

# **IDEAL GRAPHS**

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**Title**

IDEAL GRAPHS

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## ABSTRACT

In this dissertation, we explore various types of graphs that can be associated to a commutative ring with identity. In particular, if  $R$  is a commutative ring with identity, we consider a number of graphs with the vertex set being the set of proper ideals; various edge sets defined via different ideal theoretic conditions give visual insights and structure theorems pertaining to the multiplicative ideal theory of  $R$ . We characterize the interplay between the ideal theory and various properties of these graphs including diameter and connectivity.

## ACKNOWLEDGMENTS

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# CHAPTER 1. INTRODUCTION

## 1.1. Ring Theory Definitions And Elementary Results

In this study, we investigate a number of graphs that can be constructed from an arbitrary commutative ring with identity. The vertices of these graphs are the ideals of the ring, the definitions of the edges vary according to various ideal theoretic properties that we wish to highlight. Then, we elaborate upon the connections between graph properties and ideal properties. We begin with elementary definitions, examples, and theorems from graph theory [9], [7] and commutative ring theory [6] to provide the reader foundations in our primary topics. In particular, we focus on historical constructions such as the zero divisor graph (first proposed by Beck in [3]) and the irreducible divisor graph (first proposed by Coykendall and Maney in [5]).

**Definition 1.1.** A ring  $R$  is a set together with two binary operations, addition and multiplication. such that for all  $h, w, z \in R$  we have the following conditions:

- a)  $(R, +)$  is an abelian group.
- b) Multiplication is associative  $(hw)z = h(wz)$ . for all  $h, w, z \in R$
- c) The distributive laws hold in  $R$  for all  $h, w, z \in R$ .

(i)  $(h + w)z = (hz) + (wz)$

(ii)  $h(w + z) = (hw) + (hz)$

**Example 1.2.** The prototype example of a ring with identity is the integers  $\mathbb{Z}$ . More examples include the set of rational numbers  $\mathbb{Q}$ , the set of real numbers  $\mathbb{R}$ , the set of complex numbers  $\mathbb{C}$  and polynomial ring  $R[x] = \{\sum_{i=0}^{\infty} r_i x^i \mid r_i \in R\}$  where  $R$  is any ring. The ring  $2\mathbb{Z}$  is an example of a ring which does not have an identity.

From here we assume rings are commutative unless otherwise stated.

**Definition 1.3.** Let  $R$  be a ring. We say  $r \in R$  is a zero divisor, if there exists  $0 \neq a \in R$  such that  $ra = 0$ .

A nontrivial commutative ring with identity where zero is the only zero divisor is called an integral domain.

**Example 1.4.** In the ring  $\mathbb{Z}/8\mathbb{Z}$ ,  $\bar{4}$  is an example of a zero divisor element, since  $\bar{4}\bar{2} = \bar{0}$ . In the ring  $\mathbb{Z}$ , there are no nontrivial zero divisors.

**Definition 1.5.** A commutative ring  $R$  is called a field if every nonzero element is a unit (has a multiplicative inverse) . In the sense that for all  $0 \neq a \in R$  there exist  $b$  such that  $ab=1$ .

It is clear that any field is an integral domain. However, the converse is not necessarily true. The integers is an example of an integral domain that is not a field since for any two integers  $x$  and  $y$  such that  $xy=0$ , we have  $x = 0$  or  $y = 0$ . Hence zero is the only zero divisor. Also, for any  $n > 1$  the multiplicative inverse  $\frac{1}{n}$  is not an integer. Hence the integers  $\mathbb{Z}$  is an integral domain but it is not a field. The following family of rings will serve as well for illustration purposes later. We first present an elementary proposition.

**Proposition 1.6.** the ring  $\mathbb{Z}/n\mathbb{Z}$  is a field if and only if  $n$  is prime.

**Definition 1.7.** Let  $R$  be a ring. A subset  $I \subseteq R$  is called an ideal if for all  $h, w \in I$  and  $r \in R$  then

a)  $h - w \in I$

b)  $rh \in I$

We call the ideal  $M$  maximal if it is maximal among all proper ideals with respect to set containment. In other words, for all ideals  $M \subseteq I \subseteq R$ , then either  $I = M$  or  $I = R$ . Also, we call  $P$  a prime ideal if  $xy \in P$ , implies  $x \in P$  or  $y \in P$ . We recall the following theorem.



**Theorem 1.8.** Let  $R$  be a commutative ring with identity and assume  $I \subseteq R$  then;

- a)  $I$  is maximal if and only if  $R/I$  is a field.
- b)  $I$  is prime if and only if  $R/I$  is an integral domain.

It is immediate that any maximal ideal is prime. The converse is not true as the next example shows.

**Example 1.9.** The ideal  $(2, x)$  is a maximal ideal in the ring  $\mathbb{Z}[x]$ , hence it is prime. However the ideal  $(x)$  is a prime ideal in  $\mathbb{Z}[x]$ . But, it is not maximal.

Our next lemma is equivalent to the Axiom of Choice and is central in this study.

**Lemma 1.10** (Zorn's Lemma). Let  $S$  be a partially ordered set with the property that every chain in  $S$  has an upper bound in  $S$ . Then  $S$  has a maximal element.

Zorn's lemma leads to the inevitable conclusion that every commutative ring with identity has a maximal ideal. The next proposition shows something more general is true.

**Proposition 1.11.** Let  $R$  be a commutative ring with identity and let  $I \subset R$  be an ideal. Then there exists a maximal ideal  $M$  such that  $M \supset I$ .

Lastly we turn to a definition related to factorization and observation about domains.

**Definition 1.12.** Let  $D$  be an integral domain. And let  $a$  be an element in  $D$ .

- a) The element  $a \in D$  is irreducible if, for any  $a = a_1 a_2$  then either  $a_1$  or  $a_2$  is a unit in  $D$ .
- b) The element  $a \in D$  is prime if  $a|a_1 a_2$  implies  $a|a_1$  or  $a|a_2$ .

**Theorem 1.13.** Let  $D$  be an integral domain. Then every nonzero prime element is irreducible.

*Proof.* Suppose that  $a = xy$  is a nonzero prime element . Then either  $a|x$  or  $a|y$ . Without loss of generality assume that  $a|x$ . This implies  $x = da$  where  $d \in D$ . Since  $a = xy, x = dxy$ . Hence  $x - dxy = 0$  . From that we conclude  $dy = 1$ . Hence  $y$  is a unit.  $\square$

In general the converse is not true. However, if  $R$  is a unique factorization domain, then the notion of nonzero prime element and irreducible element are the same. For example, all irreducible elements in  $\mathbb{Z}$  are prime elements.

We now recall the concept of localizations.

**Definition 1.14.** Let  $R$  be a domain. A nonempty subset  $\Gamma \subseteq R$  (not containing 0) is said to be multiplicatively closed if it is closed under multiplication. In other words, if  $a, b \in \Gamma$  then  $ab \in \Gamma$ .

**Theorem 1.15.** [8] Let  $R$  be commutative ring with identity and  $I \subseteq R$  and ideal. If  $\Gamma$  is a multiplicatively closed set in  $R$  such that  $\Gamma \cap I = \emptyset$ . Then there is a prime ideal  $P$  such that  $I \subseteq P$  such that  $P \cap \Gamma = \emptyset$

**Definition 1.16.** Let  $R$  be a domain and  $\Gamma \subseteq R$  be a multiplicatively closed subset of  $R$ ; we define a localization of  $R$  at  $\Gamma$  to be  $R_\Gamma = \{\frac{r}{a} | r \in R \text{ and } a \in \Gamma\}$

We note that if  $P$  is a prime ideal then  $R_P = \{\frac{a}{b} | a, b \in R \text{ and } b \notin P\}$ . That is,  $\Gamma = R \setminus P$  which is multiplicatively closed, since  $P$  is prime.

**Definition 1.17.** A ring  $R$  is called a quasi-local ring if it has only one maximal ideal.

A prototype example of a quasi -local ring is the ring  $\mathbb{F}[[x]]$  where  $\mathbb{F}$  is a field.

If  $R$  is a domain with quotient field  $K$  and  $S \subseteq R$  is a multiplicative closed set then we always have the inclusions  $R \subseteq R_S \subseteq K$ .

The following proposition gives us an important connection between quasi-local rings and localizations.

**Proposition 1.18.** Let  $R$  be a commutative ring with identity. The ring  $R_P = \{\frac{r}{s} | r, s \in R \text{ and } s \notin P\}$  is a quasi-local ring for all prime ideals  $P$ .

We remark here that, if  $R$  is an integral domain and  $P$  be a prime ideal, then the localization over  $P$  is an integral domain as well.

**Proposition 1.19.** If  $R$  is an integral domain, then  $R_P$  is an integral domain for all prime ideals  $P$ .

The converse of the above proposition is not necessarily true. For example consider the ring  $\mathbb{Z}_6$ . The localization of  $R$  at  $P = (3)$  is an integral domain. However the ring  $\mathbb{Z}_6$  is not.

We now consider valuation domains which are of fundamental importance in multiplicative ideal theory. In some sense, they are the building blocks of integrally closed domains.

**Definition 1.20.** Let  $R$  be a commutative ring with identity then we say  $R$  is a valuation domain if given any two nonzero elements  $x, y \in R$  then either  $x|y$  or  $y|x$ . Equivalently, for all  $a \in K^*$ , either  $a$  or  $a^{-1}$  is an element of  $R$ .

We remark here if  $R$  is a valuation domain then  $R$  is quasi-local and all (prime) ideals of  $R$  are linearly ordered. The following theorem is one of the reasons that valuation domains are so central to multiplicative ideal theory.

**Theorem 1.21.** [8] Given an integral domain  $R$  with quotient field  $K$  and a proper ideal  $I \subsetneq R$ , then there exist a valuation overring  $V_I$ ,  $R \subsetneq V_I \subsetneq K$ , such that  $I$  survives in  $V_I$ .

The next theorem highlights the relation between the integral closure of a domain  $R$  and its valuation overrings  $V$ .

**Theorem 1.22.** Let  $R$  be a domain, then the integral closure of the ring  $R$  is given by  $\bar{R} = \bigcap_{R \subseteq V \subseteq K} V$ , where  $V$  ranges over all valuation overrings of  $R$ .

**Theorem 1.23.** Let  $R$  be a commutative ring with identity. Then the following are equivalent.

- a) Every ideal of  $R$  is finitely generated.
- b) Every prime ideal of  $R$  is finitely generated.
- c) Every ascending chain of ideals stabilizes.

Any ring satisfying one condition is called a noetherian ring.

**Theorem 1.24** (The Hilbert Basis Theorem). If  $R$  is commutative with identity and  $R$  is Noetherian, then so is  $R[x]$ .

There is an analogous result for power series.

**Theorem 1.25.** If  $R$  is commutative with identity and  $R$  is Noetherian, then so is  $R[[x]]$ .

**Definition 1.26.** Let  $R$  be a commutative ring with identity. We say that  $R$  is Artinian ring if it is satisfying the descending chain condition on ideals. In the sense that every strictly descending sequence of ideals stabilizes

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_q \supseteq I_{q+1} \supseteq \cdots$$

Then there exists a positive integer  $q$  such that  $I_q = I_{q+1} = \cdots$

**Definition 1.27.** A commutative ring with identity  $R$ , is zero-dimensional if every prime ideal is maximal.

The following result is classical and will be of use to us later.

**Theorem 1.28.**  $R$  is artinian if and only if  $R$  is Noetherian and zero dimensional.

We remark here that the integral domain is artinian ring if and only if the ring  $R$  is a field.

## 1.2. Graph Theory Definitions And Elementary Results

Modern graph theory has connections to many branches of mathematics such as topology, matrix theory and ring theory. In this section we give some basic concepts and theorem from graph theory. There are many books give us the basic concepts of graph theory for example [9] and [7].

**Definition 1.29.** A graph  $G$  is a pair  $(V, E)$  where  $V$  is a nonempty set of vertices and  $E$  is the set of edges.

In a graph  $G$ , If any two vertices are connected by edge then they are called adjacent, otherwise they are called disjoint. The following is an example of a graph where the number of vertices are 6 and the number of edges are 7. We can see that the vertex 5 is adjacent with vertex 4. However 5 and 3 are disjoint.

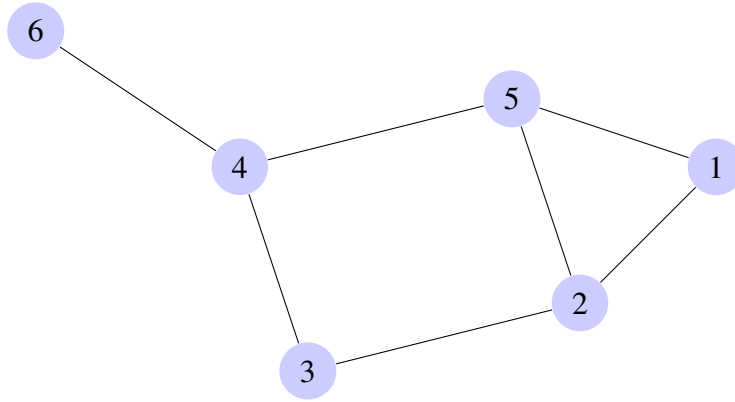


Figure 1.1: Graph  $G = (V, E)$

**Definition 1.30.** A graph  $G^* = (V^*, E^*)$  is called a subgraph of  $G$  if  $V^* \subseteq V$  and  $E^* \subseteq E$ .

In the following example the graph  $H$  is a subgraph of  $G = (V, E)$ , since  $V^* \subseteq V$  and  $E^* \subseteq E$ .

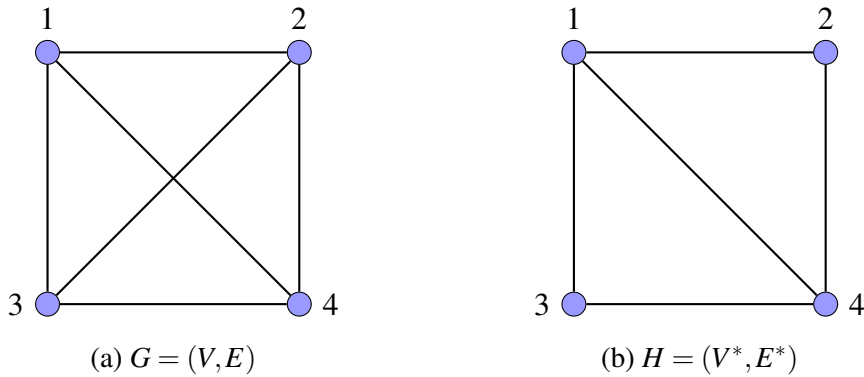


Figure 1.2: Graph  $G$  and its subgraph  $H$

**Definition 1.31.** Let  $G$  be a graph.

- (i) We say  $G$  is simple if it has no loops or parallel edges.
- (ii) We say  $G$  is a complete graph if any two vertices of graph  $G$  are adjacent. The complete graph on  $n$  vertices is denoted by  $K_n$ .
- (iii) A graph  $G$  is finite if the number of vertices and edges are finite.

The following graph  $K_6$  is complete.

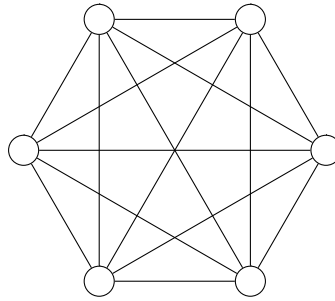


Figure 1.3: Complete graph  $K_6$

**Definition 1.32.** In the graph  $G$

- (i) A walk is an alternating sequence of vertices starting at  $v_1$  and ending at  $v_2$  and connecting edges.
- (ii) A path is a walk that does not include any repeated vertices.

**Definition 1.33.** The diameter of a graph  $G = (V, E)$  denoted by  $D(G)$  is the maximum distance among all pairs of vertices in  $G$  where the distance between any two vertices  $v_1$  and  $v_2$  is the length of the shortest path joining  $v_1$  and  $v_2$ .

In the following example the diameter of the graph  $G = (V, E)$  is 3, because if we look to the length of the shortest path joining 3 and 4 is 3 which is the maximum distance among all pairs of vertices in  $G = (V, E)$ .

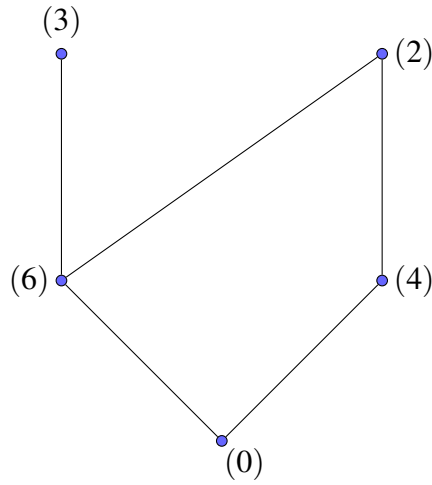


Figure 1.4: The diameter of the graph  $G = (V, E)$



Our next definition is one of the most important concepts in graph theory and it is central in our study.

**Definition 1.34.** A graph  $G$  is called connected if there is a path between any vertex to any other vertex in the graph. Otherwise it is called disconnected graph.

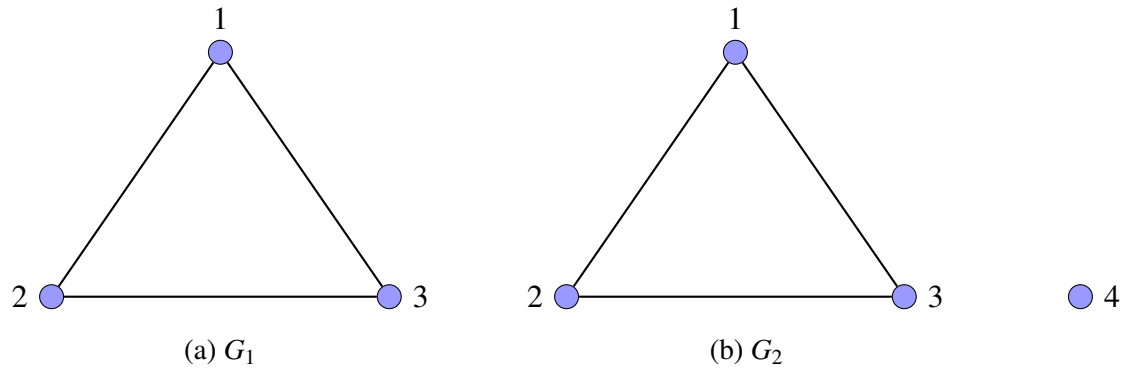


Figure 1.5: Connected graph  $G_1$  and disconnected graph  $G_2$

We remark here that any complete graph is connected. However the converse is not necessarily true. The following are examples of connected graphs but they are not complete.

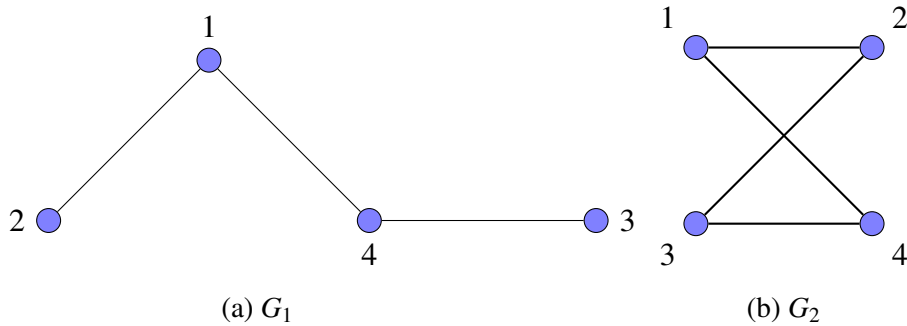


Figure 1.6:  $G_1$  and  $G_2$  are connected graphs but not complete

### 1.3. Preliminary Results Concerning Graphs And Commutative Algebra

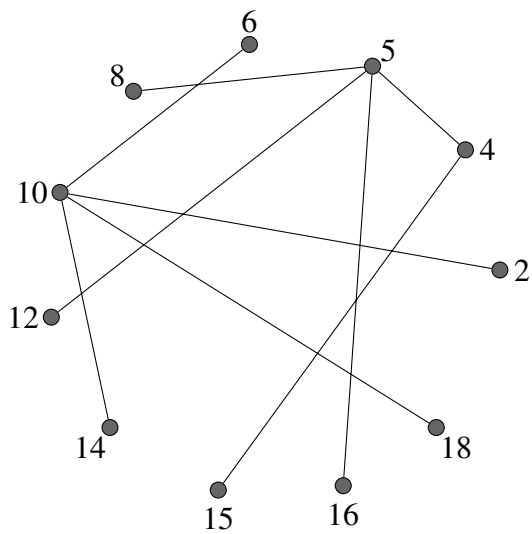
Recently there has been much attention paid to various aspects of commutative algebra and graphs that can be associated to various structures and objects. One such example that highlights the interplay between commutative algebra and graph theory is the notion of zero divisor graphs. The concept of the zero divisor graph was first proposed by Beck in [3]. Beck defined the zero divisor graph as follows.

**Definition 1.35.** Let  $R$  be a ring. We say  $Z(R)$  is a zero divisor graph if the set of the vertices are the elements of  $R$ . Two elements  $x$  and  $y$  in the ring  $R$  are adjacent if  $xy=0$ .

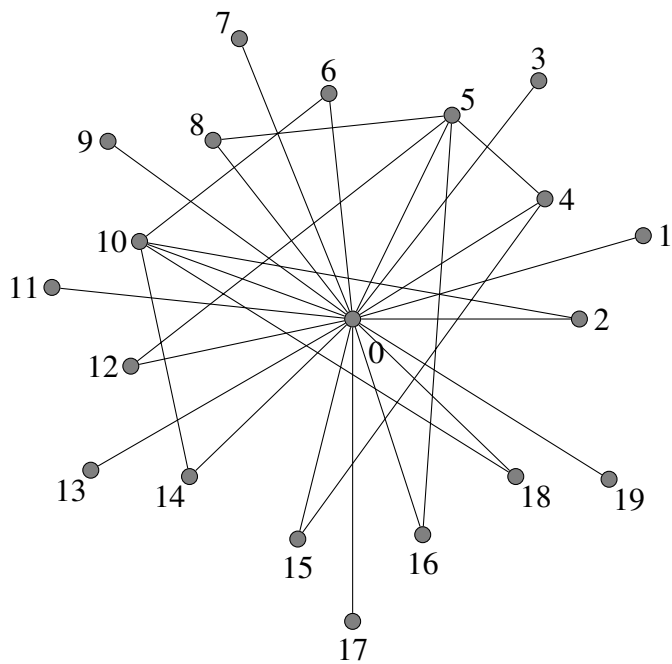
It should be noted that given two vertices  $a$  and  $b$  in the Beck zero divisor graphs there is a path from  $a$  to  $b$  via  $a - 0 - b$ . Hence every Beck zero divisor graph is connected of diameter less than or equal 2. Anderson and Livingston refined Beck's notion to get a better view of the zero divisor structure. In Anderson and Livingston's paper [1] they simplify the definition of the zero divisor graph. In the new definition the set of vertices is the set of nonzero zero divisors of the ring.

**Definition 1.36.** [Anderson and Livingston] [1] Let  $R$  be a commutative ring with identity. The zero divisor graph  $Z(R)$  is a simple graph where is the vertex set defined to be the set of nonzero divisors of  $R$ . And there is an edge between the distinct vertices  $z_1, z_2 \in R$  if and only if  $z_1 z_2 = 0$ .

In the following example we can see the difference between Anderson and Livingston's definition and Beck's definition for the ring  $Z_{20}$ .



(a) Zero divisor graph of  $Z_{20}$  by Anderson and Livingston's definition



(b) Zero divisor graph of  $Z_{20}$  by using Beck's definition

Figure 1.7: Zero divisor graphs of  $Z_{20}$

The following very interesting theorem spawned an industry of research in zero-divisor graphs.

**Theorem 1.37.** (Anderson and Livingston) [1]. Let  $R$  be a commutative ring with identity and  $Z(R)$  its zero divisor graph. Then  $Z(R)$  is connected and has diameter less than or equal to 3.

Another structure that highlights the interplay between graphs and commutative algebra is the irreducible divisor graph. The idea of irreducible divisor graph originated with Coykendall and Maney [5]. The authors of this paper used the irreducible divisor graph to characterize certain classes of domains. The authors focused on the case where  $R$  is atomic (every nonzero nonunit element of  $R$  can be written as a products of irreducibles).

**Definition 1.38.** (Coykendall and Maney) [5]. Let  $R$  be a ring

- (i)  $Irr(R)$  is the set of all irreducible elements in  $R$ .
- (ii)  $\overline{Irr}(R)$  is the set of equivalence classes of irreducibles of  $R$  modulo unit equivalence  
 $(r_1 \sim r_2 \iff r_1 = ur_2 \text{ for some } u \in U(R)).$

**Definition 1.39.** (Coykendall and Maney ) [5]. Let  $R$  be an atomic domain. Let  $a \in R$  be a nonzero nonunit element. The irreducible divisor graph of an element  $a$ , is given by  $G(a) = (V, E)$ , where the vertex set  $V$  is  $\{v \in \overline{Irr}(R) | v|a\}$ , and given  $v_1, v_2 \in \overline{Irr}(R)$ ,  $v_1 v_2 \in E$  if and only if  $v_1 \cdot v_2 | a$ .

In [5] the authors gave some examples on irreducible divisor graph. We will provide some of them.

**Example 1.40.** Let  $R := \mathbb{Z}[\sqrt{-5}]$ . Note that up to unit equivalence the only irreducible factorization of 6 are  $6 = (1 + \sqrt{-5})(1 - \sqrt{-5}) = (2)(3)$ . The irreducible divisor graph  $G(6)$  is pictured below.

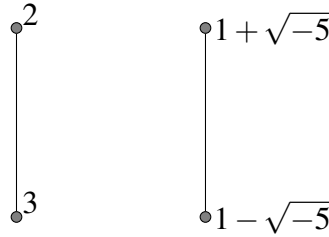


Figure 1.8:  $G(6)$

**Example 1.41.** Let  $R := \mathbb{Z}[\sqrt{-14}]$ . Up to unit equivalence the only irreducible factorizations of 81 are  $81 = (5 - 2\sqrt{-14})(5 + 2\sqrt{-14}) = 3^4$ . The irreducible divisor graph  $G(81)$  is pictured below.

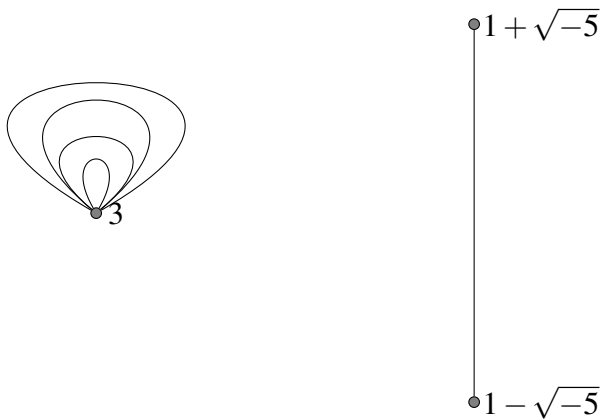


Figure 1.9:  $G(81)$

The following is one of the preliminary results of Coykendall and Maney [5]. We first need to recall the definition of a finite factorization domain.

**Definition 1.42.** Let  $R$  be an atomic domain. We call  $R$  is a finite factorization domain if every nonzero nonunit element of  $R$  has finitely many nonassociate irreducible divisors.

**Proposition 1.43.** [5] Let  $R$  be an atomic domain . Then  $R$  is an finite factorization domain if and only if the irreducible divisor graph  $G(x)$  is finite for each nonzero nonunit  $x \in R$ .

It is worth noting that in 2008, M. Axtell and J. Stickles [2] generalized the concept of irreducible divisor graph by Coykendall and Maney to commutative rings with zero divisors.

## CHAPTER 2. GRAPHS DETERMINED BY IDEAL THEORETIC PROPERTIES

### 2.1. Ideal Graphs Definitions And Elementary Results

In this section we define and justify some types of graphs that are determined by ideal theoretic properties. We investigate eight graphs constructed from an arbitrary commutative ring with identity with the vertex set being the set of proper ideals.

**Definition 2.1.** Let  $R$  be a commutative ring with identity. We define the following graphs associated to  $R$ , in all cases, the vertex set is the set of ideals of  $R$ :

- (i) For  $G_0(R)$ , we say that  $I$  and  $J$  have an edge between them if and only if  $I$  and  $J$  are adjacent.
- (ii) For  $G_1(R)$ , we say that  $I$  and  $J$  have an edge between them if and only if there is a maximal ideal  $\mathfrak{M}$  such that  $I = J\mathfrak{M}$ .
- (iii) For  $G_2(R)$ , we say that  $I$  and  $J$  have an edge between them if and only if  $(I : J) = \mathfrak{M}$
- (iv) For  $G_3(R)$ , we say that  $I$  and  $J$  have an edge between them if and only if there is a nonzero nonunit  $a \in R$  such that  $I = Ja$ .
- (v) Let  $R$  be an integral domain with quotient field  $K$ . In  $G_4(R)$ , we say that  $I$  and  $J$  have an edge between them if and only if there is a nonzero  $k \in K$  such that  $I = Jk$ .
- (vi) For  $G_5(R)$  we say that  $I$  and  $J$  have an edge between them if and only if  $I \otimes_R J = 0$
- (vii) For  $G_6(R)$  we say that  $I$  and  $J$  have an edge between them if and only if  $I \subset J$  and  $J/I$  is a finitely generated ideal of  $R/I$ .
- (viii) For  $G_7(R)$  we say that  $I$  and  $J$  have an edge between them if and only if  $I \subset J$  and  $J/I$  is a principal ideal of  $R/I$ .

We give some results and justifications for some types of graphs that are determined by ideal theoretic properties. We begin with a result that lends another perspective to  $G_5(R)$ .

**Theorem 2.2.** Let  $I, J \subseteq R$  be ideals. Then  $I \otimes_R J = 0 \iff IJ = 0$ .

*Proof.* Suppose first that  $IJ = 0$ . Then for all  $x \in I$  and  $y \in J$ , we have  $xy = 0$ . If  $x \otimes y \in I \otimes_R J$  then  $x \otimes y = 1 \otimes xy = 1 \otimes 0 = 0$ . Since  $I \otimes_R J$  is generated by tensors, then  $I \otimes_R J = 0$ . For the other direction, consider the map  $\phi : I \times J \rightarrow IJ$  such that  $\phi(x, y) = xy$ . Clearly  $\phi$  is bilinear. By the universal mapping property of tensor product, there exist  $\Phi : I \otimes_R J \rightarrow IJ$  such that  $\Phi \iota = \phi$  where  $\iota : I \times J \rightarrow I \otimes_R J$  is the canonical bilinear map given by  $\iota(x, y) = x \otimes y$ . Note that if  $x \in I$  and  $y \in J$  then  $\Phi(x \otimes y) = xy \in IJ$ . Hence, as all generators of  $IJ$  are in the image of  $\Phi$ ,  $\Phi$  is onto. Hence, as  $I \otimes_R J = 0$  and  $\Phi$  is onto,  $IJ = 0$ .  $\square$

From the above theorem we can conclude that In  $G_5(R)$  we say that  $I$  and  $J$  have an edge between them if and only if  $I \otimes_R J = 0 \iff IJ = 0$ .

**Notation 2.3.** Let  $R$  be a ring. We denote by  $G_i^*(R)$ , the subgraph of  $G_i(R)$  with the zero ideal removed from the vertex set.

**Lemma 2.4.** Let  $I, J \subseteq R$  be distinct ideals. If  $I = J\mathfrak{M}$  where  $\mathfrak{M}$  is a maximal ideal then  $(I : J) = \mathfrak{M}$ , but not conversely.

*Proof.* Note that  $\mathfrak{M}J \subseteq I$ , and hence  $\mathfrak{M} \subseteq (I : J)$ . Since  $I$  and  $J$  are distinct, we must have equality.

To see that the converse does not hold consider the domain  $K[x, y]$ , where  $K$  is any field, and the ideals  $I = (x, xy, y^2)$  and  $J = (x, y)$ . Note that  $(I : J) = J$  but  $J^2 \subsetneq I$ .  $\square$

**Lemma 2.5.** If  $I \subset J$  are adjacent, then  $(I : J)$  is maximal.

*Proof.* Let  $I \subsetneq J$  be adjacent and let  $\mathcal{C} = (I : J)$ ; we will show that  $\mathcal{C}$  is maximal. Since  $I$  is strictly contained in  $J$ , we can find an  $x \in J \setminus I$ ; additionally, we note that  $J = (I, x)$  because of the adjacency of  $I$  and  $J$ .



We now choose an arbitrary  $z \notin \mathcal{C}$  and note that  $zx \notin I$  (indeed, if  $zx \in I$  then the fact that  $J = (I, x)$  would show that  $z \in \mathcal{C}$  which is a contradiction). Hence, it must be the case that  $J = (I, zx)$ , and since  $x \in J$ , we obtain

$$x = rzx + \alpha$$

for some  $r \in R$  and  $\alpha \in I$ . Rearranging the above, we now have

$$x(1 - rz) = \alpha \in I.$$

Since  $(1 - rz)$  conducts  $x$  to  $I$  and  $J = (I, x)$ ,  $(1 - rz) \in \mathcal{C}$ . Therefore  $(\mathcal{C}, z) = R$  for all  $z \notin \mathcal{C}$  and so  $\mathcal{C}$  is maximal. □

**Lemma 2.6.** If  $I \subset J$  are adjacent, then  $J/I$  is a principal ideal of  $R/I$ .

*Proof.* Assume that  $I \subset J$  are adjacent. Since  $I$  is strictly contained in  $J$ , we can find an  $a \in J \setminus I$ ; additionally, we note that  $J = (I, a)$  because of the adjacency of  $I$  and  $J$ . Therefore  $J/I = (a + I)$  is a principal ideal of  $R/I$ . □

**Notation 2.7.** All of the graphs have the same vertex set.  $G_0(R)$  is a subgraph of  $G_2(R)$  and  $G_1(R)$  is a subgraph of  $G_2(R)$  excepting loops. If  $R$  is an integral domain then  $G_3(R)$  is a subgraph of  $G_4(R)$ .  $G_7(R)$  is a subgraph of  $G_6(R)$  and  $G_0(R)$  is a subgraph of  $G_7(R)$ .

Now we make a connectivity conclusion such as if  $G_0$  is connected then so is  $G_2$ .

**Lemma 2.8.** Suppose that  $I \subsetneq J$  are adjacent and  $\mathcal{C} := (I : J)$ . If  $\mathcal{C} \cap J = I$  then  $J \setminus I$  is a multiplicatively closed set.

*Proof.* Let  $x, y \in J \setminus I$ . Certainly  $xy \in J$ . By way of contradiction, we assume that  $xy \in I$ . Since  $I$  and  $J$  are adjacent and  $x \notin I$ ,  $(x, I) = J$ .

Now let  $j \in J$  be arbitrary. By the previous remark, we can find  $r \in R$  and  $i \in I$  such that  $j = rx + i$ . Multiplying this by  $y$  we obtain that  $yj = rxy + iy \in I$ . Hence  $y \in \mathcal{C} \cap J = I$  which is the desired contradiction.  $\square$

**Lemma 2.9.** Let  $I \subsetneq J$  be adjacent and  $A$  another ideal. Then either  $I \cap A = J \cap A$  or  $I \cap A$  and  $J \cap A$  are adjacent.

*Proof.* We will assume that  $I \cap A$  and  $J \cap A$  are distinct and suppose that  $x \in (J \cap A) \setminus (I \cap A)$ . Since  $x \notin I$ , it must be the case that  $J = (I, x)$ .

Let  $j \in J \cap A$  be arbitrary. Since  $J = (I, x)$ , we have that

$$j = rx + \alpha$$

for some  $r \in R$  and  $\alpha \in I$ . We observe further that  $rx \in J \cap A$ , and hence,  $\alpha \in I \cap A$ . We conclude that

$$J \cap A = (I \cap A, x)$$

for any  $x \in (J \cap A) \setminus (I \cap A)$ , and so  $I \cap A$  and  $J \cap A$  must be adjacent.  $\square$

**Lemma 2.10.** If  $I \subsetneq J$  are adjacent ideals and  $x \in R$ . Then the ideals  $(I, x)$  and  $(J, x)$  are either equal or adjacent.

*Proof.* Suppose that  $A$  is an ideal with  $(I, x) \subsetneq A \subseteq (J, x)$  and let  $a \in A \setminus (I, x)$ . We write  $a = j + rx$  with  $j \in J \setminus I$ . Since  $I$  and  $J$  are adjacent and  $j \notin I$ ,  $(I, j) = J$ . Hence  $A$  contains  $J$  and  $x$ ; we conclude that  $A = (J, x)$  and the proof is complete.  $\square$

**Proposition 2.11.** Let  $I \subseteq J$  be adjacent ideals and  $S \subset R$  be a multiplicatively closed set (not containing 0). Then,  $I_S \subseteq J_S$  are either adjacent or equal.

*Proof.* Since  $I \subsetneq J$  are adjacent, there is an  $x \in J \setminus I$  such that  $J = (I, x)$  (in fact, any  $x \in J \setminus I$  will do). Suppose that  $I_S \subsetneq J_S$ . So there is a  $\frac{b}{s} \in J_S \setminus I_S$ . Certainly it is the case that  $b \in J \setminus I$  and since  $I \subsetneq J$  are adjacent, we can find an  $r \in R$  and  $\alpha \in I$  such that  $b = rx + \alpha$ . In  $R_S$  we have the equation

$$\frac{b}{s} = \left(\frac{rt}{s}\right)\left(\frac{x}{t}\right) + \frac{\alpha}{s}$$

which shows that  $J_S$  is generated over  $I_S$  by any element  $\frac{x}{t} \in J_S \setminus I_S$ . Hence  $I_S$  and  $J_S$  are either equal or adjacent.  $\square$

**Proposition 2.12.** Let  $A, B \subseteq R$  be ideals containing the ideal  $I$ . Then  $A$  and  $B$  are adjacent in  $R$  if and only if  $A/I$  and  $B/I$  are adjacent in  $R/I$ .

*Proof.* Suppose  $A \subsetneq B$  both contain the ideal  $I$  and are adjacent. If there is an ideal of  $R/I$  strictly between  $A/I$  and  $B/I$ . This ideal can be written in the form  $C/I$  where  $C$  is an ideal strictly between  $A$  and  $B$ , which is a contradiction.

Now we suppose that  $A$  and  $B$  contain  $I$  with  $A/I$  and  $B/I$  adjacent. If there is an ideal (say  $C$ ) strictly between  $A$  and  $B$ , then we have the containment

$$A/I \subseteq C/I \subseteq B/I.$$

To see that the first containment is, in fact, strict, we suppose that  $A/I = C/I$ . Hence, given any  $c \in C$ , we can write  $c = a + z$  for some  $a \in A$ ,  $z \in I$ . But since  $I \subseteq A$ , we obtain that  $c \in A$  and so  $C = A$  which is a contradiction. The strictness of the second containment is shown similarly.  $\square$

**Lemma 2.13.** Let  $I \subseteq J$  be ideals such that  $J$  is finitely generated over  $I$ . Then  $I$  is contained in a finitely generated ideal if and only if  $J$  is contained in a finitely generated ideal.

*Proof.* For the backward direction, if  $J$  is contained in a finitely generated ideal, then so is  $I$ , since  $I \subseteq J$ . For the other direction, suppose that  $I$  is contained in a finitely generated

ideal, say  $N$ , where  $N$  is finitely generated. Since  $J$  is finitely generated over  $I$ , we can write  $J = (I, a_1, a_2, \dots, a_m)$ . It is clear now that  $J$  is contained in the finitely generated ideal  $(N, a_1, a_2, \dots, a_m)$ . Therefore  $J$  is contained in a finitely generated ideal.  $\square$

**Proposition 2.14.** Let  $R$  be a 1–dimensional domain.  $R$  is Noetherian if and only if  $R/I$  is Artinian for each ideal  $0 \neq I \subseteq R$ .

*Proof.* Suppose first that  $R$  is a Noetherian domain. If  $I \subseteq R$  is a nonzero ideal, then  $R/I$  is Noetherian of dimension 0 and hence is Artinian.

Now suppose that  $R/I$  is Artinian for each nonzero ideal  $I \subset R$ . It suffices to show that every ideal of  $R$  is finitely generated. Let  $J \subset R$  be an arbitrary nonzero ideal. Let  $0 \neq x \in J$  and note that by hypothesis,  $R/(x)$  is Artinian. From Lemma 2.13 it follows that  $J$  is finitely generated and hence  $R$  is Noetherian.  $\square$

## 2.2. Examples

In this section we provide the reader with some examples that give a clear view of our investigated graphs.

**Example 2.15.** Consider the ring  $R = \mathbb{Z}_{12}$ . The proper ideals of  $R = \mathbb{Z}_{12}$  are  $(0), (2), (3), (4), (6)$ . In figure 2.1 we provide an examples of  $G_0, G_1, G_2$  and  $G_3$ .

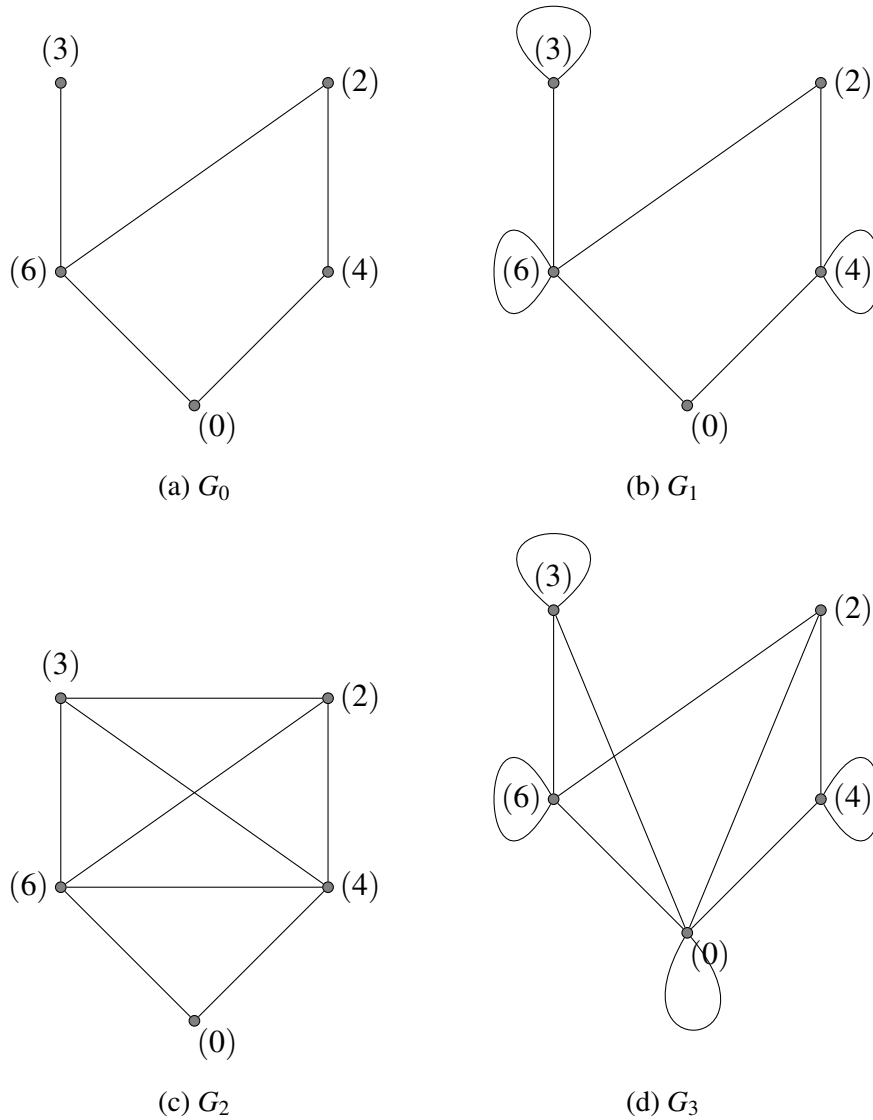


Figure 2.1: Ideal graphs  $G_0, G_1, G_2$  and  $G_3$  of the ring  $R = \mathbb{Z}_{12}$

**Example 2.16.** Consider the ring  $\mathbb{Z}_{10}$ . The set of ideals in  $\mathbb{Z}_{10}$  are  $(0)$ ,  $(2)$ ,  $(4)$  and  $(5)$ . Figure 2.2 shows the ideal graph  $G_5$  of the ring  $\mathbb{Z}_{10}$

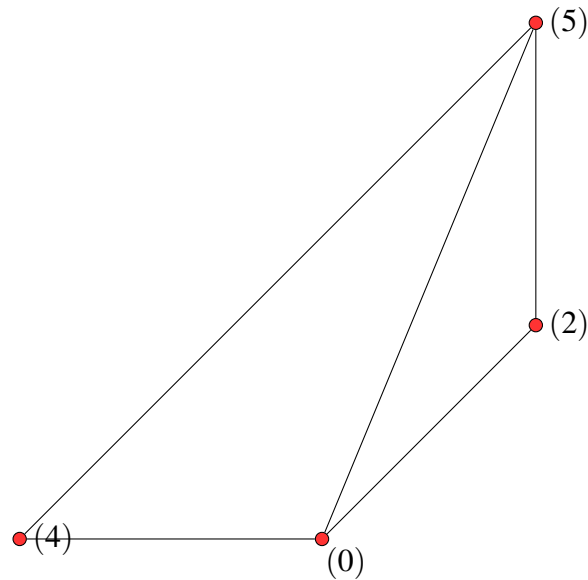
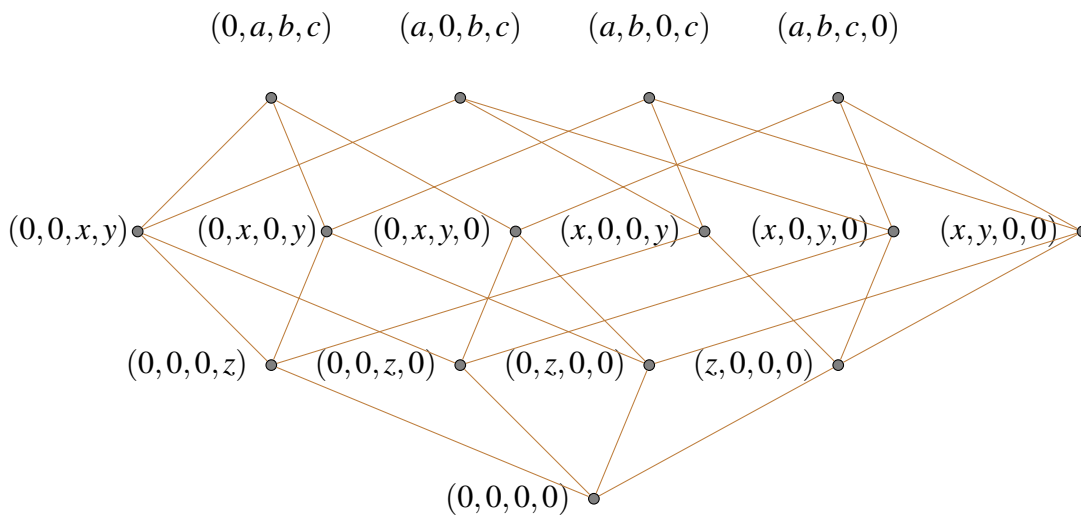


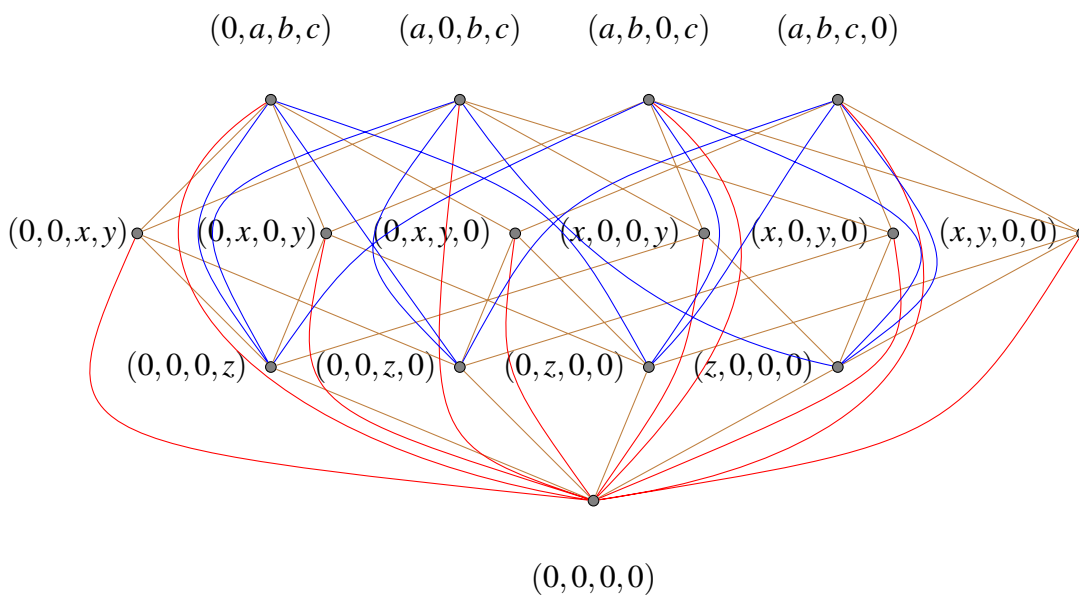
Figure 2.2:  $G_5$

**Example 2.17.** Consider the domain  $\mathbb{Z}$ . We can see that in the graph ideal  $G_4$  the ideal  $2\mathbb{Z}$  has an edge with  $4\mathbb{Z}$  where the constant  $k = 2$ . Also, the ideal  $2\mathbb{Z}$  has an edge with the ideal  $3\mathbb{Z}$  where the constant  $k$  is  $3/2$ . This graph is complete on countably many vertices.

**Example 2.18.** Consider the ring  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ . The ideal graphs  $G_6$  and  $G_7$  are pictured in the next page.



(a)  $G_7$



(b)  $G_6$

Figure 2.3: Ideal graphs  $G_6$  and  $G_7$  of the ring  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

## CHAPTER 3. THE GRAPHS AND IDEAL-THEORETIC CONSEQUENCES

### 3.1. $G_0, G_1$ And $G_2$

**Theorem 3.1.** Let  $R$  be a commutative ring with identity and  $G_0(R)$  its adjacency graph. The following are equivalent.

1.  $R$  is Artinian.
2.  $G_0(R)$  is connected.

*Proof.* The forward direction is the easier one. Indeed, suppose that  $R$  is Artinian and let  $I, J \subseteq R$  be two ideals. It suffices to show that there is a finite path from  $I$  to  $J$ , or equivalently, there is a finite sequence of adjacent ideals between  $I$  and  $J$ . To this end, it suffices to show that there is a finite sequence of adjacent ideals connecting  $I$  and  $I \cap J$ .

Note that a simple application of Zorn's lemma can be used to show that there is a (maximal) saturated chain of ideals (in the sense that if  $A$  is an ideal comparable to every element of the chain, then  $A$  is an element of the chain) between  $I$  and  $I \cap J$ . We claim that this chain of ideals is finite and that successive elements of the chain are adjacent ideals.

Suppose that there is no ideal  $I \cap J \subseteq B \subsetneq I$  such that  $B$  and  $I$  are adjacent. Starting at  $I \cap J$  we can inductively build a chain of ideals

$$I \cap J := B_0 \subsetneq B_1 \subsetneq \cdots \subsetneq B_n \subsetneq I$$

by defining  $B_{n+1} = (B_n, x_{n+1})$  where  $x_{n+1} \in I \setminus B_n$ . The fact that there is no ideal  $I \cap J \subseteq B \subsetneq I$  with  $B$  adjacent to  $I$  allows the infinite ascending chain

$$B_0 \subsetneq B_1 \subsetneq \cdots \subsetneq B_n \subsetneq \cdots .$$



Since  $R$  is not Noetherian, then it is not Artinian, which is a contradiction.

Starting with an ideal  $I_1$  which is adjacent to  $I$  and contains  $I \cap J$ , we inductively begin building the chain

$$I \supsetneq I_1 \supsetneq I_2 \supsetneq \cdots \supsetneq I_n \supsetneq I \cap J$$

with  $I_k$  and  $I_{k+1}$  adjacent for all  $1 \leq k \leq n-1$ . Since  $R$  is Artinian, this process terminates (say at  $n$  by abuse of notation). Note that  $I_n$  and  $I \cap J$  must be adjacent (else the process would not have terminated). We connect  $J$  and  $I \cap J$  similarly and the proof of this direction is complete.

For the other direction, we will assume that  $R$  is not Artinian and show that  $R$  cannot be connected. Suppose that we have the infinite descending chain of ideals

$$I_1 \supsetneq I_2 \supsetneq \cdots \supsetneq I_n \supsetneq I_{n+1} \supsetneq \cdots \supsetneq I = \bigcap_{k=1}^{\infty} I_k.$$

It will suffice to show that  $I$  cannot be connected in a finite sequence of steps (adjacencies) to any subscripted element of the chain above. We first note that  $I$  cannot have an edge with any element of the chain, since if  $I$  and  $I_n$  are adjacent, this implies that  $I_{n+2} \subsetneq I$  which is a contradiction. To set up our induction, we suppose that we have

$$I_n \leftrightarrow A \leftrightarrow I$$

where the notation " $X \leftrightarrow Y$ " means that  $X$  and  $Y$  are equal or adjacent (but in the instance of the above  $I_n \leftrightarrow A \leftrightarrow I$  we will assume that there are no equalities).

First we intersect the above with  $I_{n+1}$  to obtain

$$I_{n+1} \leftrightarrow A \cap I_{n+1} \leftrightarrow I.$$

Lemma 2.9 shows us that each successive pair of ideals is adjacent or equal. With this in hand, we first assume that  $I \subseteq A$ ; in this case we observe that

$$A \supseteq A \cap I_{n+1} \supseteq I$$

and because of the adjacency of  $A$  and  $I$ , these containments cannot both be strict. If it is the case that  $A = A \cap I_{n+1}$  then  $A \subsetneq I_{n+1}$ ; the containment must be strict else  $A = I_{n+1}$  and  $I$  is adjacent to an element of the chain. Hence we have  $I_n \supsetneq I_{n+1} \supsetneq A$  and this contradicts the adjacency of  $A$  and  $I_n$ .

On the other hand, if it is the case that  $A \cap I_{n+1} = I$  then since  $A \cap I_{n+1} \leftrightarrow I_{n+1}$ , we have the  $I$  is adjacent to  $I_{n+1}$ , which is again a contradiction. We now consider the case  $A \subseteq I$ . If we have

$$I_n \leftrightarrow A \leftrightarrow I$$

(this time with  $A \subseteq I$ ), we merely intersect with  $I_{n+1}$  to obtain

$$I_{n+1} \leftrightarrow A = A \cap I_{n+1} \leftrightarrow I.$$

Applying Lemma 2.9 again, we see that since  $I_{n+1} \subsetneq I_n$  and  $I_n$  and  $A$  are adjacent, it must be the case that  $A = I_{n+1}$  which is absurd.

With this first step completed we will assume inductively that given any infinite descending chain of ideals

$$I_1 \supsetneq I_2 \supsetneq \cdots \supsetneq I_n \supsetneq I_{n+1} \supsetneq \cdots \supsetneq I = \bigcap_{k=1}^{\infty} I_k$$

that there is no chain of  $m$  adjacent ideals connecting  $I$  to a subscripted element of the chain. Now suppose that we can find a chain of  $m + 1$  adjacent ideals

$$I = J_0 \leftrightarrow J_1 \leftrightarrow \cdots \leftrightarrow J_m \leftrightarrow J_{m+1} \leftrightarrow I_n.$$

In applying the inductive hypothesis, we first assume that  $I \subseteq J_1$ . Since

$$I \subseteq I_{n+1} \cap J_1$$

and  $I$  and  $J_1$  are adjacent, then either  $I_{n+1} \cap J_1 = J_1$  or  $I_{n+1} \cap J_1 = I$ . If  $I_{n+1} \cap J_1 = I$  then intersecting (both) chains with  $I_{n+1}$  gives a shorter path from  $I$  to an element of the descending chain  $\{I_k\}$  and we appeal to the inductive hypothesis. In the case that  $I_{n+1} \cap J_1 = J_1$ , then  $J_1 \subseteq I_{n+1}$ . Since the intersection of the descending chain  $\{I_k\}$  is  $I$ , there must be a largest index, say  $N$ , such that  $J_1 \subseteq I_N$ . Note that  $I \subseteq J_1 \cap I_{N+1} \subsetneq J_1$ . By the adjacency of  $J_1$  and  $I$ , it must be the case that  $I_{N+1} \cap J_1 = I$  and this reduces us to the previous case. This proof is over because of Proposition 3.3.  $\square$

**Corollary 3.2.** Let  $R$  be an integral domain.  $G_0^*(R)$  is connected if and only if  $R$  is Noetherian and  $\dim(R) \leq 1$ .

*Proof.* Suppose that  $R$  is 1-dimensional and Noetherian and let  $I, J \subseteq R$  be nonzero ideals. To show that  $G_0^*(R)$  is connected, it suffices to show that  $I$  and  $I \cap J$  can be connected in a finite sequence of steps. To this end, we note that since  $I \cap J$  is nonzero, the ring  $R/(I \cap J)$  is Artinian. By Theorem 3.1 there is a finite sequence of adjacent ideals (of the displayed form)

$$(I \cap J)/(I \cap J) \subset I_1/(I \cap J) \subset I_2/(I \cap J) \subset \cdots \subset I/(I \cap J)$$

connecting  $I/(I \cap J)$  to the zero ideal in  $R/(I \cap J)$ . By Proposition 2.12 this corresponds to a chain of adjacent ideals

$$(I \cap J) \subset I_1 \subset I_2 \subset \cdots \subset I$$

in  $R$  and hence  $G_0^*(R)$  is connected.

Now we suppose that  $G_0^*(R)$  is connected. Let  $I \subset R$  be an arbitrary nonzero ideal. Proposition 2.12 assures us that adjacency is preserved modulo  $I$  and so we obtain that  $G_0(R/I)$  is connected. Hence Theorem 3.1 gives us that  $R/I$  is Artinian (for any nonzero ideal  $I$ ). From Proposition 2.14 we obtain that  $R$  is Noetherian.

To see that  $R$  is 1-dimensional, we suppose that there is a chain of primes  $(0) \subsetneq \mathfrak{P} \subsetneq \mathfrak{M}$ . If we choose the ideal  $I = \mathfrak{P}$  above, we would have that  $\dim(R/\mathfrak{P}) = 1$  and hence  $R/\mathfrak{P}$  is not Artinian. We conclude that  $G_0(R/\mathfrak{P})$  is not connected, which is our desired contradiction.

□

**Proposition 3.3.** Let  $I \subseteq J$  be ideals and  $\{I_n\}_{i=0}^N$  ideals with  $I = I_0$ ,  $J = I_N$ , and  $I_k$  and  $I_{k+1}$  adjacent for each  $0 \leq i \leq N - 1$ . Then the collection of ideals can be refined to an increasing chain of ideals

$$I = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_M = J$$

with each successive pair of ideals adjacent and  $M \leq N$ .

*Proof.* Using the notation above, we say that the ideal  $I_k$  is a *hinge* ideal if  $I_k$  either properly contains both  $I_{k-1}$  and  $I_{k+1}$  or is properly contained in them both. We proceed by induction on  $m$ , the number of hinge ideals between  $I$  and  $J$ .

We first note that if there are  $m = 0$  hinge ideals, then the collection of ideals between  $I$  and  $J$  form an increasing chain and in this case, the refinement is the original collection of ideals and the conclusion holds.

We assume that for all  $k \leq m$ , if there are  $k$  hinge ideals, the chain can be refined to a strictly increasing chain with the number of elements  $M$  in the chain less than or equal  $N$ . Suppose that we now have a collection of successively adjacent ideals as in the statement of the result and we suppose that there are  $m + 1$  hinge ideals. To simplify matters, we can

assume that all of the ideals  $\{I_s\}$  are contained in  $J$  by intersecting the collection with  $J$  and applying Lemma 2.9. We will assume that the smallest index for a hinge ideal is  $t$  (that is,  $I_t$  is the first ideal in our list that is a hinge ideal).

In the first case, we assume that  $I \subseteq I_t$  (so  $I \subseteq I_1 \subseteq \cdots \subseteq I_t$  is a chain). Since  $I_t \subseteq J$  and the collection  $\{I_s\}_{s=t}^N$  has no more than  $m$  hinge ideals, we can apply the inductive hypothesis and refine this collection to an increasing chain of adjacent ideals with no more than  $N - t$  elements. Combining this with the original chain from  $I$  to  $I_t$ , we have a total increasing chain of adjacent ideals with no more than  $(N - t) + (t + 1) = N + 1$  elements and this case is established.

For the second case, we assume that the chain from  $I$  to  $I_t$  is a decreasing chain. Because of the adjacency of successive elements of the chain, we can find  $x_s \in I_s \setminus I_{s+1}$ ,  $0 \leq s \leq t - 1$  such that

$$I_s = (I_{s+1}, x_s).$$

We now make a preliminary refinement of the collection  $\{I_s\}_{s=0}^N$  by letting  $I'_s = (I_s, x_0, x_1, \dots, x_{t-1})$ . Applying Lemma 2.10  $s$  times shows that the new collection  $\{I'_s\}$  forms a chain:

$$I = I' = I'_0 = I'_1 = \cdots = I'_t \subseteq I'_{t+1}$$

and the remaining ideals  $\{I'_s\}_{s=t+2}^N$  are successively adjacent (or equal) with less than or equal to  $m$  hinge ideals. We also note that each ideal  $I'_k \subseteq J$ . Applying the inductive hypothesis, we can refine this to an increasing chain of adjacent ideals beginning at  $I = I_t$  and ending at  $J$  of length at most  $N - (t + 1) \leq N$  and this concludes the proof.

□

**Theorem 3.4.** Let  $R$  be a commutative ring with identity. The following conditions are equivalent.

1.  $G_1(R)$  is connected.
2.  $G_2(R)$  is connected.
3. There is a collection of not necessarily distinct maximal ideals  $\{\mathfrak{M}_1, \mathfrak{M}_2, \dots, \mathfrak{M}_n\}$  such that  $\mathfrak{M}_1 \mathfrak{M}_2 \cdots \mathfrak{M}_n = 0$ .

*Proof.* By notation 2.7, we have the implication (1)  $\implies$  (2). For the implication (2)  $\implies$  (3), we suppose that  $G_2(R)$  is connected and that  $\mathfrak{M}$  is a maximal ideal of  $R$ . By assumption, there is a finite path from  $\mathfrak{M}$  to the ideal (0):

$$\mathfrak{M} = I_0 \begin{smallmatrix} \supsetneq \\ \subsetneq \end{smallmatrix} I_1 \leftrightarrow I_2 \leftrightarrow \cdots \leftrightarrow I_m \begin{smallmatrix} \supsetneq \\ \subsetneq \end{smallmatrix} I_{m+1} = 0$$

where each  $\leftrightarrow$  denotes either  $\supsetneq$  or  $\subsetneq$ . In the proof of this implication, we will use the notion of hinge ideals introduced in the proof of Proposition 3.3. Note that there must be an even number of hinge ideals in the path described above, which we will denote  $H_1, H_2, \dots, H_{2t}$ . So an abbreviated version of the path described above can be expressed in the form

$$\mathfrak{M} \begin{smallmatrix} \supsetneq \\ \subsetneq \end{smallmatrix} H_1 \begin{smallmatrix} \subsetneq \\ \supsetneq \end{smallmatrix} H_2 \begin{smallmatrix} \supsetneq \\ \subsetneq \end{smallmatrix} \cdots \begin{smallmatrix} \subsetneq \\ \supsetneq \end{smallmatrix} H_{2t} \begin{smallmatrix} \supsetneq \\ \subsetneq \end{smallmatrix} 0$$

where we will have the convention that  $H_j = I_{s_j}$  for all  $1 \leq j \leq 2t$ . We also declare that  $s_0 = 0$  and  $s_{2t+1} = m + 1$ .

Since this is a path in the graph  $G_2(R)$ , then successive ideals must have maximal conductor. We will say that  $M_k = (I_{k+1} : I_k)$  if  $s_{2a} \leq k \leq s_{2a+1}$  and  $N_k = (I_k : I_{k+1})$  if  $s_{2a+1} \leq k \leq s_{2a+2}$ ,  $0 \leq a \leq t$ .

We first note that

$$H_1 = I_{s_1} \supseteq \mathfrak{M}M_0M_1 \cdots M_{s_1-1}$$

and since  $H_1 \subseteq H_2$ , we have that

$$H_2 \supseteq \mathfrak{M}M_0M_1 \cdots M_{s_1-1}.$$

In a similar fashion, we note that

$$H_3 = I_{s_3} \supseteq H_2M_{s_2}M_{s_2+1} \cdots M_{s_3-1} \supseteq \mathfrak{M}M_0M_1 \cdots M_{s_1-1}M_{s_2}M_{s_2+1} \cdots M_{s_3-1}.$$

Inductively we obtain

$$H_{2i+1} \supseteq \mathfrak{M}M_0M_1 \cdots M_{s_1-1}M_{s_2}M_{s_2+1} \cdots M_{s_3-1} \cdots M_{s_{2i}}M_{s_{2i}+1} \cdots M_{s_{2i+1}-1}.$$

In particular we obtain

$$0 = \mathfrak{M}M_0M_1 \cdots M_{s_1-1}M_{s_2}M_{s_2+1} \cdots M_{s_3-1} \cdots M_{s_{2i}}M_{s_{2i}+1} \cdots M_m$$

and hence there is a collection of maximal ideals with product equal to  $(0)$ .

For the implication  $(3) \implies (1)$ , we will assume that there is a collection of maximal ideals  $\mathfrak{M}_i$ ,  $1 \leq i \leq n$  such that  $\mathfrak{M}_1\mathfrak{M}_2 \cdots \mathfrak{M}_n = 0$ . To show that  $G_1(R)$  is connected, it suffices to show that if  $I \subset R$  is an arbitrary ideal, then there is a finite path to the zero ideal.

But note that

$$I \supseteq I\mathfrak{M}_1 \supseteq I\mathfrak{M}_1\mathfrak{M}_2 \supseteq \cdots \supseteq I\mathfrak{M}_1\mathfrak{M}_2 \cdots \supseteq \mathfrak{M}_n = 0$$

is such a path of length no more than  $n$ .

□

**Corollary 3.5.** If  $G_i(R)$ ,  $1 \leq i \leq 2$  is connected then  $\text{diam}(G_i(R)) \leq 2n$  where  $n$  is the smallest positive integer for which there is a collection of maximal ideals  $\mathfrak{M}_i$ ,  $1 \leq i \leq n$  for which  $\mathfrak{M}_1\mathfrak{M}_2 \cdots \mathfrak{M}_n = 0$ .

*Proof.* The fact that  $\text{diam}(G_1(R)) \leq 2n$  is immediate from the proof of the implication (3)  $\implies$  (1) in Theorem 3.4. The fact that  $\text{diam}(G_2(R)) \leq 2n$  follows from the fact that  $G_1(R)$  is a subgraph of  $G_2(R)$ . □

**Corollary 3.6.** If  $G_i(R)$  is connected for  $0 \leq i \leq 2$  then  $R$  is semiquasilocal and 0–dimensional.

*Proof.* By Theorem 3.1,  $G_0(R)$  is connected if and only if  $R$  is Artinian, and hence  $R$  is 0–dimensional, and, in this case, semilocal. If  $G_1(R)$  or  $G_2(R)$  is connected then Theorem 3.4 gives that  $\mathfrak{M}_1\mathfrak{M}_2 \cdots \mathfrak{M}_n = 0$  for a (not necessarily distinct) collection of maximal ideals  $\{\mathfrak{M}_1, \mathfrak{M}_2, \dots, \mathfrak{M}_n\}$ . If  $\mathfrak{M}$  is an arbitrary maximal ideal, then  $\mathfrak{M} \supseteq \mathfrak{M}_1\mathfrak{M}_2 \cdots \mathfrak{M}_n$  and hence  $\mathfrak{M} = \mathfrak{M}_k$  for some  $1 \leq k \leq n$  which shows that the list of ideals  $\{\mathfrak{M}_1, \mathfrak{M}_2, \dots, \mathfrak{M}_n\}$  contains  $\text{MaxSpec}(R)$ . Hence  $R$  is semiquasilocal.

To see that  $R$  is 0–dimensional, we appeal once again to the fact that

$$\mathfrak{M}_1\mathfrak{M}_2 \cdots \mathfrak{M}_n = 0.$$

Recalling that this collection of maximal ideals contains  $\text{MaxSpec}(R)$ , we suppose that we can find a prime ideal  $\mathfrak{P}$  such that  $\mathfrak{M}_k \not\supseteq \mathfrak{P}$ . Since  $\mathfrak{P} \supseteq \mathfrak{M}_1\mathfrak{M}_2 \cdots \mathfrak{M}_n = 0$ , we must have that  $\mathfrak{P} \supseteq \mathfrak{M}_i$  for some  $1 \leq i \leq n$ . Hence  $\mathfrak{M}_i \subsetneq \mathfrak{M}_k$  which is our desired contradiction.

□



### 3.2. $G_3$ And $G_4$

For the graph  $G_4(R)$ , we assume that  $R$  is an integral domain with quotient field  $K$ . This will usually be our focus for  $G_3(R)$  as well and so this section will carry the assumption that  $R$  is an integral domain unless specified otherwise.

It should be noted that in the case that  $R$  is an integral domain,  $G_3(R)$  is a variant on the so-called divisor graph of an integral domain studied in [4], where the ideals  $I$  and  $J$  are assumed to be principal and possess an edge between them if  $I = Ja$  where  $a \in R$  is irreducible.

For this section, it will be useful to keep in mind that  $G_3(R)$  is a subgraph of  $G_4(R)$ .

**Theorem 3.7.** The following conditions are equivalent.

1.  $G_4^*(R)$  is connected.
2.  $G_4^*(R)$  is complete.
3.  $R$  is a PID

*Proof.* For this proof, we discard the case where  $R$  is a field as all of the conditions are satisfied vacuously. Since any complete graph is connected,  $(2) \implies (1)$  is immediate. For the implication  $(1) \implies (3)$ , we let  $I \subset R$  be an arbitrary ideal and  $x \in R$  a nonzero nonunit (which exists as  $R$  is not a field). Since  $G_4(R)$  is connected, there is a sequence of ideals connecting  $(x)$  and  $I$ :

$$(x) := J_0 - J_1 - J_2 - \cdots - J_{n-1} - J_n := I.$$

Since the edges above are in the graph  $G_4(R)$ , we must have, for all  $1 \leq i \leq n$ ,  $k_i \in K$  such that  $J_i = k_i J_{i-1}$ . Note that  $(x) = k_1 k_2 \cdots k_n I$  and hence,  $I$  is principal.

Finally for the implication  $(3) \implies (2)$ , we let  $I = aR$  and  $J = bR$  be two arbitrary ideals of  $R$  (with  $a, b \neq 0$ ). Note that  $I = \frac{a}{b}J$  and hence  $G_4(R)$  is complete.  $\square$

**Theorem 3.8.**  $G_3^*(R)$  is complete if and only if  $R$  is a Noetherian valuation domain.

*Proof.* For the forward implication, we will assume that  $G_3^*(R)$  is complete. As  $G_3^*(R)$  is a subgraph of  $G_4^*(R)$ ,  $G_4^*(R)$  must also be complete. Hence  $R$  must be a PID. It now suffices to show that  $R$  is local. To this end, suppose that  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  are maximal ideals. Without loss of generality, there is a nonzero nonunit  $a \in R$  such that  $\mathfrak{M}_1 = a\mathfrak{M}_2 \subsetneq \mathfrak{M}_2$  which contradicts that maximality of  $\mathfrak{M}_1$ . Hence  $R$  is a local PID and hence a Noetherian valuation domain.

On the other hand, if  $R$  is a Noetherian valuation domain then any two nonzero proper ideals are of the form  $(\pi^n)$  and  $(\pi^m)$  where  $\pi$  is a generator of the maximal ideal and  $n, m \geq 1$ . If we say (without loss of generality) that  $n \leq m$  then  $\pi^{m-n}(\pi^n) = (\pi^m)$  and hence  $G_3^*(R)$  is complete.  $\square$

**Theorem 3.9.**  $G_3^*(R)$  is connected if and only if  $R$  is a PID. In this case,  $\text{diam}(G_3^*(R)) \leq 2$ , and  $\text{diam}(G_3^*(R)) = 1$  if and only if  $R$  is local.

*Proof.* As  $G_3^*(R)$  is a subgraph of  $G_4^*(R)$ , the fact that  $G_3^*(R)$  is connected implies that  $G_4^*(R)$  is connected. Hence, by Theorem 3.9,  $R$  must be a PID.

On the other hand, if  $R$  is a PID and  $I = aR$  and  $J = bR$  ( $a, b \neq 0$ ) are arbitrary ideals, then we can connect  $I$  and  $J$  as follows:

$$I = aR - abR - bR = J.$$

The above demonstrates the veracity of the remark that  $\text{diam}(G_3^*(R)) \leq 2$ . The fact that  $\text{diam}(G_3^*(R)) = 1$  precisely when  $R$  is local follows from Theorem 3.8 and the fact that the notions of local PID and Noetherian valuation domain are equivalent.  $\square$

**Theorem 3.10.** If  $R$  is a Dedekind domain with quotient field  $K$ , then the number of connected components of the graphs  $G_4^*(R)$  and  $G_3^*(R)$  is in one to one correspondence with the elements of the class group  $\text{Cl}(R)$ . Each connected component of  $G_4^*(R)$  is complete

and each connected component of  $G_3^*(R)$  has diameter 2 and the connected components of  $G_3^*(R)$  is a local PID.

*Proof.* If  $R$  is a Dedekind domain with quotient field  $K$ , then two ideals,  $I$  and  $J$ , are in the same class of  $\text{Cl}(R)$  if and only if  $I = Jk$  for some nonzero  $k \in K$ . Hence each connected component of  $G_4^*(R)$  is complete and these components are in one to one correspondence with the elements of  $\text{Cl}(R)$ .

For the  $G_3^*(R)$  case, we first note that if there is a path from  $I$  to  $J$  then there must be some nonzero  $k \in K$  such that  $I = Jk$ . It remains to show that if  $I$  and  $J$  are in the same ideal class, then there is a path connecting them. To this end, we note that if  $I = Jk$  for some nonzero  $k \in K$ , we can write  $I = \frac{a}{b}J$  and in a similar fashion as before, we can connect  $I$  and  $J$  via

$$I - bI = aJ - J,$$

and hence there is a path of length 2 connecting  $I$  and  $J$ .

To see that in the case of  $G_3^*(R)$  the diameter of each component is precisely 2 if  $R$  is not local, we will assume that one of the connected components of  $G_3^*(R)$  is complete. Suppose that  $R$  has at least two maximal ideals, say  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$ . We select elements  $m_1 \in \mathfrak{M}_1 \setminus \mathfrak{M}_2$  and  $m_2 \in \mathfrak{M}_2 \setminus \mathfrak{M}_1$ . If  $I$  is in the complete connected component of  $G_3^*(R)$  then there is an edge between  $I$  and  $\frac{m_1}{m_2}I$ . Hence there is a nonzero  $a \in R$  such that either  $aI = \frac{m_1}{m_2}I$  or  $I = a\frac{m_1}{m_2}I$ . Since  $R$  is Dedekind, nonzero ideals cancel and hence we have either  $am_2 = m_1$  or  $um_2 = am_1$  for some unit  $u \in R$ . But the first equation implies that  $m_1 \in \mathfrak{M}_2$  and the second implies that  $m_2 \in \mathfrak{M}_1$ . Either way, we have a contradiction..

Hence  $R$  must be local. Since a local Dedekind domain is PID( and a Noetherian valuation domain), we are done.

□

### 3.3. $G_5$

Despite the title of this section, most of our attention will be devoted to the graph  $G_5^*(R)$  and some of its variants. The reason for excluding the zero ideal is because the use of the zero ideal gives extra structure to this graph with no useful new information. It is easy to see that for any commutative ring with identity (even an integral domain) that the graph  $G_5(R)$  is connected with diameter no more than 2 if we allow use of the zero ideal. Indeed, if  $I$  and  $J$  are arbitrary ideals, then  $I - (0) - J$  is a path connecting them. So if  $R$  is an integral domain, the graph  $G_5(R)$  would be an infinite star graph with the zero ideal at the center. These extra connections muddy the waters and give no useful insights for our current purposes.

**Theorem 3.11.** If  $G_1(R)$  or  $G_2(R)$  is connected, then so is  $G_5^p(R)$  where the vertex set of  $G_5^p(R)$  is  $\{I \mid \exists J \text{ s.t. } IJ = 0\}$  and there is an edge between  $I$  and  $J$  if and only if  $IJ = 0$ . Additionally  $\text{diam}(G_5^p(R)) \leq 3$ .

*Proof.* By Theorem 3.4 there is a collection of not necessarily distinct maximal ideals  $\{\mathfrak{M}_1, \mathfrak{M}_2, \dots, \mathfrak{M}_n\}$  such that  $\mathfrak{M}_1 \mathfrak{M}_2 \cdots \mathfrak{M}_n = 0$ . We can also assume that no proper sub-product of these listed ideals is zero.

If  $I, J \subset R$  then there is a maximal ideal  $\mathfrak{M}$  containing  $I$ . Since  $\mathfrak{M} \supseteq I \supset 0 = \mathfrak{M}_1 \mathfrak{M}_2 \cdots \mathfrak{M}_n$ ,  $\mathfrak{M}$  must be  $\mathfrak{M}_i$  for some  $1 \leq i \leq n$ . We will assume without loss of generality that  $I \subseteq \mathfrak{M}_1$  and  $J \subseteq \mathfrak{M}_n$ . To see that  $G_5^p(R)$  is connected with diameter no more than three consider the path

$$I \leftrightarrow \mathfrak{M}_2 \mathfrak{M}_3 \cdots \mathfrak{M}_n \leftrightarrow \mathfrak{M}_1 \mathfrak{M}_2 \cdots \mathfrak{M}_{n-1} \leftrightarrow J$$

□

Here is an example to show that  $G_5(R)$  may be connected without  $G_1(R)$ ,  $G_2(R)$  is being connected.

**Example 3.12.** Let  $\mathbb{F}$  be a field and consider first the domain  $R := \mathbb{F}[x_1, x_2, \dots, x_n, \dots, y]$ , let  $I \subset R$  be the ideal  $I := (\{x_i^2, x_i y, y^2\}_{i \geq 1})$ , and let  $T := R/I$ .  $T$  is quasilocal, with maximal ideal  $\mathfrak{M}$ . It is easy to see that  $G_5(T)$  is connected. Indeed if  $I, J \subset T$  then we have the path  $I \leftrightarrow (y) \leftrightarrow J$ . But there is no collection of maximal ideals with product 0 since for all  $k \geq 1$ ,  $0 \neq x_1 x_2 \cdots x_k \in \mathfrak{M}^k$ .

### 3.4. $G_6$ And $G_7$

In this section we define a new property called nearly finitely (resp. principally) covered and we will show the connection between the properties of  $G_6$  and  $G_7$  and our new property.

**Definition 3.13.** We say that  $R$  is nearly finitely (resp. principally) covered if any ideal  $I$  contain an ideal  $J \subseteq I$  such that  $I$  is finitely generated over  $J$  and  $J$  is contained in a finitely generated ideal.

**Theorem 3.14.** The following conditions are equivalent.

1.  $G_6(R)$  (resp.  $G_7(R)$ ) is connected.
2.  $R$  is nearly finitely (resp. principally) covered.

Additionally if  $G_6(R)$  (resp.  $G_7(R)$ ) is connected then the diameter of the graph is not more than 6.

*Proof.* If  $R$  is nearly finitely covered and  $I_1, I_2 \subseteq R$  are ideals, we will find a path from  $I_1$  to  $I_2$  of length 6 (and establish the last remark as well). Note that  $I_1$  and  $I_2$  contain ideals ( $J_1$  and  $J_2$  respectively) that are contained in finitely generated ideals  $F_1$  and  $F_2$  respectively. We consider the path

$$I_1 \supseteq J_1 \subseteq F_1 \supseteq F_1 F_2 \subseteq F_2 \supseteq J_2 \subseteq I_2.$$

The existence of this path gives the first direction.

For the other direction let us suppose that  $I$  is an ideal that contains no ideal  $J \subseteq I$  such that  $I$  is finitely generated over  $J$  and  $J$  is contained in a finitely generated ideal (note that, in particular,  $I$  cannot be finitely generated).

If we suppose that we can connect  $(0)$  and  $I$ , we must have a finite collection of finite chains of ideals  $C_1, C_2, \dots, C_k$  where the minimal element of  $C_1$  is  $(0)$ , the maximal element of  $C_1$  coincides with the maximal element of  $C_2$  (and in general, the maximal element of  $C_i$  coincides with the maximal (resp. minimal) element of  $C_{i+1}$  if  $i$  is odd (resp. even)). Finally,  $I$  is the maximal element of  $C_k$  if  $k$  is odd, and it is minimal if  $k$  is even. We also recall that within each  $C_i$  each pair of ideals has the property that the larger one is finitely generated over the smaller.

We denote the maximal element of  $C_i$  by  $\mathfrak{M}_i$  and the minimal element by  $\mathfrak{m}_i$  and observe that  $\mathfrak{M}_i$  is finitely generated over  $\mathfrak{m}_i$ . Additionally  $\mathfrak{M}_i$  is finitely generated over  $\mathfrak{m}_{i+1}$  if  $i$  is odd and  $\mathfrak{M}_{i+1}$  is finitely generated over  $\mathfrak{m}_i$  if  $i$  is even. By successive application of Lemma 2.13, we obtain that  $I$  is contained in a finitely generated ideal and this is the desired contradiction.

□

**Corollary 3.15.** If every maximal ideal of  $R$  is finitely generated (resp. principal) then  $G_6(R)$  (resp.  $G_7(R)$ ) is connected and of diameter no more than 4.

*Proof.* If every maximal ideal is finitely generated, then clearly  $R$  is nearly finitely covered. To see the veracity of the statement concerning the diameter, let  $I, J \subseteq R$  and  $\mathfrak{M}, \mathfrak{N} \subseteq R$  be maximal ideals such that  $I \subseteq \mathfrak{M}$  and  $J \subseteq \mathfrak{N}$ . To connect  $I$  and  $J$  consider the path

$$I \subseteq \mathfrak{M} \supseteq \mathfrak{M}\mathfrak{N} \subseteq \mathfrak{N} \supseteq J.$$

□

We now present an example to show that the nearly finitely covered condition is needed.

**Example 3.16.** Let  $\mathbb{F}_2$  be a field of characteristic 2 and consider the ring

$$R := \mathbb{F}[x_1, x_2, x_3, \dots, x_n, \dots t] / (\{x_i x_j\}_{i, j \geq 1}, x_1 - x_2 t, x_2 - x_3 t, \dots, x_n - x_{n+1} t, \dots).$$

We will abuse the notation in  $R$  by letting  $x_i$  and  $t$  denote the homomorphic images of the original indeterminates under the standard projection. In  $R$  we consider the maximal ideals  $\mathfrak{M}_1 = (x_1, x_2, \dots, x_n, \dots, 1 - t)$  and  $\mathfrak{M}_2 = (t)$ . We now let  $S := (\mathfrak{M}_1 \cup \mathfrak{M}_2)^c$  and we consider the ring  $T := R_S$ .

It is clear that not all of the maximal ideals of  $T$  are finitely generated (the ideal  $\mathfrak{M}_1$  is not finitely generated). To demonstrate that  $T$  is nearly principally (and hence finitely) covered, we first consider the homomorphism

$$\phi : R \longrightarrow \mathbb{F}[t]_{S_1}$$

where  $S_1 = ((t) \cup (1 - t))^c$ , induced by the rule that  $\phi(x_n) = 0$  for all  $n \geq 1$ . We note that  $\mathbb{F}[t]_{S_1}$  is a PID.

It suffices to show that if  $I \subset R$  is a proper ideal, then it is nearly principally covered. To this end, we first suppose that  $I \subseteq \mathfrak{M}_2$ . In this case,  $(I, t) = \mathfrak{M}_2$  is finitely generated over  $I$  and we are done.

It remains to consider the case  $I \subseteq \mathfrak{M}_1 \setminus \mathfrak{M}_2$ . Since  $\mathbb{F}[t]_{S_1}$  is a PID, the ideal  $\phi(I)$  is principal and generated by some  $f(t) \in \mathbb{F}[t]_{S_1}$ . Let  $g \in I$  be such that  $\phi(g) = f(t)$ . We now claim that

$$I = ((I \cap \mathfrak{M}_2), g).$$

To establish the nontrivial containment, we suppose that  $h \in I$ . Let  $\phi(h) = f(t)a(t)$  where  $a(t) \in \mathbb{F}[t]_{S_1}$ . We note that  $h = \phi(h) + k_1$  and  $g = f(t) + k_2$  where  $k_1, k_2 \in \ker(\phi)$ . As since  $\ker(\phi) \subseteq \mathfrak{M}_2$ , a simple computation shows that

$$h - ga(t) = k_2a(t) + k_1 \in I \cap \mathfrak{M}_2$$

and hence  $h \in ((I \cap \mathfrak{M}_2), g)$  as claimed. The upshot is that  $I$  is finitely generated over  $I \cap \mathfrak{M}_2$  and  $I \cap \mathfrak{M}_2 \subseteq (t)$ . Hence  $I$  is nearly principally covered.



## CHAPTER 4. CONCLUSION

In this study we looked at new types of graphs which associated to a commutative ring with identity. In our new graphs we considered that the vertex set being the set of proper ideals. The edge set defined via different ideal theoretic properties. In chapter 2, we define and justify some types of graphs that are determined by ideal theoretic properties and gave some results and justifications for our new graphs. Also, we provided the reader with some examples that give a clear view of our investigated graphs. In chapter 3, we provided some insight about the relation between ideal graphs and ideal properties. For example, theorem (3.1 ) gives us an understanding of  $G_0$  when  $R$  is artinian. Also we proved that If  $R$  is PID then for the graph  $G_4^*(R)$ , connected and complete are the same. In section 3.4, we defined a new property called nearly finitely(resp. principally) covered and we proved that  $G_6$  (resp.  $G_7$ ) are connected under this property.

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