SEMIDUALIZING DG MODULES OVER TENSOR PRODUCTS

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DOCTOR OF PHILOSOPHY

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ABSTRACT

In this dissertation, we study rings: sets with addition, subtraction, and multiplication. One way to study a ring is by studying its modules: the algebraic objects the ring acts on. Since it is impractical to study all of its modules, I study its semidualizing modules. These modules have proven useful in the study of the composition of local ring homomorphisms of finite G-dimension and Bass numbers of local rings.

Let R be a commutative, noetherian ring with identity. A finitely generated R-module C is *semidualizing* if the homothety map $\chi_C^R : R \to \mathrm{Hom}_R(C, C)$ is an isomorphism and $\mathrm{Ext}_R^i(C, C) = 0$ for all $i > 0$. For example, the ring R is semidualizing over itself, as is a dualizing module, if R has one. In some sense the number of semidualizing modules a ring has gives a measure of the "complexity" of the ring. I am interested in that number.

More generally in this dissertation we use the definition of semidualizing differential graded (DG) module, pioneered by Christensen and Sather-Wagstaff. In particular, I construct semidualizing DG modules over the tensor product of two DG k-algebras, say A' and A''. This gives us a lower bound on the number of semidualizing DG modules over the tensor product $A' \otimes_k A''$. Therefore, as far as semidualizing DG modules can detect, the singularity of $A' \otimes_k A''$ is at least as bad as the singularities of both A' and A'' combined.

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TABLE OF CONTENTS

1. INTRODUCTION

Assumption 1.0.0.1. Throughout this dissertation, let R be a commutative, noetherian ring with identity.

Commutative algebra is the study of rings: a ring is a set R with coherent rules for addition, subtraction, and multiplication (but not necessarily division). These algebraic objects are ubiquitous in mathematics. For example, from arithmetic, the set of integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is a ring. The point here is that, given two integers m and n, the sum $m + n$, difference $m - n$, and product mn are also integers; however, the quotient m/n may not be an integer. Also, from algebra, we have polynomial rings: sets of polynomials like $x^2 + 5x + 2$ and $x^2 + y^2 - z^2 + 1$. Again, the point here is, given two polynomials f and g, the sum $f+g$, difference $f-g$, and product fg are also polynomials; however, the quotient f/g may not be a polynomial. Additionally from calculus, we have rings of continuous and differentiable functions. Moreover, algebra offers powerful tools for advancing these other areas.

This idea is especially important in geometry where rings provide a two way information conduit. For instance, if you start with an algebraic system of polynomial equations, the corresponding geometric object of interest is the solution to the system. For instance the solution set of the equation $y^2 - x^3 = 0$ is a curve, and the solution set of the equation $x^2 + y^2 + z^2 - 1 = 0$ is a sphere. On the other hand, given a geometric object, there is a natural ring associated to it. The point here is that if we can understand that ring, we can further understand the geometric object and vice versa. For example, the more complicated the ring, the more complicated the geometric object and conversely. This is the foundation of much of modern algebraic geometry.

One way to study the complexity of a ring R is by looking at its modules. In particular, modules help to understand the complexity of R as follows: if all R-modules have a simple form, then R itself has a simple form; and if R has a simple form, then all its modules have a simple form. This idea has its roots in the famous theorem of Auslander, Buchsbaum, and Serre, which states that a local ring is "regular" if and only if all of its modules have finite projective dimension; see [4] and [32]. This point of view has a huge number of significant consequences for algebra and many other areas of mathematics.

It is impossible to study all modules, so we must focus on special ones. My research focuses on "semidualizing modules." Foxby [13] introduced these modules, while Vasconcelos [34] and Golod [18] rediscovered them independently and applied them in different contexts. They are useful tools that have been applied successfully to the study of compositions of local ring homomorphisms of finite G-dimension [6, 29] and Bass numbers of local rings [30]. They have also been studied for their own sake, e.g., in [9, 17, 21, 23]

Definition 1.0.0.2. A finitely generated R-module C is *semidualizing* if the natural homothety map $\chi_C^R : R \to \mathrm{Hom}_R(C, C)$ is an isomorphism and $\mathrm{Ext}_R^i(C, C) = 0$ for all $i \geqslant 1$. Let $\mathfrak{S}_0(R)$ denote the set of isomorphism classes of semidualizing modules.

For instance, R is always a semidualizing R-module. Other examples include Grothendieck's canonical modules, used to study cohomology of algebraic varieties.

The size of $\mathfrak{S}_0(R)$ measures the severity of the singularity of a ring, specifically how close a ring is to being Gorenstein. If $\mathfrak{S}_0(R)$ is large, then R is far from being Gorenstein. If $\mathfrak{S}_0(R)$ is small, then R is in a sense close to being Gorenstein. For instance, if R is Gorenstein and local, then $|\mathfrak{S}_0(R)| = 1$. The converse of this statement holds if R has a "dualizing module."

In my research I am interested in studying ways of combining two rings, say R_1 and R_2. For instance, a standard construction is the tensor product $R_1 \otimes_k R_2$. If R_1 and R_2 come from geometric objects V_1 and V_2, this ring $R_1 \otimes_k R_2$ corresponds to the "cartesian product" $V_1 \times V_2$, which is at least as complicated as V_1 and V_2. This is impossible to visualize in general: even in the simplest case, where V_1 and V_2 are curves in the plane, the product $V_1 \times V_2$ is a surface in 4-space! Thus, algebraic methods are extremely important here. In this context, the main theorem of this dissertation states that the set $\mathfrak{S}_0(R_1 \otimes_k R_2)$ is at least as large as the cartesian product $\mathfrak{S}_0(R_1) \times \mathfrak{S}_0(R_2)$. This fits the above context, as $\mathfrak{S}_0(R_1) \times \mathfrak{S}_0(R_2)$ is at least as complicated as $\mathfrak{S}_0(R_1)$ and $\mathfrak{S}_0(R_2)$ combined.

More generally in this dissertation we use the definition of semidualizing DG module. ("DG" is short for "Differential Graded." See Chapter 2 for relevant background information.) The idea for the definition is essentially from Christensen and Sather-Wagstaff [10]; see also [25]. The DG setting comes from algebraic topology, and its use in commutative algebra was pioneered by Avramov; see, e.g., [5]. It has been useful for answering questions about rings. For instance, Nasseh and Sather-

Wagstaff [25] were able to use this to answer Vasconcelos' question [34, p. 97], showing a local ring has only finitely many isomorphism classes of semidualizing modules.

What follows is the main result of this dissertation, which is proven in 3.2.1.7. The reverse implication of part (a) of the following theorem shows that if you grab a semidualizing DG module over A' and a semidualizing DG module over A'' you can construct a semidualizing DG module over $A' \otimes_k A''$ by tensoring them together over the field k. The forward implication of part (a) shows that if you start with a semidualizing DG module over $A' \otimes_k A''$ that happens to be a tensor product, the components of the tensor product must have been semidualizing DG modules over the corresponding DG k-algebras A' and A''. This is a consequence of the "Künneth formula," properly interpreted. Part (b), on the other hand, uses some significant technology including an extension of Foxby and Christensen's Bass classes to the DG setting. Much of this dissertation is devoted to the development of this technology.

Theorem 1.0.0.3. *Let k be a field. Let A' and A'' be homologically bounded, local DG k-algebras. Let $M' \in D^f(A')$ and $M'' \in D^f(A'')$.*

(a) Then $M' \otimes_k M''$ is semidualizing over $A' \otimes_k A''$ if and only if M' is semidualizing over A' and M'' is semidualizing over A''.

(b) The map $\psi : \mathfrak{S}(A') \times \mathfrak{S}(A'') \to \mathfrak{S}(A' \otimes_k A'')$ defined by $\psi(C', C'') = C' \otimes_k C''$ is well-defined and injective.

The fact that ψ is injective gives us a lower bound on the number of semidualizing DG modules over $A' \otimes_k A''$. This fits with the big picture idea discussed above: as far as semidualizing DG modules can detect, the singularity of the ring $A' \otimes_k A''$ is at least as bad as the singularities of both A' and A'' combined.

We end this introduction with a summary of the contents of the rest of this dissertation. Chapter 2 contains significant background material on semidualizing modules, derived categories (the natural habitat for semidualizing modules), and DG algebras and modules. In Chapter 3, we develop necessary DG technology and prove our main theorem. We conclude with Chapter 4 which contains ideas for future work.

2. BACKGROUND

This chapter contains background material and some technical results needed for the proof of our main theorem.

2.1. Semidualizing Modules

2.1.1. Definitions and Examples

The following definition is due to Grothendieck, see [19].

Definition 2.1.1.1. An R-module D is *dualizing* if and only if D is a semidualizing R-module and D has finite injective dimension over R.

Definition 2.1.1.2. R is *Gorenstein* if and only if R has finite injective dimension over R.

Example 2.1.1.3. Let k be a field. The ring $k[\![X_1, X_2, \ldots, X_n]\!]$ is Gorenstein.

The forward implication of the following fact is due to Sharp [33], and the reverse implication is due to Foxby [13] and Reiten [26].

Fact 2.1.1.4. If R is local, then R is Cohen Macaulay and is a homomorphic image of a local Gorenstein ring if and only if R has a dualizing module.

The next example is due to Sather-Wagstaff [31].

Example 2.1.1.5. Let k be a field, $R' = k[\![X, Y]\!]/(X, Y)^2$ and $R'' = k[\![Z, W]\!]/(Z, W)^2$. The rings R' and R'' are artinian and hence Cohen Macaulay. Also, the local rings R' and R'' are the homomorphic image of the Gorenstein rings $k[\![X, Y]\!]$ and $k[\![Z, W]\!]$, respectively. Therefore, by Fact 2.1.1.4, the rings R' and R'' have dualizing modules D' and D'', respectively. Define $R := R' \otimes_k R''$. The semidualizing modules of R are R, $D' \otimes_k R''$, $R' \otimes_k D''$, and $D' \otimes_k D''$.

The following notion of the Bass and Auslander classes are due to Foxby; see [13].

Definition 2.1.1.6. An R-module M is in the *Bass class* $\mathcal{B}_C(R)$ if and only if the natural evaluation homomorphism $\xi_M^C : C \otimes_R \operatorname{Hom}_R(C, M) \to M$ is an isomorphism and $\operatorname{Ext}_R^i(C, M) = 0 = \operatorname{Tor}_i^R(C, \operatorname{Hom}_R(C, M))$ for all $i > 0$.

Definition 2.1.1.7. An R-module M is in the *Auslander class* $\mathcal{A}_C(R)$ if and only if the natural map $\gamma_M^C : M \to \operatorname{Hom}_R(C, C \otimes_R M)$ is an isomorphism and $\operatorname{Tor}_i^R(C, M) = 0 = \operatorname{Ext}_R^i(C, C \otimes_R M)$ for all $i > 0$.

Example 2.1.1.8. Let $C = R$. Then $\mathcal{B}_C(R) = \mathcal{A}_C(R)$ is the class of all R-modules.

Example 2.1.1.9. If C is a dualizing R-module, then the modules in $\mathcal{A}_C(R)$ are exactly the modules of finite "Gorenstein projective dimension," and the modules in $\mathcal{B}_C(R)$ are exactly the modules of finite "Gorenstein injective dimension," by [11]. In this setting the Auslander and Bass classes give very useful resolution-free characterizations of these modules.

2.1.2. Properties

Fact 2.1.2.1. If $C \in \mathfrak{S}_0(R)$, then $C \in \mathcal{B}_C(R)$.

Proof: Assume $C \in \mathfrak{S}_0(R)$. Consider the following diagram.

$$\begin{array}{ccc} C \otimes_R \operatorname{Hom}_R(C, C) & \xrightarrow{\xi_C^C} & C \\ {\scriptstyle C \otimes_R \chi_C^R} \Big\uparrow \cong & \nearrow \cong & \\ C \otimes_R R & & \end{array}$$

The diagram commutes as follows.

$$\begin{array}{ccc} c \otimes_R \chi_C^R(r) & \longmapsto & \chi_C^R(r)(c) = rc \\ \Big\uparrow & \nearrow & \\ c \otimes_R r & & \end{array}$$

Notice, the assumption $C \in \mathfrak{S}_0(R)$ implies $\operatorname{Ext}_R^i(C, C) = 0$ for all $i > 0$. Also,

$$\operatorname{Tor}_i^R(C, \operatorname{Hom}_R(C, C)) \cong \operatorname{Tor}_i^R(C, R) = 0$$

for all $i > 0$ because R is free. □

The following fact is from [13, Lemma 1.3].

Fact 2.1.2.2. Let C be a semidualizing R-module and consider the exact sequence of R-modules

$$0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0.$$

1. If two of the $M_i \in \mathcal{B}_C(R)$, then so is the third.

2. If two of the $M_i \in \mathcal{A}_C(R)$, then so is the third.

In the following two facts, the proof of implication (i) $\implies$ (ii) is from [13, Proposition 1.2], the implication (ii) $\implies$ (iii) is easy, and the implication (iii) $\implies$ (i) is from [28].

Fact 2.1.2.3. Let C be a finitely generated R-module. The following are equivalent.

(i) $C \in \mathfrak{S}_0(R)$.

(ii) The class $\mathcal{B}_C(R)$ contains every R-module of finite injective dimension.

(iii) The class $\mathcal{B}_C(R)$ contains every injective R-module.

Fact 2.1.2.4. Let C be a finitely generated R-module. The following are equivalent.

(i) $C \in \mathfrak{S}_0(R)$.

(ii) The class $\mathcal{A}_C(R)$ contains every R-module of finite flat dimension.

(iii) The class $\mathcal{A}_C(R)$ contains every flat R-module.

The next definition is from Golod [18], based on work of Auslander and Bridger [3].

Definition 2.1.2.5. Let C be a finitely generated R-module. A finitely generated R-module G is *totally C-reflexive* if the biduality map $\delta_G^C : G \to \operatorname{Hom}_R(\operatorname{Hom}_R(G, C), C)$ is an isomorphism and $\operatorname{Ext}_R^i(G, C) = 0 = \operatorname{Ext}_R^i(\operatorname{Hom}_R(G, C), C)$ for all $i > 0$.

Example 2.1.2.6. Let C be a semidualizing R-module and let P be a finitely generated projective R-module.

1. If G is totally C-reflexive, then so is $G \otimes_R P$.

2. The modules P and $C \otimes_R P$ are totally C-reflexive.

3. For each integer $n \geqslant 0$, the modules R^n and C^n are totally C-reflexive.

Fact 2.1.2.7. Let C be a semidualizing R-module and consider the exact sequence of finitely generated R-modules

$$0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$$

such that M_3 is totally C-reflexive. Then M_1 is totally C-reflexive if and only if M_2 is totally C-reflexive.

The following fact is from [16, Theorem 1.4], translated into the module setting.

Fact 2.1.2.8. Assume R is local. Let B and C be semidualizing R-modules. The following conditions are equivalent.

(i) $B \cong C$.

(ii) $B \in \mathcal{B}_C(R)$ and $C \in \mathcal{B}_B(R)$.

2.2. The Category of R-Complexes

The derived category $D(R)$ is the native habitat for semidualizing modules and semidualizing complexes. Its construction is quite technical. In the next few sections, we outline the construction and basic properties of $D(R)$, beginning with the category $\mathcal{C}(R)$.

2.2.1. Definitions and Examples

Definition 2.2.1.1. A sequence of R-module homomorphisms

$$M = \cdots \xrightarrow{\partial_{i+1}^M} M_i \xrightarrow{\partial_i^M} M_{i-1} \xrightarrow{\partial_{i-1}^M} \cdots$$

is an *R-complex* if $\partial_{i-1}^M \partial_i^M = 0$ for all i. We say that M_i is the module in *degree i* in the R-complex M. The *ith homology module* of an R-complex M is the R-module

$$\mathrm{H}_i(M) = \mathrm{Ker}(\partial_i^M)/\,\mathrm{Im}(\partial_{i+1}^M).$$

The notation $|a| = i$ means $a \in M_i$. The *supremum* and *infimum* of M are as follows:

$$\inf(M) := \inf\{n \in \mathbb{Z} | H_n(M) \neq 0\}$$
$$\sup(M) := \sup\{n \in \mathbb{Z} | H_n(M) \neq 0\}.$$

Here are some important examples to keep in mind.

Example 2.2.1.2. Let M be an R-module. We can think of M as an R-complex concentrated in degree zero: $M = 0 \to M \to 0$. An *augmented projective resolution of M* over R is an exact sequence of R-module homomorphisms

$$P^+ = \cdots \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \xrightarrow{\tau} M \to 0$$

such that each P_i is a projective R-module. The *truncated resolution of* M over R associated to P^+ is the R-complex

$$P = \cdots \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \to 0.$$

This is an R-complex such that $\mathrm{H}_i(P) \cong 0$ for all $i \neq 0$ and $\mathrm{H}_0(P) \cong M$. An *augmented injective resolution of* M over R is an exact sequence of R-module homomorphisms

$$^+I = 0 \to M \xrightarrow{\epsilon} I_0 \xrightarrow{\partial_0^I} I_{-1} \xrightarrow{\partial_{-1}^I} I_{-2} \to \cdots$$

such that each I_i is an injective R-module. The *truncated resolution of* M over R associated to ^+I is the R-complex

$$I = 0 \to I_0 \xrightarrow{\partial_0^I} I_1 \xrightarrow{\partial_{-1}^I} I_{-2} \to \cdots .$$

This is an R-complex such that $\mathrm{H}_i(I) \cong 0$ for all $i \neq 0$ and $\mathrm{H}_0(I) \cong M$.

Fact 2.2.1.3. An R-complex M is exact if and only if $\mathrm{H}_i(M) = 0$ for all i.

Definition 2.2.1.4. Let i be an integer. The *ith suspension* of an R-complex M is the R-complex $\Sigma^i M$ defined by $(\Sigma^i M)_n := M_{n-i}$ and $\partial_n^{\Sigma^i M} := (-1)^i \partial_{n-i}^M$. The scalar multiplication on $\Sigma^i M$ is defined by the formula $\mu^{\Sigma^i M}(r \otimes m) := (-1)^i \mu^M (r \otimes m)$.

Fact 2.2.1.5. Let i be an integer. The ith suspension of an R-complex M is an R-complex such that $\mathrm{H}_n(\Sigma^i M) \cong \mathrm{H}_{n-i}(M)$ for all $n \in \mathbb{Z}$.

Definition 2.2.1.6. Let M and N be R-complexes. A *chain map* $F : M \to N$ is a sequence $\{F_i : M_i \to N_i\}_{i \in \mathbb{Z}}$ making each of the rectangles in the following diagram commute.

$$\begin{array}{ccccccccc}
M & = & \cdots & \xrightarrow{\partial_{i+1}^M} & M_i & \xrightarrow{\partial_i^M} & M_{i-1} & \xrightarrow{\partial_{i-1}^M} & \cdots \\
\downarrow F & & & & \downarrow F_i & & \downarrow F_{i-1} & & \\
N & = & \cdots & \xrightarrow{\partial_{i+1}^N} & N_i & \xrightarrow{\partial_i^N} & N_{i-1} & \xrightarrow{\partial_{i-1}^N} & \cdots
\end{array}$$

Definition 2.2.1.7. Let $\mathcal{C}(R)$ denote the category with objects the R-complexes and morphisms the chain maps.

The next example is essentially the starting point for semidualizing complexes.

Example 2.2.1.8. For each $r \in R$, the homothety map $r^X : X \xrightarrow{\cdot r} X$ given by multiplication by r is a morphism. The identity map on X, denoted 1^X, is a morphism.

Fact 2.2.1.9. Let $f : M \to N$ be a chain map. For each i, the induced map $\mathrm{H}_i(f) : \mathrm{H}_i(M) \to \mathrm{H}_i(N)$ given by $\mathrm{H}_i(f)(\overline{x}) = \overline{f(x)}$ is a well-defined homomorphism.

Definition 2.2.1.10. A chain map $f : M \to N$ is a *quasiisomorphism* if for all $i \in \mathbb{Z}$ the induced map $\mathrm{H}_i(f) : \mathrm{H}_i(M) \to \mathrm{H}_i(N)$ is an isomorphism. We use the symbol $\simeq$ to identify quasiisomorphisms.

Example 2.2.1.11. Let M be an R-module with augmented projective resolution P^+ and augmented injective resolution ${}^+I$; see the notation from Example 2.2.1.2. The maps τ and ϵ induce quasiisomorphisms $P \xrightarrow{\simeq} M \xrightarrow{\simeq} I$.

2.2.2. Important Constructions

Definition 2.2.2.1. Let $f : M \to N$ be a chain map. The *mapping cone* of f is the sequence $\mathrm{Cone}(f)$ defined as follows.

$$\mathrm{Cone}(f) = \cdots \to \begin{matrix} N_i \\ \oplus \\ M_{i-1} \end{matrix} \xrightarrow{\begin{pmatrix} \partial_i^N & f_{i-1} \\ 0 & -\partial_{i-1}^M \end{pmatrix}} \begin{matrix} N_{i-1} \\ \oplus \\ M_{i-2} \end{matrix} \xrightarrow{\begin{pmatrix} \partial_{i-1}^N & f_{i-2} \\ 0 & -\partial_{i-2}^M \end{pmatrix}} \begin{matrix} N_{i-2} \\ \oplus \\ M_{i-3} \end{matrix} \to \cdots$$

Fact 2.2.2.2. Let $f : M \to N$ be a chain map.

1. The $\mathrm{Cone}(f)$ is an R-complex.
2. The following sequence is exact

$$0 \to N \xrightarrow{\iota} \mathrm{Cone}(f) \xrightarrow{\tau} \Sigma M \to 0$$

   where ι and τ are the natural injection and surjection.

3. The chain map f is a quasiisomorphism if and only if $\mathrm{Cone}(f)$ is exact.

Definition 2.2.2.3. Let M and N be R-complexes. The *Hom complex* $\mathrm{Hom}_R(M, N)$ is defined as follows. For each integer n, set $\mathrm{Hom}_R(M, N)_n := \prod_{p \in \mathbb{Z}} \mathrm{Hom}_R(M_p, N_{p+n})$ and $\partial_n^{\mathrm{Hom}_R(M,N)}(\{f_p\}) := \{\partial_{p+n}^N f_p - (-1)^n f_{p-1} \partial_p^M\}$. We sometimes write f in place of $\{f_p\}$.

Fact 2.2.2.4. The Hom complex $\mathrm{Hom}_R(M,N)$ is an R-complex such that the morphism sets $\mathcal{M}\mathrm{or}_{\mathcal{C}(R)}(M,N) := \mathrm{Ker}(\partial_0^{\mathrm{Hom}_R(M,N)})$.

Fact 2.2.2.5. Let M and N be R-modules. Let P be a projective resolution of M, let Q be a projective resolution of N, let J be an injective resolution of M, and let I be an injective resolution of N. For each i, there are isomorphisms

$$\begin{aligned}\mathrm{Ext}^i_R(M,N) &\cong \mathrm{H}_{-i}(\mathrm{Hom}_R(P,N))\\ &\cong \mathrm{H}_{-i}(\mathrm{Hom}_R(M,I))\\ &\cong \mathrm{H}_{-i}(\mathrm{Hom}_R(P,I))\\ &\cong \mathrm{H}_{-i}(\mathrm{Hom}_R(P,Q))\\ &\cong \mathrm{H}_{-i}(\mathrm{Hom}_R(J,I)).\end{aligned}$$

In other words, the modules $\mathrm{Ext}^i_R(M,N)$ can be computed using projective and/or injective resolutions of M and N, and this is independent of the choice of P, Q, J, and I.

Example 2.2.2.6. Let C be a finitely generated R-module. Let P be a projective resolution of C and I be an injective resolution of C. Then C is semidualizing if and only if the homothety map $R \to \mathrm{Hom}_R(I,I)$ is a quasiisomorphism, equivalently, if and only if the homothety map $R \to \mathrm{Hom}_R(P,P)$ is a quasiisomorphism.

Definition 2.2.2.7. Given a chain map $f : M \to N$ and an R-complex L, we define $\mathrm{Hom}_R(L,f) : \mathrm{Hom}_R(L,M) \to \mathrm{Hom}_R(L,N)$ as follows: each $\{g_p\} \in \mathrm{Hom}_R(L,M)_n$ is mapped to $\{f_{p+n}g_p\} \in \mathrm{Hom}_R(L,N)_n$. Similarily, define $\mathrm{Hom}_R(f,L) : \mathrm{Hom}_R(N,L) \to \mathrm{Hom}_R(M,L)$ by the formula $\{g_p\} \mapsto \{g_pf_p\}$.

Fact 2.2.2.8. If $f : M \to N$ is a chain map, and L is an R-complex, then $\mathrm{Hom}_R(L,f)$ and $\mathrm{Hom}_R(f,L)$ are chain maps.

Definition 2.2.2.9. Let M and N be R-complexes. The *tensor product complex* $M \otimes_R N$ is defined as follows. For each integer i, set $(M \otimes_R N)_i := \bigoplus_{p\in\mathbb{Z}} M_p \otimes_R N_{i-p}$ and let $\partial_i^{M\otimes_R N}$ be given on generators by the formula

$$\partial_i^{M\otimes_R N}(\ldots,0,m_p\otimes n_{i-p},0,\ldots) := (\ldots,0,\partial_p^M(m_p)\otimes n_{i-p},(-1)^p m_p\partial_{i-p}^N(n_{i-p}),0,\ldots).$$

Fact 2.2.2.10. Let M and N be R-complexes. The tensor product complex $M \otimes_R N$ is an R-complex.

Fact 2.2.2.11. Let M and N be R-modules, and let P be a projective resolution of M and Q be a projective resolution of N. For each i, there are isomorphisms

$$\operatorname{Tor}_i^R(M,N) \cong \mathrm{H}_i(P \otimes_R N) \cong \mathrm{H}_i(P \otimes_R Q) \cong \mathrm{H}_i(M \otimes_R Q).$$

In other words, the modules $\operatorname{Tor}_i^R(M,N)$ can be computed using a projective resolution of M and/or a projective resolution of N, and this is independent of the choice of P and Q.

Definition 2.2.2.12. Given a chain map $f : M \to N$ and an R-complex L, we define the induced chain map $L \otimes_R f : L \otimes_R M \to L \otimes_R N$ by the formula $l \otimes m \mapsto l \otimes f_{|m|}(m)$. Similarily, define the map $f \otimes_R L : M \otimes_R L \to N \otimes_R L$ by the formula $m \otimes l \mapsto f_{|m|}(m) \otimes l$.

Fact 2.2.2.13. If $f : M \to N$ is a chain map, and L is an R-complex, then $L \otimes_R f$ and $f \otimes_R L$ are chain maps.

2.2.3. Categorical Properties

The following definition of abelian categories comes from [24].

Definition 2.2.3.1. A category $\mathcal{A}$ is *additive* if every finite family of objects has a product, each set $\operatorname{Hom}_{\mathcal{A}}(A,B)$ is an abelian group, and the composition maps

$$\operatorname{Hom}_{\mathcal{A}}(A,B) \times \operatorname{Hom}_{\mathcal{A}}(B,C) \to \operatorname{Hom}_{\mathcal{A}}(A,C)$$

sending a pair (ϕ, ψ) to $\psi \circ \phi$ are bi-additive. An additive category $\mathcal{A}$ is *abelian*, if every morphism $\phi : A \to B$ has a kernel and cokernel, and if the canonical factorization

$$\begin{array}{ccccccc} \operatorname{Ker}(\phi) & \xrightarrow{\phi'} & A & \xrightarrow{\phi} & B & \xrightarrow{\phi''} & \operatorname{Coker}(\phi) \\ & & \downarrow & & \uparrow & & \\ & & \operatorname{Coker}(\phi') & \xrightarrow{\overline{\phi}} & \operatorname{Ker}(\phi'') & & \end{array}$$

of ϕ induces an isomorphism $\overline{\phi}$.

Fact 2.2.3.2. The category $\mathcal{C}(R)$ is an abelian category.

Fact 2.2.3.3. Let ψ, $\phi \in \mathcal{M}\text{or}_{\mathcal{C}(R)}(X,Y)$. Then $\psi + \phi$, $\psi - \phi \in \mathcal{M}\text{or}_{\mathcal{C}(R)}(X,Y)$.

Proof: Kernels are closed under addition and subtraction; see Fact 2.2.2.4. □

Fact 2.2.3.4. Let $\psi : X \to Y$ and $\phi : Y \to Z$ be two morphisms in $\mathcal{C}(R)$. Then the composition $\phi \circ \psi$ is a morphism in $\mathcal{C}(R)$.

Proof: By definition, $\psi \in \text{Ker}(\partial_0^{\text{Hom}_R(X,Y)})$ and $\phi \in \text{Ker}(\partial_0^{\text{Hom}_R(Y,Z)})$. Thus

$$(0) = \partial_0^{\text{Hom}_R(X,Y)}(\psi) = (\partial_i^Y \psi_i - \psi_{i-1}\partial_i^X)_i.$$

Hence $\partial_i^Y \psi_i = \psi_{i-1}\partial_i^X$ for each i. Similarly, $\partial_i^Z \phi_i = \phi_{i-1}\partial_i^Y$ for each i. Now,

$$\begin{aligned}
\partial_0^{\text{Hom}_R(X,Z)}(\phi \circ \psi) &= (\partial_i^Z(\phi \circ \psi)_i - (\phi \circ \psi)_{i-1}\partial_i^X)_i \\
&= (\partial_i^Z(\phi_i \circ \psi_i) - (\phi_{i-1} \circ \psi_{i-1})\partial_i^X)_i \\
&= (\phi_{i-1} \circ \partial_i^Y \circ \psi_i - \phi_{i-1} \circ \partial_i^Y \circ \psi_i)_i \\
&= (0)_i.
\end{aligned}$$

□

2.2.4. Homotopies

Here we describe the technology needed to define the second step of the construction of $D(R)$.

Definition 2.2.4.1. Let f, $g : X \to Y$ be chain maps between R-complexes.

1. f is *null-homotopic* (or *homotopic to* 0), denoted $f \sim 0$, if for each i there exists $s_i : X_i \to Y_{i+1}$ such that $f_i = \partial_{i+1}^Y \circ s_i + s_{i-1} \circ \partial_i^X$. We say $s = \{s_i\}$ is a *homotopy* between f and 0.
2. f and g are homotopic, denoted $f \sim g$, if $f - g \sim 0$.
3. f is a *homotopy equivalence* if there exists a chain map $h : Y \to X$ such that $f \circ h \sim \text{id}_Y$ and $h \circ f \sim \text{id}_X$ (then h is a *homotopy inverse* for f).

Fact 2.2.4.2. Let $f : X \to Y$ be a chain map.

1. If $f \sim 0$, then $\mathrm{H}_i(f) = 0$ for all i.
2. The chain map $f : X \to Y$ is null-homotopic if and only if $f \in \mathrm{Im}(\partial_1^{\mathrm{Hom}_R(X,Y)})$.
3. Homotopic is an equivalence relation on $\mathcal{M}\mathrm{or}_{\mathcal{C}(R)}(X, Y)$.

Fact 2.2.4.3. Let β, $\gamma : Y \to Z$ be chain maps such that $\beta \sim \gamma$.

1. $\mathrm{H}_i(\beta) = \mathrm{H}_i(\gamma)$ for all i.
2. β is a quasiisomorphism if and only if γ is a quasiisomorphism.
3. If β is a homotopy equivalence, then $\mathrm{H}_i(\beta)$ is an isomorphism for all i.

Fact 2.2.4.4. If $\alpha : A \to B$ is a homotopy equivalence with homotopy inverses β, $\beta' : B \to A$, then $\beta \sim \beta'$.

Notation 2.2.4.5. Given R-complexes X and Y, let

$$\begin{aligned}
\mathrm{I}(X, Y) &= \{\text{chain maps}\, f : X \to Y \mid f \sim 0\} \\
&= \mathrm{Im}(\partial_1^{\mathrm{Hom}_R(X,Y)}) \\
&\subseteq \mathrm{Ker}(\partial_0^{\mathrm{Hom}_R(X,Y)}) \\
&= \mathcal{M}\mathrm{or}_{\mathcal{C}(R)}(X, Y) \\
&\subseteq \mathrm{Hom}_R(X, Y)_0.
\end{aligned}$$

2.3. The Homotopy Category

The topic of this section is the homotopy category, which is used as a bridge from the category $\mathcal{C}(R)$ of R-complexes to the derived category $D(R)$. The derived category was introduced by Verdier to establish the tools needed to state and prove Grothendieck local duality; see [35].

2.3.1. Definitions

Definition 2.3.1.1. The *homotopy category*, $\mathcal{K}(R)$, is the category whose objects are the R-complexes with morphism sets

$$\mathcal{M}\mathrm{or}_{\mathcal{K}(R)}(X, Y) := \mathcal{M}\mathrm{or}_{\mathcal{C}(R)}(X, Y)/\, \mathrm{I}(X, Y) = \mathrm{H}_0(\mathrm{Hom}_R(X, Y)).$$

Definition 2.3.1.2. An R-complex M is *homologically degreewise finite*, denoted $M \in \mathcal{K}^{(f)}(R)$, if $\mathrm{H}_i(R)$ is finitely generated over R for all i. We say M is *homologically bounded*, denoted $M \in \mathcal{K}_{(\square)}(R)$, if $\mathrm{H}_i(M) = 0$ for $|i| \gg 0$. Additionally, M is *homologically finite* if $M \in \mathcal{K}_{(\square)}(R) \cap \mathcal{K}^{(f)}(R)$, i.e., $M \in \mathcal{K}^{(f)}_{(\square)}(R)$. We say M is *homologically bounded below*, denoted $M \in \mathcal{K}_{(\sqsupset)}(R)$, if $\inf(M) > -\infty$. Similarly, M is *homologically bounded above*, denoted $M \in \mathcal{K}_{(\sqsubset)}(A)$, if $\sup(M) < \infty$.

Fact 2.3.1.3. Let f, $g \in \mathcal{M}\mathrm{or}_{\mathcal{C}(R)}(X, Y)$ with images $\overline{f}$, $\overline{g} \in \mathcal{M}\mathrm{or}_{\mathcal{K}(R)}(X, Y)$. Then

1. $\overline{f}$ is an isomorphism in $\mathcal{K}(R)$ if and only if f is a homotopy equivalence.
2. $\overline{f} = 0$ if and only if $f \sim 0$.
3. $\overline{f} = \overline{g}$ if and only if $f \sim g$.

2.3.2. Categorical Properties

One of the subtleties of the categories $\mathcal{K}(R)$ and $D(R)$ is that they are not abelian. Instead, they are "triangulated," as defined next. This form of the definition comes from [24].

Definition 2.3.2.1 (Axioms of a triangulated category)**.** Let $\mathcal{T}$ be an additive category with an equivalence $\Sigma : \mathcal{T} \to \mathcal{T}$. A *triangle* in $\mathcal{T}$ is a sequence (α, β, γ) of morphisms $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$. A *morphism* between two triangles (α, β, γ) and $(\alpha', \beta', \gamma')$ is a triple (ϕ_1, ϕ_2, ϕ_3) of maps in $\mathcal{T}$ making the following diagram commute.

$$
\begin{CD}
X @>{\alpha}>> Y @>{\beta}>> Z @>{\gamma}>> \Sigma X \\
@V{\phi_1}VV @V{\phi_2}VV @V{\phi_3}VV @V{\Sigma\phi_1}VV \\
X' @>{\alpha'}>> Y' @>{\beta'}>> Z' @>{\gamma'}>> \Sigma X'
\end{CD}
$$

The category $\mathcal{T}$ is called *triangulated* if it is equipped with a class of distinguished triangles (called *exact triangles*) satisfying the following axioms.

(TR1) A triangle isomorphic to an exact triangle is exact. For each object X, the triangle $0 \to X \xrightarrow{\mathrm{id}} X \to 0$ is exact. Each morphism α fits into an exact triangle (α, β, γ).

(TR2) A triangle (α, β, γ) is exact if and only if $(\beta, \gamma, -\Sigma\alpha)$ is exact.

(TR3) Given two triangles (α, β, γ) and $(\alpha', \beta', \gamma')$, each pair of maps ϕ_1 and ϕ_2 satisfying $\phi_2 \circ \alpha = \alpha' \circ \phi_1$ can be completed to a morphism

$$\begin{array}{ccccccc} X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \xrightarrow{\gamma} & \Sigma X \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_3} & & \downarrow{\scriptstyle \Sigma\phi_1} \\ X' & \xrightarrow{\alpha'} & Y' & \xrightarrow{\beta'} & Z' & \xrightarrow{\gamma'} & \Sigma X' \end{array}$$

of triangles.

(TR4) Given exact triangles $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3)$, and $(\gamma_1, \gamma_2, \gamma_3)$ with $\gamma_1 = \beta_1 \circ \alpha_1$, there exists a triangle $(\delta_1, \delta_2, \delta_3)$ making the following diagram commute.

$$\begin{array}{ccccccc} X & \xrightarrow{\alpha_1} & Y & \xrightarrow{\alpha_2} & U & \xrightarrow{\alpha_3} & \Sigma X \\ \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle =} \\ X & \xrightarrow{\gamma_1} & Z & \xrightarrow{\gamma_2} & V & \xrightarrow{\gamma_3} & \Sigma X \\ & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \delta_2} & & \downarrow{\scriptstyle \Sigma\alpha_1} \\ & & W & \xrightarrow{=} & W & \xrightarrow{\beta_3} & \Sigma Y \\ & & \downarrow{\scriptstyle \beta_3} & & \downarrow{\scriptstyle \delta_3} & & \\ & & \Sigma Y & \xrightarrow{\Sigma\alpha_2} & \Sigma U & & \end{array}$$

Axiom (TR4) is known as the *octahedral axiom* because the four exact triangles can be arranged in a diagram having the shape of an octahedron.

Fact 2.3.2.2. The homotopy category $\mathcal{K}(R)$ is a triangulated category where exact triangles are those isomorphic to $X \overset{\overline{\alpha}}{\to} Y \overset{\overline{\iota}}{\to} \operatorname{Cone}(\alpha) \overset{\overline{\tau}}{\to} \Sigma X$, where α is a chain map, and $\overline{\iota}$ and $\overline{\tau}$ are induced by the natural maps from the short exact sequence in Fact 2.2.2.2.

Definition 2.3.2.3. A diagram of chain maps

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ \downarrow{\scriptstyle \gamma} & & {\scriptstyle \beta}\downarrow \\ U & \xrightarrow{\delta} & Z \end{array} \qquad (\dagger)$$

commutes up to homotopy if $\beta \circ \alpha \sim \delta \circ \gamma$.

Fact 2.3.2.4. Consider the diagram (†) from Definition 2.3.2.3 in $\mathcal{C}(R)$ and consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{\overline{\alpha}} & Y \\ \Big\downarrow{\scriptstyle \overline{\gamma}} & & {\scriptstyle \overline{\beta}}\Big\downarrow \\ U & \xrightarrow{\overline{\delta}} & Z \end{array}$$

in $\mathcal{K}(R)$. Then this diagram commutes in $\mathcal{K}(R)$ if and only if (†) commutes up to homotopy.

Fact 2.3.2.5. There exists a functor $\mathcal{G} : \mathcal{C}(R) \to \mathcal{K}(R)$ given by $\mathcal{G}(X) = X$ and $\mathcal{G}(\alpha) = \overline{\alpha}$.

2.4. The Derived Category $D(R)$

We are now prepared to define the derived category $D(R)$. It is constructed by "localizing the homotopy category" by formally inverting the quasiisomorphisms.

2.4.1. Definitions

Definition 2.4.1.1. Let M be a homologically bounded below R-complex. A *projective resolution* of M is a quasiisomorphism $P \xrightarrow{\simeq} M$ such that P is a bounded below complex of projective R-modules.

Definition 2.4.1.2. Let N be a homologically bounded above R-complex. An *injective resolution* of N is a quasiisomorphism $N \xrightarrow{\simeq} I$ such that I is a bounded above complex of injective R-modules.

Fact 2.4.1.3. If M is an R-complex such that $\inf(M) > -\infty$, then there exists a projective resolution $P \xrightarrow{\simeq} M$ such that $P_i = 0$ for all $i < \inf(M)$.

Fact 2.4.1.4. If N is an R-complex such that $\sup(M) < \infty$, then there exists an injective resolution $N \xrightarrow{\simeq} I$ such that $I_j = 0$ for all $j > \sup(N)$.

Note that the following terminology is non-standard.

Definition 2.4.1.5. Let $X, Y \in \mathcal{K}(R)$. An *inj-diagram* from X to Y is a sequence of morphisms $X \xrightarrow{\overline{\alpha}} U \underset{\simeq}{\xleftarrow{\overline{\beta}}} Y$.

Example 2.4.1.6. If $\alpha : X \to Y$ is a chain map, then $X \xrightarrow{\overline{\alpha}} Y \underset{\simeq}{\xleftarrow{\overline{1_Y}}} Y$ is an inj-diagram.

Fact 2.4.1.7. Let I be a bounded above complex of injective R-modules. For each quasiisomorphism $X \underset{\simeq}{\xrightarrow{\alpha}} Y$ and for each chain map $X \xrightarrow{\beta} I$, there exists a chain map $Y \xrightarrow{\gamma} I$ such that

$(\gamma \circ \alpha) \sim \beta$, i.e., the following diagram commutes in $\mathcal{K}(R)$.

$$\begin{array}{ccc} X & \xrightarrow[\overline{\alpha}]{\simeq} & Y \\ {\scriptstyle \overline{\beta}}\big\downarrow & \swarrow{\scriptstyle \overline{\gamma}} & \\ I & & \end{array}$$

Moreover, $\overline{\gamma}$ is unique in $\mathcal{K}(R)$.

Definition 2.4.1.8. The *composition* of inj-diagrams is defined as follows. Let $X \xrightarrow{\overline{\alpha}} V \xleftarrow[\simeq]{\overline{\beta}} Y$ and $Y \xrightarrow{\overline{\gamma}} U \xleftarrow[\simeq]{\overline{\delta}} Z$ be two inj-diagrams. If there exists a commutative diagram in $\mathcal{K}(R)$ of the following form

$$\begin{array}{ccccccccc} X & \xrightarrow{\overline{\alpha}} & V & \xleftarrow[\overline{\beta}]{\simeq} & Y & \xrightarrow{\overline{\gamma}} & U & \xleftarrow[\overline{\delta}]{\simeq} & Z \\ & & & {\scriptstyle \overline{\varpi}}\searrow & & \swarrow{\scriptstyle \overline{\sigma}}\,{\scriptstyle \simeq} & & & \\ & & & & J & & & & \end{array}$$

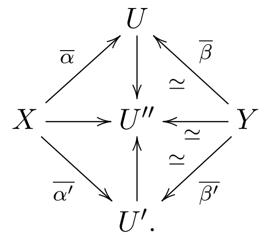
then we set $(Y \xrightarrow{\overline{\gamma}} U \xleftarrow[\simeq]{\overline{\delta}} Z) \circ (X \xrightarrow{\overline{\alpha}} V \xleftarrow[\simeq]{\overline{\beta}} Y) := (X \xrightarrow{\overline{\varpi}\,\overline{\alpha}} J \xleftarrow[\simeq]{\overline{\sigma}\overline{\delta}} Z)$.

Remark 2.4.1.9. In the notation of Definition 2.4.1.8, there exists a composition when U has an injective resolution over R. Indeed, let $U \xrightarrow[\simeq]{\sigma} J$ be an injective resolution over R. By Fact 2.4.1.7 we have the following commutative in $\mathcal{K}(R)$.

$$\begin{array}{ccccccccc} X & \xrightarrow{\overline{\alpha}} & V & \xleftarrow[\overline{\beta}]{\simeq} & Y & \xrightarrow{\overline{\gamma}} & U & \xleftarrow[\overline{\delta}]{\simeq} & Z \\ & & & {\scriptstyle \overline{\varpi}}\searrow & & \swarrow{\scriptstyle \overline{\sigma}}\,{\scriptstyle \simeq} & & & \\ & & & & J & & & & \end{array}$$

Example 2.4.1.10. Consider chain maps $A \xrightarrow{f} B \xrightarrow{g} C$ with associated inj-diagrams $A \xrightarrow{\overline{f}} B \xleftarrow{=} B$ and $B \xrightarrow{\overline{g}} C \xleftarrow{=} C$. Their composition is the inj-diagram $A \xrightarrow{\overline{g}\,\overline{f}} C \xleftarrow{=} C$. Thus compositions in $\mathcal{C}(R)$ are compatible with compositions of inj-diagrams.

Definition 2.4.1.11. Two inj-diagrams $X \xrightarrow{\overline{\alpha}} U \underset{\simeq}{\xleftarrow{\overline{\beta}}} Y$ and $X \xrightarrow{\overline{\alpha'}} U' \underset{\simeq}{\xleftarrow{\overline{\beta'}}} Y$ are *equivalent* if there exists a commutative diagram in $\mathcal{K}(R)$



Fact 2.4.1.12. The equivalence of inj-diagrams in Definition 2.4.1.11 forms an equivalence relation on the class of inj-diagrams.

Fact 2.4.1.13. The composition of inj-diagrams is well-defined up to equivalence.

Definition 2.4.1.14. The *bounded above derived category* $D_{-}(R)$ is the category with objects the homologically bounded above R-complexes and morphisms the equivalence classes of inj-diagrams. Moreover, the identity morphism equivalence class is $\mathrm{id}_X : (X \xrightarrow{=} X \xleftarrow{=} X)$. The minus sign indicates that the homology of the complexes in this category lives mostly in negative degrees.

Definition 2.4.1.15. An R-complex M is *homologically bounded*, denoted $M \in D_b(R)$, if $\mathrm{H}_i(M) = 0$ for $|i| \gg 0$. An R-complex $M \in D_{-}(R)$ is *homologically degreewise finite*, denoted $M \in D_{-}^{f}(R)$, if $\mathrm{H}_i(R)$ is finitely generated over R for all i. An R-complex M is *homologically finite* if $M \in D_b(R) \cap D^f(R)$, i.e., $M \in D_b^f(R)$.

Notation 2.4.1.16. The equivalence class of $X \xrightarrow{\overline{\alpha}} U \underset{\simeq}{\xleftarrow{\overline{\beta}}} Y$ in $D_{-}(R)$ is denoted by $\overline{\alpha}/\overline{\beta}$. The equivalence class of $\Sigma X \xrightarrow{\Sigma\overline{\alpha}} \Sigma U \underset{\simeq}{\xleftarrow{\Sigma\overline{\beta}}} \Sigma Y$ is denoted by $\Sigma(\overline{\alpha}/\overline{\beta})$.

Fact 2.4.1.17. The composition of two morphisms in $D_{-}(R)$ is a well-defined morphism in $D_{-}(R)$.

Fact 2.4.1.18. The bounded above derived category $D_{-}(R)$ is a triangulated category where exact triangles are those isomorphic to $X \xrightarrow{\overline{f}} Y \xrightarrow{\overline{\iota}} \mathrm{Cone}\,(f) \xrightarrow{\overline{\tau}} \Sigma X$ where f is a chain map and $\overline{\iota}$ and $\overline{\tau}$ are induced by the natural maps from the short exact sequence in Fact 2.2.2.2. In other words, exact triangles in $D_{-}(R)$ are those coming from exact triangles in $\mathcal{K}_{(\sqsubset)}(R)$.

Fact 2.4.1.19. There exists a functor $\mathcal{F} : \mathcal{K}_{(\sqsubset)}(R) \to D_{-}(R)$ of triangulated categories given by $\mathcal{F}(X) = X$ and $\mathcal{F}(\overline{\alpha}) = \overline{\alpha}/\overline{1}$.

Remark 2.4.1.20. We can similarly define the "bounded below derived category" $D_+(R)$ as the triangulated category with objects the homologically bounded below R-complexes and morphisms equivalence classes of "proj-diagrams" (similar to inj-diagrams). We can also define the unbounded derived category $D(R)$, but it is more technical. The difficulty comes in defining compositions. We need a version of injective resolutions for arbitrary complexes in $\mathcal{C}(R)$. This notion exists, roughly as: for all R-complexes X there exists a quasiisomorphism $X \underset{\rho}{\xrightarrow{\simeq}} I$ such that I is a complex of injective R-modules satisfying the "lifting lemma". Such a complex I is "semiinjective," see Section 2.5. Similarly, the unbounded derived category $D(R)$ can be defined by use of "semiprojective" complexes, see Section 2.5. The existence of "semiinjective" and "semiprojective" complexes is discussed in [7].

2.4.2. Derived Functors

Definition 2.4.2.1. If M, $N \in D(R)$, and $P \xrightarrow{\simeq} M$ is a semiprojective resolution over R, then

$$\mathbf{R}\mathrm{Hom}_R(M,N) := \mathrm{Hom}_R(P,N) \qquad \mathrm{Ext}^i_R(M,N) := \mathrm{H}_{-i}(\mathbf{R}\mathrm{Hom}_R(M,N))$$

$$M \otimes^{\mathbf{L}}_R N := P \otimes_R N \qquad \mathrm{Tor}^R_i(M,N) := \mathrm{H}_i(M \otimes^{\mathbf{L}}_R N).$$

Fact 2.4.2.2. The complexes $\mathbf{R}\mathrm{Hom}_R(M,N)$ and $M \otimes^{\mathbf{L}}_R N$ are well-defined objects up to isomorphism in $D(R)$, i.e., independent of choice of P. This is a direct consequence of the lifting property that defines semiprojectivity.

Fact 2.4.2.3 (Balance for $\mathbf{R}$Hom and Ext)**.** If $M \in D(R)$ with semiprojective resolution $P \xrightarrow{\simeq} M$ over R and $N \in D(R)$ with semiinjective resolution $N \xrightarrow{\simeq} I$, then we have isomorphisms

$$\mathrm{Hom}_R(P,N) \simeq \mathrm{Hom}_R(P,I) \simeq \mathrm{Hom}_R(M,I)$$

in $D(R)$. Thus, $\mathbf{R}\mathrm{Hom}_R(M,N)$ is represented by any/all of the displayed R-complexes.

2.4.3. Semidualizing Complexes

Fact 2.4.3.1. Let $N \in D_+(R)$. Then there exists a well-defined morphism $\chi^R_N : R \to \mathbf{R}\mathrm{Hom}_R(N,N)$ in $D(R)$ represented by $\overline{\chi^R_P}/\overline{1} : R \xrightarrow{\overline{\chi^R_P}} \mathrm{Hom}_R(P,P) \xleftarrow{\overline{1}} \mathrm{Hom}_R(P,P)$ where $P \xrightarrow{\simeq} N$ is a projective resolution over R.

Fact 2.4.3.2. Let $M \in D_-(R)$. Then the morphism $\chi^R_M : R \to \mathbf{R}\mathrm{Hom}_R(M,M)$ in $D(R)$ is represented by $\overline{\chi^R_I}/\overline{1} : R \xrightarrow{\overline{\chi^R_I}} \mathrm{Hom}_R(I,I) \xleftarrow{\overline{1}} \mathrm{Hom}_R(I,I)$ where $M \xrightarrow{\simeq} I$ is an injective resolution over R.

Christensen [8] was the first to work with semidualizing complexes in general. This built significantly on the research of Hartshorne and Grothendieck [20] on dualizing complexes, and the work of Avramov and Foxby [6] on relative dualizing complexes.

Definition 2.4.3.3. Let $C \in D^f_b(R)$. Then C is a *semidualizing complex* if the homothety morphism $\chi^R_C : R \to \mathbf{R}\mathrm{Hom}_R(C,C)$ is an isomorphism in $D(R)$.

Example 2.4.3.4. A semidualizing R-module is the same as a semidualizing complex concentrated in degree 0, up to isomorphism in $D(R)$.

2.5. DG Algebras and DG Modules

We now turn our attention to the more general context of our main theorem. Recall from the introduction that "DG" is short for "Differential Graded." In short, a DG R-algebra is a commutative graded R-algebra with a differential that is compatible with the algebra structure. See [5, 7, 12] for thorough introductions to this area.

2.5.1. DG Algebras

Definition 2.5.1.1. A *commutative differential graded algebra over* R ("DG R-algebra" for short) is an R-complex A equipped with a chain map $\mu^A : A \otimes_R A \to A$ with $ab := \mu^A(a \otimes b)$ that is:

associative: for all $a, b, c \in A$ we have $(ab)c = a(bc)$;

unital: there is an element $1 \in A_0$ such that for all $a \in A$ we have $1a = a$;

graded commutative: for all $a, b \in A$ we have $ab = (-1)^{|a||b|}ba$ and $a^2 = 0$ when $|a|$ is odd; and

positively graded: $A_i = 0$ for $i < 0$.

The map μ^A is the *product* on A. Given a DG R-algebra A, the *underlying algebra* is the graded commutative R-algebra $A^\natural = \oplus_{i=0}^{\infty} A_i$. A *morphism* of DG R-algebras is a chain map $f : A \to B$ between DG R-algebras respecting products and multiplicative identities: $f(aa') = f(a)f(a')$ and $f(1) = 1$.

Example 2.5.1.2. The ring R is a DG R-algebra concentrated in degree 0, as is any commutative R-algebra.

Assumption 2.5.1.3. For the rest of this chapter, A is a DG R-algebra, and k is a field.

Fact 2.5.1.4. The fact that the product on A is a chain map says that ∂^A satisfies the *Leibniz rule*: $\partial^A_{|a|+|b|}(ab) = \partial^A_{|a|}(a)b + (-1)^{|a|}a\partial^B_{|b|}(b)$. Also, we know that $\mathrm{H}_0(A)$ is an R-algebra, and each $\mathrm{H}_i(A)$ is an $\mathrm{H}_0(A)$-module.

Definition 2.5.1.5. We say that A is *weakly noetherian* if $H_0(A)$ is noetherian and the $H_0(A)$-module $H_i(A)$ is finitely generated for all $i \geqslant 0$. We say that A is *local* if it is weakly noetherian, R is local, and the ring $\mathrm{H}_0(A)$ is a local R-algebra.

Fact 2.5.1.6. Assume R is local with maximal ideal $\mathfrak{m}_R$. Assume that A is a local DG R-algebra, and let $\mathfrak{m}_{\mathrm{H}_0(A)}$ be the maximal ideal of $\mathrm{H}_0(A)$. The assumption that A is local implies $A \not\simeq 0$. The composition $A \to \mathrm{H}_0(A) \to \mathrm{H}_0(A)/\mathfrak{m}_{\mathrm{H}_0(A)}$ is a surjective morphism of DG R-algebras with kernel of the form $\mathfrak{m}_A = \cdots \xrightarrow{\partial^A_2} A_1 \xrightarrow{\partial^A_1} \mathfrak{m}_0 \to 0$ for some maximal ideal $\mathfrak{m}_0 \subsetneq A_0$. The quotient $A/\mathfrak{m}_A$ is isomorphic to $\mathrm{H}_0(A)/\mathfrak{m}_{\mathrm{H}_0(A)}$. Since $\mathrm{H}_0(A)$ is a local R-algebra, we have $\mathfrak{m}_R A_0 \subseteq \mathfrak{m}_0$.

Definition 2.5.1.7. Assume that R is local. Given a local DG R-algebra A, the subcomplex $\mathfrak{m}_A$ from Fact 2.5.1.6 is the *augmentation ideal* of A.

2.5.2. DG Modules and the DG Category

In the transition from rings to DG algebras, modules and complexes are replaced by DG modules, defined next.

Definition 2.5.2.1. A *differential graded module over* A ("DG A-module" for short) is an R-complex M equipped with a chain map $\mu^M : A \otimes_R M \to M$ such that the rule $am := \mu^M(a \otimes m)$ is associative and unital. The map μ^M is *scalar multiplication* on M. The *underlying* $A^\natural$*-module* associated to M is the $A^\natural$ -module $M^\natural = \oplus_{i=-\infty}^{\infty} M_i$.

Example 2.5.2.2. A DG R-module is simply an R-complex.

Fact 2.5.2.3. Let M be a DG A-module. The fact that scalar multiplication on M is a chain map says that ∂^M satisfies the Leibniz rule: $\partial^A_{|a|+|m|}(am) = \partial^A_{|a|}(a)m + (-1)^{|a|}a\partial^M_{|m|}(m)$. The R-module M_i is an A_0-module and $\mathrm{H}_i(M)$ is an $\mathrm{H}_0(A)$-module for each i.

Definition 2.5.2.4. Let i be an integer. The *ith suspension* of a DG A-module M is the DG A-module $\Sigma^i M$ defined by $(\Sigma^i M)_n := M_{n-i}$ and $\partial^{\Sigma^i M}_n := (-1)^i \partial^M_{n-i}$. The scalar multiplication on $\Sigma^i M$ is defined by the formula $\mu^{\Sigma^i M}(a \otimes m) := (-1)^{i|a|}\mu^M(a \otimes m)$.

Definition 2.5.2.5. Let $\mathrm{DG}(A)$ denote the category with objects the DG A-modules and morphisms the chain maps $f : M \to N$ between DG A-modules that respect scalar multiplication: $f(am) = af(m)$.

Definition 2.5.2.6. A *quasiisomorphism* of DG A-modules is a morphism $M \to N$ such that each induced map $\mathrm{H}_i(M) \to \mathrm{H}_i(N)$ is an isomorphism. We use the symbol $\simeq$ to identify quasiisomorphisms.

2.5.3. Important Constructions

Much of what follows mirrors our treatment of R-complexes above.

Definition 2.5.3.1. Given an integer i, a *DG A-module homomorphism of degree n* is an element $f \in \mathrm{Hom}_R(M,N)_n$ such that $f_{i+j}(am) = (-1)^{ni} a f_j(m)$ for all $a \in A_i$ and $m \in M_j$. The subcomplex of $\mathrm{Hom}_R(M,N)$ consisting of all DG A-module homomorphisms $M \to N$ is denoted $\mathrm{Hom}_A(M,N)$. Note that this uses the following: $\partial^{\mathrm{Hom}_R(M,N)}(\mathrm{Hom}_A(M,N)) \subseteq \mathrm{Hom}_A(M,N)$, so $\mathrm{Hom}_A(M,N)$ is a complex with $\partial^{\mathrm{Hom}_A(M,N)}$ induced by $\partial^{\mathrm{Hom}_R(M,N)}$. Also, it is a DG A-module via $(af)_{|m|}(m) := a(f_{|m|}(m)) = (-1)^{|a||f|} f_{j+|a|}(am)$ such that $\mathcal{M}\mathrm{or}_{\mathrm{DG}(A)}(M,N) := \mathrm{Ker}(\partial_0^{\mathrm{Hom}_A(M,N)})$.

Definition 2.5.3.2. Given a morphism $f : L \to M$ of DG A-modules, we define the map $\mathrm{Hom}_A(N,f) : \mathrm{Hom}_A(N,L) \to \mathrm{Hom}_A(M,L)$ as follows: each sequence $\{g_p\} \in \mathrm{Hom}_A(N,L)_n$ is mapped to $\{f_{p+n}g_p\} \in \mathrm{Hom}_A(N,M)_n$. Similarily, define the map $\mathrm{Hom}_A(f,N) : \mathrm{Hom}_A(M,N) \to \mathrm{Hom}_A(L,N)$ by the formula $\{g_p\} \mapsto \{g_p f_p\}$.

Fact 2.5.3.3. If $f : L \to M$ is a morphism of DG A-modules, then $\mathrm{Hom}_A(N,f)$ and $\mathrm{Hom}_A(f,N)$ are morphisms of DG A-modules.

Definition 2.5.3.4. The *tensor product* $M \otimes_A N$ is the quotient $(M \otimes_R N)/U$ where U is the graded submodule of $M \otimes_R N$ generated over R by the elements of the form $(am) \otimes n - (-1)^{|a||m|} m \otimes (an)$. Note that this uses the following: $\partial^{M \otimes_R N}(U) \subseteq U$ so, $M \otimes_A N$ is a complex with $\partial^{M \otimes_A N}$ induced by $\partial^{M \otimes_R N}$. Given an element $m \otimes n \in M \otimes_R N$, we denote the image in $M \otimes_A N$ as $m \otimes n$. Also, $M \otimes_A N$ is a DG A-module via $a(m \otimes n) := (am) \otimes n = (-1)^{|a||m|} m \otimes (an)$.

What follows is central for our main theorem.

Fact 2.5.3.5. Let A' and A'' be DG R-algebras. Let N' be a DG A'-module and N'' be a DG A''-module. Then one has the following.

(a) $A' \otimes_R A''$ is a DG R-algebra via the multiplication $(a' \otimes a'')(b' \otimes b'') = (-1)^{|a''||b'|}(a'b') \otimes (a''b'')$.

(b) The complex $N' \otimes_R N''$ is a DG $A' \otimes_R A''$-module via the multiplication $(a' \otimes a'')(n' \otimes n'') = (-1)^{|a''||n'|}(a'n') \otimes (a''n'')$.

(c) Assume A' and A'' are weakly noetherian. In general, it does not necessarily follow that $A' \otimes_k A''$ is weakly noetherian. For instance, if $\mathrm{H}_0(A'), \mathrm{H}_0(A'') \cong R[\![x]\!]$, then

$$\mathrm{H}_0(A' \otimes_R A'') \cong \mathrm{H}_0(A') \otimes_R \mathrm{H}_0(A'') \cong R[\![x]\!] \otimes_R R[\![x]\!]$$

is not noetherian. However, if $\mathrm{H}_0(A')$ or $\mathrm{H}_0(A'')$ is essentially of finite type over R, then $A' \otimes_R A''$ is weakly noetherian.

Definition 2.5.3.6. A bounded below DG A-module M is *semifree* if $M^\natural$ is a free graded $A^\natural$-module; when this is the case, a *semibasis* of M is a subset $E \subseteq M$ that forms a basis of $M^\natural$ over $A^\natural$. A *semifree resolution* of a DG A-module N is a quasiisomorphism $F \xrightarrow{\simeq} N$ such that F is semifree over A. Over a local DG algebra $(A, \mathfrak{m}_A)$, we say a semifree resolution F is *minimal* if each semibasis of F is finite in each degree and $\partial^F(F) \subseteq \mathfrak{m}_A F$. We say that a DG A-module M is *semiprojective* if $\mathrm{Hom}_A(M, -)$ respects surjective quasiisomorphisms. A *semiprojective resolution* of a DG A-module N is a quasiisomorphism $P \xrightarrow{\simeq} N$ such that P is semiprojective over A. The existence of a semiprojective resolution is from [7].

Remark 2.5.3.7. When M is not bounded below, the definition of semifree is much more technical; see [7, 3.7.5 and 3.8.1(0)]. However, this level of generality is unnecessary for the results of this dissertation.

Definition 2.5.3.8. A DG A-module M is *semiinjective* if $\mathrm{Hom}_A(-, M)$ converts injective quasiisomorphisms into surjective quasiisomorphisms. A *semiinjective resolution* of a DG A-module M is a quasiisomorphism $M \xrightarrow{\simeq} I$ such that I is semiinjective over A. The existence of a semiinjective resolution is from [7].

Definition 2.5.3.9. Let M be a DG A-module. Given an integer n, the *nth left soft truncation of* M is the complex

$$M_{(\subseteq n)} := 0 \to M_n / \mathrm{Im}(\partial^M_{n+1}) \to M_{n-1} \to M_{n-2} \to \cdots$$

and the *nth left hard truncation of* M is the complex

$$M_{(\leqslant n)} := 0 \to M_n \to M_{n-1} \to M_{n-2} \to \cdots .$$

The complexes $M_{(\supseteq n)}$ and $M_{(\geqslant n)}$ are defined similarily.

Fact 2.5.3.10. The truncations $M_{(\subseteq n)}$, $M_{(\leqslant n)}$, $M_{(\supseteq n)}$, and $M_{(\geqslant n)}$ are DG A-modules with structure induced by that of M, and there exist natural morphisms $M_{(\leqslant n)} \to M \to M_{(\subseteq n)}$ and $M_{(\supseteq n)} \to M \to M_{(\geqslant n)}$.

2.6. The Derived Category $D(A)$

As with the derived category $D(R)$, the derived category of a DG R-algebra A, denoted $D(A)$, is formed from the homotopy category $\mathcal{K}(A)$, by formally inverting all the quasiisomorphisms. The objects of $D(A)$ are DG A-modules and the morphisms are equivalence classes of inj-diagrams. Isomorphisms in $D(A)$ are identified by the symbol $\simeq$.

2.6.1. Important Constructions

Definition 2.6.1.1. If M, $N \in D(A)$, and $P \xrightarrow{\simeq} M$ is a semiprojective resolution over A, then

$$\mathbf{R}\mathrm{Hom}_A(M,N) := \mathrm{Hom}_A(P,N) \qquad \mathrm{Ext}^i_A(M,N) := \mathrm{H}_{-i}(\mathbf{R}\mathrm{Hom}_A(M,N))$$
$$M \otimes^{\mathbf{L}}_A N := P \otimes_A N \qquad \mathrm{Tor}^A_i(M,N) := \mathrm{H}_i(M \otimes^{\mathbf{L}}_A N).$$

Fact 2.6.1.2. The DG A-modules $\mathbf{R}\mathrm{Hom}_A(M,N)$ and $M \otimes^{\mathbf{L}}_A N$ are well-defined objects up to isomorphism in $D(A)$, i.e., independent of choice of P.

Fact 2.6.1.3 (Balance for $\mathbf{R}$Hom and Ext)**.** If $M \in D(A)$ with semiprojective resolution $P \xrightarrow{\simeq} M$ over A and $N \in D(A)$ with semiinjective resolution $N \xrightarrow{\simeq} I$, then we have isomorphisms

$$\mathrm{Hom}_A(P,N) \simeq \mathrm{Hom}_A(P,I) \simeq \mathrm{Hom}_A(M,I)$$

in $D(A)$. Thus $\mathbf{R}\mathrm{Hom}_R(M,N)$ is represented by any/all of the displayed DG A-modules.

Definition 2.6.1.4. A DG A-module M is *homologically degreewise finite*, denoted $M \in D^f(A)$, if $\mathrm{H}_i(M)$ is finitely generated over $\mathrm{H}_0(A)$ for all i. We say M is *homologically bounded*, denoted

$M \in D_b(A)$, if $\mathrm{H}_i(M) = 0$ for $|i| \gg 0$. Additionally, M is *homologically finite* if $M \in D_b(A) \cap D^f(A)$, i.e., $M \in D_b^f(A)$. We say M is *homologically bounded below*, denoted $M \in D_+(A)$, if $\inf(M) > -\infty$. Similarly, M is *homologically bounded above*, denoted $M \in D_-(A)$, if $\sup(M) < \infty$.

Fact 2.6.1.5. Let M and N be DG A-modules.

(a) If $M \in D_+(A)$, then there exists a semifree resolution $F \xrightarrow{\simeq} M$, by [7, 5.2.1].

(b) If A is weakly noetherian and $M \in D_+^f(A)$, then there exists a semifree resolution $F \xrightarrow{\simeq} M$ with semibasis E such that $|E \cap M_n| < \infty$ for all n, by [7, 5.2.2(1)]. We call such a resolution a "degreewise finite semifree resolution."

(c) If F is semifree over A, then F is semiprojective over A, by [7, 3.3.5].

2.6.2. Semidualizing DG Modules

Definition 2.6.2.1. Let A be a homologically bounded DG R-algebra. A *semidualizing* DG A-module is a homologically finite DG A-module C that admits a degreewise finite semifree resolution over A such that the homothety morphism $\chi_C^A : A \to \mathbf{R}\mathrm{Hom}_A(C, C)$ is an isomorphism in the derived category $D(A)$. Let $\mathfrak{S}(A)$ denote the set of shift-isomorphism classes of semidualizing DG A-modules in $D(A)$.

Example 2.6.2.2. If A is homologically bounded, then $A \in \mathfrak{S}(A)$.

Remark 2.6.2.3. In Definition 2.6.2.1, we assume that A is homologically bounded. For homologically unbounded DG algebras, it is not clear what the correct definition of "semidualizing DG module" should be. For instance, if one uses the naive definition (simply removing the homologically bounded assumption of A in 2.6.2.1), then one runs up against the following problem: if A is homologically unbounded, it does not admit a semidualizing DG module. (Indeed, if C is in $D_b^f(A)$, then $\mathbf{R}\mathrm{Hom}_A(C, C)$ is in $D_-(A)$. So if $A \simeq \mathbf{R}\mathrm{Hom}_A(C, C) \in D_-(A)$, then $A \in D_-(A) \cap D_+(A)$, i.e., A is homologically bounded.) Even worse, the trivial DG module A is not semidualizing in this situation. Versions of Definitions 2.6.2.6–2.6.2.10 are even more subtle.

Assumption 2.6.2.4. For the remainder of this section, assume A is homologically bounded.

Remark 2.6.2.5. For many applications, we focus on weakly noetherian algebras. Over such algebras, the "homologically finite" assumption in Definition 2.6.2.1 implies the "degreewise finite semifree resolution" assumption; see Fact 2.6.1.5(b). On the other hand, if A is not weakly noetherian, then a DG module $M \in D_b^f(A)$ need not have such a resolution. In this dissertation, we choose not to focus exclusively on the weakly noetherian situation because the algebra $A' \otimes_k A''$ from Fact 2.5.3.5(a) need not be weakly noetherian; see Fact 2.5.3.5(c). Note that the requirement of a degreewise finite semifree resolution fits with the definition of semidualizing in the non-noetherian ring situation from [22].

The following notion was defined for dualizing R-modules by Foxby [14] and for an arbitrary semidualizing R-module or R-complex by Christensen [9].

Definition 2.6.2.6. Let $C \in \mathfrak{S}(A)$ and $M \in D_b(A)$. Then M is in the *Bass class* $\mathcal{B}_C(A)$ if the natural evaluation morphism $\xi_M^C : C \otimes_A^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_A(C, M) \to M$ is an isomorphism in $D(A)$ and $\mathbf{R}\mathrm{Hom}_A(C, M) \in D_b(A)$.

The next two facts are from [9].

Fact 2.6.2.7. If we are working over R, with $C \in \mathfrak{S}_0(R)$, then the modules in $\mathcal{B}_C(R)$ are exactly the modules satisfying the definition in 2.1.1.6.

Similarly, we have the following notions of the Auslander class and derived reflexive DG modules.

Definition 2.6.2.8. Let $C \in \mathfrak{S}(A)$ and $M \in D_b(A)$. Then M is in the *Auslander class* $\mathcal{A}_C(A)$ if the natural morphism $\gamma_M^C : M \to \mathbf{R}\mathrm{Hom}_A(C, C \otimes_A^{\mathbf{L}} M)$ is an isomorphism in $D(A)$ and $C \otimes_A^{\mathbf{L}} M \in D_b(A)$.

Fact 2.6.2.9. If we are working over R, with $C \in \mathfrak{S}_0(R)$, then the modules in $\mathcal{A}_C(R)$ are exactly the modules satisfying the definition in 2.1.1.7.

Definition 2.6.2.10. Let $C \in \mathfrak{S}(A)$ and $M \in D_b^f(A)$. Then M is *derived C-reflexive* if the natural biduality morphism $\delta_M^C : M \to \mathbf{R}\mathrm{Hom}_A(\mathbf{R}\mathrm{Hom}_A(M, C), C)$ is an isomorphism in $D(A)$ and $\mathbf{R}\mathrm{Hom}_A(M, C) \in D_b^f(A)$.

Fact 2.6.2.11. If we are working over R, with $C \in \mathfrak{S}_0(R)$ and M a finitely generated R-module, then M has a bounded resolution by totally C-reflexive R-modules, see Definition 2.1.2.5. This follows as in [36].

Example 2.6.2.12. It is straightforward to show that $\mathcal{B}_A(A) = D_b(A) = \mathcal{A}_A(A)$. Also, if $C \in \mathfrak{S}(A)$, then $C \in \mathcal{B}_C(A)$, $A \in \mathcal{A}_C(A)$, and A is derived C-reflexive.

Notation 2.6.2.13. Let $\overline{\mathfrak{S}}(A)$ denote the set of equivalence classes in $\mathfrak{S}(A)$ under the equivalence relation $\approx$, where $B \approx C$ if $B \in \mathcal{B}_C(A)$ and $C \in \mathcal{B}_B(A)$.

See e.g. [1] and [25] for properties and applications of the next construction.

Definition 2.6.2.14. Assume A is local and $M \in D_+^f(A)$. For each integer i, the *ith Betti number of M* is

$$\beta_i^A(M) := \operatorname{rank}_k(\operatorname{Tor}_i^A(k, M))$$

where $k = A/\mathfrak{m}_A$. The *Poincaré series* of A is the formal Laurent series

$$P_A^M(t) := \sum_{i \in \mathbb{Z}} \beta_i^A(M) t^i.$$

The Poincaré series is a bookkeeping tool. However, it is surprisingly powerful, as the next result begins to show; see also the proof of the subsequent result.

Lemma 2.6.2.15. *Assume $(A, \mathfrak{m}_A)$ is local, set $k = A/\mathfrak{m}_A$, and let $M \in D_+^f(A)$. If $P_A^M(t) = t^e$ for some e, then $M \simeq \Sigma^e A$ in $D(A)$.*

Proof: Notice, $P_A^M(t) = t^e$ for some e implies

$$\operatorname{rank}_k(\operatorname{Tor}_i^A(k, M)) = \begin{cases} 1 & \text{if } i = e \\ 0 & \text{if } i \neq e. \end{cases}$$

Let $F \xrightarrow{\simeq} M$ be a minimal semifree resolution over A, and let E be a semibasis of F; see Definition 2.5.3.6 and [1, Proposition 1]. By definition, we have $F^\natural \cong \oplus_j \Sigma^j (A^\natural)^{\gamma_j}$ for integers $\gamma_j = |E \cap F_j|$ and $\partial^{k \otimes_A F} = 0$. Thus, $\gamma_j = \operatorname{rank}_k(\operatorname{Tor}_i^A(k, M))$ for all j. Therefore, the above display implies that F has a single semibasis element x in degree e. Now, since A is positively graded, we have $F_i = 0$ for all $i < e$. It follows that $\partial_e^F(x) = 0$. From this it is straightforward to show $F \cong \Sigma^e A$. Therefore, we have $M \simeq F \cong \Sigma^e A$. $\square$

The next result is a DG version of a result of Araya et al. [2, (5.3)]; see also [15, Lemma 3.2].

Lemma 2.6.2.16. *Assume A is local and let B, $C \in \mathfrak{S}(A)$. Then $B \approx C$ if and only if $B \simeq \Sigma^e C$ in $D(A)$, for some integer e.*

Proof: One implication is straightforward since $C \in \mathcal{B}_C(A)$ and $B \in \mathcal{B}_B(A)$.

For the other implication, assume $B \approx C$. Hence, $B \in \mathcal{B}_C(A)$ and $C \in \mathcal{B}_B(A)$. Thus, $B \simeq C \otimes^{\mathbf{L}}_A \mathbf{R}\mathrm{Hom}_A(C, B)$ and

$$C \simeq B \otimes^{\mathbf{L}}_A \mathbf{R}\mathrm{Hom}_A(B, C) \simeq C \otimes^{\mathbf{L}}_A \mathbf{R}\mathrm{Hom}_A(C, B) \otimes^{\mathbf{L}}_A \mathbf{R}\mathrm{Hom}_A(B, C).$$

This yields the following equality of Poincaré series:

$$P^C_A(t) = P^C_A(t) \cdot P^{\mathbf{R}\mathrm{Hom}_A(C,B)}_A(t) \cdot P^{\mathbf{R}\mathrm{Hom}_A(B,C)}_A(t).$$

It follows that $1 = P^{\mathbf{R}\mathrm{Hom}_A(C,B)}_A(t) \cdot P^{\mathbf{R}\mathrm{Hom}_A(B,C)}_A(t)$. Therefore, we have $P^{\mathbf{R}\mathrm{Hom}_A(C,B)}_A(t) = t^e$ and $P^{\mathbf{R}\mathrm{Hom}_A(B,C)}_A(t) = t^{-e}$ for some integer e. By Lemma 2.6.2.15 we have $\mathbf{R}\mathrm{Hom}_A(C, B) \simeq \Sigma^e A$ and $\mathbf{R}\mathrm{Hom}_A(B, C) \simeq \Sigma^{-e} A$ for some integer e. Thus $B \simeq C \otimes^{\mathbf{L}}_A \Sigma^e A \simeq \Sigma^e C$. □

2.6.3. k-Complexes

The remainder of this section focuses on k-complexes.

Lemma 2.6.3.1. *Let L be a k-complex. Then L is semiprojective over k.*

Proof: Notice that $L^\natural$ is free over $k = k^\natural$, therefore projective. Now by [7, 3.9.1] $\partial^k = 0$ implies $\mathrm{H}(L)$ and $\mathrm{Im}(\partial^L)$ are DG k-modules. Also, $\mathrm{H}(L)$ and $\mathrm{Im}(\partial^L)$ are projective over k. Thus, by [7, 3.9.7], L is semiprojective over k. □

Remark 2.6.3.2. Let B' and B'' be k-complexes. By Lemma 2.6.3.1, the complexes B' and B'' are semiprojective over k. Thus $B' \otimes_k B'' \simeq B' \otimes^{\mathbf{L}}_k B''$.

Fact 2.6.3.3. Let B', B'', C', C'' be k-vector spaces, and let $\alpha' : B' \to C'$ and $\alpha'' : B'' \to C''$ be k-module homomorphisms. If α' and α'' are both isomorphisms, then $\alpha' \otimes_k \alpha''$ is an isomorphism. If B', $B'' \neq 0$ or C', $C'' \neq 0$, then the converse holds.

What follows is a special case of the Künneth formula, see [27, 10.81].

Fact 2.6.3.4. Let X' and X'' be k-complexes. Then

$$\mathrm{H}_i(X' \otimes_k X'') \cong \bigoplus_{p+q=i} \mathrm{H}_p(X') \otimes_k \mathrm{H}_q(X'').$$

Moreover, if $\alpha' : X' \to Y'$ and $\alpha'' : X'' \to Y''$ are chain maps over k, then $\bigoplus_{p+q=i} \mathrm{H}_p(\alpha') \otimes_k \mathrm{H}_q(\alpha'')$ is identified with $\mathrm{H}_i(\alpha' \otimes_k \alpha'')$ under the above isomorphism and the corresponding isomorphism for $\mathrm{H}_i(Y' \otimes_k Y'')$.

The next few lemmas show how to detect various properties for tensor products of k-complexes.

Lemma 2.6.3.5. *Let X', X'', Y', Y'' be k-complexes, and let $\alpha' : X' \to Y'$ and $\alpha'' : X'' \to Y''$ be chain maps over k. If α' and α'' are isomorphisms, then $\alpha' \otimes_k \alpha'' : X' \otimes_k X'' \to Y' \otimes_k Y''$ is an isomorphism. If X', $X'' \neq 0$ or Y', $Y'' \neq 0$, then the converse holds.*

Proof: The forward implication is standard. For the reverse implication, assume X', X'', Y', $Y'' \neq 0$ and $\alpha' \otimes_k \alpha''$ is an isomorphism. By definition we have $(\alpha' \otimes_k \alpha'')_i = \bigoplus_{p+q=i}(\alpha'_p \otimes_k \alpha''_q)$, so $\alpha'_p \otimes_k \alpha''_q$ is an isomorphism for all p, q.

It remains to show that α'_p and α''_q are isomorphisms for all p, q. Let $X'_{p_0} \neq 0$, $X''_{q_0} \neq 0$, $Y'_{p_1} \neq 0$, and $Y''_{q_1} \neq 0$. Suppose $X'_p = 0$. Then $0 = X'_p \otimes_k X''_{q_1} \xrightarrow[\cong]{\alpha'_p \otimes_k \alpha_{q_1}} Y'_p \otimes_k Y''_{q_1}$. Therefore, $Y'_p \otimes_k Y''_{q_1} = 0$. Since $Y''_{q_1} \neq 0$, we have $Y'_p = 0$. So α'_p is an isomorphism. By a similar argument $Y'_p = 0$ implies α'_p is an isomorphism. Assume $X'_p, Y'_p \neq 0$. The assumption $X''_{q_0} \neq 0$ implies that $Y''_{q_0} \neq 0$. Therefore, $\alpha'_p \otimes_k \alpha''_{q_0}$ is an isomorphism such that $X'_p, Y'_p, X''_{q_0}, Y''_{q_0} \neq 0$. Thus, Fact 2.6.3.3 implies α'_p and α''_{q_0} are isomorphisms.

A symmetric argument shows that α''_q is an isomorphism for all q. □

Lemma 2.6.3.6. *Let A', A'', B', B'' be k-complexes, and let $\alpha' : A' \to B'$ and $\alpha'' : A'' \to B''$ be chain maps over k. If α' and α'' are quasiisomorphisms, then $\alpha' \otimes_k \alpha''$ is a quasiisomorphism. If A', $A'' \not\simeq 0$ or B', $B'' \not\simeq 0$, then the converse holds.*

Proof: This follows from Facts 2.6.3.3-2.6.3.4 and Lemma 2.6.3.5. □

Lemma 2.6.3.7. *Let M' and M'' be k-complexes. If M' and M'' are homologically bounded, then $M' \otimes_k M''$ is homologically bounded. If $M', M'' \not\simeq 0$, then the converse holds.*

Proof: For the forward implication, set $t = \sup(M')$, $w = \sup(M'')$, $s = \inf(M')$, and $l = \inf(M'')$.

Case 1: If $p+q > t+w$, then $\mathrm{H}_p(M') \otimes_k \mathrm{H}_q(M'') = 0$ because $p+q > t+w$ implies $p > t$ or $q > w$. Therefore, by Fact 2.6.3.4, $\mathrm{H}_i(M' \otimes_k M'') = \bigoplus_{p+q=i}(\mathrm{H}_p(M') \otimes_k \mathrm{H}_q(M'')) = 0$ for $i > t+w$.

Case 2: If $p+q < s+l$, then $\mathrm{H}_p(M') \otimes_k \mathrm{H}_q(M'') = 0$ because $p+q < s+l$ implies $p < s$ or $q < l$. Therefore, $\mathrm{H}_i(M' \otimes_k M'') = 0$ for $i < s+l$.

For the reverse implication suppose $\mathrm{H}_{p_j}(M') \neq 0$ for infinitely many indices j. Since $M'' \not\simeq 0$ there exists an integer b such that $\mathrm{H}_b(M'') \neq 0$. Now, $\mathrm{H}_{p_j}(M') \otimes_k \mathrm{H}_b(M'') \neq 0$ for infinitely many indices j. However, $\mathrm{H}_{p_j}(M') \otimes_k \mathrm{H}_b(M'') \subseteq \mathrm{H}_{p_j+b}(M' \otimes_k M'')$. Hence, there is an infinite number of $\sigma = p_j + b$ such that $\mathrm{H}_\sigma(M' \otimes_k M'') \neq 0$ which is a contradiction since $M' \otimes_k M''$ is homologically bounded. Thus $\mathrm{H}_{p_j}(M') \neq 0$ for only finitely many j. Hence M' is homologically bounded. By a similar argument, M'' is homologically bounded. $\square$

3. KEY RESULTS

The results of this chapter are all new. Section 3.1 consists of technical results for use in Section 3.2, which contains our main theorem.

3.1. DG Tensor Products

Assumption 3.1.0.1. In this section A' and A'' are DG R-algebras and $A := A' \otimes_R A''$.

3.1.1. Two Lemmas

We begin with a result augmenting Subsection 2.6.3.

Lemma 3.1.1.1. *Assume that $R = k$ is a field. Let M' and M'' be DG A'- and A''-modules respectively. If M' and M'' are homologically degreewise finite over A' and A'', respectively, then $M' \otimes_k M''$ is homologically degreewise finite over A under any of the following conditions:*

(1) M' is homologically bounded,

(2) M'' is homologically bounded,

(3) M' and M'' are homologically bounded below, or

(4) M' and M'' are homologically bounded above.

Proof: (1) By Fact 2.6.3.4, for all i we have $\mathrm{H}_i(M' \otimes_k M'') \cong \bigoplus_{p+q=i} \mathrm{H}_p(M') \otimes_k \mathrm{H}_q(M'')$. Note that this direct sum is finite because $M' \in D_b(A')$.

Now, $\mathrm{H}_p(M')$ is finitely generated over $\mathrm{H}_0(A')$ for all p, and $\mathrm{H}_q(M'')$ is finitely generated over $\mathrm{H}_0(A'')$ for all q, by our assumption. It follows that $\mathrm{H}_p(M') \otimes_k \mathrm{H}_q(M'')$ is finitely generated over $\mathrm{H}_0(A') \otimes_k \mathrm{H}_0(A'')$ for all p and q. Hence $\bigoplus_{p+q=i} \mathrm{H}_p(M') \otimes_k \mathrm{H}_q(M'')$ is finitely generated for all i.

The proofs of parts (2)–(4) are similar to the proof of part (1). Notice that in each case the assumptions guarantee the direct sum $\bigoplus_{p+q=i} \mathrm{H}_p(M') \otimes_k \mathrm{H}_q(M'')$ is finite. □

The next result gives us some flexibility for understanding how DG A'- and A''-modules yield DG A-modules.

Lemma 3.1.1.2. *Let X' and X'' be DG A'- and A''-modules, respectively. The map*

$$\alpha^{X'}_{X''} : X' \otimes_R X'' \to (A \otimes_{A'} X') \otimes_A (A \otimes_{A''} X'')$$

given by $x' \otimes x'' \mapsto (1 \otimes x') \otimes (1 \otimes x'')$ is an isomorphism of DG A-modules.

Proof: The given map is the composition of the following sequence of isomorphisms.

$$\begin{aligned} X' \otimes_R X'' &\cong (X' \otimes_{A'} (A' \otimes_R A'')) \otimes_{A''} X'' \\ &\cong ((A' \otimes_R A'') \otimes_{A'} X') \otimes_{A''} X'' \\ &\cong (A \otimes_{A'} X') \otimes_{A''} X'' \\ &\cong (A \otimes_{A'} X') \otimes_A (A \otimes_{A''} X'') \end{aligned}$$

It is straightforward to show that $\alpha^{X'}_{X''}$ is A-linear. □

3.1.2. Semiprojective Tensor Products

Lemma 3.1.2.1. *If P' is a semiprojective DG A'-module and P'' is semiprojective DG A''-module, then $P' \otimes_R P''$ is semiprojective over A.*

Proof: By Lemma 3.1.1.2, we have $P' \otimes_R P'' \cong (A \otimes_{A'} P') \otimes_A (A \otimes_{A''} P'')$ as DG A-modules.

The fact that P' is semiprojective over A' implies that $A \otimes_{A'} P'$ is semiprojective over A because

$$\mathrm{Hom}_A(A \otimes_{A'} P', -) \cong \mathrm{Hom}_{A'}(P', \mathrm{Hom}_A(A, -)) \cong \mathrm{Hom}_{A'}(P', -).$$

Similarly, $A \otimes_{A''} P''$ is semiprojective over A. Now X, Y semiprojective over A implies $X \otimes_A Y$ is semiprojective over A because $\mathrm{Hom}_A(X \otimes_A Y, -) \cong \mathrm{Hom}_A(Y, \mathrm{Hom}_A(X, -))$. Therefore, $A \otimes_{A'} P'$ semiprojective over A and $A \otimes_{A''} P''$ semiprojective over A imply that $(A \otimes_{A'} P') \otimes_A (A \otimes_{A''} P'') \cong P' \otimes_R P''$ is semiprojective. □

Lemma 3.1.2.2. *Assume that $R = k$ is a field. Let M' and M'' be DG A'- and A''-modules respectively. If $P' \xrightarrow[\simeq]{\alpha'} M'$ and $P'' \xrightarrow[\simeq]{\alpha''} M''$ are semiprojective resolutions over A' and A'', respectively, then $P' \otimes_k P'' \xrightarrow[\simeq]{\alpha' \otimes_k \alpha''} M' \otimes_k M''$ is a semiprojective resolution over A.*

Proof: Notice $P' \otimes_k P''$ is semiprojective over A and $P' \otimes_k P'' \xrightarrow{\alpha' \otimes_k \alpha''} M' \otimes_k M''$ is a quasiisomorphism by Lemmas 3.1.2.1 and 2.6.3.6. □

3.1.3. Two Useful Isomorphisms

Remark 3.1.3.1. In this subsection, we use tildes to distinguish between morphisms on derived constructions (i.e., $\otimes^{\mathbf{L}}$ and $\mathbf{R}\mathrm{Hom}$) and maps defined on complexes representing such constructions. See, e.g., γ and $\tilde{\gamma}$ in the next two results.

The next result is similar in flavor to Lemma 3.1.1.2.

Lemma 3.1.3.2. *Let X', Y' and X'', Y'' be DG A'- and A''-modules, respectively. The map*

$$\tilde{\gamma}^{X',X''}_{Y',Y''} : (X' \otimes_{A'} Y') \otimes_R (X'' \otimes_{A''} Y'') \to (X' \otimes_R X'') \otimes_A (Y' \otimes_R Y'')$$

given by $(x' \otimes y') \otimes (x'' \otimes y'') \mapsto (-1)^{|y'||x''|}(x' \otimes x'') \otimes (y' \otimes y'')$ is an isomorphism of DG A-modules.

Proof: Lemma 3.1.1.2 gives the first and last isomorphisms in the following display. The second and third isomorphisms are by associativity, commutativity, etc. of tensor products.

$$\begin{aligned}(X' \otimes_R X'') \otimes_A (Y' \otimes_R Y'') &\cong [(A \otimes_{A'} X') \otimes_A (A \otimes_{A''} X'')] \otimes_A [(A \otimes_{A'} Y') \otimes_A (A \otimes_{A''} Y'')] \\ &\cong (A \otimes_{A'} X') \otimes_A (A \otimes_{A'} Y') \otimes_A (A \otimes_{A''} X'') \otimes_A (A \otimes_{A''} Y'') \\ &\cong (A \otimes_{A'} (X' \otimes_{A'} Y')) \otimes_A (A \otimes_{A''} (X'' \otimes_{A''} Y'')) \\ &\cong (X' \otimes_{A'} Y') \otimes_R (X'' \otimes_{A''} Y'')\end{aligned}$$

It is straightforward to show that $\tilde{\gamma}^{X',X''}_{Y',Y''}$ is the composition of the displayed isomorphisms and is A-linear. □

Lemma 3.1.3.3. *Assume that $R = k$ is a field. Let X', Y' and X'', Y'' be DG A'- and A''-modules, respectively. Let $P' \xrightarrow{\simeq} X'$, $P'' \xrightarrow{\simeq} X''$, $Q' \xrightarrow{\simeq} Y'$, and $Q'' \xrightarrow{\simeq} Y''$ be semiprojective resolutions over A' and A'' as appropriate. Then the morphism*

$$\gamma^{X',X''}_{Y',Y''} : (X' \otimes^{\mathbf{L}}_{A'} Y') \otimes_k (X'' \otimes^{\mathbf{L}}_{A''} Y'') \to (X' \otimes_k X'') \otimes^{\mathbf{L}}_A (Y' \otimes_k Y'')$$

induced by the map $\tilde{\gamma}^{P',P''}_{Q',Q''}$ from Lemma 3.1.3.2 is an isomorphism in $D(A)$.

Proof: By Lemma 3.1.2.2, the maps $P' \otimes_k P'' \xrightarrow{\simeq} X' \otimes_k X''$ and $Q' \otimes_k Q'' \xrightarrow{\simeq} Y' \otimes_k Y''$ are semiprojective resolutions over A. By Lemma 3.1.3.2 the map

$$\tilde{\gamma}^{P',P''}_{Q',Q''} : (P' \otimes_{A'} Q') \otimes_k (P'' \otimes_{A''} Q'') \to (P' \otimes_k P'') \otimes_A (Q' \otimes_k Q'')$$

is an isomorphism of DG A-modules. Therefore, $\gamma^{X',X''}_{Y',Y''}$ is an isomorphism in $D(A)$. $\square$

The remainder of this section is devoted to understanding $\mathbf{R}\mathrm{Hom}_A(N, M)$ for DG A-modules M and N constructed as above.

Definition 3.1.3.4. Let N', M' and N'', M'' be DG A'- and A''-modules, respectively. Consider elements $f' \in \mathrm{Hom}_{A'}(N', M')$ and $f'' \in \mathrm{Hom}_{A''}(N'', M'')$. Let $f' \boxtimes f'' : N' \otimes_R N'' \to M' \otimes_R M''$ be given by $(f' \boxtimes f'')_{|x' \otimes x''|}(x' \otimes x'') = (-1)^{|f''||x'|} f'_{|x'|}(x') \otimes f''_{|x''|}(x'')$.

Remark 3.1.3.5. With notation as in Definition 3.1.3.4, the map $f' \boxtimes f''$ is well-defined and A-linear.

Example 3.1.3.6. For R-complexes X' and X'', we have $\partial^{X' \otimes_R X''} = (\partial^{X'} \boxtimes \mathrm{id}) + (\mathrm{id} \boxtimes \partial^{X''})$.

Definition 3.1.3.7. Let N', M' and N'', M'' be DG A'- and A''-modules, respectively. Let

$$\tilde{\eta}^{N',N''}_{M',M''} : \mathrm{Hom}_{A'}(N', M') \otimes_R \mathrm{Hom}_{A''}(N'', M'') \to \mathrm{Hom}_A(N' \otimes_R N'', M' \otimes_R M'')$$

be given by $f' \otimes f'' \mapsto f' \boxtimes f''$.

Remark 3.1.3.8. The map $\tilde{\eta}^{N',N''}_{M',M''}$ is a well-defined morphism of DG A-modules.

Proposition 3.1.3.9. *Assume that $R = k$ is a field. If N', N'' are homologically degreewise finite, semifree DG A'- and A''-modules, respectively, and M', M'' are bounded above DG A'- and A''-modules, respectively, then the map $\tilde{\eta}^{N',N''}_{M',M''}$ is an isomorphism of DG A-modules.*

Proof: The injectivity and surjectivity of $\tilde{\eta}^{N',N''}_{M',M''}$ are independent of the differentials on A', A'', M', M'', N', and N''. Therefore, without loss of generality, assume that all differentials are 0. Thus, there are integers p_0, q_0 such that $N' \cong \bigoplus_{p \geqslant p_0} \Sigma^p (A')^{\beta'_p}$ and $N'' \cong \bigoplus_{q \geqslant q_0} \Sigma^q (A'')^{\beta''_q}$ for some integers β'_p, $\beta'_q \geqslant 0$.

Special case: Assume $N' = A'$ and $N'' = A''$. Set $\tilde{\eta} = \tilde{\eta}^{A',A''}_{M',M''}$. It is straightforward to show that the following diagram commutes.

$$\begin{array}{ccc} \operatorname{Hom}_{A'}(A', M') \otimes_k \operatorname{Hom}_{A''}(A'', M'') & \xrightarrow{\cong} & M' \otimes_k M'' \\ \Big\downarrow \tilde{\eta} & \nearrow_{\cong} & \\ \operatorname{Hom}_A(A, M' \otimes_k M'') & & \end{array}$$

Hence $\tilde{\eta}$ is an isomorphism in this case.

General case: Set $\tilde{\eta}' = \tilde{\eta}^{N',N''}_{M',M''}$. First we have

$$N' \otimes_k N'' \cong \bigoplus_{p \geqslant p_0} \bigoplus_{q \geqslant q_0} \Sigma^{p+q} (A' \otimes_k A'')^{\beta'_p \beta''_q}.$$

Now, for all $m \in \mathbb{Z}$, our boundedness condition on M' implies that

$$\begin{aligned} \operatorname{Hom}_{A'}(N', M')_m &\cong \operatorname{Hom}_{A'} \left(\bigoplus_{p \geqslant p_0} \Sigma^p (A')^{\beta'_p}, M' \right)_m \\ &\cong \prod_{p \geqslant p_0} \operatorname{Hom}_{A'}(\Sigma^p (A')^{\beta'_p}, M')_m \\ &= \bigoplus_{p \geqslant p_0} \operatorname{Hom}_{A'}(\Sigma^p (A')^{\beta'_p}, M')_m. \end{aligned}$$

Similarly, for all $n \in \mathbb{Z}$, we have

$$\operatorname{Hom}_{A''}(N'', M'')_n \cong \operatorname{Hom}_{A''} \left(\bigoplus_{q > q_0} \Sigma^q (A'')^{\beta''_q}, M'' \right)_n \cong \bigoplus_{q \geqslant q_0} \operatorname{Hom}_{A''}(\Sigma^q (A'')^{\beta''_q}, M'')_n.$$

The domain of $\tilde{\eta}'_i$ decomposes as follows.

$$\begin{aligned} &[\operatorname{Hom}_{A'}(N', M') \otimes_k \operatorname{Hom}_{A''}(N'', M'')]_i \\ &\quad \cong \bigoplus_{m+n=i} \left[\operatorname{Hom}_{A'} \left(\bigoplus_{p \geqslant p_0} \Sigma^p (A')^{\beta'_p}, M' \right)_m \otimes_k \operatorname{Hom}_{A''} \left(\bigoplus_{q \geqslant q_0} \Sigma^q (A'')^{\beta''_q}, M'' \right)_n \right] \\ &\quad \cong \bigoplus_{m+n=i} \bigoplus_{p \geqslant p_0} \bigoplus_{q \geqslant q_0} \left[\operatorname{Hom}_{A'}(\Sigma^p A', M')_m^{\beta'_p} \otimes_k \operatorname{Hom}_{A''}(\Sigma^q A'', M'')_n^{\beta''_q} \right] \\ &\quad \cong \bigoplus_{m+n=i} \bigoplus_{p \geqslant p_0} \bigoplus_{q \geqslant q_0} \Sigma^{-p-q} [\operatorname{Hom}_{A'}(A', M')_m \otimes_k \operatorname{Hom}_{A''}(A'', M'')_n]^{\beta'_p \beta''_q} \end{aligned}$$

$$\cong \bigoplus_{p\geqslant p_0}\bigoplus_{q\geqslant q_0}\Sigma^{-p-q}\left[\bigoplus_{m+n=i}\operatorname{Hom}_{A'}(A',M')_m\otimes_k\operatorname{Hom}_{A''}(A'',M'')_n\right]^{\beta'_p\beta''_q}$$
$$\cong \bigoplus_{p\geqslant p_0}\bigoplus_{q\geqslant q_0}\Sigma^{-p-q}[\operatorname{Hom}_{A'}(A',M')\otimes_k\operatorname{Hom}_{A''}(A'',M'')]_i^{\beta'_p\beta''_q}.$$

Next, we consider the codomain in degree i.

$$\begin{aligned}\operatorname{Hom}_A(N'\otimes_k N'',M'\otimes_k M'')_i &\cong \operatorname{Hom}_A\left(\bigoplus_{p\geqslant p_0}\bigoplus_{q\geqslant q_0}\Sigma^{p+q}(A'\otimes_k A'')^{\beta'_p\beta''_q},M'\otimes_k M''\right)_i\\ &\cong \bigoplus_{p\geqslant p_0}\bigoplus_{q\geqslant q_0}\operatorname{Hom}_A(\Sigma^{p+q}(A'\otimes_k A'')^{\beta'_p\beta''_q},M'\otimes_k M'')_i\\ &\cong \bigoplus_{p\geqslant p_0}\bigoplus_{q\geqslant q_0}\Sigma^{-p-q}\operatorname{Hom}_A(A'\otimes_k A'',M'\otimes_k M'')_i^{\beta'_p\beta''_q}.\end{aligned}$$

It is straightforward to show that $\tilde{\eta}$ is compatible with direct sums and shifts. Therefore, we have $\tilde{\eta}' = \bigoplus_{p\geqslant p_0}\bigoplus_{q\geqslant q_0}\Sigma^{-p-q}\tilde{\eta}$. Since $\tilde{\eta}$ is an isomorphism by our special case, we conclude that $\tilde{\eta}'$ is an isomorphism. $\square$

Remark 3.1.3.10. Assume that $R = k$ is a field. Let N' and N'' be DG A'- and A''-modules respectively. Let $P' \xrightarrow{\simeq} N'$ and $P'' \xrightarrow{\simeq} N''$ be semiprojective resolutions over A' and A'', respectively. By Lemma 3.1.2.2, we have that $P'\otimes_k P'' \xrightarrow{\simeq} N'\otimes_k N''$ is a semiprojective resolution over A. Therefore, $\tilde{\eta}^{P',P''}_{M',M''}:\operatorname{Hom}_{A'}(P',M')\otimes_k\operatorname{Hom}_{A''}(P'',M'')\to\operatorname{Hom}_A(P'\otimes_k P'',M'\otimes_k M'')$ represents a well-defined morphism

$$\eta^{N',N''}_{M',M''}:\mathbf{R}\operatorname{Hom}_{A'}(N',M')\otimes_k\mathbf{R}\operatorname{Hom}_{A''}(N'',M'')\to\mathbf{R}\operatorname{Hom}_A(N'\otimes_k N'',M'\otimes_k M'')$$

in $D(A)$.

For the next result, notice if A' and A'' are weakly noetherian, then DG modules $N'\in D^f_+(A')$ and $N''\in D^f_+(A'')$ admit degreewise finite semifree resolutions, by Fact 2.6.1.5.

Proposition 3.1.3.11. *Assume that $R = k$ is a field. Let $N'\in D^f_+(A')$ and $N''\in D^f_+(A'')$ admit degreewise finite semifree resolutions over A' and A'', respectively, and let $M'\in D_-(A')$ and $M''\in D_-(A'')$. Then $\eta^{N',N''}_{M',M''}$ is an isomorphism in $D(A)$.*

Proof: Notice that M' and M'' homologically bounded above imply there exist bounded above L' and L'' with quasiisomorphisms $\alpha' : M' \xrightarrow{\simeq} L'$ and $\alpha'' : M'' \xrightarrow{\simeq} L''$; here L' and L'' are obtained by "soft truncations" of M' and M'', respectively. Therefore, we can replace M' and M'' by L' and L'' to assume that M' and M'' are bounded above. By assumption, there exist semifree resolutions $P' \xrightarrow{\simeq} N'$ and $P'' \xrightarrow{\simeq} N''$ such that P', P'' are bounded below and degreewise finite. Therefore, we can replace N' and N'' by P' and P'' respectively to assume that N' and N'' are semifree and degreewise finite. The result now follows from Proposition 3.1.3.9. □

3.2. Semidualizing DG Modules

In this section we prove the main result of this dissertation and document a few corollaries.

Assumption 3.2.0.1. In this section k is a field, A' and A'' are homologically bounded DG k-algebras such that $A' \not\simeq 0 \not\simeq A''$, and $A := A' \otimes_k A''$.

3.2.1. The Main Theorem

The next two results are the keys for proving Theorem 1.0.0.3 from the introduction.

Theorem 3.2.1.1. *If M' and M'' are semidualizing DG modules over A' and A'', respectively, then $M' \otimes_k M''$ is a semidualizing DG module over A. If M' and M'' admit degreewise finite semifree resolutions over A' and A'', respectively, and $M' \in D^f(A')$ and $M'' \in D^f(A'')$, then the converse holds. For instance, if A' and A'' are weakly noetherian and $M' \in D^f(A')$ and $M'' \in D^f(A'')$, then the converse holds.*

Proof: Step 1. Note that $A', A'' \not\simeq 0$, by Assumption 3.2.0.1. Thus we have $A \not\simeq 0$, e.g., by Fact 2.6.3.4.

Step 2. If $M' \in \mathfrak{S}(A')$, then $M' \not\simeq 0$ because $\mathbf{R}\mathrm{Hom}_{A'}(M', M') \simeq A' \not\simeq 0$. On the other hand, if $M' \otimes_k M'' \in \mathfrak{S}(A)$, then $M' \otimes_k M'' \not\simeq 0$, so $M' \not\simeq 0$. Thus, we assume for the remainder of the proof that $M' \not\simeq 0$ and similarly, $M'' \not\simeq 0$.

Step 3. In the forward implication we assume $M' \in \mathfrak{S}(A')$ and $M'' \in \mathfrak{S}(A'')$, therefore we have $M' \in D^f_b(A')$ and $M'' \in D^f_b(A'')$. In the reverse implication the assumption $M' \otimes_k M'' \in D_b(A)$ implies $M' \in D_b(A)$ and $M'' \in D_b(A'')$, by Lemma 2.6.3.7. Thus, we assume for the remainder of the proof that $M' \in D^f_b(A')$ and $M'' \in D^f_b(A'')$.

Step 4. We assume for the remainder of the proof that M' and M'' admit degreewise finite semifree resolutions. Notice, in the forward implication, the conditions $M' \in \mathfrak{S}(A')$ and $M'' \in \mathfrak{S}(A'')$ guarantee that such resolutions exist. It is worth noting that in the converse, if A' and A'' are weakly noetherian and $M' \in D^f_b(A')$ and $M'' \in D^f_b(A'')$, then Fact 2.6.1.5(b) guarantees that such resolutions exist. Note that it follows that $M' \otimes_k M'' \in D^f_b(A)$ has such a resolution over A; see Lemmas 2.6.3.7 and 3.1.1.1.

Step 5: Consider the following commutative diagram in $D(A)$.

$$\begin{array}{ccc} A = A' \otimes_k A'' & \xrightarrow{\chi^{A'}_{M'} \otimes_k \chi^{A''}_{M''}} & \mathbf{R}\operatorname{Hom}_{A'}(M', M') \otimes_k \mathbf{R}\operatorname{Hom}_{A''}(M'', M'') \\ & \searrow^{\chi^{A}_{M' \otimes_k M''}} & \simeq \downarrow \eta^{M',M''}_{M',M''} \\ & & \mathbf{R}\operatorname{Hom}_A(M' \otimes_k M'', M' \otimes_k M'') \end{array}$$

Notice that the morphism $\eta^{M',M''}_{M',M''}$ in this diagram is an isomorphism in $D(A)$, by Proposition 3.1.3.11.

In the forward implication, the morphism $\chi^{A'}_{M'}$ is an isomorphism in $D(A')$ and $\chi^{A''}_{M''}$ is an isomorphism in $D(A'')$, so $\chi^{A'}_{M'} \otimes_k \chi^{A''}_{M''}$ is an isomorphism in $D(A)$ by Lemma 2.6.3.6. Therefore, the commutative diagram implies that $\chi^{A}_{M' \otimes_k M''}$ is an isomorphism in $D(A)$.

In the reverse implication, our commutative diagram with $\eta^{M',M''}_{M',M''}$ and $\chi^{A}_{M' \otimes_k M''}$ isomorphisms in $D(A)$ imply that $\chi^{A'}_{M'} \otimes_k \chi^{A''}_{M''}$ is an isomorphism in $D(A)$. Thus, Lemma 2.6.3.6 implies that $\chi^{A'}_{M'}$ is an isomorphism in $D(A')$ and $\chi^{A''}_{M''}$ is an isomorphism in $D(A'')$. □

Theorem 3.2.1.2. *Fix $M' \in \mathfrak{S}(A')$ and $M'' \in \mathfrak{S}(A'')$, and let $N' \in D(A')$ and $N'' \in D(A'')$. If $N' \in \mathcal{B}_{M'}(A')$ and $N'' \in \mathcal{B}_{M''}(A'')$, then $N' \otimes_k N'' \in \mathcal{B}_{M' \otimes_k M''}(A)$. If $N', N'' \not\simeq 0$, then the converse holds.*

Proof: We prove the converse. The proof of the forward implication is similar and easier. Therefore, assume for the rest of the proof that $N', N'' \not\simeq 0$.

Step 1: Notice, $N' \otimes_k N'' \not\simeq 0$ by our assumption.

Step 2: By Lemma 2.6.3.7, the condition $N' \otimes_k N'' \in D_b(A)$ implies $N' \in D_b(A')$ and $N'' \in D_b(A'')$.

Step 3: We need to show if $\mathbf{R}\mathrm{Hom}_A(M' \otimes_k M'', N' \otimes_k N'') \in D_b(A)$, then $\mathbf{R}\mathrm{Hom}_{A'}(M', N') \in D_b(A')$ and $\mathbf{R}\mathrm{Hom}_{A''}(M'', N'') \in D_b(A'')$. Notice, by Proposition 3.1.3.11 we have the isomorphism

$$\eta^{M',M''}_{N',N''} : \mathbf{R}\mathrm{Hom}_{A'}(M',N') \otimes_k \mathbf{R}\mathrm{Hom}_{A''}(M'',N'') \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_{A}(M' \otimes_k M'', N' \otimes_k N'') \tag{3.1}$$

in $D(A)$.

Claim: One has $\mathbf{R}\mathrm{Hom}_{A'}(M',N') \not\simeq 0 \not\simeq \mathbf{R}\mathrm{Hom}_{A''}(M'',N'')$.

Proof of Claim: We have $0 \not\simeq N' \otimes_k N'' \in \mathcal{B}_{M' \otimes_k M''}(A)$, so

$$0 \not\simeq (N' \otimes_k N'') \simeq N' \otimes_k N'' \otimes_A \mathbf{R}\mathrm{Hom}_A(M' \otimes_k M'', N' \otimes_k N'').$$

Therefore, $\mathbf{R}\mathrm{Hom}_{A'}(M',N') \not\simeq 0 \not\simeq \mathbf{R}\mathrm{Hom}_{A''}(M'',N'')$.

Notice, $\mathbf{R}\mathrm{Hom}_{A'}(M',N') \not\simeq 0 \not\simeq \mathbf{R}\mathrm{Hom}_{A''}(M'',N'')$ and $\mathbf{R}\mathrm{Hom}_A(M' \otimes_k M'', N' \otimes_k N'') \in D_b(A)$, implies $\mathbf{R}\mathrm{Hom}_{A'}(M',N') \in D_b(A')$ and $\mathbf{R}\mathrm{Hom}_{A''}(M'',N'') \in D_b(A'')$ by Lemma 2.6.3.7 and the isomorphism (3.1).

Step 4: We need to show that if $\xi^{M' \otimes_k M''}_{N' \otimes_k N''}$ is an isomorphism in $D(A)$, then $\xi^{M'}_{N'}$ and $\xi^{M''}_{N''}$ are isomorphisms in $D(A')$ and $D(A'')$, respectively.

Consider the following commutative diagram in $D(A)$.

$$\begin{array}{ccc}
(M' \otimes^{\mathbf{L}}_{A'} \mathbf{R}\mathrm{Hom}_{A'}(M',N')) \otimes_k (M'' \otimes^{\mathbf{L}}_{A''} \mathbf{R}\mathrm{Hom}_{A''}(M'',N'')) & & \\
\gamma^{M',M''}_{\mathbf{R}\mathrm{Hom}_{A'}(M',N'),\mathbf{R}\mathrm{Hom}_{A''}(M'',N'')} \Big\downarrow \simeq & \searrow^{\xi^{M'}_{N'} \otimes_k \xi^{M''}_{N''}} & \\
(M' \otimes_k M'') \otimes^{\mathbf{L}}_A (\mathbf{R}\mathrm{Hom}_{A'}(M',N') \otimes_k \mathbf{R}\mathrm{Hom}_{A''}(M'',N'')) & & N' \otimes_k N'' \\
(M' \otimes_k M'') \otimes^{\mathbf{L}}_A \eta^{M',M''}_{N',N''} \Big\downarrow \simeq & \nearrow_{\xi^{M' \otimes_k M''}_{N' \otimes_k N''}} & \\
(M' \otimes_k M'') \otimes^{\mathbf{L}}_A \mathbf{R}\mathrm{Hom}_A(M' \otimes_k M'', N' \otimes_k N'') & &
\end{array}$$

Notice that the morphisms $\gamma^{M',M''}_{\mathbf{R}\mathrm{Hom}_{A'}(M',N'),\mathbf{R}\mathrm{Hom}_{A''}(M'',N'')}$ and $(N' \otimes_k N'') \otimes^{\mathbf{L}}_A \eta^{M',M''}_{N',N''}$ are isomorphisms by Lemma 3.1.3.3 and (3.1). Thus, $\xi^{M' \otimes_k M''}_{N' \otimes_k N''}$ an isomorphism in $D(A)$ implies $\xi^{M'}_{N'} \otimes_k \xi^{M''}_{N''}$ is an isomorphism in $D(A)$. Therefore, by Lemma 2.6.3.6, since N', $N'' \not\simeq 0$, the fact that $\xi^{M'}_{N'} \otimes_k \xi^{M''}_{N''}$ is an isomorphism in $D(A)$ implies $\xi^{M'}_{N'}$ and $\xi^{M''}_{N''}$ are isomorphisms in $D(A')$ and $D(A'')$, respectively.

□

The next two results are proved similarly to Theorem 3.2.1.2.

Theorem 3.2.1.3. *Fix $M' \in \mathfrak{G}(A')$ and $M'' \in \mathfrak{G}(A'')$ and let $N' \in D(A')$ and $N'' \in D(A'')$. If $N' \in \mathcal{A}_{M'}(A')$ and $N'' \in \mathcal{A}_{M''}(A'')$, then $N' \otimes_k N'' \in \mathcal{A}_{M' \otimes_k M''}(A)$. If N', $N'' \not\simeq 0$, then the converse holds.*

Theorem 3.2.1.4. *Fix $M' \in \mathfrak{G}(A')$ and $M'' \in \mathfrak{G}(A'')$ and let $N' \in D^f(A')$ and $N'' \in D^f(A'')$. If N' is derived M'-reflexive over A' and N'' is derived M''-reflexive over A'', then $N' \otimes_k N''$ is derived $M' \otimes_k M''$-reflexive over A. If N', $N'' \not\simeq 0$, then the converse holds.*

In the next result, we use the notation of 2.6.2.13.

Theorem 3.2.1.5. *Assume $M', N' \in \mathfrak{G}(A')$ and $M'', N'' \in \mathfrak{G}(A'')$. Then $M' \approx N'$ and $M'' \approx N''$ if and only if $M' \otimes_k M'' \approx N' \otimes_k N''$.*

Proof: This is a consequence of Theorem 3.2.1.2, since M', M'', N', $N'' \not\simeq 0$. □

Theorem 3.2.1.6. *The map $\psi : \overline{\mathfrak{G}}(A') \times \overline{\mathfrak{G}}(A'') \to \overline{\mathfrak{G}}(A)$ given by $\psi(C', C'') = C' \otimes_k C''$ is well-defined and injective.*

Proof: This follows from Theorems 3.2.1.1 and 3.2.1.5. For instance, assume $\psi(M', M'') = \psi(N', N'')$. Then $M' \otimes_k M'' \approx N' \otimes_k N''$. Thus, $M' \approx N'$ and $M'' \approx N''$ by Theorem 3.2.1.5. □

3.2.1.7 (Proof of Theorem 1.0.0.3)**.** Note that A', A'' local implies that A', $A'' \not\simeq 0$ by 2.5.1.6.

(a): This follows from Theorem 3.2.1.1.

(b): The map ψ being well-defined is due to part (a). The map ψ being injective is a special case of Theorem 3.2.1.6 due to Lemma 2.6.2.16. □

3.2.2. Three Corollaries and a Question

We conclude by documenting some special cases of the above results and a natural question.

Corollary 3.2.2.1. *Let R_i be a local k-algebra for $i = 1, 2$. Let X_i be a finitely generated R_i-module for $i = 1, 2$.*

1. *One has $X_1 \otimes_k X_2 \in \mathfrak{G}_0(R_1 \otimes_k R_2)$ if and only if $X_i \in \mathfrak{G}_0(R_i)$ for $i = 1, 2$.*
2. *The map $\psi : \mathfrak{G}_0(R_1) \times \mathfrak{G}_0(R_2) \to \mathfrak{G}_0(R_1 \otimes_k R_2)$ given by $\psi(C_1, C_2) = C_1 \otimes_k C_2$ is well-defined and injective.*

Corollary 3.2.2.2. *Let R_i be a local k-algebra for $i = 1, 2$. Let $X_i \in D^f(R_i)$ for $i = 1, 2$.*

1. *One has $X_1 \otimes_k X_2 \in \mathfrak{S}(R_1 \otimes_k R_2)$ if and only if $X_i \in \mathfrak{S}(R_i)$ for $i = 1, 2$.*
2. *The map $\psi : \mathfrak{S}(R_1) \times \mathfrak{S}(R_2) \to \mathfrak{S}(R_1 \otimes_k R_2)$ given by $\psi(C_1, C_2) = C_1 \otimes_k C_2$ is well-defined and injective.*

Corollary 3.2.2.3. *Let R_i be a k-algebra for $i = 1, 2$. Let $X_i \in D^f(R_i)$ for $i = 1, 2$. Then the map $\psi : \overline{\mathfrak{S}}(R_1) \times \overline{\mathfrak{S}}(R_2) \to \overline{\mathfrak{S}}(R_1 \otimes_k R_2)$ given by $\psi(C_1, C_2) = C_1 \otimes_k C_2$ is well-defined and injective.*

Question 3.2.2.4. Is the map ψ in Theorem 1.0.0.3 surjective?

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