MAXIMALLY EDGE-COLORED DIRECTED GRAPH ALGEBRAS

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ABSTRACT

Graph C^*-algebras are constructed using projections corresponding to the vertices of the graph, and partial isometries corresponding to the edges of the graph. Here, we use the gauge-invariant uniqueness theorem to first establish that the C^*-algebra of a graph composed of a directed cycle with finitely many edges emitting away from that cycle is $M_{n+k}(C(\mathbb{T}))$, where n is the length of the cycle and k is the number of edges emitting away. We use this result to establish the main results of the thesis, which pertain to maximally edge-colored directed graphs. We show that the C^*-algebra of any finite maximally edge-colored directed graph is $*_{M_n(\mathbb{C})}\{M_n(C(\mathbb{T}))\}_k$, where n is the number of vertices of the graph and k depends on the structure of the graph. Finally, we show that this algebra is in fact isomorphic to $M_n(*_{\mathbb{C}}\{C(\mathbb{T})\}_k)$.

ACKNOWLEDGEMENTS

I would like to sincerely thank my advisor, Dr. Benton Duncan, for dedicating a significant amount of time to this project over the past several years. His guidance and support were critical to my successful completion of this thesis.

DEDICATION

This dissertation is dedicated first to my wonderfully supportive husband, Jace Brownlee, who has been by my side from start to finish of this journey. Next, to my amazing children, Emilia and Dashel, without whom life wouldn't be complete... though without whom I would have finished this several years earlier. And finally, to our loving family, who has let me get away with saying "I'll be done in three years" for at least the past six.

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1. INTRODUCTION AND BACKGROUND

1.1. Introduction

This research is in the area of *graph algebras*, which are C^*-algebras related to row-finite directed graphs. Operator algebraists have been studying these for several decades now, since finding graph algebras to be a rich source of examples of C^*-algebras. Also, several algebras which were already being studied can now be regarded as graph algebras, opening doors to other methods of study, such as matrix algebras and the Cuntz algebras [12]. Furthermore, there are algebraic structures in these algebras that coincide with properties of the directed graphs they were generated from. For example, if the graph has a finite number of vertices, the graph algebra will be unital. There are also structural indications that indicate whether or not the graph algebra will be simple [12]. Finally, in considering amalgamated free products, it is hoped that a better understanding of these algebras can occur by relating them to graphs.

1.2. C^*-algebras

Definition 1.2.1. A C^**-algebra* is a Banach algebra $\mathscr{A}$ together with an involution (see below) such that $||a^*a|| = ||a||^2$ for all $a \in \mathscr{A}$. An *involution* is a map from $\mathscr{A}$ into $\mathscr{A}$ mapping $a \mapsto a^*$ such that $(a^*)^* = a$, $(ab)^* = b^*a^*$, and $(\lambda a + b)^* = \overline{\lambda}a^* + b^*$ for all $a, b \in \mathscr{A}$ and $\lambda \in \mathbb{C}$.

The following are some examples of C^*-algebras:

1. $\mathbb{C}$ is a commutative, unital C^*-algebra, with $z^* := \overline{z}$ and $||z|| := |z|$.

2. $C(X)$, the collection of complex-valued functions on X, where X is a compact, Hausdorff space, is a commutative, unital C^*-algebra, with $f^*(x) := \overline{f(x)}$ and $||f(x)|| := ||f(x)||_\infty$.

3. $C_0(\mathbb{R})$, the collections of functions $f : \mathbb{R} \to \mathbb{C}$ such that $\lim_{x\to\pm\infty} f(x) = 0$, is a commutative, nonunital C^*-algebra with $||f||$ and f^* as above.

4. The set of bounded operators on a Hilbert space $\mathscr{H}$,

$$\mathscr{B}(\mathscr{H}) = \{T : \mathscr{H} \to \mathscr{H} \,|\, T \text{ is bounded, linear}\},$$

is a unital, noncommutative C^*-algebra, with $||T|| = \inf\{M : ||Tx|| \leq M||x|| \text{ for all } x \in \mathscr{H}\}$ and T^* is the operator adjoint (for every $T \in \mathscr{B}(\mathscr{H})$, there is a unique $T^* \in \mathscr{B}(\mathscr{H})$ such that for every $h, g \in \mathscr{H}$, $\langle Th, g\rangle = \langle h, T^*g\rangle$. T^* is called the *operator adjoint* of T).

5. $M_n(\mathbb{C})$, the algebra of $n \times n$ matrices whose entries are complex numbers, and $M_n(C(\mathbb{T}))$, the algebra of $n \times n$ matrices whose entries are continuous functions on the unit circle of the complex plane, are C^*-algebras when considered as operators over the Hilbert spaces

$$\mathscr{H} = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} : a_i \in \mathbb{C} \right\} \text{ and } \mathscr{H} = \left\{ \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} : f_i \in L^2(\mathbb{T}) \right\}, \text{ respectively.}$$

Here, the operator adjoint is $M^* := (\overline{M})^T$, and $||M||$ is the usual operator norm.

For a more in-depth study of their features, or for more examples of C^*-algebras, see [4, 8].

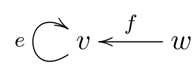
Our focus on C^*-algebras will mainly have to do with Example (5) above. Note that we think of these matrix algebras as operators; there are specifically two types of operators that we need to be familiar with in order to continue. First, an operator P is a *projection* if it satisfies the equality $P^2 = P = P^*$. We can see quickly that, for example, $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is a projection in $M_2(\mathbb{C})$. The second type of operator needed is a *partial isometry*; S is a partial isometry if SS^* and S^*S are both projections. We may recall that U is an *isometry* if $U^*U = I$, where I is the identity operator in the C^*-algebra. Thus, a partial isometry S is an isometry of its initial space onto its range space in $\mathscr{H}$. A simple example of a partial isometry is $S = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Observe that $S^* = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, so that $SS^* = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $S^*S = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are both projections.

Lastly, a bit of C^*-algebra terminology we will see throughout these results. We call $\rho : \mathscr{A} \to \mathscr{B}$, where $\mathscr{A}, \mathscr{B}$ are C^*-algebras, a $*$-*homomorphism* if it is a linear, multiplicative map for which $\rho(A^*) = \rho(A)^*$ for all $A \in \mathscr{A}$. If ρ is bijective, then it is a $*$-*isomorphism.* A $*$-*representation* π of a C^*-algebra $\mathscr{A}$ on a Hilbert space $\mathscr{H}$ is a $*$-homomorphism of $\mathscr{A}$ into $\mathscr{B}(\mathscr{H})$. The Gelfand-Naimark theorem (see, for instance, [8]) established that every C^*-algebra is isometrically $*$-isomorphic to a closed $*$-subalgebra of $\mathscr{B}(\mathscr{H})$ for some choice of $\mathscr{H}$. Hence, we can think of a $*$-representation as being simply a $*$-homomorphism for the purposes of this paper.

1.3. Directed graphs and their graph algebras

The following description is a summary of material provided in [12]; details of many of the facts stated below can be found in that reference. A *directed graph* E is a collection of vertices and edges such that each edge has a *source (vertex)* and a *range (vertex)*. In general, the notation for the collection of vertices is E^0, the collection of edges is E^1, the collection of paths of length n is E^n, and the collection of all finite-length paths in E is E^*. Taking into account that each edge has a source and a range, we define maps $s, r : E^1 \to E^0$ where $s(e)$ is the source vertex of edge e and $r(e)$ is the range vertex of edge e. For example, consider the following directed graph:

$E:$



Here, $s(e) = r(e) = r(f) = v$, and $s(f) = w$. The vertex $w \in E^0$ is an example of a *source*, which is a vertex which receives no edges.

Next, we describe our framework for assigning C^*-algebra operators to the edges and vertices of our graphs. Let E be a row-finite directed graph and $\mathscr{H}$ a Hilbert space (a *row-finite* directed graph is one in which no vertex receives infinitely many edges; the name is derived from the graph's corresponding adjacency matrix).

Definition 1.3.1. A *Cuntz-Krieger E-family* $\{S, P\}$ on $\mathscr{H}$ is a collection $\{S_e : e \in E^1\}$ of partial isometries and $\{P_v : v \in E^0\}$ of mutually orthogonal projections such that

(CK1) $S_e^* S_e = P_{s(e)}$ for all $e \in E^1$, and

(CK2) $P_v = \sum_{r(e)=v} S_e S_e^*$ for all $v \in E^0$ where v is not a source.

Here, what is being required in CK1 is that the initial space of S_e is all of $P_v\mathscr{H}$ if $s(e) = v$. In CK2 we require that the range space of P_v is the direct sum of all of the range spaces $S_e\mathscr{H}$, where $r(e) = v$. The outcome of the requirements of CK1 and CK2, known as the *Cuntz-Krieger relations*, is that moving along paths in the graph will be consistent with finding nonzero products of elements in $\{S, P\}$. We will demonstrate this shortly. Now, however, having only these requirements and properties of projections and partial isometries, the following facts are true [12]:

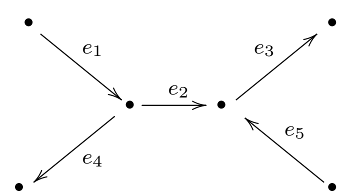
Fact 1.3.2. *For any edge $e \in E^1$, $S_e = P_{r(e)}S_e = S_e P_{s(e)}$.*

This follows from the fact that S_e is an isometry of $P_{s(e)}\mathscr{H}$ onto a closed subspace of $P_{r(e)}\mathscr{H}$. Observe that what this says is that the projections will act like identity operators when being multiplied by the appropriate partial isometry on the appropriate side. It turns out that if a projection is multiplied by any other operator in $\{S, P\}$ the result will be the zero operator.

Fact 1.3.3. *Every non-zero finite product of the partial isometries S_e and S_f^* has the form $S_\mu S_\nu^*$ for some $\mu, \nu \in E^*$ with $s(\mu) = s(\nu)$.*

This fact is not obvious, and follows from a number of propositions and corollaries. However, this result is consistent with our claim that the Cuntz-Krieger relations CK1 and CK2 essentially require that E-families behave in a way that corresponds to moving along paths in a graph.

Example 1.3.4. As an illustration of what we mean by E-families behaving in a way that corresponds to moving along paths in a graph, we consider the following graph G:



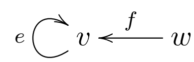
Notice that we have left out any labels for the vertices, as the facts above eliminate the need to label them in this case. Now, if one looks to the Cuntz-Krieger relations and the facts above, one would find that the product $S_{e_3}S_{e_2}S_{e_4}^*$ must be nonzero, since we can follow e_4 backwards to the source of e_2, and then move along the path e_3e_2 in graph G (two observations here: the operator adjoint corresponds to moving backwards along an edge, and, when looking at a product of partial isometries or a path in a graph, we read the edges in reverse order; this is due to the convention of operator use). Similarly, $S_{e_3}S_{e_2}S_{e_5}^* = 0$ must be the case, since it does not make sense to follow e_5 backwards and then go forward along path e_3e_2.

Finally, we are concerned with building a C^*-algebra from a Cuntz-Krieger E-family; in general, $C^*(S, P)$ is the C^*-algebra generated by the Cuntz-Krieger E-family $\{S, P\}$. Because of the $*$-algebraic consequences of the Cuntz-Krieger relations, a series of corollaries to those and to the facts above lead to the main corollary below [12, Corollary 1.16]:

Corollary 1.3.5. *If $\{S, P\}$ is a Cuntz-Krieger E-family for a row-finite graph E, then* $C^*(S, P) = \overline{span}\{S_\mu S_\nu^* : \mu, \nu \in E^*, s(\mu) = s(\nu)\}$.

Example 1.3.6. One example of such a C^*-algebra is for the graph E seen previously:

$E :$



We can define a Cuntz-Krieger E-family on $\mathscr{H} = \ell^2$ for $\vec{x} = (x_0, x_1, x_2, \dots)$ as follows:

$$P_v(\vec{x}) = (0, x_1, x_2, x_3, \dots), P_w(\vec{x}) = (x_0, 0, 0, 0, \dots),$$

$$S_e(\vec{x}) = (0, 0, x_1, x_2, \dots), \text{ and } S_f(\vec{x}) = (0, x_0, 0, 0, 0, \dots).$$

With S_e and S_f defined this way, one can check that $S_e^*(\vec{x}) = (0, x_2, x_3, x_4, \dots)$ and $S_f^*(\vec{x}) = (x_1, 0, 0, 0, \dots)$. The Cuntz-Krieger relations require that $S_e^* S_e = P_v$, $S_f^* S_f = P_w$, and $P_v = S_e S_e^* + S_f S_f^*$. These are straight-forward to check; for example,

$$\begin{aligned} S_e^* S_e(\vec{x}) &= S_e^*(0, 0, x_1, x_2, \dots) \\ &= (0, x_1, x_2, x_3, \dots) \\ &= P_v(\vec{x}). \end{aligned}$$

Thus, the set $\{S, P\}$ is a Cuntz-Krieger E-family. It can be shown that in fact $S_e + S_f$ is enough to generate all of $C^*(S, P)$, since all four of the operators above can be recovered from this single operator (for example, check that that $(S_e + S_f)(S_e + S_f)^* = P_v$). Hence, $C^*(S, P) = C^*(S_e + S_f)$.

There are natural questions that arise at this point: (1) Can there be (several) different Cuntz-Krieger E-families for a directed graph E, and, if yes, (2) will these E-families generate isomorphic C^*-algebras? We can answer both questions affirmatively, although the answer to question (2) is not yes in all cases. To begin to answer this question, we need to introduce a C^*-algebra $C^*(E)$ that is universal for C^*-algebras generated by Cuntz-Krieger E-families, and which is always generated by an E-family labeled $\{S, P\}$. The following proposition is not obvious, and the proof can be found in [12].

Proposition 1.3.7. *For any row-finite directed graph E, there is a C^*-algebra $C^*(E)$ generated by a Cuntz-Krieger E-family $\{S, P\}$ such that for every Cuntz-Krieger E-family $\{T, Q\}$ in a C^*-algebra $\mathscr{B}$, there is a homomorphism $\pi_{T,Q}$ of $C^*(E)$ into $\mathscr{B}$ satisfying $\pi_{T,Q}(S_e) = T_e$ for every $e \in E^1$ and $\pi_{T,Q}(P_v) = Q_v$ for every $v \in E^0$.*

The C^*-algebra $C^*(E)$ is called the *C^*-algebra of the graph E*, and generically is called the *graph algebra*. The following corollary justifies our use of the word *the* to describe the graph algebra, demonstrating that it is unique up to isomorphism. Once again, the proof can be found in [12].

Corollary 1.3.8. *Suppose E is a row-finite directed graph, and $\mathscr{C}$ is a C^*-algebra generated by a Cuntz-Krieger E-family $\{W, R\}$ such that for every Cuntz-Krieger E-family $\{T, Q\}$ in a C^*-algebra $\mathscr{B}$, there is a homomorphism $\rho_{T,Q}$ of $\mathscr{C}$ into $\mathscr{B}$ satisfying $\rho_{T,Q}(W_e) = T_e$ for every $e \in E^1$ and $\rho_{T,Q}(R_v) = Q_v$ for every $v \in E^0$. Then, there is an isomorphism ϕ of $C^*(E)$ onto $\mathscr{C}$ such that $\phi(S_e) = W_e$ for every $e \in E^1$ and $\phi(P_v) = R_v$ for every $v \in E^0$.*

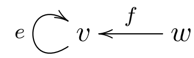
1.4. Uniqueness theorems

Now, because of Corollary 1.3.8, we know that the C^*-algebra of a graph E has a universal property. Hence, we can prove that a C^*-algebra $\mathscr{B}$ is isomorphic to $C^*(E)$ by finding a Cuntz-Krieger E-family $\{T, Q\}$ which generates $\mathscr{B}$ and has the universal property. Fortunately, the following results tell us that it is often not necessary to check that $\{T, Q\}$ has the universal property. The first is limited to only certain types of graphs, and is essentially due to Cuntz and Krieger, as the name implies [12, Theorem 2.4][7].

Theorem 1.4.1 (The Cuntz-Krieger uniqueness theorem)**.** *Suppose E is a row-finite directed graph in which every cycle has an entry, and $\{T, Q\}$ is a Cuntz-Krieger E-family in a C^*-algebra $\mathscr{B}$ such that $Q_v \neq 0$ for every $v \in E^0$. Then the homomorphism $\pi_{T,Q} : C^*(E) \to \mathscr{B}$ is an isomorphism of $C^*(E)$ onto $C^*(T, Q)$.*

The first thing to note here is that we require that "every cycle [in E] has an entry." A *cycle* in a directed graph is any path which starts at and returns to the same vertex, and where no edges in the cycle share the same source vertex; an edge e is an *entry* to a cycle if e is not part of the cycle, and it has the same range vertex as an edge in the cycle. This is actually quite restrictive, but if a graph satisfies this requirement, then *any* Cuntz-Krieger E-family will satisfy the universal property. Thus, we need only to find one Cuntz-Krieger E-family; its corresponding C^*-algebra will be $C^*(E)$. For example, again, back to our familiar graph E:

E :



Recall that we found a Cuntz-Krieger E-family, $\{S, P\}$, and $C^*(S, P) = C^*(S_e + S_f)$. Since the only cycle in E is e, and f is an entry into that cycle, by the Cuntz-Krieger uniqueness theorem we know that $C^*(E) = C^*(S_e + S_f)$.

Suppose, however, that a graph has a cycle which has no entry; then the Cuntz-Krieger uniqueness theorem is not useful to us. Fortunately, there is a fix for that as well. We begin by describing what is known as a *gauge action*. In general, an *action* of a locally compact group G on a C^*-algebra $\mathscr{A}$ is a homomorphism $s \mapsto \alpha_s$ of G into the group $\operatorname{Aut}\mathscr{A}$ of automorphisms of $\mathscr{A}$ such that $s \mapsto \alpha_s(a)$ is continuous for each fixed $a \in \mathscr{A}$. The gauge action is a particular action of $\mathbb{T}$ on $C^*(E)$, and is described in the following proposition, then used in the main theorem below [12, Theorem 2.2]:

Proposition 1.4.2. *Let E be a row-finite directed graph with $C^*(E)$ generated by $\{S, P\}$. Then there is an action γ of $\mathbb{T}$ on $C^*(E)$ such that for all $w \in \mathbb{T}$, $\gamma_w(S_e) = wS_e$ for every $e \in E^1$ and $\gamma_w(P_v) = P_v$ for every $v \in E^0$.*

Theorem 1.4.3 (The gauge-invariant uniqueness theorem)**.** *Let E be a row-finite directed graph, and suppose that $\{T, Q\}$ is a Cuntz-Krieger E-family in a C^*-algebra $\mathscr{B}$ with each $Q_v \neq 0$. If there is a continuous action $\beta : \mathbb{T} \to Aut\mathscr{B}$ such that for all $w \in \mathbb{T}$, $\beta_w(T_e) = wT_e$ for every $e \in E^1$ and $\beta_w(Q_v) = Q_v$ for every $v \in E^0$, then $\pi_{T,Q}$ is an isomorphism of $C^*(E)$ onto $C^*(T, Q)$.*

To summarize what we just saw, the proposition guarantees that there is a gauge action on the graph algebra for any row-finite directed graph E. The gauge-invariant uniqueness theorem tells us that if we can find such an action on a Cuntz-Krieger E-family for our graph, this E-family *must* generate a C^*-algebra isomorphic to the graph algebra. That is, the C^*-algebra $C^*(S, P)$, where $\{S, P\}$ is the aforementioned E-family, is the graph algebra $C^*(E)$, up to isomorphism. This theorem was originally stated in [1] and is restated and proved in full in [12].

The following are a few well-known results that we will need throughout the course of the thesis. The proof of the first is a good example of the use of Theorem 1.4.1, and the next uses Theorem 1.4.3 in a way that is similar to the method we will use to prove Proposition 2.1.1:

Proposition 1.4.4. *Let G be a rooted tree (that is, there is a vertex acting as a root, and all edges are directed away from the root) with n vertices. Then $C^*(G) = M_n(\mathbb{C})$.*

Proof. Begin by labeling the vertices of G as $v_1, v_2, \ldots, v_n$, and edges $e_1, e_2, \ldots, e_{n-1}$ in any fashion (order is not important). Define the projections and partial isometries as follows: for each vertex v_i, let $P_{v_i} = e_{ii}$; define $S_{e_i} = e_{jk}$ where $s(e_i) = v_k$ and $r(e_i) = v_j$. We claim that $\{S, P\}$ is a Cuntz-Krieger G-family.

That $P_{v_i} = e_{ii}$ is a projection for every i has been established, and these are mutually orthogonal (check that $P_{v_i}P_{v_j} = 0$ for all i, j with $i \neq j$). Next, $S_{e_i} = e_{jk}$ means $S_{e_i}^* = e_{kj}$ so that $S_{e_i}^* S_{e_i} = e_{kk} = P_{v_k}$ and $S_{e_i} S_{e_i}^* = e_{jj} = P_{v_j}$ are projections as well. Thus, we have a family of projections and partial isometries. We now check the Cuntz-Krieger relations on $\{S, P\}$. First, $S_{e_i}^* S_{e_i} = e_{kk} = P_{v_k} = P_{s(e_i)}$ for all i, so CK1 is satisfied. For CK2, consider all vertices which are not a source (notice that the root of the tree will be a source). For each of these vertices, $S_{e_i} S_{e_i}^* = e_{jj} = P_{v_j} = P_{r(e_i)}$, so CK2 is satisfied as well. Thus, $\{S, P\}$ is a Cuntz-Krieger G-family.

Now, G does not have any cycles since G is a tree, so G satisfies the requirements of the Cuntz-Krieger uniqueness theorem. Therefore, since $\{S, P\}$ is a Cuntz-Krieger G-family with $P_v \neq 0$ for all $v \in G^0$, we have that $C^*(G) = C^*(S, P)$. It is easy to check that $\{S, P\}$ will generate all of $M_n(\mathbb{C})$. Hence, $C^*(G) = M_n(\mathbb{C})$. $\square$

Proposition 1.4.5. *Let G be a directed cycle of length n. Then $C^*(G) = M_n(C(\mathbb{T}))$.*

Proof. Let the vertices $\{v_1, v_2, \ldots, v_n\}$ and edges $\{e_1, e_2, \ldots, e_n\}$ be labeled in such a way that $s(e_i) = v_i$, and $r(e_i) = v_{i+1}$, with $r(e_n) = v_1$. Consider the C^*-algebra $M_n(C(\mathbb{T}))$; we set $P_{v_i} = 1e_{ii}$, $S_{e_i} = 1e_{(i+1)i}$ for $i < n$, and $S_{e_n} = ze_{1n}$, where z is the unitary generator of $C(\mathbb{T})$, and $z^* := \overline{z}$. Then we claim that $\pi_{S,P} : C^*(G) \to M_n(C(\mathbb{T}))$ is an isomorphism.

First, we check that $\{S, P\}$ is a Cuntz-Krieger G-family. Certainly $P_{v_i} = 1e_{ii}$ is a projection for every i, and these are mutually orthogonal (check that $P_{v_i}P_{v_j} = 0$ for all i, j with $i \neq j$); with $S_{e_i} = 1e_{(i+1)i}$ for $i < n$, we have $S_{e_i}^* = 1e_{i(i+1)}$ so that $S_{e_i}^* S_{e_i} = P_{v_i}$ and $S_{e_i} S_{e_i}^* = P_{v_{i+1}}$ are projections for all $i < n$. Also, since $z\overline{z} = 1$, we have $S_{e_n}^* S_{e_n} = P_{v_n}$ and $S_{e_n} S_{e_n}^* = P_{v_1}$ are projections as well. Thus, we have a family of projections and partial isometries. We now check the Cuntz-Krieger relations on $\{S, P\}$. Since, by construction, $s(e_i) = v_i$ for all $i \leq n$, we have already seen above that $S_e^* S_e = P_{s(e)}$ for all $e \in G^1$, so CK1 is satisfied. For CK2, notice that v_i is not a source for any i, but also that $v_i = r(e)$ for exactly one edge e for each i. By construction, $r(e_i) = v_{i+1}$ for all $i < n$, and we see above that $S_{e_i} S_{e_i}^* = P_{v_{i+1}}$ for $i < n$. Finally, $S_{e_n} S_{e_n}^* = P_{v_1}$ was also established above, and $r(e_n) = v_1$. Hence, $\{S, P\}$ is a Cuntz-Krieger G-family.

Next, we claim that the range of $\pi_{S,P}$ contains all of $M_n(C(\mathbb{T}))$. Since e_{ij} can be factored as a product involving arbitrarily many copies of $e_{1n} e_{n(n-1)} \cdots e_{21}$, we have that every matrix of the form $z^m e_{ij}$ is in $C^*(S, P)$ for all $m \in \mathbb{Z}$ (taking adjoints for $m < 0$). Thus, the range of $\pi_{S,P}$ contains all matrices of trigonometric polynomials. We use the sup norm topology on $C(\mathbb{T})$. The unit circle is compact, and the trigonometric polynomials separate points of $C(\mathbb{T})$ in this topology. Hence, by Stone-Weierstrauss, the trigonometric polynomials are dense in $C(\mathbb{T})$. Thus, the range of $\pi_{S,P}$ contains all of $M_n(C(\mathbb{T}))$.

We now wish to find a gauge action on $M_n(C(\mathbb{T}))$. For fixed $w \in \mathbb{T}$, let $U_w \in M_n(\mathbb{C})$ be defined as $U_w := \sum_{j=1}^n w^j e_{jj}$, and define β_w by

$$\beta_w(f_{ij}(z)) = U_w(f_{ij}(w^n z))U_w^*$$

(that $\beta_w \in \mathrm{Aut}(M_n(C(\mathbb{T})))$ is immediate, where $\beta_w^{-1}(f_{ij}(z)) = U_w(f_{ij}(w^{-n} z))U_w^*$). Then, since e_{ii} commutes with U_w for all i,

$$\beta_w(P_{v_i}) = \beta_w(1e_{ii}) = U_w(1e_{ii})U_w^* = U_w U_w^*(1e_{ii}) = 1e_{ii} = P_{v_i}.$$

Next, for $i < n$, we have,

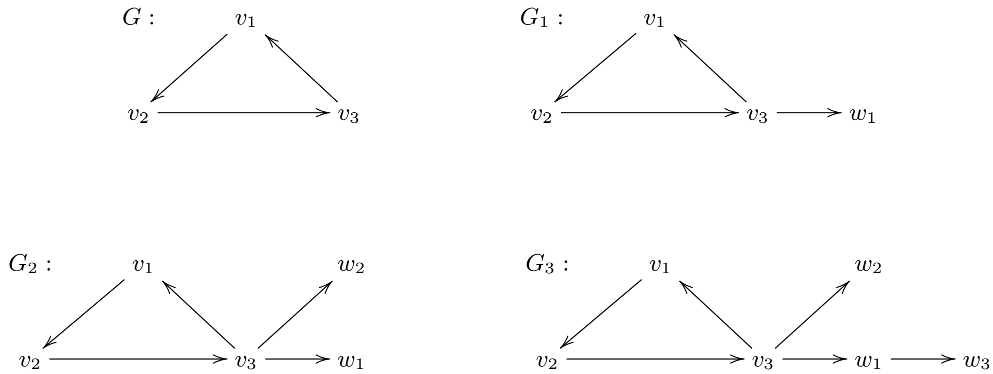
$$\begin{aligned}\beta_w(S_{e_i}) = \beta_w(1e_{(i+1)i}) &= \sum_{j=1}^{n} w^j e_{jj}(1e_{(i+1)i})U_w^* \\ &= w^{i+1}(1e_{(i+1)i})\sum_{k=1}^{n} w^{-k}e_{kk} \\ &= w^{i+1}(1e_{(i+1)i})w^{-i} \\ &= w(1e_{(i+1)i}) \\ &= wS_{e_i}.\end{aligned}$$

Finally, the w^n in the evaluation of the f_{ij} comes into play when we check $\beta_w(S_{e_n})$:

$$\begin{aligned}\beta_w(S_{e_n}) = \beta_w(ze_{1n}) &= \sum_{j=1}^{n} w^j e_{jj}(w^n ze_{1n})U_w^* \\ &= w^{n+1}ze_{1n}\sum_{k=1}^{n} w^{-k}e_{kk} \\ &= wze_{1n} \\ &= wS_{e_n}.\end{aligned}$$

Thus, we have shown that $\beta : \mathbb{T} \to \mathrm{Aut}(M_n(C(\mathbb{T}))$ is a continuous action such that $\beta_w(S_e) = wS_e$ for all $e \in G^1$ and $\beta_w(P_v) = P_v$ for all $v \in G^0$. Therefore, by the gauge-invariant uniqueness theorem, $\pi_{S,P}$ is an isomorphism of $C^*(G)$ onto $C^*(S,P) = M_n(C(\mathbb{T}))$. Hence, the C^*-algebra for a directed n-cycle is $M_n(C(\mathbb{T}))$. □

And finally, some terminology which will need to be used as well: Let E be a directed graph. We will say that we are *adding an outward pointing edge* to E when we mean that a new edge e has been added, with $s(e) = v$ for a vertex $v \in E^0$, and $r(e) = w$ for a new vertex w which was not in E^0. Furthermore, we will say that we have *iterated the process of adding an outward pointing edge to E k times* when we mean that we have added an outward pointing edge to E, and then added an additional outward pointing edge to the resulting graph (iterated two times thus far), and added an additional outward pointing edge to the result (three times), etc., until k new edges have been added. An illustration follows below:



In this illustration, we begin with the 3-cycle G. The graph G_1 has been built by adding an outward pointing edge to G. Next, an outward pointing edge is added to G_1, so the process of adding an outward pointing edge to G has been iterated twice. Finally, we can build G_3 by iterating the process of adding an outward pointing edge to G three times. To be clear, at each step, the outward pointing edge could have been added at *any* of the vertices which were already present.

1.5. Edge-colored graph algebras

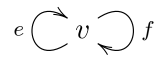
Much of the following background material was introduced by Duncan [10], and when it comes from elsewhere it will be clearly referenced. Here we give precise definitions pertaining to edge-colored graph algebras, and state a few theorems which will be needed as foundation for the results of Chapter 3.

We begin by building on the definition of a Cuntz-Krieger E-family.

Definition 1.5.1. Let S be a collection of partial isometries, P be a collection of pairwise orthogonal projections, and $f : S \to \mathbb{N}$ a function correlating to an edge-coloring f on E (we interchange f to mean an edge-coloring and a coloring of partial isometries). We say that $\{S, P, f\}$ is an *edge-colored Cuntz-Krieger E-family* on $\mathscr{H}$ if $\{f^{-1}(n), P\}$ is a Cuntz-Krieger family on $\mathscr{H}$ for each $n \in \mathbb{N}$.

We observe that any Cuntz-Krieger family will be an edge-colored Cuntz-Krieger family if we just color all of the edges the same color (that is, for example, $f(S_e) = 1$ for all $e \in E^1$). However, it is not the case that every edge-colored Cuntz-Krieger family is a Cuntz-Krieger family.

Consider the following graph:

$E:$ 

Define S_e and S_f to be partial isometries such that $S_e^*S_e = S_f^*S_f = S_eS_e^* = S_fS_f^* = P_v$, with $f(S_e) = 1$ and $f(S_f) = 2$. Then $\{S, P, f\}$ is an edge-colored Cuntz-Krieger E-family but is not a Cuntz-Krieger E-family (since CK2 is not satisfied when the colors are taken away).

Next, we define a universal property for an algebra. Notice that given an edge-colored Cuntz-Krieger family $\{S, P, f\}$ associated to an edge-colored directed graph $\{G, f\}$, it will generate a C^*-algebra, which we will call $C^*(S, P, f)$.

Definition 1.5.2. We say that a C^*-algebra $\mathscr{A}$ is *universal* for an edge-colored directed graph $\{G, f\}$ if

- $\mathscr{A}$ is generated by an edge-colored Cuntz-Krieger family $\{S, P, f\}$ associated to $\{G, f\}$, and
- given any edge-colored Cuntz-Krieger family $\{T, Q, g\}$ associated to $\{G, f\}$, there is a $*$-representation $\pi : \mathscr{A} \to C^*(T, Q, g)$.

If such a universal algebra exists, we will call it $C^*(G, f)$.

Before establishing the existence of such an algebra, the following definitions are necessary. These are a restatement of the definitions cited in [2], which they have credited to Voiculescu [13]; for further reading beyond these texts, see also [3].

Definition 1.5.3. The *reduced amalgamated (free) product* $(\mathscr{A}, \Phi)$ of a nonempty family $(\mathscr{A}_i, \Phi_i)_{i\in I}$ of unital C^*-algebras containing a unital subalgebra $\mathscr{A}_0$ with conditional expectations $\Phi_i : \mathscr{A}_i \to \mathscr{A}_0$ is uniquely determined by the following conditions:

1. $\mathscr{A}$ is a unital C^*-algebra, and there are unital $*$-homomorphisms $\sigma_i : \mathscr{A}_i \to \mathscr{A}$ such that $\sigma_i|_{\mathscr{A}_0} = \sigma_j|_{\mathscr{A}_0}$ for all $i, j \in I$. Moreover, the map $\sigma_i|_{\mathscr{A}_0}$ is injective and we identify $\mathscr{A}_0$ with its image in $\mathscr{A}$ through this map.

2. $\mathscr{A}$ is generated by $\bigcup_{i\in I} \sigma_i(\mathscr{A}_i)$.

3. $\Phi : \mathscr{A} \to \mathscr{A}_0$ is a conditional expectation such that $\Phi \circ \sigma_i = \Phi_i$ for all $i \in I$.

4. For $(i_1, \ldots, i_n) \in \Lambda(I)$ and $a_j \in \ker \Phi_{i_j}$, we have $\Phi(\sigma_{i_1}(a_1) \cdots \sigma_{i_n}(a_n)) = 0$. Here, $\Lambda(I)$ denotes the set of all finite tuples $(i_1, \ldots, i_n)$ with $i_j \in I$ for all j such that $i_j \neq i_{j+1}$ for $j = 1, \ldots, n-1$ (hence, for example, $(2,3,1,3,1,2) \in \Lambda(I)$ for $I = \{1,2,3\}$).

5. If $c \in \mathscr{A}$ such that $\Phi(a^*c^*ca) = 0$ for all $a \in \mathscr{A}$, then $c = 0$.

Definition 1.5.4. The *full amalgamated (free) product* $*_{\mathscr{A}_0}\mathscr{A}_i$ satisfies (1) and (2) above.

The general notation of the free product of unital C^*-algebras $\mathscr{A}_i$ for $i \in I$ amalgamated over $\mathscr{A}_0$, where $\mathscr{A}_0$ is a subalgebra of $\mathscr{A}_i$ for all i, is $*_{\mathscr{A}_0}\mathscr{A}_i$. By (1), there is a unique $*$-homomorphism $\xi : *_{\mathscr{A}_0}\mathscr{A}_i \to \mathscr{A}$ such that $\sigma_i = \xi \circ \gamma_i$, where $\gamma_i : \mathscr{A}_i \to *_{\mathscr{A}_0}\mathscr{A}_i$ are the canonical maps, and by (2) this map is surjective [2]. Observe that if $\mathscr{B}$ is any other algebra satisfying (1-5) (or (1-2)), then the free product maps onto $\mathscr{B}$. In other words, $\mathscr{A}$ is the largest C^*-algebra satisfying (1-5) (or (1-2)).

We remind the reader here that a *conditional expectation* from $\mathscr{A}$ onto $\mathscr{B}$ (where $\mathscr{B} \subset \mathscr{A}$ are C^*-algebras) is a contractive completely positive projection ρ such that $\rho(bxb') = b\rho(x)b'$ for every $x \in \mathscr{A}$ and $b, b' \in \mathscr{B}$. In practice, the following theorem, due to Tomiyama, is often applied to verify a linear map is a conditional expectation [5, Theorem 1.5.10]:

Theorem 1.5.5. *Let $\mathscr{B} \subset \mathscr{A}$ be C^*-algebras and ρ be a projection from $\mathscr{A}$ onto $\mathscr{B}$. Then, the following are equivalent:*

a. ρ is a conditional expectation;

b. ρ is a contractive completely positive map;

c. ρ is contractive.

The following theorem now establishes that such a universal algebra does exist [10, Theorem 1].

Theorem 1.5.6. *Given an edge-colored directed graph $\{G, f\}$, the algebra $C^*(G, f)$ exists. In particular, given an edge-colored directed graph $\{G, f\}$ there is an edge-colored Cuntz-Krieger family associated to $\{G, f\}$.*

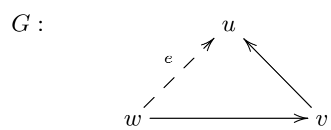
Sketch of proof. Let G_i denote the directed graph $\{G^0, f^{-1}(i), r, s\}$ where r, s are restrictions of the range and source maps of G; then $G = \cup G_i$. If P_i denotes the collection of projections in G_i

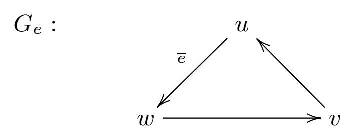
associated to its vertices, then we see a natural $*$-isomorphism between the P_i's, and will call this subalgebra P. We claim that $C^*(G, f) = *_P C^*(G_i)$, and denote the usual Cuntz-Krieger family for $C^*(G_i)$ by $\{S_i, P\}$. Define an edge-colored Cuntz-Krieger family $\{\cup S_i, P, f\}$ where $f(S_e) = i$ for $S_e \in S_i$. Then the graph associated to $\{\cup S_i, P, f\}$ will be $\{G, f\}$ and the result follows by applying universal properties for the free product to verify the universal property listed above. $\square$

Finally, one more proposition we will need to reference in Chapter 3; this result is due to Duncan [11, Proposition 3]:

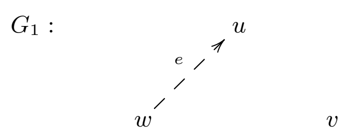
Proposition 1.5.7. *Let $\{G, f\}$ be an edge-colored directed graph with $e \in G^1$. Construct a new graph by reversing the edge e, with $\bar{e}$ the resulting edge; call that graph G_e. Define a new coloring f_e by $f_e(g) := f(g)$ for all $g \in G^1 \setminus \{e\}$, and $f_e(e) := k + 1$, where $k = max\{f(g) : r(g) = r(\bar{e})\}$. If $f(e) \neq f(g)$ for any edge g with $r(g) = r(e)$, then $C^*(G, f)$ is isomorphic to $C^*(G_e, f_e)$.*

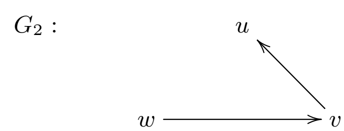
Example 1.5.8. To illustrate Proposition 1.5.7, consider the following graphs G and G_e:





Let $D = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} : a, b, c \in \mathbb{C} \right\}$. Then D is the C^*-algebra generated by the projections $e_{ii} \in M_3(\mathbb{C})$, which are the projections corresponding to vertices $u, v, w \in G^0$. Consider graphs G_1 and G_2 (notice G_i is as described in the proof of Theorem 1.5.6 above for each i):





The graph algebra for G_1 is $\begin{bmatrix} M_2(\mathbb{C}) & 0 \\ 0 & \mathbb{C} \end{bmatrix}$, where this denotes all matrices of the form $\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix}$ where $a, b, c, d, e \in \mathbb{C}$ (see, for example, [10, Page 5]). The graph algebra for G_2 is $M_3(\mathbb{C})$, as seen above in Proposition 1.4.4. Hence, as seen in the proof of Theorem 1.5.6 above, we have

$$C^*(G, f) = \begin{bmatrix} M_2(\mathbb{C}) & 0 \\ 0 & \mathbb{C} \end{bmatrix} *_D M_3(\mathbb{C}).$$

We know from Proposition 1.4.5 that $C^*(G_e, f_e) = M_3(C(\mathbb{T}))$. Thus, by Proposition 1.5.7, we have that $C^*(G, f) \cong C^*(G_e, f_e)$, or more specifically,

$$\begin{bmatrix} M_2(\mathbb{C}) & 0 \\ 0 & \mathbb{C} \end{bmatrix} *_D M_3(\mathbb{C}) \cong M_3(C(\mathbb{T})).$$

Observe that we see here a concrete example of how edge-colored graph algebras might give us a better understanding of amalgamated free products.

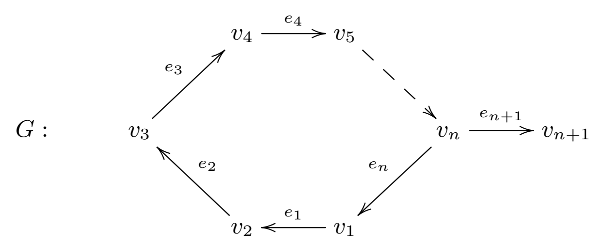
2. DIRECTED GRAPH ALGEBRAS CONTAINING A SINGLE CYCLE

The results in this chapter do not involve any edge-colorings. Here, we generalize the graph algebra for a graph consisting of a single directed cycle with outward pointing edges added finitely many times. This result will be used to prove one of the main results of Chapter 3.

2.1. Graph algebra of a single cycle plus one edge

Lemma 2.1.1. *Let C_n be the directed cycle of length n, and let G be the graph composed of C_n with the addition of an outward pointing edge. Then $C^*(G) \cong M_{n+1}(C(\mathbb{T}))$.*

Proof. Let the vertices $\{v_1, v_2, \ldots, v_n, v_{n+1}\}$ and edges $\{e_1, e_2, \ldots, e_n, e_{n+1}\}$ be labeled in such a way that $s(e_i) = v_i$ for $i = 1, \ldots, n$, $s(e_{n+1}) = v_n$, $r(e_i) = v_{i+1}$ for $i = 1, \ldots, n-1$, $r(e_n) = v_1$, and $r(e_{n+1}) = v_{n+1}$.



We will now consider the C^*-algebra $M_{n+1}(C(\mathbb{T}))$; we set $P_{v_i} = 1e_{ii}$ for all i, $S_{e_i} = 1e_{(i+1)i}$ for all $i < n$, $S_{e_n} = ze_{1n}$ (where z is the unitary generator of $C(\mathbb{T})$ with $z^* = \overline{z}$), and $S_{e_{n+1}} = 1e_{(n+1)n}$. We claim that $\pi_{S,P} : C^*(G) \to M_{n+1}(C(\mathbb{T}))$ is an isomorphism.

First, we check that $\{S, P\}$ is a Cuntz-Krieger G-family. Certainly $P_{v_i} = 1e_{ii}$ is a projection for every i, and these are mutually orthogonal (check that $P_{v_i}P_{v_j} = 0$ for all i, j with $i \neq j$). With $S_{e_i} = 1e_{(i+1)i}$ for $i < n$, we have $S_{e_i}^* = 1e_{i(i+1)}$ so that $S_{e_i}^* S_{e_i} = P_{v_i}$ and $S_{e_i} S_{e_i}^* = P_{v_{i+1}}$ are projections for all $i < n$. We can see similarly that $S_{e_{n+1}}^* S_{e_{n+1}} = P_{v_n}$ and $S_{e_{n+1}} S_{e_{n+1}}^* = P_{v_{n+1}}$ are projections. Also, since $z\overline{z} = 1$, we have $S_{e_n}^* S_{e_n} = P_{v_n}$ and $S_{e_n} S_{e_n}^* = P_{v_1}$ are projections as well. Thus, we have a family of projections and partial isometries. We now check the Cuntz-Krieger relations on $\{S, P\}$. Since, by construction, $s(e_i) = v_i$ for all $i \leq n$ and $s(e_{n+1}) = v_n$, we have already seen above that $S_e^* S_e = P_{s(e)}$ for all $e \in G^1$, so CK1 is satisfied. For CK2, notice that v_i is not a source for any i, but also that $v_i = r(e)$ for exactly one edge e for each i. By construction,

$r(e_i) = v_{i+1}$ for all $i < n$, and we see above that $S_{e_i}S_{e_i}^* = P_{v_{i+1}}$ for $i < n$. Also, $S_{e_{n+1}}S_{e_{n+1}}^* = P_{v_{n+1}}$, and we have $r(e_{n+1}) = v_{n+1}$. Finally, $S_{e_n}S_{e_n}^* = P_{v_1}$ was also established, and $r(e_n) = v_1$. Hence, $\{S, P\}$ is a Cuntz-Krieger G-family.

Next, we claim that the range of $\pi_{S,P}$ contains all of $M_{n+1}(C(\mathbb{T}))$. Since e_{ij} can be factored as a product involving arbitrarily many copies of $e_{1(n+1)}e_{(n+1)n}\cdots e_{21}$, we have that every matrix of the form $z^m e_{ij}$ is in $C^*(S, P)$ for all $m \in \mathbb{Z}$ (taking adjoints for $m < 0$). Thus, the range of $\pi_{S,P}$ contains all matrices of trigonometric polynomials. We use the sup norm topology on $C(\mathbb{T})$. The unit circle is compact, and the trigonometric polynomials separate points of $C(\mathbb{T})$ in this topology. Hence, by Stone-Weierstrauss, the trigonometric polynomials are dense in $C(\mathbb{T})$. Thus, the range of $\pi_{S,P}$ contains all of $M_{n+1}(C(\mathbb{T}))$.

We now wish to find a gauge action on $M_{n+1}(C(\mathbb{T}))$. For fixed $w \in \mathbb{T}$, let $U_w \in M_{n+1}(C(\mathbb{T}))$ be defined as $U_w := \sum_{j=1}^{n+1} w^j e_{jj}$, and define β_w by

$$\beta_w(f_{ij}(z)) = U_w(f_{ij}(w^n z))U_w^*$$

(that $\beta_w \in \mathrm{Aut}(M_{n+1}(C(\mathbb{T})))$ is immediate, where $\beta_w^{-1}(f_{ij}(z)) = U_w(f_{ij}(w^{-n}z))U_w^*$).
Then, since e_{ii} commutes with U_w for all i,

$$\beta_w(P_{v_i}) = \beta_w(1e_{ii}) = U_w(1e_{ii})U_w^* = U_wU_w^*(1e_{ii}) = 1e_{ii} = P_{v_i}.$$

Next, for $i < n$, we have,

$$\begin{aligned}
\beta_w(S_{e_i}) = \beta_w(1e_{(i+1)i}) &= \sum_{j=1}^{n+1} w^j e_{jj}(1e_{(i+1)i})U_w^* \\
&= w^{i+1}(1e_{(i+1)i})\sum_{k=1}^{n+1} w^{-k}e_{kk} \\
&= w^{i+1}(1e_{(i+1)i})w^{-i} \\
&= w(1e_{(i+1)i}) \\
&= wS_{e_i}.
\end{aligned}$$

Similarly, the result above follows for $S_{e_{n+1}}$. Finally, the w^n in the evaluation of the f_{ij} comes into play when we check $\beta_w(S_{e_n})$:

$$\begin{aligned}\beta_w(S_{e_n}) = \beta_w(ze_{1n}) &= \sum_{j=1}^{n} w^j e_{jj}(w^n ze_{1n})U_w^* \\ &= w^{n+1} ze_{1n} \sum_{k=1}^{n} w^{-k} e_{kk} \\ &= wze_{1n} \\ &= wS_{e_n}.\end{aligned}$$

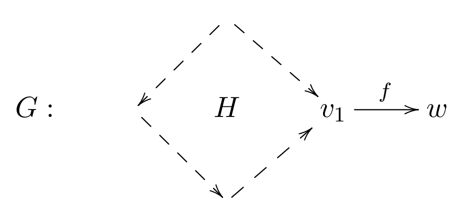
Thus, we have shown that $\beta : \mathbb{T} \to \mathrm{Aut}(M_{n+1}(C(\mathbb{T})))$ is a continuous action (in the strong operator topology) such that $\beta_w(S_e) = wS_e$ for all $e \in G^1$ and $\beta_w(P_v) = P_v$ for all $v \in G^0$. Therefore, by the gauge-invariant uniqueness theorem, $\pi_{S,P}$ is an isomorphism of $C^*(G)$ onto $C^*(S, P) = M_{n+1}(C(\mathbb{T}))$. Hence, the C^*-algebra for graph G as described above is $M_{n+1}(C(\mathbb{T}))$. □

2.2. Representing one graph algebra in terms of another

The following two results involve representing one graph algebra in terms of another graph algebra, where one graph is a specially chosen subgraph of the other graph. The latter will be used to prove the main result of the chapter.

Proposition 2.2.1. *Let H be a row-finite directed graph with vertices $\{v_i\}$ and edges $\{e_j\}$. Let G be composed of the graph H with the addition of an outward pointing edge f to vertex w. Suppose $C^*(G)$ is generated by Cuntz-Krieger G-family $\{T, Q\}$. Then $C^*(H) \cong (\sum Q_{v_i})C^*(G)(\sum Q_{v_i})$.*

Proof. We have labeled the outward pointing edge f with range vertex w, and without loss of generality we assume the vertices of H are labeled in such a way that $s(f) = v_1$. Below is an illustration of this to be referred to if needed; note that H has no restrictions, and the direction and placement of the dashed arrows is irrelevant:



Consider first $C^*(G)$; by Proposition 1.4.2, we know that there is a Cuntz-Krieger G-family $\{T, Q\}$ and an action γ such that for $z \in \mathbb{T}$, $\gamma_z(T_e) = zT_e$ for all $e \in G^1$, and $\gamma_z(Q_v) = Q_v$ for all $v \in G^0$. Now, suppose T_f is the partial isometry associated to the outward pointing edge f, and Q_w is the projection associated with its range vertex w. Then by construction $\{T \setminus \{T_f\}, Q \setminus \{Q_w\}\}$ will be a Cuntz-Krieger H-family, and $\gamma|_{\{T\setminus\{T_f\}, Q\setminus\{Q_w\}\}}$ will be a gauge action on the family. Therefore, by the gauge-invariant uniqueness theorem, $C^*(H) \cong C^*(T \setminus \{T_f\}, Q \setminus \{Q_w\})$. Since $\{T \setminus \{T_f\}, Q \setminus \{Q_w\}\} \subset \{T, Q\}$, we have $C^*(H) \cong C^*(T \setminus \{T_f\}, Q \setminus \{Q_w\}) \subset C^*(T, Q) = C^*(G)$. We claim that $C^*(T \setminus \{T_f\}, Q \setminus \{Q_w\}) = (\sum Q_{v_i})C^*(G)(\sum Q_{v_i})$.

($\subseteq$): Let $X \in C^*(T \setminus \{T_f\}, Q \setminus \{Q_w\}) = C^*(\{T_{e_j}\}, \{Q_{v_i}\})$. Recall that if E is any row-finite directed graph, then $C^*(E)$ is generated by the set

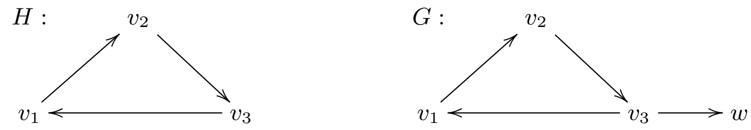
$$\{S_\mu S_\nu^* : \mu, \nu \in E^*, \, s(\mu) = s(\nu)\},$$

where E^* is the set of all finite paths in E (see Corollary 1.3.5). Thus, we can assume $X = T_\mu T_\nu^*$ for some $\mu, \nu \in H^*$ with $s(\mu) = s(\nu)$ (since $\{v_i\}$ are the vertices of H and $\{e_j\}$ are the edges of H). As H is a subgraph of G, μ and ν are also in G^*. Hence, $T_\mu T_\nu^* \in C^*(G)$. Recall that $\sum Q_{v_i}$ is the multiplicative identity in $C^*(\{T_{e_j}\}, \{Q_{v_i}\})$ (see, for example, Remark 1.7 in [12]). Hence, $T_\mu T_\nu^* \in C^*(G)$ implies that $T_\mu T_\nu^* \in (\sum Q_{v_i})C^*(G)(\sum Q_{v_i})$. Therefore, $C^*(T \setminus \{T_f\}, Q \setminus \{Q_w\}) \subseteq (\sum Q_{v_i})C^*(G)(\sum Q_{v_i})$.

($\supseteq$): Let $Y \in (\sum Q_{v_i})C^*(G)(\sum Q_{v_i})$. Again, by the same argument as above, we can assume $Y = (\sum Q_{v_i})(T_{\mu'} T_{\nu'}^*)(\sum Q_{v_i})$ for $\mu', \nu' \in G^*$. If $T_{\mu'} T_{\nu'}^* \in C^*(T \setminus \{T_f\}, Q \setminus \{Q_w\}) = C^*(\{T_{e_j}\}, \{Q_{v_i}\})$, we are done (since certainly $\sum Q_{v_i} \in C^*(\{T_{e_j}\}, \{Q_{v_i}\})$). Thus, we suppose that $T_{\mu'} T_{\nu'}^* \in C^*(G) \setminus C^*(\{T_{e_j}\}, \{Q_{v_i}\})$. Since $C^*(G) = C^*(T, Q)$, we know $C^*(G) \setminus C^*(\{T_{e_j}\}, \{Q_{v_i}\}) = C^*(T_f, Q_w)$. Thus, $\mu' = f = \nu'$; then $T_{\mu'} T_{\nu'}^* = T_f T_f^* = Q_w$. If this is the case, then $Y = (\sum Q_{v_i})Q_w(\sum Q_{v_i}) = 0$ since the projections are mutually orthogonal. Therefore, $Y \in C^*(T \setminus \{T_f\}, Q \setminus \{Q_w\})$.

Now, since $C^*(T \setminus \{T_f\}, Q \setminus \{Q_w\}) = (\sum Q_{v_i})C^*(G)(\sum Q_{v_i})$, we know from above that $C^*(H) \cong (\sum Q_{v_i})C^*(G)(\sum Q_{v_i})$. □

Example 2.2.2. To illustrate Proposition 2.2.1, consider the following graphs:



By Proposition 1.4.5 we know that $C^*(H) = M_3(C(\mathbb{T}))$, and by Lemma 2.1.1 we have $C^*(G) = M_4(C(\mathbb{T}))$. Then $\sum P_{v_i} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in C^*(H)$, but when we move to $C^*(G)$, we think of this sum as a block matrix in a 4×4 matrix. That is, $\sum P_{v_i} \cong \sum Q_{v_i} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in C^*(G)$.

Hence, here $C^*(H)$ sits inside of $C^*(G)$ in the sense that

$$C^*(H) \cong \begin{bmatrix} M_3(C(\mathbb{T})) & 0 \\ 0 & 0 \end{bmatrix} = \left(\sum Q_{v_i}\right) M_4(C(\mathbb{T})) \left(\sum Q_{v_i}\right) \subset M_4(C(\mathbb{T})) = C^*(G)$$

(see Example 1.5.8 for another example of the matrix notation seen here).

Proposition 2.2.3. *Let H be a row-finite directed graph with Cuntz-Krieger H-family $\{S, P\}$, and let G be composed of the graph H with the addition of an outward pointing edge f. Let T_f be the partial isometry in the Cuntz-Krieger G-family corresponding to edge f. Then,*

$$C^*(G) \cong \begin{bmatrix} C^*(H) & C^*(H)T_f^* \\ T_f C^*(H) & T_f C^*(H) T_f^* \end{bmatrix}.$$

Proof. Let $\alpha : \mathbb{T} \to \text{Aut}(C^*(H))$ be the gauge action on the C^*-algebra for graph H, which is guaranteed to exist by Proposition 1.4.2. We'll let μ_i, ν_i, γ_i, and δ_i be paths in H, with $S_{\mu_i}, S_{\nu_i}, S_{\gamma_i}$, and S_{δ_i} their corresponding partial isometries. Denote by $\mathscr{A}$ the space $\begin{bmatrix} C^*(H) & C^*(H)T_f^* \\ T_f C^*(H) & T_f C^*(H) T_f^* \end{bmatrix}$ (for a discussion of a matrix representation such as this, see [6, Chapter 3]). We define an action

$\beta : \mathbb{T} \to \operatorname{Aut}(\mathscr{A})$ by

$$\beta_w \left(\begin{bmatrix} S_{\mu_1} S_{\nu_1}^* & S_{\mu_2} S_{\nu_2}^* T_f^* \\ T_f S_{\mu_3} S_{\nu_3}^* & T_f S_{\mu_4} S_{\nu_4}^* T_f^* \end{bmatrix} \right) := \begin{bmatrix} \alpha_w(S_{\mu_1} S_{\nu_1}^*) & \overline{w} \alpha_w(S_{\mu_2} S_{\nu_2}^*) T_f^* \\ w T_f \alpha_w(S_{\mu_3} S_{\nu_3}^*) & T_f \alpha_w(S_{\mu_4} S_{\nu_4}^*) T_f^* \end{bmatrix}.$$

We first need to show that β_w is an automorphism. Notice that β_w is well-defined, since α_w is an automorphism.

The action is multiplicative:

$$\beta_w \left(\begin{bmatrix} S_{\mu_1} S_{\nu_1}^* & S_{\mu_2} S_{\nu_2}^* T_f^* \\ T_f S_{\mu_3} S_{\nu_3}^* & T_f S_{\mu_4} S_{\nu_4}^* T_f^* \end{bmatrix} \right) \beta_w \left(\begin{bmatrix} S_{\gamma_1} S_{\delta_1}^* & S_{\gamma_2} S_{\delta_2}^* T_f^* \\ T_f S_{\gamma_3} S_{\delta_3}^* & T_f S_{\gamma_4} S_{\delta_4}^* T_f^* \end{bmatrix} \right)$$

$$= \begin{bmatrix} \alpha_w(S_{\mu_1} S_{\nu_1}^*) & \overline{w} \alpha_w(S_{\mu_2} S_{\nu_2}^*) T_f^* \\ w T_f \alpha_w(S_{\mu_3} S_{\nu_3}^*) & T_f \alpha_w(S_{\mu_4} S_{\nu_4}^*) T_f^* \end{bmatrix} \begin{bmatrix} \alpha_w(S_{\gamma_1} S_{\delta_1}^*) & \overline{w} \alpha_w(S_{\gamma_2} S_{\delta_2}^*) T_f^* \\ w T_f \alpha_w(S_{\gamma_3} S_{\delta_3}^*) & T_f \alpha_w(S_{\gamma_4} S_{\delta_4}^*) T_f^* \end{bmatrix}$$

$$= \left[\begin{matrix} \alpha_w(S_{\mu_1} S_{\nu_1}^*) \alpha_w(S_{\gamma_1} S_{\delta_1}^*) + \overline{w} \alpha_w(S_{\mu_2} S_{\nu_2}^*) T_f^* w T_f \alpha_w(S_{\gamma_3} S_{\delta_3}^*) \\ w T_f \alpha_w(S_{\mu_3} S_{\nu_3}^*) \alpha_w(S_{\gamma_1} S_{\delta_1}^*) + T_f \alpha_w(S_{\mu_4} S_{\nu_4}^*) T_f^* w T_f \alpha_w(S_{\gamma_3} S_{\delta_3}^*) \end{matrix} \right.$$

$$\left. \begin{matrix} \alpha_w(S_{\mu_1} S_{\nu_1}^*) \overline{w} \alpha_w(S_{\gamma_2} S_{\delta_2}^*) T_f^* + \overline{w} \alpha_w(S_{\mu_2} S_{\nu_2}^*) T_f^* T_f \alpha_w(S_{\gamma_4} S_{\delta_4}^*) T_f^* \\ w T_f \alpha_w(S_{\mu_3} S_{\nu_3}^*) \overline{w} \alpha_w(S_{\gamma_2} S_{\delta_2}^*) T_f^* + T_f \alpha_w(S_{\mu_4} S_{\nu_4}^*) T_f^* T_f \alpha_w(S_{\gamma_4} S_{\delta_4}^*) T_f^* \end{matrix} \right]$$

Next, we use the following important observations:

- α_w is a homomorphism, so it is linear and multiplicative;
- $T_f T_f^* = Q_x$ where $r(f) = x$ and Q_x is the corresponding projection in $C^*(G)$, and $T_f^* T_f = P_v$, where $v = s(f) \in H^0$ and P_v is its corresponding projection in $C^*(H)$. These act like identities if the products in question are nonzero; and,
- $w\overline{w} = 1$ since w is a complex scalar.

We will use these facts again when showing β_w is linear. Continuing,

$$= \begin{bmatrix} \alpha_w(S_{\mu_1}S^*_{\nu_1}S_{\gamma_1}S^*_{\delta_1}) + \alpha_w(S_{\mu_2}S^*_{\nu_2}S_{\gamma_3}S^*_{\delta_3}) & \overline{w}\alpha_w(S_{\mu_1}S^*_{\nu_1}S_{\gamma_2}S^*_{\delta_2})T^*_f + \overline{w}\alpha_w(S_{\mu_2}S^*_{\nu_2}S_{\gamma_4}S^*_{\delta_4})T^*_f \\ wT_f\alpha_w(S_{\mu_3}S^*_{\nu_3}S_{\gamma_1}S^*_{\delta_1}) + wT_f\alpha_w(S_{\mu_4}S^*_{\nu_4}S_{\gamma_3}S^*_{\delta_3}) & T_f\alpha_w(S_{\mu_3}S^*_{\nu_3}S_{\gamma_2}S^*_{\delta_2})T^*_f + T_f\alpha_w(S_{\mu_4}S^*_{\nu_4}S_{\gamma_4}S^*_{\delta_4})T^*_f \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_w(S_{\mu_1}S^*_{\nu_1}S_{\gamma_1}S^*_{\delta_1} + S_{\mu_2}S^*_{\nu_2}S_{\gamma_3}S^*_{\delta_3}) & \overline{w}\alpha_w(S_{\mu_1}S^*_{\nu_1}S_{\gamma_2}S^*_{\delta_2} + S_{\mu_2}S^*_{\nu_2}S_{\gamma_4}S^*_{\delta_4})T^*_f \\ wT_f\alpha_w(S_{\mu_3}S^*_{\nu_3}S_{\gamma_1}S^*_{\delta_1} + S_{\mu_4}S^*_{\nu_4}S_{\gamma_3}S^*_{\delta_3}) & T_f\alpha_w(S_{\mu_3}S^*_{\nu_3}S_{\gamma_2}S^*_{\delta_2} + S_{\mu_4}S^*_{\nu_4}S_{\gamma_4}S^*_{\delta_4})T^*_f \end{bmatrix}$$

$$= \beta_w\left(\begin{bmatrix} S_{\mu_1}S^*_{\nu_1}S_{\gamma_1}S^*_{\delta_1} + S_{\mu_2}S^*_{\nu_2}S_{\gamma_3}S^*_{\delta_3} & (S_{\mu_1}S^*_{\nu_1}S_{\gamma_2}S^*_{\delta_2} + S_{\mu_2}S^*_{\nu_2}S_{\gamma_4}S^*_{\delta_4})T^*_f \\ T_f(S_{\mu_3}S^*_{\nu_3}S_{\gamma_1}S^*_{\delta_1} + S_{\mu_4}S^*_{\nu_4}S_{\gamma_3}S^*_{\delta_3}) & T_f(S_{\mu_3}S^*_{\nu_3}S_{\gamma_2}S^*_{\delta_2} + S_{\mu_4}S^*_{\nu_4}S_{\gamma_4}S^*_{\delta_4})T^*_f \end{bmatrix}\right)$$

$$= \beta_w\left(\begin{bmatrix} S_{\mu_1}S^*_{\nu_1} & S_{\mu_2}S^*_{\nu_2}T^*_f \\ T_fS_{\mu_3}S^*_{\nu_3} & T_fS_{\mu_4}S^*_{\nu_4}T^*_f \end{bmatrix}\begin{bmatrix} S_{\gamma_1}S^*_{\delta_1} & S_{\gamma_2}S^*_{\delta_2}T^*_f \\ T_fS_{\gamma_3}S^*_{\delta_3} & T_fS_{\gamma_4}S^*_{\delta_4}T^*_f \end{bmatrix}\right).$$

The action is linear: Let c and d be scalars. Then,

$$c\beta_w\left(\begin{bmatrix} S_{\mu_1}S^*_{\nu_1} & S_{\mu_2}S^*_{\nu_2}T^*_f \\ T_fS_{\mu_3}S^*_{\nu_3} & T_fS_{\mu_4}S^*_{\nu_4}T^*_f \end{bmatrix}\right) + d\beta_w\left(\begin{bmatrix} S_{\gamma_1}S^*_{\delta_1} & S_{\gamma_2}S^*_{\delta_2}T^*_f \\ T_fS_{\gamma_3}S^*_{\delta_3} & T_fS_{\gamma_4}S^*_{\delta_4}T^*_f \end{bmatrix}\right)$$

$$= c\begin{bmatrix} \alpha_w(S_{\mu_1}S^*_{\nu_1}) & \overline{w}\alpha_w(S_{\mu_2}S^*_{\nu_2})T^*_f \\ wT_f\alpha_w(S_{\mu_3}S^*_{\nu_3}) & T_f\alpha_w(S_{\mu_4}S^*_{\nu_4})T^*_f \end{bmatrix} + d\begin{bmatrix} \alpha_w(S_{\gamma_1}S^*_{\delta_1}) & \overline{w}\alpha_w(S_{\gamma_2}S^*_{\delta_2})T^*_f \\ wT_f\alpha_w(S_{\gamma_3}S^*_{\delta_3}) & T_f\alpha_w(S_{\gamma_4}S^*_{\delta_4})T^*_f \end{bmatrix}$$

$$= \begin{bmatrix} c\alpha_w(S_{\mu_1}S^*_{\nu_1}) + d\alpha_w(S_{\gamma_1}S^*_{\delta_1}) & c\overline{w}\alpha_w(S_{\mu_2}S^*_{\nu_2})T^*_f + d\overline{w}\alpha_w(S_{\gamma_2}S^*_{\delta_2})T^*_f \\ cwT_f\alpha_w(S_{\mu_3}S^*_{\nu_3}) + dwT_f\alpha_w(S_{\gamma_3}S^*_{\delta_3}) & cT_f\alpha_w(S_{\mu_4}S^*_{\nu_4})T^*_f + dT_f\alpha_w(S_{\gamma_4}S^*_{\delta_4})T^*_f \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_w(cS_{\mu_1}S^*_{\nu_1} + dS_{\gamma_1}S^*_{\delta_1}) & \overline{w}\alpha_w(cS_{\mu_2}S^*_{\nu_2} + dS_{\gamma_2}S^*_{\delta_2})T^*_f \\ wT_f\alpha_w(cS_{\mu_3}S^*_{\nu_3} + dS_{\gamma_3}S^*_{\delta_3}) & T_f\alpha_w(cS_{\mu_4}S^*_{\nu_4} + dS_{\gamma_4}S^*_{\delta_4})T^*_f \end{bmatrix}$$

$$= \beta_w\left(\begin{bmatrix} cS_{\mu_1}S^*_{\nu_1} + dS_{\gamma_1}S^*_{\delta_1} & (cS_{\mu_2}S^*_{\nu_2} + dS_{\gamma_2}S^*_{\delta_2})T^*_f \\ T_f(cS_{\mu_3}S^*_{\nu_3} + dS_{\gamma_3}S^*_{\delta_3}) & T_f(cS_{\mu_4}S^*_{\nu_4} + dS_{\gamma_4}S^*_{\delta_4})T^*_f \end{bmatrix}\right)$$

$$= \beta_w\left(c\begin{bmatrix} S_{\mu_1}S^*_{\nu_1} & S_{\mu_2}S^*_{\nu_2}T^*_f \\ T_fS_{\mu_3}S^*_{\nu_3} & T_fS_{\mu_4}S^*_{\nu_4}T^*_f \end{bmatrix} + d\begin{bmatrix} S_{\gamma_1}S^*_{\delta_1} & S_{\gamma_2}S^*_{\delta_2}T^*_f \\ T_fS_{\gamma_3}S^*_{\delta_3} & T_fS_{\gamma_4}S^*_{\delta_4}T^*_f \end{bmatrix}\right).$$

We now need to show that β_w is one-to-one and onto. We will do this by defining a function ξ_w on $\mathscr{A}$ and showing that it is the inverse of β_w. Note that α_w is an automorphism, so it is one-to-one and onto, and hence has an inverse. We define

$$\xi_w\left(\begin{bmatrix} S_{\mu_1}S^*_{\nu_1} & S_{\mu_2}S^*_{\nu_2}T^*_f \\ T_fS_{\mu_3}S^*_{\nu_3} & T_fS_{\mu_4}S^*_{\nu_4}T^*_f \end{bmatrix}\right) := \begin{bmatrix} \alpha_w^{-1}(S_{\mu_1}S^*_{\nu_1}) & w\alpha_w^{-1}(S_{\mu_2}S^*_{\nu_2})T^*_f \\ \overline{w}T_f\alpha_w^{-1}(S_{\mu_3}S^*_{\nu_3}) & T_f\alpha_w^{-1}(S_{\mu_4}S^*_{\nu_4})T^*_f \end{bmatrix}$$

and we will show that $(\xi_w \circ \beta_w)(X) = X$ and $(\beta_w \circ \xi_w)(X) = X$ for $X \in \mathscr{A}$. Observe,

$$\begin{aligned} &\xi_w\left(\beta_w\left(\begin{bmatrix} S_{\mu_1}S^*_{\nu_1} & S_{\mu_2}S^*_{\nu_2}T^*_f \\ T_fS_{\mu_3}S^*_{\nu_3} & T_fS_{\mu_4}S^*_{\nu_4}T^*_f \end{bmatrix}\right)\right) \\ &= \xi_w\left(\begin{bmatrix} \alpha_w(S_{\mu_1}S^*_{\nu_1}) & \overline{w}\alpha_w(S_{\mu_2}S^*_{\nu_2})T^*_f \\ wT_f\alpha_w(S_{\mu_3}S^*_{\nu_3}) & T_f\alpha_w(S_{\mu_4}S^*_{\nu_4})T^*_f \end{bmatrix}\right) \\ &= \begin{bmatrix} \alpha_w^{-1}(\alpha_w(S_{\mu_1}S^*_{\nu_1})) & w\overline{w}\alpha_w^{-1}(\alpha_w(S_{\mu_2}S^*_{\nu_2}))T^*_f \\ \overline{w}wT_f\alpha_w^{-1}(\alpha_w(S_{\mu_3}S^*_{\nu_3})) & T_f\alpha_w^{-1}(\alpha_w(S_{\mu_4}S^*_{\nu_4}))T^*_f \end{bmatrix} \\ &= \begin{bmatrix} S_{\mu_1}S^*_{\nu_1} & S_{\mu_2}S^*_{\nu_2}T^*_f \\ T_fS_{\mu_3}S^*_{\nu_3} & T_fS_{\mu_4}S^*_{\nu_4}T^*_f \end{bmatrix}, \text{ and,} \end{aligned}$$

$$\beta_w\left(\xi_w\left(\begin{bmatrix} S_{\mu_1}S^*_{\nu_1} & S_{\mu_2}S^*_{\nu_2}T^*_f \\ T_fS_{\mu_3}S^*_{\nu_3} & T_fS_{\mu_4}S^*_{\nu_4}T^*_f \end{bmatrix}\right)\right)$$

$$= \beta_w \left(\begin{bmatrix} \alpha_w^{-1}(S_{\mu_1}S_{\nu_1}^*) & w\alpha_w^{-1}(S_{\mu_2}S_{\nu_2}^*)T_f^* \\ \overline{w}T_f\alpha_w^{-1}(S_{\mu_3}S_{\nu_3}^*) & T_f\alpha_w^{-1}(S_{\mu_4}S_{\nu_4}^*)T_f^* \end{bmatrix} \right)$$

$$= \begin{bmatrix} \alpha_w(\alpha_w^{-1}(S_{\mu_1}S_{\nu_1}^*)) & \overline{w}w\alpha_w(\alpha_w^{-1}(S_{\mu_2}S_{\nu_2}^*))T_f^* \\ w\overline{w}T_f\alpha_w(\alpha_w^{-1}(S_{\mu_3}S_{\nu_3}^*)) & T_f\alpha_w(\alpha_w^{-1}(S_{\mu_4}S_{\nu_4}^*))T_f^* \end{bmatrix}$$

$$= \begin{bmatrix} S_{\mu_1}S_{\nu_1}^* & S_{\mu_2}S_{\nu_2}^*T_f^* \\ T_fS_{\mu_3}S_{\nu_3}^* & T_fS_{\mu_4}S_{\nu_4}^*T_f^* \end{bmatrix}.$$

Therefore, $(\xi_w \circ \beta_w)(X) = X$ and $(\beta_w \circ \xi_w)(X) = X$ for $X \in \mathscr{A}$, so β_w is one-to-one and onto, and is thus an automorphism. As it is a $*$-homomorphism on C^*-algebras which is one-to-one, it is norm-continuous [9].

Now, since α_w is a $*$-homomorphism, we have the following:

$$\begin{aligned}
\beta_w \left(\begin{bmatrix} S_{\mu_1}S_{\nu_1}^* & S_{\mu_2}S_{\nu_2}^*T_f^* \\ T_fS_{\mu_3}S_{\nu_3}^* & T_fS_{\mu_4}S_{\nu_4}^*T_f^* \end{bmatrix}^* \right) &= \beta_w \left(\begin{bmatrix} (S_{\mu_1}S_{\nu_1}^*)^* & (T_fS_{\mu_3}S_{\nu_3}^*)^* \\ (S_{\mu_2}S_{\nu_2}^*T_f^*)^* & (T_fS_{\mu_4}S_{\nu_4}^*T_f^*)^* \end{bmatrix} \right) \\
&= \beta_w \left(\begin{bmatrix} S_{\nu_1}S_{\mu_1}^* & S_{\nu_3}S_{\mu_3}^*T_f^* \\ T_fS_{\nu_2}S_{\mu_2}^* & T_fS_{\nu_4}S_{\mu_4}^*T_f^* \end{bmatrix} \right) \\
&= \begin{bmatrix} \alpha_w(S_{\nu_1}S_{\mu_1}^*) & \overline{w}\alpha_w(S_{\nu_3}S_{\mu_3}^*)T_f^* \\ wT_f\alpha_w(S_{\nu_2}S_{\mu_2}^*) & T_f\alpha_w(S_{\nu_4}S_{\mu_4}^*)T_f^* \end{bmatrix} \\
&= \begin{bmatrix} \alpha_w(S_{\mu_1}S_{\nu_1}^*)^* & \overline{w}\alpha_w(S_{\mu_3}S_{\nu_3}^*)^*T_f^* \\ wT_f\alpha_w(S_{\mu_2}S_{\nu_2}^*)^* & T_f\alpha_w(S_{\mu_4}S_{\nu_4}^*)^*T_f^* \end{bmatrix} \\
&= \begin{bmatrix} (\alpha_w(S_{\mu_1}S_{\nu_1}^*))^* & (wT_f\alpha_w(S_{\mu_3}S_{\nu_3}^*))^* \\ (\overline{w}\alpha_w(S_{\mu_2}S_{\nu_2}^*)T_f^*)^* & (T_f\alpha_w(S_{\mu_4}S_{\nu_4}^*)T_f^*)^* \end{bmatrix} \\
&= \begin{bmatrix} \alpha_w(S_{\mu_1}S_{\nu_1}^*) & \overline{w}\alpha_w(S_{\mu_2}S_{\nu_2}^*)T_f^* \\ wT_f\alpha_w(S_{\mu_3}S_{\nu_3}^*) & T_f\alpha_w(S_{\mu_4}S_{\nu_4}^*)T_f^* \end{bmatrix}^* \\
&= \left(\beta_w \left(\begin{bmatrix} S_{\mu_1}S_{\nu_1}^* & S_{\mu_2}S_{\nu_2}^*T_f^* \\ T_fS_{\mu_3}S_{\nu_3}^* & T_fS_{\mu_4}S_{\nu_4}^*T_f^* \end{bmatrix} \right) \right)^*.
\end{aligned}$$

Thus, β_w is a $*$-automorphism.

Lastly, we need to check that β_w acts appropriately on each of the partial isometries and on each of the projections. For each edge e in H^1, since α_w is the gauge action for $C^*(H)$, we have

$$\beta_w(S_e) = \beta_w\left(\begin{bmatrix} S_e & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} \alpha_w(S_e) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} wS_e & 0 \\ 0 & 0 \end{bmatrix} = w\begin{bmatrix} S_e & 0 \\ 0 & 0 \end{bmatrix} = wS_e.$$

Similarly, we will show that $\beta_w(T_f) = wT_f$, $\beta_w(P_v) = P_v$ for all $v \in H^0$, and $\beta_w(Q_x) = Q_x$.

$$\begin{aligned}
\beta_w(T_f) &= \beta_w\left(\begin{bmatrix} 0 & 0 \\ T_f & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ wT_f\alpha_w(I) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ wT_f & 0 \end{bmatrix} = w\begin{bmatrix} 0 & 0 \\ T_f & 0 \end{bmatrix} = wT_f, \\
\beta_w(P_v) &= \beta_w\left(\begin{bmatrix} P_v & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} \alpha_w(P_v) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} P_v & 0 \\ 0 & 0 \end{bmatrix} = P_v, \text{ and} \\
\beta_w(Q_x) &= \beta_w\left(\begin{bmatrix} 0 & 0 \\ 0 & Q_x \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & T_f\alpha_w(I)T_f^* \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & Q_x \end{bmatrix} = Q_x.
\end{aligned}$$

Thus, we have shown that $\beta : \mathbb{T} \to \text{Aut}(\mathscr{A})$ is a gauge action such that

$$\pi_{(\{S,T_f\},\{P,Q_x\})} \circ \sigma_w = \beta_w \circ \pi_{(\{S,T_f\},\{P,Q_x\})}$$

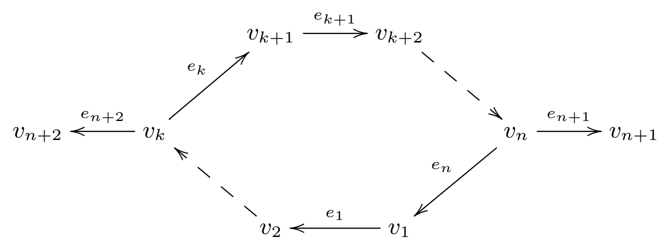
for the gauge action $\sigma : \mathbb{T} \to \text{Aut}(C^*(G))$. Therefore, by the gauge-invariant uniqueness theorem, $\pi_{(\{S,T_f\},\{P,Q_x\})}$ is an isomorphism. Hence, the C^*-algebra for the graph G with subgraph H is $\mathscr{A}$. $\square$

2.3. Graph algebra of a single cycle plus two edges

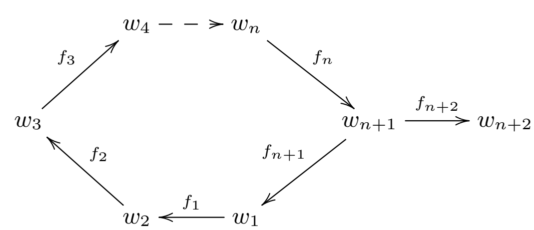
Lemma 2.3.1. *Let H be a directed cycle of length n and G be constructed by iterating the process of adding an outward pointing edge to H two times. Then $C^*(G) \cong M_{n+2}(C(\mathbb{T}))$.*

Proof. Without loss of generality, label the vertices and edges of G as follows: the vertices and edges of the cycle are, respectively, $v_1, \dots, v_n$ and $e_1, \dots, e_n$ where $s(e_i) = v_i$, with the two outward pointing edges e_{n+1} and e_{n+2}; let $s(e_{n+1}) = v_n$, $r(e_{n+1}) = v_{n+1}$, $s(e_{n+2}) = v_k$ for some $1 \le k \le n+1$, and $r(e_{n+2}) = v_{n+2}$ (see illustration below). Now we define a series of other graphs. Let H_1 be the subgraph of G composed of the directed n-cycle with only the first outward pointing edge e_{n+1}, and with all vertices and edges labeled as in G. Next, let H_2 be the directed $(n+1)$-cycle C_{n+1}, whose vertices will be labeled w_i and edges f_i $(1 \leq i \leq n+1)$ such that $s(f_i) = w_i$. Finally, we will define G_2 to be the graph H_2 with added outward pointing edge f_{n+2} whose source is w_{n+1} and range is a new vertex w_{n+2} (see illustration below).

e.g. G :



G_2 :



In the diagrams above, the diagram for H_1 would be that of G with edge e_{n+2} and vertex v_{n+2} removed, and for H_2 would be that of G_2 with edge f_{n+2} and vertex w_{n+2} removed. We will let the Cuntz-Krieger G-family be denoted by $\{S, P\}$, and let the H_1-family be the subset $\{S \setminus \{S_{e_{n+2}}\}, P \setminus \{P_{v_{n+2}}\}\}$. The Cuntz-Krieger G_2-family will be referred to by $\{T, Q\}$, and the H_2-family will be the subset $\{T \setminus \{T_{f_{n+2}}\}, Q \setminus \{Q_{w_{n+2}}\}\}$.

Observe that, by Lemma 2.1.1, since H_1 is a directed n-cycle with one additional outward pointing edge, $C^*(H_1) \cong M_{n+1}(C(\mathbb{T}))$, and $C^*(H_2) \cong M_{n+1}(C(\mathbb{T}))$ as well since H_2 is a directed cycle of length $n+1$; thus $C^*(H_1) \cong C^*(H_2)$. We'll let $\sigma : C^*(H_1) \to C^*(H_2)$ be the corresponding *-isomorphism.

Let μ_i, ν_i, γ_i, and δ_i be paths in H_1 with corresponding partial isometries S_{μ_i}, S_{ν_i}, S_{γ_i} and S_{δ_i}, and let the partial isometry in $C^*(G)$ corresponding to edge e_{n+2} be denoted by $S_{e_{n+2}}$. Let the partial isometry in $C^*(G_2)$ corresponding to edge f_{n+2} be denoted by $T_{f_{n+2}}$. Then we know from

Proposition 2.2.3 that

$$C^*(G) \cong \begin{bmatrix} C^*(H_1) & C^*(H_1)S^*_{e_{n+2}} \\ S_{e_{n+2}}C^*(H_1) & S_{e_{n+2}}C^*(H_1)S^*_{e_{n+2}} \end{bmatrix},$$

$$C^*(G_2) \cong \begin{bmatrix} C^*(H_2) & C^*(H_2)T^*_{f_{n+2}} \\ T_{f_{n+2}}C^*(H_2) & T_{f_{n+2}}C^*(H_2)T^*_{f_{n+2}} \end{bmatrix},$$

and we also know from Lemma 2.1.1 that $C^*(G_2) \cong M_{n+2}(C(\mathbb{T}))$.

Finally, define $\pi : C^*(G) \to C^*(G_2)$ such that

$$\pi\left(\begin{bmatrix} S_{\mu_1}S^*_{\nu_1} & S_{\mu_2}S^*_{\nu_2}S^*_{e_{n+2}} \\ S_{e_{n+2}}S_{\mu_3}S^*_{\nu_3} & S_{e_{n+2}}S_{\mu_4}S^*_{\nu_4}S^*_{e_{n+2}} \end{bmatrix}\right) = \begin{bmatrix} \sigma(S_{\mu_1}S^*_{\nu_1}) & \sigma(S_{\mu_2}S^*_{\nu_2})T^*_{f_{n+2}} \\ T_{f_{n+2}}\sigma(S_{\mu_3}S^*_{\nu_3}) & T_{f_{n+2}}\sigma(S_{\mu_4}S^*_{\nu_4})T^*_{f_{n+2}} \end{bmatrix}.$$

We claim that π is a $*$-isomorphism.

The map is multiplicative:

$$\pi\left(\begin{bmatrix} S_{\mu_1}S^*_{\nu_1} & S_{\mu_2}S^*_{\nu_2}S^*_{e_{n+2}} \\ S_{e_{n+2}}S_{\mu_3}S^*_{\nu_3} & S_{e_{n+2}}S_{\mu_4}S^*_{\nu_4}S^*_{e_{n+2}} \end{bmatrix}\right)\pi\left(\begin{bmatrix} S_{\gamma_1}S^*_{\delta_1} & S_{\gamma_2}S^*_{\delta_2}S^*_{e_{n+2}} \\ S_{e_{n+2}}S_{\gamma_3}S^*_{\delta_3} & S_{e_{n+2}}S_{\gamma_4}S^*_{\delta_4}S^*_{e_{n+2}} \end{bmatrix}\right)$$

$$= \begin{bmatrix} \sigma(S_{\mu_1}S^*_{\nu_1}) & \sigma(S_{\mu_2}S^*_{\nu_2})T^*_{f_{n+2}} \\ T_{f_{n+2}}\sigma(S_{\mu_3}S^*_{\nu_3}) & T_{f_{n+2}}\sigma(S_{\mu_4}S^*_{\nu_4})T^*_{f_{n+2}} \end{bmatrix}\begin{bmatrix} \sigma(S_{\gamma_1}S^*_{\delta_1}) & \sigma(S_{\gamma_2}S^*_{\delta_2})T^*_{f_{n+2}} \\ T_{f_{n+2}}\sigma(S_{\gamma_3}S^*_{\delta_3}) & T_{f_{n+2}}\sigma(S_{\gamma_4}S^*_{\delta_4})T^*_{f_{n+2}} \end{bmatrix}$$

$$= \begin{bmatrix} \sigma(S_{\mu_1}S^*_{\nu_1})\sigma(S_{\gamma_1}S^*_{\delta_1}) + \sigma(S_{\mu_2}S^*_{\nu_2})T^*_{f_{n+2}}T_{f_{n+2}}\sigma(S_{\gamma_3}S^*_{\delta_3}) & \sigma(S_{\mu_1}S^*_{\nu_1})\sigma(S_{\gamma_2}S^*_{\delta_2})T^*_{f_{n+2}} + \sigma(S_{\mu_2}S^*_{\nu_2})T^*_{f_{n+2}}T_{f_{n+2}}\sigma(S_{\gamma_4}S^*_{\delta_4})T^*_{f_{n+2}} \\ T_{f_{n+2}}\sigma(S_{\mu_3}S^*_{\nu_3})\sigma(S_{\gamma_1}S^*_{\delta_1}) + T_{f_{n+2}}\sigma(S_{\mu_4}S^*_{\nu_4})T^*_{f_{n+2}}T_{f_{n+2}}\sigma(S_{\gamma_3}S^*_{\delta_3}) & T_{f_{n+2}}\sigma(S_{\mu_3}S^*_{\nu_3})\sigma(S_{\gamma_2}S^*_{\delta_2})T^*_{f_{n+2}} + T_{f_{n+2}}\sigma(S_{\mu_4}S^*_{\nu_4})T^*_{f_{n+2}}T_{f_{n+2}}\sigma(S_{\gamma_4}S^*_{\delta_4})T^*_{f_{n+2}} \end{bmatrix}$$

Next, we use the following important observations:

- σ is a homomorphism, so it is linear and multiplicative;

- $T_{f_{n+2}}T^*_{f_{n+2}} := Q_{n+2}$ and $T^*_{f_{n+2}}T_{f_{n+2}} := Q_{n+1}$; these act like identities if the products in question are nonzero.

We will use these facts again when showing that π is linear. Continuing,

$$= \begin{bmatrix} \sigma(S_{\mu_1}S^*_{\nu_1}S_{\gamma_1}S^*_{\delta_1} + S_{\mu_2}S^*_{\nu_2}S_{\gamma_3}S^*_{\delta_3}) & \sigma(S_{\mu_1}S^*_{\nu_1}S_{\gamma_2}S^*_{\delta_2} + S_{\mu_2}S^*_{\nu_2}S_{\gamma_4}S^*_{\delta_4})T^*_{f_{n+2}} \\ T_{f_{n+2}}\sigma(S_{\mu_3}S^*_{\nu_3}S_{\gamma_1}S^*_{\delta_1} + S_{\mu_4}S^*_{\nu_4}S_{\gamma_3}S^*_{\delta_3}) & T_{f_{n+2}}\sigma(S_{\mu_3}S^*_{\nu_3}S_{\gamma_2}S^*_{\delta_2} + S_{\mu_4}S^*_{\nu_4}S_{\gamma_4}S^*_{\delta_4})T^*_{f_{n+2}} \end{bmatrix}$$

$$= \pi\left(\begin{bmatrix} S_{\mu_1}S^*_{\nu_1}S_{\gamma_1}S^*_{\delta_1} + S_{\mu_2}S^*_{\nu_2}S_{\gamma_3}S^*_{\delta_3} & (S_{\mu_1}S^*_{\nu_1}S_{\gamma_2}S^*_{\delta_2} + S_{\mu_2}S^*_{\nu_2}S_{\gamma_4}S^*_{\delta_4})S^*_{e_{n+2}} \\ S_{e_{n+2}}(S_{\mu_3}S^*_{\nu_3}S_{\gamma_1}S^*_{\delta_1} + S_{\mu_4}S^*_{\nu_4}S_{\gamma_3}S^*_{\delta_3}) & S_{e_{n+2}}(S_{\mu_3}S^*_{\nu_3}S_{\gamma_2}S^*_{\delta_2} + S_{\mu_4}S^*_{\nu_4}S_{\gamma_4}S^*_{\delta_4})S^*_{e_{n+2}} \end{bmatrix}\right)$$

$$= \pi\left(\begin{bmatrix} S_{\mu_1}S^*_{\nu_1} & S_{\mu_2}S^*_{\nu_2}S^*_{e_{n+2}} \\ S_{e_{n+2}}S_{\mu_3}S^*_{\nu_3} & S_{e_{n+2}}S_{\mu_4}S^*_{\nu_4}S^*_{e_{n+2}} \end{bmatrix}\begin{bmatrix} S_{\gamma_1}S^*_{\delta_1} & S_{\gamma_2}S^*_{\delta_2}S^*_{e_{n+2}} \\ S_{e_{n+2}}S_{\gamma_3}S^*_{\delta_3} & S_{e_{n+2}}S_{\gamma_4}S^*_{\delta_4}S^*_{e_{n+2}} \end{bmatrix}\right)$$

The map is linear: Let c and d be scalars. Then,

$$c\,\pi\left(\begin{bmatrix} S_{\mu_1}S^*_{\nu_1} & S_{\mu_2}S^*_{\nu_2}S^*_{e_{n+2}} \\ S_{e_{n+2}}S_{\mu_3}S^*_{\nu_3} & S_{e_{n+2}}S_{\mu_4}S^*_{\nu_4}S^*_{e_{n+2}} \end{bmatrix}\right) + d\,\pi\left(\begin{bmatrix} S_{\gamma_1}S^*_{\delta_1} & S_{\gamma_2}S^*_{\delta_2}S^*_{e_{n+2}} \\ S_{e_{n+2}}S_{\gamma_3}S^*_{\delta_3} & S_{e_{n+2}}S_{\gamma_4}S^*_{\delta_4}S^*_{e_{n+2}} \end{bmatrix}\right)$$

$$= \begin{bmatrix} c\sigma(S_{\mu_1}S^*_{\nu_1}) + d\sigma(S_{\gamma_1}S^*_{\delta_1}) & c\sigma(S_{\mu_2}S^*_{\nu_2})T^*_{f_{n+2}} + d\sigma(S_{\gamma_2}S^*_{\delta_2})T^*_{f_{n+2}} \\ cT_{f_{n+2}}\sigma(S_{\mu_3}S^*_{\nu_3}) + dT_{f_{n+2}}\sigma(S_{\gamma_3}S^*_{\delta_3}) & cT_{f_{n+2}}\sigma(S_{\mu_4}S^*_{\nu_4})T^*_{f_{n+2}} + dT_{f_{n+2}}\sigma(S_{\gamma_4}S^*_{\delta_4})T^*_{f_{n+2}} \end{bmatrix}$$

$$= \begin{bmatrix} \sigma(cS_{\mu_1}S^*_{\nu_1} + dS_{\gamma_1}S^*_{\delta_1}) & \sigma(cS_{\mu_2}S^*_{\nu_2} + dS_{\gamma_2}S^*_{\delta_2})T^*_{f_{n+2}} \\ T_{f_{n+2}}\sigma(cS_{\mu_3}S^*_{\nu_3} + dS_{\gamma_3}S^*_{\delta_3}) & T_{f_{n+2}}\sigma(cS_{\mu_4}S^*_{\nu_4} + dS_{\gamma_4}S^*_{\delta_4})T^*_{f_{n+2}} \end{bmatrix}$$

$$= \pi\left(\begin{bmatrix} cS_{\mu_1}S^*_{\nu_1} + dS_{\gamma_1}S^*_{\delta_1} & (cS_{\mu_2}S^*_{\nu_2} + dS_{\gamma_2}S^*_{\delta_2})S^*_{e_{n+2}} \\ S_{e_{n+2}}(cS_{\mu_3}S^*_{\nu_3} + dS_{\gamma_3}S^*_{\delta_3}) & S_{e_{n+2}}(cS_{\mu_4}S^*_{\nu_4} + dS_{\gamma_4}S^*_{\delta_4})S^*_{e_{n+2}} \end{bmatrix}\right)$$

$$= \pi\left(c\begin{bmatrix} S_{\mu_1}S^*_{\nu_1} & S_{\mu_2}S^*_{\nu_2}S^*_{e_{n+2}} \\ S_{e_{n+2}}S_{\mu_3}S^*_{\nu_3} & S_{e_{n+2}}S_{\mu_4}S^*_{\nu_4}S^*_{e_{n+2}} \end{bmatrix} + d\begin{bmatrix} S_{\gamma_1}S^*_{\delta_1} & S_{\gamma_2}S^*_{\delta_2}S^*_{e_{n+2}} \\ S_{e_{n+2}}S_{\gamma_3}S^*_{\delta_3} & S_{e_{n+2}}S_{\gamma_4}S^*_{\delta_4}S^*_{e_{n+2}} \end{bmatrix}\right).$$

Therefore, π is a homomorphism.

We next show that π is one-to-one and onto by showing that π has an inverse. Note that since σ is an isomorphism, σ has an inverse σ^{-1}. For the following, we will now assume that

γ_i and δ_i are paths in H_2, rather than H_1 as above (with partial isometries T_{γ_i}, T_{δ_i}). Define $\xi : C^*(G_2) \to C^*(G)$ by

$$\xi\left(\begin{bmatrix} T_{\gamma_1}T^*_{\delta_1} & T_{\gamma_2}T^*_{\delta_2}T^*_{f_{n+2}} \\ T_{f_{n+2}}T_{\gamma_3}T^*_{\delta_3} & T_{f_{n+2}}T_{\gamma_4}T^*_{\delta_4}T^*_{f_{n+2}} \end{bmatrix}\right) = \begin{bmatrix} \sigma^{-1}(T_{\gamma_1}T^*_{\delta_1}) & \sigma^{-1}(T_{\gamma_2}T^*_{\delta_2})S^*_{e_{n+2}} \\ S_{e_{n+2}}\sigma^{-1}(T_{\gamma_3}T^*_{\delta_3}) & S_{e_{n+2}}\sigma^{-1}(T_{\gamma_4}T^*_{\delta_4})S^*_{e_{n+2}} \end{bmatrix}.$$

Then,

$$\begin{aligned} &\pi\left(\xi\left(\begin{bmatrix} T_{\gamma_1}T^*_{\delta_1} & T_{\gamma_2}T^*_{\delta_2}T^*_{f_{n+2}} \\ T_{f_{n+2}}T_{\gamma_3}T^*_{\delta_3} & T_{f_{n+2}}T_{\gamma_4}T^*_{\delta_4}T^*_{f_{n+2}} \end{bmatrix}\right)\right) \\ &\quad = \pi\left(\begin{bmatrix} \sigma^{-1}(T_{\gamma_1}T^*_{\delta_1}) & \sigma^{-1}(T_{\gamma_2}T^*_{\delta_2})S^*_{e_{n+2}} \\ S_{e_{n+2}}\sigma^{-1}(T_{\gamma_3}T^*_{\delta_3}) & S_{e_{n+2}}\sigma^{-1}(T_{\gamma_4}T^*_{\delta_4})S^*_{e_{n+2}} \end{bmatrix}\right) \\ &\quad = \begin{bmatrix} \sigma(\sigma^{-1}(T_{\gamma_1}T^*_{\delta_1})) & \sigma(\sigma^{-1}(T_{\gamma_2}T^*_{\delta_2}))T^*_{f_{n+2}} \\ T_{f_{n+2}}\sigma(\sigma^{-1}(T_{\gamma_3}T^*_{\delta_3})) & T_{f_{n+2}}\sigma(\sigma^{-1}(T_{\gamma_4}T^*_{\delta_4}))T^*_{f_{n+2}} \end{bmatrix} \\ &\quad = \begin{bmatrix} T_{\gamma_1}T^*_{\delta_1} & T_{\gamma_2}T^*_{\delta_2}T^*_{f_{n+2}} \\ T_{f_{n+2}}T_{\gamma_3}T^*_{\delta_3} & T_{f_{n+2}}T_{\gamma_4}T^*_{\delta_4}T^*_{f_{n+2}} \end{bmatrix}, \text{ and,} \end{aligned}$$

$$\begin{aligned} &\xi\left(\pi\left(\begin{bmatrix} S_{\mu_1}S^*_{\nu_1} & S_{\mu_2}S^*_{\nu_2}S^*_{e_{n+2}} \\ S_{e_{n+2}}S_{\mu_3}S^*_{\nu_3} & S_{e_{n+2}}S_{\mu_4}S^*_{\nu_4}S^*_{e_{n+2}} \end{bmatrix}\right)\right) \\ &\quad = \xi\left(\begin{bmatrix} \sigma(S_{\mu_1}S^*_{\nu_1}) & \sigma(S_{\mu_2}S^*_{\nu_2})T^*_{f_{n+2}} \\ T_{f_{n+2}}\sigma(S_{\mu_3}S^*_{\nu_3}) & T_{f_{n+2}}\sigma(S_{\mu_4}S^*_{\nu_4})T^*_{f_{n+2}} \end{bmatrix}\right) \\ &\quad = \begin{bmatrix} \sigma^{-1}(\sigma(S_{\mu_1}S^*_{\nu_1})) & \sigma^{-1}(\sigma(S_{\mu_2}S^*_{\nu_2}))S^*_{e_{n+2}} \\ S_{e_{n+2}}\sigma^{-1}(\sigma(S_{\mu_3}S^*_{\nu_3})) & S_{e_{n+2}}\sigma^{-1}(\sigma(S_{\mu_4}S^*_{\nu_4}))S^*_{e_{n+2}} \end{bmatrix} \\ &\quad = \begin{bmatrix} S_{\mu_1}S^*_{\nu_1} & S_{\mu_2}S^*_{\nu_2}S^*_{e_{n+2}} \\ S_{e_{n+2}}S_{\mu_3}S^*_{\nu_3} & S_{e_{n+2}}S_{\mu_4}S^*_{\nu_4}S^*_{e_{n+2}} \end{bmatrix}. \end{aligned}$$

Therefore, π is an isomorphism. The piece that remains to be shown is that π is a $*$-isomorphism.

As σ is a $*$-isomorphism, we observe,

$$\begin{aligned}
\pi\left(\begin{bmatrix} S_{\mu_1}S_{\nu_1}^* & S_{\mu_2}S_{\nu_2}^*S_{e_{n+2}}^* \\ S_{e_{n+2}}S_{\mu_3}S_{\nu_3}^* & S_{e_{n+2}}S_{\mu_4}S_{\nu_4}^*S_{e_{n+2}}^* \end{bmatrix}^*\right) &= \pi\left(\begin{bmatrix} S_{\nu_1}S_{\mu_1}^* & S_{\nu_3}S_{\mu_3}^*S_{e_{n+2}}^* \\ S_{e_{n+2}}S_{\nu_2}S_{\mu_2}^* & S_{e_{n+2}}S_{\nu_4}S_{\mu_4}^*S_{e_{n+2}}^* \end{bmatrix}\right) \\
&= \begin{bmatrix} \sigma(S_{\nu_1}S_{\mu_1}^*) & \sigma(S_{\nu_3}S_{\mu_3}^*)T_{f_{n+2}}^* \\ T_{f_{n+2}}\sigma(S_{\nu_2}S_{\mu_2}^*) & T_{f_{n+2}}\sigma(S_{\nu_4}S_{\mu_4}^*)T_{f_{n+2}}^* \end{bmatrix} \\
&= \begin{bmatrix} \sigma(S_{\mu_1}S_{\nu_1}^*)^* & \sigma(S_{\mu_3}S_{\nu_3}^*)^*T_{f_{n+2}}^* \\ T_{f_{n+2}}\sigma(S_{\mu_2}S_{\nu_2}^*)^* & T_{f_{n+2}}\sigma(S_{\mu_4}S_{\nu_4}^*)^*T_{f_{n+2}}^* \end{bmatrix} \\
&= \begin{bmatrix} \sigma(S_{\mu_1}S_{\nu_1}^*) & \sigma(S_{\mu_2}S_{\nu_2}^*)T_{f_{n+2}}^* \\ T_{f_{n+2}}\sigma(S_{\mu_3}S_{\nu_3}^*) & T_{f_{n+2}}\sigma(S_{\mu_4}S_{\nu_4}^*)T_{f_{n+2}}^* \end{bmatrix}^* \\
&= \left(\pi\left(\begin{bmatrix} S_{\mu_1}S_{\nu_1}^* & S_{\mu_2}S_{\nu_2}^*S_{e_{n+2}}^* \\ S_{e_{n+2}}S_{\mu_3}S_{\nu_3}^* & S_{e_{n+2}}S_{\mu_4}S_{\nu_4}^*S_{e_{n+2}}^* \end{bmatrix}\right)\right)^*.
\end{aligned}$$

Hence, we have shown that π is a $*$-isomorphism, so it has been shown that $C^*(G) \cong C^*(G_2)$, and therefore, $C^*(G) \cong M_{n+2}(C(\mathbb{T}))$. □

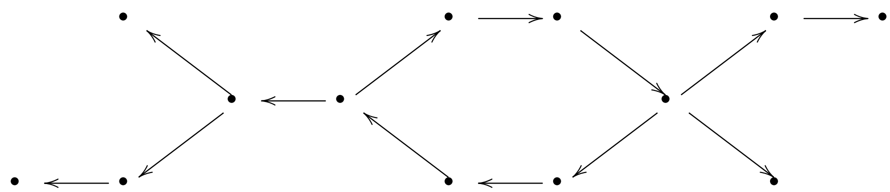
2.4. Main result for a graph containing a single cycle

Theorem 2.4.1. *Let H be a cycle of length n. Let G be a graph constructed by iterating the process of adding an outward pointing edge to H k times. Then $C^*(G) \cong M_{n+k}(C(\mathbb{T}))$.*

Proof. We know the claim holds for $k = 1, 2$ by Lemmas 2.1.1 and 2.3.1. Suppose the claim holds for all j such that $1 \leq j < k$.

Define H_1 to be a directed n-cycle with the process of adding an outward pointing edge iterated $k-1$ times; then by assumption we have that $C^*(H_1) \cong M_{n+k-1}(C(\mathbb{T}))$. Let G be the graph H_1 with one additional outward pointing edge; thus, we want to show $C^*(G) \cong M_{n+k}(C(\mathbb{T}))$. Define also H_2 to be an $(n+k-1)$-cycle, and G_2 to be H_2 with one added outward pointing edge. Then $C^*(H_1) \cong C^*(H_2)$, and $C^*(G_2) \cong M_{n+k}(C(\mathbb{T}))$ by Lemma 2.1.1. Now, following exactly the construction of the proof of Lemma 2.3.1, we have that $C^*(G) \cong M_{n+k}(C(\mathbb{T}))$. □

As an illustration of this result, consider the following graph G:



Observe that G has exactly one cycle, a 6-cycle, which has had added to it seven outward-pointing edges; thus, $C^*(G) \cong M_{13}(C(\mathbb{T}))$. One might observe that, with G described as in Theorem 2.4.1, the number $n + k$ will always equal the number of vertices of G.

3. MAXIMALLY EDGE-COLORED DIRECTED GRAPH ALGEBRAS

The following results on edge-colored directed graphs assume that f is always a one-to-one edge coloring; this allows us to avoid conflicts with CK2. Complications in finding graph algebras arise when two edges have the same range vertex because of CK2. By ensuring that any two edges with the same range vertex have a different color, we can avoid the complication.

In fact, one might notice that, because of the one-to-one nature of the coloring, these results can be extended to undirected graphs. One needs only to take such an undirected graph and assign a source and range vertex to each edge (choose a direction), and then assign a coloring in such a way that no two edges receive the same color. In this sense, the study of undirected graph C^*-algebras is equivalent to the study of certain edge-colored directed graph C^*-algebras.

Finally, as we are only trying to avoid the conflict with CK2, one could choose instead a "minimal" coloring. In this case, the number of colors required would be the maximum number of edges whose range is the same vertex. As long as no two edges have the same range vertex and the same color, the coloring is arbitrary. Hence, there are many ways to color a graph to get the results that follow.

Before continuing, we also explain the notation used here for amalgamated free products (to refer to the formal definition of the free product, see Definition 1.5.3). Recall from the referenced definition that the general notation of the free product of unital C^*-algebras $\mathscr{A}_i$ for $i \in I$ amalgamated over $\mathscr{A}_0$, where $\mathscr{A}_0$ is a subalgebra of $\mathscr{A}_i$ for all i, is $*_{\mathscr{A}_0}\mathscr{A}_i$. A general element of this algebra is a linear combination of elements of the form $a_{i_1} * a_{i_2} * \cdots * a_{i_n}$, where $i_j \in I$ for all j and $i_j \neq i_k$ if $k = j+1$ (the stars can be omitted here if preferred). The amalgamation essentially allows us to move elements of $\mathscr{A}_0$ across the product. For example, suppose that in the product $a_{i_1} * a_{i_2} * \cdots * a_{i_n}$ we have $a_{i_1} \in \mathscr{A}_0$. Then $a_{i_1} * a_{i_2} * \cdots * a_{i_n} = 1 * a_{i_1}a_{i_2} * \cdots * a_{i_n}$, for instance. In that sense, no product containing elements of the common subalgebra has a unique representation.

Lastly, we will be considering generators of these algebras. Again, because of the amalgamation, the generators will not always have unique representations, but we will use the following representations, in general:

- A generator will have the form $a_1 * a_2 * \cdots * a_n$, where $a_i \in \mathscr{A}_i$ for all $i \in I$;
- Suppose $\mathscr{A}_0$ is generated by elements $\{x_k\}$. Then $x_k * 1 * \cdots * 1$ is in the generating set for $*_{\mathscr{A}_0}\mathscr{A}_i$ for all k;
- Suppose $\mathscr{A}_i$ is generated by elements $\{y_j\}$. Then $1 * 1 * \cdots * 1 * y_j * 1 * \cdots * 1$, with y_j in the i^{th} position, is in the generating set for $*_{\mathscr{A}_0}\mathscr{A}_i$, for all j such that $y_j \notin \{x_k\}$.

3.1. Representing $C^*(G,f)$ as a free product over the algebra of a spanning tree

In the following theorem, we define $\{T, f\}$ to be a spanning tree in our graph $\{G, f\}$; because the edge-coloring is one-to-one, we can take the spanning tree to disregard the direction of the edges. That is, there is no need for there to be a root vertex.

Theorem 3.1.1. *Let $\{G, f\}$ be a finite edge-colored directed graph with vertex set G^0 and edge set G^1, where $f : G^1 \to \mathbb{N}$ is a one-to-one edge-coloring. Define $\{T, f\}$ to be a spanning tree in $\{G, f\}$, and the set $\{e_i\} = G^1 \setminus T$. Let $\{G_i, f\}$ be the edge-colored directed graph with vertices G^0 and edges $G^1_i = T^1 \cup e_i$, with f, s, and r as restrictions of the edge-coloring, source and range maps of $\{G, f\}$, respectively. Then $C^*(G, f) = *_{C^*(T)}\{C^*(G_i, f)\}$.*

Proof. Let $C^*(G, f)$ be generated by an edge-colored Cuntz-Krieger family $\{S, P, f\}$. Notice that $\{G_i, f\} := \{G^0, G^1_i, f\}$ has associated edge-colored Cuntz-Krieger family $\{S_i, P, f\}$ with S_i the partial isometries corresponding to the edges in G^1_i, and that $C^*(G_i, f) = C^*(S_i, P, f)$. Observe that $S = \cup S_i$, so it is clear that $\{\cup S_i, P, f\}$ has associated graph $\{G, f\}$. Next, $C^*(T)$ is a subalgebra of $C^*(G_i, f)$ for all i, and is generated by $\{S_T, P, f\}$, where S_T is the appropriate subset of S.

Define $\sigma_i : C^*(G_i, f) \to C^*(G, f)$ as an inclusion mapping on the generators $\{S_i, P, f\}$; we show here that σ_i satisfies the requirements for the full amalgamated product, as seen in Section 1.5. First, let $a \in C^*(T)$ be a generator, so that $a \in \{S_T, P, f\}$. Observe that $S_T \subseteq S_i$ for all i. As σ_i is an inclusion map on the generators $\{S_i, P, f\}$, and $\{S_T, P, f\} \subset \{S_i, P, f\}$, σ_i is also an inclusion map on $\{S_T, P, f\}$. In particular, $\sigma_i(a) = a = \sigma_j(a)$ for all i, j. Since this is true on the generators, it is true on all of $C^*(T)$, so we have $\sigma_i|_{C^*(T)} = \sigma_j|_{C^*(T)}$. Observe that inclusion mappings are injective by construction, as well as linear and multiplicative, and will preserve the $*$-operation. Hence, $\sigma_i|_{C^*(T)}$ is an injective $*$-homomorphism for all i. Since the generators for $C^*(G,f)$ are $\{\cup S_i, P, f\}$, and σ_i is an inclusion mapping, we see that $C^*(G,f)$ is generated by $\cup\,\sigma_i(C^*(G_i,f))$. Thus, by the universal properties of the full amalgamated product, we have a surjective $*$-homomorphism $\xi : *_{C^*(T)}\{C^*(G_i,f)\} \to C^*(G,f)$. We note here that ξ satisfies $\sigma_i = \xi \circ \gamma_i$ for all $i \in I$, where $\gamma_i : C^*(G_i,f) \to *_{C^*(T)}\{C^*(G_i,f)\}$ are the canonical maps [2].

Next, define a $*$-homomorphism $\pi : C^*(G,f) \to *_{C^*(T)}\{C^*(G_i,f)\}$ by sending all $S_e \in S_T$ to $S_e * 1 * \cdots * 1$, all $P_v \in P$ to $P_v * 1 * \cdots * 1$ (observe, these are the generators of $C^*(T)$), and all S_{e_i} to $1 * 1 * \cdots * S_{e_i} * 1 * \cdots * 1$ (recall, $e_i \in G_i^1 \setminus T$). Then π is onto a generating set, so π is surjective.

Finally, we need to show that ξ and π are inverses of each other. We'll show that they are on the generating sets, and therefore in their entirety. First, let X be a generator of $C^*(G,f)$; then $X = P_v$ for some $v \in G^0$, $X = S_e \in S_T$, or $X = S_{e_i}$ for some $i \in I$. If $X = P_v$ or $X = S_e$, then $\xi(\pi(X)) = \xi(X * 1 * \cdots * 1)$, and the identity $\sigma_i = \xi \circ \gamma_i$ forces the result that $\xi(X * 1 * \cdots * 1) = X$. Similarly, if $X = S_{e_i}$, then $\xi(\pi(X)) = \xi(1 * 1 * \cdots * X * \cdots * 1)$, and again, $\sigma_i = \xi \circ \gamma_i$ forces $\xi(1 * 1 * \cdots * X * \cdots * 1) = X$. Thus, $\xi(\pi(X)) = X$ for all $X \in \{S, P, f\}$, and so too for all $X \in C^*(G,f)$. Next, let Y be a generator of $*_{C^*(T)}\{C^*(G_i,f)\}$; then $Y = P_v * 1 * \cdots * 1$ for some $v \in G^0$, or $Y = S_e * 1 * \cdots * 1$ for some $e \in T^1$, or $Y = 1 * 1 * \cdots * S_{e_i} * \cdots * 1$ for some $i \in I$. We continue to make use of the identity $\sigma_i = \xi \circ \gamma_i$; If $Y = P_v * 1 * \cdots * 1$ or $Y = S_e * 1 * \cdots * 1$, then the identity forces $\xi(Y) = P_v$ or $\xi(Y) = S_e$, respectively. Thus, $\pi(\xi(Y)) = Y$ by the construction of π. Similarly, if $Y = 1 * 1 * \cdots * S_{e_i} * \cdots * 1$ for some $i \in I$, then $\xi(Y) = S_{e_i}$, and again, $\pi(\xi(Y)) = Y$ by construction. Thus, $\pi(\xi(Y)) = Y$ for all Y generating $*_{C^*(T)}\{C^*(G_i,f)\}$, and so too for all $Y \in *_{C^*(T)}\{C^*(G_i,f)\}$. Hence, ξ and π are inverses of each other, so it follows that $C^*(G,f) \cong *_{C^*(T)}\{C^*(G,f)\}$. □

3.2. Graph algebras of maximally colored trees and graphs containing single cycles

Proposition 3.2.1. *Let $\{T, f\}$ be a finite graph containing no cycles (directed or otherwise), with $f : T^1 \to \mathbb{N}$ a one-to-one edge-coloring. Then $C^*(T,f) \cong M_n(\mathbb{C})$, where n is the number of vertices of T.*

Proof. Observe that $\{T, f\}$ is an edge-colored tree that disregards direction. By Proposition 1.5.7, we can reverse the direction of an edge and color it any color without changing the graph algebra, so long as that new edge doesn't have the same color as another edge with the same range vertex (this is true because the one-to-one edge-coloring ensures that the coloring of two edges with the same range vertex couldn't have been the same to begin with). Assign a root vertex r for T and let $k \in \mathbb{N}$, where $k = 1 + \max\{f(e) \mid e \in T^1\}$. Choose any edge e adjacent to the root vertex; if $s(e) = r$, define $g(e) = k$ (observe that no other edge could have color k and have the same range vertex as e), and if $r(e) = r$, apply Proposition 1.5.7 to flip the direction and recolor, so that $s(e) = r$ and $g(e) = k$. Continue this process by completing all edges adjacent to the root first, then moving outward toward the leaves of the tree. The result will be a single-colored graph T_1 (we can omit the coloring g since T_1 is effectively uncolored) with $r(e) \neq r(h)$ for all $e, h \in T_1^1$. It is known that $C^*(T_1) = M_n(\mathbb{C})$ as stated above in Proposition 1.4.4. Thus, by Proposition 1.5.7, $C^*(T, f) \cong M_n(\mathbb{C})$. □

Proposition 3.2.2. *Let $\{C, f\}$ be a finite graph composed of exactly a single cycle with any number of branches stretching out of the cycle (disregarding direction in all cases), with $f : C^1 \to \mathbb{N}$ a one-to-one edge-coloring. Then $C^*(C, f) \cong M_n(C(\mathbb{T}))$, where n is the number of vertices of C.*

Proof. Again, we will make use of Proposition 1.5.7. Assign a starting vertex v for C and let $k \in \mathbb{N}$, where $k = 1 + \max\{f(e) \mid e \in C^1\}$. Choose either edge e adjacent to the starting vertex; if $s(e) = v$, define $g(e) = k$ (observe that no other edge could have color k and have the same range vertex as e), and if $r(e) = v$, apply Proposition 1.5.7 to flip the direction and recolor, so that $s(e) = v$ and $g(e) = k$. Continue this process by moving next to the edge e_2 with $s(e_2) = r(e)$ (note that $r(e) \neq v$ since the edge has been flipped already), and assigning $g(e_2) = k$. Keep moving around the cycle similarly. Next, if there are any remaining edges, they must be connected to the cycle by vertices on the cycle. Treat each of these vertices as a root for the trees that must branch off of the cycle, and for each of these complete the process as described in the proof of Proposition 3.2.1. The result will be a single-colored graph C_1 (we can omit the coloring g since C_1 is effectively uncolored) with $r(e) \neq r(h)$ for all $e, h \in C_1^1$. It is known that $C^*(C_1) = M_n(C(\mathbb{T}))$ as stated and shown above in Theorem 2.4.1. Thus, by Proposition 1.5.7, $C^*(C, f) \cong M_n(C(\mathbb{T}))$. □

3.3. Main results for full amalgamated free products

Corollary 3.3.1. *Let $\{G, f\}$ be a finite edge-colored directed graph with $f : G^1 \to \mathbb{N}$ a one-to-one edge-coloring. Then $C^*(G, f) \cong *_{M_n(\mathbb{C})}\{M_n(C(\mathbb{T}))\}$, where n is the number of vertices of G.*

Proof. Observe that $\{G, f\}$ is the same as the graph described in Theorem 3.1.1. Then, with $\{G, f\}$ described in the same way, we have already seen that $C^*(G, f) = *_{C^*(T)}\{C^*(G_i, f)\}$. From Proposition 3.2.1 we know that $C^*(T) \cong M_n(\mathbb{C})$. Also, by Proposition 3.2.2 we have that $C^*(G_i, f) \cong M_n(C(\mathbb{T}))$, since adding an edge to a spanning tree will create exactly one cycle in the graph. Thus, by substitution, $C^*(G, f) \cong *_{M_n(\mathbb{C})}\{M_n(C(\mathbb{T}))\}$. □

Theorem 3.3.2. $*_{M_n(\mathbb{C})}\{M_n(C(\mathbb{T}))\}_k \cong M_n(*_{\mathbb{C}}\{C(\mathbb{T})\}_k)$

Proof. Denote the h^{th} copy of $M_n(C(\mathbb{T}))$ in the free product on the left above by $M_n(C(\mathbb{T}))_{(h)}$. For each copy $M_n(C(\mathbb{T}))_{(h)}$ for $h = 1, \ldots, k$, let the generator of $C(\mathbb{T})_{(h)}$ be denoted by z_h. Then $M_n(C(\mathbb{T}))_{(h)}$ is generated by the set $\{e_{ii}\,(i = 1, \ldots, n), e_{(i+1)i}\,(i < n), z_h e_{1n}\}$ (see the proof of Lemma 2.1.1 for discussion). Define

$$\sigma_h : M_n(C(\mathbb{T}))_{(h)} \to M_n(*_{\mathbb{C}}\{C(\mathbb{T})\}_k)$$

as an inclusion map on the generators of $M_n(C(\mathbb{T}))_{(h)}$; specifically, $\sigma_h(e_{ij}) = (1 * 1 * \cdots * 1)f_{ij}$ and $\sigma_h(z_h e_{ij}) = (1 * \cdots 1 * z_h * 1 * \cdots * 1)f_{ij}$, where z_h is in the h^{th} slot. Then by construction we have $\sigma_g|_{M_n(\mathbb{C})} = \sigma_h|_{M_n(\mathbb{C})}$ for all g, h, and $\sigma_h|_{M_n(\mathbb{C})}$ is an injective $*$-homomorphism. And, since σ_h is an inclusion map for all h, we observe that $M_n(*_{\mathbb{C}}\{C(\mathbb{T})\}_k)$ is generated by $\cup\sigma_h(M_n(C(\mathbb{T}))_{(h)})$. Thus, by the universal properties of the free product, we have a surjective $*$-homomorphism

$$\xi : *_{M_n(\mathbb{C})}\{M_n(C(\mathbb{T}))\}_k \to M_n(*_{\mathbb{C}}\{C(\mathbb{T})\}_k).$$

From [2] we know that ξ satisfies the identity $\sigma_h = \xi \circ \gamma_h$ for all h, where $\gamma_h : M_n(C(\mathbb{T}))_{(h)} \to *_{M_n(\mathbb{C})}\{M_n(C(\mathbb{T}))\}_k$ are the canonical maps.

Now, define a map $\alpha_h : C(\mathbb{T})_{(h)} \to M_n(C(\mathbb{T}))_{(h)}$ by sending the generator $z_h \in C(\mathbb{T})_{(h)}$ to $z_h e_{11}$, and thus scalars $c \in \mathbb{C}$ to ce_{11}. We claim that α_h is a $*$-homomorphism. The map is linear: Let $\beta, \delta \in \mathbb{C}$. Then,

$$\begin{aligned} \alpha_h(\beta f(z_h) + \delta g(z_h)) = [\beta f(z_h) + \delta g(z_h)]e_{11} &= \beta[f(z_h)e_{11}] + \delta[g(z_h)e_{11}] \\ &= \beta\,\alpha_h(f(z_h)) + \delta\,\alpha_h(g(z_h)). \end{aligned}$$

The map is multiplicative:

$$\alpha_h(f(z_h)\,g(z_h)) = [f(z_h)\,g(z_h)]e_{11} = [f(z_h)e_{11}][g(z_h)e_{11}] = \alpha_h(f(z_h))\,\alpha_h(g(z_h)).$$

Finally, the map respects the $*$-operation:

$$\alpha_h(f(z_h)^*) = \alpha_h(\overline{f(z_h)}) = \overline{f(z_h)})e_{11} = \overline{f(z_h)e_{11}} = [\alpha_h(f(z_h))]^*.$$

Thus, α_h is a $*$-homomorphism for all h.

Next, we define another map $\pi : *_{\mathbb{C}}\{C(\mathbb{T})\}_k \to *_{M_n(\mathbb{C})}\{M_n(C(\mathbb{T}))\}_k$ by

$$\pi(x) = \pi(x_1 * x_2 * \cdots * x_r) := \alpha_{h_1}(x_1) * \alpha_{h_2}(x_2) * \cdots * \alpha_{h_r}(x_r)$$

where $x_i \in C(\mathbb{T})_{h_i}$. Since α_{h_j} is a $*$-homomorphism for all j, we know that, by construction, π will be a $*$-homomorphism as well.

Finally, define $\phi : M_n(*_{\mathbb{C}}\{C(\mathbb{T})\}_k) \to *_{M_n(\mathbb{C})}\{M_n(C(\mathbb{T}))\}_k$ by

$$\phi([x_{ij}]) = \sum_{i=1}^{n}\sum_{j=1}^{n} e_{i1}\pi(x_{ij})e_{1j}.$$

Then ϕ is linear since π is linear and summations respect linearity. Next we consider multiplication:

$$\begin{aligned}
\phi([x_{ij}][y_{pq}]) &= \phi\left(\left[\sum_{l=1}^{n} x_{il}\,y_{lq}\right]_{iq}\right) \\
&= \sum_{i=1}^{n}\sum_{q=1}^{n} e_{i1}\,\pi\left(\sum_{l=1}^{n} x_{il}\,y_{lq}\right)e_{1q} \\
&= \sum_{i=1}^{n}\sum_{q=1}^{n}\sum_{l=1}^{n} e_{i1}\,\pi(x_{il}\,y_{lq})e_{1q} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{p=1}^{n}\sum_{q=1}^{n} e_{i1}\,\pi(x_{ij})e_{1j}\,e_{p1}\,\pi(y_{pq})e_{1q} \\
&= \left(\sum_{i=1}^{n}\sum_{j=1}^{n} e_{i1}\,\pi(x_{ij})e_{1j}\right)\left(\sum_{p=1}^{n}\sum_{q=1}^{n} e_{p1}\,\pi(y_{pq})e_{1q}\right) \\
&= \phi([x_{ij}])\phi([y_{pq}]).
\end{aligned}$$

Thus, ϕ is a homomorphism. The map ϕ also respects the $*$-operation:

$$\begin{aligned}
\phi\left([x_{ij}]^*\right) &= \phi\left([\overline{x_{ij}}]^T\right) \\
&= \phi\left([\overline{x_{ji}}]\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} e_{j1}\pi(\overline{x_{ji}})e_{1i} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} e_{j1}\pi(x_{ji}^*)e_{1i} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} e_{j1}(\pi(x_{ji}))^* e_{1i} \\
&= \left[\sum_{i=1}^{n}\sum_{j=1}^{n} e_{i1}\pi(x_{ij})e_{1j}\right]^* \\
&= [\phi([x_{ij}])]^* .
\end{aligned}$$

Hence, ϕ is a $*$-homomorphism. It is also surjective, since by construction ϕ is onto a generating set.

Lastly, we need to show that ϕ and ξ are inverses of each other. Begin by letting X be a generator for $*_{M_n(\mathbb{C})}\{M_n(C(\mathbb{T}))\}_k$. Then $X = e_{ij} * I * \cdots * I$ for some $i, j \leq n$, or $X = I * \cdots * I * z_h e_{ij} * I * \cdots * I$ for some $h \leq k$ and $i, j \leq n$, with $z_h e_{ij}$ in the the h^{th} slot; notice that because of the amalgamation over $M_n(\mathbb{C})$ these representations of the generators are not unique. We need to show that $\phi(\xi(X)) = X$. Observe, the identity $\sigma_h = \xi \circ \gamma_h$ forces $\xi(e_{ij} * I * \cdots * I) = (1 * 1 * \cdots * 1)f_{ij}$ and $\xi(I * \cdots * I * z_h e_{ij} * I \cdots * I) = (1 * \cdots * 1 * z_h * 1 * \cdots * 1)f_{ij}$. Then, using the definition of ϕ,

$$\begin{aligned}
\phi(\xi(e_{ij} * I * \cdots * I)) &= \phi((1 * 1 * \cdots * 1)f_{ij}) \\
&= \phi\left(\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & 1 * 1 * \cdots * 1 & \vdots \\ 0 & \cdots & 0 \end{bmatrix}\right) \\
&= e_{i1}\pi(1 * 1 * \cdots * 1)e_{1j} \\
&= e_{i1} * e_{11} * e_{11} * \cdots * e_{11} * e_{1j} \\
&= e_{ij} * I * \cdots * I, \qquad (3.1)
\end{aligned}$$

where the equality in line (3.1) is due to the amalgamation over $M_n(\mathbb{C})$. Similarly,

$$\begin{aligned}
\phi(\xi(I * \cdots * I * z_h e_{ij} * I * \cdots * I)) &= \phi((1 * \cdots * 1 * z_h * 1 * \cdots * 1) f_{ij}) \\
&= \phi\left(\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & 1 * \cdots * 1 * z_h * 1 * \cdots * 1 & \vdots \\ 0 & \cdots & 0 \end{bmatrix}\right) \\
&= e_{i1}\pi(1 * \cdots * 1 * z_h * 1 * \cdots * 1)e_{1j} \\
&= e_{i1} * e_{11} * \cdots * e_{11} * z_h e_{11} * e_{11} * \cdots * e_{11} * e_{1j} \\
&= I * \cdots * I * z_h e_{ij} * I * \cdots * I, \qquad (3.2)
\end{aligned}$$

where, again, the equality in line (3.2) is due to the amalgamation over $M_n(\mathbb{C})$. Hence, we have shown that $\phi(\xi(X)) = X$ for all generators X of $*_{M_n(\mathbb{C})}\{M_n(C(\mathbb{T}))\}_k$.

Finally, we need to show that if Y is a generator for $M_n(*_{\mathbb{C}}\{C(\mathbb{T})\}_k)$, then $\xi(\phi(Y)) = Y$. If Y is a generator, then $Y = (1 * 1 * \cdots * 1)f_{ij}$ for some $i, j \leq n$, or $Y = (1 * \cdots * 1 * z_h * 1 * \cdots * 1)f_{ij}$ for some $h \leq k$ and $i, j \leq n$, with z_h in the h^{th} slot. Then,

$$\begin{aligned}
\xi(\phi((1 * 1 * \cdots * 1)f_{ij})) &= \xi\left(\phi\left(\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & 1 * 1 * \cdots * 1 & \vdots \\ 0 & \cdots & 0 \end{bmatrix}\right)\right) \\
&= \xi(e_{i1}\pi(1 * 1 * \cdots * 1)e_{1j}) \\
&= \xi(e_{i1} * e_{11} * e_{11} * \cdots * e_{11} * e_{1j} \\
&= \xi(e_{ij} * I * \cdots * I)) \qquad (3.3) \\
&= (1 * 1 * \cdots * 1)f_{ij},
\end{aligned}$$

where the equality in line (3.3) is due to the amalgamation over $M_n(\mathbb{C})$. Similarly,

$$\begin{aligned}
\xi(\phi((1*\cdots*1*z_h*1*\cdots*1)f_{ij})) &= \xi\left(\phi\left(\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & 1*\cdots*1*z_h*1*\cdots*1 & \vdots \\ 0 & \cdots & 0 \end{bmatrix}\right)\right) \\
&= \xi(e_{i1}\pi(1*\cdots*1*z_h*1*\cdots*1)e_{1j}) \\
&= \xi(e_{i1}*e_{11}*\cdots*e_{11}*z_h e_{11}*e_{11}*\cdots*e_{11}*e_{1j}) \\
&= \xi(I*\cdots*I*z_h e_{ij}*I*\cdots*I) \qquad (3.4) \\
&= (1*\cdots*1*z_h*1*\cdots*1)f_{ij},
\end{aligned}$$

where, again, the equality in line (3.4) is due to the amalgamation over $M_n(\mathbb{C})$. Hence, we have that $\xi(\phi(Y)) = Y$ for all generators Y of $M_n(*_{\mathbb{C}}\{C(\mathbb{T})\}_k)$. Since these two maps are inverses of each other on the generators of their corresponding domains, we have that they are inverses on their domains. Therefore, we have $*_{M_n(\mathbb{C})}\{M_n(C(\mathbb{T}))\}_k \cong M_n(*_{\mathbb{C}}\{C(\mathbb{T})\}_k)$. □

3.4. Afterword

Future work could be done in the area of reduced free products; for example, we would like to show that a variant of Theorem 3.3.2 holds for the reduced free product as well. We plan to show this in forthcoming work.

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