SUBFRACTALS INDUCED BY SUBSHIFTS

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ABSTRACT

In this thesis, a subfractal is the subset of points in the attractor of an iterated function system in which every point in the subfractal is associated with an allowable word from a subshift on the underlying symbolic space. In the case in which (1) the subshift is a subshift of finite type with an irreducible adjacency matrix, (2) the iterated function system satisfies the open set condition, and (3) contractive bounds exist for each map in the iterated function system, we find bounds for both the Hausdorff and box dimensions of the subfractal, where the bounds depend both on the adjacency matrix and the contractive bounds on the maps. We extend this result to sofic subshifts, a more general subshift than a subshift of finite type, and to allow the adjacency matrix to be reducible. The structure of a subfractal naturally defines a measure on $\mathbb{R}^n$. For an iterated function system which satisfies the open set condition and in which the maps are similitudes, we construct an invariant measure supported on a subfractal induced by a subshift of finite type. For this specific measure, we calculate the local dimension for almost every point, and hence calculate the Hausdorff dimension for the measure.

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1. CHAPTER ONE - INTRODUCTION

The idea of a fractal was developed to help mathematically define objects that exist in nature. Often times in mathematics, one considers smooth curves, spheres, boxes, or other familiar geometric shapes; however, natural objects can rarely be described using conventional geometric shapes. Consider a snowflake or coastline, both of which are natural objects with highly irregular structures. The term “fractal” was used to describe these chaotic structures, and an entire branch of rich mathematics has been developed to study fractals. While the rise in popularity of fractal geometry is relatively recent, mathematical fractal-like structures were studied as early as the late 1800’s.

1.1. Background

The first fractal in the sense of the modern definition is the everywhere continuous and nowhere differentiable function on $\mathbb{R}$ defined as

$$f(x) = \sum_{k=1}^{\infty} b^k \cos(\pi a^n x),\ a \in \mathbb{Z}^+,\ 0 < b < 1.$$

The function was constructed in 1872 by Karl Weierstraß [27]. Soon after in 1883, as part of his investigation of the set theory and the cardinality of infinite sets, Georg Cantor introduced the famous Cantor Set [3]. While Weierstraß’ function was based on analytic construction, in 1904 Helge von Koch used only geometric methods to construct another everywhere continuous and nowhere differentiable function, which we know today as the Koch curve [18]. Inspired by this work, in 1915 Waclaw Sierpiński constructed the well-known Sierpiński Gasket and Sierpiński Carpet [24]. In 1918, while investigating the global structures of iterations of analytic functions, Gaston Julia and Pierre Fatou independently discovered what is known today as the Julia Set in their papers submitted for the Grand Prix of the French Academy of Sciences [12–14, 17].

Although these works constitute the earliest results in fractal geometry, none of these mathematicians realized that the objects they studied have self-similarity properties and scaling structure. This was partly due to the irregular structure of the objects and the need for deeper computations to reveal their true fractal nature. Another reason was due to the way they were presented, which was not fashionable at the time of their introduction. Hence, for approximately 60 years such mathematical objects were largely ignored. It was due to the pioneering work of Benoit Mandelbrot around the late 1970's that these and many other fractals were brought back to the mathematical realm, and their systematic study began [20, 21]. The major works that laid the foundations and fundamental properties of fractal geometry as we know today were done by J.E. Hutchinson [16], K. Falconer [10, 11] and R.D. Mauldin and S.C. Williams [22] in the early 1980's.

By a fortunate coincidence, while early examples of fractals were being introduced, some other mathematicians were developing the theory of dimension, that would come very handy later on in the structural investigation of fractals. Felix Hausdorff, in his research on the topology of metric spaces, extended the definition of topological dimension to allow for sets to have non-integer dimension values [15]. Later, via covering theorems of Abram Besicovitch, this definition was extended to geometric measure theory resulting in what we know today as the Hausdorff measure and Hausdorff dimension. Following these, the box dimension was introduced by Georges Bouligand around 1939 [2] and the packing dimension was introduced by Claude Tricot around 1982 [26].

1.2. Hausdorff space

As mentioned in the previous section, one can identify or describe a fractal by using properties like self-similarity or recursion, but a more precise process is needed to formally define a fractal. For example, the middle-third Cantor set can be described as follows: start with an interval of length one, cut the interval into three equal pieces and remove the center piece. Repeat this process with the two remaining intervals of length $\frac{1}{3}$. Continue this process. This explanation is easy to visualize, but is too vague to fully understand and investigate all of the unique and interesting properties of the Cantor set. Hence, a more rigorous mathematical approach was developed using an *iterated function system (IFS)* to define not only the Cantor set, but also a wide class of fractal sets. In order to properly define an IFS, one must first understand the space in which these types of fractals reside.

Let $\mathcal{K}$ be a compact metric space with metric d. For purposes of this thesis, we will assume that $\mathcal{K}$ is a compact subset of $\mathbb{R}^n$ equipped with the Euclidean metric, but it should be noted that in many cases we can extend the theory to a more general metric space setting.

Definition 1.2.1. Let $A \subset \mathcal{K}$ and $r > 0$. We define the open *r-neighborhood* of A as

$$N_r(A) = \{y : d(x, y) < r \text{ for some } x \in A\}.$$

Definition 1.2.2. Let $A, B \subseteq \mathcal{K}$ and $r > 0$. The *Hausdorff distance* $D_H : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ is defined as

$$D_H(A, B) = \inf\{r : A \subseteq N_r(B) \text{ and } B \subseteq N_r(A)\}.$$

If we examine D_H on our space $\mathcal{K}$ defined above, then D_H fails to meet all of the properties of a well defined metric, in particular when A or B is not closed. Since an IFS will be defined on compact sets, we will let $\mathscr{H}$ denote the collection of all nonempty, compact subsets of $\mathcal{K}$.

Proposition 1.2.3. *The Hausdorff metric D_H is a metric on the set $\mathscr{H}$.*

A proof of this proposition can be found in [1]. Essentially, a fractal is the limiting set of a convergent sequence of sets, so it will be useful to work in a complete metric space. The metric space $(\mathscr{H}, D_H)$, commonly referred to as the Hausdorff space, will be used to define fractals, so the following theorem will be useful for work with fractals.

Theorem 1.2.4. *$(\mathscr{H}, D_H)$ is a complete metric space.*

A proof of this theorem can be found in [1] or [6].

1.3. Iterated function systems

An iterated function system (IFS) is used to define a class of fractals in a compact metric space, which is typically a compact subset of $\mathbb{R}^n$. Many of the standard definitions and terminology we will state below can be found in [1, 6, 9, 16]. Hutchinson provided the most foundational work with IFSs, and Falconer gives a comprehensive overview of IFSs and their properties [8, 9, 16].

Definition 1.3.1. An *iterated function system* is a finite collection of maps $f = \{f_i : 1 \leq i \leq N\}$, where each map $f_i : \mathcal{K} \to \mathcal{K}$ is well-defined on $\mathcal{K}$ for $1 \leq i \leq N$.

The IFS is applied to $\mathcal{K}$, and the first iteration set is the union of the images of each map in the IFS. More specifically,

$$f(\mathcal{K}) = \bigcup_{i=1}^{N} f_i(\mathcal{K}).$$

This is the first step taken to define a fractal. To continue the process, take repeated iterations of the IFS on $\mathcal{K}$. The fractal is formed by considering the limiting case,

$$\mathcal{F} = \lim_{k \to \infty} f^k(\mathcal{K}),$$

where f^k denotes the composition of f with itself $k-1$ times, for $k \geq 1$. It is natural to ask whether this limit exists, and the answer to that question depends on the maps $\{f_i\}_{i=1}^N$. In the simplest case, one can assume that each f_i is a *similarity*, so that for each f_i there exists a constant $0 < c_i < 1$ such that

$$d(f_i(x), f_i(y)) = c_i d(x, y),$$

for all $x, y \in \mathcal{K}$, $1 \leq i \leq N$. Notice that by letting $r = c_{max} = \max\{c_i : 1 \leq i \leq N\}$, we have $\{f_i(x) : 1 \leq i \leq N\} \subset N_r(\{f_i(y) : 1 \leq i \leq N\})$ and $\{f_i(y) : 1 \leq i \leq N\} \subset N_r(\{f_i(x) : 1 \leq i \leq N\})$ for all $x, y \in \mathcal{K}$. Hence, $D_H(f(A), f(B)) \leq c_{max} D_H(A, B)$, where f denotes the IFS, and for all $A, B \in \mathscr{H}$. Therefore, in this case the IFS f itself is a contractive map. By the contractive mapping principle, the limit exists and is non-empty. In this thesis, we consider a more general case in which each f_i is *hyperbolic*, which means for each f_i, there exist constants $0 < s_i \leq \bar{s}_i < 1$ such that

$$s_i d(x, y) \leq d(f_i(x), f_i(y)) \leq \bar{s}_i d(x, y)$$

for all $x, y \in \mathcal{K}$ and $1 \leq i \leq N$. Since the resulting IFS is contractive, again it follows that the limit (and hence, the fractal) exists.

Now, consider a slightly different case in which the IFS contains a finite set of contractive maps and also the identity map. This specific IFS variant is called a tree-IFS (TIFS). A tree fractal is much like the trees we observe in nature; it has a base and branches, which continue to break into smaller and smaller branches. The identity function in the TIFS ensures that the "branches" are included in the entire fractal, as each finite iteration appears in the fractal. Due to the inclusion of the identity map, it is not immediately obvious that the TIFS is a contractive map. Hence, we must prove that a tree fractal actually exists without using the contractive mapping principle. This will be discussed explicitly in the Section 1.5, but we need some tools from symbolic dynamics before we can prove that statement.

1.4. Symbolic dynamics

Each finite IFS, say $\{f_1, \dots f_N\}$ is naturally associated with a finite alphabet $\mathcal{A} = \{1, \dots, N\}$, where each letter from $\mathcal{A}$ is associated with a map from the IFS. As mentioned in the previous section, while constructing a fractal we consider all compositions of the maps from the IFS in all possible orders. This can be tedious to write out, so we use symbolic dynamics with this alphabet to simplify the process.

Let $\mathcal{A} = \{1, \dots, N\}$ be a finite alphabet and consider $X = \mathcal{A}^{\mathbb{N}}$, the collection of all infinite one-sided strings in which each coordinate is occupied by a letter from $\mathcal{A}$. This space is often times called a *shift space* because it is typically equipped with both a metric and the shift map (to be defined below). First, let us define a metric on X. While it is not the only metric on X, one of the most frequently used metrics is defined as follows: let $\omega, \tau \in X$, where we denote specific coordinates by $\omega = \omega_1\omega_2\omega_3\dots, \tau = \tau_1\tau_2\tau_3\dots$, and

$$d_X(\omega,\tau) = \begin{cases} \frac{1}{2^k}, \text{ where } k = \min\{i : \omega_i \neq \tau_i\} & \text{for } \omega \neq \tau \\ 0 & \text{for } \omega = \tau. \end{cases}$$

By the definition of d_X, we immediately obtain (1) $d_X(\omega,\omega) = 0$, (2) $d(\omega,\tau) = 0$ implies that $\omega = \tau$, and (3) $d_X(\omega,\tau) = d_X(\tau,\omega)$. Let $\omega, \tau, \xi \in X$ and suppose $d_X(\omega,\xi) = \frac{1}{2^{d_1}}$ and $d_X(\xi,\tau) = \frac{1}{2^{d_2}}$. Now, suppose $d_X(\omega,\tau) = \frac{1}{2^{d_0}}$ so that $\omega_i = \tau_i$ for $1 \leq i < d_0$. Suppose $d_1 \leq d_0$. Then the inequality $\frac{1}{2^{d_0}} \leq \frac{1}{2^{d_1}} + \frac{1}{2^{d_2}}$ follows immediately. Now, assume that $d_1 > d_0$. This means that $\omega_i = \xi_i$ for $1 \leq i < d_1$, which implies that $\omega_{d_0} = \xi_{d_0}$. Hence, $\tau_{d_0} \neq \xi_{d_0}$ and $d_0 = d_2$. Therefore, $\frac{1}{2^{d_0}} \leq \frac{1}{2^{d_1}} + \frac{1}{2^{d_2}}$. Thus, the map d_X satifies $d_X(\omega,\tau) \leq d_X(\omega,\xi) + d_X(\xi,\tau)$; that is, d_X is indeed a metric on X.

The shift map $\sigma : X \to X$ is defined as, for $\omega_1\omega_2 \dots \in X$,

$$\sigma(\omega_1\omega_2\omega_3\dots) = \omega_2\omega_3\omega_4\dots.$$

Notice that for $\varepsilon > 0$ there exists an $m > 1$ such that $\frac{1}{2^m} < \varepsilon$. If we choose $\delta = \frac{1}{2^{m+1}}$ and select $\omega, \tau \in X$ with $d_X(\omega,\tau) < \delta$, then $d_X(\sigma(\omega),\sigma(\tau)) < \frac{1}{2^m} < \varepsilon$. Hence, the shift map is continuous. It should also be noted that σ is an onto function, so that $\sigma(X) = \{\sigma(\omega) : \omega \in X\} = X$. The shift map will be used extensively in Chapter 3.

The following are common notations used in symbolic dynamics. A finite word of length n is of the form $\omega = \omega_1\omega_2\ldots\omega_n$ where $\omega_i \in \mathcal{A}$ for $1 \le i \le n$, and we will use $\ell(\omega)$ to denote the length of the word ω. Let $B_n(X) = \{\omega = \omega_1 \ldots \omega_n : \omega_i \in \mathcal{A}, 1 \le i \le n\}$ denote the set of all finite words of length n and $B_*(X) = \bigcup_{n\ge1} B_n(X)$ denote the set of all words of finite length. For $\omega \in B_n(X), \tau \in B_m(X)$, and $\xi \in X$, we will use the following notations:

$$\omega\tau = \omega_1 \ldots \omega_n\tau_1 \ldots \tau_m$$

$$\omega^- = \omega_1 \ldots \omega_{n-1}$$

$$\xi|_k = \xi_1 \ldots \xi_k \text{ for all } k \ge 1.$$

In the more familiar metric space $\mathbb{R}^n$, open balls are the "building blocks" of the space, in the sense that any set can be approximated by open balls with minimal error. Fortunately, an analogous structure exists in X, called a cylinder set, to serve as the open ball and help in understanding the structure of X. In particular, it is instrumental in constructing measures on this space. A *cylinder set* with base $\omega \in B_n(X)$ is the set

$$[\![\omega]\!] = \{\tau \in X : \tau_i = \omega_i \text{ for } 1 \le i \le n\}.$$

Fix a cylinder set $[\![\omega]\!]$ and let $\tau \in [\![\omega]\!]$. Suppose $\ell(\omega) = n$ and let $N > n$. Now, consider another point $\xi \in X$ such that $d_X(\tau, \xi) < \frac{1}{2^N}$. Then, $\tau_i = \xi_i$ for $1 \le i < N$, and in particular, for $1 \le i \le n$. Hence, $\xi \in [\![\omega]\!]$ and the cylinder set $[\![\omega]\!]$ is open. Next, let $\{\tau^1, \tau^2, \ldots\} \subset [\![\omega]\!]$ be a convergent sequence, say $\tau^i \to \tau$. Again, choose $N > n$. Since the sequence is convergent, there exists τ^k in the sequence such that $d_X(\tau^k, \tau) < \frac{1}{2^N}$. Hence, $\tau_i = \tau_i^k$ for $1 \le i < N$, and in particular for $1 \le i \le n$. Hence, $\tau \in [\![\omega]\!]$ and the cylinder set $[\![\omega]\!]$ is closed. Therefore, cylinder sets in X are both open and closed. More information on shift spaces can be found in [19].

Now, this language can be used to define a fractal using an IFS. To simplify the notation, for $\omega \in B_n(X)$ we will use the notation $f_\omega = f_{\omega_n} \circ f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_1}$. The n-th iteration of an IFS can now be written as

$$f^n(\mathcal{K}) = \bigcup_{\omega \in B_n(X)} f_\omega(\mathcal{K}),$$

and hence, the attractor of the IFS can be written as

$$\lim_{n\to\infty} \bigcup_{\omega\in B_n(X)} f_\omega(\mathcal{K}).$$

Since we will frequently work in both X and on the attractor $\mathcal{F}$, it will be helpful to define the associated coding map $\pi : X \to \mathcal{F}$ by $\pi(\omega) = \lim_{n\to\infty} f_{\omega|_n}(\mathcal{K})$.

1.5. Examples

Example 1.5.1 (Cantor Set)**.** One of the most simple examples of a fractal is the Cantor set. Let $\mathcal{K} = [0, 1]$, and define the following maps in the IFS:

$$f_0(x) = \frac{1}{3}x \qquad \text{and} \qquad f_1(x) = \frac{1}{3}x + \frac{2}{3}.$$

The first four iterations of the Cantor set are pictured below.

Figure 1.1. Cantor Set

Notice that every end point of every interval in any iteration is a point in the Cantor set. For example, $\frac{1}{3}$ and $\frac{8}{9}$ are points in the Cantor set. It may be tempting to assume that all points in the Cantor set are rational, but some quick analysis using symbolic dynamics reveals that irrational points exist in the Cantor set. Consider a point x in the Cantor set with $\pi(\omega) = x$, where π denotes the coding map and for some $\omega \in \{0, 1\}^{\mathbb{N}}$. Now, replace each 1 in ω with a 2 so that $\omega \in \{0, 2\}^{\mathbb{N}}$. We can calculate the location of the point as follows,

$$x = \sum_{i=1}^{\infty} \frac{\omega_i}{3^i}.$$

For any periodic or eventually periodic sequence ω, x will be a rational number. However, if

ω is non-repeating, x will be an irrational number. It can also be shown that the Cantor set is uncountable, but has zero Lebesgue measure (or length). For this reason, the Cantor set is a popular counterexample for many problems in analysis. It is also an example of a totally disconnected set, which means that the only connected subsets of the Cantor set are singletons.

Example 1.5.2 (Sierpiński's Triangle)**.** Another well-known and frequently used fractal is Sierpiński's triangle. Let $\mathcal{K}$ be a compact set in $\mathbb{R}^2$. For simplicity, we may assume that $\mathcal{K}$ is an equilateral triangle of side length 1, with the left-most vertex at the origin. Define the maps in the IFS as follows,

$$f_0(x,y) = \left(\frac{1}{2}x, \frac{1}{2}y\right), \quad f_1(x,y) = \left(\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y\right), \text{ and } \quad f_2(x,y) = \left(\frac{1}{2}x + \frac{1}{4}, \frac{1}{2}y + \frac{\sqrt{3}}{2}\right).$$

The first four iterations of Sierpiński's triangle are pictured below.

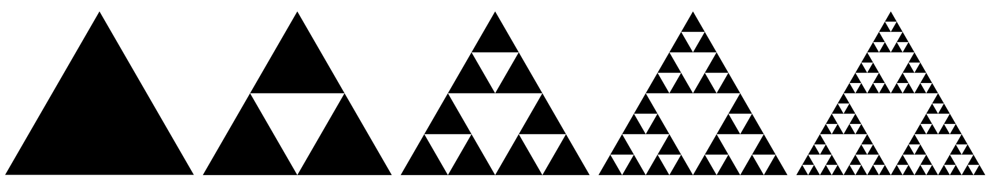

Figure 1.2. Sierpiński's triangle

Unlike the Cantor set, Sierpiński's triangle is not totally disconnected. In fact, the boundary of every triangle that appears in every finite iteration is part of Sierpiński's triangle, meaning that line segments exist within Sierpiński's triangle. However, the Lebesgue measure (or area) of Sierpiński's triangle is zero.

1.6. Tree fractals

Let $A_0 \subset \mathcal{K}$ be a compact, connected set with finite boundary, and consider a collection of contractive maps $f_i : \mathcal{K} \to \mathcal{K}$ defined by

$$f_i(x) = c_i x + \alpha_i, \text{ where } 0 < c_i < 1 \text{ and } \alpha_i \in \mathcal{K} \text{ for } 1 \le i \le N.$$

Define $f = \{f_0, f_1, \ldots, f_N\}$, where $f_0 : X \to X$ is given by $f_0(x) = x$, and let $f(A_0) = \bigcup_{i=0}^{N} f_i(A_0)$.

We will choose α_i for $1 \leq i \leq N$ such that $f(A_0)$ is a connected set. The iterated function system $f = \{f_0, f_1, \ldots, f_N\}$ is called the *tree iterated function system* (TIFS).

Since f_0 is the identity map, and f_i is a contractive map for $1 \leq i \leq N$, then for a large enough choice of k, the Hausdorff distance between $f^k(A_0)$ and $f^{k+1}(A_0)$ be can arbitrarily small. Since $f : \mathscr{H} \to \mathscr{H}$ is "almost" a contractive map, but not necessarily a contractive map, one needs to check that the sequence $\{f^k(A_0)\}_{k=0}^{\infty}$ converges.

Proposition 1.6.1. *Let $A_0 \in \mathscr{H}$ and f be a TIFS. The sequence $\{f^k(A_0)\}_{k\geq 0}$ converges in $(\mathscr{H}, D)$.*

Proof. Let $f = \{f_i\}_{i=0}^{N}$ where f_0 is the identity map, and each f_i is a contractive map with contractive factor c_i for $1 \leq i \leq N$. First, notice that $f^k(A_0) \subset f^{k+1}(A_0)$. Also, notice that if $\omega \in B_k(X)$ contains 0 as an entry, then f_ω contains the identity map, and hence, can be represented by f_τ for some $\tau \in B_m(X)$ with $m < k$. Hence, $f_\omega(A_0) \subset f^m(A_0) \subseteq f^{k-1}(A_0)$. So, we can write $f^{k+1}(A_0) = f^k(A_0) \cup \left(\bigcup_{\omega \in B_{k+1}(Y)} f_\omega(A_0) \right)$, where Y is the full shift with respect to the alphabet $\hat{\mathcal{A}} = \{1, 2, \ldots, N\}$.

Using this observation, we obtain $D_H(f^k(A_0), f^{k+1}(A_0)) = D_H(f^k(A_0), \bigcup_{\omega\in B_{k+1}(Y)} f_\omega(A_0))$. Let $c_{max} = \max\{c_i : 1 \leq i \leq N\}$ and $d_0 = |A_0| = \text{diam}(A_0)$. Then, $D_H(f^k(A_0), \bigcup_{\omega\in B_{k+1}(Y)} f_\omega(A_0))$ $\leq c_{max}^{k+1} d_0$. Since $c_{max} < 1$, then we can choose k large enough such that $D_H(f^k(A_0), f^{k+1}(A_0))$ can be made arbitrarily small. Hence, for $\varepsilon > 0$, we can find M such that $\dfrac{c_{max}^{M+1}}{1 - c_{max}} d_0 < \varepsilon$. Then, for $m, n \geq M$ with $m < n$, we have:

$$\begin{aligned} D_H(f^m(A_0), f^n(A_0)) &\leq \sum_{l=0}^{n-m-1} D_H(f^{m+l}(A_0), f^{m+l+1}(A_0)) \leq \sum_{l=0}^{n-m-1} c_{max}^{m+l+1} d_0 \\ &= \frac{c_{max}^m - c_{max}^n}{1 - c_{max}} c_{max} d_0 < \frac{c_{max}^{m+1}}{1 - c_{max}} d_0 \leq \frac{c_{max}^{M+1}}{1 - c_{max}} d_0 < \varepsilon. \end{aligned}$$

This implies that the sequence $\{f^k(A_0)\}_{k=0}^{\infty}$ is a Cauchy sequence in the space $(\mathscr{H}, D_H)$. Hence, the sequence $\{f^k(A_0)\}_{k=0}^{\infty}$ must be convergent since $(\mathscr{H}, D_H)$ is a complete metric space [16]. $\square$

By Proposition 1.6.1, $\mathcal{F} = \lim_{k\to\infty} f^k(A_0)$, is well-defined. The set $\mathcal{F}$ is the attractor of the TIFS, and we will call $\mathcal{F}$ the *(full) tree fractal.* The attractor $\mathcal{F}$ depends on the set A_0, since the contractive mapping principle does not apply to the TIFS.

A tree fractal is interesting on its own, but the canopy is much more interesting due to its fractal-like structure. In order to identify the canopy of a tree fractal, which is the attractor of an IFS $\hat{f} = \{f_1, f_2, \ldots, f_N\}$, we observe that the main role of the identity map is preserving the "trunk" of the tree; hence, the "branches" of the tree are built by the remaining contractions $f_1, \ldots, f_N$, and then preserved under the identity map. Notice that f is not a contractive map on $(\mathscr{H}, D)$. However, $\hat{f}$ is an IFS, just like the IFSs that were introduced in section 1.3. Therefore, we know an attractor exists with respect to $\hat{f}$. We define the *canopy* of the TIFS to be $\mathcal{F}_c = \lim_{n\to\infty} \hat{f}^n(A_0)$.

Example 1.6.2 (T-tree)**.** Let $X_0 = \{(0, y) : 0 \leq y \leq 1\} \cup \{(x, 1) : -1 \leq x \leq 1\}$ be our initial set and $\{f_0, f_1, f_2\}$ be the IFS with

$$f_0(x, y) = (x, y), \quad f_1(x, y) = \left(\frac{1}{2}x - 1, \frac{1}{2}y + 1\right) \quad \text{and} \quad f_2(x, y) = \left(\frac{1}{2}x + 1, \frac{1}{2}y + 1\right).$$

Notice that in the case of the T-tree, the maps f_i will not map from $X_0 \to X_0$, but we can find a compact set $\mathcal{K}$ such that $\mathcal{F} \subset \mathcal{K}$ so that $f_i : \mathcal{K} \to \mathcal{K}$ for all $0 \leq i \leq 2$. However, we are interested in the image of X_0. The first two iterations of the T-tree are displayed below.

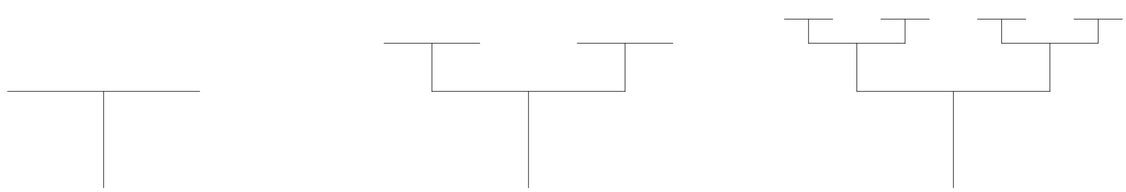

Figure 1.3. T-tree

The contractive factors chosen for this example (both $\frac{1}{2}$) create an interesting situation, where the canopy of this T-tree is the full line segment $\{(x, 2) : -2 \leq x \leq 2\}$. By varying the contractive ratios, other interesting examples arise. For instance, take the similarity ratios for both f_1 and f_2 to be $\frac{1}{3}$. In this case, the canopy will form a Cantor set. By choosing different contractive ratios for each map, say $\frac{1}{3}$ for f_1 and $\frac{1}{2}$ for f_2, the resulting canopy will not longer be a subset of a line, but will rather be scattered in a diagonal-like pattern.

Example 1.6.3 (Sierpiński's Triangle Tree)**.** Let $X_0 = l_1 \cup l_2 \cup l_3$, where each l_i is a line segment of length 1, emanating from the origin with $\frac{2\pi}{3}$ angle of separation between each line. (We will assume

l_1 has endpoints $(0,0)$ and $(0,1)$, l_2 has endpoints $(0,0)$ and $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$, and l_3 has endpoints $(0,0)$ and $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$.) Next, let

$$f_1(x,y) = \left(\frac{1}{2}x, \frac{1}{2}y + 1\right),\ f_2(x,y) = \left(\frac{1}{2}x - \frac{\sqrt{3}}{2}, \frac{1}{2}y - \frac{1}{2}\right),\ \text{and } f_3(x,y) = \left(\frac{1}{2}x + \frac{\sqrt{3}}{2}, \frac{1}{2}y - \frac{1}{2}\right).$$

In this case, the canopy of the tree fractal will form Sierpiński's triangle. The first two iterations are pictured below.

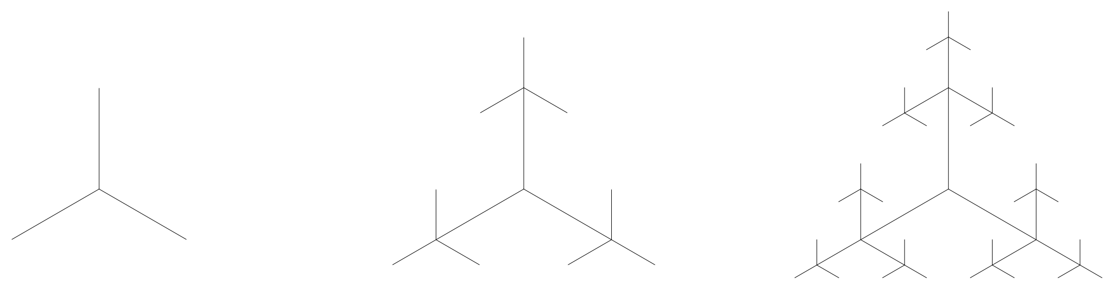

Figure 1.4. Sierpiński Triangle Tree

2. CHAPTER TWO - FRACTAL DIMENSIONS

Since fractals behave in a way that defies our normal conventions for measurement, like countability and Lebesgue measure, the concept of fractal dimension was developed as a way to quantify the space filled by a fractal. Although fractal dimensions can be defined in vastly different ways, they each serve a purpose for different types of problems. In this chapter, we will formally define three types of fractal dimensions and will discuss the foundations for fractal dimension calculations for IFSs.

2.1. Fractal dimension definitions

In order for a fractal dimension to make sense, there are desirable properties for such a dimension to satisfy. Let us consider the following properties for a fractal dimension, denoted by $\dim(\cdot)$ [9].

- *Monotonicity:* For any subset $A \subset B$, the dimension satisfies $\dim(A) \leq \dim(B)$.
- *Countable stability:* If $\{E_1, E_2, \ldots\}$ is a countable collection of sets, then $\dim\left(\bigcup_{i=1}^{\infty} E_i\right) = \sup_{i\geq 1}\{\dim(E_i)\}$.
- *Countable sets:* Let A be a countable set. Then, $\dim(A) = 0$.
- *Open sets in $\mathbb{R}^n$:* Let $A \subset \mathbb{R}^n$ be an open set. Then, $\dim(A) = n$.

Although this is not an exhaustive list of desirable properties, it gives basic expectations for a dimension to satisfy.

One of the most frequently used fractal dimensions is Hausdorff dimension. The popularity of Hausdorff dimension is due to the fact it is defined by a measure, which means many problems involving Hausdorff dimension can be solved by exploiting properties of the measure. However, in practice, Hausdorff dimension can be quite difficult to calculate. In order to formally define Hausdorff dimension, we must first define Hausdorff measure. Let $E \subseteq \mathcal{K}$. Letting $\overline{\mathcal{H}}^s_\varepsilon(E) = \inf_{\mathcal{U}\in\mathcal{O}} \sum_{U\in\mathcal{U}} (\mathrm{diam}(U))^s$, where $\mathcal{O}$ is the collection of all open ε-covers of E and $s \geq 0$ is fixed, the s-dimensional Hausdorff outer measure is defined to be $\overline{\mathcal{H}}^s = \lim_{\varepsilon\to 0} \overline{\mathcal{H}}^s_\varepsilon$. Restricting the outer measure to measurable sets, one defines the s-dimensional Hausdorff measure, $\mathcal{H}^s$. The *Hausdorff dimension of* E, denoted $\dim_H(E)$, is defined as the unique value of h such that:

$$\mathcal{H}^s(E) = \begin{cases} 0, & s > h \\ \infty, & s < h. \end{cases}$$

Hausdorff dimension satisfies a number of properties, including all of the properties listed at the beginning of this section [9].

Another commonly used fractal dimension is box dimension. Box dimension is popular because it is easy to define and to use in practice. For example, scientists estimate box dimension to quantify the complexity of a natural fractal-like object, such as a coast line. For a non-empty, bounded set $E \subset \mathcal{K}$, let $N_r(E)$ denote the smallest number of sets of diameter at most r that can cover E. The *lower and upper box dimensions of* E are defined, respectively, as

$$\underline{\dim}_B(E) = \liminf_{r \to 0} \frac{\log N_r(E)}{-\log r} \text{ and } \overline{\dim}_B(E) = \limsup_{r \to 0} \frac{\log N_r(E)}{-\log r}.$$

In the case in which the limit exists so that the upper and lower box dimensions are equal, we refer to this as the *box dimension* [9], and denote it by $\dim_B(\cdot)$

Box dimension is relatively easy to use in calculations, but is sometimes difficult to build theory upon due to its definition. Another major disadvantage with box dimension is that it is not countably stable. It can be shown that for a set E, $\dim_B(E) = \dim_B(\bar{E})$, where $\bar{E}$ denotes the closure of E [9]. Consequently, a dense subset has the same box dimension as the set in which it is dense. For example, $\mathbb{Q} \cap [0,1]$ is dense in $[0,1]$, and hence $\dim_B(\mathbb{Q} \cap [0,1]) = \dim_B([0,1]) = 1$. The set $\mathbb{Q} \cap [0,1]$ is countable with $\dim_B(x) = 0$ for all $x \in \mathbb{Q} \cap [0,1]$. Therefore, box dimension cannot be countably stable.

The definitions of Hausdorff and box dimensions are vastly different, with Hausdorff dimension relying on a measure and box dimension relying on a covering. Another fractal dimension, called *packing dimension*, was developed by defining a measure using a packing (instead of a covering) of disjoint balls of varying diameters. More formally, let a δ-packing of E be a countable collection of disjoint balls, $\{B_i\}$, of radii at most $\delta > 0$ with centers in E. For $\delta > 0$, define functions

$$P_\delta^s(E) = \sup\left\{\sum_{i=1}^{\infty}(\mathrm{diam}(B_i))^s : \{B_i\}_{i\geq 1} \text{ is a } \delta\text{-packing of } E\right\}$$

and $P_0^s(E) = \lim_{\delta\to 0} P_\delta^s(E)$. Now, the s-dimensional packing measure of E is defined as

$$P^s(E) = \inf\left\{\sum_{i=1}^{\infty} P_0^s(E_i) : E \subset \bigcup_{i=1}^{\infty} E_i\right\}.$$

The packing dimension of E, denoted $\dim_P(E)$, is the unique value of s such that:

$$P^r(E) = \begin{cases} 0, & r > s \\ \infty, & r < s. \end{cases}$$

For $E \subset \mathcal{K}$, the following inequalities are well-known [6, 9]

$$\dim_H(E) \leq \dim_P(E) \leq \overline{\dim}_B(E) \text{ and } \dim_H(E) \leq \underline{\dim}_B \leq \overline{\dim}_B(E).$$

2.2. Fractal dimension of IFS attractor

Given the definitions of fractal dimensions, one may assume that actual dimension calculations would be difficult. However, in certain cases for IFSs, the fractal dimensions of the attractors can be obtained via relatively straight-forward calculations, provided they satisfy a well-known condition.

Definition 2.2.1. An IFS $\{\mathcal{K}; f_1, \ldots, f_N\}$ satisfies the *open set condition (OSC)* if there exists some non-empty, bounded, open subset $U \subset \mathcal{K}$ such that $\bigcup_{i=1}^{N} f_i(U) \subset U$, where the union is disjoint.

The following result exists for IFSs containing similarities which satisfy the OSC [9, 16].

Proposition 2.2.2. *Let $\{\mathcal{K}; f_1, \ldots, f_N\}$ be an IFS satisfying the OSC, where each f_i is a similarity with similarity ratio $0 < c_i < 1$ for $1 \leq i \leq N$. Then, $\dim_H(\mathcal{F}) = s$, where $\mathcal{F}$ is the attractor and s satisfies*

$$\sum_{i=1}^{N} c_i^s = 1.$$

If each f_i is a contractive mapping instead of a similarity with $c_i d(x,y) \leq d(f_i(x), f_i(y)) \leq \bar{c}_i d(x,y)$ for each $1 \leq i \leq N$ and all $x, y \in \mathcal{K}$, then $s \leq \dim_H(\mathcal{F}) \leq \bar{s}$ where $\sum_{i=1}^{N} c_i^s = 1 = \sum_{i=1}^{N} \bar{c}_i^{\bar{s}}$ [9].

One variation of an IFS attractor is a *Markov attractor* of an IFS, which is defined via N contractive functions and an associated $N \times N$ matrix A. The Markov attractor of an IFS only contains those points associated with an *admissible* sequence with respect to matrix A, which is a sequence of integers $(i_l)_{l \geq 1}$ such that $(A)_{i_l, i_{l+1}} \neq 0$ for all $l \geq 1$. A matrix A is *primitive* if there exists some integer M such that $(A^M)_{ij} > 0$ for all $1 \leq i, j \leq N$, where $(A^M)_{ij}$ denotes the ij-entry of A^M.

Let $\mathcal{F}_A$ denote the Markov attractor, i.e. the collection of all points in $\mathcal{F}$ which are associated with an admissible sequence with respect to A. An IFS is called *disjoint* if $f_i(\mathcal{F}) \cap f_j(\mathcal{F}) = \emptyset$ for all $i \neq j$ and $1 \leq i, j \leq N$, which is a stronger condition than the OSC. In [7], Ellis and Branton proved the following theorem about the Hausdorff dimension of a Markov attractor of an IFS.

Theorem 2.2.3. *Let A be a primitive $N \times N$ (0,1)-matrix and $\mathcal{F}_A$ be the Markov attractor of a disjoint IFS $(\mathcal{K}; f_1, \ldots, f_N)$. Suppose that $s_i d(x,y) \leq d(f_i(x), f_i(y)) \leq \bar{s}_i d(x,y)$, for all $x, y \in \mathcal{K}$, $1 \leq i \leq N$, and for some constants $0 < s_i \leq \overline{s_i} < 1$. Then,*

$$\dim_H(\mathcal{F}_A) \leq u, \quad \textit{for the value } u \textit{ which satisfies } \rho(A\bar{S}^u) = 1,$$

where $\rho(\cdot)$ denotes the spectral radius and $\bar{S}^u$ is the diagonal matrix with diag$(\bar{s_1}^u, \ldots, \bar{s_n}^u)$.

In the same paper [7], Ellis and Branton made the following conjecture for the lower bound: $\dim_H(\mathcal{F}_A) \geq l$ where $\rho(AS^l) = 1$ and S^l is a diagonal matrix with $\mathrm{diag}({s_1}^l, \ldots, {s_n}^l)$.

An $N \times N$ matrix A is called *irreducible* if given i, j, $1 \leq i, j \leq N$, there exists some M such that $(A^M)_{ij} \neq 0$. Every primitive matrix is irreducible, but there exist matrices which are irreducible and not primitive [19]. Roychowdhury proved the conjecture proposed by Ellis and Branton, and also generalized Theorem 2.2.3 by allowing the matrix A to be irreducible and requiring the IFS only satisfy the OSC [23].

In Chapter 3, we will relate the attractors of the systems in these results to subfractals. In Chapter 4, we will extend the results of Roychowdhury to include an analogous result for attractors induced by reducible matrices and more general subfractals.

2.3. Examples

Example 2.3.1 (Cantor Set)**.** Recall the construction of the Cantor set in Chapter 1. Each map in the IFS for the Cantor set is a similitude with similarity ratio $\frac{1}{3}$. Also, the IFS is disjoint, so by applying Theorem 2.2.2, we are looking for the value h which satisfies

$$\frac{1}{3}^h + \frac{1}{3}^h = 1.$$

Hence, the Hausdorff dimension of the Cantor set is given by $\dim_H(\mathcal{F}) = \frac{\log 2}{\log 3}$.

Using the known relationships between different fractal dimensions, $\frac{\log 2}{\log 3}$ is a lower bound for the box dimension of the Cantor set. Notice at the first iteration that we have two intervals, each of length $\frac{1}{3}$, and at the second iteration we have four intervals, each of length $\frac{1}{9}$. Continuing this pattern, we notice that at iteration k, we have 2^k intervals, each of length $\frac{1}{3^k}$. Let $0 < r < 1$ be given and choose k such that $\frac{1}{3^{k+1}} \leq r \leq \frac{1}{3^k}$. Notice for this value of r, it must be true that $N_r(\mathcal{F}) < 2^{k+1}$. Then, we obtain

$$\overline{\dim}_B(\mathcal{F}) = \lim_{r \to 0} \frac{\log(N_r(\mathcal{F}))}{-\log r} \leq \lim_{k \to \infty} \frac{\log(2^{k+1})}{\log(3^k)} = \frac{\log 2}{\log 3}.$$

Hence, the box dimension of the Cantor set is given by $\dim_B(\mathcal{F}) = \frac{\log 2}{\log 3}$.

Example 2.3.2 (Sierpiński's Triangle)**.** Sierpiński's Triangle also has an IFS containing three similitudes, each with similarity ratio $\frac{1}{2}$. The IFS satisfies the OSC in this case, so again by Theorem 2.2.2, we obtain $\dim_H(\mathcal{F}) = \frac{\log 3}{\log 2}$. Following the same arguments as in the Cantor set example, the box dimension of Sierpiński's triangle is also given by $\dim_B(\mathcal{F}) = \frac{\log 3}{\log 2}$.

Example 2.3.3 (Markov attractor of Cantor set IFS)**.** Now, let the Cantor set be as defined above, but now let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Let $\mathcal{F}_A$ denote the set of all points in the Cantor set which correspond to an admissible sequence with respect to A. Since A is a primitive matrix (and therefore irreducible matrix), we can apply Theorem 2.2.3. First, let us calculate the matrices

$$S = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \text{ and } AS^{(h)} = \begin{bmatrix} \frac{1}{3}^h & \frac{1}{3}^h \\ \frac{1}{3}^h & 0 \end{bmatrix}$$

necessary for the theorem. We are looking for the value of h such that $\rho(AS^{(h)}) = 1$. Notice that the eigenvalues of $AS^{(h)}$ are $\lambda_1 = \frac{1}{3}^h \cdot \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1}{3}^h \cdot \frac{1-\sqrt{5}}{2}$. Hence, by Theorem 2.2.3, the Hausdorff dimension is given by

$$\dim_H(\mathcal{F}_A) = \frac{\log(\frac{1+\sqrt{5}}{2})}{\log 3}.$$

It should be noted here that Example 2.3.3 is not surprising, given that each map in the IFS has the same contractive factor. The matrix A chosen for this example is related to a special subshift, called the Golden Mean Shift, and we see that the spectral radius of A is the golden ratio, $\frac{1+\sqrt{5}}{2}$. The true flavor of Theorem 2.2.3 appears in examples in which the IFS has varying contractive ratios for different maps. The reader can find more interesting examples later in Chapter 4, after a more thorough explanation of subshifts is given.

3. CHAPTER THREE - SUBFRACTALS

Finding a definition for a subfractal is not a trivial task. Due to the recursive behavior of fractals, an arbitrary subset of a fractal will likely inherit most of the properties of the fractal as a whole. In particular, the Hausdorff and box dimensions of the subset may be equal to the Hausdorff and box dimensions, respectively, of the whole fractal. Is there a way to define a "subfractal" such that the set is genuinely different than the whole fractal, in the sense that the fractal dimensions of the subfractal and whole fractal are not equal?

The answer to that question is yes, but we must first develop a deeper understanding of the associated symbolic space. In particular, we will examine different types of subshifts of the full shift space, X, and will eventually define a subfractal using specific types of subshifts.

3.1. Subshift of finite type

Let (X, σ) be a shift space. Let us consider a subset $Y \subseteq X$ that is *shift invariant*, which means that $\sigma(Y) = \{\sigma(\omega) : \omega \in Y\} = Y$.

Definition 3.1.1. A *subshift* of X is a subset $Y \subseteq X$ that is both shift invariant and closed.

One way to define a subshift is by using a list of forbidden words. Given a subshift $Y \subset X$, a *forbidden word* is a finite string $\omega \in B_*(X)$ that does not appear anywhere in τ for all $\tau \in Y$.

Definition 3.1.2. A *subshift of finite type (SFT)* is a subshift which can be described with a finite list of forbidden words.

SFTs are not only easy to describe, but also have valuable connections with matrices and graphs that help us understand its properties in detail.

In order to formally describe the connection between SFTs and matrices, we must first define an operation on finite words. Let $\omega, \xi \in B_{k-1}(X)$. The word ω is *compatible with* ξ if $\omega_2 \ldots \omega_{k-1} = \xi_1 \ldots \xi_{k-2}$. A compatible pair is a pair $(\omega, \xi) \in B_{k-1}(X) \times B_{k-1}(X)$, where ω is compatible with ξ. Let $(B_{k-1}(X) \times B_{k-1}(X))_{comp}$ denote the collection of all compatible pairs $(\omega, \xi) \in B_{k-1}(X) \times B_{k-1}(X)$. Define an operation $\odot : (B_{k-1}(X) \times B_{k-1}(X))_{comp} \to B_k(X)$ by $\omega \odot \xi = \omega_1 \omega_2 \ldots \omega_{k-1} \xi_{k-1}$ $(= \omega_1 \xi_1 \xi_2 \ldots \xi_{k-1})$.

Let X be the full shift with alphabet $\mathcal{A} = \{1, \ldots, m\}$. Let X_F be an SFT with forbidden words $F = \{\tau^1, \ldots, \tau^l\}$. Suppose that $\ell(\tau^i) < \ell(\tau^j)$ for some $1 \leq i, j \leq l$. If $\ell(\tau^j) = n$, then we can replace τ^i with finitely many words, each of length n, which contain the string τ^i. Without loss of generality, we can assume $\tau^i \in B_k(X)$ for all $1 \leq i \leq l$. Let $N = m^{k-1}$, where $m = |\mathcal{A}|$. We will construct an N x N adjacency matrix A as follows. Label the rows with all possible words (both allowable and forbidden) of length $k-1$, i.e. label the rows with $\{\omega^1, \ldots, \omega^N\} = B_{k-1}(X)$. Label the corresponding columns similarly. Let the entry be $a_{ij} = 0$ if ω^i is not compatible with ω^j or $a_{ij} = 0$ if ω^i is compatible with ω^j but $\omega^i \odot \omega^j \in F$. The entry $a_{ij} = 1$ if ω^i is compatible with ω^j and $\omega^i \odot \omega^j \in B_k(X_F)$.

For the sake of clarity, consider the following examples. First, consider the SFT on the alphabet $\mathcal{A} = \{0, 1\}$ with forbidden word $F_1 = \{11\}$. The forbidden word has length 2, and therefore the adjacency matrix will be 2×2 since $|B_1(X)| = 2$. The adjacency matrix will be of the form:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Next, let us consider an SFT on the same alphabet $\mathcal{A} = \{0, 1\}$ but with forbidden word list $F_2 = \{001, 100, 111\}$. Since each forbidden word has length 3, we will need to consider a 4 x 4 matrix since $|B_2(X)| = 4$. We will choose the following labeling of rows: $R_1 \to 00, R_2 \to 01, R_3 \to 10, R_4 \to 11$. The corresponding matrix will be of the form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Here, the entries $a_{12} = a_{31} = a_{44} = 0$ correspond to the forbidden words 001, 100, 111, respectively. The entries $a_{13} = a_{14} = a_{21} = a_{22} = a_{33} = a_{34} = a_{41} = a_{42} = 0$ correspond to pairs which are not compatible. The 1's in the matrix all correspond to compatible pairs which are also allowable words. For the remainder of the paper, we will use X_A to denote the subshift, where A is the adjacency matrix.

To each such N x N adjacency matrix, we can associate a directed graph $G_A = (V, E)$ where $V = \{v_1, v_2, \ldots, v_N\}$ and $E = \{e_{ij}\}_{i,j=1}^{N}$ where e_{ij} is an edge from v_i to v_j if the entry $a_{ij} = 1$ from A. A directed graph $G = (V, E)$ is called *strongly connected* if for any two vertices $v_i, v_j \in V$, there exists a path from v_i to v_j.

Proposition 3.1.3. *A matrix A is irreducible iff it is associated with a graph G_A which is strongly connected.*

For details on Proposition 3.1.3, see [19]. By Perron-Frobenius Theorem, we know that if A is an irreducible matrix, then A has a positive eigenvector $\mathbf{v}_A$ corresponding to a positive eigenvalue $\lambda_A \in \mathbb{R}$ such that $|\mu| \leq \lambda_A$ where μ is any eigenvalue of A [19]. For any non-negative irreducible N x N matrix A with a positive eigenvector and corresponding positive maximal eigenvalue λ, there exist constants $k_1, k_2 > 0$ independent of n such that

$$k_1 \lambda^n \leq \sum_{i,j=1}^{N} (A^n)_{ij} \leq k_2 \lambda^n,$$

for all $n \geq 1$. We will utilize this consequence of the Perron-Frobenius Theorem while calculating the fractal dimensions for subfractals in Chapter 4.

Example 3.1.4 (Golden Mean Shift)**.** Let us take a closer look at the Golden Mean Shift (GMS), one of the most famous SFTs. Let $\mathcal{A} = \{0, 1\}$ be an alphabet and $F = \{11\}$ be the forbidden word list. The adjacency matrix for this subshift, as briefly mentioned above, is

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0. \end{bmatrix}.$$

The graph associated with this SFT is displayed below.

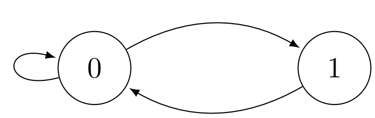

Figure 3.1. GMS graph

Example 3.1.5. Another SFT was defined above to illustrate the construction of an adjacency matrix. Let $\mathcal{A} = \{0,1\}$ and $F = \{001, 100, 111\}$. The graph associated with this SFT is displayed below.

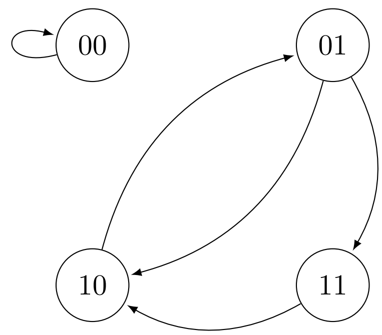

Figure 3.2. SFT graph

It should be noted here that A is a reducible matrix, and hence this graph presentation of this subshift is not strongly connected. However, there exist two strongly connected subgraphs, a fact we will utilize later in Chapter 4.

3.2. Sofic Subshifts

We now turn our attention to a wider class of subshifts called *sofic subshifts*. Every SFT is a sofic subshift, but there exist sofic subshifts which are not SFTs [19]. Much like an SFT, there exists a graphical presentation for each sofic subshift. However, the graphs used to represent sofic subshifts are *labeled graphs*, which means each edge on the graph carries a label. The graphs used to describe SFTs had no labels on the edges but the vertices represented the letters (or finite words) associated with the symbolic space.

Let $\mathcal{G} = (G, \mathcal{L})$ be a labeled graph, consisting of a graph G on finitely many vertices with edge set $\mathcal{E}$ and a labeling $\mathcal{L} : \mathcal{E} \to \mathcal{A}$, where $\mathcal{A}$ is the finite alphabet. Given a path $\pi = e_1 e_2 \dots e_n$ on G, we define the *label of path* π as $\mathcal{L}(\pi) = \mathcal{L}(e_1)\mathcal{L}(e_2)\dots\mathcal{L}(e_n)$, for any $n \geq 1$. If ξ is an infinite path, say $\xi = e_1 e_2 e_3 \dots$, similarly we define the *label of the path* as $\mathcal{L}_\infty(\xi) = \mathcal{L}(e_1)\mathcal{L}(e_2)\mathcal{L}(e_3)\dots$. The set of all labels of infinite paths on G is denoted by

$$X_{\mathcal{G}} = \{x \in X : x = \mathcal{L}_\infty(\xi) \text{ for some infinite path } \xi \text{ on } G\}.$$

Definition 3.2.1. A subset $Y \subseteq X$ is a **sofic subshift** if $Y = X_{\mathcal{G}}$ for some labeled graph $\mathcal{G}$.

We emphasize that the definition only requires that Y is a subset, not necessarily a subshift. It can be shown by using the definition that the subset Y must be a subshift if it has a labeled graph presentation. Given a sofic subshift, a labeled graph $\mathcal{G}$ which represents the subshift (i.e. $Y = X_{\mathcal{G}}$) is not necessarily unique. However, a little more can be said about the properties of such a graph.

It is known that every sofic shift has a *right-resolving* graph presentation, which means that for each vertex v in G, all edges leaving v have different labels [19]. Hence, if $X_{\mathcal{G}}$ is a sofic subshift, we will assume that $\mathcal{G}$ is a right-resolving presentation. A *minimal* right-resolving presentation of a sofic subshift Y is a presentation with the fewest vertices among all right-resolving presentations of Y. A minimal right-resolving presentation is not necessarily unique, but it allows one to fix the number of vertices for a presentation of the subshift Y. Notice that a minimal right-resolving presentation $\mathcal{G}$ has k total vertices for some fixed value k. Now, define a $k \times k$ adjacency matrix $M_{\mathcal{G}}$ by defining the entries as $m_{ij} = \sum_{e_{ij}} \mathcal{L}(e_{ij})$, where e_{ij} is an edge from vertex v_i to v_j in the graph G. For more information on sofic subshifts and associated graphs, see [19].

Other characterizations of sofic subshifts have been discovered, each of which has a purpose for different types of problems. A common characterization of a sofic subshift (Y, σ) is that it must be a factor of some SFT, say (X, σ). That is, there exists a continuous map $\psi : X \to Y$ such that $\sigma \circ \psi = \psi \circ \sigma$. Another helpful characterization involves the number of follower sets for a subshift. The *follower set* of a word $\omega \in B_*(Y)$ for some subshift Y is the set of all finite words that can follow ω, denoted by $F_Y(\omega) = \{\tau \in B_*(Y) : \omega\tau \in B_*(Y)\}$. For example, the follower set of any finite word $\xi \in B_*(X)$ for the full shift X is $F_X(\xi) = B_*(X)$. A subshift is sofic if and only if it has a finite number of distinct follower sets.

Example 3.2.2 (Golden Mean Shift)**.** In the previous section, we showed that the Golden Mean shift is an SFT, and hence must also be sofic. Let $\mathcal{A} = \{0, 1\}$ and $F = \{11\}$. Let X_A denote the Golden Mean shift. Notice that the language of X_A (or collection of all finite words in X_A) is given by

$$B_*(X_A) = \{0, 1, 00, 01, 10, 000, 001, 010, 100, 101, \ldots\}.$$

Next, notice that the follower sets of 0 and 1 are given by

$$F_{X_A}(0) = \{0, 1, 00, 01, 10, 000, 001, 010, 100, 101, \ldots\} = B_*(X_A) \text{ and}$$
$$F_{X_A}(1) = \{0, 00, 01, 000, 001, 010, 0000, 0001, 0010, 0100, 0101, \ldots\}.$$

These two follower sets determine the follower set of every finite word since $F_{X_A}(\omega 0) = F_{X_A}(0)$ and $F_{X_A}(\omega 1) = F_{X_A}(1)$ for any $\omega \in B_*(X_A)$. Therefore, the Golden Mean shift has two distinct follower sets.

Example 3.2.3 (Even Shift)**.** Let $\mathcal{A} = \{0, 1\}$ and $F = \{101, 10001, 1000001, \ldots\}$ be the list of forbidden words. In this subshift, allowable words satisfy the property that consecutive 1's are separated by an even number of 0's, hence the name "Even Shift". Notice the list of forbidden words is not finite and it is impossible to find a finite list of forbidden words to represent this subshift, meaning that this subshift is not an SFT. However, a graph presentation for this subshift exists and is displayed below.

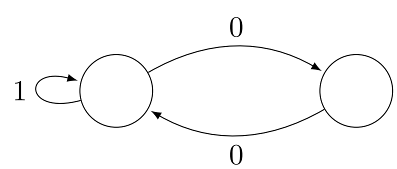

Figure 3.3. Even shift graph

Therefore, the even shift is a sofic subshift. This graph is of particular interest because it resembles the graph for the Golden Mean Shift.

Next, let us find the follower sets for the even shift. Let

$$B_*(Y) = \{0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 110, 111, \ldots\}$$

be the language of Y, the even shift. First, consider the follower set of 0, denoted by $F_Y(0)$. Notice that any word $\omega \in B_*(Y)$ in the language is allowed to follow 0, i.e. $0\omega \in B_*(Y)$. Therefore, the follower set of 0 is given by $F_Y(0) = B_*(Y)$. However, the follower set of 1 is not the entire language, as $01 \in B_*(Y)$ but $101 \in F$. We find that

$$F_Y(1) = \{0, 1, 00, 10, 11, 000, 001, 100, 110, 111, \ldots\}.$$

Next, notice that any word ending in 1 will have a follower set equal to $F_Y(1)$. Also notice that any finite string of 0's will have the same follower set as $F_Y(0)$. The only words that will produce a new follower set are of the form $\omega 10^n$ for any $\omega \in B_*(Y)$ and any $n \geq 1$. We find that

$$F_Y(10) = \{0, 00, 01, 000, 010, 011, \ldots\}.$$

Now, notice that for any $\omega \in B_*(Y)$,

$$F_Y(\omega 10^n) = F_Y(1) \text{ for } n \text{ even },$$

$$F_Y(\omega 10^n) = F_Y(10) \text{ for } n \text{ odd }.$$

Hence, the even shift has three distinct follower sets, $F_Y(0), F_Y(1)$, and $F_Y(10)$.

Example 3.2.4 (S-gap shifts)**.** S-gap shifts are a class of subshifts defined by a subset $S \subset \mathbb{N} \cup \{0\}$. Let $\mathcal{A} = \{0, 1\}$, and let $X = \mathcal{A}^{\mathbb{Z}}$ be the collection of all two-sided infinite sequences from $\mathcal{A}$. Allowable words follow the rule that any consecutive pair of 1's is separated by n 0's, where $n \in S$. It should be noted that if the set S is infinite, then to ensure that the subshift is closed it must also contain all strings that start or end with an infinite string of 0's. Both the golden mean shift and the even shift are examples of S-gap shifts, with $S_{GMS} = \{1, 2, 3, \ldots\}$ for the golden mean shift and $S_{even} = \{0, 2, 4, \ldots, 2n, \ldots\}$ for the even shift.

An S-gap shift can be an SFT, sofic, or neither. An S-gap shift is sofic if the gaps between consecutive elements of S are eventually periodic [4]. More precisely, let $S = \{s_0, s_1, s_2, \ldots\}$ with $s_i < s_{i+1}$ for all $i \geq 0$. Let $D = \{s_0, d_1, d_2, d_3, \ldots\}$ where $d_i = s_i - s_{i-1}$. If D is eventually periodic, then the S-gap shift is sofic.

3.3. Entropy of a subshift

In Chapter 1, we discussed the necessity of having a property to distinguish fractals, and the answer to that problem was fractal dimensions. In symbolic dynamics, there is a similar problem with finding a way to quantify a subshift, but in this case, the answer is entropy. It is, in some sense, a measure of the complexity of the system. Entropy is often used to characterize a dynamical system because it is invariant under conjugacy, which is an important feature in ergodic theory. Defining entropy can be a tedious process in a general dynamical system, but we can simplify this process with the following basic definition for symbolic spaces.

Definition 3.3.1. The **entropy** of a subshift Y is given by

$$h(Y) = \lim_{n\to\infty} \frac{1}{n} \log |B_n(Y)|,$$

where $|\cdot|$ denotes cardinality.

Although this is a simple definition, actual calculations of entropy for subshifts can be complicated. In the case of SFTs and sofic subshifts, the adjacency matrices along with the Perron-Frobenius Theorem help to simplify entropy calculations.

First, consider an SFT, X_A, with forbidden words of length $k > 1$ and $N \times N$ adjacency matrix A. Then,

$$|B_n(X_A)| = \sum_{i,j=1}^{N} (A^{n+k-1})_{ij}.$$

Assuming that A is irreducible, one may apply the Perron-Frobenius Theorem, as seen in Section 3.1. Hence, it must be the case that the entropy of an SFT X_A is given by

$$h(X_A) = \log(\lambda_A),$$

where λ_A is the maximal eigenvalue of A.

In the sofic case, an adjacency matrix can be constructed similarly. Every sofic subshift has a labeled graph presentation, say $\mathcal{G} = (G, \mathcal{L})$. Let A_G denote the adjacency matrix with respect to the underlying graph G. If A_G is an irreducible matrix, the entropy of a sofic subshift is given by $h(X_{\mathcal{G}}) = \log(\lambda_{A_G})$, where λ_{A_G} denotes the maximal eigenvalue of A_G [19]. Later in this chapter, we will discuss the connection between the entropy of a subshift and the Hausdorff dimension of a subfractal induced by that subshift.

Example 3.3.2 (Golden Mean Shift). The eigenvalues of A, the adjacency matrix given in Example 3.1.3, are $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$. Hence, the entropy of the golden mean shift is $h(X_A) = \log(\frac{1+\sqrt{5}}{2})$.

Example 3.3.3 (Even Shift). Notice that the underlying graph with respect to the labeled graph of the even shift (given in Example 3.2.2) is identical to the graph presentation of the golden mean shift. Therefore, the entropy of the even shift is also given by $h(X) = \log(\frac{1+\sqrt{5}}{2})$.

It is tempting to assume that the even shift and the golden mean shift are conjugate because they have equal entropy values, but it is not true. Even though entropy is invariant with respect to conjugacy, two distinct and very different subshifts may have the same entropy.

Example 3.3.4 (S-gap shifts). The entropy of an S-gap shift is given by $\log(\lambda)$, where λ is the unique positive solution of

$$\sum_{n \in S} \frac{1}{x^{n+1}} = 1.$$

See [25] for details.

3.4. Subfractals induced by SFTs or sofic subshifts

Let $\{\mathcal{K}; f_1, \ldots f_m\}$ be a hyperbolic IFS, $\mathcal{F}$ denote the attractor of the IFS, and $c_i, \bar{c}_i$ denote the contractive bounds on the functions $c_i d(x,y) \leq d(f_i(x), f_i(y)) \leq \bar{c}_i d(x,y)$ for $1 \leq i \leq m$. Let $\mathcal{A} = \{1, \ldots, m\}$ be an alphabet, and let X denote the full shift with alphabet $\mathcal{A}$. Define the associated coding map $\pi : X \to \mathcal{F}$ by $\pi(\omega) = \lim_{n \to \infty} f_{\omega|_n}(\mathcal{K})$.

For each such IFS, we can define a subfractal of $\mathcal{F}$ induced by subshift X_A by only considering the points associated with an allowable word from the subshift. Let X_A be an SFT and define $\mathcal{F}_{X_A} = \{\pi(\omega) : \omega \in X_A\}$.

As defined in section 2, fix an $N \times N$ adjacency matrix A. Let $B_{k-1}(X) = \{\tau^1, \tau^2, \ldots, \tau^N\}$, where $N = m^{k-1}$. Define two other $N \times N$ matrices, S_0 and S, as follows:

$$S_0 = \begin{bmatrix} c_{\tau^1} & 0 & \cdots & 0 \\ 0 & c_{\tau^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{\tau^N}. \end{bmatrix} \text{ and } S = \begin{bmatrix} c_{i_1} & 0 & \cdots & 0 \\ 0 & c_{i_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{i_N} \end{bmatrix},$$

where $i_j \in \mathcal{A}$ for all $1 \leq j \leq N$ and the order of the $i_j's$ is chosen so that

$$\sum_{i,j=1}^{N} (S_0 A S)_{i,j} = \sum_{\omega \in B_k(X_A)} c_\omega.$$

Similarly, we define

$$\bar{S}_0 = \begin{bmatrix} \bar{c}_{\mathcal{T}^1} & 0 & \cdots & 0 \\ 0 & \bar{c}_{\mathcal{T}^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{c}_{\mathcal{T}^N} \end{bmatrix} \text{ and } \bar{S} = \begin{bmatrix} \bar{c}_{i_1} & 0 & \cdots & 0 \\ 0 & \bar{c}_{i_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{c}_{i_N} \end{bmatrix}.$$

Example 3.4.1 (Cantor subfractal-Golden Mean Shift)**.** Let $\{\mathcal{K}; f_1, f_2\}$ be the IFS with $\mathcal{K} = [0,1]$, $f_0(x) = \frac{1}{3}x$, and $f_1(x) = \frac{1}{3}x + \frac{2}{3}$. Let X_A be the Golden Mean Shift (from Example 3.1.4) with forbidden word list $\{11\}$. The first four iterations of the subfractal induced by this IFS and the Golden Mean Shift are pictured below.

Figure 3.4. GMS Cantor subfractal

Example 3.4.2 (Another Cantor subfractal)**.** Let the IFS be the same as in Example 3.4.1, and let X_A be the subshift from Example 3.1.5 with forbidden word list $\{001, 100, 111\}$. The first four images of the subfractal induced by the IFS and X_A are pictured below.

Figure 3.5. Cantor subfractal

Example 3.4.3 (Sierpiński's Triangle Subfractal)**.** Let $\{\mathcal{K}; f_1, f_2, f_3\}$ be the IFS with $\mathcal{K}$ as in Example 1.4.2, $f_0(x, y) = (\frac{1}{2}x, \frac{1}{2}y)$, $f_1(x, y) = (\frac{1}{2}x + \frac{1}{2}, \frac{1}{2})$, and $f_2(x, y) = (\frac{1}{2}x + \frac{1}{4}, \frac{1}{2}y + \frac{\sqrt{3}}{2})$. Let X_A be a SFT with forbidden word list $\{12, 21\}$. The first three iterations of the subfractal are pictured below.

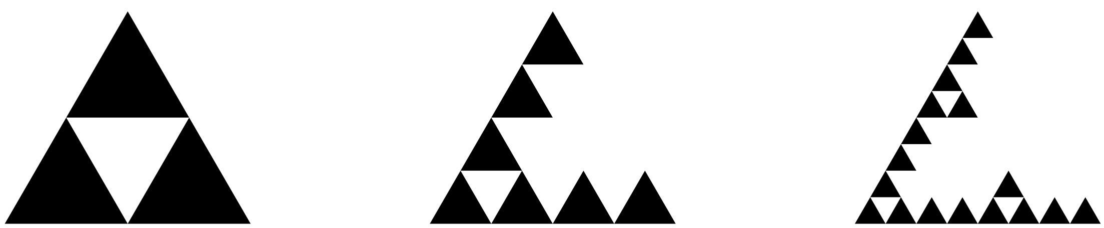

Figure 3.6. Sierpiński's triangle subfractal

Recall the Markov attractor of an IFS used by Ellis, Branton, and Roychowdhury in Chapter 2. A Markov attractor of an IFS is defined via m contractive maps and an $m \times m$ matrix A, where the Markov attractor only contained those points associated with an admissible sequence with respect to matrix A. Notice that a system of this form coincides with a subfractal induced by an SFT which is defined by a forbidden word list in which each word has length 2.

Now, let us consider a subfractal induced by a sofic subshift. Let $\{\mathcal{K}; f_1, \ldots, f_m\}$ be a hyperbolic IFS with $c_i d(x, y) \leq d(f_i(x), f_i(y)) \leq \overline{c}_i d(x, y)$ for $1 \leq i \leq m$ and all $x, y \in \mathcal{K}$. For purposes of dimension calculations, we introduce a real-valued variable $t \in \mathbb{R}$. We will define two $k \times k$ matrices, $A_{\mathcal{G},t}$ and $\overline{A}_{\mathcal{G},t}$ similar to the matrix $M_{\mathcal{G}}$ in Section 3.2. Let $A_{\mathcal{G},t}$ be defined by the entries $a_{ij}^{(t)} = \sum_{e_{ij}} c^t_{(\mathcal{L}(e_{ij}))}$, where $a_{ij}^{(t)}$ denotes the (i,j)-th entry of $A_{\mathcal{G},t}$. Let $\overline{A}_{\mathcal{G},t}$ be defined by the entries $\overline{a}_{ij}^{(t)} = \sum_{e_{ij}} \overline{c}^t_{(\mathcal{L}(e_{ij}))}$.

Recall that a finite word from a sofic subshift, say $X_{\mathcal{G}}$, is associated with a finite labeled path in the graph $\mathcal{G}$, but that path is not necessarily unique. For this reason, we require that the right-resolving graph presentation of the sofic subshift be minimal. In Chapter 4, specifically in Lemma 3.2.1, we will see that

$$\frac{1}{k}\sum_{i,j=1}^{k}(A^n_{\mathcal{G},t})_{i,j} \leq \sum_{\omega\in B_n(X_{\mathcal{G}})} c^t_{\omega} \leq \sum_{i,j=1}^{k}(A^n_{\mathcal{G},t})_{i,j},$$

for any $n \geq 1$, where k denotes the number of vertices in G.

Example 3.4.4 (Cantor subfractal - Even shift)**.** Let $\{\mathcal{K}; f_1, f_2\}$ be the IFS of the Cantor set and $X_{\mathcal{G}}$ be the even shift (as seen in Example 3.1.5). The first four iterations of the subfractal induced by this IFS and subshift are pictured below.

Figure 3.7. Even shift Cantor subfractal

4. CHAPTER 4 - FRACTAL DIMENSION OF SUBFRACTALS

In this chapter, we will develop a technique that enables us to calculate some fractal dimensions of subfractals. For certain types of IFSs, a topological pressure function is related to the Hausdorff dimension of the attractor. In particular, the zero of the topological pressure function is equal to the Hausdorff dimension for some select classes of IFSs. For example, consider the attractor of a disjoint IFS containing m similarity maps with similarity ratios c_i for $1 \leq i \leq m$. Finding the zero of a topological pressure function associated with such an IFS simplifies to finding the value the value h which satisfies

$$\sum_{i=1}^{m} c_i^h = 1,$$

as we saw in Chapter 1. In this chapter, we will formally define a more general topological pressure function. Then, using the zeros of topological pressure functions, we will find bounds for the Hausdorff and upper box dimension of subfractals, for the specific classes discussed in Chapter 3.

4.1. Topological Pressure

In the broadest sense, pressure functions serve various purposes in ergodic theory, often in problems associated with entropy. For purposes of this paper, we define two specific topological pressure functions associated with an IFS $\{\mathcal{K};, f_1, \ldots, f_m\}$ and SFT X_A as follows.

Definition 4.1.1. The *lower topological pressure function* of $\mathcal{F}_{X_A}$ is given by

$$P(t) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\omega \in B_n(X_A)} c_\omega^t \right),$$

where c_i is the lower contractive bound on the map f_i for $1 \leq i \leq m$. Similarly, we define the *upper topological pressure function* by

$$\bar{P}(t) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\omega \in B_n(X_A)} \bar{c}_\omega^t \right),$$

where $\bar{c}_i$ is the the upper contractive bound on f_i for $1 \leq i \leq m$.

Proposition 4.1.2. *The lower and upper topological pressure functions $P(t)$ and $\bar{P}(t)$ are strictly decreasing, convex, and continuous on $\mathbb{R}$.*

Proof. We will show the proof for $P(t)$. The proof for $\bar{P}(t)$ follows similarly. Let $\delta > 0$. By using the fact that $c_\omega \leq c_{max}^n$ for all $\omega \in B_n(X_A)$, where $c_{max} = \max_{1 \leq i \leq m}\{c_i\}$, we have

$$\begin{aligned}
P(t+\delta) &= \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\omega \in B_n(X_A)} c_\omega^{t+\delta} \right) \leq \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\omega \in B_n(X_A)} c_\omega^t c_{max}^{n\delta} \right) \\
&= \lim_{n\to\infty} \frac{1}{n} \log \left(c_{max}^{n\delta} \sum_{\omega \in B_n(X_A)} c_\omega^t \right) = \lim_{n\to\infty} \frac{1}{n} [n\delta \log(c_{max})] + P(t) \\
&= \delta \log(c_{max}) + P(t) < P(t),
\end{aligned}$$

since $0 < c_{max} < 1$. Hence, $P(t)$ is strictly decreasing. If $t_1, t_2 \in \mathbb{R}$ and $a_1, a_2 > 0$ with $a_1 + a_2 = 1$, then, by Hölder's inequality, we have

$$\begin{aligned}
P(a_1 t_1 + a_2 t_2) &= \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\omega \in B_n(X_A)} (c_\omega)^{a_1 t_1 + a_2 t_2} \right) \\
&= \lim_{n\to\infty} \frac{1}{n} \log \left[\sum_{\omega \in B_n(X_A)} ((c_\omega)^{t_1})^{a_1} ((c_\omega)^{t_2})^{a_2} \right] \\
&\leq \lim_{n\to\infty} \frac{1}{n} \log \left[\sum_{\omega \in B_n(X_A)} (c_\omega)^{t_1} \right]^{a_1} \left[\sum_{\omega \in B_n(X_A)} (c_\omega)^{t_2} \right]^{a_2} \\
&= a_1 P(t_1) + a_2 P(t_2).
\end{aligned}$$

Hence, $P(t)$ is a convex function and strictly decreasing, and thus must be continuous.

□

Proposition 4.1.3. *There is a unique value $h \in [0, \infty)$ such that $P(h) = 0$.*

Proof. If $t = 0$, then

$$P(0) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\omega \in B_n(X_A)} c_\omega^0 \right) = \lim_{n\to\infty} \frac{1}{n} \log(|B_n(X_A)|) \geq 0.$$

Next, in the case in which $t \to \infty$, we obtain

$$\begin{aligned}
P(t) &= \lim_{n\to\infty} \frac{1}{n} \log\left(\sum_{\omega\in B_n(X_A)} c_\omega^t\right) \leq \lim_{n\to\infty} \frac{1}{n} \log\left(\sum_{\omega\in B_n(X_A)} c_{max}^{nt}\right) \\
&= t\log(c_{\max}) + \lim_{n\to\infty} \frac{1}{n} \log(|B_n(X_A)|) \leq t\log(c_{max}) + \lim_{n\to\infty} \frac{1}{n} \log(m^n) \\
&= t\log(c_{max}) + \log(m).
\end{aligned}$$

Since $0 < c_{max} < 1$, we must have $[t\log(c_{max}) + \log(m)] \to -\infty$ as $t \to \infty$, and hence $\lim\limits_{t\to\infty} P(t) = -\infty$. By Proposition 4.1.2, there exists a unique value h such that $P(h) = 0$. $\square$

Following the same steps as in the proof above, we obtain the following proposition.

Proposition 4.1.4. *There is a unique value $H \in [0, \infty)$ such that $\bar{P}(H) = 0$.*

Proposition 4.1.5. *Let h and H be the unique values such that $P(h) = 0 = \bar{P}(H)$. Then, $h \leq H$.*

Proof. Assume that $h > H$. Then, $\bar{P}(h) < \bar{P}(H) = 0$. We also know that $c_\omega \leq \bar{c}_\omega$ for all $\omega \in B_n(X_A)$. Hence,

$$0 = P(h) = \lim_{n\to\infty} \frac{1}{n} \log\left(\sum_{\omega\in B_n(X_A)} c_\omega^h\right) \leq \lim_{n\to\infty} \frac{1}{n} \log\left(\sum_{\omega\in B_n(X_A)} \bar{c}_\omega^h\right) = \bar{P}(h) < 0,$$

which is a contradiction. Hence, $h \leq H$. $\square$

Recall the definitions of S and S_0 from Chapter 3. For $t \in \mathbb{R}$, define

$$S^{(t)} = \begin{bmatrix} c_{i_1}^t & 0 & \cdots & 0 \\ 0 & c_{i_2}^t & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{i_N}^t \end{bmatrix},$$

and define $S_0^{(t)}$, $\bar{S}^{(t)}$, and $\bar{S}_0^{(t)}$ similarly.

Lemma 4.1.6. *Let $\{\mathcal{K}; f_i : 1 \leq i \leq m\}$ be a hyperbolic IFS and X_A be a subshift of the full shift, X, on alphabet $\mathcal{A} = \{1, \ldots, m\}$. Let S_0 and S be matrices associated with the subfractal $\mathcal{F}_{X_A}$, as* *above. Then, the associated lower and upper topological pressure functions $P(t)$ and $\bar{P}(t)$ can be written, respectively, as*

$$P(t) = \lim_{n\to\infty} \frac{1}{n} \log\left(\sum_{i,j=1}^{N} [S_0^{(t)}(AS^{(t)})^{n-k+1}]_{i,j}\right) \quad \textit{and}$$
$$\bar{P}(t) = \lim_{n\to\infty} \frac{1}{n} \log\left(\sum_{i,j=1}^{N} [\bar{S}_0^{(t)}(A\bar{S}^{(t)})^{n-k+1}]_{i,j}\right).$$

Proof. Recall that if F is a list of forbidden words, all of length k, then A is an N x N matrix, where $N = |B_{k-1}(X)| = m^{k-1}$. We will prove the assertion by induction. First, the nonzero entries of A correspond to the allowable words of length k. Hence, by definition of $A, S_0,$ and S, we have

$$\sum_{i,j=1}^{N} [S_0AS]_{ij} = \sum_{\omega\in B_k(X_A)} c_\omega.$$

Now, assume that $\sum_{i,j=1}^{N} [S_0(AS)^n]_{ij} = \sum_{\omega\in B_{n+k-1}(X_A)} c_\omega$ for some $n > 1$. The entries of $S_0(AS)^n$ consist of sums of contractive factors associated with allowable words of length $n + k - 1$. Now, consider the matrix $S_0(AS)^n(AS)$. By the definition of A and S, this multiplication will result in entries consisting of sums of contractive factors associated with allowable words of length $n+k$. Since $S_0(AS)^n$ contains all allowable words of length $n + k - 1$, then we must have $\sum_{i,j=1}^{N} [S_0(AS)^{n+1}]_{ij} = \sum_{\omega\in B_{n+k}(X_A)} c_\omega$. Hence,

$$P(t) = \lim_{n\to\infty} \frac{1}{n} \log\left(\sum_{\omega\in B_n(X_A)} c_\omega^t\right) = \lim_{n\to\infty} \frac{1}{n} \log\left(\sum_{i,j=1}^{N} [S_0^{(t)}(AS^{(t)})^{n-(k-1)}]_{ij}\right).$$

The proof follows similarly for the upper topological pressure function.

□

4.2. Main Result for SFTs

We begin with a technical lemma that will provide bounds needed for the main result.

Lemma 4.2.1. *Let S_0, A, and S be defined as in Chapter 3, where A is an irreducible matrix.*

Then, for any $t > 0$, there exist positive constants K, L such that

$$c_{min}^{(k-1)t} K\lambda_{AS^{(t)}}^n \leq \sum_{i,j=1}^{N} [S_0^{(t)}(AS^{(t)})^n]_{ij} \leq c_{max}^{(k-1)t} L\lambda_{AS^{(t)}}^n,$$

where $c_{min} = \min_{1\leq i\leq m}\{c_i\}$, $c_{max} = \max_{1\leq i\leq m}\{c_i\}$, $\lambda_{AS^{(t)}}$ is the maximal eigenvalue of $AS^{(t)}$.

Proof. Notice that for every non-zero entry of S_0, we have $c_{min}^{k-1} \leq (S_0)_{ij} \leq c_{max}^{k-1}$, for all $1 \leq i, j \leq N$. Hence, by the Perron-Frobenius Theorem, we have constants K and L such that

$$\begin{aligned} c_{min}^{(k-1)t} K\lambda_{AS^t}^n \leq c_{min}^{(k-1)t} \sum_{i,j=1}^{N} [(AS^{(t)})^n]_{ij} &\leq \sum_{i,j=1}^{N} [S_0^{(t)}(AS^{(t)})^n]_{ij} \\ &\leq c_{max}^{(k-1)t} \sum_{i,j=1}^{N} [(AS^{(t)})^n]_{ij} \leq c_{max}^{(k-1)t} L\lambda_{AS^{(t)}}^n \end{aligned}$$

□

Remark 1. *By Lemma 4.2.1, one can show that, for fixed value $t \in [0, \infty]$,*

$$\begin{aligned} P(t) &= \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{i,j=1}^{N} [S_0^{(t)}(AS^{(t)})^n]_{ij} \right) \\ &\leq \lim_{n\to\infty} \frac{1}{n} \log(c_{max}^{(k-1)t} L\lambda_{AS^{(t)}}^n) = \log(\lambda_{AS^{(t)}}) = \log(\rho(AS^{(t)}), \end{aligned}$$

where $\rho(AS^{(t)})$ denotes the spectral radius of $AS^{(t)}$. Similarly, we can show that $\log(\rho(AS^{(t)})) \leq P(t)$, and hence $P(t) = \log(\rho(AS^{(t)})$. Therefore, the unique value h such that $P(h) = 0$ is also the value of h such that $\rho(AS^{(h)}) = 1$. Analogously, we can show that the value H such that $\bar{P}(H) = 0$ is also the value of H that satisfies $\rho(A\bar{S}^{(H)}) = 1$.

Proposition 4.2.2. *Let h be the unique zero of the lower topological pressure function. There exist positive constants K_0, L_0 such that*

$$K_0 \leq \sum_{\omega\in B_n(X_A)} c_\omega^h \leq L_0,$$

for all $n \geq 1$.

Proof. Let $s < h$. Then, $P(s) > P(h) = 0$. So, we have

$$0 < P(s) = \lim_{p\to\infty} \frac{1}{np} \log \left(\sum_{\omega \in B_{np}(X_A)} c_\omega^s \right) \le \lim_{p\to\infty} \frac{1}{np} \log \left(\sum_{\omega \in B_n(X_A)} c_\omega^s \right)^p$$
$$= \frac{1}{n} \log \left(\sum_{\omega \in B_n(X_A)} c_\omega^s \right).$$

Hence, $\sum_{\omega \in B_n(X_A)} c_\omega^s > 1$, and it follows that $\sum_{\omega \in B_n(X_A)} c_\omega^h \ge 1$.
Now, assume that $s > h$. Then, $0 = P(h) > P(s)$. So, by Lemma 4.2.1, we have

$$0 > P(s) = \lim_{p\to\infty} \frac{1}{np} \log \left(\sum_{\omega \in B_{np}(X_A)} c_\omega^s \right) = \lim_{p\to\infty} \frac{1}{np} \log \left(\sum_{i,j=1}^{N} [S_0^{(s)}(AS^{(s)})^{np}]_{i,j} \right)$$
$$\ge \lim_{p\to\infty} \frac{1}{np} \log \left(c_{min}^{(k-1)s} K \lambda_{AS^{(s)}}^{np} \right) = \frac{1}{n} \log(\lambda_{AS^{(s)}}^n)$$
$$\ge \frac{1}{n} \log \left(\frac{1}{Lc_{max}^{(k-1)s}} \sum_{i,j=1}^{N} [S_0^{(s)}(AS^{(s)})^{n}]_{i,j} \right) = \frac{1}{n} \log \left(\frac{1}{Lc_{max}^{(k-1)s}} \sum_{\omega \in B_n(X_A)} c_\omega^s \right).$$

Hence, $\sum_{\omega \in B_n(X_A)} c_\omega^s < Lc_{max}^{(k-1)s}$, which implies that $\sum_{\omega \in B_n(X_A)} c_\omega^h \le Lc_{max}^{(k-1)h}$. ☐

Following similar steps in the proof of Proposition 4.2.2, we obtain the following proposition.

Proposition 4.2.3. *Let H be the unique zero of the upper topological pressure function. There exist positive constants K_1, L_1 such that*

$$K_1 \le \sum_{\omega \in B_n(X_A)} \bar{c}_\omega^H \le L_1.$$

In order to show that h is a lower bound for $\dim_H(\mathcal{F}_{X_A})$, we will utilize the uniform mass distribution principle from Falconer [9]. Hence, we must define an appropriate Borel probability measure to satisfy the principle. Let h be the unique value such that $P(h) = 0$. Let $\omega \in B_n(X)$ and let $[\![\omega]\!] = \{\tau \in X : \tau_i = \omega_i, 1 \le i \le n\}$ be the cylinder set with base ω. We will use the fact that $c_{\omega\tau} = c_\omega c_\tau$. Define

$$\nu_n(\llbracket\omega\rrbracket) = \frac{\displaystyle\sum_{\omega\tau \in B_{n+\ell(\omega)}(X_A)} c_{\omega\tau}^h}{\displaystyle\sum_{\tau \in B_{n+\ell(\omega)}(X_A)} c_\tau^h}.$$

For all $n \geq 1$ and any $\omega \in B_*(X_A)$, we have by Proposition 4.2.2,

$$0 \leq \frac{\displaystyle\sum_{\omega\tau \in B_{n+\ell(\omega)}(X_A)} c_{\omega\tau}^h}{L_0} \leq \nu_n(\llbracket\omega\rrbracket) \leq \frac{c_\omega^h \displaystyle\sum_{\tau \in B_n(X_A)} c_\tau^h}{\displaystyle\sum_{\tau \in B_{n+\ell(\omega)}(X_A)} c_\tau^h} \leq \frac{L_0}{K_0} c_\omega^h < \infty.$$

Hence, for all $\omega \in B_*(X_A)$, $\mathrm{Lim}_{n\to\infty}\nu_n(\llbracket\omega\rrbracket)$ exists, where Lim denotes the Banach limit. Let $\nu(\llbracket\omega\rrbracket) = \mathrm{Lim}_{n\to\infty}\nu_n(\llbracket\omega\rrbracket)$. Also, notice that

$$\begin{aligned}\sum_{i=1}^m \nu(\llbracket\omega i\rrbracket) &= \mathrm{Lim}_{n\to\infty} \sum_{i=1}^m \frac{\displaystyle\sum_{\omega i\tau \in B_{n+\ell(\omega i)}(X_A)} c_{\omega i\tau}^h}{\displaystyle\sum_{\tau \in B_{n+\ell(\omega i)}(X_A)} c_\tau^h} \\ &= \mathrm{Lim}_{n\to\infty} \frac{\displaystyle\sum_{\omega\tau \in B_{n+1+\ell(\omega)}(X_A)} c_{\omega\tau}^h}{\displaystyle\sum_{\tau \in B_{n+1+\ell(\omega)}(X_A)} c_\tau^h} = \nu(\llbracket\omega\rrbracket).\end{aligned}$$

Hence, by applying Kolmogorov extension theorem, we can extend ν to a unique Borel probability measure γ on X_A. Let $\mu_h = \gamma \circ \pi^{-1}$, where π is the coding map. Hence, μ_h is supported on $\mathcal{F}_{X_A}$.

Corollary 4.2.4. *There exist constants $K_0, L_0 > 0$ such that*

$$\mu_h(f_\omega(\mathcal{K})) \leq \frac{L_0}{K_0} c_\omega^h.$$

Proof. By definition of μ_h and Proposition 4.2.2, we have

$$\mu_h(f_\omega(\mathcal{K})) = \nu(\llbracket\omega\rrbracket) = \frac{\displaystyle\sum_{\omega\tau \in B_n(X_A)} c^h_{\omega\tau}}{\displaystyle\sum_{\tau \in B_{n+\ell(\omega)}(X_A)} c^h_\tau} \leq \frac{c^h_\omega \displaystyle\sum_{\tau \in B_n(X_A)} c^h_\tau}{\displaystyle\sum_{\tau \in B_{n+\ell(\omega)}(X_A)} c^h_\tau} \leq c^h_\omega \frac{L_0}{K_0}.$$

□

Proposition 4.2.5. *For $0 < r < 1$ and $x \in \mathcal{F}_{X_A}$, the ball $B(x, r)$ intersects at most M elements of $\mathcal{U}_r = \{f_\omega(\mathcal{K}) : |f_\omega(\mathcal{K})| \leq r < |f_{\omega^-}(\mathcal{K})|\}$, where M is finite and independent of r.*

Proof. Let $0 < r < 1$ and $x \in \mathcal{F}_{X_A}$. Let $W_r = \{\omega \in B_*(X_A) : f_\omega(\mathcal{K}) \cap B(x, r) \neq \emptyset, f_\omega(\mathcal{K}) \in \mathcal{U}_r\}$ and $|W_r| = M$. Let $y \in B(x, r)$ and $z \in f_\omega(\mathcal{K})$ where $\omega \in W_r$. Notice that

$$d(y, z) \leq |B(x, r)| + |f_\omega(\mathcal{K})| \leq 3r.$$

Hence, $\{f_\omega(\mathcal{K}) : \omega \in W_r\} \subset B(x, 3r)$. For any $f_\omega(\mathcal{K}) \in \mathcal{U}_r$, we have

$$|f_\omega(\mathcal{K})| \geq c_{min}|f_{\omega^-}(\mathcal{K})| > c_{min}r.$$

Due to the open set condition, there exists a ball B_a of radius $a > 0$ such that $B_a \subset \mathcal{K}$ and $f_\omega(B_a) \cap f_\tau(B_a) = \emptyset$ for $\omega, \tau \in W_r$. For each $\omega \in W_r$, we have $f_\omega(B_a) \subset f_\omega(\mathcal{K})$. Let m denote Lebesgue measure on $\mathcal{K}$. Since the balls are disjoint and contained in $B(x, 3r)$, we have

$$\sum_{\omega \in W_r} m(f_\omega(B_a)) \leq m(B(x, 3r)).$$

Using the fact that $|f_\omega(\mathcal{K})| > c_{min}r$, we have

$$M \cdot m(B(x, ac_{min}r)) \leq m(B(x, 3r)).$$

Hence, $M \leq \dfrac{m(B(x,3r)}{m(B(x, ac_{min}r))}$. Since the ratio compares concentric balls, each with a radius equal to a constant multiple of r, we can let $M \leq \left\lceil \frac{m(B(x,3r))}{m(B(x,c_{min}r))} \right\rceil < \infty$, which satisfies the assertion of the proposition.

$\square$

Theorem 4.2.6. *Let h, H be the unique values such that $P(h) = 0 = \bar{P}(H)$. Then, $h \leq \dim_H(\mathcal{F}_{X_A}) \leq H$.*

Proof. Let $\mathcal{U}_n = \{f_\omega(\mathcal{K}) : \omega \in B_n(X_A)\}$. Notice that $\mathcal{U}_n$ is a cover for all $n \geq 1$. Hence, by Proposition 4.2.3, we have

$$\begin{aligned}\mathcal{H}^H(\mathcal{F}_{X_A}) &= \lim_{\varepsilon\to 0}\inf_{\mathcal{U}} \sum_{E\in\mathcal{U}} |E|^H \leq \lim_{n\to\infty} \sum_{\omega\in B_n(X_A)} |f_\omega(\mathcal{K})|^H \\ &\leq \lim_{n\to\infty} \sum_{\omega\in B_n(X_A)} |\mathcal{K}|^H \bar{c}_\omega^H \leq |\mathcal{K}|^H \cdot L_1 < \infty,\end{aligned}$$

where $\mathcal{U}$ denotes an arbitrary ε-cover of $\mathcal{F}_{X_A}$.

Thus, $\dim_H(\mathcal{F}_{X_A}) \leq H$. Let $r > 0$ and $B(x, r)$ be a ball centered at $x \in \mathcal{F}_{X_A}$. By Proposition 4.2.5, $B(x, r)$ intersects at most M elements of the cover $\mathcal{U}_r$. Let $\mathcal{U}_M$ denote the subset of $\mathcal{U}_r$ consisting of all elements that intersect $B(x, r)$ and W_M denote all allowable words associated with an element of $\mathcal{U}_M$. By Corollary 4.2.4, we have

$$\begin{aligned}\frac{\mu_h(B(x,r))}{r^h} &\leq \frac{\sum_{f_\omega(\mathcal{K})\in\mathcal{U}_M} \mu_h(f_\omega(\mathcal{K}))}{r^h} \leq \frac{\sum_{\omega\in W_M} \frac{L_0}{K_0} c_\omega^h}{r^h} \\ &\leq \frac{M\frac{L_0}{K_0}|\mathcal{K}|^{-h} r^h}{r^h} = M\frac{L_0}{K_0}|\mathcal{K}|^{-h}.\end{aligned}$$

Hence, $\limsup_{r\to 0} \dfrac{\mu_h(B(x,r))}{r^h} \leq M\dfrac{L_0}{K_0}|\mathcal{K}|^{-h} < \infty$. By the uniform mass distribution principle [9], we have $\mathcal{H}^h(\mathcal{F}) \geq \dfrac{M\frac{L_0}{K_0}|\mathcal{K}|^{-h}}{\mu_h(\mathcal{F}_{X_A})} > 0$. Thus, $\dim_H(\mathcal{F}_{X_A}) \geq h$. $\square$

Theorem 4.2.7. *Let h, H be the unique values such that $P(h) = 0 = \bar{P}(H)$. Then, $h \leq \overline{\dim}_B(\mathcal{F}_{X_A}) \leq H$.*

Proof. The following relationship between Hausdorff and box dimensions is well-known:

$$\dim_H(\mathcal{F}) \leq \underline{\dim}_B(\mathcal{F}) \leq \overline{\dim}_B(\mathcal{F}).$$

Hence, it suffices to show that $\overline{\dim}_B(\mathcal{F}_{X_A}) \leq H$. Let $\mathcal{U}_r = \{f_\omega(\mathcal{K}) : |f_\omega(\mathcal{K})| \leq r < |f_{\omega^-}(\mathcal{K})|\}$,

$k = \min\{|\omega| : f_\omega(\mathcal{K}) \in \mathcal{U}_r\}$, and $\mathcal{O}_k = \{f_\omega(\mathcal{K}) : \omega \in B_k(X_A)\}$. Notice that $\bigcup_{f_\omega(\mathcal{K})\in\mathcal{U}_r} f_\omega(\mathcal{K}) \subseteq \bigcup_{f_\omega(\mathcal{K})\in\mathcal{O}_k} f_\omega(\mathcal{K})$. Hence, by Proposition 4.2.4, we have

$$\sum_{f_\omega(\mathcal{K})\in\mathcal{U}_r} |f_\omega(\mathcal{K})|^H \leq \sum_{f_\omega(\mathcal{K})\in\mathcal{O}_k} |f_\omega(\mathcal{K})|^H \leq |\mathcal{K}|^H \sum_{\omega\in B_k(X_A)} \bar{c}_\omega^H \leq |\mathcal{K}|^H L_1.$$

Also, for $f_\omega(\mathcal{K}) \in \mathcal{U}_r$,

$$|f_\omega(\mathcal{K})| \geq |f_{\omega^-}(\mathcal{K})| \cdot c_{min} > rc_{min}.$$

Let $N_r(\mathcal{F}_{X_A})$ denote the smallest number of sets of diameter at most r which form a cover of $\mathcal{F}_{X_A}$. Then,

$$(rc_{min})^H N_r(\mathcal{F}_{X_A}) \leq |f_\omega(\mathcal{K})|^H N_r(\mathcal{F}_{X_A}) \leq \sum_{f_\omega(\mathcal{K})\in\mathcal{U}_r} |f_\omega(\mathcal{K})|^H \leq |\mathcal{K}|^H L_1.$$

Hence, $N_r(\mathcal{F}_{X_A}) \leq (rc_{min})^{-H}|\mathcal{K}|^H L_1$, and thus

$$\frac{\log(N_r(\mathcal{F}_{X_A}))}{-\log(r)} \leq \frac{\log(L_1|\mathcal{K}|^H) - H\log(rc_{min})}{-\log(r)} = \frac{\log(L_1|\mathcal{K}|^H)}{-\log(r)} + \frac{H\log(c_{min})}{\log(r)} + H.$$

By the definition of upper box dimension, we have

$$\overline{\dim}_B(\mathcal{F}_{X_A}) = \limsup_{r\to 0} \frac{\log(N_r(\mathcal{F}_{X_A}))}{-\log(r)} \leq \limsup_{r\to 0} \left[\frac{\log(L_1|\mathcal{K}|^H)}{-\log(r)} + \frac{H\log(c_{min})}{\log(r)}\right] + H = H.$$

□

Remark 2. *As seen in Chapter 2, the following inequalities are well-known for $E \subset \mathcal{K}$:*

$$\dim_H(E) \leq \dim_P(E) \leq \overline{\dim}_B(E) \text{ and } \dim_H(E) \leq \underline{\dim}_B \leq \overline{\dim}_B(E).$$

Hence, we have also shown that

$$h \leq \dim_P(\mathcal{F}_{X_A}) \leq H \text{ and } h \leq \underline{\dim}_B(\mathcal{F}_{X_A}) \leq H,$$

and in the case that $h = H$,

$$\dim_H(\mathcal{F}_{X_A}) = \dim_P(\mathcal{F}_{X_A}) = \overline{\dim}_B(\mathcal{F}_{X_A}) = \underline{\dim}_B(\mathcal{F}_{X_A}).$$

4.3. Main result for sofic subshifts

In this section, we will extend the assertions from Theorem 4.2.6 and Theorem 4.2.7 to sofic subshifts.

Lemma 4.3.1. *Let $X_{\mathcal{G}}$ be a sofic subshift, where $\mathcal{G} = (G, \mathcal{L})$. If G has k vertices, then*

$$\frac{1}{k}\sum_{i,j=1}^{k}[A^n_{\mathcal{G},t}]_{i,j} \leq \sum_{\omega \in B_n(X_{\mathcal{G}})} c^t_\omega \leq \sum_{i,j=1}^{k}[A^n_{\mathcal{G},t}]_{i,j}.$$

Proof. Let $\omega \in B_n(X_{\mathcal{G}})$ for some $n \geq 1$. Notice that there may be more than one path for ω in $\mathcal{G}$. Since $\sum_{i,j=1}^{k}[A^n_{\mathcal{G}}]_{ij}$ sums contractive factors related to all labeled paths of length n in $\mathcal{G}$, then $\sum_{\omega \in B_n(X_{\mathcal{G}})} c^t_\omega \leq \sum_{i,j=1}^{k}[A^n_{\mathcal{G},t}]_{ij}$. Now, if G has k vertices, then $\mathcal{G}$ also has k vertices. By assumption, $\mathcal{G}$ is right-resolving, meaning no two edges leaving the same vertex have the same label. Hence, any $\omega \in B_n(X_{\mathcal{G}})$ can have at most k paths in $\mathcal{G}$. Therefore, for fixed value k, $\frac{1}{k}\sum_{i,j=1}^{k}[A^n_{\mathcal{G},t}]_{i,j} \leq \sum_{\omega \in B_n(X_{\mathcal{G}})} c^t_\omega$. □

Theorem 4.3.2. *Let $\{\mathcal{K}; f_1, \ldots, f_m\}$ be a hyperbolic IFS with $c_i d(x,y) \leq d(f_i(x), f_i(y)) \leq \overline{c}_i d(x,y)$ for $1 \leq i \leq m$ and all $x, y \in \mathcal{K}$. Let $X_{\mathcal{G}}$ be a sofic subshift on the alphabet $\mathcal{A} = \{1, \ldots, m\}$ and $\mathcal{F}_{\mathcal{G}}$ be the subfractal defined by the IFS and $X_{\mathcal{G}}$. Suppose $A_{\mathcal{G}}$ is irreducible. If $P(h) = 0$ and $\bar{P}(H) = 0$, then*

$$h \leq \dim_H(\mathcal{F}_{\mathcal{G}}) \leq H \text{ and } h \leq \overline{\dim}_B(\mathcal{F}_{\mathcal{G}}) \leq H.$$

Proof. By Lemma 4.3.1, we can rewrite the lower and upper topological pressure functions as

$$P(t) = \lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{\omega \in B_n(X_{\mathcal{G}})} c^t_\omega\right) = \lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{i,j=1}^{k}[A^n_{\mathcal{G},t}]_{i,j}\right) \text{ and}$$

$$\overline{P}(t) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{i,j=1}^{k} [\overline{A}_{\mathcal{G},t}^{n}]_{i,j} \right).$$

The remainder of the proof follows as in Theorem 4.2.6 and Theorem 4.2.7. $\square$

Remark 3. *Similar to Remark 1, the values of h and H such that $P(h) = 0 = \bar{P}(H)$ also satisfy $\rho(A_{\mathcal{G},h}) = 1 = \rho(\overline{A}_{\mathcal{G},H})$.*

Remark 4. *Similar to Remark 2, due to known relationships between Hausdorff, packing, upper and lower box dimensions, we also have*

$$h \leq \dim_P(\mathcal{F}_{\mathcal{G}}) \leq H \text{ and } h \leq \underline{\dim}_B(\mathcal{F}_{\mathcal{G}}) \leq H.$$

4.4. Generalization to reducible matrices

In this section, we will eliminate the irreducibility condition on the matrices in the case of Hausdorff dimension. Consider the case where $A_{\mathcal{G}}$ (or A_G if we have an SFT) is a reducible matrix. Let A be a reducible $m \times m$ (0,1)- matrix, and $\mathcal{G}$ be the associated graph. Since A is a reducible matrix, the graph $\mathcal{G}$ is not strongly connected, but it contains a finite number of strongly connected components, say $C_1, \ldots, C_k$. To each component, we can associate a submatrix $A_1, \ldots A_k$ where A_i is irreducible for $1 \leq i \leq k$. Now, we can simultaneously permute the rows and columns of A to obtain:

$$\tilde{A} = \begin{bmatrix} A_k & 0 & 0 & \cdots & 0 \\ * & A_{k-1} & 0 & \cdots & 0 \\ * & * & A_{k-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & A_1 \end{bmatrix}.$$

For further details on this process, see [19].

The process used to obtain $\tilde{A}$ from A will not affect the characteristic polynomial, and hence A and $\tilde{A}$ have the same eigenvalues. Also, by simultaneously interchanging rows and columns of A, each entry of A^n will appear in $\tilde{A}^n$, although possibly in a different entry position. Hence, we can assume $\sum_{i,j=1}^{m}(A^n)_{ij} = \sum_{i,j=1}^{m}(\tilde{A}^n)_{ij}$ [19]. Without loss of generality, we will assume that A is in the form of $\tilde{A}$.

If $A_{\mathcal{G}}$ is a reducible $m \times m$ matrix with irreducible components $A_1, \ldots, A_k$, let A_i be an $m_i \times m_i$ matrix for $1 \leq i \leq k$. For $l < k$, we define the set

$$trn(A_l, A_p) = \{a_{ij} \neq 0 : \sum_{s=l+1}^{k} m_s \leq i \leq \sum_{s=l}^{k} m_s, \sum_{s=p+1}^{k} m_s \leq j \leq \sum_{s=p}^{k} m_s\}$$

of all non-zero entries from $A_{\mathcal{G}}$ corresponding to a transitional edge in $\mathcal{G}$ from component C_l (associated with A_l) to the component C_p (associated with A_p). Let $B_{trn}(X_{\mathcal{G}}) \subset B_*(X_{\mathcal{G}})$ denote all finite words corresponding to a transitional edge from the graph $\mathcal{G}$.

Each strongly connected component of the graph, C_i, corresponds to an irreducible submatrix, A_i, and a subshift X_{A_i}, $1 \leq i \leq k$. For simplicity, we will talk about the construction of words in $X_{\mathcal{G}}$ by using the strongly connected components C_i, $1 \leq i \leq k$ from $\mathcal{G}$. Given the structure of the entire graph $\mathcal{G}$ and direction of the transitional edges, words in $X_{\mathcal{G}}$ must begin in a component C_i, move through components C_j, and end in component C_l where $1 \leq i \leq j \leq l \leq k$. To formalize this in the subshift setting, we introduce the following notation. For $1 \leq i < j \leq k$, let

$$X_{A_i} \circledast X_{A_j} = \{\omega \in X_{\mathcal{G}} : \omega = \tau a \xi, \text{ where } \tau \in B_*(X_{A_i}), a \in B_{trn}(X_{\mathcal{G}}), \xi \in X_{A_j}\}.$$

Similarly, for $1 \leq i_1 < i_2 < \cdots < i_l \leq k$, we define $X_{A_{i_1}} \circledast \cdots \circledast X_{A_{i_l}} = \{\omega \in X_{\mathcal{G}} : \omega = \tau_1 a_1 \tau_2 \cdots a_{l-1}\xi, \text{ where } \tau_j \in B_*(X_{A_{i_j}}), a_j \in B_{trn}(X_{\mathcal{G}}) \text{ for } 1 \leq j < l, \xi \in X_{A_{i_l}}\}$.

Lemma 4.4.1. *If $\mathcal{G}$ has k irreducible components for $k \geq 2$, then*

$$X_{\mathcal{G}} = \left(\bigcup_{i=1}^{k} X_{A_i}\right) \cup \left(\bigcup_{j=2}^{k} \bigcup_{i_1,\ldots,i_j=1}^{k} X_{A_{i_1}} \circledast \cdots \circledast X_{A_{i_j}}\right),$$

where $i_l < i_{l+1}$ for $1 \leq l < j$.

Proof. We will use induction for this argument. If $\mathcal{G}$ has two strongly connected components, C_1 and C_2, with at least one transitional edge from C_1 to C_2 then it follows that $X_{A_1} \cup X_{A_2} \cup (X_{A_1} \circledast X_{A_2}) \subseteq X_{\mathcal{G}}$. Now, let $\omega \in X_{\mathcal{G}}$. Then, ω must begin in either C_1, C_2, or on a transitional edge. If ω starts in C_2, then $\omega \in X_{A_2}$ because there are no transitional edges leaving C_2 in $\mathcal{G}$. If ω starts on a transitional edge, then $\omega \in (X_{A_1} \circledast X_{A_2})$ because it is of the form $\omega = \tau a \xi$ where τ is the empty word from $B_*(X_{A_1})$. If ω starts in C_1, then either $\omega \in X_{A_1}$ or $\omega \in (X_{A_1} \circledast X_{A_2})$. Hence, we must have

$$X_{A_1} \cup X_{A_2} \cup (X_{A_1} \circledast X_{A_2}) = X_{\mathcal{G}}.$$

Now, assume $\mathcal{G}$ is a connected graph with k strongly connected components, and consider the subgraph, say $\mathcal{G}|_{(k-1)}$, consisting of the first $k-1$ components. Assume that

$$X_{\mathcal{G}|_{(k-1)}} = (\bigcup_{i=1}^{k-1} X_{A_i}) \cup (\bigcup_{j=2}^{k-1} \bigcup_{i_1,\dots i_j=1}^{k-1} X_{A_{i_1}} \circledast \cdots \circledast X_{A_{i_j}}).$$

By comparing the graphs $\mathcal{G}$ and $\mathcal{G}|_{(k-1)}$ and their corresponding subshifts $X_{\mathcal{G}}$ and $X_{\mathcal{G}|_{(k-1)}}$, we can conclude that any word in $X_{\mathcal{G}} - X_{\mathcal{G}|_{(k-1)}}$ will end in C_k. Hence,

$$X_{\mathcal{G}} = X_{\mathcal{G}|_{(k-1)}} \cup X_{A_k} \cup \left(\bigcup_{i_1,\dots i_j=1}^{k-1} X_{A_{i_1}} \circledast \cdots \circledast X_{A_{i_j}} \circledast X_{A_k} \right),$$

which satisfies the assertion. $\square$

Proposition 4.4.2. *Let $A_{\mathcal{G}}$ be a reducible matrix with irreducible components $A_1, \dots, A_k$. Then,*

$$h_{i_j} \leq \dim_H(\mathcal{F}_{X_{A_{i_1}} \circledast \cdots \circledast X_{A_{i_j}}}) \leq H_{i_j},$$

where

$$h_{i_j} \leq \dim_H(\mathcal{F}_{X_{A_{i_j}}}) \leq H_{i_j}$$

and h_{i_j}, H_{i_j} are the bounds from Theorem 4.3.2.

Proof. Consider a finite word $\tau_1 a_1 \tau_2 a_2 \dots \tau_{j-1} a_{j-1}$ where $\tau_l \in B_*(X_{A_{i_l}})$ and $a_l \in trn(A_{i_l}, A_{i_l+1})$ for $1 \leq l \leq j-1$. For any $n \geq 1$, there are finitely many words $\tau_l \in B_*(X_{A_{i_l}})$ with $\ell(\tau_l) \leq n$, for all $1 \leq l \leq j$. Hence, there are finitely many words of the form $\tau_1 a_1 \cdots \tau_{j-1} a_{j-1}$ of length n. So, the collection $S = \{\tau_1 a_1 \cdots \tau_{j-1} a_{j-1} : \tau_l \in B_*(X_{A_{i_l}})$ for $1 \leq l \leq j-1, a_l \in trn(A_{i_l}, A_{i_l+1}), \ell(\tau_1 a_1 \cdots \tau_{j-1} a_{j-1}) < \infty\}$ is at most countable since $B_*(X_{A_{i_l}})$ is countable for

$i \leq i_l \leq k$. For $\omega \in S$, let $\omega X_{A_{i_j}} = \{\omega\xi \in X_{\mathcal{G}} : \xi \in X_{A_{i_j}}\}$. Then, $\dim_H((\mathcal{F}_{X_{A_{i_1 1}} \circledast \cdots \circledast X_{A_{i_j}}}) = \sup_{\omega \in S} \dim_H(\mathcal{F}_{\omega X_{A_{i_j}}})$.

Next, notice that $\mathcal{F}_{\omega X_{A_i}} = \{f_{\omega\xi}(x) : \xi \in X_{A_i} \text{ and } x \in \mathcal{K}\}$ for any $1 \leq i \leq k$. Recall that $f_{\omega\xi}(x) = f_\xi \circ f_\omega(x)$ and $f_\omega(x) \in \mathcal{K}$ for all $x \in \mathcal{K}$. Hence, $\mathcal{F}_{\omega X_{A_i}} \subseteq \mathcal{F}_{X_{A_i}}$. Hence,

$$\dim_H(\mathcal{F}_{\omega X_{A_i}}) \leq \dim_H(\mathcal{F}_{X_{A_i}}) \leq H_i,$$

where H_i is the bound from Theorem 4.3.2.

Let $\omega \in S$ with $\ell(\omega) = m$ and A_i be an irreducible block in A. Consider $\dim_H(\mathcal{F}_{\omega X_{A_i}})$. Although ωX_{A_i} is not necessarily a subshift itself, we can apply similar techniques used to prove Theorem 4.3.2 to show that the zero of the lower topological pressure function $P_{\omega,i}(t) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\tau \in B_n(\omega X_{A_i})} c_\tau^t \right)$ is a lower bound for $\dim_H(\mathcal{F}_{\omega X_{A_i}})$. Notice that

$$\begin{aligned}
P_{\omega,i}(t) &= \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\tau \in B_n(\omega X_{A_i})} c_\tau^t \right) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\tau \in B_{n-m}(X_{A_i})} c_\omega^t c_\tau^t \right) \\
&= \lim_{n\to\infty} \frac{1}{n} \left[\log(c_\omega^t) + \log \left(\sum_{\tau \in B_{n-m}(X_{A_i})} c_\tau^t \right) \right] \\
&= \lim_{n\to\infty} \frac{1}{n-m} \log \left(\sum_{\tau \in B_{n-m}(X_{A_i})} c_\tau^t \right) \\
&= P_i(t),
\end{aligned}$$

where $P_i(t)$ is the lower topological pressure function associated with the subfractal $\mathcal{F}_{X_{A_i}}$. Hence,

$$h_i \leq \dim_H(\mathcal{F}_{\omega X_{A_i}}).$$

Thus,

$$\dim_H(\mathcal{F}_{X_{A_{i_1 1}} \circledast \cdots \circledast X_{A_{i_j}}}) = \sup_{\omega \in S} \dim_H(\mathcal{F}_{\omega X_{A_{i_j}}}) \leq H_{i_j} \text{ and}$$

$$h_{i_j} \leq \dim_H(\mathcal{F}_{X_{A_{i_1 1}} \circledast \cdots \circledast X_{A_{i_j}}}),$$

where h_{i_j} and H_{i_j} are the zeros of the upper and lower topological pressure functions $P_{i_j}(t)$ and $\overline{P}_{i_j}(t)$ with respect to the subfractal $\mathcal{F}_{X_{A_{i_j}}}$ for some $1 \le i_j \le k$. $\square$

For a similar statement about subshifts with a reducible matrix A, we have, by Lemma 4.4.1,

$$\mathcal{F}_{X_{\mathcal{G}}} = \left(\bigcup_{i=1}^{k} \mathcal{F}_{X_{A_i}} \right) \cup \left(\bigcup_{j=2}^{k} \bigcup_{1 \le i_1 < \cdots < i_j \le k} \mathcal{F}_{X_{A_{i_1}} \circledast \cdots \circledast X_{A_{i_j}}} \right).$$

Thus, by Proposition 4.4.2, we have the following theorem.

Theorem 4.4.3. *Let $X_{\mathcal{G}}$ be a sofic subshift with associated matrix $A_{\mathcal{G}}$. Assume $A_{\mathcal{G}}$ has irreducible components $A_1, \ldots, A_k$. Let $\mathcal{F}_{X_{\mathcal{G}}}$ and $\mathcal{F}_{X_{A_i}}$ denote the subfractals associated with the subshifts $X_{\mathcal{G}}$ and X_{A_i}, respectively. Then,*

$$\max_{1 \le i \le k} \{h_i\} \le \dim_H(\mathcal{F}_{X_{\mathcal{G}}}) \le \max_{1 \le i \le k} \{H_i\},$$

where $P_i(h_i) = 0 = \bar{P}_i(H_i)$ given in Theorem 4.3.2 for all $1 \le i \le k$.

Theorem 4.4.3 extends the results of Theorem 4.3.2 in the case of Hausdorff dimension only. Recall from Chapter 2 that box dimension is not countably stable, which is vital to the proof of Proposition 4.4.2 with respect to Hausdorff dimension. Therefore, the results were not extended in the case of box dimension and it remains an open question whether the bounds exist for box dimension of a subfractal induced by a reducible matrix.

Example 4.4.4. Consider the subfractal of the Cantor set introduced in Example 3.4.2. Since the maps are similarities with the same similarity ratio of $\frac{1}{3}$, we can calculate the Hausdorff dimension of the subfractal. Recall that the adjacency matrix, A, for this subshift is reducible with two irreducible components, A_1 and A_2, and no transitional edges, where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Notice that the subfractal induced by X_{A_1} is a single point, and hence has zero Hausdorff dimension.

By Theorem 4.4.3,

$$\dim_H(\mathcal{F}_{X_A}) = \dim_H(\mathcal{F}_{X_{A_2}}).$$

Next, construct that matrices $S_0 = \begin{bmatrix} \frac{1}{9} & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{9} \end{bmatrix}$ and $S = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$. We find that

$$\dim_H(\mathcal{F}_{X_A}) = \frac{\log(\lambda)}{\log 3},$$

where λ denotes the maximal eigenvalue of A.

Example 4.4.4 should not be surprising, knowing that the entropy of the subshift X_A is given by $h(X_A) = \log(\lambda)$. If an IFS is composed of similarity ratios, each of which was the same similarity ratio c, the topological pressure function simplifies to

$$P(t) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\omega \in B_n(X_A)} c_\omega^t \right) = \lim_{n\to\infty} \frac{1}{n} \log(|B_n(X_A)| c^{nt}) = h(X_A) + \log(c^t).$$

Therefore, the Hausdorff dimension of the subfractal will be given by

$$\dim_H(\mathcal{F}_{X_A}) = \frac{h(X_A)}{-\log c},$$

where $h(X_A)$ denotes the entropy of the subshift X_A. Because of this, we are more interested in IFSs which contain maps with different similarity ratios (or different contractive bounds). To emphasize this point, let us consider a variation of the previous example.

Example 4.4.5. Let X_A be the subshift given in Example 4.4.4, but let the IFS be a variation of the Cantor set with $f_0(x) = \frac{1}{3}x$ and $f_1(x) = \frac{1}{2}x + \frac{1}{2}$. As we saw in the previous example, the Hausdorff dimension of the subfractal will be determined by the subfractal induced by X_{A_2}.

So, we construct the matrices

$$S_0 = \begin{bmatrix} \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}, \quad S = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \quad \text{and} \quad A_2 S = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}.$$

The Hausdorff dimension of the subfractal is given by the value of h such that $\rho(AS^{(h)}) = 1$.

5. CHAPTER FIVE - MULTIFRACTAL ANALYSIS

Thus far, we have studied attractors of IFSs (fractals) and subsets of the attractor of an IFS (subfractals). If a weight is assigned to each map in the IFS, the resulting attractor now contains more information than the usual attractor of an IFS. One way to visualize this is to think of the weights as values on a grayscale so that the weights determine how dark parts of the attractor appear. An illustration of this type of fractal would appear with varying levels of darkness. Consider a measure in $\mathbb{R}^n$ that is defined by the weights and that is supported on the fractal. What kinds of properties does the measure satisfy?

Alternatively, one can define a measure using information from a weighted IFS (or in our case, a weighted subfractal), and then study the support of the measure. By varying the weights, the "density" of the measure will vary for different points in the support set. This method allows one to study an array of fractals for the price of one measure. A modification of this process was developed for a *recurrent IFS* by replacing the weights on each map with a row-stochastic probability matrix and initial distribution vector. In this chapter, we will build our understanding of self-similar measures, measures supported on the attractor of a recurrent IFS, and measures supported on subfractals defined in previous chapters. We will also examine different properties of these measures, including local dimension and Hausdorff dimension.

5.1. Self-similar measures

Let $\{\mathcal{K}; f_1, \dots f_m\}$ be a disjoint IFS of similitudes with similarity ratios c_i for $1 \le i \le m$. Assign to each map a probability p_i such that $\sum_{i=1}^m p_i = 1$. The goal is to construct a measure which is supported on the attractor of the IFS, using the probabilities assigned to each map.

Let $\mathcal{A} = \{1, \dots, m\}$ be an alphabet with full shift space (X, σ). Define a measure ν on X such that

$$\nu([\![\omega]\!]) = p_{\omega_1} \cdots p_{\omega_n},$$

for any cylinder set $[\![\omega]\!]$. Notice that

$$\sum_{i=1}^{m} \nu([\![\omega i]\!]) = \sum_{i=1}^{m} p_{\omega_1} \cdots p_{\omega_n} p_i = p_{\omega_1} \cdots p_{\omega_n} \sum_{i=1}^{m} p_i$$
$$= p_{\omega_1} \cdots p_{\omega_n} = \nu([\![\omega]\!]).$$

Hence, by the Kolmogorov extension theorem, ν can be extended to a measure on all of X. Now, using the coding map $\pi : X \to \mathcal{F}$, define a measure μ on $\mathbb{R}^n$ by $\mu = \nu \circ \pi^{-1}$. Let $\mathcal{F}$ be the attractor of the IFS, and let $\mathcal{F}_\omega = f_\omega(\mathcal{F})$ for any $\omega \in B_n(X)$. Notice that the measure μ on $\mathbb{R}^n$ satisfies

$$\mu(\mathcal{F}_\omega) = p_{\omega_1} \cdots p_{\omega_n}.$$

Now, consider the support of the measure. In particular, we would like to calculate the "density" of the measure at certain points in the attractor of the IFS. The *upper* and *lower local dimensions* μ with respect to $x \in \mathcal{F}$ are defined as [8]

$$\underline{\dim}_{\mathrm{loc}}\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \text{ and } \overline{\dim}_{\mathrm{loc}}\mu(x) = \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

If the upper and lower local dimensions of μ at x are equal, we call this value the *local dimension* of μ at x and denote it by $\dim_{\mathrm{loc}} \mu(x)$. Local dimension gives a value to describe how concentrated a measure is at a specific point x, where a greater value corresponds to a less dense concentration of the measure at that point. For example, if μ is a measure supported on the attractor of an IFS, say $\mathcal{F}$, notice that $\dim_{\mathrm{loc}} \mu(x) = \infty$ for $x \notin \mathcal{F}$.

Recall that the Hausdorff dimension is an important property for distinguishing sets in fractal geometry. One reason for the special interest in local dimension of a measure is that it has a strong connection with Hausdorff dimension of a set. More precisely, the following is an equivalent definition for Hausdorff dimension of a set [8]:

$$\dim_H(E) = \sup\{s : \text{ there exists } \mu \text{ with } 0 < \mu(E) < \infty \text{ and}$$
$$\underline{\dim}_{\mathrm{loc}}\mu(x) \geq s \text{ for } \mu\text{-almost every } x \in E\}.$$

Another way to quantify the complexity of a measure supported on a fractal is to define the *Hausdorff dimension of a measure* μ as follows

$$\dim_H(\mu) = \sup\{s : \underline{\dim}_{\text{loc}}\mu(x) \geq s \text{ for } \mu\text{-almost every x}\},$$

or equivalently

$$\dim_H(\mu) = \inf\{\dim_H(E) : E \text{ is a Borel set with } \mu(E) > 0\}.$$

The following theorem from L.-S. Young highlights the connection between the Hausdorff dimension of a measure and the local dimension of a measure μ at a point x [28].

Theorem 5.1.1. *Let $\Lambda \subset \mathbb{R}^n$ be a measurable set with $\mu(\Lambda) > 0$. Suppose that for every $x \in \Lambda$,*

$$\underline{\delta} \leq \liminf_{\rho \to 0} \frac{\log \mu(B(x,\rho))}{\log \rho} \leq \limsup_{\rho \to 0} \frac{\log \mu(B(x,\rho))}{\log \rho} \leq \overline{\delta}.$$

Then,

$$\underline{\delta} \leq \dim_H(\mu) \leq \overline{\delta}.$$

To emphasize the importance of this theorem, let us consider the following example of a self-similar measure on $[0,1] \subset \mathbb{R}$.

Example 5.1.2. Let $\{f_0, f_1\}$ be an IFS with $f_0(x) = \frac{1}{2}x$ and $f_1(x) = \frac{1}{2}x + \frac{1}{2}$ for all $x \in I = [0,1]$ and $\mathcal{A} = \{0,1\}$ be the alphabet with full shift (X,σ). Now, let $0 \leq p_0 \leq \frac{1}{2}$ be the probability associated with f_0 and $p_1 = 1 - p_0$ be the probability associated with f_1, and let μ be the measure defined by $\mu(I_\omega) = p_\omega$, where $I_\omega = f_\omega(I)$ for any $\omega \in B_*(X)$, and extended to all of I in the usual way. For μ-a.e. $x \in [0,1]$, the local dimension of μ is given by

$$\dim_{\text{loc}} \mu(x) = s(p_0,p_1), \quad \text{where } s(p_0,p_1) = \frac{p_0 \log p_0 + p_1 \log p_1}{\log 2}.$$

So, by Theorem 5.1.1, the Hausdorff dimension of the measure is given by $\dim_H(\mu) = s(p_0,p_1)$. For more details on this example, see [8]. Observe that one can define uncountably many distinct measures of this form.

5.2. Measure supported on a recurrent IFS attractor

Let $\{f_i : 1 \leq i \leq M\}$ be a collection of similitudes with similarity ratios c_i for $1 \leq i \leq M$. Let $P = (p_{ij})_{i,j=1}^M$ be an $M \times M$ row stochastic matrix, i.e. $\sum_{j=1}^M p_{ij} = 1$ for all $1 \leq i \leq M$. Then, $\{\mathcal{K}; p_{ij}, f_i : 1 \leq i, j \leq M\}$ is called a *recurrent IFS*. If P is an irreducible matrix, then there exists a unique initial distribution vector $\mathbf{m} = (m_1, \ldots, m_M)$ such that

$$\sum_{i=1}^M m_i p_{ij} = m_j$$

for all $1 \leq j \leq M$.

Now, define a probability measure Q on cylinder sets from the symbolic space as follows: for $\omega = \omega_1 \ldots \omega_n \in B_n(X)$,

$$Q([\![\omega]\!]) = m_{\omega_n} p_{\omega_n \omega_{n-1}} \ldots p_{\omega_2 \omega_1}.$$

Since Q is well-defined on cylinder sets, by standard arguments one can extend Q to the entire symbolic space. Then, for a Borel set $B \subset \mathbb{R}^n$, define a measure μ as follows:

$$\mu(B) = Q\{\omega \in X : \pi(\omega) \in B\}.$$

Notice that μ is invariant with respect to $\{f_1, \ldots, f_M\}$. For a measure μ constructed in this way, A. Deliu, J.S. Geronimo, R. Shonkwiler, and D. Hardin proved the following [5]:

Theorem 5.2.1. *Let $\{P, f_i; 1 \leq i \leq M\}$ be a recurrent IFS with invariant measure μ as constructed above. Suppose that*

1. *f_i is a similitude for $1 \leq i \leq M$ with $0 < c_1 \leq c_2 \leq \ldots \leq c_M < 1$*
2. *the open set condition is satisfied and*
3. *the row stochastic matrix P is irreducible.*

Then, for μ-a.e. $x \in \mathbb{R}^n$,

$$\lim_{\rho \to 0} \frac{\log \mu(B(x, \rho))}{\log \rho} = \alpha,$$

where

$$\alpha = \frac{\sum_{i=1}^M \sum_{j=1}^M m_i p_{ij} \log(p_{ij})}{\sum_{i=1}^M m_i \log(c_i)}.$$

Now, combining Theorem 5.2.1 with Theorem 5.1.1, it follows immediately that

$$\dim_H(\mu) = \alpha,$$

for the value α given in Theorem 5.2.1.

5.3. Measure supported on subfractal induced by SFT

Let X_A be an SFT with forbidden words all of length k, for some $k \geq 2$. We will construct a sliding block code from the full shift space (X, σ) to a new subshift space (Y, σ) with the alphabet $\bar{\mathcal{A}} = \{\beta_1, \dots \beta_N\}$, where $N = M^{k-1}$ and each letter β_i corresponds to a word $\omega^i \in B_{k-1}(X)$. Let $\Phi : B_{k-1}(X) \to \bar{\mathcal{A}}$ be a block code which assigns each string of length $k-1$ from alphabet $\mathcal{A}$ to a letter in $\bar{\mathcal{A}}$. Since we have exactly one letter in $\beta_i \in \bar{\mathcal{A}}$ for each word $\omega^i \in B_{k-1}$, the map Φ must be bijective. Now, let $\phi : X \to Y$ be the sliding block code given by

$$\omega_1 \omega_2 \dots \mapsto \Phi(\omega_1 \dots \omega_{k-1}) \Phi(\omega_2 \dots \omega_k) \Phi(\omega_3 \dots \omega_{k+1}) \dots \quad ,$$

where $Y = \phi(X) \subset \bar{\mathcal{A}}^{\mathbb{N}}$. For more information on sliding block codes, see [19]. The map Φ^{-1} is clearly well-defined, but the map ϕ^{-1} requires a more detailed construction.

We can extend the operation $\odot$ defined in Chapter 3 so that $\odot : (B_n(X) \times B_{k-1}(X))_{comp} \to B_{n+1}$ for all $n \geq k \geq 2$, where a word $\omega \in B_n(X)$ is *compatible* with $\xi \in B_{k-1}$ if $\omega_{n-k+2} \dots \omega_n = \xi_1 \dots \xi_{k-2}$. In this case, $\omega \odot \xi = \omega_1 \dots \omega_n \xi_{k-1}$ for all compatible pairs (ω, ξ). Now, we can define $\phi^{-1} : Y \to X$ by

$$\phi^{-1}(\tau_1 \tau_2 \dots) = \Phi^{-1}(\tau_1) \odot \Phi^{-1}(\tau_2) \odot \Phi^{-1}(\tau_3) \odot \cdots .$$

We have defined both ϕ and ϕ^{-1} on infinite strings from X and Y, respectively. Both maps can be applied naturally to finite words $B_n(X)$ and $B_q(Y)$, for $n \geq k-1$ and $q \geq 1$, as follows:

$$\phi(\omega) = \Phi(\omega_1 \dots \omega_{k-1}) \dots \Phi(\omega_{n-k+1} \dots \omega_n)$$
$$\phi^{-1}(\xi) = \Phi^{-1}(\xi_1) \odot \dots \odot \Phi^{-1}(\xi_q)$$

for all $n \geq k$ and all $q \geq 1$.

Now, let $P = (p_{ij})_{1 \leq i,j \leq N}$ be an $N \times N$ row stochastic probability matrix, i.e. $\sum_{j=1}^{N} p_{ij} = 1$ for $1 \leq i \leq N$, with $p_{ij} = 0$ if and only if $a_{ij} = 0$ for entries from the adjacency matrix $A = (a_{ij})_{1 \leq i,j \leq N}$. If P is an irreducible matrix, then there exists a unique initial probability distribution vector $\mathbf{m} = (m_1, \ldots, m_N)$ such that $\mathbf{m}P = \mathbf{m}$.

Proposition 5.3.1. *Let X, Y, and ϕ be as above. Then, there exists an invariant probability measure on X.*

Proof. Let $\xi = \xi_1 \ldots \xi_n \in B_n(Y)$ and $[\![\xi]\!]$ denote the cylinder set with base ξ. Define

$$\nu([\![\xi]\!]) = m_{\xi_n} p_{\xi_n \xi_{n-1}} \ldots p_{\xi_2 \xi_1}.$$

First, we will show that ν is an invariant measure with respect to cylinder sets. Using the fact that P is row stochastic, we obtain

$$\begin{aligned} \nu(\sigma^{-1}([\![\xi]\!])) = \sum_{\beta_i \in \bar{\mathcal{A}}} \nu([\![\beta_i \xi]\!]) &= \sum_{\beta_i \in \bar{\mathcal{A}}} m_{\xi_n} p_{\xi_n \xi_{n-1}} \cdots p_{\xi_2 \xi_1} p_{\xi_1 \beta_i} \\ &= m_{\xi_n} p_{\xi_n \xi_{n-1}} \cdots p_{\xi_2 \xi_1} = \nu([\![\xi]\!]), \end{aligned}$$

as desired. Next, we will extend the measure to the entire symbolic space. Using the fact that $\sum_{\beta_i \in \bar{\mathcal{A}}} m_i p_{ij} = m_j$, we obtain

$$\begin{aligned} \sum_{\beta_i \in \bar{\mathcal{A}}} \nu([\![\xi \beta_i]\!]) &= \sum_{\beta_i \in \bar{\mathcal{A}}} m_{\beta_i} p_{\beta_i \xi_n} p_{\xi_n \xi_{n-1}} \cdots p_{\xi_2 \xi_1} \\ &= p_{\xi_n \xi_{n-1}} \cdots p_{\xi_2 \xi_1} \sum_{\beta_i \in \bar{\mathcal{A}}} m_{\beta_i} p_{\beta_i \xi_n} \\ &= m_{\xi_n} p_{\xi_n \xi_{n-1}} \cdots p_{\xi_2 \xi_1} \\ &= \nu([\![\xi]\!]). \end{aligned}$$

By applying the Kolmogorov extension theorem, we can extend ν to a measure defined on all of Y. Now, let $\gamma = \nu \circ \phi$, where ϕ is the sliding block code. $\square$

Using the invariant probability measure γ from Proposition 5.3.1, we can construct a measure μ on $\mathbb{R}^n$ which is supported on the subfractal. For Borel set $B \subset \mathbb{R}^n$, we define a measure μ as follows:

$$\mu(B) = \gamma\{\omega \in X : \pi(\omega) \in B\}.$$

5.4. Result for measures supported on a subfractal

For the remainder of this paper, we will assume that for given SFT X_A, the forbidden words are all of length k, for fixed $k \geq 2$. A common technique in fractal dimension calculations is to use a Moran covering of the form $\mathcal{O}_\rho = \{f_\omega(O) : |f_\omega(O)| < \rho \leq |f_{\omega^-}(O)|\}$, where O is a compact set such that $\mathcal{F} \subset O$, where $\mathcal{F}$ denotes the attractor of the IFS and $|\cdot|$ denotes the diameter of a set. For $\rho > 0$, we will define and utilize a variation of a Moran cover of the form

$$\begin{aligned} \mathcal{U}_\rho = &\{f_\omega(\mathcal{K}) : \omega = \omega_1 \ldots \omega_{q(k-1)} \text{ and} \\ &|f_{\omega_1 \ldots \omega_{q(k-1)}}(\mathcal{K})| < \rho \leq |f_{\omega_1 \ldots \omega_{(q-1)(k-1)}}(\mathcal{K})| \text{ for some } q \in \mathbb{Z}^+\}. \end{aligned}$$

Lemma 5.4.1. *For $0 < \rho < 1$ and $x \in \mathcal{F}$, the ball $B(x, \rho)$ intersects at most L elements of $\mathcal{U}_\rho$, where L is finite and independent of ρ.*

Notice that Lemma 5.4.1 is almost identical to Proposition 4.2.5., with the difference being in the definitions of $\mathcal{U}_r$ and $\mathcal{U}_\rho$. The proof follows almost exactly as the same steps for the proof of Proposition 4.2.5, except we find that $L \leq \left\lceil \frac{m_L(B(x,3\rho))}{m_L(B(x,ac_{min}^{(k-1)}\rho))} \right\rceil$, where m_L denotes Lebesgue measure. Hence, we will omit the details.

Now, let $\mathcal{U}_\rho$ be as in Lemma 5.4.1, μ be the measure from Section 5.3 and let $M_{\rho,x} = \{\omega \in B_*(X_A) : f_\omega(\mathcal{K}) \in \mathcal{U}_\rho$ and $f_\omega(\mathcal{K}) \cap B(x, \rho) \neq \emptyset\}$.

Lemma 5.4.2. *Let X_A be an SFT as described above with associated IFS $\{f_i : 1 \leq i \leq M\}$, each f_i a similitude with similarity ratio c_i, respectively, $1 \leq i \leq M$. Suppose the OSC is satisfied, and that probability matrix P is irreducible. Then, there exist positive constants K_1, K_2 such that for any $\rho > 0$ and $x \in \mathcal{F}$,*

$$K_1^\alpha \rho^\alpha \frac{m_{(\phi(\omega|_q))_{\ell(\phi(\omega|_q))}} p_{(\phi(\omega|_q))}}{c_{(\omega|_q)}^\alpha} \leq \mu(B(x, \rho)) \leq K_2^\alpha \rho^\alpha \sum_{\xi \in M_{\rho,x}} \frac{m_{\phi(\xi)_{\ell(\phi(\xi))}} p_{\phi(\xi)}}{c_\xi^\alpha},$$

where $\pi(\omega) = x$ *and*

$$\alpha = \frac{(k-1)\sum_{i=1}^{N}\sum_{j=1}^{N} m_i p_{ij}\log(p_{ij})}{\sum_{i=1}^{N} m_i \log(c_{\phi^{-1}(\beta_i)})}.$$

Proof. Without loss of generality, we will assume $\mathcal{K} = \mathcal{F}$. We will show the lower bound first. Let $\omega \in X_A$ such that $\pi(\omega) = x$ for some $x \in \mathcal{F}$. Let q be the least integer such that $f_{\omega_1 \dots \omega_{q(k-1)}}(\mathcal{K}) \subset B(x,\rho)$. If $\phi(\omega) = \tau$, then $\phi(\omega_1 \dots \omega_{q(k-1)}) = \tau_1 \dots \tau_l$ where $l = q(k-1)-(k-2)$. Hence, we obtain

$$\mu(B(x,\rho)) \geq \mu(f_{\omega_1 \dots \omega_{q(k-1)}}(\mathcal{K})) = m_{\tau_l} p_{\tau_l \tau_{l-1}} \cdots p_{\tau_2 \tau_1}.$$

Also, by our choice of q, we must have

$$c_{\omega_1} \cdots c_{\omega_{q(k-1)}} |\mathcal{K}| \leq 2\rho < c_{\omega_1} \cdots c_{\omega_{(q-1)(k-1)}} |\mathcal{K}|.$$

Therefore, if we let $c_{min} = \min_{1 \leq i \leq M}\{c_i\}$, we have

$$2\rho c_{min}^{k-1} < c_{\omega_1} \cdots c_{\omega_{q(k-1)}} |\mathcal{K}|.$$

Using these two inequalities, we obtain

$$\begin{aligned}
\mu(B(x,\rho)) &\geq m_{\tau_l} p_{\tau_l \tau_{l-1}} \cdots p_{\tau_2 \tau_1} \cdot \frac{(2\rho c_{min}^{(k-1)})^{\alpha}}{(c_{\omega_1} \cdots c_{\omega_{q(k-1)}} |\mathcal{K}|)^{\alpha}} \\
&= K_1^{\alpha} \rho^{\alpha} \cdot \frac{m_{\tau_l} p_{\tau_l \tau_{l-1}} \cdots p_{\tau_2 \tau_1}}{(c_{\omega_1} \cdots c_{\omega_{q(k-1)}})^{\alpha}} \\
&= K_1^{\alpha} \rho^{\alpha} \frac{m_{(\phi(\omega|_{q(k-1)}))_{\ell(\phi(\omega|_{q(k-1)}))}} p_{(\phi(\omega|_{q(k-1)}))}}{c^{\alpha}_{(\omega|_{q(k-1)})}},
\end{aligned}$$

where $K_1 = \dfrac{2c_{min}^{(k-1)}}{|\mathcal{K}|}$.

Now, we will prove the upper bound. Notice that the measure μ is supported on $\mathcal{F}$. For $\tilde{M}_{\rho,x} = \{f_{\omega}(\mathcal{K}) : \omega \in M_{\rho,x}\}$, we have

$$
\begin{aligned}
\mu(B(x,\rho)) \leq \mu(\tilde{M}_{\rho,x}) &= \sum_{\omega \in M_{\rho,x}} \mu(f_\omega(\mathcal{K})) \\
&= \sum_{\omega \in M_{\rho,x}} m_{\phi(\omega)_{\ell(\phi(\omega))}} p_{\phi(\omega)} \\
&= \sum_{\omega \in M_{\rho,x}} \frac{m_{\phi(\omega)_{\ell(\phi(\omega))}} p_{\phi(\omega)}}{c_\omega^\alpha} \cdot c_\omega^\alpha \\
&\leq K_2^\alpha \rho^\alpha \sum_{\omega \in M_{\rho,x}} \frac{m_{\phi(\omega)_{\ell(\phi(\omega))}} p_{\phi(\omega)}}{c_\omega^\alpha},
\end{aligned}
$$

where $K_2 = |\mathcal{K}|^{-1}$. □

Lemma 5.4.3. *For μ-a.e. $\omega \in B_*(Y)$,*

$$\lim_{q\to\infty} \frac{1}{q} \log(p_{\omega_q\omega_{q-1}} \ldots p_{\omega_2\omega_1}) = \sum_{i=1}^N \sum_{j=1}^N m_i p_{ij} \log(p_{ij})$$

and

$$\lim_{q\to\infty} \frac{1}{q} \log(c_{\omega_1\cdots\omega_{q(k-1)}}) = \sum_{i=1}^N m_i \log(c_{\Phi^{-1}(\beta_i)}).$$

Proof. Since P is irreducible and our system is ergodic, then by the Ergodic Theorem, for μ-a.e. $\omega \in B_*(Y)$, we have

$$
\begin{aligned}
&\lim_{q\to\infty} \frac{1}{q} \log(p_{\omega_q\omega_{q-1}} \cdots p_{\omega_2\omega_1}) \\
&= \lim_{q\to\infty} \frac{1}{q} \sum_{i=1}^{q-1} \log(p_{\omega_{i+1}\omega_i}) \\
&= \sum_{i=1}^N \sum_{j=1}^N m_i p_{ij} \log(p_{ij}).
\end{aligned}
$$

Again, by the Ergodic Theorem, we have

$$\begin{aligned}
&\lim_{q\to\infty} \frac{1}{q} \log(c_{\omega_1\omega_2\ldots\omega_{q(k-1)}}) \\
&= \lim_{q\to\infty} \frac{1}{q} \sum_{i=0}^{q-1} \log(c_{\omega_{i(k-1)+1}} \ldots c_{\omega_{(i+1)(k-1)}}) \\
&= \sum_{i=1}^{N} m_i \log(c_{\Phi^{-1}(\beta_i)}),
\end{aligned}$$

where $\beta_i \in \bar{\mathcal{A}}, 1 \le i \le N$. □

Recall that $\mathcal{U}_\rho = \{f_\omega(\mathcal{K}) : |f_{\omega_1\ldots\omega_{q(k-1)}}(\mathcal{K})| < \rho \le |f_{\omega_1\ldots\omega_{(q-1)(k-1)}}(\mathcal{K})| \text{ for some } q \in \mathbb{Z}^+\}$, and hence as $\rho \to 0$, it must be the case that $q \to \infty$.

Lemma 5.4.4. *Let $M_{\rho,x}$ be as above and q be the integer such that $\omega = \omega_1 \ldots \omega_{q(k-1)} \in M_{\rho,x}$ for some $x \in \mathcal{F}$. Then,*

$$\lim_{\rho\to 0} \frac{\log(\rho)}{q} = \sum_{i=1}^{N} m_i \log(c_{\Phi^{-1}(\beta_i)}).$$

Proof. By the definition of $M_{\rho,x}$, q is the least integer such that

$$c_{\omega_1} \ldots c_{\omega_{q(k-1)}} |\mathcal{K}| < \rho \le c_{\omega_1} \ldots c_{\omega_{(q-1)(k-1)}} |\mathcal{K}|.$$

Hence,

$$c_{min}^{(k-1)} \rho \le c_{\omega_1\ldots\omega_{q(k-1)}} |\mathcal{K}| < \rho.$$

By this inequality and Lemma 5.4.3, we obtain

$$\lim_{\rho\to 0} \frac{\log \rho}{q} = \sum_{i=1}^{N} m_i \log(c_{\Phi^{-1}(\beta_i)}).$$

□

Theorem 5.4.5. *Let $\{f_i : 1 \le i \le M\}$ be an IFS satisfying the OSC and containing similitudes f_i with similarity ratios c_i, respectively, for $1 \le i \le M$. Let X_A be an SFT on the alphabet $\mathcal{A} = \{1, \ldots, M\}$ and P be an $N \times N$ irreducible, row stochastic probability matrix corresponding to*

X_A *where* $N = M^{k-1}$*. Then for* μ*-a.e.* $x \in \mathbb{R}^n$*,*

$$\lim_{\rho\to 0} \frac{\log(\mu(B(x,\rho))}{\log \rho} = \alpha,$$

where

$$\alpha = \frac{(k-1)\sum_{i=1}^{N}\sum_{j=1}^{N} m_i p_{ij}\log(p_{ij})}{\sum_{i=1}^{N} m_i \log(c_{\phi^{-1}(\beta_i)})}.$$

Proof. Recall that $|M_{\rho,x}| \leq L$ for all $\rho > 0$ and $x \in \mathcal{F}$. Since μ is supported on $\mathcal{F}$, then we will only consider $\omega \in M_{\rho,x}$ such that $\gamma([\![\omega]\!]) > 0$, where γ is the measure constructed in Section 5.3. Since L is independent of ρ, there must exist some $\xi \in X_A$ satisfying $\frac{p_{\phi(\omega)}}{c_\omega} \leq \frac{p_{\phi(\xi|_{q(k-1)})}}{c_{\xi|_{q(k-1)}}}$ for all $\omega \in M_{\rho,x}$ with $\ell(\omega) = q(k-1)$, and which also satisfies the assertion of Lemma 5.4.3. Hence, it follows that

$$\lim_{\rho\to 0}\log\left(\sum_{\omega\in M_{\rho,x}} \frac{p_{\phi(\omega)}}{c_\omega}\right) \leq \lim_{\rho\to 0}\log\left(L\cdot\frac{p_{\phi(\xi|_{q(k-1)})}}{c_{\xi|_{q(k-1)}}}\right).$$

Using Lemma 5.4.3 and Lemma 5.4.4, and the fact that L is independent of q, we have

$$\begin{aligned}
&\lim_{\rho\to 0}\frac{\log\left(\sum_{\omega\in M_{\rho,x}}\frac{p_{\phi(\omega)}}{c_\omega^\alpha}\right)}{\log\rho} \leq \lim_{\rho\to 0}\frac{\log\left(L\cdot\frac{p_{\phi(\xi|_{q(k-1)})}}{c^\alpha_{\xi|_{q(k-1)}}}\right)}{\log\rho}\\
&\leq \lim_{q\to\infty}\frac{(k-1)\log(p_{\tau_1\ldots\tau_{q(k-1)}})}{q(k-1)\sum_{i=1}^{N} m_i\log(c_{\Phi^{-1}(\beta_i)})} - \lim_{q\to\infty}\frac{\log(c^\alpha_{\xi_1\ldots\xi_{q(k-1)}})}{q\sum_{i=1}^{N} m_i\log(c_{\Phi^{-1}(\beta_i)})}\\
&= \frac{(k-1)\sum_{i=1}^{N}\sum_{j=1}^{N} m_i p_{ij}\log(p_{ij})}{\sum_{i=1}^{N^{k-1}} m_i\log(c_{\Phi^{-1}(\beta_i)})} - \alpha = 0.
\end{aligned}$$

We also have

$$\lim_{\rho\to 0}\log\left(\sum_{\omega\in M_{\rho,x}}\frac{p_{\phi(\omega)}}{c_\omega}\right) \geq \lim_{\rho\to 0}\log\left(\frac{p_{\phi(\omega)}}{c_\omega}\right),$$

for any $\omega \in M_{\rho,x}$. Therefore, it follows that

$$\begin{aligned}
&\lim_{\rho\to 0}\frac{\log\left(\sum_{\omega\in M_{\rho,x}}\frac{p_{\phi(\omega)}}{c_\omega^\alpha}\right)}{\log\rho}\\
&\quad\geq \lim_{q\to\infty}\frac{\log(p_{\tau_1\ldots\tau_{q(k-1)}})}{q\sum_{i=1}^{N} m_i\log(c_{\Phi^{-1}(\beta_i)})} - \lim_{q\to\infty}\frac{\log(c^\alpha_{\omega_1\ldots\omega_{q(k-1)}})}{q\sum_{i=1}^{N} m_i\log(c_{\Phi^{-1}(\beta_i)})} = 0.
\end{aligned}$$

Thus, by Lemma 5.4.2, we have

$$\lim_{\rho \to 0} \frac{\log(\mu(B(x,\rho)))}{\log \rho}$$
$$\leq \lim_{\rho \to 0} \frac{\log\left(\rho^{\alpha} \sum_{M_{\rho,x}} \frac{m_{\xi_q} p_{\xi}}{c^{\alpha}_{\phi^{-1}(\xi)}}\right)}{\log \rho} = \alpha.$$

Similarly, by Lemma 5.4.2, we obtain

$$\lim_{\rho \to 0} \frac{\log(\mu(B(x,\rho))}{\log \rho} \geq \alpha.$$

□

The following Corollary follows immediately from Theorem 5.4.5 and Theorem 5.1.1.

Corollary 5.4.6. *Let μ be the measure in Theorem 5.4.5. Then,*

$$\dim_H(\mu) = \alpha,$$

where α is the value given in Lemma 5.4.2.

Example 5.4.7. In this example, we will illustrate for a specific choice of values for the entries of P, we have

$$\dim_H(\mu) = \dim_H(\mathcal{F}),$$

where $\mathcal{F}$ is the attractor of an IFS in which we only consider allowable points with respect to an SFT on the symbolic space. By Theorem 4.2.6, the Hausdorff dimension of the attractor of this form is given by the nonnegative real number h such that

$$\rho(AS^{(h)}) = 1,$$

where A is the adjacency matrix, S is a diagonal matrix with the contractive factors on the diagonal arranged with respect to the matrix A, and $S^{(h)}$ contains corresponding entries of S raised to the h power. Let $\mathcal{A} = \{0, 1\}$ and X_A be an SFT with forbidden word list $\{\tau^1, \ldots, \tau^l\}$ where $\ell(\tau_i) = 3$ for $1 \leq i \leq l$. We will assume A is an irreducible 4×4 matrix. Let h be the value such that

$\rho(AS^{(h)}) = 1$. Through standard calculations, we can show that the characteristic polynomial of $S_0^{(h/2)}A$ is equal to the characteristic polynomial of $AS^{(h)}$, where S_0 is a diagonal matrix with contractive factors associated with strings of length 2 arranged on the diagonal with respect to A. Hence, $\rho(S_0^{(h/2)}A) = 1$. Since $S_0^{(h/2)}A$ is irreducible, then by the Perron-Frobenius theorem there exists a positive eigenvector $\mathbf{v} = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$ associated with eigenvalue 1.

Let $p_{ij} = \frac{v_j}{v_i} c_{\omega^i}^{(h/2)} a_{ij}$ where $c_{\omega^i}^{(h/2)} a_{ij}$ is the ij-th entry of $S_0^{(h/2)}A$ for $1 \le i, j \le 4$. Using the fact that $\mathbf{v}$ is an eigenvector associated with eigenvalue 1, we have

$$\sum_{j=1}^{4} \frac{v_j}{v_i} c_{\omega^i}^{(h/2)} a_{ij} = 1,$$

so that P is row stochastic. We also notice that $p_{ij}\log(a_{ij}) = 0$ for all $1 \le i, j \le 4$ and

$$\sum_{i=1}^{4}\sum_{j=1}^{4} m_i p_{ij} \log(v_j) = \sum_{j=1}^{4} m_j \log(v_j) = \sum_{i=1}^{4}\sum_{j=1}^{4} m_i p_{ij} \log(v_i).$$

Hence, by Theorem 5.4.5, we obtain

$$\begin{aligned}
\alpha &= \frac{2\sum_{i=1}^{4}\sum_{j=1}^{4} m_i p_{ij}\log(p_{ij})}{\sum_{i=1}^{4} m_i \log(c_{\omega^i})} \\
&= \frac{2\sum_{i=1}^{4}\sum_{j=1}^{4} m_i p_{ij}[\log(v_j) - \log(v_i) + \frac{h}{2}\log(c_{\omega^i}) + \log(a_{ij})]}{\sum_{i=1}^{4} m_i \log(c_{\omega^i})} \\
&= h.
\end{aligned}$$

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