_j \psi'(\beta_1 q_j)}{l_n q_n \psi'(\beta_1 q_n)} = 1$. Similarly, $\dfrac{\sum_{j=1}^{n-1} l_j q_j \psi'(\beta_2 q_j)}{l_n q_n \psi'(\beta_2 q_n)} = 1$.

Denote $\frac{l_j q_j}{l_n q_n} := \alpha_j > 0$ for all j, and $F(\beta) = \sum_{j=1}^{n-1} \alpha_j \frac{\psi'(\beta q_j)}{\psi'(\beta q_n)}$. Hence, there exist $\beta_1 \neq \beta_2$ such that $F(\beta_1) = F(\beta_2) = 1$. It follows that there exists $\beta_0 \in (\beta_1, \beta_2)$ such that $F'(\beta_0) = 0$. This implies that $\sum_{j=1}^{n-1} \alpha_j \frac{d}{d\beta}\left(\frac{\psi'(\beta q_j)}{\psi'(\beta q_n)}\right) |_{\beta=\beta_0} = 0$, where, $q_j < q_n$ for all $j \in \{1 \dots n-1\}$.

Since $\alpha_j > 0$ for all $j \in \{1 \dots n-1\}$, we conclude that $\frac{d}{d\beta}\left(\frac{\psi'(\beta q_j)}{\psi'(\beta q_n)}\right)|_{\beta=\beta_0}$ is not sign definite. Hence we obtain a contradiction to assumption 2, which says that $\frac{d}{d\beta}\left(\frac{\psi'(\beta p)}{\psi'(\beta q)}\right)$, where $p \neq q$ is sign definite. Therefore, if $q_n > q_j$ for all $j \in \{1 \dots n-1\}$, then $g_1'(\beta)$ has at most one root in the interval $(0, \beta_{max})$.

Moreover, assume that $\lim_{\beta \to 0} g_1'(\beta) \geq 0$, where $g_1'(\beta) = \sum_{j=1}^{n-1} l_j q_j \psi'(\beta q_j) - l_n q_n \psi'(\beta q_n)$. Using remark 3.6.5 and first part of this lemma, it is easy to see that $g_1(\beta)$ is non-decreasing on the interval $(0, \beta_{max})$.

□

Remark 3.6.7. In lemmas 3.6.4 and 3.6.6, we assumed that the last component is negative. Similar results hold true for side orthant with first or second component negative. These results will be used in the proof of the case of two side orthants.

Now, we show the main result of the section.

Proposition 3.6.8. *The function $f(x) = \sum_{i=1}^{n} c_i \phi(x_i)$ has at most one point of local maximum in the side orthant.*

Proof. Recall that $g_1(\beta) = \sum_{j=1}^{n-1} l_j \psi(\beta q_j) - l_n \psi(\beta q_n)$, and $g_1'(\beta) = \sum_{j=1}^{n-1} l_j q_j \psi'(\beta q_j) - l_n q_n \psi'(\beta q_n)$.

<u>Case 1</u>: Suppose that $q_n > q_j$ for all $j \in \{1 \dots n-1\}$. Using remark 3.6.5, we get that $g_1'(\beta) \to \infty$, as $\beta \to \beta_{max}$. From lemma 3.6.6, we obtain that that if $q_n > q_j$ for all $j \in \{1 \dots n-1\}$, then $g_1'(\beta)$ has at most one root. Hence, for a given $b \in \mathbb{R}$, there exists at most one value of β such that $g_1(\beta) = b$, and $g_1'(\beta) \geq 0$. This completes the proof of Case 1.

<u>Case 2</u>: Suppose that $q_n < q_{j0}$ where $j0 \neq n$. Using lemma 3.6.4, we get that $g_1'(\beta)$ has at most two roots in the interval $(0, \beta_{max}]$. In addition, we showed in lemma 3.6.4 that $g_1'(\beta) \to -\infty$, as $\beta \to \beta_{max}$. Hence, for a given $b \in \mathbb{R}$, there exists at most one $\beta \in (0, \beta_{max}]$ such that $g_1(\beta) = b$,

and $g_1'(\beta) \geq 0$. Therefore $f(x)$ has at most one point of local maximum in the side orthant. Case 2 is completed. □

We saw that the function $f(x)$ has at most one point of local maximum in the main orthant, as well as in a side orthant with exactly one negative coordinate. It might happen that there are two points of local maxima, one in the main orthant and another in a side orthant, with one component negative. But in the next section we will show that $f(x)$ does not have point of local maxima in both the main orthant and a side orthant.

3.7. Points of Local Maxima in Main and Side Orthant

In this section we will show that the function $f(x) = \sum_{i=1}^{n} c_i \phi(x_i)$ does not have points of local maxima in both the main orthant and a side orthant. In the main orthant $f(x)$ is of the form

$$f_1(\beta) = \sum_{i=1}^{n} l_i \psi(\beta q_j), \tag{3.24}$$

and in the side orthant, it is of the form

$$g_1(\beta) = \sum_{i=1}^{n-1} l_i \psi(\beta q_i) - l_n \psi(\beta q_n). \tag{3.25}$$

First, we show two auxiliary results. These will be used to show that $f(x)$ does not have points of local maxima in both the main orthant and a side orthant.

Lemma 3.7.1. *Suppose that $f_1(\beta)$ and $g_1(\beta)$ are defined as above. Then the following are true:*

(i) $f_1(\beta) \geq g_1(\beta)$ for all $\beta \in (0, \beta_{max}]$, and

(ii) $f_1'(\beta) < 0$ for all $\beta \in (0, \beta_{max}]$.

Proof. It can be easily checked that

$$f_1'(\beta) = \sum_{j=1}^{n} l_j q_j \psi'(\beta q_j), \tag{3.26}$$

and ,

$$g_1'(\beta) = \sum_{j=1}^{n-1} l_j q_j \psi'(\beta q_j) - l_n q_n \psi'(\beta q_n). \tag{3.27}$$

where $\psi'(\beta q) = \frac{d\psi}{ds}\mid_{s=\beta q}$.

(i) First, we show that $f_1(\beta) \geq g_1(\beta)$ for all $\beta \in (0, \beta_{max}]$. Using assumption 1, we have $\phi'(x) \geq 0$ for all x. Hence, $\psi(x) > 0$ for all x. Moreover, it is easy to see that the expressions for $f_1(\beta)$ and $g_1(\beta)$ are identical except for the term $l_n\psi(\beta q_n)$. Since $l_n\psi(\beta q_n) > 0$, we obtain $f_1(\beta) \geq g_1(\beta)$ for all $\beta \in (0, \beta_{max}]$. This completes the proof of (i).

(ii) The proof follows from the fact that $\psi'(x) < 0$ for all x (assumption 1).

□

Next, we present the main result of this section.

Proposition 3.7.2. *The function $f(x) = \sum_{i=1}^{n} c_i\phi(x_i)$ has at most one point of local maximum in the union of the main orthant (i.e. $x_j > 0$ for all $j \in \{1 \dots n\}$) and the side orthant (i.e. $x_j > 0$ for all $j \in \{1 \dots n-1\}, x_n < 0$).*

Proof. We need to show that $f(x)$ does not have points of local maxima in both the main orthant and the side orthant. Using proposition 3.5.2 and corollary 3.6.3, we need to show that for a given $b \in \mathbb{R}$, there do not exist $\beta_1 \neq \beta_2 \in (0, \beta_{max}]$, such that $f_1(\beta_1) = b$, and $g_1(\beta_2) = b$, $g_1'(\beta_2) \geq 0$. We will prove the above result using two cases.

Case 1: Let $q_n > q_j$ for all $j \in \{\dots n-1\}$. Using assumption 1, we get that $\psi'(y) < 0$ and $\psi(y) \in [0, \infty)$ where $y \in (0, \phi'(0)]$. This implies that $\lim_{y \to \phi'(0)} \psi(y) = 0$ Define $y := \beta q$. Then $\lim_{\beta \to \beta_{max}} \psi(\beta q_n) = 0$, which in turn implies that $\lim_{\beta \to \beta_{max}} l_n\psi(\beta q_n) = 0$. Hence

$$f_1(\beta_{max)}) = g_1(\beta_{max}) \tag{3.28}$$

Now we will look at the following possibilities.

Sub Case (i): Suppose that $\lim_{\beta \to 0} g_1'(\beta) < 0$. Since $g_1'(\beta) \to \infty$, as $\beta \to \beta_{max}$, there exists $\overline{\beta} \in (0, \beta_{max})$, such that $g_1'(\overline{\beta}) = 0$. To summarize, $f_1(\beta)$ and $g_1(\beta)$ satisfy the following conditions:

(a) $f_1'(\beta) < 0$, and $f_1(\beta) > g_1(\beta)$ for all $\beta \in (0, \beta_{max})$ (lemma (3.7.1)),

(b) $g_1'(\beta) < 0$ on the interval $(0, \overline{\beta})$ and $g_1'(\beta) > 0$ on the interval $(\overline{\beta}, \beta_{max}]$.

(c) $f_1(\beta_{max}) = g_1(\beta_{max})$.

Using the above conditions, we can deduce that, for a given $b \in \mathbb{R}$, there do not exist $\beta_1 \neq \beta_2$ such that $f_1(\beta_1) = b$ and $g_1(\beta_2) = b$, $g_1'(\beta_2) \geq 0$. Hence, $f(\cdot)$ has at most one point of local maximum, either in the main orthant or in the side orthant with last component negative. This completes the proof of Sub Case (i).

Sub Case (ii): Suppose that $\lim_{\beta \to 0} g_1'(\beta) \geq 0$. From lemma 3.6.6, we get that $g_1(\beta)$ is a non-decreasing function. Using lemma 3.7.1, $f_1(\beta) > g_1(\beta)$ for all $\beta \in (0, \beta_{max})$, and $f_1'(\beta) < 0$ for all β. At the same time, $g_1(\beta_{max}) = f_1(\beta_{max})$. Therefore, for a given $b \in \mathbb{R}$, there exists at most one $\beta \in (0, \beta_{max})$ such that either $f_1(\beta) = b$, or $g_1(\beta) = b$, $g_1'(\beta) \geq 0$. Proof of Sub Case (ii) is completed.

Combining the results of Sub Case (i) and Sub Case (ii), we conclude that $f(x)$ does not have points of local maxima in both the main orthant and the side orthant. This completes the proof of Case 1.

Case 2: Suppose $q_n < q_{j0}$, where $j0 \in \{1 \dots n-1\}$, and $q_{j0} = \max(q_j)_{j=1}^{n}$. We will go over this case by contradiction. Suppose that the function $f(x)$ has two points of local maxima, one in each main and side orthant. This implies that there exist $\beta_1 \neq \beta_2$, such that for a given $b \in \mathbb{R}$, $f_1(\beta_1) = g_1(\beta_2) = b$, and $g_1'(\beta_2) \geq 0$. Notice that β_1 and β_2 lie in $(0, \beta_{max}]$. Recall that

$$f_1(\beta) = \sum_{j=1}^{n} l_j \psi(\beta q_j), \tag{3.29}$$

and

$$g_1(\beta) = \sum_{j=1}^{n-1} l_j \psi(\beta q_j) - l_n \psi(\beta q_n). \tag{3.30}$$

Using part (ii) of lemma 3.6.4, $g_1'(\beta) \to -\infty$, as $\beta \to \beta_{max}$. Assume that $g_1'(\beta_2) > 0$. The case when $g_1'(\beta_2) = 0$, will be analyzed later. Since $g_1'(\beta_2) > 0$ there exists $\beta' \in (\beta_2, \beta_{max})$ such that $g_1'(\beta') = 0$, and $g_1(\beta') > b = f_1(\beta_1) \geq f_1(\beta_{max})$ (lemma 3.7.1).

Since $g_1'(\beta') = 0$, we get $l_n = \frac{1}{q_n \psi'(\beta' q_n)} \sum_{j=1}^{n-1} l_j q_j \psi'(\beta' q_j)$. This implies

$$g_1(\beta') = \sum_{j=1}^{n-1} l_j \Big(\psi(\beta' q_j) - \frac{\psi(\beta' q_n)}{q_n \psi'(\beta' q_n)} q_j \psi'(\beta' q_j) \Big). \tag{3.31}$$

In addition, $f_1(\beta) = \sum_{j=1}^{n-1} l_j \Big(\psi(\beta q_j) + \frac{\psi(\beta q_n)}{q_n \psi'(\beta' q_n)} q_j \psi'(\beta' q_j) \Big)$. We get

$$f_1(\beta_{max}) = \sum_{j=1}^{n-1} l_j \Big(\psi(\beta_{max} q_j) + \frac{\psi(\beta_{max} q_n)}{q_n \psi'(\beta' q_n)} q_j \psi'(\beta' q_j) \Big). \tag{3.32}$$

Since $g_1(\beta') > f_1(\beta_{max})$, we obtain

$$\sum_{j=1}^{n-1} l_j \Big(\psi(\beta' q_j) - \frac{\psi(\beta' q_n)}{q_n \psi'(\beta' q_n)} q_j \psi'(\beta' q_j) \Big) > \sum_{j=1}^{n-1} l_j \Big(\psi(\beta_{max} q_j) + \frac{\psi(\beta_{max} q_n)}{q_n \psi'(\beta' q_n)} q_j \psi'(\beta' q_j) \Big). \tag{3.33}$$

We saw in section 4 that $l_j > 0$ for all $j \in \{1 \ldots n\}$. Hence there exists $j \in \{1 \ldots n-1\}$ such that

$$\psi(\beta' q_j) - \frac{\psi(\beta' q_n)}{q_n \psi'(\beta' q_n)} q_j \psi'(\beta' q_j) > \psi(\beta_{max} q_j) + \frac{\psi(\beta_{max} q_n)}{q_n \psi'(\beta' q_n)} q_j \psi'(\beta' q_j) \tag{3.34}$$

We will show that the above inequality is not true. In other words, we will show that for all q_j, where $j \in \{1 \ldots n-1\}$,

$$\psi(\beta' q_j) - \frac{\psi(\beta' q_n)}{q_n \psi'(\beta' q_n)} q_j \psi'(\beta' q_j) \leq \psi(\beta_{max} q_j) + \frac{\psi(\beta_{max} q_n)}{q_n \psi'(\beta' q_n)} q_j \psi'(\beta' q_j) \tag{3.35}$$

Pick arbitrary q_j, where $j \in \{1 \ldots n-1\}$. We will consider the following possibilities.

Sub Case (i): Assume that $q_n > q_j$. Using assumption 2, we have $\frac{\psi'(\beta p)}{\psi'(\beta q)}$, $p > q$ is an increasing function of β. This implies $-\frac{q_j}{q_n} \psi(\beta q_n) \frac{d}{d\beta} \Big(\frac{\psi'(\beta q_j)}{\psi'(\beta q_n)} \Big) > 0$ (since $\psi(\cdot) > 0$). Notice that

$$\begin{aligned} \frac{d}{d\beta} \Big(\psi(\beta q_j) - \frac{\psi(\beta q_n)}{q_n \psi'(\beta q_n)} q_j \psi'(\beta q_j) \Big) &= q_j \psi'(\beta q_j) - \frac{q_j}{q_n} \psi(\beta q_n) \frac{d}{d\beta} \Big(\frac{\psi'(\beta q_j)}{\psi'(\beta q_n)} \Big) - \frac{q_j}{q_n} \frac{\psi'(\beta q_j)}{\psi'(\beta q_n)} q_n \psi'(\beta q_n) \\ &= -\frac{q_j}{q_n} \psi(\beta q_n) \frac{d}{d\beta} \Big(\frac{\psi'(\beta q_j)}{\psi'(\beta q_n)} \Big) \\ &> 0. \end{aligned} \tag{3.36}$$

The left hand side of inequality (3.35) is an increasing function of β. Hence we obtain

$$\begin{aligned} \psi(\beta' q_j) - \frac{\psi(\beta' q_n)}{q_n \psi'(\beta' q_n)} q_j \psi'(\beta' q_j) &< \psi(\beta_{max} q_j) - \frac{\psi(\beta_{max} q_n)}{q_n \psi'(\beta_{max} q_n)} q_j \psi'(\beta_{max} q_j) \\ &< \psi(\beta_{max} q_j) + \frac{\psi(\beta_{max} q_n)}{q_n \psi'(\beta' q_n)} q_j \psi'(\beta' q_j). \end{aligned} \tag{3.37}$$

since $\dfrac{\psi(\beta q_n)}{q_n\psi'(\beta q_n)} q_j\psi'(\beta q_j) > 0$ for all β.

Therefore inequality (3.35) holds true. This completes the proof of Sub Case (i).

Sub Case (ii): Assume that $q_n < q_j$. Using assumption 4 we have $\frac{d}{d\beta}\left(\frac{\psi(\beta q)}{\psi(\beta p)}\right) > 0$, where $p > q$. Then we obtain the following sequence of inequalities,

$$\begin{aligned}
&\frac{d}{d\beta}\Big(\ln\frac{\psi(\beta q)}{\psi(\beta p)}\Big) > 0,\\
&\frac{d}{d\beta}\Big(\ln(\psi(\beta q))\Big) > \frac{d}{d\beta}\Big(\ln(\psi(\beta p))\Big),\\
&\frac{1}{\psi(\beta p)}\frac{d}{d\beta}(\psi(\beta p)) < \frac{1}{\psi(\beta q)}\frac{d}{d\beta}(\psi(\beta q)),\\
&\psi(\beta p) < \psi(\beta q)\frac{(\psi'(\beta p))p}{(\psi'(\beta q))q}.
\end{aligned} \tag{3.38}$$

since $\psi'(x) < 0$, and $\psi(x) > 0$.

Let $p = q_j$, and $q := q_n$. We get $\psi(\beta q_j) - \frac{\psi(\beta q_n)}{q_n\psi'(\beta q_n)} q_j\psi'(\beta q_j) < 0$. Therefore, the left hand side of inequality (3.35) is negative.

We have seen earlier that $\psi(x) \geq 0$ for all $x \in (0, \phi'(0)]$, and $\psi'(x) < 0$ for all $x \in (0, \phi'(0))$. This implies that $\psi(\beta_{max} q_j) + \frac{\psi(\beta_{max} q_n)}{q_n\psi'(\beta' q_n)} q_j\psi'(\beta' q_j) > 0$. Hence inequality (3.35) is true. This completes the proof of Sub Case (ii).

Sub Case (iii): Assume that $q_n = q_j$. Then it is easy to see that $\psi(\beta' q_j) - \frac{\psi(\beta' q_n)}{q_n\psi'(\beta' q_n)} q_j\psi'(\beta' q_j) = 0$. The left hand side of inequality (3.35) is zero. The right hand side of inequality (3.35) is equal to $2\psi(\beta_{max} q_n)$. From assumption 1, we know that $\psi(\cdot) \geq 0$, which in turn implies that inequality (3.35) holds true. Sub case (iii) is completed.

Next, assume that $g_1'(\beta_2) = 0$. This implies that $g_1(\beta_2) = f_1(\beta_1) \geq f_1(\beta_{max})$. If $g_1(\beta_2) > f_1(\beta_{max})$, replace β' by β_2. Then the proof will proceed in a similar manner as above. But if $\beta_1 = \beta_{max}$ we get $g_1(\beta_2) = f_1(\beta_{max}) = b$. This implies that

$$\sum_{j=1}^{n-1} l_j\Big(\psi(\beta_2 q_j) - \frac{\psi(\beta_2 q_n)}{q_n\psi'(\beta_2 q_n)} q_j\psi'(\beta_2 q_j)\Big) - \sum_{j=1}^{n-1} l_j\Big(\psi(\beta_{max} q_j) + \frac{\psi(\beta_{max} q_n)}{q_n\psi'(\beta_2 q_n)} q_j\psi'(\beta_2 q_j)\Big) = 0. \tag{3.39}$$

Since $l_j > 0$ for all $j \in \{1 \dots n-1\}$, there are two possibilities. Either there exists $j \in$

$\{1 \ldots n-1\}$ such that

$$\psi(\beta_2 q_j) - \frac{\psi(\beta_2 q_n)}{q_n \psi'(\beta_2 q_n)} q_j \psi'(\beta_2 q_j) > \psi(\beta_{max} q_j) + \frac{\psi(\beta_{max} q_n)}{q_n \psi'(\beta_2 q_n)} q_j \psi'(\beta_2 q_j), \tag{3.40}$$

or, for all $j \in \{1, \ldots, n-1\}$, we get

$$\psi(\beta_2 q_j) - \frac{\psi(\beta_2 q_n)}{q_n \psi'(\beta_2 q_n)} q_j \psi'(\beta_2 q_j) = \psi(\beta_{max} q_j) + \frac{\psi(\beta_{max} q_n)}{q_n \psi'(\beta_2 q_n)} q_j \psi'(\beta_2 q_j). \tag{3.41}$$

If the first possibility occurs, then replace β' by β_2, and the problem is reduced to a similar form as above.

Now, consider the other possibility. Suppose for all $j \in \{1 \ldots n-1\}$, we have

$$\psi(\beta_2 q_j) - \frac{\psi(\beta_2 q_n)}{q_n \psi'(\beta_2 q_n)} q_j \psi'(\beta_2 q_j) = \psi(\beta_{max} q_j) + \frac{\psi(\beta_{max} q_n)}{q_n \psi'(\beta_2 q_n)} q_j \psi'(\beta_2 q_j). \tag{3.42}$$

But we saw in Sub case(i), Case 2 that if $q_n > q_j$, then we obtain a strict inequality and the second possibility cannot happen.

Therefore, we obtain a contradiction to our assumption. Combining the results of Case 1 and Case 2, we see that $f(x)$ does not have points of local maxima in the main orthant and the side orthant with last component negative. □

We have seen that the function $f(x) = \sum_{i=1}^{n} c_i \phi(x_i), c_i \neq 0$ for all $i \in \{1 \ldots n\}$ does not have points of local maxima in the main and side orthants. Now, we will show similar result for two side orthants.

3.8. Points of Local Maxima in Two Side Orthants

In this section, we will show that the function $f(x) = \sum_{i=1}^{n} c_i \phi(x_i)$, $c_i \neq 0$ for all n, does not have points of local maxima in two side orthants. We will show the proof for the case when the first and second components are negative. Other cases can be analyzed similarly.

In the first side orthant, $f(x)$ takes the form

$$g_1(\beta) = \sum_{j=2}^{n} l_j \psi(\beta q_j) - l_1 \psi(\beta q_1), \tag{3.43}$$

and in the second side orthant $f(x)$ is of the form

$$g_2(\beta) = \sum_{j=1, j\neq 2}^{n} l_j \psi(\beta q_j) - l_2 \psi(\beta q_2). \tag{3.44}$$

It can be easily checked that

$$g_1'(\beta) = \sum_{j=2}^{n} l_j q_j \psi'(\beta q_j) - l_1 q_1 \psi'(\beta q_1), \tag{3.45}$$

and

$$g_2'(\beta) = \sum_{j=1, j\neq 2}^{n} l_j q_j \psi'(\beta q_j) - l_2 q_2 \psi'(\beta q_2). \tag{3.46}$$

First, we will present the necessary condition for the existence of two points of local maxima; one in each side orthant. This condition will be expressed as an inequality. Next, we will show that the inequality does not hold true. For the present discussion we will assume that $q_1 > q_2$. The proof for the other case is identical.

Lemma 3.8.1. *Suppose $q_1 > q_2$ and $g_1'(\beta_1) = g_2'(\beta_2) = 0$ for some β_1, β_2. Then $\beta_1 > \beta_2$.*

Proof. Since $g_1'(\beta_1) = 0$, and $g_2'(\beta_2) = 0$, we obtain

$$l_1 q_1 \psi'(\beta_1 q_1) = l_2 q_2 \psi'(\beta_1 q_2) + \sum_{j=3}^{n} l_j q_j \psi'(\beta_1 q_j), \tag{3.47}$$

and

$$l_2 q_2 \psi'(\beta_2 q_2) = l_1 q_1 \psi'(\beta_2 q_1) + \sum_{j=3}^{n} l_j q_j \psi'(\beta_2 q_j) \tag{3.48}$$

Solving for l_1 and l_2 we get

$$l_1 = \frac{1}{q_1} \sum_{j=3}^{n} l_j q_j \Big(\frac{\psi'(\beta_1 q_2)\psi'(\beta_2 q_j) + \psi'(\beta_2 q_2)\psi'(\beta_1 q_j)}{\psi'(\beta_1 q_1)\psi'(\beta_2 q_2) - \psi'(\beta_2 q_1)\psi'(\beta_1 q_2)} \Big), \tag{3.49}$$

and,

$$l_2 = \frac{1}{q_2} \sum_{j=3}^{n} l_j q_j \Big(\frac{\psi'(\beta_2 q_1)\psi'(\beta_1 q_j) + \psi'(\beta_1 q_1)\psi'(\beta_2 q_j)}{\psi'(\beta_1 q_1)\psi'(\beta_2 q_2) - \psi'(\beta_2 q_1)\psi'(\beta_1 q_2)} \Big) \tag{3.50}$$

For simplicity, $\Delta := \psi'(\beta_1 q_1)\psi'(\beta_2 q_2) - \psi'(\beta_2 q_1)\psi'(\beta_1 q_2) := \Delta$. Since $l_j > 0$ for all j, and $\psi'(\cdot) < 0$, we get $\Delta > 0$. We get

$$\frac{\psi'(\beta_1 q_1)}{\psi'(\beta_1 q_2)} > \frac{\psi'(\beta_2 q_1)}{\psi'(\beta_2 q_2)}. \tag{3.51}$$

Suppose by contradiction, that $\beta_1 < \beta_2$. From assumption 2, we get $\frac{\psi'(\beta p)}{\psi'(\beta q)}$, $p > q$ is an increasing function of β. Using assumption 2, and the assumption that $\beta_2 > \beta_1$, we get $\frac{\psi'(\beta_1 q_1)}{\psi'(\beta_1 q_2)} < \frac{\psi'(\beta_2 q_1)}{\psi'(\beta_2 q_2)}$. Here $p = q_1$, and $q = q_2$. This contradicts (3.51). Hence, $\beta_1 > \beta_2$. □

Next, we will present another auxiliary result.

Lemma 3.8.2. *Consider the functions $g_1(\beta)$, and $g_2(\beta)$ defined above. If $g_1'(\beta) \geq 0$ for some $\beta \in (0, \beta_{max})$ then $g_2'(\beta) < 0$.*

Proof. Suppose $g_1'(\beta) \geq 0$ for some $\beta \in (0, \beta_{max})$. We have $l_1 q_1 \psi'(\beta q_1) \leq l_2 q_2 \psi'(\beta q_2) + \sum_{j=3}^{n} l_j q_j \psi'(\beta q_j)$. Since $\psi'(x) < 0$ for all x, we get $l_1 q_1 \psi'(\beta q_1) < l_2 q_2 \psi'(\beta q_2)$. Similarly, $g_2'(\beta) \geq 0$ implies that $l_1 q_1 \psi'(\beta q_1) > l_2 q_2 \psi'(\beta q_2)$. Hence we obtain contradiction. □

In the next subsection, we develop a necessary condition for existence of two points of local maxima in the side orthants.

3.8.1. Necessary Condition for Points of Local Maxima in Both Side Orthants

Notice that, if $q_1 = \max(q_j)_{j=1}^n$, then $g_1'(\beta) \to \infty$, and $g_2'(\beta) \to -\infty$, as $\beta \to \beta_{max}$(remark 3.6.7). We also know that $g_1(\cdot)$ has at most one stationary point, and $g_2(\cdot)$ has at most two stationary points in the interval $(0, \beta_{max})$.

However, if $q_{j0} = \max(q_j)_{j=1}^n$, where $j0 \neq 1, 2$, then $g_1'(\beta) \to -\infty$ and $g_2'(\beta) \to -\infty$, as $\beta \to \beta_{max}$. Moreover, both $g_1'(\beta)$ and $g_2'(\beta)$ have at most two roots in the interval $(0, \beta_{max})$.

The above mentioned facts will be used throughout this subsection. Before developing the necessary conditions for the existence of optimality, we present another auxiliary result.

Lemma 3.8.3. *Suppose that $q_1 > q_2$. In addition, assume that for a given $b \in \mathbb{R}$ there exist $\beta' \neq \beta''$ such that $g_1(\beta') = g_2(\beta'') = b, g_1'(\beta') \geq 0$, and $g_2'(\beta'') \geq 0$. Then $\beta' > \beta''$.*

Proof. We will show this result using two cases.

Case 1: Suppose that $q_1 = \max(q_j)_{j=1}^n$.

Assume that $g_1'(\beta') \geq 0$ and $g_2'(\beta'') \geq 0$. Since $g_1(\beta)$ has at most one stationary point, we obtain that $g_1'(\beta) \geq 0$ for all $\beta \in [\beta', \beta_{max})$. Using lemma 3.8.2, we get $g_2'(\beta) < 0$ for all

$\beta \in [\beta', \beta_{max})$. But we assumed that $g_2'(\beta'') \geq 0$. Therefore $\beta' > \beta''$.

Case 2: Let $q_{j0} = \max(q_j)_{j=1}^n$ where $j0 \neq 1, 2$.

Suppose that $\beta' \leq \beta''$ and $g_1'(\beta') \geq 0$, $g_2'(\beta'') \geq 0$. Notice that if $\beta' = \beta''$ then, $g_1'(\beta') \geq 0$ implies $g_2'(\beta'') < 0$ (lemma 3.8.2). But we assumed that $g_2'(\beta'') \geq 0$. Hence we obtain contradiction. In addition, it is easy to see that if $g_1'(\beta') = g_2'(\beta'') = 0$, we get $\beta' > \beta''$(lemma 3.8.1). Therefore, we will consider the case when $\beta' < \beta''$ and $g_1'(\beta') > 0$ and $g_2'(\beta'') \geq 0$.

Since $g_1'(\beta) \to -\infty$ as $\beta \to \beta_{max}$ and $g_1(\beta)$ has at most two roots, there exists $\beta_3 \in (\beta', \beta_{max})$ such that $g_1'(\beta) > 0$ for all $\beta \in [\beta', \beta_3)$. Using a similar reasoning, there exists $\beta_4 \in [\beta'', \beta_{max})$ such that $g_2'(\beta) \geq 0$ for all $\beta \in [\beta'', \beta_4]$. Notice that if $g_2'(\beta'') = 0$ then $\beta'' = \beta_4$.

From lemma 3.8.1, we get $\beta_3 > \beta_4$. Then we obtain that $\beta' < \beta'' \leq \beta_4 < \beta_3$. But we saw that $g_1'(\beta) \geq 0$ for all $\beta \in [\beta', \beta_3]$. Since $\beta'' \in (\beta', \beta_3)$ we get $g_2'(\beta'') < 0$ (lemma 3.8.2). Hence we obtain a contradiction to our earlier assumption.

□

Lemma 3.8.4. *Consider the functions $g_1(\beta)$ and $g_2(\beta)$, defined as above. Assume that $q_1 > q_2$. Then $f(x) = \sum_{i=1}^n c_i \phi(x_i)$ has two points of local maxima in two side orthants only if $g_1(\beta_1) - g_2(\beta_2) \leq 0$, where β_1 and β_2 are critical points for the functions $g_1(\beta)$ and $g_2(\beta)$ respectively.*

Proof. Suppose that $f(x)$ has two points of local maxima, one in each side orthant. This implies that for a given $b \in \mathbb{R}$ there exist $\beta' \neq \beta''$ such that we have $g_1(\beta') = g_2(\beta'') = b$, $g_1'(\beta') \geq 0$, and $g_2'(\beta'') \geq 0$.

Since $g_2'(\beta'') \geq 0$ we get $g_1'(\beta'') < 0$ (see lemma 3.8.2). Using the fact that $g_1'(\beta') \geq 0$ and $\beta' > \beta''$ (lemma 3.8.3), we obtain that there exists $\beta_1 \in (\beta'', \beta']$ such that $g_1'(\beta_1) = 0$ and $g_1'(\beta) \geq 0$ for all $\beta \in [\beta_1, \beta']$. This implies that $g_1(\beta_1) \leq g_1(\beta')$.

We saw above that $g_1'(\beta) \geq 0$ for all $\beta \in [\beta_1, \beta']$. Therefore, using lemma 3.8.2 we get $g_2'(\beta) < 0$ for all $\beta \in [\beta_1, \beta']$. Since $g_2'(\beta'') \geq 0$, there exists $\beta_2 \in [\beta'', \beta_1)$ such that $g_2'(\beta_2) = 0$. Moreover, $g_2'(\beta) \geq 0$ for all $\beta \in [\beta'', \beta_2]$. This implies that $g_2(\beta_2) \geq g_2(\beta'')$. Since $g_1(\beta') = g_2(\beta'')$, we obtain that $g_1(\beta_1) \leq g_2(\beta_2)$. □

In the previous lemma, we developed the necessary inequality for existence of two points of local maxima, (one in each side orthant) for our cost function $f(x)$. Next, we will show that this inequality does not hold true. To this end, we will first rewrite the inequality $g_1(\beta_1) - g_2(\beta_2) \leq 0$ in a more suitable form. This is the subject of discussion of the next subsection.

3.8.2. Three Point Problem

Lemma 3.8.5. *Consider the functions $g_1(\cdot)$ and $g_2(\cdot)$, as defined above. Suppose that β_1, and β_2 are critical points for $g_1(\cdot)$ and $g_2(\cdot)$ respectively. Moreover, assume that $g_1(\beta_1) - g_2(\beta_2) \leq 0$. Then, the following inequality holds:*

$$\begin{aligned}
&\frac{\beta_2}{2\beta_1^2}\Big[\frac{\psi_\beta(\beta_2 q_3)}{\psi_\beta(\beta_1 q_3)}\Big(\frac{\psi(\beta_1 q_1)}{\psi_\beta(\beta_1 q_1)} - \frac{\psi(\beta_1 q_2)}{\psi_\beta(\beta_1 q_2)}\Big)\\
&+ \frac{\psi_\beta(\beta_2 q_3)}{\psi_\beta(\beta_1 q_3)} \cdot \frac{\psi_\beta(\beta_2 q_1)}{\psi_\beta(\beta_1 q_1)}\Big(\frac{\psi(\beta_2 q_1)}{\psi_\beta(\beta_2 q_1)} - \frac{\psi(\beta_2 q_2)}{\psi_\beta(\beta_2 q_2)}\Big)\\
&+ \frac{\psi_\beta(\beta_2 q_2)}{\psi_\beta(\beta_1 q_2)}\Big(\frac{\psi(\beta_1 q_1)}{\psi_\beta(\beta_1 q_1)} - \frac{\psi(\beta_1 q_3)}{\psi_\beta(\beta_1 q_3)}\Big)\\
&+ \frac{\psi_\beta(\beta_2 q_2)}{\psi_\beta(\beta_1 q_2)} \cdot \frac{\psi_\beta(\beta_2 q_1)}{\psi_\beta(\beta_1 q_1)}\Big(\frac{\psi(\beta_2 q_1)}{\psi_\beta(\beta_2 q_1)} - \frac{\psi(\beta_2 q_3)}{\psi_\beta(\beta_2 q_3)}\Big)\\
&+ \frac{\psi_\beta(\beta_2 q_1)}{\psi_\beta(\beta_1 q_1)}\Big(\frac{\psi(\beta_1 q_3)}{\psi_\beta(\beta_1 q_3)} - \frac{\psi(\beta_1 q_2)}{\psi_\beta(\beta_1 q_2)}\Big)\\
&+ \Big(\frac{\psi(\beta_2 q_2)}{\psi_\beta(\beta_2 q_2)} - \frac{\psi(\beta_2 q_3)}{\psi_\beta(\beta_2 q_3)}\Big)\Big(\frac{\psi_\beta(\beta_2 q_3)}{\psi_\beta(\beta_1 q_3)} \cdot \frac{\psi_\beta(\beta_2 q_1)}{\psi_\beta(\beta_1 q_1)} - \frac{\psi_\beta(\beta_2 q_1)}{\psi_\beta(\beta_1 q_1)} \cdot \frac{\psi_\beta(\beta_2 q_2)}{\psi_\beta(\beta_1 q_2)} - \frac{\psi_\beta(\beta_2 q_3)}{\psi_\beta(\beta_1 q_3)} \cdot \frac{\psi_\beta(\beta_2 q_2)}{\psi_\beta(\beta_1 q_2)}\Big)\Big]\\
&\leq 0.
\end{aligned} \tag{3.52}$$

Here, $\psi_\beta(\beta_i q_j) := \frac{d}{d\beta}(\psi(\beta q_j))|_{\beta=\beta_i}$, and $i \in \{1,2\}, j \in \{1,2,3\}$.

Proof. Notice that the proof is quite technical. We will first analyze the case $g_1(\beta_1) - g_2(\beta_2) < 0$. Since $g_1'(\beta_1) = 0$, and $g_2'(\beta_2) = 0$, we get

$$l_1 q_1 \psi'(\beta_1 q_1) = l_2 q_2 \psi'(\beta_1 q_2) + \sum_{j=3}^{n} l_j q_j \psi'(\beta_1 q_j), \tag{3.53}$$

and

$$l_2 q_2 \psi'(\beta_2 q_2) = l_1 q_1 \psi'(\beta_2 q_1) + \sum_{j=3}^{n} l_j q_j \psi'(\beta_2 q_j). \tag{3.54}$$

Solving for l_1 and l_2 we get

$$l_1 = \frac{1}{q_1}\sum_{j=3}^{n} l_j q_j \Big(\frac{\psi'(\beta_1 q_2)\psi'(\beta_2 q_j) + \psi'(\beta_2 q_2)\psi'(\beta_1 q_j)}{\psi'(\beta_1 q_1)\psi'(\beta_2 q_2) - \psi'(\beta_2 q_1)\psi'(\beta_1 q_2)}\Big), \tag{3.55}$$

and,

$$l_2 = \frac{1}{q_2}\sum_{j=3}^{n} l_j q_j \Big(\frac{\psi'(\beta_2 q_1)\psi'(\beta_1 q_j) + \psi'(\beta_1 q_1)\psi'(\beta_2 q_j)}{\psi'(\beta_1 q_1)\psi'(\beta_2 q_2) - \psi'(\beta_2 q_1)\psi'(\beta_1 q_2)}\Big). \tag{3.56}$$

For simplicity, denote $\Delta = \psi'(\beta_1 q_1)\psi'(\beta_2 q_2) - \psi'(\beta_2 q_1)\psi'(\beta_1 q_2)$. Since $l_j > 0$ for all j, and $\psi'(\cdot) < 0$, we have $\Delta > 0$.

Using equations (3.55) and (3.56), we can rewrite the expressions for $g_1(\beta_1)$ and $g_2(\beta_2)$ as

$$\begin{aligned} g_1(\beta_1) = \sum_{j=3}^{n} l_j \Big[& q_j \frac{\psi(\beta_1 q_2)}{q_2} \Big(\frac{\psi'(\beta_2 q_1)\psi'(\beta_1 q_j) + \psi'(\beta_2 q_j)\psi'(\beta_1 q_1)}{\Delta}\Big) \\ & - \frac{q_j \psi(\beta_1 q_1)}{q_1} \Big(\frac{\psi'(\beta_1 q_2)\psi'(\beta_2 q_j) + \psi'(\beta_2 q_2)\psi'(\beta_1 q_j)}{\Delta}\Big) + \psi(\beta_1 q_j)\Big], \end{aligned} \tag{3.57}$$

and,

$$\begin{aligned} g_2(\beta_2) = \sum_{j=3}^{n} l_j \Big[& q_j \frac{\psi(\beta_2 q_1)}{q_1} \Big(\frac{\psi'(\beta_1 q_2)\psi'(\beta_2 q_j) + \psi'(\beta_1 q_j)\psi'(\beta_2 q_2)}{\Delta}\Big) \\ & - \frac{q_j \psi(\beta_2 q_2)}{q_2} \Big(\frac{\psi'(\beta_2 q_1)\psi'(\beta_1 q_j) + \psi'(\beta_1 q_1)\psi'(\beta_2 q_j)}{\Delta}\Big) + \psi(\beta_2 q_j)\Big]. \end{aligned} \tag{3.58}$$

Then,

$$\begin{aligned} & g_1(\beta_1) - g_2(\beta_2) = \\ & \quad \sum_{j=3}^{n} l_j \Big[- \Big(\frac{\psi'(\beta_1 q_2)\psi'(\beta_2 q_j) + \psi'(\beta_2 q_2)\psi'(\beta_1 q_j)}{\Delta}\Big)\Big(\frac{q_j \psi(\beta_1 q_1)}{q_1} + \frac{q_j \psi(\beta_2 q_1)}{q_1}\Big) \\ & \quad + \Big(\frac{\psi'(\beta_2 q_1)\psi'(\beta_1 q_j) + \psi'(\beta_1 q_1)\psi'(\beta_2 q_j)}{\Delta}\Big)\Big(\frac{q_j \psi(\beta_1 q_2)}{q_2} + \frac{q_j \psi(\beta_2 q_2)}{q_2}\Big) \\ & \quad + (\psi(\beta_1 q_j) - \psi(\beta_2 q_j))\Big] < 0. \end{aligned} \tag{3.59}$$

Multiplying both sides of the above inequality by $4\beta_1\beta_2 q_1 q_2 \Delta$, we obtain

$$\sum_{j=3}^{n} l_j \Big[-\Big(4\beta_1\beta_2 q_2 q_j \psi'(\beta_1 q_2)\psi'(\beta_2 q_j) + 4\beta_1\beta_2 q_2 q_j \psi'(\beta_2 q_2)\psi'(\beta_1 q_j)\Big)(\psi(\beta_1 q_1) + \psi(\beta_2 q_1))$$
$$+ \Big(4\beta_1\beta_2 q_1 q_j \psi'(\beta_2 q_1)\psi'(\beta_1 q_j) + 4\beta_1\beta_2 q_1 q_j \psi'(\beta_2 q_j)\psi'(\beta_1 q_1)\Big)(\psi(\beta_1 q_2) + \psi(\beta_2 q_2))$$
$$+ 4\beta_1\beta_2 q_1 q_2 \Delta(\psi(\beta_1 q_j) - \psi(\beta_2 q_j))\Big] < 0. \tag{3.60}$$

Since $l_j > 0$ for all j, at least one of the coefficients in the above sum should be negative. Without loss of generality let $j = 3$. Then,

$$- 4\beta_1\beta_2 q_2 q_3 \Big[\psi'(\beta_1 q_2)\psi'(\beta_2 q_3) + \psi'(\beta_2 q_2)\psi'(\beta_1 q_3)\Big)\Big(\psi(\beta_1 q_1) + \psi(\beta_2 q_1)\Big)$$
$$+ 4\beta_1\beta_2 q_1 q_3 \Big(\psi'(\beta_2 q_1)\psi'(\beta_1 q_3) + \psi'(\beta_2 q_3)\psi'(\beta_1 q_1)\Big)\Big(\psi(\beta_1 q_2) + \psi(\beta_2 q_2)\Big)$$
$$+ 4\beta_1\beta_2 q_1 q_2 \Delta\Big(\psi(\beta_1 q_3) - \psi(\beta_2 q_3)\Big)\Big] < 0. \tag{3.61}$$

Divide both sides of above inequality by $-8\beta_1^3 q_1 q_2 q_3 \psi'(\beta_1 q_1)\psi'(\beta_1 q_2)\psi'(\beta_1 q_3)$ to get

$$\frac{1}{2\beta_1 q_1 \psi'(\beta_1 q_1)}\Big(\frac{2\beta_2 q_2 \psi'(\beta_2 q_2)}{2\beta_1 q_2 \psi'(\beta_1 q_2)} + \frac{2\beta_2 q_3 \psi'(\beta_2 q_3)}{2\beta_1 q_3 \psi'(\beta_1 q_3)}\Big)(\psi(\beta_1 q_1) + \psi(\beta_2 q_1))$$
$$- \frac{1}{2\beta_1 q_2 \psi'(\beta_1 q_2)}\Big(\frac{2\beta_2 q_1 \psi'(\beta_2 q_1)}{2\beta_1 q_1 \psi'(\beta_1 q_1)} + \frac{2\beta_2 q_3 \psi'(\beta_2 q_3)}{2\beta_1 q_3 \psi'(\beta_1 q_3)}\Big)(\psi(\beta_1 q_2) + \psi(\beta_2 q_2))$$
$$- \frac{1}{2\beta_1 q_3 \psi'(\beta_1 q_3)}\Big(\frac{2\beta_2 q_2 \psi'(\beta_2 q_2)}{2\beta_1 q_2 \psi'(\beta_1 q_2)} - \frac{2\beta_2 q_1 \psi'(\beta_2 q_1)}{2\beta_1 q_1 \psi'(\beta_1 q_1)}\Big)(\psi(\beta_1 q_3) - \psi(\beta_2 q_3)) < 0. \tag{3.62}$$

After distributing the terms on the left side, we obtain

$$
\begin{aligned}
&\frac{\beta_2}{2\beta_1^2}\Big[\Big(\frac{\psi_\beta(\beta_2 q_2)}{\psi_\beta(\beta_1 q_2)}+\frac{\psi_\beta(\beta_2 q_3)}{\psi_\beta(\beta_1 q_3)}\Big)\Big(\frac{\psi(\beta_1 q_1)}{\psi_\beta(\beta_1 q_1)}+\frac{\psi(\beta_2 q_1)}{\psi_\beta(\beta_1 q_1)}\Big)\\
&+\Big(\frac{\psi_\beta(\beta_2 q_1)}{\psi_\beta(\beta_1 q_1)}+\frac{\psi_\beta(\beta_2 q_3)}{\psi_\beta(\beta_1 q_3)}\Big)\Big(-\frac{\psi(\beta_1 q_2)}{\psi_\beta(\beta_1 q_2)}-\frac{\psi(\beta_2 q_2)}{\psi_\beta(\beta_1 q_2)}\Big)\\
&+\Big(\frac{\psi_\beta(\beta_2 q_2)}{\psi_\beta(\beta_1 q_2)}-\frac{\psi_\beta(\beta_2 q_1)}{\psi_\beta(\beta_1 q_1)}\Big)\Big(-\frac{\psi(\beta_1 q_3)}{\psi_\beta(\beta_1 q_3)}+\frac{\psi(\beta_2 q_3)}{\psi_\beta(\beta_1 q_3)}\Big)\Big]\\
&=\frac{\beta_2}{2\beta_1^2}\Big[\frac{\psi_\beta(\beta_2 q_3)}{\psi_\beta(\beta_1 q_3)}\Big(\frac{\psi(\beta_1 q_1)}{\psi_\beta(\beta_1 q_1)}-\frac{\psi(\beta_1 q_2)}{\psi_\beta(\beta_1 q_2)}\Big)\\
&+\frac{\psi_\beta(\beta_2 q_2)}{\psi_\beta(\beta_1 q_2)}\Big(\frac{\psi(\beta_1 q_1)}{\psi_\beta(\beta_1 q_1)}-\frac{\psi(\beta_1 q_3)}{\psi_\beta(\beta_1 q_3)}\Big)\\
&+\frac{\psi_\beta(\beta_2 q_1)}{\psi_\beta(\beta_1 q_1)}\Big(\frac{\psi(\beta_1 q_3)}{\psi_\beta(\beta_1 q_3)}-\frac{\psi(\beta_1 q_2)}{\psi_\beta(\beta_1 q_2)}\Big)\\
&+\frac{\psi_\beta(\beta_2 q_2)}{\psi_\beta(\beta_1 q_2)}\cdot\frac{\psi(\beta_2 q_1)}{\psi_\beta(\beta_1 q_1)}+\frac{\psi_\beta(\beta_2 q_2)}{\psi_\beta(\beta_1 q_2)}\cdot\frac{\psi(\beta_2 q_3)}{\psi_\beta(\beta_1 q_3)}\\
&+\frac{\psi_\beta(\beta_2 q_3)}{\psi_\beta(\beta_1 q_3)}\cdot\frac{\psi(\beta_2 q_1)}{\psi_\beta(\beta_1 q_1)}-\frac{\psi_\beta(\beta_2 q_3)}{\psi_\beta(\beta_1 q_3)}\cdot\frac{\psi(\beta_2 q_2)}{\psi_\beta(\beta_1 q_2)}\\
&-\frac{\psi_\beta(\beta_2 q_1)}{\psi_\beta(\beta_1 q_1)}\cdot\frac{\psi(\beta_2 q_2)}{\psi_\beta(\beta_1 q_2)}-\frac{\psi_\beta(\beta_2 q_1)}{\psi_\beta(\beta_1 q_1)}\cdot\frac{\psi(\beta_2 q_3)}{\psi_\beta(\beta_1 q_3)}\Big]<0. \qquad (3.63)
\end{aligned}
$$

We can see that the first three terms out of nine terms in the latter sum of inequality (3.63) are monotonic in terms of q. Next we express the remaining six terms in similar form. We have

$$
\begin{aligned}
&\frac{\beta_2}{2\beta_1^2}\Big[\frac{\psi_\beta(\beta_2 q_3)}{\psi_\beta(\beta_1 q_3)}\cdot\frac{\psi(\beta_2 q_1)}{\psi_\beta(\beta_1 q_1)}\cdot\frac{\psi_\beta(\beta_2 q_1)}{\psi_\beta(\beta_2 q_1)}+\frac{\psi_\beta(\beta_2 q_2)}{\psi_\beta(\beta_1 q_2)}\cdot\frac{\psi(\beta_2 q_1)}{\psi_\beta(\beta_1 q_1)}\cdot\frac{\psi_\beta(\beta_2 q_1)}{\psi_\beta(\beta_2 q_1)}\\
&-\frac{\psi_\beta(\beta_2 q_1)}{\psi_\beta(\beta_1 q_1)}\cdot\frac{\psi(\beta_2 q_2)}{\psi_\beta(\beta_1 q_2)}\cdot\frac{\psi_\beta(\beta_2 q_2)}{\psi_\beta(\beta_2 q_2)}-\frac{\psi_\beta(\beta_2 q_3)}{\psi_\beta(\beta_1 q_3)}\cdot\frac{\psi(\beta_2 q_2)}{\psi_\beta(\beta_1 q_2)}\cdot\frac{\psi_\beta(\beta_2 q_2)}{\psi_\beta(\beta_2 q_2)}\\
&+\frac{\psi_\beta(\beta_2 q_2)}{\psi_\beta(\beta_1 q_2)}\cdot\frac{\psi(\beta_2 q_3)}{\psi_\beta(\beta_1 q_3)}\cdot\frac{\psi_\beta(\beta_2 q_3)}{\psi_\beta(\beta_2 q_3)}-\frac{\psi_\beta(\beta_2 q_1)}{\psi_\beta(\beta_1 q_1)}\cdot\frac{\psi(\beta_2 q_3)}{\psi_\beta(\beta_1 q_3)}\cdot\frac{\psi_\beta(\beta_2 q_3)}{\psi_\beta(\beta_2 q_3)}\Big]. \qquad (3.64)
\end{aligned}
$$

For brevity, denote $x_j=\frac{\psi_\beta(\beta_2 q_j)}{\psi_\beta(\beta_1 q_j)}$, and $y_j=\frac{\psi(\beta_2 q_j)}{\psi_\beta(\beta_2 q_j)}$, where $j\in\{1,2,3\}$. With the new notation, the above expression can be expressed as

$$\frac{\beta_2}{2\beta_1^2}\Big(x_3x_1y_1+x_2x_1y_1-x_1x_2y_2-x_3x_2y_2+x_2x_3y_3-x_1x_3y_3\Big). \qquad (3.65)$$

Adding and subtracting the terms $\frac{\beta_2}{2\beta_1^2}x_3x_1y_2$ and $\frac{\beta_2}{2\beta_1^2}x_1x_2y_3$ to (3.65), we obtain

$$\begin{aligned}&\frac{\beta_2}{2\beta_1^2}\Big(x_3x_1y_1 - x_3x_1y_2 + x_3x_1y_2 + x_2x_1y_1 - x_2x_1y_3 + x_2x_1y_3 - x_1x_2y_2 - x_3x_2y_2 + x_3x_2y_3 - x_1x_3y_3\Big)\\ &= \frac{\beta_2}{2\beta_1^2}\Big(x_3x_1(y_1 - y_2) + x_2x_1(y_1 - y_3) + (y_2 - y_3)(x_3x_1 - x_1x_2 - x_3x_2)\Big). \qquad (3.66)\end{aligned}$$

Using the definition of x_j, and y_j, the above sum is the same as

$$\begin{aligned}&\frac{\beta_2}{2\beta_1^2}\Big[\frac{\psi_\beta(\beta_2q_3)}{\psi_\beta(\beta_1q_3)}\cdot\frac{\psi_\beta(\beta_2q_1)}{\psi_\beta(\beta_1q_1)}\Big(\frac{\psi(\beta_2q_1)}{\psi_\beta(\beta_2q_1)} - \frac{\psi(\beta_2q_2)}{\psi_\beta(\beta_2q_2)}\Big) + \frac{\psi_\beta(\beta_2q_2)}{\psi_\beta(\beta_1q_2)}\cdot\frac{\psi_\beta(\beta_2q_1)}{\psi_\beta(\beta_1q_1)}\Big(\frac{\psi(\beta_2q_1)}{\psi_\beta(\beta_2q_1)} - \frac{\psi(\beta_2q_3)}{\psi_\beta(\beta_2q_3)}\Big)\\ &+\Big(\frac{\psi(\beta_2q_2)}{\psi_\beta(\beta_2q_2)} - \frac{\psi(\beta_2q_3)}{\psi_\beta(\beta_2q_3)}\Big)\Big(\frac{\psi_\beta(\beta_2q_3)}{\psi_\beta(\beta_1q_3)}\cdot\frac{\psi_\beta(\beta_2q_1)}{\psi_\beta(\beta_1q_1)} - \frac{\psi_\beta(\beta_2q_1)}{\psi_\beta(\beta_1q_1)}\cdot\frac{\psi_\beta(\beta_2q_2)}{\psi_\beta(\beta_1q_2)} - \frac{\psi_\beta(\beta_2q_3)}{\psi_\beta(\beta_1q_3)}\cdot\frac{\psi_\beta(\beta_2q_2)}{\psi_\beta(\beta_1q_2)}\Big)\Big].\end{aligned} \qquad (3.67)$$

Combining the above expression with the first three monotonic terms of inequality (3.63), we obtain

$$\begin{aligned}&\frac{\beta_2}{2\beta_1^2}\Big[\frac{\psi_\beta(\beta_2q_3)}{\psi_\beta(\beta_1q_3)}\Big(\frac{\psi(\beta_1q_1)}{\psi_\beta(\beta_1q_1)} - \frac{\psi(\beta_1q_2)}{\psi_\beta(\beta_1q_2)}\Big) + \frac{\psi_\beta(\beta_2q_3)}{\psi_\beta(\beta_1q_3)}\cdot\frac{\psi_\beta(\beta_2q_1)}{\psi_\beta(\beta_1q_1)}\Big(\frac{\psi(\beta_2q_1)}{\psi_\beta(\beta_2q_1)} - \frac{\psi(\beta_2q_2)}{\psi_\beta(\beta_2q_2)}\Big)\\ &+\frac{\psi_\beta(\beta_2q_2)}{\psi_\beta(\beta_1q_2)}\Big(\frac{\psi(\beta_1q_1)}{\psi_\beta(\beta_1q_1)} - \frac{\psi(\beta_1q_3)}{\psi_\beta(\beta_1q_3)}\Big) + \frac{\psi_\beta(\beta_2q_2)}{\psi_\beta(\beta_1q_2)}\cdot\frac{\psi_\beta(\beta_2q_1)}{\psi_\beta(\beta_1q_1)}\Big(\frac{\psi(\beta_2q_1)}{\psi_\beta(\beta_2q_1)} - \frac{\psi(\beta_2q_3)}{\psi_\beta(\beta_2q_3)}\Big)\\ &+\frac{\psi_\beta(\beta_2q_1)}{\psi_\beta(\beta_1q_1)}\Big(\frac{\psi(\beta_1q_3)}{\psi_\beta(\beta_1q_3)} - \frac{\psi(\beta_1q_2)}{\psi_\beta(\beta_1q_2)}\Big) + \Big(\frac{\psi(\beta_2q_2)}{\psi_\beta(\beta_2q_2)} - \frac{\psi(\beta_2q_3)}{\psi_\beta(\beta_2q_3)}\Big)\Big(\frac{\psi_\beta(\beta_2q_3)}{\psi_\beta(\beta_1q_3)}\cdot\frac{\psi_\beta(\beta_2q_1)}{\psi_\beta(\beta_1q_1)}\\ &-\frac{\psi_\beta(\beta_2q_1)}{\psi_\beta(\beta_1q_1)}\cdot\frac{\psi_\beta(\beta_2q_2)}{\psi_\beta(\beta_1q_2)} - \frac{\psi_\beta(\beta_2q_3)}{\psi_\beta(\beta_1q_3)}\cdot\frac{\psi_\beta(\beta_2q_2)}{\psi_\beta(\beta_1q_2)}\Big)\Big] < 0. \qquad (3.68)\end{aligned}$$

Next, suppose $g_1(\beta_1) - g_2(\beta_2) = 0$. Then inequality (3.60) reduces to an equality given by

$$\begin{aligned}&\sum_{j=3}^{n} l_j\Big[-\Big(4\beta_1\beta_2q_2q_j\psi'(\beta_1q_2)\psi'(\beta_2q_j) + 4\beta_1\beta_2q_2q_j\psi'(\beta_2q_2)\psi'(\beta_1q_j)\Big)(\psi(\beta_1q_1) + \psi(\beta_2q_1))\\ &+\Big(4\beta_1\beta_2q_1q_j\psi'(\beta_2q_1)\psi'(\beta_1q_j) + 4\beta_1\beta_2q_1q_j\psi'(\beta_2q_j)\psi'(\beta_1q_1)\Big)(\psi(\beta_1q_2) + \psi(\beta_2q_2))\\ &+4\beta_1\beta_2q_1q_2\Delta(\psi(\beta_1q_j) - \psi(\beta_2q_j))\Big] = 0. \qquad (3.69)\end{aligned}$$

Since $l_j > 0$ for all j, we can have two possibilities: either there exists $j_0 \in \{3, \ldots, n\}$ such that corresponding term in the sum is negative, or every term in the sum is zero. If first case occurs, then we get the result obtained in (3.68). But if all terms in sum are zero then, we get equality. $\square$

Denote $x_j = \frac{\psi_\beta(\beta_2 q_j)}{\psi_\beta(\beta_1 q_j)}$, $y_j = \frac{\psi(\beta_2 q_j)}{\psi_\beta(\beta_2 q_j)}$, and $z_j = \frac{\psi(\beta_1 q_j)}{\psi_\beta(\beta_1 q_j)}$. Using the new notation, the above inequality can be expressed as

$$\frac{\beta_2}{2\beta_1^2}\Big[x_3(z_1 - z_2) + x_3 x_1(y_1 - y_2) + x_2(z_1 - z_3) + x_2 x_1(y_1 - y_3) + x_1(z_3 - z_2)$$
$$+ (y_3 - y_2)(x_1 x_2 + x_3 x_2 - x_3 x_1)\Big] \leq 0. \tag{3.70}$$

Next, we check whether there exist positive numbers β_1, β_2, q_1, q_2, and q_3 where $q_1 > q_2$, such that inequality (3.70) is satisfied. We call this problem as Three point problem. This is the subject of discussion in the following section.

3.8.3. Solution to Three Point Problem

We present the solution to Three point problem using two different cases. For Case I, we assume that $q_3 < \max(q_1, q_2)$. Recall that $\max(q_1, q_2) = q_1$. We show that if $q_1 = \max(q_j)_{j=1}^3$, then (3.70) is not satisfied. To this end, we present some auxiliary results.

Lemma 3.8.6. *Suppose that $q_1 > q_3 > q_2$. Then the following inequalities are satisfied.*

(i) $z_2 < z_3 < z_1 < 0$.

(ii) $y_2 < y_3 < y_1 < 0$.

(iii) $x_2 > x_3 > x_1 > 0$.

Proof. Notice, $x_j = \frac{\psi_\beta(\beta_2 q_j)}{\psi_\beta(\beta_1 q_j)}$, $y_j = \frac{\psi(\beta_2 q_j)}{\psi_\beta(\beta_2 q_j)}$, and $z_j = \frac{\psi(\beta_1 q_j)}{\psi_\beta(\beta_1 q_j)}$.

(i) We show that $z_1 > z_2$. The remaining inequalities can be shown similarly.

We need to show that $\frac{\psi(\beta_1 q_1)}{\psi_\beta(\beta_1 q_1)} > \frac{\psi(\beta_1 q_2)}{\psi_\beta(\beta_1 q_2)}$. From assumption 4, we know that $\frac{d}{d\beta}\left(\frac{\psi(\beta p)}{\psi(\beta q)}\right) < 0$, where $p > q$. This implies that $\psi_\beta(\beta p)\psi(\beta q) < \psi(\beta p)\psi_\beta(\beta q)$. Put $\beta = \beta_1$, $p = q_1$, and $q = q_2$. Then, we obtain $\psi_\beta(\beta_1 q_1)\psi(\beta_1 q_2) < \psi(\beta_1 q_1)\psi_\beta(\beta_1 q_2)$, which in turn implies that $\frac{\psi(\beta_1 q_1)}{\psi_\beta(\beta_1 q_1)} > \frac{\psi(\beta_1 q_2)}{\psi_\beta(\beta_1 q_2)}$. Similarly, we can show other inequalities. This completes the proof of (i).

(ii)The proof is identical to (i).

(iii) We will show that $x_2 > x_3$. Remaining inequalities can be shown similarly.

We need to show that $\frac{\psi_\beta(\beta_2 q_2)}{\psi_\beta(\beta_1 q_2)} > \frac{\psi_\beta(\beta_2 q_3)}{\psi_\beta(\beta_1 q_3)}$. Using assumption 2 and the fact that $\beta_1 > \beta_2$, we obtain $\frac{d}{dq}\left(\frac{\psi_\beta(\beta_2 q)}{\psi_\beta(\beta_1 q)}\right) < 0$. Hence we obtain $\frac{\psi_\beta(\beta_2 q_2)}{\psi_\beta(\beta_1 q_2)} > \frac{\psi_\beta(\beta_2 q_3)}{\psi_\beta(\beta_1 q_3)}$. Moreover, since $\psi'(x) < 0$, we get $x_j > 0$ for all $j \in \{1, 2, 3\}$. This completes the proof of (iii). □

Lemma 3.8.7. *Suppose $q_1 > q_2 > q_3$. Then the following inequalities hold true.*

(i) $x_3(y_2 - y_3) < z_2 - z_3$.

(ii) $0 < x_1 < x_2 < x_3$.

(iii) $z_3 < z_2 < z_1 < 0$.

(iv) $y_3 < y_2 < y_1 < 0$.

Proof. We show the proof for part (i). The proofs for the remaining inequalities are similar to lemma 3.8.6.

Using assumption 2, we get $\frac{d}{d\beta}\left(\frac{\psi'(\beta p)}{\psi'(\beta q)}\right) > 0, p > q$. Recall that $\psi'(\beta q) = \frac{d}{ds}\psi(s)|_{s=\beta q}$. This implies that $\frac{p}{q}\frac{d}{d\beta}\left(\frac{\psi'(\beta p)}{\psi'(\beta q)}\right) > 0$, which in turn implies $\frac{d}{d\beta}\left(\frac{\psi_\beta(\beta p)}{\psi_\beta(\beta q)}\right) > 0$. Hence $\frac{d}{d\beta}\left(\frac{\psi_\beta(\beta q)}{\psi_\beta(\beta p)}\right) < 0$, where $p > q$. Here $p = q_2$, and $q = q_3$. Since $\psi(\cdot)$ is a non-negative function, we can rewrite the inequality $\frac{d}{d\beta}\left(\frac{\psi_\beta(\beta q)}{\psi_\beta(\beta p)}\right) < 0$ as

$$\psi_\beta(\beta q_3) - \psi_\beta(\beta q_2)\frac{\psi_\beta(\beta q_3)}{\psi_\beta(\beta q_2)} - \psi(\beta q_2)\frac{d}{d\beta}\Big(\frac{\psi_\beta(\beta q_3)}{\psi_\beta(\beta q_2)}\Big) > 0 \tag{3.71}$$

The left hand side of the above inequality is the derivative of $\psi(\beta q_3) - \psi_\beta(\beta q_3)\frac{\psi(\beta q_2)}{\psi_\beta(\beta q_2)}$. Since $\beta_1 > \beta_2$ we obtain following sequence of inequalities

$$\begin{aligned}
&\psi(\beta_1 q_3) - \psi_\beta(\beta_1 q_3)\frac{\psi(\beta_1 q_2)}{\psi_\beta(\beta_1 q_2)} > \psi(\beta_2 q_3) - \psi_\beta(\beta_2 q_3)\frac{\psi(\beta_2 q_2)}{\psi_\beta(\beta_2 q_2)},\\
&\psi_\beta(\beta_2 q_3)\frac{\psi(\beta_2 q_2)}{\psi_\beta(\beta_2 q_2)} - \psi(\beta_2 q_3) > \psi_\beta(\beta_1 q_3)\frac{\psi(\beta_1 q_2)}{\psi_\beta(\beta_1 q_2)} - \psi(\beta_1 q_3),\\
&\frac{\psi_\beta(\beta_2 q_3)}{\psi_\beta(\beta_1 q_3)}\Big(\frac{\psi(\beta_2 q_2)}{\psi_\beta(\beta_2 q_2)} - \frac{\psi(\beta_2 q_3)}{\psi_\beta(\beta_2 q_3)}\Big) < \frac{\psi(\beta_1 q_2)}{\psi_\beta(\beta_1 q_2)} - \frac{\psi(\beta_1 q_3)}{\psi_\beta(\beta_1 q_3)}.
\end{aligned} \tag{3.72}$$

This implies $x_3(y_2 - y_3) < z_2 - z_3$. □

Next, we use the above auxiliary results to show that inequality (3.70) is not satisfied. Recall $x_j = \frac{\psi_\beta(\beta_2 q_j)}{\psi_\beta(\beta_1 q_j)}$, $y_j = \frac{\psi(\beta_2 q_j)}{\psi_\beta(\beta_2 q_j)}$, and $z_j = \frac{\psi(\beta_1 q_j)}{\psi_\beta(\beta_1 q_j)}$.

Lemma 3.8.8. *Suppose that* $q_1 = \max(q_j)_{j=1}^3$. *Then inequality* (3.70) *does not hold.*

Proof. (i) Suppose that $q_1 > q_3 > q_2$. We need to show that the sum

$$x_3(z_1-z_2)+x_3x_1(y_1-y_2)+x_2(z_1-z_3)+x_2x_1(y_1-y_3)+x_1(z_3-z_2)+(y_3-y_2)(x_1x_2+x_3x_2-x_3x_1) \tag{3.73}$$

is positive. From (i) and (ii) in lemma 3.8.6, we obtain

$$x_3(z_1-z_2)+x_3x_1(y_1-y_2)+x_2(z_1-z_3)+x_2x_1(y_1-y_3)+x_1(z_3-z_2) > 0 \tag{3.74}$$

It remains to show that $(y_3 - y_2)(x_1x_2 + x_2x_3 - x_3x_1) > 0$. From (ii) and (iii) in lemma 3.8.6, we obtain $y_3 > y_2$ and $x_1(x_2 - x_3) > 0$ respectively. Hence

$$x_3(z_1-z_2)+x_3x_1(y_1-y_2)+x_2(z_1-z_3)+x_2x_1(y_1-y_3)+x_1(z_3-z_2)+(y_3-y_2)(x_1x_2+x_3x_2-x_3x_1) > 0 \tag{3.75}$$

(ii) Next, suppose that $q_1 > q_2 > q_3$. We need to show that the sum

$$x_3(z_1-z_2)+x_3x_1(y_1-y_2)+x_2(z_1-z_3)+x_2x_1(y_1-y_3)+x_1(z_3-z_2)+(y_3-y_2)(x_1x_2+x_3x_2-x_3x_1) \tag{3.76}$$

is positive.

Using parts (ii) and (iii) of lemma 3.8.7, we get $x_2(z_1 - z_3) > 0$ and $x_1(z_3 - z_2) < 0$. But it can be checked that

$$x_2(z_1-z_3)+x_1(z_3-z_2) = x_2(z_1-z_2+z_2-z_3)+x_1(z_3-z_2)$$
$$= x_2(z_1-z_2)+(x_1-x_2)(z_3-z_2) > 0. \tag{3.77}$$

We can rewrite expression (3.76) as

$$
\begin{aligned}
&x_3(z_1 - z_2) + x_3x_1(y_1 - y_2) + x_2x_1(y_1 - y_3) \\
&+ x_2(z_1 - z_2) + (x_1 - x_2)(z_3 - z_2) + (y_3 - y_2)(x_1x_2 + x_3x_2 - x_3x_1) \\
&= x_3(z_1 - z_2) + x_3x_1(y_1 - y_2) + x_1x_2(y_1 - y_2) \\
&+ x_1x_2(y_2 - y_3) + x_2(z_1 - z_2) + (x_1 - x_2)(z_3 - z_2) + (y_3 - y_2)(x_1x_2 + x_3x_2 - x_3x_1) \\
&= (z_1 - z_2)(x_2 + x_3) + (y_1 - y_2)(x_1x_3 + x_1x_2) \\
&+ (x_1 - x_2)\Big((z_3 - z_2) + x_3(y_2 - y_3)\Big).
\end{aligned} \tag{3.78}
$$

Using parts (ii), (iii), and (iv) of lemma 3.8.7, we deduce that

$$(z_1 - z_2)(x_2 + x_3) + (y_1 - y_2)(x_1x_3 + x_1x_2) > 0. \tag{3.79}$$

We saw in part (ii) of lemma 3.8.7 that $x_1 - x_2 < 0$. Hence, we need to show that $(z_3 - z_2) + x_3(y_2 - y_3) < 0$. This is obvious from part (i) of lemma 3.8.7. Therefore, expression (3.76) is positive. This completes the proof of part (ii).

Summarizing the results from parts (i) and (ii), we see that if $q_3 < \max(q_1, q_2)$, then the sum $x_3(z_1-z_2)+x_3x_1(y_1-y_2)+x_2(z_1-z_3)+x_2x_1(y_1-y_3)+x_1(z_3-z_2)+(y_3-y_2)(x_1x_2+x_3x_2-x_3x_1)$ is positive.

□

Next, for Case II, we assume that $q_3 > \max(q_1, q_2) = q_1$. We need to show that

$$x_3(z_1-z_2)+x_1x_3(y_1-y_2)+x_2(z_1-z_3)+x_2x_1(y_1-y_3)+x_1(z_3-z_2)+(y_3-y_2)(x_1x_2+x_2x_3-x_1x_3) > 0. \tag{3.80}$$

Before going over the proof, we present some auxiliary results.

Lemma 3.8.9. *Suppose $q_3 > q_1 > q_2$. Then, the following inequalities hold true.*

(i) $x_2 > x_1 > x_3 > 0$.

(ii) $z_2 < z_1 < z_3 < 0$.

(iii) $y_2 < y_1 < y_3 < 0$.

(iv) $\frac{z_1}{x_1} > \frac{z_2}{x_2}$.

Proof. (i) From assumption 2, we know that $\frac{d}{d\beta}\left(\frac{\psi'(\beta p)}{\psi'(\beta q)}\right) > 0$, where $p > q$. We get $\frac{d}{dq}\left(\frac{\psi_\beta(\beta_2 q)}{\psi_\beta(\beta_1 q)}\right) < 0$, where $\beta_2 < \beta_1$(lemma 3.8.1). This implies that $\frac{\psi_\beta(\beta_2 q_2)}{\psi_\beta(\beta_1 q_2)} > \frac{\psi_\beta(\beta_2 q_1)}{\psi_\beta(\beta_1 q_1)}$, and we get $x_2 > x_1$. Similarly, we can show that $x_1 > x_3$. Hence we get $x_2 > x_1 > x_3 > 0$.

(ii) Using assumption 4, we have $\frac{d}{d\beta}\left(\frac{\psi(\beta p)}{\psi(\beta q)}\right) < 0$ for all $p > q$. This implies

$$\psi_\beta(\beta q)\psi(\beta p) > \psi_\beta(\beta p)\psi(\beta q). \tag{3.81}$$

Put $\beta = \beta_1$, $p = q_1$, and $q = q_2$. We have $\frac{\psi(\beta_1 q_1)}{\psi_\beta(\beta_1 q_1)} > \frac{\psi(\beta_1 q_2)}{\psi_\beta(\beta_1 q_2)}$. This implies $z_1 > z_2$. Similarly, we get $z_3 > z_1$. Hence $z_3 > z_1 > z_2$.

(iii) The proof is similar to (ii).

(iv) We need to show that $\frac{\psi(\beta_1 q_1)}{\psi_\beta(\beta_2 q_1)} > \frac{\psi(\beta_1 q_2)}{\psi_\beta(\beta_2 q_2)}$. Since $\psi_\beta(\cdot) < 0$, this is equivalent to showing that $\frac{\psi(\beta_1 q_1)}{\psi(\beta_1 q_2)} < \frac{\psi_\beta(\beta_2 q_1)}{\psi_\beta(\beta_2 q_2)}$.

Using assumption 2, we get $\frac{d}{d\beta}\left(\frac{\psi_\beta(\beta p)}{\psi_\beta(\beta q)}\right) > 0$, where $p > q$. This implies

$$\frac{\psi_\beta(\beta_2 q_1)}{\psi_\beta(\beta_2 q_2)} > \lim_{\beta_2 \to 0^+} \frac{\psi_\beta(\beta_2 q_1)}{\psi_\beta(\beta_2 q_2)}. \tag{3.82}$$

We show that $\lim_{\beta_2 \to 0^+} \frac{\psi_\beta(\beta_2 q_1)}{\psi_\beta(\beta_2 q_2)}$ exists.

Since $\frac{d}{d\beta}\left(\frac{\psi_\beta(\beta p)}{\psi_\beta(\beta q)}\right) > 0$, we obtain that $\left(\frac{\psi_\beta(\beta p)}{\psi_\beta(\beta q)}\right)$ is a decreasing function of β, as $\beta \to 0^+$. Hence, $\{\frac{\psi_\beta(\beta p)}{\psi_\beta(\beta q)}\}$ has an upper bound for small but positive values of β. Moreover, since $\psi_\beta(\cdot)$ is negative, $\{\frac{\psi_\beta(\beta p)}{\psi_\beta(\beta q)}\}$ has a lower bound too. We have a monotonic and bounded sequence, hence $\lim_{\beta_2 \to 0^+} \frac{\psi_\beta(\beta_2 q_1)}{\psi_\beta(\beta_2 q_2)}$ exists.

Since $\psi(y) \to \infty$, as $y \to 0^+$, we get $\lim_{\beta_2 \to 0^+} \frac{\psi(\beta_2 q_1)}{\psi(\beta_2 q_2)} \overset{LH}{=} \lim_{\beta_2 \to 0^+} \frac{\psi_\beta(\beta_2 q_1)}{\psi_\beta(\beta_2 q_2)}$. From assumption 4, we get $\frac{d}{d\beta}\left(\frac{\psi(\beta p)}{\psi(\beta q)}\right) < 0$, $p > q$. Since $\beta_1 > \beta_2$, we obtain $\lim_{\beta_2 \to 0^+} \frac{\psi(\beta_2 q_1)}{\psi(\beta_2 q_2)} > \frac{\psi(\beta_1 q_1)}{\psi(\beta_1 q_2)}$. Hence we obtain the following chain of inequalities:

$$\frac{\psi_\beta(\beta_2 q_1)}{\psi_\beta(\beta_2 q_2)} > \lim_{\beta_2 \to 0^+} \frac{\psi_\beta(\beta_2 q_1)}{\psi_\beta(\beta_2 q_2)} \overset{LH}{=} \lim_{\beta_2 \to 0^+} \frac{\psi(\beta_2 q_1)}{\psi(\beta_2 q_2)} > \frac{\psi(\beta_1 q_1)}{\psi(\beta_1 q_2)} \tag{3.83}$$

Therefore, we get $\frac{\psi(\beta_1 q_1)}{\psi(\beta_1 q_2)} < \frac{\psi_\beta(\beta_2 q_1)}{\psi_\beta(\beta_2 q_2)}$. Proof for (iv) is complete. □

Lemma 3.8.10. *Suppose that $q_3 > q_1 > q_2$. Then inequality* (3.80) *holds true.*

Proof. The left hand side of (3.80) can be rewritten as

$$\begin{aligned}
& x_3(z_1 - z_2) + x_1x_3(y_1 - y_2) + x_2z_1 - x_2z_3 \\
& + x_1(z_3 - z_2) + x_1x_2(y_1 - y_2 + y_2 - y_3) \\
& + (y_3 - y_2)x_1x_2 + (y_3 - y_2)(x_2x_3 - x_1x_3) \\
& = x_3(z_1 - z_2) + x_1x_3(y_1 - y_2) + z_3(x_1 - x_2) \\
& + x_2z_1 - x_1z_2 + x_1x_2(y_1 - y_2) + (y_3 - y_2)x_3(x_2 - x_1). \qquad (3.84)
\end{aligned}$$

Using parts (i), (ii), and (iii) of lemma 3.8.9, we get

$$x_3(z_1 - z_2) > 0, x_1x_3(y_1 - y_2) > 0, z_3(x_1 - x_2) > 0, (y_3 - y_2)x_3(x_2 - x_1) > 0. \qquad (3.85)$$

Therefore, we get

$$\begin{aligned}
& x_3(z_1 - z_2) + x_1x_3(y_1 - y_2) + z_3(x_1 - x_2) + x_2z_1 - x_1z_2 + x_1x_2(y_1 - y_2) + (y_3 - y_2)x_3(x_2 - x_1) \\
& > x_2z_1 - x_1z_2 + (x_1x_2)(y_1 - y_2). \qquad (3.86)
\end{aligned}$$

Next, we show that $x_2z_1 - x_1z_2 + (x_1x_2)(y_1 - y_2) > 0$. From part (iii) of lemma 3.8.9, we get $x_1x_2(y_1 - y_2) > 0$. Similarly, using part (iv) of lemma 3.8.9 we have $x_2z_1 - x_1z_2 > 0$. Hence

$$\begin{aligned}
& x_3(z_1 - z_2) + x_1x_3(y_1 - y_2) + x_2z_1 - x_2z_3 \\
& + x_1(z_3 - z_2) + x_1x_2(y_1 - y_2 + y_2 - y_3) \\
& + (y_3 - y_2)x_1x_2 + (y_3 - y_2)(x_2x_3 - x_1x_3) > 0. \qquad (3.87)
\end{aligned}$$

This completes the proof for the case when $q_3 > q_1 > q_2$. □

This completes the solution to Three Point problem. Now we have all the required results to present the main result of this section.

Theorem 3.8.11. *The function $f(x) = \sum_{i=1}^{n} c_i\phi(x_i)$ does not have points of local maxima in two side orthants.*

3.9. Special Case

In previous sections, we have analyzed the behavior of the function $f(x) = \sum_{i=1}^{n} c_i\phi(x_i)$ only in the open orthant (i.e. $x_j \neq 0$ for all $j \in \{1 \dots n\}$). For sake of brevity, denote by Θ the union of open orthants, and let $\Gamma = \mathbb{R}^n \; /\Theta$.

We have shown that $f(x)$ can have at most one point of local maximum in intersection of Θ and the hyperplane P. But it might happen that the function $f(\cdot)$ has two points of local maxima, one in an open orthant and another in the intersection of Γ and P. In this section, we will show that this cannot happen.

First, notice that if the function f has a point of local maximum on the boundary of an orthant, then it must be an isolated point. Suppose that $f(\cdot)$ has a point of local maximum x_0 on the boundary of an orthant, on hyperplane P. Assume that x_0 is not an isolated point. Then, there exists a sequence $\{x_i\}_{i=1}^{\infty}$, where x_i is a point of local maximum for $f(x)$ for all i, such that $\|x_0 - x_i\| \to 0$ as $i \to \infty$. Since x_i lies on the boundary, then for all i, we obtain $x_i^j = 0$ for at least one $j \in \{1 \dots n\}$. Using theorem 3.4.1, we get for all i, $x_i^j < 0$ for at most one j, where $j \in \{1, \dots, n\}$. This implies that $g(\beta_k) = b$, for the sequence $\{\beta_k\}_{k=1}^{\infty}$, where $g(\beta) := \sum_{j=1}^{n} \pm l_j\psi(\beta\frac{l_j}{c_j})$. Recall that $\psi(\cdot) = \phi'(\cdot)^{-1}$. Using the fact that $\phi(\cdot)$ is an analytic function, we obtain that $g(\beta) \equiv b$. This implies that $g(\cdot)$ is a constant function. Next, we show that this is not true.

Without loss of generality, assume that for all j such that, $1 \leq j \leq k$, $x_i^j \neq 0$ and $x_i^j = 0$ for $j \in \{k+1, \dots, n\}$. Since x_i has at most one negative coordinate, function $g(\beta)$ is either of the form $g(\beta) = \sum_{j=1}^{k} l_j\psi(\beta\frac{l_j}{c_j})$ or $g(\beta) = \sum_{j=1}^{k-1} l_j\psi(\beta\frac{l_j}{c_j}) - l_k\psi(\beta\frac{l_k}{c_k})$. But we know that in the main orthant $g(\cdot)$ is a decreasing function, and in the side orthant $g(\beta)$ has at most two critical points. Therefore $g(\beta)$ is not a constant function, which, in turn implies that the points of local maxima on the boundary of the orthant are isolated.

Lemma 3.9.1. *Suppose that x_0 is an isolated point of local maximum for the function $f(\cdot)$ on the hyperplane $P = \{x : l^T x = b\}$. Then, for all $\varepsilon > 0$ there exists $\delta_0 > 0$, $\tilde{l}$, $\tilde{b}$ such that $\|l - \tilde{l}\| < \delta_0$, $|b - \tilde{b}| < \delta_0$, and a point $\tilde{x}_0$ with all nonzero components, such that $\tilde{x}_0$ is a point of local maximum of the function $f(\cdot)$ over $\tilde{P} = \{x : \tilde{l}x = \tilde{b}\} \cap \|x_0 - \tilde{x_0}\| < \varepsilon$.*

Proof. Since x_0 is an isolated point of local maximum for $f(\cdot)$ on P, there exists a sufficiently small $\varepsilon_0 > 0$, such that if $x \in P$, $x \neq x_0$, and $\|x - x_0\| < \varepsilon_0$, then $f(x) < f(x_0)$.

Fix a positive number $\varepsilon < \varepsilon_0$. Since x_0 is a point of local maximum, there exists $\beta \in (0, \infty)$ such that $x_0^j = \pm\psi(\beta\frac{l_j}{c_j})$ for all $j \in \{1 \ldots n\}$. If all components of x_0 are nonzero, then we are done. Suppose that x_0 has zero components. Without loss of generality, let $x_0^1 = \ldots = x_0^m = 0$. Then, using assumption 1, we obtain $\beta\frac{l_k}{c_k} = \phi'(0)$ for all $k \in \{1 \ldots m\}$.

Since $f(\cdot)$ is a continuous function, there exists a positive number δ_0 such that $f(\cdot)$ has a point of local maximum over the set $\{x : \tilde{l}x = \tilde{b}\} \cap \{x : \|x - x_0\| < \varepsilon\}$ for all vectors $\tilde{l}$, and numbers $\tilde{b}$ such that $\|l - \tilde{l}\| < \delta_0$, and $|b - \tilde{b}| < \delta_0$.

Fix positive numbers δ, δ_1, such that $\delta_1 < \delta$. Denote $\tilde{l}_1 = l_1 - \delta_1$, $\tilde{l}_j = l_j - \delta$, for $2 \leq j \leq m$, $\tilde{l}_j = l_j$ for $j > m$. Denote by $\tilde{x_0}$ the point with coordinates $\tilde{x_0}^1 = \pm\psi(\beta\frac{l_1-\delta_1}{c_1})$, $\tilde{x_0}^j = \pm\psi(\beta\frac{l_j-\delta}{c_j})$ for $2 \leq j$, m, $\tilde{x_0}^j = x_0^j$ for $j > m$. Denote $\tilde{b} := \tilde{l}^T\tilde{x_0}$. Assume that $\frac{\tilde{l}_1}{c_1} > \frac{\tilde{l}_j}{c_j}$ for all $j \in \{2 \ldots n\}$.

We will choose δ such that $\|l - \tilde{l}\| < \delta_0$, and $|b - \tilde{b}| < \delta_0$. Then f has a point of local maximum, $\tilde{x}$, over the set $\tilde{P} = \{x : \tilde{l}^Tx = \tilde{b}\} \cap \{x : \|x - x_0\| < \varepsilon\}$. There exists $\tilde{\beta} > 0$ such that $\tilde{x}^j = \pm\psi(\tilde{\beta}\frac{\tilde{l}_j}{c_j})$ for all $j \in \{1 \ldots n\}$. If all components of $\tilde{x}$ are nonzero, then we are done.

Assume that at least one component of $\tilde{x}$ is zero. Due to the choice of δ_1, it should be the first component $\tilde{x}^1$. This implies that $\tilde{\beta}\frac{\tilde{l}_1}{c_1} = \phi'(0)$, and we get $\tilde{l}\tilde{x} = \tilde{b}$. Next, consider a positive number $\hat{\delta}$ and a vector $\hat{l}$ with components $\hat{l}_1 = \tilde{l}_1$, $\hat{l}_j = \tilde{l}_j - \hat{\delta}$ for $j \in \{2 \ldots n\}$. Define the vector $\hat{x}$ with components $\hat{x}^j = \pm\psi(\tilde{\beta}\frac{\hat{l}_j}{c_j})$, for all $j \in \{1 \ldots n\}$. The number $\hat{\delta}$ is chosen in such way that $\|\hat{l} - l\| < \delta_0$, $\hat{l}^T\hat{x} \neq \tilde{b}$, and $\frac{\hat{l}_1}{c_1} > \frac{\hat{l}_j}{c_j}$ for all $j \in \{2 \ldots n\}$. Consider the hyperplane $\hat{P} = \{x : \hat{l}^Tx = \tilde{b}\}$. There exists a point of local maximum, y_1, of the function $f(\cdot)$ over $\hat{P} \cap \|y - x_0\| < \varepsilon$. Therefore, there exists a positive number $\hat{\beta}$ such that $y_1^j = \pm\psi(\hat{\beta}\frac{\hat{l}_j}{c_j})$ for all $j = 1 \ldots n$. Since $\hat{l}^T\hat{x} \neq \tilde{b}$, we get $\hat{x} \notin \hat{P}$.

If all components of y_1 are nonzero, then we are done. Assume y_1 has a zero component. Due to choice of $\delta_1, \hat{\delta}$, we get $y_1^1 = 0, y_1^j \neq 0$ for all $j \in \{2 \ldots n\}$. This implies that $\frac{\hat{\beta}\hat{l}_1}{c_1} = \frac{\tilde{\beta}\hat{l}_1}{c_1} = \phi'(0)$. Therefore $\hat{\beta} = \tilde{\beta}$. But in this case $y_1 = \hat{x}$, and therefore vector y_1 does not belong to the hyperplane $\hat{P}$. Hence we get a contradiction. Therefore, all components of vector y_1 are nonzero. Thus, y_1 is a point of local maximum of the function f over $\hat{P}$, such that $\|\hat{l} - l\| < \delta_0$, $|b - \hat{b}| < \delta_0$, and $\|y - x_0\| < \varepsilon$. □

Corollary 3.9.2. *The function $f(x)$ cannot have one point of local maximum in $\Theta \cap P$, and one point of local maximum in $\Gamma \cap P$.*

Proof. Suppose that x_0 and x_1 are two points of local maxima for the function $f(\cdot)$. Without loss of generality, we can assume that x_0 is in $\Theta \cap P$(i.e. $x_0^j \neq 0$ for all $j \in \{1 \ldots n\}$). Since $f(\cdot)$ has at most one point of local maximum in open orthant, x_0 is an isolated point. Using lemma 3.9.1, we can find another point of local maximum $\hat{x}_0$ over hyperplane $\hat{P}$, such that all the components of $\hat{x}_0$ are nonzero.

Using similar reasoning as above, we can also find another point of local maximum $\hat{x}_1$ such that all the components of $\hat{x}_1$ are nonzero. Then we end up with two points of local maxima in $\Theta \cap \hat{P}$. Hence we obtain a contradiction. □

3.10. Main Result

In Section 3, assumptions about the function $\phi(\cdot)$ were listed. Those assumptions turned out to be sufficient conditions for the above results to be true. Therefore, we obtain the following result.

Theorem 3.10.1. *Consider the function $f(x) = \sum_{i=1}^{n} c_i \phi(x_i)$, where $c_i \neq 0$ for all i. Suppose that the function $\phi(\cdot)$ satisfies the conditions $(1) - (5)$ stated in Section 3. The function $f(x)$ has at most one point of local maximum on the hyperplane P defined by $\{x : l^T x = b\}$ for some $b \in \mathbb{R}$.*

Proof. The proof follows from the previous results. □

3.11. Summary

We have studied the existence of points of local maxima for the function $f(x) = \sum_{i=1}^{n} c_i \phi(x_i)$, over a hyperplane. We have found conditions, imposed on the function $\phi(\cdot)$, which guarantee the existence of at most one point of local maximum for the function on the hyperplane. Those conditions are satisfied by a wide range of neuron transfer functions. In the appendix, we will present two examples of neuron transfer functions which satisfy the properties mentioned in section 3.

4. CONCLUSION

We study the stability problem of discrete time nonlinear dynamical systems, which describe dynamics of recurrent neural networks (RNN). To date, the strongest stability criterion for such systems is given by theory of absolute stability. This theory provides necessary and sufficient conditions for existence of quadratic Lyapunov function for systems with nonlinearities, satisfying certain quadratic constraints. Non-existence of such quadratic Lyapunov functions does not imply instability of the systems. As the number of nonlinearities in the system increases, the gap between set of systems for which absolute stability criterion hold true, and the entire set of stable systems becomes wider. Since RNN is an essentially nonlinear system, it was necessary to develop an alternative stability criterion. Another stability criterion based on method of reduction of dissipativity domain (MRDD), has been developed. In particular, it provides necessary and sufficient conditions for existence of a convex Lyapunov function for nonlinear dynamical systems. Hence, it is more general as compared to criterion for absolute stability.

The main difficulty encountered in implementation of MRDD method consists of finding points of local maxima for the cost function, $f(x) = \sum_{i=1}^{n} c_i \phi(x_i)$, over a convex polytope. The function $\phi(\cdot)$ is nonconcave, hence, f can have multiple points of local maxima. Since optimal points exist on the edge of the polytope, it makes sense to analyze the behavior of function f over a hyperplane. Therefore, we studied possible number of points of local maxima for the function f, over an arbitrary hyperplane. First, we identified the possible orthants, where the points of local maxima might lie. Next, we showed that if the function $\phi(\cdot)$ satisfies a certain set of conditions, then the function f has at most one point of local maximum over an arbitrary hyperplane. Those conditions are satisfied by a wide range of neuron transfer functions.

Future work may involve finding the points of local maxima for function f over planes of lower dimension, using nonlinear constraints to describe the sets, finding points of local maxima for systems with nonzero bias, and generalizations of these results to infinite dimensional space.

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APPENDIX

A.1. Overview

In Chapter 3, assumptions regarding function $\phi(\cdot)$ were presented. We also showed that those assumptions are sufficient conditions for existence of at most one point of local maximum on the hyperplane. Here, we present equivalent statements for those assumptions. Next, we will check these properties for two functions, namely $\tanh(\cdot)$, and $\arctan(\cdot)$.

A.2. Reformulation

Lemma A.2.1. *Define* $g(x) := \ln|\psi'(e^x)|$. *Then,* $\frac{d}{dx}\Big(x\frac{d}{dx}(\ln|\psi'(x)|)\Big) > 0$ *for all* $x \in (0, \phi'(0))$ *if and only if* $g''(x) > 0$ *for all* $x \in (-\infty, \ln\phi'(0))$.

Proof. Notice that $|\psi'(x)| = e^{g(\ln x)}$. With the change of variable $y = \ln(x)$ we get

$$\begin{aligned} &\tfrac{d}{dx}\Big(x\tfrac{d}{dx}(\ln|\psi'(x)|)\Big) = \tfrac{d}{dx}(g(\ln x)) + x\tfrac{d}{dx}\Big(\tfrac{d}{dx}(g(\ln x))\Big) = \\ &= \tfrac{d}{e^y dy}(g(y)) + e^y\tfrac{d}{e^y dy}\Big(\tfrac{d}{e^y dy}(g(y))\Big) = \tfrac{g'(y)}{e^y} + \tfrac{d}{dy}\Big(\tfrac{g'(y)}{e^y}\Big) \\ &= \tfrac{g''(y)e^y + e^y g'(y) - e^y g'(y)}{e^y} = g''(y). \end{aligned} \tag{A.1}$$

□

Similarly, the assumptions 4 and 5 can be reformulated as discussed in the following lemma.

Lemma A.2.2. *Define* $f(x) := \ln(\psi(e^x))$.

(a) Suppose that $p > q > 0$. *Then,* $\frac{d}{d\beta}\Big(\frac{\psi(\beta p)}{\psi(\beta q)}\Big) < 0$ *if and only if* $f''(x) < 0$.

(b) $\frac{d}{dx}\Big(x\frac{d}{dx}\Big(\frac{\psi(x)}{x\psi'(x)}\Big)\Big) \geq 0$ *for all* $x > 0$ *if and only if the function* $(\frac{1}{f'(\cdot)})$ *is convex.*

Proof. (a) *Necessity*: Notice that $\psi(x) = e^{f(\ln x)}$. Since $\frac{d}{d\beta}\Big(\frac{\psi(\beta p)}{\psi(\beta q)}\Big) < 0$, we get

$$\psi_\beta(\beta p)\psi(\beta q) < \psi(\beta p)\psi_\beta(\beta q). \tag{A.2}$$

Then,

$$\frac{d}{d\beta}\Big(\ln(\psi(q\beta))\Big) - \frac{d}{d\beta}\Big(\ln(\psi(\beta p))\Big) = \frac{d}{d\beta}\Big(f(\ln\beta + \ln q)\Big) - \frac{d}{d\beta}\Big(f(\ln\beta + \ln p)\Big) > 0. \tag{A.3}$$

Define $y = \ln \beta$. We get

$$\frac{d}{e^y dy}\Big(f(y+\ln q)\Big) - \frac{d}{e^y dy}\Big(f(y+\ln p)\Big) = \frac{d}{dy}\Big(f(y+\ln q)\Big) - \frac{d}{dy}\Big(f(y+\ln p)\Big) > 0. \tag{A.4}$$

for all y. This implies that $f''(x) < 0$ for all $x > 0$.

Sufficiency: Pick two positive numbers, p, q such that $p > q$. We have

$$\frac{d}{dy}\Big(f(y+\ln q)\Big) - \frac{d}{dy}\Big(f(y+\ln p)\Big) > 0. \tag{A.5}$$

for all y, which in turn implies that $\frac{d}{e^y dy}\Big(f(y+\ln q)\Big) - \frac{d}{e^y dy}\Big(f(y+\ln p)\Big) > 0$.

Define $\beta := e^y$. Then, we can retrace the steps for the proof of the necessary condition.

(b)*Necessity*: We know $\frac{d}{dx}\Big(x\frac{d}{dx}\Big(\frac{\psi(x)}{x\psi'(x)}\Big)\Big) \geq 0$. Using the definition of $\psi(x)$, we get

$$\frac{d}{dx}\Big(x\frac{d}{dx}\Big(\frac{e^{f(\ln x)}}{xe^{f(\ln x)}f'(\ln x)\frac{1}{x}}\Big)\Big) \geq 0. \tag{A.6}$$

Define $y := \ln(x)$. We get $\frac{d}{e^y dy}\Big(-\frac{f''(y)}{(f'(y))^2}\Big) \geq 0$. Hence, $\frac{d^2}{dy^2}\Big(\frac{1}{f'(y)}\Big) \geq 0$ for all y, and $(\frac{1}{f'(\cdot)})$ is a convex function.

Sufficiency: We can retrace the steps for proof of the necessary condition. □

Assumption 3 also may be formulated in terms of the function g.

Lemma A.2.3. *Assume that the function ϕ satisfies assumptions 1 and 2. The function $\frac{\partial}{\partial\beta}\Big|\frac{h_\beta(\beta,q_j,q_n)}{h_\beta(\beta,q_l,q_n)}\Big|$ is not equal to zero for all positive β, and positive numbers q_j, q_n,q_l such that $q_j < q_n < q_l$ if and only if function*

$$g'(x+a) - g'(x+c) + \frac{\partial}{\partial x}\ln\Big|\frac{g'(x+a)-g'(x+b)}{g'(x+c)-g'(x+b)}\Big|$$

is not equal to zero for all numbers x and a, b, c such that $a < b < c$.

Proof. Since $|\psi'(x)| = e^{g(\ln x)}$ and $h(\beta, q_j, q_n) = \frac{\psi'(\beta q_j)}{\psi'(\beta q_n)}$, we have

$$h(\beta, q_j, q_n) = e^{g(\ln\beta+\ln q_j)-g(\ln\beta+\ln q_n)},$$

$$\frac{\partial}{\partial\beta}h(\beta, q_j, q_n) = \frac{1}{\beta}h(\beta, q_j, q_n)[g'(\ln\beta+\ln q_j) - g'(\ln\beta+\ln q_n)],$$

and

$$\beta\frac{\partial}{\partial\beta}\ln|\frac{\frac{\partial}{\partial\beta}h(\beta,q_j,q_n)}{\frac{\partial}{\partial\beta}h(\beta,q_l,q_n)}| = \\ = g'(\ln\beta+\ln q_j) - g'(\ln\beta+\ln q_l) + \beta\frac{\partial}{\partial\beta}\ln|\frac{g'(\ln\beta+\ln q_j)-g'(\ln\beta+\ln q_n)}{g'(\ln\beta+\ln q_l)-g'(\ln\beta+\ln q_n)}|. \tag{A.7}$$

Denote $x = \ln\beta$, $a = \ln q_j$, $b = \ln q_n$, $c = \ln q_l$. Then, the right hand side of (A.7) is equal to

$$g'(x+a) - g'(x+c) + \frac{\partial}{\partial x}\ln|\frac{g'(x+a)-g'(x+b)}{g'(x+c)-g'(x+b)}|. \tag{A.8}$$

□

Next, we show the relation between $f(\cdot)$ and $g(\cdot)$, defined in lemma A.2.1 and lemma A.2.2.

Lemma A.2.4. *Using the definitions of $f(\cdot)$ and $g(\cdot)$, we get $\psi(x) = e^{f(\ln\ x)}$, and $|\frac{d}{dx}(\psi(x))| = e^{g(\ln x)}$. Then, $g(y) = f(y) + \ln|\frac{df}{dy}| - y$.*

Proof. From lemma A.2.1, we have $|\psi'(x)| = e^{g(\ln\ x)}$. We get

$$\frac{d\psi}{dx}|_{x=e^y} = e^{g(y)} \tag{A.9}$$

In addition, from lemma A.2.2, we have $\psi(x) = e^{f(\ln\ x)}$. We obtain

$$\frac{d\psi}{dx}(x) = e^{f(\ln\ x)}f'(\ln\ x)\cdot(\frac{1}{x}) \tag{A.10}$$

Hence,

$$|\frac{d\psi}{dx}|_{x=e^y}| = |e^{f(y)}f'(y)\frac{1}{e^y}| \tag{A.11}$$

Using equations (A.9), and (A.11) we get $e^{g(y)} = e^{f(y)}|f'(y)|\frac{1}{e^y}$. This implies

$$g(y) = f(y) + \ln|\frac{df}{dy}| - y. \tag{A.12}$$

□

A.3. Verification of Assumptions

First, we check the properties of assumption 1.

(a) $\phi(\cdot) \in C^2$, $\phi(-x) = -\phi(x)$, $\phi'(x) > 0$, $x\phi''(x) < 0$, for all $x \neq 0$, and $\lim_{x\to\infty} \phi(x) < \infty$.

(i) Suppose $\phi(\cdot) = \tanh(\cdot)$. Notice that $\phi''(x) = -\frac{2\tanh(x)}{\cosh^2(x)}$. It is easy to see that $x\phi''(x) < 0$. The remaining properties can be easily checked.

(ii)) Now we will consider the case $\phi(\cdot) = \arctan(\cdot)$. Notice that $\phi''(x) = -\frac{2x}{(1+x^2)^2}$. Remaining properties can be checked easily.

(b) The function $\phi'(\cdot)$ is invertible.

(i) Suppose that $\phi(x) = \tanh(x)$. Then $\phi'(x) = \text{sech}^2(x)$, where $x \geq 0$. Denote $\text{sech}^2(x) = y$. This implies that $\psi(x) = \text{arctanh}(\sqrt{1-x})$, where $\psi(\cdot) := \phi^{-1}(\cdot)$.

(ii) Next, consider $\phi(x) = \arctan(x)$. This implies that $\phi'(x) = \frac{1}{1+x^2}$. Hence $\psi(x) = \sqrt{\frac{1}{x} - 1}$.

(c) For all $x > 0$, $\psi(x)$ is decreasing function.

(i) Suppose $\psi(x) = \text{arctanh}(\sqrt{1-x})$. Using the definition of $\text{arctanh}(x)$, we get

$$\text{arctanh}(\sqrt{1-x}) = \frac{1}{2}\ln\Big(\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}\Big). \tag{A.13}$$

Now, we evaluate $\psi'(x)$. It can be checked that

$$\psi'(x) = \frac{1}{2x}\Big(\frac{-1+\sqrt{1-x}}{2\sqrt{1-x}} - \frac{1+\sqrt{1-x}}{2\sqrt{1-x}}\Big) = -\frac{1}{2x\sqrt{1-x}} < 0. \tag{A.14}$$

Notice that

$$f(x) = \ln|\psi(e^x)| = \ln(\text{arctanh}(\sqrt{1-e^x})). \tag{A.15}$$

and

$$g(x) = \ln|\psi'(e^x)| = -\ln 2 - x - \frac{1}{2}\ln|e^x - 1|. \tag{A.16}$$

(ii) Similarly, for the case of $\psi(x) = \sqrt{\frac{1}{x} - 1}$, we obtain $\psi'(x) = -\frac{1}{2x^{\frac{3}{2}}\sqrt{1-x}} < 0$. Notice that

$$f(x) = \ln|\psi(e^x)| = \frac{1}{2}(\ln|e^x - 1| - x), \tag{A.17}$$

and

$$g(x) = -\ln 2 - \frac{3}{2}x - \frac{1}{2}\ln|e^x - 1|. \tag{A.18}$$

Thus, all the conditions of Assumption 1 are fulfilled for the functions $\phi(x) = \tanh(x)$ and $\phi(x) = \arctan(x)$.

Now consider Assumption 2. According to lemma A.2.1, we need to check the inequality $g''(x) > 0$.

(i) Suppose that $\phi(x) = \tanh(x)$. Then

$$g''(x) = \Big(-1 - \frac{1}{2}\frac{e^x}{e^x - 1}\Big)' = \frac{e^x}{2(e^x - 1)^2} > 0. \tag{A.19}$$

(ii) Suppose that $\phi(x) = \arctan(x)$. Then

$$g''(x) = \Big(-\frac{3}{2} - \frac{1}{2}\frac{e^x}{e^x - 1}\Big)' = \frac{e^x}{2(e^x - 1)^2} > 0. \tag{A.20}$$

All conditions of Assumption 2 are fulfilled for functions $\phi(x) = \tanh(x)$ and $\phi(x) = \arctan(x)$.

Now consider Assumption 3. According to lemma A.2.3, we need to check the inequality

$$g'(x+a) - g'(x+c) + \frac{\partial}{\partial x}\ln|\frac{g'(x+a) - g'(x+b)}{g'(x+c) - g'(x+b)}| \neq 0 \tag{A.21}$$

for all x and $a < b < c$.

(i) Suppose $\phi(x) = \tanh(x)$. Then $g'(x) = -\frac{3}{2} - \frac{1}{2(e^x-1)}$. Therefore

$$\ln\Big|\frac{g'(x+a) - g'(x+b)}{g'(x+c) - g'(x+b)}\Big| = \ln\Big|\frac{e^a - e^b}{e^b - e^c}\Big| + \ln\Big|\frac{e^{x+c} - 1}{e^{x+a} - 1}\Big|, \tag{A.22}$$

and

$$g'(x+a) - g'(x+c) + \frac{\partial}{\partial x}\ln\Big|\frac{g'(x+a) - g'(x+b)}{g'(x+c) - g'(x+b)}\Big| = \frac{3}{2}\Big(\frac{1}{e^{x+c} - 1} - \frac{1}{e^{x+a} - 1}\Big)$$
$$\neq 0. \tag{A.23}$$

(ii) Suppose that $\phi(x) = \arctan(x)$. Then, the right hand side of equation (A.22) is exactly the same, and therefore the next inequality is also true.

All the conditions of Assumption 3 are fulfilled for the functions $\phi(x) = \tanh(x)$ and $\phi(x) = \arctan(x)$.

Assumption 4 is equivalent to the inequality $f''(x) < 0$ for all x, due to lemma A.2.2.

(i) Suppose that $\phi(x) = \tanh(x)$. Then $f(x) = \ln \operatorname{arctanh}\sqrt{1 - e^x}$, and

$$f'(x) = \frac{-1}{2\sqrt{1 - e^x}\operatorname{arctanh}\sqrt{1 - e^x}}. \tag{A.24}$$

Hence, $f''(x) = \frac{-\sqrt{1-e^x} - e^x \operatorname{arctanh}\sqrt{1-e^x}}{4(1-e^x)^{3/2}\operatorname{arctanh}^2\sqrt{1-e^x}} < 0$.

(ii) For the case $\phi(x) = \arctan(x)$ we have $f(x) = \frac{1}{2}(\ln|1 - e^x| - x)$, and therefore

$$f''(x) = \frac{-1}{2(e^x - 1)^2} < 0. \tag{A.25}$$

All the conditions of Assumption 4 are fulfilled for the functions $\phi(x) = \tanh(x)$ and $\phi(x) = \arctan(x)$.

According to lemma A.2.2, Assumption 5 is equivalent to the inequality $(1/f'(x))'' \geq 0$.

(i) Suppose $\phi(x) = \tanh(x)$. Then $\frac{1}{f'(x)} = -2\sqrt{1 - e^x}\operatorname{arctanh}\sqrt{1 - e^x}$.

Therefore,

$$(\frac{1}{f'(x)})'' = \frac{e^x}{2(1-e^x)}\Big[2\sqrt{1-e^x}\text{arctanh}\sqrt{1-e^x} + \frac{\text{arctanh}\sqrt{1-e^x}}{\sqrt{1-e^x}} - 1\Big]$$
$$> 0. \tag{A.26}$$

(ii) For the case $\phi(x) = \arctan(x)$ we have $f'(x) = \frac{1}{2(e^x-1)}$ and therefore $(\frac{1}{f'(x)})'' = 2e^x > 0$.

Thus, all Assumptions 1 - 5 are satisfied for the functions $\phi(x) = \arctan(x)$ and $\phi(x) = \tanh(x)$.

A.4. Summary

We showed that the assumptions listed in chapter 3 hold true for two neuron transfer functions, namely $\tanh(\cdot)$, and $\arctan(\cdot)$. Therefore, the cost function $f(x) = \sum_{i=1}^{n} c_i\phi(x_i)$, where $\phi(\cdot) = \tanh(\cdot)$ or $\arctan(\cdot)$, has at most one point of local maximum on an arbitrary hyperplane.