

GEOMETRIC ASPECTS OF INTERCONNECTION AND
DAMPING ASSIGNMENT - PASSIVITY-BASED CONTROL

by

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Für Claudia...

Abstract

This dissertation deals with smooth feedback stabilization of control-affine systems via Interconnection and Damping Assignment - Passivity-Based Control (IDA-PBC). The IDA-PBC methodology is a feedback control design technique that aims to establish or manipulate a port-Hamiltonian structure of the closed-loop system. For a mechanical control system, a port-Hamiltonian system is a natural description of the dynamics, and several effective controller designs have been presented for this class of systems. In other fields of engineering, the development of such controller design is an active area of research. In particular, applications of IDA-PBC techniques prove to be difficult in practice for process control applications where the concept of energy is usually ill-defined. This thesis seeks to extend the application of the IDA-PBC methodology beyond mechanical control systems. This is achieved by following three directions of research. First, we establish conditions under which a port-Hamiltonian system can be written as a feedback interconnection of two port-Hamiltonian system. We identify such an interconnection structure for linear control systems based on their intrinsic properties. Second, as observed in application of IDA-PBC to non-mechanical systems, several additional assumptions on the structure of the desired port-Hamiltonian system can effectively reduce the complexity of the matching problem. We establish a unified approach that considers these additional assumptions.

Third, we connect the matching problem to the classical feedback equivalence approach. We show that feedback equivalence between control-affine systems can be employed to construct some feasible interconnection and damping structures.

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Chapter 1

Introduction

Passivity-based control (PBC) can be used to stabilize continuous-time non-linear control-affine systems. A passivity-based control design methodology consists in finding a feedback law such that the closed-loop system is rendered passive with respect to a desired storage function. The scope of this thesis is restricted to a special passivity-based control methodology, called Interconnection and Damping Assignment (IDA). The IDA-PBC methodology aims to establish or manipulate the so called port-Hamiltonian structure of the closed-loop system [Ortega et al., 2002b]. A port-Hamiltonian structure consists of an energy storage function, and an interconnection and damping structure. The interconnection and damping structure represent respectively the distribution of energy between the states of the system, and the dissipation of energy. The port-Hamiltonian structure often has physical meaning, especially in the realm of mechanical and electromechanical systems. Consequently, IDA-PBC has found a wide range of applications; in particular, numerous applications have been made to underactuated mechanical systems; see Ortega et al. [2002a], Acosta et al. [2005] and references therein. In these applications, a particular version of

IDA-PBC, called non-parametric IDA-PBC is considered. It assumes that desired interconnection and damping structures for the closed-loop system are known, and that a Hamiltonian function is obtained as a solution to a set of partial differential equations (PDEs), which are parameterized by the aforementioned structures [Ortega et al., 2002b]. The effect of the interconnection and damping structure on the qualitative behaviour of the closed-loop system is disturbed by the Hamiltonian function. This has been illustrated for instance in an example in Kotyczka and Lohmann [2009]. Hence, it is important to understand how the choice of the desired interconnection and damping structure affects the closed-loop system. For control problems in which physical intuition is not apparent, it is also of value to have guidelines for the choice of the the interconnection and damping structure. In addition, it allows the introduction of a generalized concept of energy, which is valid in different fields of engineering. Our goal is to establish such guidelines based on three perspectives. The first perspective is based on intrinsically-defined properties of control systems. The second is based on simplification and reduction of the problem, and the third is based on feedback equivalence.

Our first objective is to show that linear port-Hamiltonian systems can be decomposed based on their controllability matrices. This decomposition defines an interconnection structure intrinsically. Our second objective is to establish guidelines to design an interconnection and damping structure under the frequently made assumption of “constant” structures. In the proposed approach, the degrees of freedom available in the IDA-PBC design are used to achieve some additional desired properties of the closed-loop Hamiltonian function. Our third objective, in the final part of this thesis, is to study the IDA-PBC methodology as a feedback equivalence

[Gardner, 1989a] and to determine new interconnection and damping structures via existing feedback equivalences.

1.1 Statement of Contributions

The following list constitutes the contributions that this thesis makes to the body of knowledge in systems control.

1. Section 3.2 establishes the novel concept of decomposition for linear port-Hamiltonian systems, which allows us to study the “interconnection topology” of the system.
2. Theorem 4.2.2 establishes necessary and sufficient conditions on the existence of a port-Hamiltonian representation based on exterior differential calculus.
3. Section 4.4 introduces a splitting of the interconnection and damping structure based on the equilibrium manifold of the control system. The splitting restricts the existence conditions for a Hamiltonian function to conditions on a substructure of the interconnection and damping structure.
4. Section 5.3 describes the matching problem as a feedback equivalence problem. Corollary 5.3.3 presents a solution of the matching problem generated by an existing feedback equivalence.
5. Section 5.4 presents an application of the IDA-PBC methodology to a non-isothermal continuous stirred-tank reactor (CSTR) via feedback equivalence, giving a mechanical interpretation of the states of the CSTR.

1.2 Organization of the Dissertation

CHAPTER 2 In Chapter 2 we introduce necessary mathematical preliminaries and present a detailed exposition of the IDA-PBC methodology. A novel description of port-Hamiltonian systems based on the notation of tensor fields is introduced and the “matching problem” for IDA-PBC is defined.

CHAPTER 3 We consider the IDA-PBC methodology for linear control systems. The objective here is to establish conditions for an intrinsic decomposition of the interconnection structure. The approach is based on the concepts of abstractions of control systems and achievable Dirac structures. A simple LC-circuit motivates and illustrates the results in this chapter. This result has been accepted to the 2011 American Control Conference [Höfner and Guay, 2011a].

CHAPTER 4 We establish an alternative description of port-Hamiltonian systems, which represents a unified approach to solving the matching problem, in Chapter 4. Furthermore, we consider the set of admissible equilibria to establish a local decomposition of the interconnection and damping structure. This result has been submitted to the 18th IFAC World Conference [Höfner and Guay, 2011b].

CHAPTER 5 In this chapter, we establish a geometric interpretation of feedback equivalence, as treated by Gardner and Shadwick [1990a, 1992] and collaborators, and formulate the matching problem of the IDA-PBC methodology in this context. An application of the transitivity property shows, in example of a non-isothermal CSTR, that feedback equivalence can be used to generate a passivity-based controller, which allow to understand the closed-loop system in terms of a simple mechanical system.

This result has been presented at the 8th IFAC Symposium on Nonlinear Control Systems [Höffner and Guay, 2010].

CHAPTER 6 We summarize the new results of the dissertation in this chapter and suggest some avenues for future work based on these results.

Chapter 2

Literature Review

The purpose of this chapter is twofold. First, we establish the necessary mathematical preliminaries. Second, we review the literature on the IDA-PBC methodology. In particular, we classify different versions of IDA-PBC, review open problems of the, so called, non-parameterized version of IDA-PBC. Then, we discuss the literature on existing solutions to the main open problem in non-parameterized IDA-PBC.

2.1 Mathematical Preliminaries

In this section, we introduce basic definitions and establish notation that are used throughout the thesis. If not stated otherwise, it is assumed that all objects (*e.g.*, maps, functions and manifolds) are smooth. For more information, we refer to Abraham et al. [1988], Bullo and Lewis [2004].

Local vector bundles

Consider finite-dimensional real vector spaces $\mathbf{E}_1, \dots, \mathbf{E}_k$ and \mathbf{F} . Let

$$L(\mathbf{E}_1, \dots, \mathbf{E}_k; \mathbf{F})$$

denote the vector space of k -multilinear maps of $\mathbf{E}_1 \times \dots \times \mathbf{E}_k$ to \mathbf{F} .

Definition 2.1.1. Let V be a finite-dimensional real vector space and let \mathcal{X} be an open subset of \mathbb{R}^n . We call the Cartesian product $E = \mathcal{X} \times V$ a *local vector bundle of rank* $\dim(V)$. We call \mathcal{X} the *base space* of E , which can be identified with the *zero section* $\mathcal{X} \times \{0\}$. For $p \in \mathcal{X}$, $E_p = \{p\} \times V$ is called the *fiber over* p , which we endow with the vector space structure of V . The map $\pi : E \rightarrow \mathcal{X}$ given by $\pi(p, v) = p$ is called the *projection* of E (Thus, the fiber over $p \in \mathcal{X}$ is $\pi^{-1}(p)$).

Definition 2.1.2. Let $E = \mathcal{X} \times V$ and $E' = \mathcal{X}' \times V'$ be local vector bundles, with $\text{rank } E' = k$. We say that E' is a *subbundle* of E if $V' \subset V$, $\pi' = \pi|_{V'}$, and $E \cap V' = U \times \mathbb{R}^k \times \{0\}$.

In particular, the tangent bundle $T\mathcal{X}$ of an open subset $\mathcal{X} \subset \mathbb{R}^n$ is a local vector bundle. Furthermore, a constant-rank distribution is a subbundle of $T\mathcal{X}$ and therefore also a local vector bundle of \mathcal{X} , possibly after restriction to an open subset of \mathcal{X} and a change of coordinates.

Definition 2.1.3. Let $E = \mathcal{X} \times V$ and $E' = \mathcal{X}' \times V'$ be local vector bundles. A map $\phi : E \rightarrow E'$ is called a *local vector bundle map* if it has the form $\phi(p, v) = (\phi_1(p), \phi_2(p)(v))$, where $\phi_1 : \mathcal{X} \rightarrow \mathcal{X}'$ and $\phi_2 : \mathcal{X} \rightarrow L(V; V')$. A local vector bundle map that has an inverse, which is also a local vector bundle map is called *local vector bundle isomorphism*.

A particular example of a local vector bundle map is the tangent map Tf of a map $f : \mathcal{X} \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ on \mathcal{X} , which is given by $Tf(p, v) = (f(p), Df(p)(v))$, where $Df(p)$ is the derivative of f at p .

Definition 2.1.4. Suppose $E = \mathcal{X} \times V$ and $E' = \mathcal{X} \times V'$ are local vector bundles. The *homomorphism bundle* $\text{Hom}(E; E')$ is the local vector bundle $\mathcal{X} \times L(V; V')$. In other words, the fiber over $p \in \mathcal{X}$ is

$$\text{Hom}(E; E')_p = L(E_p; E'_p).$$

The rank of $\text{Hom}(E; E')$ is the product of the ranks of E and E' , since the dimension of $L(V; V')$ is $\dim(V) \cdot \dim(V')$.

If $S \subset \mathbf{E}$, we denote by $\text{ann}(S)$ the *annihilator* of S , which is defined by $\text{ann}(S) = \{\alpha \in \mathbf{E}^* \mid \alpha(v) = 0, v \in S\}$, where $\mathbf{E}^* = L(\mathbf{E}; \mathbb{R})$ is the dual space to \mathbf{E} . Similarly, for $T \subset \mathbf{E}^*$ we define the *coannihilator* of T to be the subspace of \mathbf{E} defined by $\text{coann}(T) = \{v \in \mathbf{E} \mid \alpha(v), \alpha \in T\}$. Let Δ be a constant-rank distribution on \mathcal{X} . Then $\text{ann}(\Delta)$ denotes the annihilator of Δ , *i.e.*, $(\text{ann}(\Delta(p))) = (\text{ann}(\Delta))(p)$ for all $p \in \mathcal{X}$. We make use of the following local vector bundle construction in Chapter 4. We define, after shrinking \mathcal{X} and changing the coordinates if necessary, the local vector bundle $\text{Hom}(\Delta) = \text{Hom}(\Delta; T\mathcal{X})$. We similarly define $\text{Hom}(\text{ann}(\Delta)) = \text{Hom}(T\mathcal{X}; (\text{ann}(\Delta))^*)$, where $(\text{ann}(\Delta))^*$ is the dual space to $\text{ann}(\Delta)$.

Tensor bundles

Definition 2.1.5. For a finite-dimensional real vector space \mathbf{E} and its dual \mathbf{E}^* we let

$$T_s^r(\mathbf{E}) = L(\mathbf{E}^*, \dots, \mathbf{E}^*, \mathbf{E}, \dots, \mathbf{E}; \mathbb{R})$$

(r copies of \mathbf{E}^* and s copies of \mathbf{E}). Elements of $T_s^r(\mathbf{E})$ are called *tensors on \mathbf{E} , contravariant of order r and covariant of order s* ; or simply of type (r, s) .

The most common type of tensors that we work with in this thesis are of type $(0, 2)$ and $(2, 0)$; these can be identified with linear maps from \mathbf{E} to \mathbf{E}^* and from \mathbf{E}^* to \mathbf{E} , respectively. That is $T_0^2(\mathbf{E}) = L(\mathbf{E}; \mathbf{E}^*)$, $T_2^0(\mathbf{E}) = L(\mathbf{E}^*; \mathbf{E})$. We denote the local vector bundle with fiber $T_s^r(T_p\mathcal{X})$ over p by $T_s^r(\mathcal{X})$, the *bundle of tensors of type (r, s)* or *tensor bundle of type (r, s)* . In particular, we make the following identification of local vector bundles $T_0^2(\mathcal{X}) = \text{Hom}(T\mathcal{X}; T^*\mathcal{X})$ and $T_2^0(\mathcal{X}) = \text{Hom}(T^*\mathcal{X}; T\mathcal{X})$.

Let \mathbf{E} be a real finite dimensional vector space. Let $\{e_1, \dots, e_n\}$ be a basis for \mathbf{E} and $\{e^1, \dots, e^n\}$ be a basis for \mathbf{E}^* dual to a $\{e_1, \dots, e_n\}$. If $t \in T_s^r(\mathbf{E})$, the real numbers

$$t_{j_1 \dots j_s}^{i_1 \dots i_r} = t(e^{i_1}, \dots, e^{i_r}, e_{j_1}, \dots, e_{j_s})$$

are called the *components of t relative to the basis $\{e_1, \dots, e_n\}$ and $\{e^1, \dots, e^n\}$* . For example, an inner product $\langle\langle \cdot, \cdot \rangle\rangle$ on \mathbf{E} is a symmetric tensor of type $(0, 2)$. Its matrix $[G^{ij}]$ with respect the basis $\{e_i\}$ has components $G_{ij} = \langle\langle e_i, e_j \rangle\rangle$. Thus the matrix $[G_{ij}]$ is symmetric and positive definite. Its inverse is written as $[G^{ij}]$.

Sections of a local vector bundle

Next, we define sections of local vector bundles.

Definition 2.1.6. Let $E = \mathcal{X} \times V$ be a local vector bundle. A section of E is a map $\sigma : \mathcal{X} \rightarrow E$, such that $\pi \circ \sigma = \text{id}_{\mathcal{X}}$, where $\text{id}_{\mathcal{X}}$ is the identity map on \mathcal{X} .

We denote the set of all sections of a local vector bundle E by $\Gamma(E)$. This set can be endowed with a natural vector space structure, with addition and scalar multiplication performed pointwise.

Let $E = \mathcal{X} \times V$ and $E' = \mathcal{X} \times V'$ be local vector bundle with the same base space. From Definition 2.1.4 and Definition 2.1.3 we can deduce that a section σ of $\text{Hom}(E; E')$ can be identified with local vector bundle map ϕ with $\phi_1 = \text{id}_{\mathcal{X}}$. Hence, if ϕ is a local vector bundle isomorphism, then its inverse ϕ^{-1} can be identified with a section of $\text{Hom}(E'; E)$, we denote this section by σ^{-1} .

Tensor fields

Tensor fields of type (r, s) are sections of the local vector bundle of tensors of type (r, s) . In local coordinates (x^1, \dots, x^n) , a section σ of $T_s^r(\mathcal{X})$ can be written as

$$\sigma = \sigma_{i_1 \dots i_s}^{j_1 \dots j_r} \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_r}} \otimes dx^{i_1} \otimes \dots \otimes dx^{i_s},$$

with standard notation as for example in Abraham et al. [1988]. The functions $\sigma_{i_1 \dots i_s}^{j_1 \dots j_r}$ are called the *component functions* of σ in these coordinates.

Next, we define the interior product of a vector field (resp., a one-form) with a (r, s) -tensor field. The interior product will be used to define the drift vector field of a port-Hamiltonian system in the following section.

Definition 2.1.7. Let $\mathcal{X} \subset \mathbb{R}^n$ be open. The *interior product* of a vector field $X \in \Gamma(T\mathcal{X})$ (resp., a one-form $\alpha \in \Gamma(T^*\mathcal{X})$) with a (r, s) -tensor field $t \in \Gamma(T_s^r(\mathcal{X}))$ is the tensor field of type $(r, s - 1)$ (resp., $(r - 1, s)$) defined by

$$\begin{aligned}(tX)(\alpha^1, \dots, \alpha^r, X_1, \dots, X_{s-1}) &= t(\alpha^1, \dots, \alpha^r, X, X_1, \dots, X_{s-1}) \\ (t\alpha)(\alpha^1, \dots, \alpha^{r-1}, X_1, \dots, X_s) &= t(\alpha, \alpha^1, \dots, \alpha^{r-1}, X, X_1, \dots, X_s).\end{aligned}$$

Note that we deviate from the usual notation of the interior product given by $i_X t$.

Let us consider the coordinate expression of two interior products that we make use of in Chapter 4. Let t be $(0, 2)$ -tensor field on $\mathcal{X} \subset \mathbb{R}^n$ and let (x^1, \dots, x^n) be coordinates on \mathcal{X} . Then t has the local expression $t = t^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$. Let α be a one-form with local expression $\alpha_i dx^i$. Then $t\alpha = t^{ij} \alpha_i \frac{\partial}{\partial x^j}$ is a vector field on \mathcal{X} . Similarly, if $X = X^i \frac{\partial}{\partial x^i}$ is a vector field and σ a $(2, 0)$ -tensor field, then $\sigma X = \sigma_{ij} X^i dx^j$ is a one-form. Furthermore, if $t(s\beta) = \beta$ for all one-forms β , $s(p)$ is the inverse of $t(p)$ for all $p \in \mathcal{X}$.

Also, let us first recall the following definition of the derivative of a map [Bullo and Lewis, 2004].

Definition 2.1.8. Let $U \subset \mathbb{R}^n$ be an open set and let $f : U \rightarrow \mathbb{R}^m$ be a map. The map

$$\begin{aligned}Df : U &\rightarrow L(\mathbb{R}^n, \mathbb{R}^m) \\ u &\mapsto Df(u)\end{aligned}$$

is called *derivative* of f . Proceeding inductively we define

$$D^r f = (D(Dr^{r-1}f)) : U \rightarrow L^r(\mathbb{R}^n; \mathbb{R}^m),$$

where we have identified $L(\mathbb{R}^n, L^{r-1}(\mathbb{R}^n; \mathbb{R}^m))$ with $L^r(\mathbb{R}^n; \mathbb{R}^m)$. Furthermore, if $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$ then we denote $\text{Hess } f(p) = D^2 f(p)$ the *Hessian* of f at p .

Control systems

Definition 2.1.9. Let \mathcal{X} and \mathcal{U} be open subsets of \mathbb{R}^n and \mathbb{R}^m , respectively. Let $n \geq m \geq 1$ and $0_{\mathbb{R}^m} \in \mathcal{U}$. Let f, g_1, \dots, g_m be vector fields on \mathcal{X} . A *control-affine system* is a triple $\Sigma = (\mathcal{X}, \{f, g_1, \dots, g_m\}, \mathcal{U})$, where \mathcal{X} is called the *state space*, \mathcal{U} is the *control space*, the vector field f is called the *drift vector field* and g_1, \dots, g_m are called the *control vector fields*. We call the distribution g generated by the control vector fields the *control distribution*. That is, $g(p) = \text{span} \{g_1(p), \dots, g_m(p)\}$.

Let Σ be a control-affine system, a point $p_* \in \mathcal{X}$ is called an *admissible equilibrium* of Σ if there exists a $u = (u^1, \dots, u^m) \in \mathcal{U}$ such that $0_{\mathbb{R}^n} = f(p_*) + g_i(p_*)u^i$. The following assumption, will be imposed on the class of control-affine systems studied throughout this thesis.

Assumption 2.1.10. If Σ is a control-affine system, its control distribution is of rank m . To emphasize the role of the distribution, denote the control-affine system by $\Sigma = (\mathcal{X}, f, g, \mathcal{U})$.

Definition 2.1.11. Consider a control-affine system Σ and $T \in \mathbb{R}_{>0}$. Let $u : [0, T] \rightarrow \mathcal{U}$

be a control input such that the solution c of the ODE

$$\frac{dc}{dt}(t) = \dot{c}(t) = f(c(t)) + g_i(c(t))u^i(t).$$

exists on $[0, T]$. The pair (c, u) is called a *solution* (or trajectory) of Σ .

Passivity-based control

Consider a control-affine system $\Sigma = (\mathcal{X}, f, g, \mathcal{U})$ and an output map $y : \mathcal{X} \rightarrow \mathbb{R}^m$. The system Σ is said to be *passive* with respect to y if there exists a function $H \in C^1(\mathcal{X})$ with $H(x) \geq 0$ for all $x \in \mathcal{X}$ and a nonnegative function $W \in C^1(\mathcal{X})$ such that

$$\dot{H}(c(t)) = \langle f, dH \rangle(c(t)) \leq -W(c(t)) + u(t)^\top y(c(t)) \quad (2.1)$$

for every solution (c, u) of Σ . The system Σ is said to be *strictly passive* if W is positive definite. The map y is then called *passive output* and H is called the *storage function*. In order to stabilize Σ to a desired admissible equilibrium x_* , passivity-based control aims to render the closed-loop system passive with respect to a storage function with minimum at the desired equilibrium. The following proposition justifies this aim.

Proposition 2.1.12. [Ortega et al., 2002b] Let Σ be a passive control-affine system with passive output $y : \mathcal{X} \rightarrow \mathbb{R}^m$ and storage function $H \in C^1(\mathcal{X})$ and let $x_* \in \mathcal{X}$ be an admissible equilibrium of Σ . If we can find a map $\beta : \mathcal{X} \rightarrow \mathcal{U}$ such that the partial differential equations

$$[f(x) + g(x)\beta(x)]^\top \frac{\partial H}{\partial x}(x) = -[g(x)\beta(x)]^\top \frac{\partial H}{\partial x}(x), \quad \forall x \in \mathcal{X}$$

can be solved for H_a , and the function $H_d = H + H_a$ has a minimum at x_* , then the feedback $u = \beta$ is an *energy-balancing stabilizer* for the equilibrium x_* . That is, the Lyapunov function H_d satisfies

$$H_d(x(t)) = H(x(t)) - \int_0^t u^\top(s)y(s) ds + \kappa$$

for some constant κ determined by the initial conditions.

Port-Hamiltonian systems

Port-Hamiltonian systems result from network modeling of energy-conserving lumped-parameter systems with independent storage functions, and are descriptions of a wide range of physical systems [van der Schaft, 1999]. One of the most prominent appearances of port-Hamiltonian systems in control theory is in passivity-based controller design, since the inequality (2.1) takes a particularly simple form for this class of systems. We first introduce a novel description of the interconnection and damping structure and then define port-Hamiltonian systems based on this description.

The bundle of structure tensors

The interconnection structure in port-Hamiltonian systems in mechanics is a symplectic structure, *i.e.*, a non degenerate closed two-form on the cotangent bundle of the configuration space (see Abraham and Marsden [1978]). One generalization of symplectic structures are Dirac structures, which are introduced in Chapter 3. A different perspective that we want to consider here is to represent the interconnection and damping structure by tensor fields.

Definition 2.1.13. Let $\mathcal{X} \subset \mathbb{R}^n$ be open and let $T_0^2(\mathcal{X})$ be the local vector bundle of tensors of type $(2, 0)$. A section F of $T_0^2(\mathcal{X})$ such that

- i) it is local vector bundle isomorphism for $T^*\mathcal{X}$ and $T\mathcal{X}$, and
- ii) $\text{Sym } F_p \leq 0$ for all $p \in \mathcal{X}$; where $\text{Sym } F_p$ denotes the symmetrization of F_p [Lee, 2003].

is called a *structure tensor field*.

Every structure tensor field can be written as the difference between the symmetric tensor field $R = -\text{Sym } F$, called the *damping structure (tensor field)* and the skew-symmetric tensor field $J = F + \text{Sym } F$, called the *interconnection structure (tensor field)*. Next, we define port-Hamiltonian systems.

Definition 2.1.14. Let \mathcal{X} and \mathcal{U} be open subsets of \mathbb{R}^n and \mathbb{R}^m , respectively, with $n \geq m \geq 1$ and $0_{\mathbb{R}^m} \in \mathcal{U}$. Let

- F be a structure tensor field on \mathcal{X} ,
- H be a function on \mathcal{X} , and
- g_1, \dots, g_m be vector fields on \mathcal{X} .

A *port-Hamiltonian system* is a 5-tuple $\Sigma = (\mathcal{X}, F, H, g, \mathcal{U})$. It is a control-affine system with drift vector field FdH . The function H is called the *Hamiltonian function*.

Consider the port-Hamiltonian system Σ . Let (x^1, \dots, x^n) be coordinates on \mathcal{X} . In these coordinates, we have $dH = \frac{\partial H}{\partial x^i} dx^i$ and $F = F^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$. Then we can write the drift vector field of Σ in coordinates

$$f^j \frac{\partial}{\partial x^j} = F^{ij} \frac{\partial H}{\partial x^i} \frac{\partial}{\partial x^j}. \quad (2.2)$$

Example 2.1.15. We consider the local description of a simple mechanical control system as an example of a port-Hamiltonian system. Let Q be an n -dimensional manifold with local coordinate chart $(U, q = (q^1, \dots, q^n))$ and let $(q, p) = (q^1, \dots, q^n, p^1, \dots, p^n)$ be coordinates on $T^*U \subset T^*Q$. The Euler–Lagrange equations for a trajectory¹ $((q, p), u)$ of a simple mechanical control system are

$$\begin{aligned}\dot{q}(t) &= \frac{\partial H^\top}{\partial p}(q(t), p(t)) \\ \dot{p}(t) &= -\frac{\partial H^\top}{\partial q}(q(t), p(t)) + B(q(t))u(t) \\ y(t) &= B^\top(q(t))\frac{\partial H}{\partial p}(q(t), p(t)),\end{aligned}$$

with $B : U \rightarrow \mathbb{R}^{n \times m}$. The function H is the sum of kinetic energy function

$$\begin{aligned}K : T^*U &\rightarrow \mathbb{R} \\ (q, p) &\mapsto \frac{1}{2}p^\top M^{-1}(q)p,\end{aligned}$$

where $M(q) = M^\top(q) > 0$ for all $q \in U$ and potential energy function $V : U \rightarrow \mathbb{R}$, which is bounded from below. Hence, the system is in the form of equation (2.2) with $x = (q, p)$ and $J(x) = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ for all $x \in T^*U$, where I_n denotes the $n \times n$ identity matrix.

Structure preserving maps

We define maps that preserve structure tensor fields between open sets of two vector spaces with possibly different dimensions. We make use of this construction in

¹With slight abuse of notation we use (q, p) also to denote a solution.

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Definition 2.1.16. Let F and G be structure tensor fields on $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^l$, respectively, and let $\Phi : U \rightarrow V$ be a map. Then Φ is said to be *structure preserving* if

$$(\forall q \in \Phi^{-1}(V))(\forall \alpha, \beta \in T_{\Phi(q)}^*V) \quad F_q(\Phi^*(\alpha), \Phi^*(\beta)) = G_{\Phi(q)}(\alpha, \beta),$$

where Φ^* is the pull back map of Φ [Bullo and Lewis, 2004].

For example, if $\text{Sym } F = 0$ and $\text{Sym } G = 0$, *i.e.*, they can be identified with symplectic structure, then a structure preserving maps is called a symplectomorphism (see also Tabuada and Pappas [2003a] for similar discussion).

2.2 Interconnection and Damping Assignment – Passivity Based Control

The development of the IDA-PBC methodology originated as a method for stabilization of underactuated mechanical systems invoking the physically motivated principles of energy shaping and damping injection. The desired closed-loop system is chosen to be a port-Hamiltonian system, which encodes the desired features such as stability and convergence [Ortega et al., 2002b]. We denote this by a port-Hamiltonian system $\Sigma_d = (\mathcal{X}, J_d, R_d, H_d, g, \mathcal{U})$. In this section, we give a literature review on IDA-PBC. First, we formally describe the IDA-PBC methodology, then we formulate the matching problem and classify approaches for solutions of the problem. Finally, we summarize different solutions to the matching problem that have been considered in

the literature and discuss their advantages and shortcomings. The following result establishes sufficient conditions for local stabilization of a control-affine system Σ .

Proposition 2.2.1 (Ortega and Garcia-Canseco [2004]). Consider the control-affine system $\Sigma = (\mathcal{X}, f, g, \mathcal{U})$. Assume that there exists matrix-valued maps $g^\perp : \mathcal{X} \rightarrow \mathbb{R}^{n-m}$ such that $g^\perp(x)g(x) = 0$, for all $x \in \mathcal{X}$, a skew-symmetric $(0, 2)$ -tensor field J_d , a symmetric positive semi-definite $(0, 2)$ -tensor field R_d and a function $H_d : \mathcal{X} \rightarrow \mathbb{R}$ such that H_d is a solution to the set of PDEs

$$g^\perp(x)f(x) = g^\perp(x)(J_d - R_d)dH_d(x), \quad (2.3)$$

where H_d is such that

$$x_\star = \arg \min_{x \in \mathcal{X}} H_d(x) \quad (2.4)$$

with $x_\star \in \mathcal{X}$ the admissible equilibrium to be stabilized. Then, the closed-loop system with $u = \beta : \mathcal{X} \rightarrow \mathcal{U}$, where

$$\beta(x) = (g^\top(x)g(x))^{-1}g^\top(x)((J_d - R_d)(dH_d)(x) - f(x)) \quad (2.5)$$

is a port-Hamiltonian system $\Sigma_d = (\mathcal{X}, J_d, R_d, H_d, g, \mathcal{U})$ with x_\star a (locally) stable equilibrium. It is asymptotically stable if, in addition x_\star is an isolated minimum of H_d and the largest invariant set under the closed-loop dynamics contained in

$$\{x \in \mathcal{X} \mid R_d(dH_d, dH_d)(x) = 0\} \quad (2.6)$$

is equal to $\{x_\star\}$. Moreover, an estimate of its domain of attraction is given by the

largest bounded level set $\{x \in \mathcal{X} \mid H_d(x) \leq c\}$.

Hence, the key step in the IDA-PBC methodology is the solution of the equations (2.3) which we will refer to as *matching equations*. A *matching problem* is to find conditions under which a computable solution exists. It has been noted in Ortega and Garcia-Canseco [2004] that for the matching equations:

1. J_d and R_d are free, up to the constraint of skew-symmetry and positive semidefiniteness, respectively;
2. H_d may be totally, or partially, fixed as long as (2.4) can be ensured;
3. There is an additional degree of freedom in g^\perp which is not uniquely defined by g . As reported in Acosta et al. [2005], Duindam [2006], this degree of freedom can be used to linearize the nonlinear kinetic energy PDEs that appears in mechanical systems.

Remark 2.2.2. There exists a solution to the matching equation if there exists a feedback that asymptotically stabilizes the control-affine system Ortega et al. [2002b].

There are at least three conceptually different approaches to solve the matching equations:

(Non-Parameterized IDA) The desired interconnection and damping structure and g^\perp are fixed. This yields a set of PDEs whose solutions define the admissible energy functions H_d . Among the family of solutions, the one that satisfies (2.4), if it exists, is selected [Ortega et al., 2002b].

(Algebraic IDA) The desired energy function is fixed, then (2.3) becomes an algebraic equation in J_d , R_d and g^\perp (see Fujimoto et al. [2001]).

(Parameterized IDA) For some physical systems it is desirable to restrict the desired energy function to a certain class, for instance, for mechanical systems the desired total energy should be the sum of a potential energy, which depends only on the generalized positions and the kinetic energy, which should be quadratic in the generalized momentum [Ortega et al., 2002a]. Fixing the structure of the energy function yields a new set of PDEs for its unknown terms and, at the same time, imposes some constraints on the interconnection and damping structure.

Remark 2.2.3. Note that the algebraic IDA can be solved by algebraic manipulation and parametrized IDA has been applied to mechanical systems, and several solutions have been presented. In this thesis we only consider non-parameterized IDA.

Next, we define the matching problem that is subject of this thesis.

Problem 2.2.4 ((Non-parameterized IDA) Matching problem for Σ). Find a structure tensor F_d such that there exists a solution H_d to (5.4) which satisfies (2.4).

2.2.1 Matching problem

In this section, we review the results on the matching problem. First, the development of necessary and sufficient conditions for the existence of a solution of the matching equations is presented. Then, the computation of explicit solutions for individual cases is reviewed.

Necessary and sufficient conditions

In Cheng et al. [2005], conditions for the existence of a feedback that transforms a control-affine system into a port-Hamiltonian system or generalized gradient system is studied. Two approaches to the solution of the matching equations for non-parameterized IDA are presented, which are shown to be equivalent. The first approach aims to find a local static state feedback $u = \beta : \mathcal{X} \rightarrow \mathcal{U}$ directly. It makes use of Poincaré's lemma [Lee, 2003], which states that every closed differential form is locally exact. Under the assumption that the desired structures are non-singular, this approach results in $\frac{n}{2}(n-1)$ partial differential equations in terms of the feedback control β and the existence of a solution is then necessary and sufficient for a solution to the matching problem. The Hamiltonian function of the desired closed-loop system is found by simple integration, but is not necessary for the implementation of the controller. The second approach defines a set of PDEs in terms of the desired Hamiltonian function, as in (5.4). If a solution can be found then the corresponding controller is the stabilizer (2.5). Furthermore, verifiable conditions for the existence of a solution to the matching equations are given and the minimal number of partial differential equations that must be solved are determined in Cheng et al. [2005]. The derivation of these conditions are mainly motivated by results presented in Tabuada and Pappas [2003b], in which necessary and sufficient conditions for the feedback equivalence between control-affine systems and port-Hamiltonian systems on symplectic manifolds are presented.

Simplification and Solutions

The simplification of a set of PDEs is the ability to rewrite the PDEs in a form that can be solved using standard PDE or ODE techniques such as the method of characteristics [Arnol'd, 1988]. All results on simplification of the matching equations are restricted to simple mechanical control systems. Hence, we briefly review the literature on energy balancing and IDA-PBC for mechanical control systems and focus on special types of mechanical system for which an explicit solution can be found.

For mechanical control systems, the total energy is the sum of kinetic and potential energy, and it can be shown that the matching equations can be decomposed into a *kinetic energy PDE* and a *potential energy PDE*. Underactuated mechanical control systems with one degree of under actuation have received special attention. These systems are such that the kinetic energy PDE reduces to an ODE under some assumptions [Gomez-Estern et al., 2001]. It has also been shown in Acosta et al. [2005] that the resulting ODE can be solved explicitly under additional assumptions. The explicit solution can be found by means of parameterization of the interconnection structure and hence reducing the space of all closed-loop kinetic energy matrices such that their derivatives with respect to the unactuated coordinate are in the image of g . This was extended in Gomez-Estern and van der Schaft [2004] to include natural damping in the open-loop system. We also consider the possibility of a change of coordinates to simplify the matching equation. In Fujimoto and Sugie [2000], it is shown that the class of port-Hamiltonian systems is invariant under change of coordinates of the state space. This allows us to consider a coordinate-free description of port-Hamiltonian systems. Simplification of the matching equation via a change

of coordinates for mechanical systems with degree one of underactuation has been studied in Viola et al. [2007] and Viola [2008]. The study shows that the forcing term in the kinetic energy PDE can be eliminated by an appropriate choice of coordinates, which generates a homogeneous linear PDE. Hence, a change of coordinates could also be beneficial if we consider port-Hamiltonian systems other than mechanical control systems.

2.2.2 Applications

Many practical control design techniques based on IDA-BPC have been reported in the literature. For example, magnetic levitation systems [Ortega et al., 2001, Fujimoto et al., 2001] and bipedal dynamic robot [Viola, 2008]. Motivated by the success of the IDA-PBC methodology for mechanical and other lumped-parameter modeled systems, researchers have also considered to develop an extension to other classes of physical systems. To do so, the concept of energy is generalized to a conserved quantity, *i.e.*, a function that remains constant evaluated along trajectories of the system (if no dissipation is assumed). In Johnsen et al. [2008], robustness properties of IDA-PBC have been investigated on the control of a biochemical fermenter model. The application of IDA-PBC to a four-tank system has been studied in Johnsen and Allgöwer [2007]. In Hangos et al. [2004, 2001], a systematic approach for Hamiltonian representation of process control systems was introduced. A notable difference with other approaches is that, unlike in mechanical systems and other lumped parameter models, a static relationship between states and co-states is assumed. The total energy of the process system can be decomposed in a kinetic and potential part, hence mimicking closely the structure of a mechanical control system. In the context of

irreversible thermodynamics, the concept of Hamiltonian systems has been extended to the thermodynamic phase space modelled as a contact manifold [Eberard et al., 2007]. Furthermore, control of mass-balance systems by output feedback based on a damping assignment has been studied in Ortega et al. [2000, 1999] on the example of a predator-prey system.

Other relevant contributions

In Ramírez and Sbárbaro [2008], a comparative analysis was carried out that shows that the IDA-PBC approach provides considerable advantages over synthetic-output linearization in stabilization of a CSTR model. The method for the synthesis of the IDA-PBC controller is based on the developments of non-exact matching IDA-PBC. The advantage gained by non-exact matching is that an explicit solution of a set of PDEs is not required. An in-depth study of non-exact matching as well as the application of the IDA-PBC methodology to nonlinear process system has recently been presented in Ramírez et al. [2009]. Using a related approach, a result by Acosta and Astolfi [2009] proposes the application of a dynamic controller to compensate for the error introduced by an approximate solution. In Scherpen and van der Schaft [2008], a reduction of port-Hamiltonian systems has been presented which reduces a non-observable and non-strongly accessible to an observable and strongly accessible port-Hamiltonian system. This result shows that the port-Hamiltonian structure is invariant under the state decomposition in observable and strongly accessible part, under some additional assumptions. The decomposition is also invariant under feedback, indicating that the Hamiltonian decomposition should also result in a decomposition of the matching problem. In Belabbas [2009], energy conserving feedback of

port-Hamiltonian systems on symplectic manifolds have been studied and an explicit expression of the achievable closed-loop symplectic forms under lossless interconnection has been derived. It has been shown that the achievable closed-loop symplectic forms depend on the Lie brackets of the Hamiltonian input vector fields. It indicates that similar dependencies should appear in the more general case of port-Hamiltonian systems.

2.3 Control by Interconnection

The second passivity-based control technique considered is Control by Interconnection (CbI) [Ortega et al., 2001, 2002b, Castanos et al., 2009, Dalsmo and van der Schaft, 1998], which is applicable to port-Hamiltonian system. The control objective in CbI is to render the closed-loop system passive. Passivity of the closed-loop is established by choosing the controller to be a port-Hamiltonian system and considering feedback as a power conserving interconnection between plant and controller [Dalsmo and van der Schaft, 1998]. As a consequence of the power-conserving interconnection, the sum of the plant and controller Hamiltonian function is equal to the Hamiltonian function of the closed-loop system. The steady-state and the transient behavior are determined by the closed-loop Hamiltonian function, the interconnection structure and damping structure. It is necessary to relate the states of the plant and the controller via the generation of additional invariant sets defined by, so-called, Casimir functions [Dalsmo and van der Schaft, 1998], which are functions of the plant and controller states independent of the Hamiltonian function. In its basic formulation, CbI assumes that only the plant output is measurable and considers the classical output feedback interconnection. In this case, the Casimir functions are fully determined by

the plant. This imposes a severe restriction on the plant dissipative structure [Ortega et al., 2008]. In particular, this means, that in its basic form, CbI is not amenable for systems with dissipation in the coordinates directly influenced by the control inputs. This dissipation obstacle restricts CbI to applications like mechanical systems, in which the coordinates to be shaped are typically positions, which are unaffected by dissipation through friction. To overcome the dissipation obstacle and make the CbI widely applicable, several extensions of CbI have been proposed Ortega et al. [2008]. Two extensions are considered, one extension proposes the generation of new cyclo-passive outputs (with new storage functions) by exploiting the non-uniqueness of the port-Hamiltonian representation of the system. Applying CbI through these new port variables overcomes the dissipation obstacle, but rules out several interesting physical examples. For the second extension, the output feedback is replaced by a suitably defined state-modulated interconnection [van der Schaft, 1999], which is a given by a state feedback. In this case, the existence conditions for the Casimir functions can be further relaxed, thus improving the applicability of CbI to a larger class of port-Hamiltonian systems. A detailed analysis of CbI and its extensions is presented in Ortega et al. [2008], where the connection to different types of IDA-PBC and other passivity based control techniques were investigated. The following observations were made:

1. If a port-Hamiltonian systems can be controlled by any variation of CbI, *i.e.*, if a closed-loop Hamiltonian function with the desired properties can be computed, then the system can also be controlled using the IDA-PBC methodology, but the converse is not necessarily true.
2. The static feedback law obtained by the IDA-PBC methodology is the restricted

application of the dynamic feedback CbI controller to the invariant sets defined by the Casimir functions.

3. If a given plant can be stabilized by CbI, then it can also be stabilized by the IDA-PBC technique, showing that in this case there is no advantage in considering dynamic feedback.

Hence, IDA-PBC includes, in the sense mentioned above, all forms of CbI. If we can find a solution to the matching problem in IDA-PBC, then this allows us to find Casimir functions for the corresponding CbI controller.

Summary

In this chapter, we presented an overview of recent developments in IDA-PBC. A common stumbling block in all developments is the absence of a constructive procedure to obtain explicit solutions of the matching equations, required for the controller synthesis. We highlight the following:

1. Literature that presents explicit solutions to the matching problem is limited to mechanical control systems.
2. The degrees of freedom in the choice of the interconnection and damping structures in non-parameterized IDA-PBC are used to simplify the matching equations.

The following chapters present novel approaches, which are applicable to control-affine system. Furthermore, we develop new approaches to solve the matching equations and unify existing approaches by analyzing their common assumption on the control-affine system and the interconnection and damping structures.

Chapter 3

The Matching Problem for Linear Systems

The main objective of this chapter is to establish that linear port-Hamiltonian systems can be effectively decomposed into reduced port-Hamiltonian systems. This decomposition provides more insight into the intrinsic interconnection structure of the linear port-Hamiltonian system. In particular, we show that certain linear port-Hamiltonian systems can be decomposed based on their controllability matrices. For these linear port-Hamiltonian systems, decomposition could be a viable approach for solving the matching problem.

3.1 Introduction

The concept of decomposing a port-Hamiltonian system has received limited attention in the literature. The notable exceptions are Ortega et al. [1998] and Cervera et al. [2007]. In the former reference, a decomposition result for mechanical system

is presented. This is based on the Lagrangian formulation and assumes that two distinct Lagrangians have been identified *a priori*. In contrast, we do not assume *a priori* knowledge of a decomposition the Hamiltonian function. The latter reference is concerned with Dirac structures, which can represent port-Hamiltonian systems. In the conclusions section of Cervera et al. [2007], it is noted that an explicit algorithm for the minimal representation of a network of Dirac structures is profitable in the context of modelling interconnected systems. In this chapter, we denote by Σ a linear system $\dot{x} = Ax + Bu$ with $x \in \mathcal{X} = \mathbb{R}^n$ and $u \in \mathcal{U} = \mathbb{R}^m$, respectively, where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. In this context, Assumption 2.1.10 is equivalent to the following assumption, valid throughout this chapter.

Assumption 3.1.1. The matrix B has full column rank m .

Denote by B_\perp a full rank $n \times (n - m)$ matrix such that $\text{im } B_\perp$ is a complement of $\text{im } B$, *i.e.* $\text{im } B \oplus \text{im } B_\perp = \mathbb{R}^n$. Let $B^\perp = B_\perp^\top$, then it is a full row rank $(n - m) \times n$ matrix that annihilates B , *i.e.*, $B^\perp B = 0$.

3.1.1 Problem statement

In this section, we consider the matching problem for linear control systems. We then present existing solutions to this “linear matching problem” and discuss their advantages and shortcomings. This motivates the approach proposed in this chapter for decomposition of linear port-Hamiltonian systems.

Problem 3.1.2 (Linear Matching Problem (LMP)). Given a linear system Σ , find matrices $J_d = -J_d^\top, R_d = R_d^\top \geq 0, Q_d = Q_d^\top > 0$ and a linear state feedback $K : \mathcal{X} \rightarrow \mathcal{U}$

such that the following equation holds:

$$A + BK = (J_d - R_d)Q_d. \quad (3.1)$$

For linear systems, the partial differential equations we referred to as matching equations in the general control-affine case, reduce to a system of algebraic equations and inequalities. Here, just as in non-parametric IDA-PBC, we are interested in finding matrices J_d and R_d such that there exists a matrix Q_d that solves (3.1). If this is the case, then $H_d(x) = \frac{1}{2}x^\top Q_d x$ is a Hamiltonian function for the closed-loop port-Hamiltonian system with interconnection and damping structure J_d and R_d , respectively. Following standard arguments, one can write a linear state feedback K that satisfies (3.1), in terms of J_d , R_d and Q_d as

$$K = (B^\top B)^{-1} B^\top ((J_d - R_d)Q_d - A). \quad (3.2)$$

We abbreviate the matrix $J_d - R_d$ by F_d . The requirement that $F_d + F_d^\top \leq 0$, is equivalent to the requirements that $J_d = -J_d^\top$ and $R_d = R_d^\top \geq 0$. Hence, we refer to a pair (F_d, Q_d) that satisfies (3.1) with K as in (3.2), and $F_d + F_d^\top \leq 0$ and $Q_d = Q_d^\top > 0$ as a *solution* to the LMP.

Remark 3.1.3. The closed-loop system is stable if a solution (F_d, Q_d) to the LMP exists. It is asymptotically stable if, in addition, the origin is the largest invariant set contained in

$$\mathcal{O} = \{x \in \mathbb{R}^n \mid x^\top Q_d F_d Q_d x = 0\}.$$

Next, we review existing solutions to the LMP.

3.1.2 Existing solutions to the LMP

Prajna et al. [2002] give two constructive conditions for the existence of a solution, based on the solution of linear matrix inequalities (LMI) [Boyd et al., 1994]. We state both results, which were presented for linear port-Hamiltonian system, for general linear systems.

Proposition 3.1.4. Denote $B^\perp A$ by E^\perp and let $E_\perp = (E^\perp)^\top$. Then there exists a solution to the linear matching problem if and only if we can find a solution $X = X^\top \in \mathbb{R}^{n \times n}$ of the following LMIs:

$$X > 0 \tag{3.3}$$

$$-(E^\perp X B_\perp + B^\perp X E_\perp) \geq 0. \tag{3.4}$$

Given such a matrix X , we can compute F_d as follows:

$$F_d = \begin{bmatrix} B_\perp \\ B^\top \end{bmatrix}^{-1} \begin{bmatrix} E^\perp X \\ -B^\top X E_\perp (B^\perp B_\perp)^{-1} B^\perp \end{bmatrix},$$

and the matrices J_d, R_d and Q_d are given for instance by

$$J_d = \frac{1}{2}(F_d - F_d^\top), \quad R_d = -\frac{1}{2}(F_d + F_d^\top) \quad \text{and} \quad Q_d = X^{-1}.$$

Note that the conditions (3.3) and (3.4) in Proposition 3.1.4 are independent of F_d . It is of interest for practical purposes to have some influence on the structure of

F_d . Hence, a second, slightly stronger, set of conditions has been proposed in Prajna et al. [2002], which are formulated in terms of F_d that imposes a specific structure on F_d . For the next proposition, we need the following definition.

Definition 3.1.5. The linear system Σ is *controllable at $s = 0$* if it has no uncontrollable poles at $s = 0$.

Note that a linear system is controllable at $s = 0$ if and only if $\text{rank}[A, B] = n$.

Proposition 3.1.6. Suppose that Σ is controllable at $s = 0$ and denote $B_{\perp}A$ by E_{\perp} . Furthermore, let $E \in \mathbb{R}^{n \times m}$ be a full column rank matrix that is annihilated by E^{\perp} . There exists a solution to the linear matching problem if and only if there exists a solution $F_d \in \mathbb{R}^{n \times n}$ of the following LMI:

$$B^{\perp}F_dE_{\perp} + E^{\perp}F_d^{\top}B_{\perp} > 0 \quad (3.5)$$

$$-(F_d + F_d^{\top}) \geq 0, \quad (3.6)$$

together with the linear constraint

$$B^{\perp}F_dE_{\perp} - E^{\perp}F_d^{\top}B_{\perp} = 0. \quad (3.7)$$

Remark 3.1.7. The linear constraint (3.7) is the linear version of the existence condition for a pseudo-gradient system as proposed by Cheng et al. [2005], which is an integrability condition restricted to the domain of B_{\perp} . We establish similar existence conditions in Chapter 4. Furthermore, (3.5) establishes that the matrix Q_d is positive definite and (3.6) guarantees asymptotic stability if the inequality is strict.

Remark 3.1.8. Gharesifard and Lewis [2010] recently introduced a geometric formulation of stabilization via energy shaping for simple linear mechanical control systems. The contribution is formulated in terms of the kinetic and potential energy PDE and gives a constructive proof for energy balancing.

Propositions 3.2.15 and 3.2.17 formulate the conditions for a solution of the LMP as a set of LMIs, respectively. A more control-theoretic sufficient condition is presented next.

Proposition 3.1.9 (Prajna et al. [2002]). The set of LMI (3.3)–(3.4) is solvable if the system Σ is (weakly) stabilizable; in the sense that the uncontrollable part of Σ is stable.

The proof of this proposition is based on the following lemma. It uses the solution of the associated Lyapunov equation as Hamiltonian function and the interconnection and damping structure are then constructed accordingly.

Lemma 3.1.10 (Prajna et al. [2002]). The (asymptotically) stable linear system $\dot{x} = Ax$, can be written in port-Hamiltonian form, with positive definite Q and positive semi-definite (positive definite) R .

Prajna et al. [2002] consider the linear matching problem where the given system Σ is a linear port-Hamiltonian system. In this case, the objective is to alter the interconnection and damping structure $J - R$ and/or the Hamiltonian function (determined by its Hessian, Q). Proposition 3.1.6 allows us to specify only a subset of the elements of $J - R$ and solve the LMI for the remaining elements (See example in Prajna et al. [2002], Section 4). The motivation behind such an approach is clear when the interconnection and damping structure have physical meaning, for example

springs and dampers in mechanical systems. Here, J in some sense represents the “topology” of the physical system. Further consideration leads to the question raised in Prajna et al. [2002]: What is the minimal amount of change in the interconnection structure for a given system to achieve stabilization? In the matching problem considered in this chapter it is not assumed that Σ is a port-Hamiltonian system. As a result, no physical intuition is available to determine which elements of $J_d - R_d$ should be fixed. We propose a framework that allows us to analyse the “topology” of the linear system based on control-theoretic considerations. The main thrust of this research comes from the concept of linear abstractions and C -related control systems proposed in Pappas et al. [2000].

The remainder of this chapter is structured as follows. In Section 3.2, we review the concept of Dirac structures and abstractions. It is shown that port-Hamiltonian systems can be realized by interconnection of the abstraction with a “virtual controller”-port-Hamiltonian system. In the final section, implications and extensions of the proposed framework are discussed.

3.2 Decomposition of Linear Port-Hamiltonian Systems

It is known that power-conserving interconnections of port-Hamiltonian system are again port-Hamiltonian systems (see for example Duindam et al. [2009]). Hence, under this general type of interconnection, the port-Hamiltonian structure of control-affine systems is preserved. Furthermore, the interconnection of two port-Hamiltonian systems enjoys several desirable properties. Its Hamiltonian function is the sum of

the individual Hamiltonians and it is passive if the individual port-Hamiltonian systems are passive. One question of interest in this chapter is whether this process of interconnection can be reversed. More precisely, given a port-Hamiltonian system Σ , when can one write it as an interconnection of “reduced” port-Hamiltonian systems? We first define what a reduced port-Hamiltonian system is with respect to Σ . For this we use the notion of abstraction of a linear control system. Then, we develop a framework that allows us to determine when an abstraction of Σ is also a port-Hamiltonian system. With this as a backdrop, we can now state the primary objective of this section, which is to develop conditions under which a port-Hamiltonian system can be written as an interconnection of an abstraction with an additional port-Hamiltonian system. It is conceptually preferable, for our purpose, to work with Dirac structures, which are introduced in the following section, to represent linear port-Hamiltonian systems.

3.2.1 Dirac structures

Dirac structures are generalizations of symplectic and Poisson structures [Libermann and Marle, 1987], and can be used to model the interconnection structure of port-Hamiltonian systems. They are algebraic structures on vector spaces, and can be extended naturally to (differentiable) Dirac structures on manifolds [Dalsmo and van der Schaft, 1998]. We are interested in Dirac structures that are induced by port-Hamiltonian systems. The following discussion is based on Pasumarthly [2006].

Definition 3.2.1. Let \mathcal{F} be a finite-dimensional real vector space with dual \mathcal{F}^* . A *Dirac structure on \mathcal{F}* is a subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ such that $\dim \mathcal{D} = \dim \mathcal{F}$ and $\langle e, f \rangle = 0 \forall (f, e) \in \mathcal{D} \times \mathcal{D}^*$, where $\langle e, f \rangle$ denotes the value of the linear functional $e \in \mathcal{F}^*$ acting

on $f \in \mathcal{F}$.

For a Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$, \mathcal{F} is called the *flow space* of \mathcal{D} and \mathcal{F}^* is called the *effort space*. Furthermore, we denote by *power* the dual product $\langle e, f \rangle$ for $(f, e) \in \mathcal{F} \times \mathcal{F}^*$. Then “power conservation” is one of the defining property of a Dirac structure \mathcal{D} , since all elements of \mathcal{D} have zero power. Let $\mathcal{F}_1, \dots, \mathcal{F}_r$ be finite-dimensional real vector spaces, $\mathcal{F} = \mathcal{F}_1 \times \dots \times \mathcal{F}_r$, and $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ a Dirac structure. We denote $\mathcal{F}_i \times \mathcal{F}_i^* \cap \mathcal{D}$ the *i*th *port space* and its elements are called the port variables. The Dirac structure \mathcal{D} “links” various port variables such that the total power considering all port variables is zero, *i.e.*, if $(f, e) = (f_1, \dots, f_r, e_1, \dots, e_r) \in \mathcal{D}$, then

$$\langle e, f \rangle = e_1^\top f_1 + \dots + e_r^\top f_r = 0.$$

Next, we consider how linear port-Hamiltonian systems determine Dirac structures in a natural way. In this context, a Dirac structure \mathcal{D} is interconnected via the *internal ports* (f_x, e_x) and (f_R, e_R) to an energy storage element, and a resistive element, respectively. These two elements represent the storage or Hamiltonian function and the damping structure, respectively. In addition, we define (external) control ports (f_c, e_c) by which the port-Hamiltonian system is connected to other port-Hamiltonian systems.

Energy storage element For the linear port-Hamiltonian system Σ , the *energy storage element* (\mathcal{X}, H) consists of the state space \mathcal{X} , which denotes the space of *energy-variables* and the Hamiltonian function H . The flow variables of the energy storage element are the time derivatives \dot{x} and the effort variables are given by $\frac{\partial H}{\partial x}$. The energy storage element satisfies the total energy balance $\dot{H} = \langle \frac{\partial H}{\partial x}, \dot{x} \rangle = 0$. Then

the equations

$$\begin{aligned} f_x &= -\dot{x}, \\ e_x &= \frac{\partial H}{\partial x} \end{aligned}$$

are called the *interconnection* of \mathcal{D} with the energy storage element (\mathcal{X}, H) , via the internal port (f_x, e_x) .

Resistive element The second internal port $\mathcal{F}_R \times \mathcal{F}_R^* \cap \mathcal{D}$ with port variables (f_R, e_R) is interconnected to a *static resistive relation* given by $\mathcal{R} : \mathcal{F} \times \mathcal{F}^* \rightarrow \mathbb{R}$ such that

$$\mathcal{R}(f_R, e_R) = 0,$$

with the property that for all $(f_R, e_R) \in \mathcal{F} \times \mathcal{F}^*$ satisfying the above equation, we have $\langle e_R, f_R \rangle \leq 0$. Equivalently, all (f_R, e_R) have to satisfy

$$R_f f_R + R_e e_R = 0,$$

where R_f and R_e are square matrices satisfying

$$R_f R_e^\top = R_e R_f^\top \geq 0$$

$$\text{rank} [R_f, R_e] = \dim \mathcal{F}_R.$$

External ports Now, let us denote the (external) control ports $\mathcal{F}_c \times \mathcal{F}_c^* \cap \mathcal{D}$ with port variables (f_c, e_c) , which are available for controller action. Hence, for a Dirac structure with storage element, resistive element and control ports, the flow variables

are $f = [f_x, f_R, f_c]^\top$ and the effort variables are $e = [e_x, e_R, e_c]^\top$. Then the defining power-conservation equation $\langle e, f \rangle = 0$ reads:

$$e_s^\top f_s + e_R^\top f_R + e_c^\top f_c = 0.$$

Matrix kernel representation

Dirac structures admit different representations, see Bloch and Crouch [1999], Courant [1990], van der Schaft [1999], Cervera et al. [2007] for further details. We only require the following representation. Let $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ be a Dirac structure on \mathcal{F} . Then we can write

$$\mathcal{D} = \{(e, f) \in \mathcal{F} \times \mathcal{F}^* \mid Ff + Ee = 0\},$$

for some $n \times n$ matrices F and E satisfying

$$EF^\top + FE^\top = 0$$

$$\text{rank} [F, E] = n.$$

The pair (E, F) is called the *matrix kernel representation* of \mathcal{D} . If the image of F and E have dimensions larger than the dimension of \mathcal{F} , then (E, F) is called *relaxed*.

Example 3.2.2. A linear port-Hamiltonian system Σ without damping $R = 0$ defines

a Dirac structure with energy storage and control ports given by

$$\mathcal{D} = \left\{ (f_x, u, e_x, y) \in \mathcal{X} \times \mathcal{F} \times \mathcal{X}^* \times \mathcal{F}^* \mid \begin{bmatrix} I_n \\ 0 \end{bmatrix} f_x + \begin{bmatrix} J \\ B^\top \end{bmatrix} e_x + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ -I_m \end{bmatrix} y = 0 \right\}.$$

Hence, we have

$$F = \begin{bmatrix} I_n & B \\ 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} J & 0 \\ B^\top & -I_m \end{bmatrix}.$$

We use the following notation for the matrices involved in the matrix kernel representation of port-Hamiltonian systems:

$$F_x = \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \quad F_c = \begin{bmatrix} B \\ 0 \end{bmatrix}, \\ E_x = \begin{bmatrix} J \\ B^\top \end{bmatrix}, \quad E_c = \begin{bmatrix} 0 \\ -I_m \end{bmatrix}.$$

Interconnection of port-Hamiltonian systems and Dirac structures

The following discussion can be found in Cervera et al. [2007]. We consider two types of composition of Dirac structures, composition and gyrative composition. We study the composition of two Dirac structures with partially shared variables. Consider the Dirac structure \mathcal{D}_A on a product space $\mathcal{F}_1 \times \mathcal{F}_2$ of two finite-dimensional vector spaces \mathcal{F}_1 and \mathcal{F}_2 , and another Dirac structure \mathcal{D}_B on $\mathcal{F}_2 \times \mathcal{F}_3$, with \mathcal{F}_3 being an additional finite-dimensional vector space. The space \mathcal{F}_2 is called the space of *shared*

flow variables, and \mathcal{F}_2^* the space of *shared effort variables*. We make the following definitions:

Definition 3.2.3. Let \mathcal{D}_A and \mathcal{D}_B be two Dirac structures on $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$ and $\mathcal{F}_2 \times \mathcal{F}_2^* \times \mathcal{F}_3 \times \mathcal{F}_3^*$, respectively. The *composition* (or canonical interconnection) of \mathcal{D}_A and \mathcal{D}_B is defined as

$$\begin{aligned} \mathcal{D}_A \parallel \mathcal{D}_B = & \{(f_1, e_1, f_3, e_3) \in \mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_3 \times \mathcal{F}_3^* \mid \\ & \exists (f_2, e_2) \in \mathcal{F}_2 \times \mathcal{F}_2^* \text{ s.t.} \\ & (f_1, e_1, f_2, e_2) \in \mathcal{D}_A \text{ and } (-f_2, e_2, f_3, e_3) \in \mathcal{D}_B\}. \end{aligned}$$

Definition 3.2.4. Let \mathcal{D}_A and \mathcal{D}_B be two Dirac structures on $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$ and $\mathcal{F}_2 \times \mathcal{F}_2^* \times \mathcal{F}_3 \times \mathcal{F}_3^*$, respectively. The *gyrative composition* of \mathcal{D}_A and \mathcal{D}_B is defined as

$$\begin{aligned} \mathcal{D}_A \times \mathcal{D}_B = & \{(f_1, e_1, f_3, e_3) \in \mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_3 \times \mathcal{F}_3^* \mid \\ & \exists (f_2, e_2) \in \mathcal{F}_2 \times \mathcal{F}_2^* \text{ s.t.} \\ & (f_1, e_1, f_2, e_2) \in \mathcal{D}_A \text{ and } (e_2, -f_2, f_3, e_3) \in \mathcal{D}_B\}. \end{aligned}$$

Remark 3.2.5. Note that $\mathcal{D}_A \times \mathcal{D}_B$ can also be constructed via the composition with the symplectic Dirac structure

$$\mathcal{D}_I = \{(f_{IA}, e_{IA}, f_{IB}, e_{IB}) \mid f_{IA} = -e_{IB}, f_{IB} = e_{IA}\},$$

such that $\mathcal{D}_A \parallel \mathcal{D}_I \parallel \mathcal{D}_B = \mathcal{D}_A \times \mathcal{D}_B$.

It can be shown that, if \mathcal{D}_A and \mathcal{D}_B are two Dirac structures as defined in Definition 3.2.4, then $\mathcal{D}_A \parallel \mathcal{D}_B$ and $\mathcal{D}_A \times \mathcal{D}_B$ are Dirac structures with respect to the canonical

bilinear form on $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_3 \times \mathcal{F}_3^*$. Furthermore, if Σ_1 and Σ_2 are two port-Hamiltonian systems and \mathcal{D}_1 and \mathcal{D}_2 are their Dirac structures, constructed as in Example 3.2.2, then $\mathcal{D}_1 \times \mathcal{D}_2$ is again a port-Hamiltonian system and is the feedback interconnection of Σ_1 and Σ_2 . In particular,

$$\begin{aligned}\Sigma_1 : \dot{x}_1 &= F_1 Q_1 x_1 + B_1 u_1 \\ \Sigma_2 : \dot{x}_2 &= F_2 Q_2 x_2 + B_2 u_2\end{aligned}$$

then their feedback interconnection denoted by $\Sigma = \Sigma_1 \times \Sigma_2$ takes the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} F_1 & B_1 B_2^\top \\ -B_2 B_1^\top & F_2 \end{bmatrix} \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

We have the following proposition, due to Cervera et al. [2007], that presents a matrix kernel representation of the composition of two Dirac structures in terms of the matrix kernel representation of the individual Dirac structures.

Proposition 3.2.6. Let $\mathcal{F}_i, i = 1, 2, 3$ be a finite-dimensional linear space with $\dim \mathcal{F}_i = n_i$. Consider the Dirac structures

$$\begin{aligned}\mathcal{D}_A &\subset \mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_2 \times \mathcal{F}_2^*, & n_A &= \dim \mathcal{F}_1 \times \mathcal{F}_2 = n_1 + n_2 \\ \mathcal{D}_B &\subset \mathcal{F}_2 \times \mathcal{F}_2^* \times \mathcal{F}_3 \times \mathcal{F}_3^*, & n_B &= \dim \mathcal{F}_2 \times \mathcal{F}_3 = n_2 + n_3,\end{aligned}$$

given by the relaxed matrix kernel/image representation

$$\begin{aligned}(F_A, E_A) &= ([F_1, F_{2A}], [E_1, E_{2A}]), \\ (F_B, E_B) &= ([F_{2B}, F_3], [E_{2B}, E_3])\end{aligned}$$

$n'_A \times n_A$ matrices and $n'_B \times n_B$ matrices, respectively with $n'_A \geq n_A$ and $n'_B \geq n_B$. Define the $(n'_A + n'_B) \times 2n_2$ matrix

$$M = \begin{bmatrix} F_{2A} & E_{2A} \\ -F_{2B} & E_{2B} \end{bmatrix}$$

and let L_A, L_B be $m \times n'_A$, respectively $m \times n'_B$, matrices with

$$L = [L_A, L_B], \quad \ker L = \text{im } M.$$

Then

$$F = [L_A F_1, L_B F_3], \quad E = [L_A E_1, L_B E_3],$$

is a relaxed matrix kernel representation of $\mathcal{D}_A \parallel \mathcal{D}_B$.

Similar, we have for the gyrative composition $\mathcal{D}_A \times \mathcal{D}_B$.

Proposition 3.2.7. Let $\mathcal{F}_i, i = 1, 2, 3$ and $\mathcal{D}_A, \mathcal{D}_B$ as in Proposition 3.2.6. Define the $(n'_A + n'_B) \times 2n_2$ matrix

$$M = \begin{bmatrix} E_{2A} & F_{2A} \\ -F_{2B} & E_{2B} \end{bmatrix}$$

and let L_A, L_B be $m \times n'_A$, respectively $m \times n'_B$, matrices with $L = [L_A, L_B]$ and $\ker L = \text{im } M$. Then

$$F = [L_A F_1, L_B F_3], \quad E = [L_A E_1, L_B E_3],$$

is a relaxed matrix kernel representation of $\mathcal{D}_A \times \mathcal{D}_B$.

Proof: Let $\mathcal{D}_A \times \mathcal{D}_B = \mathcal{D}_A \parallel \mathcal{D}_I \parallel \mathcal{D}_B$ the gyrative composition of \mathcal{D}_A and \mathcal{D}_B . Then the proof follows the proof of Theorem 4 in Cervera et al. [2007] with matrix representation of the shared flow and effort variables

$$M = \begin{bmatrix} E_{2A} & F_{2A} \\ -F_{2B} & E_{2B} \end{bmatrix}.$$

■

Furthermore, we define regularity of a composition.

Definition 3.2.8. Given two Dirac structures \mathcal{D}_A and \mathcal{D}_B defined as above. Their composition is said to be *regular* if the values of the power variables in $\mathcal{F}_2 \times \mathcal{F}_2^*$ are uniquely determined by the values in the power variables in $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_3 \times \mathcal{F}_3^*$; that is, the following implication holds:

$$\begin{aligned} (f_1, e_1, f_2, e_2) \in \mathcal{D}_A \text{ and } (-f_2, e_2, f_3, e_3) \in \mathcal{D}_B \\ (f_1, e_1, \tilde{f}_2, \tilde{e}_2) \in \mathcal{D}_A \text{ and } (-\tilde{f}_2, e_2, \tilde{f}_3, e_3) \in \mathcal{D}_B \\ \Rightarrow f_2 = \tilde{f}_2, \quad e_2 = \tilde{e}_2 \end{aligned}$$

3.2.2 Linear abstraction

Next, we introduce the concepts of linear abstractions as presented in Pappas et al. [2000], where linear abstractions have been introduced for hierarchical control in which the high level control system is modelled by aggregating the details of the lower control systems.

Definition 3.2.9. Consider the linear control systems

$$\Sigma_1: \dot{x}_1 = A_1x_1 + B_1u_1$$

$$\Sigma_2: \dot{x}_2 = A_2x_2 + B_2u_2$$

with $\mathcal{X}_1 = \mathbb{R}^{n_1}$, $\mathcal{U}_1 = \mathbb{R}^{m_1}$ and $\mathcal{X}_2 = \mathbb{R}^{n_2}$, $\mathcal{U}_2 = \mathbb{R}^{m_2}$, respectively. Let $C: \mathcal{X}_1 \rightarrow \mathcal{X}_2$ be a surjective linear map. The linear systems are said to be *C-related* if for all trajectories (x_1, u_1) there exists a control action u_2 such that

$$C(A_1x_1 + B_1u_1) = A_2Cx_1 + B_2u_2. \quad (3.8)$$

Now, consider a linear system Σ_1 and surjective map C , the following proposition lets us construct a linear system which is *C-related* to Σ_1 .

Proposition 3.2.10 (Pappas et al. [2000]). Consider the linear system

$$\Sigma_1: \dot{x} = A_1x + B_1u$$

and a linear surjective map $C: \mathcal{X} \rightarrow \mathcal{X}_2 = \mathbb{R}^{n_2}$, $x \mapsto y = Cx$. Let

$$\Sigma_2: \dot{y} = A_2y + B_2v$$

be a linear system on \mathcal{X}_2 where

$$\begin{aligned} A_2 &= CAC^+, \\ B_2 &= [CB, CA_1v_1, \dots, CA_1v_r] \end{aligned}$$

with C^+ the pseudo-inverse of C (i.e., $C^+ = C^\top(CC^\top)^{-1}$) and v_1, \dots, v_r such that $\text{span}\{v_1, \dots, v_r\} = \ker C$. Then Σ_1 and Σ_2 are C -related.

The system Σ_2 is called a *linear abstraction* of Σ_1 with respect to C . For the linear port-Hamiltonian systems we make the following definition:

Definition 3.2.11. Consider the linear port-Hamiltonian system Σ and surjective map $C : \mathcal{X} \rightarrow \mathcal{X}_A = \mathbb{R}^{n_A}$. Denote a linear abstraction Σ with respect to C by Σ_A , then Σ_A is called a *linear port-Hamiltonian abstraction* of Σ . One can show that it is a linear port-Hamiltonian system with structure tensor $F_A = CFC^\top$ and $Q_A = (C^+)^\top QC^+$.

3.2.3 Decomposition

In this section, we apply results on composition of Dirac structures to abstractions of linear port-Hamiltonian systems. We present conditions that a linear port-Hamiltonian system can be written as an interconnection of an abstraction and another linear systems. We motivate this with the simple example of an LC circuit [Polyuga and van der Schaft, 2008].

Example 3.2.12. Consider a controlled LC-circuit (see Fig. 3.1) consisting of two inductors with magnetic energies $H_1(\phi_1), H_2(\phi_2)$, where ϕ_1 and ϕ_2 are the magnetic

flux linkages, and a capacitor with electrical energy $H_3(q)$, where q is the charge. Assume the components of the LC-circuit are linear, then their respective total energies are

$$H_1(\phi_1) = \frac{1}{2L_1}\phi_1^2, \quad H_2(\phi_2) = \frac{1}{2L_2}\phi_2^2 \quad \text{and} \quad H_3(q) = \frac{1}{2C}q^2,$$

respectively. Furthermore, let $V = u$ denote a voltage source, then Kirchhoff's laws yield the linear port-Hamiltonian system

$$\begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \\ 0 & 0 & \frac{1}{C} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ q \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u, \quad (3.9)$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \\ 0 & 0 & \frac{1}{C} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ q \end{bmatrix}, \quad (3.10)$$

with Hamiltonian function $H = H_1 + H_2 + H_3$ and natural passive output y . An

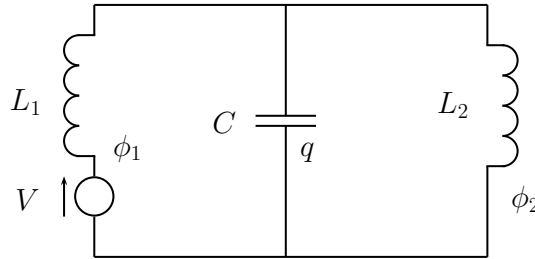


Figure 3.1: LC circuit

alternative way to establish this representation is to consider each element as a linear

port-Hamiltonian system. Let us define

$$\begin{aligned} \Sigma_1 : \quad \dot{\phi}_1 &= u_1 \\ y_1 &= \frac{\partial H_1}{\partial \phi_1} = \frac{\phi_1}{L_1} \\ \Sigma_2 : \quad \dot{\phi}_2 &= u_2 \\ y_2 &= \frac{\partial H_2}{\partial \phi_2} = \frac{\phi_2}{L_2} \\ \Sigma_3 : \quad \dot{q} &= u_3 \\ y_3 &= \frac{\partial H_3}{\partial q} = \frac{q}{C}, \end{aligned}$$

where u_i and y_i , $i = 1, 2, 3$ are the voltages and currents of each element, respectively.

The systems Σ_2 and Σ_3 are then interconnected via the following rule

$$\begin{aligned} u_2 &= y_3 + v_2 \\ u_3 &= -y_2 + v_3, \end{aligned}$$

which can also be expressed by a symplectic Dirac structure. This yields the intermediate linear port-Hamiltonian system

$$\begin{aligned} \Sigma' : \quad \begin{bmatrix} \dot{\phi}_2 \\ \dot{q} \end{bmatrix} &= \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} v_2 \\ v_3 \end{bmatrix} \\ \begin{bmatrix} y_2 \\ y_3 \end{bmatrix} &= \begin{bmatrix} \frac{\partial H_2}{\partial \phi_2} \\ \frac{\partial H_3}{\partial q} \end{bmatrix} = \begin{bmatrix} \frac{\phi_2}{L_2} \\ \frac{q}{C} \end{bmatrix}, \end{aligned}$$

where we define a new input $v' = [v_2, v_3]^\top$ and new output $y' = [y_2, y_3]^\top$. Furthermore, the Hamiltonian function of Σ' is the sum of H_2 and H_3 . Next, we interconnect Σ' to

Σ_1 via the feedback interconnection

$$\begin{aligned} u_1 &= \begin{bmatrix} 0 & 1 \end{bmatrix} y' + u \\ v' &= - \begin{bmatrix} 0 \\ 1 \end{bmatrix} y_1, \end{aligned}$$

which also can be expressed by symplectic Dirac structure. The interconnection is then the linear port-Hamiltonian system Σ .

Next, we want to analyze if we can deduce an intrinsic interconnection structure given only the system (3.9), assuming only that the final system was constructed using lossless feedback interconnection. We begin with the state that is directly influenced by the control input, we assume that the subsystem (consisting of the ϕ_1 -dynamics) is lossless with quadratic storage function $H_1(\phi_1)$, given by $H_1 = \frac{1}{2}B^\top QB\phi_1^2$. If the system is not lossless, it is clear that we can find a preliminary feedback that cancels damping in this state. Next, we need to construct the subsystem consisting of the remaining states and the interconnection with the ϕ_1 -dynamics.

First, let us define the subsystem consisting of the remaining states. Motivated by the idea of linear abstractions, we quotient the states space by $\text{im}(B)$ and define the quotient space as the reduced state space of the linear abstraction. We denote the projection onto the quotient space by C , then the reduced dynamics take the form

$$\begin{bmatrix} \dot{\phi}_2 \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L_2} & 0 \end{bmatrix} \begin{bmatrix} \phi_2 \\ q \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L_1}\phi_1 \end{bmatrix},$$

where the last term can be considered as a “virtual port” for the reduced dynamics.

Note that this only specifies a virtual port, not a virtual control input, which is used, for example, in backstepping [Sepulchre et al., 1997]. The reduced system must have a scalar input \bar{u} (to be determined) and a scalar output \bar{y} dual to \bar{u} . Moreover, the input \bar{u} should be a linear function of ϕ_1 .

We are therefore left with determination of \bar{u} and \bar{y} . By the assumption of interconnection we have that $\bar{u} = y_1 = B_1 \frac{\partial H_1}{\partial \phi_1} = \frac{1}{L_1} \phi_1$. Hence, $B_2 = [0, 1]^\top$ and we can solve $A_{12} = -B_2^\top F_2 Q_2$ with $F_2 = CFC^\top$ for Q_2 . Hence, the reduced system can be written as a linear port-Hamiltonian system.

We are interested in formalizing this approach using the idea of achievable Dirac structures, which we introduce below, and linear abstraction. An existence condition for a C -relation between two linear port-Hamiltonian systems Σ and Σ_A is given by the following lemma.

Lemma 3.2.13. Let Σ and Σ_A be linear port-Hamiltonian systems with $\dim \mathcal{U} < \dim \mathcal{U}_A$. There exists a linear surjective map $C : \mathcal{X} \rightarrow \mathcal{X}_A$ such that Σ and Σ_A are C -related if there exists a linear port-Hamiltonian system Σ_B such that $\Sigma = \Sigma_A \times \Sigma_B$.

Proof: Let us write

$$\Sigma_A : \dot{x}_A = F_A Q_A x_A + B_A u_A$$

$$\Sigma_B : \dot{x}_B = F_B Q_B x_B + B_B u_B$$

then their feedback interconnection $\Sigma = \Sigma_A \times \Sigma_B$ takes the form

$$\begin{bmatrix} \dot{x}_A \\ \dot{x}_B \end{bmatrix} = \begin{bmatrix} F_A & B_A B_B^\top \\ -B_B B_A^\top & F_B \end{bmatrix} \begin{bmatrix} Q_A & 0 \\ 0 & Q_B \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} + \begin{bmatrix} B_A & 0 \\ 0 & B_B \end{bmatrix} \begin{bmatrix} u_A \\ u_B \end{bmatrix}.$$

Hence, the control systems Σ and Σ_A are C -related with $C = [I_{n_A}, 0]$ since

$$C(FQx + Bu) = F_A Q_A Cx + B_A (-B_B^T Q_B x_B + u_A).$$

■

Next, we introduce the concept of achievable Dirac structures [Cervera et al., 2007].

Achievability of a Dirac structure arising from the standard feedback interconnection of a “plant” port-Hamiltonian system and a “controller” port-Hamiltonian system can be understood as a control by interconnection problem (see Chapter 2) of port-Hamiltonian systems restricted to the investigation of the achievable Dirac structures of the closed-loop system. Necessary and sufficient conditions for achievability are given in the following proposition. We make use this in the proofs of the main propositions of this section.

Proposition 3.2.14 (Cervera et al. [2007]). Consider a Dirac structure \mathcal{D}_P with port variables (f_1, e_1, f_2, e_2) , and a Dirac structure \mathcal{D} with port variables (f_1, e_1, f_3, e_3) . Then, there exists a controller Dirac structure \mathcal{D}_C such that $\mathcal{D} = \mathcal{D}_P || \mathcal{D}_C$ if and only if the following two equivalent conditions are satisfied

$$\mathcal{D}_P^0 \subset \mathcal{D}^0$$

$$\mathcal{D}^\pi \subset \mathcal{D}_P^\pi$$

where

$$\mathcal{D}_P^0 = \{(f_1, e_1) \mid (f_1, e_1, 0, 0) \in \mathcal{D}_P\}$$

$$\mathcal{D}_P^\pi = \{(f_1, e_1) \mid \exists (f, e) \text{ s.t. } (f_1, e_1, f, e) \in \mathcal{D}_P\}$$

$$\mathcal{D}^0 = \{(f_1, e_1) \mid (f_1, e_1, 0, 0) \in \mathcal{D}\}$$

$$\mathcal{D}^\pi = \{(f_1, e_1) \mid \exists (f, e) \text{ s.t. } (f_1, e_1, f, e) \in \mathcal{D}\}.$$

We define a “copy” of a Dirac structure \mathcal{D} by

$$\mathcal{D}^* = \{(f_1, e_1, f, e) \mid (-f_1, e_1, -f, e) \in \mathcal{D}\}.$$

The proof of this proposition is based on the copy \mathcal{D}_P^* of the plant Dirac structure \mathcal{D}_P . One possible controller Dirac structure is then constructed as $\mathcal{D}_C = \mathcal{D}_P^* \parallel \mathcal{D}$.

Proposition 3.2.15. Let Σ be a linear port-Hamiltonian system without dissipation and let C be a linear surjective map. Denote by Σ_A the linear port-Hamiltonian abstraction of Σ with respect to C . Then there exists a Dirac structure \mathcal{D}_B such that $\mathcal{D} = \mathcal{D}_A \parallel \mathcal{D}_B$, where \mathcal{D} and \mathcal{D}_A are the Dirac structures canonically associated to the linear system.

Proof: Let $x \in \mathcal{X}$, $x_A \in \mathcal{X}_A$ and $u \in \mathcal{U}$, $u_A \in \mathcal{U}_A$, respectively. We define the following vector spaces accordingly

$$\mathcal{F}_1 = \mathcal{X}_A \times \mathcal{U}$$

$$\mathcal{F}_2 = \mathbb{R}^k$$

$$\mathcal{F}_3 = \mathcal{X} / \mathcal{X}_A,$$

where $\mathcal{U}_A = \mathcal{U} \times \mathbb{R}^k$. To show that there exists a Dirac structure \mathcal{D}_B such that $\mathcal{D} = \mathcal{D}_A \|\mathcal{D}_B$, we verify that $\mathcal{D}^\pi \subset \mathcal{D}_A^\pi$, which is necessary and sufficient for the existence of \mathcal{D}_B by Proposition 3.2.14. Denote the matrix kernel representation of \mathcal{D} by $([F_x, F_c], [E_x, E_c])$. Assume $(f_1, e_1) \in \mathcal{D}^\pi$, then there exists (f_3, e_3) such that $(f_1, e_1, f_3, e_3) = (f_x, e_x, u, y)$ satisfies

$$F_x f_x + F_c u + E_x e_x + E_c y = 0. \quad (3.11)$$

Here f_3 is defined by $f = [f_3, f_2]^\top = \Lambda [f_x, u]^\top$, with the non-singular matrix

$$\Lambda = \begin{bmatrix} C_\perp & 0 \\ C & 0 \\ 0 & I_m \end{bmatrix}.$$

Premultiplying equation (3.11) by Λ implies that $C(f_x + Bu + J e_x) = 0$. Since Σ and Σ_A are C -related, there exists a $u_A \in \mathcal{U}_A$ such that

$$f_x^A + B_A u_A + J_A e_x^A = 0.$$

Let $y_A = [y, \bar{y}_A]^\top$ and $B_A = [CB, \bar{B}_A]$, with this notation, let $\bar{y}_A = \bar{B}_A^\top e_x^A$. We get

$$F_x^A f_x^A + F_c^A u_A + E_x^A e_x^A + E_c^A y^A = 0,$$

and therefore there exists a pair (f_2, e_2) such that $(f_1, e_1, f_2, e_2) = (f_x^A, u_A, e_x^A, y_A) \in \mathcal{D}_A$. ■

Remark 3.2.16. One possible implementation for the Dirac structure \mathcal{D}_B is $\mathcal{D}_A^* \|\mathcal{D}$.

We establish a similar result for the gyrative composition, here we require additional conditions.

Proposition 3.2.17. Let Σ be a linear port-Hamiltonian system without dissipation and let C be a linear surjective map. Denote by Σ_A a linear port-Hamiltonian abstraction of Σ with respect to C , and denote $B_A = [CB, \bar{B}_A]$. There exists a Dirac structure \mathcal{D}_B such that $\mathcal{D} = \mathcal{D}_A \times \mathcal{D}_B$, where \mathcal{D} and \mathcal{D}_A are Dirac structures associated to Σ and Σ_A , respectively if and only if

$$\text{im } C_{\perp} J + \bar{B}_A^{\top} (C^+)^{\top} \subset \text{im } C_{\perp} B, \quad (3.12)$$

$$CJ - J_A (C^+)^{\top} - \bar{B}_A C_{\perp} = 0. \quad (3.13)$$

Proof: Necessity can be shown by straight forward computation. Sufficiency is shown as follows. Assume that Σ and Σ_A are C -related linear port-Hamiltonian systems. Let $x \in \mathcal{X}$, $x_A \in \mathcal{X}_A$ and $u \in \mathcal{U}$, $u_A \in \mathcal{U}_A$, respectively. Let us also define two Dirac structures canonically associated to each control system, denoted by \mathcal{D} and \mathcal{D}_A . We define the vector spaces $\mathcal{F}_1, \mathcal{F}_2$ and \mathcal{F}_3 as in Proposition 3.2.15. Define furthermore $\bar{\mathcal{D}}_A = \mathcal{D}_A \parallel \mathcal{D}_I$, where \mathcal{D}_I is a symplectic Dirac structure, such that $\bar{\mathcal{D}}_A$ is a regular composition. Then we have to show that $\mathcal{D}^{\pi} \subset \bar{\mathcal{D}}_A^{\pi}$ to establish the existence of \mathcal{D}_B . Assume $(f_1, e_1) \in \mathcal{D}^{\pi}$, then there exists (f_3, e_3) such that $(f_1, e_1, f_3, e_3) = (f_x, e_x, u, y)$ satisfies

$$F_x f_x + F_c u + E_x e_x + E_c y = 0.$$

By definition $(f_1, e_1, e_2, e_2) \in \bar{\mathcal{D}}_A$ if there exists a pair (\bar{f}_2, \bar{e}_2) such that

$$(f_1, e_1, \bar{f}_2, \bar{e}_2) \in \mathcal{D}_A \quad (3.14)$$

$$\text{and } (-\bar{f}_2, \bar{e}_2, f_2, e_2) \in \mathcal{D}_I, \quad (3.15)$$

where (3.15) is equivalent to $\bar{f}_2 = e_3$ and $\bar{e}_2 = f_3$. Hence, we have, using the same notation to define f_3 as in the proof of Proposition 3.2.15, that

$$\bar{f}_2 = e_3 = C_1 e_x, \quad (3.16)$$

$$\bar{e}_2 = f_3 = -C_1 \dot{x}, \quad (3.17)$$

and it remains to show that for this choice equation (3.14) holds. The determining equations for Σ_A are

$$\dot{x}_A = J_A e_x^A + B_A u_A$$

$$y_A = B_A^\top e_x^A,$$

let $u_A = [u, \bar{u}_A]^\top$ and $y_A = [y, \bar{y}_A]$, then substituting (3.16) and (3.17) yields

$$\begin{aligned} \dot{x}_A &= J_A e_x^A + C B u + \bar{B}_A C_\perp e_x \\ -C_1 (J e_x + B u) &= \bar{B}_A^\top e_x^A. \end{aligned}$$

By definition of C -relation we have that

$$C J e_x - J_A e_x^A = \bar{B}_A^\top C_\perp e_x. \quad (3.18)$$

Furthermore, since Σ_A is a linear port-Hamiltonian abstraction we have $e_x^A = (C^+)^\top e_x$, then it follows that (3.18) holds since by assumption we have

$$(CJ - J_A(C^+)^\top - \bar{B}_A C_\perp) e_x = 0.$$

Furthermore, we have that if $(C_\perp J + \bar{B}_A^\top (C^+)^\top) e_x \in \text{im } C_\perp B$, then there exists a linear map $L : \mathcal{X} \rightarrow \mathcal{U}$ such that

$$\bar{e}_2 = \bar{B}_A^\top e_x^A = -C_\perp (J e_x + B L x)$$

which implies that $(f_1, e_1, f_2, e_2) \in \bar{\mathcal{D}}_A$. ■

Construction of \mathcal{D}_B Next, we compute the matrix kernel representation for the Dirac structure $\mathcal{D}_B = \mathcal{D}_A^* \parallel \mathcal{D}$. Note that Dirac structures represent generalized port-Hamiltonian systems [Dalsmo and van der Schaft, 1998], that may include additional algebraic constraints on the dynamic system. Before we continue we need the following technical lemma.

Lemma 3.2.18. Let \mathcal{D}_A and \mathcal{D}_B be two Dirac structures, then $(\mathcal{D}_A \parallel \mathcal{D}_B)^* = \mathcal{D}_B^* \parallel \mathcal{D}_A^*$.

Proof: We show $\mathcal{D}_B^* \parallel \mathcal{D}_A^* \subset (\mathcal{D}_A \parallel \mathcal{D}_B)^*$ and $(\mathcal{D}_A \parallel \mathcal{D}_B)^* \subset \mathcal{D}_B^* \parallel \mathcal{D}_A^*$. If $(f_1, e_1, f_3, e_3) \in (\mathcal{D}_A \parallel \mathcal{D}_B)^*$ then $(-f_1, e_2, -f_3, e_3) \in \mathcal{D}_A \parallel \mathcal{D}_B$ which implies that there exists $(-f_2, e_2)$ such that $(-f_1, e_1, -f_2, e_2) \in \mathcal{D}_A$ and $(f_2, e_2, -f_3, e_3) \in \mathcal{D}_B$. Hence, $(f_1, e_1, f_3, e_3) \in \mathcal{D}_B^* \parallel \mathcal{D}_A^*$. Now, assume $(f_1, e_1, f_3, e_3) \in \mathcal{D}_B^* \parallel \mathcal{D}_A^*$ then there exists (f_2, e_2) such that $(-f_1, e_1, -f_2, e_2) \in \mathcal{D}_A$ and $(f_2, e_2, -f_3, e_3) \in \mathcal{D}_B$. It follows that $(-f_1, e_1, -f_3, e_3) \in \mathcal{D}_A \parallel \mathcal{D}_B$ and hence $(f_1, e_1, f_3, e_3) \in (\mathcal{D}_A \parallel \mathcal{D}_B)^*$. ■

In the context of Proposition 3.2.17, we refer to $\mathcal{D}_B = \mathcal{D}_I^* \|\mathcal{D}_A^*\| \mathcal{D}$ as the *virtual controller Dirac structure*.

Let us revisit Example 3.2.12 to illustrate our findings.

Example 3.2.19. The abstraction of the LC circuit is generated by the projection C such that $C_\perp = B^\top$, hence $C_\perp B = a \neq 0$. Hence, equation (3.12) in Proposition 3.2.17 is satisfied. The second condition yields the matrix equation

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{v_1}{L_1} & 0 & 0 \end{bmatrix}$$

where $v_1 = \mathbb{R}/\{0\}$ parameterizes the kernel of C . Hence, for $v_1 = L_1$ this condition is also satisfied and the abstraction is a linear port-Hamiltonian system.

3.3 Intrinsic Decompositions

So far, we have assumed that a surjective map is given. In this section we consider a family of linear surjective maps associated to linear systems.

3.3.1 Flag of abstractions

Given a linear system Σ , there exists a canonical set of C -abstraction of Σ .

Definition 3.3.1. Let Σ denote linear system and consider the set of linear surjective maps $C^{(k)}$ with

$$\ker C^{(k)} = \text{im}[B, AB, \dots, A^k B]$$

for $0 \leq k \leq n - 1$. The k -th controllability abstraction of Σ is given by

$$\begin{aligned} A^{(k)} &= C^{(k)} A (C^{(k)})^+, \\ B^{(k)} &= [C^{(k)} B, C^{(k)} A v_1, \dots, C^{(k)} A v_{r^{(k)}}]. \end{aligned}$$

Let Σ be a linear port-Hamiltonian system, then we can define $C^{(k)}$ related structures

$$F^{(k)} = C^{(k)} F (C^{(k)})^\top,$$

such that the controllability abstractions are also linear port-Hamiltonian system. If we assume that $F + F^\top = 0$, *i.e.*, no dissipation in the system, we can apply Proposition 3.2.17 to each of the controllability abstractions to check if Σ can be decomposed. In this way we can identify different reduced systems, if the conditions of Proposition 3.2.17 are satisfied, such that we can consider to keep this reduced system invariant for the matching equation. For example, if the reduced system is already passive then its port-Hamiltonian structure does not need to be changed in order to achieve stability.

3.4 Conclusion

We present a short summary of this chapter and discuss future directions of research on this topic in details.

3.4.1 Summary

We established a methodology to decompose linear port-Hamiltonian systems based on the concept of linear abstractions and achievable Dirac structures. We showed how the decomposition can be constructed based on the standard composition of the associated Dirac structures. A second type of decomposition has been considered motivated by power-conserving feedback interconnection of port-Hamiltonian system. The condition for the existence in this case are stronger. Furthermore, the decomposition is applicable to the controllability abstractions of a linear port-Hamiltonian system, which possibly yields reduced port-Hamiltonian systems without relying on physical insight.

3.4.2 Extensions

The two main propositions in this chapter assume no damping structure of the linear port-Hamiltonian system. In order to also include damping structures, the following proposition should be used instead of Proposition 3.2.14 in the proofs of both propositions.

Proposition 3.4.1 (Cervera et al. [2007]). Given a plant Dirac structure with dissipation \mathcal{DR}_P with port variables $f_1, e_1, f_{R1}, e_{R1}, f_c, e_c$ and a desired Dirac structure with dissipation \mathcal{DR} with port variables $(f_1, e_1, f_{R1}, e_{R1}, f_2, e_2, f_{R2}, e_{R2})$. Here (f_1, e_1) and (f_{R1}, e_{R1}) respectively denote the flow and effort variables corresponding to the energy storing elements and the energy dissipating elements of the plant system. Then there exists a controller system \mathcal{DR}_C such that $\mathcal{DR} = \mathcal{DR}_P \parallel \mathcal{DR}_C$ if and only if

the following two conditions are satisfied

$$\begin{aligned}\mathcal{DR}_P^0 &\subset \mathcal{DR}^0, \\ \mathcal{DR}^\pi &\subset \mathcal{DR}_P^\pi,\end{aligned}$$

where

$$\begin{aligned}\mathcal{DR}_P^0 &= \{(f_{R1}, e_{R1}) \mid (f_{R1}, e_{R1}, 0, 0) \in \mathcal{DR}_P\} \\ \mathcal{DR}_P^\pi &= \{(f_{R1}, e_{R1}) \mid \exists (f_2, e_2) \text{ s.t. } (f_{R1}, e_{R1}, f_2, e_2) \in \mathcal{DR}_P\} \\ \mathcal{DR}^0 &= \{(f_{R1}, e_{R1}) \mid (f_{R1}, e_{R1}, 0, 0) \in \mathcal{DR}\} \\ \mathcal{DR}^\pi &= \{(f_{R1}, e_{R1}) \mid \exists (f_{R3}, e_{R3}) \text{ s.t. } (f_{R1}, e_{R1}, f_{R3}, e_{R3}) \in \mathcal{DR}\}.\end{aligned}$$

Note that in this case, both conditions, which are not equivalent have to be satisfied, where as in Proposition 3.2.14 both conditions are equivalent. This will result in additional conditions that have to be satisfied for decomposition of linear port-Hamiltonian system with non-zero damping structure.

Another direction of research is to extend decomposition of port-Hamiltonian systems to control-affine systems. For this, several obstacles must be overcome. First, the extension of abstractions cannot guarantee that the abstractions are in control-affine form [Tabuada and Pappas, 2005] but the concept of port-Hamiltonian systems relies on this control-affine form. One possible option would be to consider partial feedback linearizable systems.

Chapter 4

The Constant Structure Matching Problem

This chapter presents a detailed study of the matching problem for control-affine systems. The study leads to a solution approach for a particular class of systems, where additional assumptions are imposed on the interconnection and damping structure. In recent years, the IDA-PBC methodology has been applied to control problems in various fields of engineering [Dörfler et al., 2009, Ortega et al., 1999, 2001]. In most applications, the systems treated are such that one can compute a specific Hamiltonian function based on some heuristics. In particular, one common assumption is that the functions determining the interconnection and damping structure are constant. Based on this observation, we develop specific results for the case of constant interconnection and damping structure. We show how assumptions on constant interconnection and damping structure simplify the matching problem.

4.1 Introduction

In this section, we define the constant structure matching problem, and establish some preliminary results. We identify two common themes in the literature. The first theme is the assumption that the desired interconnection and damping structure is constant. This assumption arises in a number of applications of IDA-PBC where an explicit solution is needed. The second theme arising in the literature is the assumption that the homogeneous solutions of the matching equation are linearly parameterized by constant parameters to achieve equilibrium assignment and stability of the closed-loop system.

In Section 4.1.2, we establish an alternative description of port-Hamiltonian systems that represents the different themes in a unified form. This description is then used to establish new results on the construction of a solution to the matching problem.

4.1.1 Literature review

In the following, we detail existing solutions to specific matching problems. Following the two mentioned themes in the previous section.

Constant structure assumption In the context of process control systems, the matching problem has been considered by Johnsen and Allgöwer [2007] and Dörfler et al. [2009]. In Johnsen and Allgöwer [2007], the IDA-PBC methodology is applied to a 4-tank fluid storage system. The initial choice of a Hamiltonian function for this system is chosen such that the corresponding interconnection and damping structures are constant. In Dörfler et al. [2009], several application of IDA-PBC to

different process control problems were considered. A standing assumption was that the interconnection and damping structure are restricted to be constant in the given coordinates to simplify the controller design. In Kotyczka and Lohmann [2009], a general design framework for IDA-PBC based on the linearization of the control system at the desired equilibrium is presented. The linearization of the system and the desired constant interconnection and damping structure together with the restriction to a quadratic Hamiltonian function define a linear matching equation. The main motivation here is to formulate the linearized matching problem, which is algebraic, and hence easier to solve.

Parameterization of the homogenous solution Another commonality found in several applications of IDA-PBC is a parameterization of the homogeneous solutions to the matching equations for equilibrium assignment and/or stabilization. In Johnsen and Allgöwer [2007], a coordinate transformation is applied such that the homogeneous solution of the matching equation only depends on a subset of the transformed coordinates. The parameterization of the homogeneous solution is chosen such that the solution is a quadratic function of the error states, which results in a simple controller structure. Furthermore, the linear term of the homogenous solution is used for equilibrium assignment.

4.1.2 Preliminaries

In this section, we define constant structure tensor fields and the constant structure matching problem. The concept of affine connections on manifolds will be introduced to define constant structure tensors in a precise manner. To be more specific, one needs to provide a coordinate free definition of a constant structure tensor. The

following example provides some intuition on the construction required to define constant objects such as vector fields.

Example 4.1.1. Consider \mathbb{R}^2 with $(U_1, \phi_1 = (x, y))$ as standard Cartesian coordinate systems and $(U_2, \phi_2 = (r, \theta))$ as polar coordinates, and $U_1 \cap U_2 \neq \emptyset$. Let X be a vector field on \mathbb{R}^2 , given in the first coordinates by $X = \frac{\partial}{\partial x}$. In polar coordinates, this would be expressed as

$$X = \cos \theta \frac{\partial}{\partial r} - \frac{1}{r} \sin \theta \frac{\partial}{\partial \theta}$$

it would be misleading to refer to the vector field X as constant, if we base the definition on a specific coordinate representation.

To understand when a vector field is constant, one needs to develop a means to compare elements of the tangent spaces at two different points of the state space. Since open subsets of \mathbb{R}^n are considered here one could use the standard identification of the tangent space at each point with \mathbb{R}^n itself. It is more natural to consider the state space of a control-affine system as a smooth manifold to keep track of all identification and provide a coordinate-free representation. The tool that will allow us to do this is an affine connection. In particular, we consider the affine connection associated with a Riemannian metric induced on $\mathcal{X} \subset \mathbb{R}^n$ by inclusion from the Euclidian metric on \mathbb{R}^n .

Definition 4.1.2 (Bullo and Lewis [2004]). An affine connection on a manifold \mathcal{X} assigns to the pair $(X, Y) \in \Gamma(T\mathcal{X}) \times \Gamma(T\mathcal{X})$ a vector field $\nabla_X Y \in \Gamma(T\mathcal{X})$, and the assignment satisfies

- the map $(X, Y) \mapsto \nabla_X Y$ is \mathbb{R} -linear,

- $\nabla_{fX}Y = f\nabla_XY$ for each $X, Y \in \Gamma(T\mathcal{X})$ and $f \in \mathcal{C}^\infty(M)$, and
- $\nabla_X fY = f\nabla_XY + (\mathcal{L}_X f)Y$ for each $X, Y \in \Gamma(T\mathcal{X})$ and $f \in \mathcal{C}^\infty(\mathcal{X})$,

where $\mathcal{L}_X f$ denotes the Lie derivative of f along X .

The vector field $\nabla_X Y$ is called the *covariant derivative* of Y with respect to X . The covariant derivative of a tensor field t of type (r, s) with respect to X is define by

$$\begin{aligned} (\nabla_X t)(\alpha^1, \dots, \alpha^r, X_1, \dots, X_s) &= \mathcal{L}_X(t(\alpha^1, \dots, \alpha^r, X_1, \dots, X_s)) \\ &\quad - \sum_{i=1}^r t(\alpha^1, \dots, \nabla_X \alpha^i, \dots, \alpha^r, X_1, \dots, X_s) \\ &\quad - \sum_{j=1}^s t(\alpha^1, \dots, \alpha^r, X_1, \dots, \nabla_X X_j, \dots, X_s). \end{aligned}$$

Definition 4.1.3. If $(\mathcal{X}, \mathbb{G})$ is a Riemannian manifold, then there exists a unique affine connection ∇ on \mathcal{X} , called the *Levi-Civita connection*, such that ∇ is a metric connection (*i.e.*, $\nabla \mathbb{G} = 0$), and $\nabla_X Y - \nabla_Y X = [X, Y]$ for all $X, Y \in \Gamma(T\mathcal{X})$.

Definition 4.1.4. Let $(\mathcal{X}, \mathbb{G})$ be a Riemannian manifold, let X be a vector field on $(\mathcal{X}, \mathbb{G})$. The vector field X is said to be *parallel* or *constant* with respect to the Levi-Civita connection ∇ if $\nabla_Y X = 0$ for all vector fields Y . We use the short hand $\nabla X = 0$ to denote constant vector fields.

Revisiting example 4.1.1, we see that $\nabla X = 0$. Similarly, we can define a constant tensor field as follows:

Definition 4.1.5. Let $(\mathcal{X}, \mathbb{G})$ be a Riemannian manifold, let t be a tensor field on $(\mathcal{X}, \mathbb{G})$. t is said to be *parallel* or *constant* if $\nabla_Y t = 0$ for all vector fields Y . We shall also use the short hand $\nabla t = 0$ for this.

Constant Parameter Matching Problem

Equipped with the necessary notations, we define the constant structure matching problem.

Problem 4.1.6 (Constant Structure Matching Problem (CSMP)). Given a control-affine system $\Sigma = (\mathcal{X}, f, g, \mathcal{U})$ and an admissible equilibrium p_* of Σ , find a constant structure tensor field F_d and a function H_d such that

$$g^\perp F_d dH_d = g^\perp f, \quad (4.1)$$

where g^\perp is a section of $\text{Hom}(\text{ann } g)$. Furthermore, H_d has to satisfy

$$dH_d(p_*) = 0 \quad \text{and} \quad \text{Hess } H_d(p_*) = 0.$$

For port-Hamiltonian system, local stability is equivalent to the following two conditions on the Hamiltonian function H_d ,

$$dH_d(p_*) = 0 \quad (4.2)$$

$$\text{and} \quad \text{Hess } H_d(p_*) > 0. \quad (4.3)$$

Notice that condition (4.2) ensures H_d has an extremum at p_* , while the Lyapunov stability condition (4.3) shows that it is an isolated minimum. We refer to the former condition as the *equilibrium assignment condition* and to the latter as *stability condition*. The remainder of this chapter is structured as follows. In Section 4.2, we present a checkable existence condition based on exterior differential systems for a fixed structure tensor. The equilibrium assignment and stability condition on the

solution of the matching equation are addressed in Section 4.3. In Section 4.4, we present additional results based on the definition of an equilibrium manifold. An application to a simple heat exchanger is presented in Section 4.5. Further generalizations and extensions are discussed in Section 4.6. Background information on exterior differential systems are summarized for the readers convenience in Appendix 4.7.

4.2 Existence Conditions

We present results for the existence of a solution to the matching equations. It is assumed that a fixed structure tensor field is known *a priori*. We first establish necessary and sufficient conditions for existence of a solution to the matching equation based on an exterior differential system description.

4.2.1 Exterior differential system approach

We describe the existence of a solution to the CSMP in terms of exterior differential systems. To motivate this approach, let us consider a PDE for H_d given by $F_d dH_d = f$ where F_d is a structure tensor field and f a vector field. We write equivalently $dH_d = F_d^{-1} f$, which defines the subset $\{p = F_d^{-1} f\}$ of the first jet [Saunders, 1989] with coordinates (x, p, u) . The canonical contact form on the first jet, restricted to this subset, is $\theta = du - p_i dx^i = du - F_d^{-1} f dx$. The solutions to the EDS generated by θ , with independence condition $dx^1 \wedge \dots \wedge dx^n$, are solutions to the PDE (see Ivey and Landsberg [2003]). Furthermore, there exists a solution if and only if θ is integrable. In the following paragraphs, we establish similar conditions for the overdetermined

set of PDEs, given by the matching equations.

Let us define the codistribution $F_d^{-1}g = \text{span} \{\omega_i\}_{i=1}^m$ with $\omega_i = F_d^{-1}g_i$, which represents all cotangent directions directly influenced by the controls under F_d^{-1} . We first derive an alternative condition for a solution to the PDE $g^\perp F_d dH = g^\perp f$.

Proposition 4.2.1. Consider the control-affine system Σ and let F_d be a structure tensor. There exists a solution to the matching equation if and only if $dH_d - F_d^{-1}f \in F_d^{-1}g$.

Proof: We show that $F_d^{-1}g = \text{ann } g^\perp F_d$ then the claim follows since

$$g^\perp F_d dH_d = g^\perp f \Leftrightarrow g^\perp F_d (dH_d - F_d^{-1}f) = 0.$$

Let α be a one form, assume $\alpha \in F_d^{-1}g$, *i.e.*, there exist functions $\alpha^1, \dots, \alpha^m$ such that $\alpha = \alpha^i F_d^{-1}g_i$. Then $g^\perp F_d \alpha = g^\perp F_d \alpha^i F_d^{-1}g_i = \alpha^i g^\perp F_d F_d^{-1}g_i = \alpha^i g^\perp g_i = 0$. On the other hand, if $\alpha \in \text{ann } g^\perp F_d$ then $F_d \alpha \in g$ and since $(F_d)_p$ is a bijection for all $p \in \mathcal{X}$, $F_d^{-1}F_d \alpha \in F_d^{-1}g$, which implies that $\alpha \in F_d^{-1}g$. ■

Next, we give condition for the existence of a solution to the matching equations for a fixed F_d .

Theorem 4.2.2. Consider the control-affine system Σ and let F_d be a structure tensor field. Define the following one forms on $\mathcal{X} \times \mathbb{R}$

$$\begin{aligned} \omega_i &= F_d^{-1}g_i, \quad i = 1, \dots, m \\ \theta &= dz - \pi^*(F_d^{-1}f), \end{aligned}$$

where $\pi : \mathcal{X} \times \mathbb{R} \rightarrow \mathcal{X}$ is the projection onto the first factor. Define $I_0 = \{\omega_i\}$,

$I = \{\pi^*\omega_i, \theta\}$, and the Pfaffian systems [Bryant et al., 1990] \mathcal{I}_0 and \mathcal{I} generated by algebraically by I_0 and I , respectively. Then locally there exists a solution to the matching equation (5.4) if and only if

$$\dim \mathcal{I}^{(\infty)} = \dim \mathcal{I}_0^{(\infty)} + 1.$$

Proof: Assume there exists a solution H to the matching equation, by Proposition 4.2.1 H satisfies $dH - F_d^{-1}f \in \mathcal{I}_0$. Define $\tilde{\theta} = dz - dH$ on $\mathcal{X} \times \mathbb{R}$, with coordinates (x, z) , such that $\theta \equiv \tilde{\theta} \pmod{\mathcal{I}_0}$ and $\{\tilde{\theta}, \omega_1, \dots, \omega_m\}$ is a set of generators for \mathcal{I}_0 . Furthermore, $d\tilde{\theta} = 0$ implies that $\tilde{\theta} \in \mathcal{I}_0^{(\infty)}$. Also, using the fact that exterior differentiation and pull back commute, we have that $d\omega \equiv \alpha_\omega \pmod{\mathcal{I}_0}$ implies that $d\pi^*\omega \equiv \pi^*\alpha_\omega \pmod{\mathcal{I}}$ for all $\omega \in \mathcal{I}_0$, and similar arguments can be made for the derived systems such that $\eta_i \in \mathcal{I}_0^{(\infty)}$ implies that $\pi^*\eta_i \in \mathcal{I}^{(\infty)}$ for all $\eta_i \in \mathcal{I}_0^{(\infty)}$. It follows that $\dim \mathcal{I}^{(\infty)} = \dim \mathcal{I}_0^{(\infty)} + 1$.

Now assume $r = \dim \mathcal{I}^{(\infty)} = \dim \mathcal{I}_0^{(\infty)} + 1$. Let dy^1, \dots, dy^r be generators for $\mathcal{I}_0^{(\infty)}$ and let $\{d\tilde{H}, \pi^*dy^1, \dots, \pi^*dy^r\}$ be a set of generator for $\mathcal{I}^{(\infty)}$. Then $d\tilde{H} \in \mathcal{I}$ and we can write $d\tilde{H} = a^i\omega_i + b\theta$ for some functions a^i , $i = 1, \dots, m$ and a function b . The distribution $\text{coann}(\mathcal{I}_0^{(\infty)})$ has rank $n - r + 1$, then $\text{coann}(\pi^*(\mathcal{I}_0^{(\infty)}))$ has rank $n - r + 1 + \dim(\ker T\pi) = n - r + 2$. Assume $b = 0$, then $\frac{\partial}{\partial z}$ is annihilated by $\mathcal{I}^{(\infty)}$, then $\text{coann}(\mathcal{I}^{(\infty)})$ has rank $n - r + 1$, but by assumption this is equal to the rank of $\text{coann}(\pi^*(\mathcal{I}_0^{(\infty)}))$, which is a contradiction. Finally, b is given in coordinates by $b = \frac{\partial \tilde{H}}{\partial z}$. By the implicit function theorem, there exists a function H on \mathcal{X} such that \tilde{H} is its graph, *i.e.*, $z = H(x)$. Furthermore, $d\tilde{H} = a^i\omega_i + b(dz - F_d^{-1}f) = a^i\omega_i + b(dH - F_d^{-1}f) = 0$. Hence,

$$dH - F_d^{-1}f = \frac{1}{b}a^i\omega_i,$$

which is that $dH - F_d^{-1}f \in F_d^{-1}g$, as required. \blacksquare

Remark 4.2.3. Theorem 4.2.2 establishes a connection to classical results on the existence of solutions to PDEs. It is dual to the existence result presented in Cheng et al. [2005], which is motivated by the results presented in Tabuada and Pappas [2003b].

If F_d is constant we can derive the following result.

Corollary 4.2.4. Consider the control-affine system Σ and let F_d be a constant structure tensor field. Let $\text{ann}(g) = \text{span}\{\eta^m + 1, \dots, \eta^n\} = \eta$ and assume that

$$\nabla_{F_d\eta^i}\eta^j - \nabla_{F_d\eta^j}\eta^i \in \eta, \quad \text{for all } i, j \in \{m+1, \dots, n\}. \quad (4.4)$$

Then locally there exists a solution to the matching equation (5.4) if and only if

$$d\theta \equiv 0 \pmod{\mathcal{I}}. \quad (4.5)$$

Proof: If we show that \mathcal{I}_0 is integrable, then Theorem 4.2.2 implies that (4.5) is equivalent to the existence of a solution to the matching equation, since then $\mathcal{I}_0 = \mathcal{I}_0^{(\infty)}$ and $d\theta \in \mathcal{I}$ implies $\mathcal{I} = \mathcal{I}^{(\infty)}$. First we show that $\text{coann}(F_d^{-1}g) = F_d\eta$, which is that $F_d^{-1}g_i(F_d\eta^j) = 0$ for all $i = 1, \dots, m$, $j = m+1, \dots, n$. We have by definition that $F_d^{-1}g_i(F_d\eta^j) = F_d^{-1}(g_i, F_d\eta^j) = \eta^j(g_i) = 0$. Next, we show that $d(F_d^{-1}g_i)$ also

annihilates $F_d\eta$, which establishes that $F_d^{-1}g$ is integrable [Lee, 2003]. Consider

$$\begin{aligned} d(F_d^{-1}g_k)(F_d\eta^i, F_d\eta^j) &= F_d\eta^i(F_d^{-1}g_k(F_d\eta^j)) - F_d\eta^j(F_d^{-1}g_k(F_d\eta^i)) - F_d^{-1}g_k([F_d\eta^i, F_d\eta^j]) \\ &= F_d^{-1}g_k([F_d\eta^i, F_d\eta^j]) \\ &= F_d^{-1}g_k(\nabla_{F_d\eta^i}F_d\eta^j - \nabla_{F_d\eta^j}F_d\eta^i) \end{aligned}$$

If F_d is constant then $\nabla_{F_d\eta^i}F_d\eta^j = (\nabla_{F_d\eta^i}F_d)\eta^j + F_d(\nabla_{F_d\eta^i}\eta^j) = F_d(\nabla_{F_d\eta^i}\eta^j)$. Hence,

$$\begin{aligned} d(F_d^{-1}g_k)(F_d\eta^i, F_d\eta^j) &= F_d^{-1}g_k(F_d\nabla_{F_d\eta^i}\eta^j - F_d\nabla_{F_d\eta^j}\eta^i) \\ &= \nabla_{F_d\eta^i}\eta^j(g_k) - \nabla_{F_d\eta^j}\eta^i(g_k) \\ &= (\nabla_{F_d\eta^i}\eta^j - \nabla_{F_d\eta^j}\eta^i)(g_k). \end{aligned}$$

Hence, $d(F_d^{-1}g_k)(F_d\eta^i, F_d\eta^j) = 0$ which is that $\mathcal{I}_0 = F_d^{-1}g$ is integrable. \blacksquare

Remark 4.2.5. If Σ is such that $m = n - 1$, then (4.4) is trivially satisfied.

Example 4.2.6. Let F_d be a constant structure tensor field. Assume that the control vector fields of Σ are constant. Control-affine systems that satisfy this assumption are considered in Johnsen and Allgöwer [2007] and Kotyczka and Lohmann [2009]. Clearly, condition (4.4) is satisfied. Let (x^1, \dots, x^n) be coordinates on \mathcal{X} . Then we can define new coordinates $y^j = \sum x^i[(F_d)_{ij}]$, where $(F_d)_{ij}$ is the matrix of component functions of F_d in coordinates. Then $\omega_k = \sum g_k^j dy^j$ and $\mathcal{I}_0 = \{\omega_k\}_{k=1}^m$. Note that θ is given by

$$\theta = dz - f^1 dy^1 - \dots - f^n dy^n$$

in the adapted coordinates. The exterior derivative of θ yields

$$\begin{aligned} d\theta &= -\frac{\partial f^1}{\partial y^i} dy^i \wedge dy^1 + \dots + \frac{\partial f^n}{\partial y^i} dy^i \wedge dy^n \\ &\equiv -\sum_{\substack{i,j=1 \\ i < j}}^n \left(\frac{\partial f^j}{\partial y^i} - \frac{\partial f^i}{\partial y^j} \right) dy^i \wedge dy^j \pmod{\mathcal{I}}. \end{aligned}$$

Hence, if $\frac{\partial f^j}{\partial y^i} = \frac{\partial f^i}{\partial y^j}$ for $i, j = m+1, \dots, n$, then there exists a solution to the matching equation.

Remark 4.2.7. The idea of adapting the coordinates based on the constant structure tensor was also used in Kotyczka and Lohmann [2009] to simplify the matching problem.

4.2.2 Homotopy operator approach

Next, we present an alternative and constructive sufficient condition for the existence of a solution to the matching equation, based on the homotopy operator [Edelen, 2005, Hudon et al., 2008]. We introduce a homotopy operator as a linear operator on the space of differential forms on \mathcal{X} , it satisfies the identity

$$\omega = d(\mathbb{H}\omega) + \mathbb{H}d\omega, \tag{4.6}$$

where ω is a differential form on \mathcal{X} . The first term on the right-hand side of (4.6) is called the *exact part* ω_e of ω , since it is an exact one-form, *i.e.*, the differential of a function. The second term is called the *anti-exact part* ω_a of ω . A homotopy operator is defined on an open star-shaped subset centered at a point $p \in \mathcal{X}$. An open subset of S of \mathbb{R}^n is said to be *star-shaped* centered at p if the following conditions hold:

- S is contained in a coordinate neighborhood U of p .
- The coordinate chart ϕ is such that $\phi(p) = 0_{\mathbb{R}^n}$.
- If q is any point in S with coordinates $x = (x^1, \dots, x^n)$, then the set of points $(q + \lambda(q - p))$ with $\lambda \in [0, 1]$ belongs to S .

We define the following vector field on S

$$\mathfrak{X}(x) = x^i \frac{\partial}{\partial x^i}.$$

In these coordinates the homotopy operator \mathbb{H} is given by

$$\mathbb{H}\omega(x) = \int_0^1 \mathfrak{X}(x) \lrcorner \omega(\lambda x) \lambda^{k-1} d\lambda,$$

where \lrcorner denotes the interior product [Lee, 2003]. For a system Σ and a structure tensor field F_d , we have established that $\nu = F_d^{-1}f$ is a one form on \mathcal{X} . We use the identity (4.6) to compute the exact part ν_e and the anti-exact part ν_a of ν . We can derive a simple sufficient condition for the existence of a solution to the matching equation.

Proposition 4.2.8. Consider F_d and Σ with admissible equilibrium p_* , denote $\nu = F_d^{-1}f$, if $\nu_a \in F_d^{-1}g$ then, locally, on a star-shaped set centered at p_* there exists a solution to the matching equation.

Proof: We write $F_d^{-1}f = \nu = \nu_e + \nu_a$, then $\nu_a = \nu - F_d^{-1}f$ and $d(\mathbb{H}\nu) = \nu_e$. Hence, $d(\mathbb{H}\nu) - F_d^{-1}f \in F_d^{-1}g$, which is equivalent to the existence of a solution to the matching equation. ■

The function $\mathbb{H}\nu$ is the Hamiltonian function of the port-Hamiltonian system with structure tensor F_d if we apply the energy-stabilizing feedback of the form (2.5).

4.3 Equilibrium Assignment and Stability Condition

Based on the results of the previous section, the equilibrium assignment and the stability condition are considered in this section. If a solution to the matching equation is known and satisfies the equilibrium assignment and stability condition then it is a solution to the matching problem. Here, we assume that these conditions are not satisfied and study the extend to which the homogeneous solution of the matching equation can be used to satisfy them.

4.3.1 Homogeneous solution

For a system Σ and F_d , let \mathcal{I}_0 be the exterior differential system generated by $\omega = F_d^{-1}g$, then the bottom derived system $\mathcal{I}_0^{(\infty)}$ is generated by a set of exact generators $\{dh_1, \dots, dh_r\}$ with $0 \leq r \leq m$. Next, we show that the generators of $\mathcal{I}_0^{(\infty)}$ are solutions of the homogeneous matching equations.

Proposition 4.3.1. A function h is a solution of the homogenous matching equation $g^\perp F_d dh = 0$ if and only if $dh \in \mathcal{I}_0^{(\infty)}$.

Proof: If h is a solution to the homogenous solution, then $dh \in F_d^{-1}g$ and in addition dh is closed, which is equivalent to $dh \in \mathcal{I}_0^{(\infty)}$. ■

Remark 4.3.2. The computation of $\mathcal{I}_0^{(\infty)}$, necessary to check the existence of a

solution to the matching problem, can be directly used to compute a homogeneous solution.

Next, we establish a simple quadratic parameterization of the homogenous solution, which gives us additional degrees of freedom to satisfy the stability condition on the desired Hamiltonian function. Assume that there exists a solution H_d that satisfies $dH_d(p_*) = 0$ and let h^1, \dots, h^r be a set of linear independent homogenous solutions. Define $h_*^i = h^i(p_*)$, *i.e.*, the value of h^i at a desired admissible equilibrium, and the error functions $\tilde{h}^i = h^i - h_*^i$ for all $i = 1, \dots, r$. Consider the function

$$\psi = \frac{1}{2} P_{ij} \tilde{h}^i \tilde{h}^j + l_i \tilde{h}^i$$

with $P_{ij} = P_{ji} \in \mathbb{R}$, $[P_{ij}] > 0$ and $l_i \in \mathbb{R}$ for $i = 1, \dots, r$. The partial derivatives of ψ with respect to some coordinates x^1, \dots, x^n are given by

$$\frac{\partial \psi}{\partial x^k} = P_{ij} \frac{\partial \tilde{h}^i}{\partial x^k} \tilde{h}^j + l_i \frac{\partial \tilde{h}^i}{\partial x^k}, \quad k = 1, \dots, n.$$

Then $H_d + \psi$ is also a solution to the matching equation. We use this parameterization in the following section.

4.3.2 Equilibrium assignment

Next, we derive necessary and sufficient conditions for equilibrium assignment via a homogenous solution.

Proposition 4.3.3. Let H_d be a solution to the matching equation. There exists a homogenous solution h such that $d\tilde{H}_d = d(H_d + h)(p_*) = 0$ if and only if $dH_d(p_*) \in$

$\mathcal{I}_0^{(\infty)}(p_*)$, where $\mathcal{I}_0^{(\infty)}(p_*)$ denotes the subspace of $T_{p_*}\mathcal{X}$ spanned by a set of generators of $\mathcal{I}_0^{(\infty)}$ at p_* .

Proof: If $\tilde{H}_d = H_d$ then $d\tilde{H}_d(p_*) = dH_d(p_*) = 0 \in \mathcal{I}_0^{(\infty)}(p_*)$. Now assume, $dH_d(p_*) \in \mathcal{I}_0^{(\infty)}(p_*)$. Note that at any admissible equilibrium p_* , we have $g_{p_*}^\perp f(p_*) = 0$. Hence, any solution H_d is such that $g_{p_*}^\perp F_d dH_d(p_*) = 0$. Let $\{dh^1, \dots, dh^r\}$ be a set of generator of $\mathcal{I}_0^{(\infty)}$. Then there exists constants l_1, \dots, l_m such that $dH_d(p_*) = l_i dh^i(p_*)$. Hence, we can chose $\tilde{H}_d = H_d + l_i h^i$ such that $d\tilde{H}_d(p_*) = 0$. ■

We immediately get the following corollary for constant structure tensors.

Corollary 4.3.4. If F_d is a constant structure tensor field and the control distribution satisfies (4.4), then equilibrium assignment via homogenous solutions is always possible.

Proof: By Corollary 4.2.4, we have that $\mathcal{I}_0 = \mathcal{I}_0^{(\infty)}$, which is that $F_d^{-1}g$ is integrable and of rank $n - m$. If dH_d is a solution to the matching equation then $dH_d(p_*) \in \text{ann}(g^\perp F_d)(p_*) = F_d^{-1}g(p_*) = \mathcal{I}_0^{(\infty)}(p_*)$. ■

An example in which equilibrium assignment via the homogeneous solution is not possible is given next.

Example 4.3.5. Consider the control-affine system determined by

$$f = \begin{bmatrix} f^1(x^1) \\ f^2(x^2) \\ 1 \\ 0 \end{bmatrix}, \quad g = \begin{bmatrix} -x^2 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the fixed structure tensor $F = -I_4$. Then, $\omega_1 = -x^2 dx^1 + dx^3, \omega_2 = dx^4$ and $\mathcal{I}_0 = \{\omega_1, \omega_2\}$. A set of homogeneous solutions are integrals of the bottom derived

system $\mathcal{I}_0^{(\infty)} = \{\omega_2\}$. Hence $h^1 = x^4$ is a homogenous solution. Furthermore, there exists a solution to the matching equation since

$$d\theta = 0$$

with $\theta = dz - f^1 dx^1 - f^2 dx^2 - dx^3$. Hence, $H_d = \int f^1 dx^1 + \int f^2 dx^2 + x^3$ is a solution to the matching equation. Next, we consider the equilibrium assignment, the set of admissible equilibria is described by

$$\Gamma = \{x \in \mathcal{X} \mid f^2(x^2) = 0, f^1(x^1) = -x^2\}.$$

We evaluate dH_d at an desired equilibrium, which gives us

$$\left. \frac{\partial H_d}{\partial x} \right|_{x_*}^\top = \begin{bmatrix} -x_*^2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \neq 0.$$

Hence, equilibrium assignment via the homogenous solutions is not possible in this example.

4.3.3 Stability condition

Next, we focus our attention on the stability condition (4.3). Consider the control-affine system Σ and a structure tensor field F_d . Let H_d be a solution to the matching

equations defined by Σ and F_d such that $dH(p_\star) = 0$. If

$$\text{Hess } H_d(p_\star) > 0, \quad (4.7)$$

then p_\star is a locally stable equilibrium of the closed-loop system, since H_d is non-increasing along trajectories of the closed-loop [Ortega and Garcia-Canseco, 2004]. Next, we show that we can give equivalent conditions for stability depending on Σ and F_d , *i.e.*, not explicitly on H_d . We consider neighborhood around an admissible equilibrium p_\star and Cartesian coordinate.

Let us denote $V = T_{p_\star} \mathcal{X}$, $W = (\text{ann}(g)(p_\star))^*$, then we define $G^\perp = g_{p_\star}^\perp : V \rightarrow W$ with transpose $G_\perp = (G^\perp)^\top$. Also define $\mathbf{F}_d = (F_d)_{p_\star} : V^* \rightarrow V$, $\mathbf{f} = f(p_\star)$, $A = Df(p_\star)$, $B = Dg^\perp(p_\star) : \mathcal{X} \rightarrow L(\mathbb{R}^n; L(V; W)) = L^2(\mathbb{R}^n; W)$. We abbreviate

$$L_{p_\star} = G^\perp(\mathbf{A}\mathbf{F}_d^\top + \mathbf{F}_d^\top \mathbf{A}^\top)G_\perp + B(\mathbf{f}\mathbf{F}_d^\top G_\perp + G^\perp \mathbf{f}\mathbf{F}_d^\top)B^\top.$$

We require the following result:

Proposition 4.3.6. Consider Σ and the constant structure tensor field F_d . Let H_d be a solution to the matching equation defined by Σ and F_d such that $dH_d(p_\star) = 0$. If

$$B\mathbf{F}_d DH_d(p_\star) \mathbf{F}_d^\top G_\perp + G^\perp \mathbf{F}_d DH_d(p_\star) \mathbf{F}_d^\top B^\top < L_{p_\star}, \quad (4.8)$$

then p_\star is a locally stable equilibrium.

Proof: We can equivalently to (4.7) require that

$$\frac{1}{2}(D^2H_d(p_\star) + D^2H_d^\top(p_\star)) > 0. \quad (4.9)$$

Furthermore, we have that H_d is a solution to the matching equation. If we take the derivative on both sides of the matching equation we have

$$B\mathbf{F}_dDH_d + G^\perp\mathbf{F}_dD^2H_d = B\mathbf{f} + G^\perp A,$$

since F_d is constant we have $DF_d(p) = 0$ for all $p \in \mathcal{X}$. Hence, by linearity, we get

$$G^\perp\mathbf{F}_dD^2H_d(p_\star) = G^\perp A + B(\mathbf{f} - \mathbf{F}_dDH_d(p_\star)). \quad (4.10)$$

If we premultiply (4.9) by $G^\perp\mathbf{F}_d$ and postmultiply by $(G^\perp\mathbf{F}_d)^\top$ and substitute (4.10), we get

$$G^\perp(\mathbf{A}\mathbf{F}_d^\top + \mathbf{F}_d\mathbf{A}^\top)G_\perp + B(\mathbf{f} - \mathbf{F}_dDH_d(p_\star))\mathbf{F}_d^\top G_\perp + G^\perp(\mathbf{f} - \mathbf{F}_dDH_d(p_\star))\mathbf{F}_d^\top B^\top > 0$$

after rearranging the terms this yields (4.8), as required. ■

If the control vector fields of Σ are constant then $B = 0$ and the stability condition (4.9) reduces to

$$L_{p_\star} = G^\perp(\mathbf{A}\mathbf{F}_d^\top + \mathbf{F}_d\mathbf{A}^\top)G_\perp > 0, \quad (4.11)$$

which is a linear matrix inequality (LMI) for the constant structure tensor F_d at p_\star . The LMI can be efficiently solved using numerical tools [Boyd et al., 1994], see

also Chapter 3. Since F_d is constant, \mathbf{F}_d defines F_d on the coordinate chart that is considered here.

Constant structure tensor and constant control distribution We summarize the results of this section for control-affine systems and constant structure tensor fields in the following proposition.

Proposition 4.3.7. Consider the control-affine system Σ with an admissible equilibrium p_* and constant control vector fields. There exists a solution to the constant parameter matching problem if there exists a solution F_d to the LMI

$$G^\perp(\mathbf{A}\mathbf{F}_d^\top + \mathbf{F}_d\mathbf{A}^\top)G_\perp > 0.$$

such that

$$\frac{\partial f^j}{\partial y^i} = \frac{\partial f^i}{\partial y^j}, \quad \text{for all } i, j = m+1, \dots, n, i < j.$$

with $y^j = \sum[(F_d)_{ij}]x^i$.

4.4 Horizontal and Vertical Structure Tensor

In this section, we reduce the set of structure tensors that are available for a solution to the matching problem based on the set of admissible equilibria of Σ . We establish a decomposition of the structure tensor, considered here as a linear map, into two linear maps.

4.4.1 Equilibrium manifold

The key step in the proposed approach is the definition of two distinct linear maps whose direct sum reflects the interior product of a structure tensor with a one-form. The construction is based on the definition of an equilibrium manifold. The following has been presented in Niemiec and Kravaris [2003]. The set of admissible equilibria of the system Σ is defined by

$$\Gamma = \{p \in \mathcal{X} \mid \exists u \in \mathcal{U} \text{ such that } f(p) + g_i(p)u^i = 0\}.$$

Let $x = (x^1, \dots, x^n)$ be local coordinates. A point $p_* \in \Gamma$ is called *regular* if

$$\det \left(\left. \frac{\partial f}{\partial x} \right|_{x_*} + u_*^i \left. \frac{\partial g_i}{\partial x} \right|_{x_*} \right) \neq 0$$

where $u_* = (u_*^1, \dots, u_*^m)$ are real numbers that satisfy $f(p_*) + g(p_*)u_* = 0$.

Remark 4.4.1. Note that if an admissible equilibrium is regular, then the linearization of Σ at that equilibrium is “controllable at $s = 0$ ”, as defined in Chapter 3.

Proposition 4.4.2 (Niemiec and Kravaris [2003]). Let $w_j, j = m+1, \dots, n$, be $(n-m)$ linear independent vector fields on \mathcal{X} , which are orthogonal to $g_i, i.e.,$

$$\mathbb{G}(w_j, g_i) = 0, \quad j = m+1, \dots, n; \quad i = 1, \dots, m.$$

Also define the functions

$$h^j = \mathbb{G}(w_j, f), \quad j = m+1, \dots, n.$$

Then, in a neighbourhood of a regular point on Γ :

1. h^{m+1}, \dots, h^n are linearly independent functions.
2. Γ is exactly the intersection of the $(n - m)$ level surfaces $h^{m+1} = \dots = h^n = 0$.

Hence, Σ is a local embedded submanifold of \mathcal{X} . We denote the tangent space of Γ , a distribution on \mathcal{X} , by \mathcal{V} . At each point $p \in \mathcal{X}$, the tangent space $T_p\mathcal{X}$ may be written as a direct sum $T_p\mathcal{X} = \mathcal{V}_p \oplus \mathcal{H}_p$, where \mathcal{H}_p is the complement of \mathcal{V}_p with respect to \mathbb{G} , see Fig. 4.2. Furthermore, this decomposition also defines a decomposition of the cotangent space $T_p^*\mathcal{X} = \mathcal{V}_p^* \oplus \mathcal{H}_p^*$ dual to $\mathcal{V}_p \oplus \mathcal{H}_p$ for all $p \in \mathcal{X}$, where \mathcal{H}^* is locally spanned by dh^{m+1}, \dots, dh^n . A structure tensor F_d at a point p can be considered as a

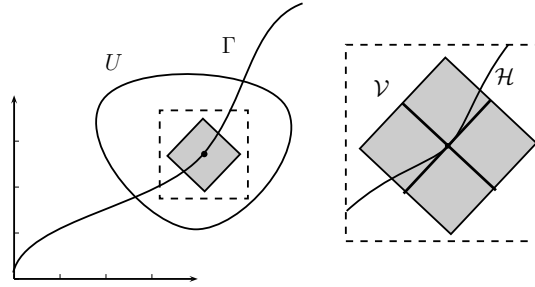


Figure 4.1: Horizontal and vertical subspace of $T_p\mathcal{X}$ at $p \in \Gamma$

linear map from $T_p^*\mathcal{X}$ to $T_p\mathcal{X}$ by the identification of $(0, 2)$ -tensor fields with sections of the homomorphism bundle, see Chapter 2. Furthermore, we have established the decomposition of $T_p\mathcal{X}$ into \mathcal{V}_p and \mathcal{H}_p such that $T_p\mathcal{X} = \mathcal{V}_p \oplus \mathcal{H}_p$ for all $p \in \mathcal{X}$. This decomposition induces a decomposition of the map $(F_d)_p$ into

$$(F_d^{\mathcal{V}})_p : \mathcal{V}_p^* \rightarrow T_p\mathcal{X} \quad \text{and} \quad (F_d^{\mathcal{H}})_p : \mathcal{H}_p^* \rightarrow T_p\mathcal{X}$$

in the following way. For any $v = (v^{\mathcal{V}}, v^{\mathcal{H}}) \in T_p^* \mathcal{X} = (\mathcal{H}^* \oplus \mathcal{V}^*)(p)$, we have that

$$F_d v = (F_d^{\mathcal{V}})_p(0, v^{\mathcal{V}}) + (F_d^{\mathcal{H}})_p(v^{\mathcal{H}}, 0).$$

Coframe derivatives

We require additional notation, which can also be found in Olver [1995]. Let M be an n -manifold, let (x^1, \dots, x^n) be local coordinates on a neighborhood U and let $\eta = \{\omega^i\}_{i=1}^n$ be a local coframe, *i.e.*, $\{\omega^1(p), \dots, \omega^n(p)\}$ is a basis of $T_p U$ for all $p \in U$. Each ω^i can be expressed in terms of the coordinated coframe $\{dx^i\}_{i=1}^n$ by $\omega^j = a_{ij} dx^i$, where $A = [a_{ij}]$ is a non-singular matrix of functions in local coordinates. The dual frame to η is given by the set of linear independent vector fields $\frac{\partial}{\partial \omega^i} = b^{ij}(p) \frac{\partial}{\partial x^i}$ $i = 1, \dots, n$, such that $[b^{ij}] = B = A^{-1}$.

Definition 4.4.3. Let M be an n -dimensional manifold and let $\omega = \{\omega^i\}_{i=1}^n$ be a local coframe on a neighborhood U . Let η be a one-form on M , then $\eta|_U$ can be written as $\eta|_U = \eta_{\eta^i} \omega^i$. The uniquely defined functions η_{η^i} are called the *component functions of η with respect to the coframe ω* .

Note that if η is the differential of a function f and ω is a coordinate coframe, then the component functions are the partial derivatives of f with respect to the coordinates whose differentials define the coordinate coframe. In general, we express the differential of a function in coframe-adapted form as follows. Let (x^1, \dots, x^n) be a local coordinates on M . Then, the differential of a function f can locally be written as

$$df = f_{\omega^i} \omega^i,$$

where the component functions $f_{\omega^i} = b_i^j \frac{\partial f}{\partial x^j}$ are called *coframe derivatives* of f with respect to the coframe ω . Let $A \in GL(n, \mathbb{R})$ be the transformation matrix on the coframe that adapts the coordinate coframe $dx = (dx^1, \dots, dx^n)$ to the coframe ω , *i.e.*, $\omega = Adx$, such that $\omega = (\omega^{\mathcal{V}}, \omega^{\mathcal{H}})$ with $\omega_p^{\mathcal{V}}$ a basis for \mathcal{V}_p^* for all $p \in \mathcal{X}$. We construct the adapted structure tensor \bar{F}_d given by $\bar{F}_d = F_d A^{-1}$. Hence, we can write the matching equations as

$$\begin{aligned} g^\perp F_d dH_d &= g^\perp f \\ g^\perp \bar{F}_d (H_d)_\omega \omega &= g^\perp f \\ g^\perp (\bar{F}_d^{\mathcal{V}} (H_d)_{\omega^{\mathcal{V}}} \omega^{\mathcal{V}} + \bar{F}_d^{\mathcal{H}} (H_d)_{\omega^{\mathcal{H}}} \omega^{\mathcal{H}}) &= g^\perp f, \end{aligned}$$

where $(H_d)_\omega$, $(H_d)_{\omega^{\mathcal{V}}}$ and $(H_d)_{\omega^{\mathcal{H}}}$ are vectors of the component functions of the differential dH_d with respect to the coframe ω and partial coframes $\omega^{\mathcal{V}}$ and $\omega^{\mathcal{H}}$, respectively.

4.4.2 Homogeneous solution

The main result of this section shows that, for a subset of the structure tensor fields determined by conditions on the vertical structure tensor, a homogeneous solution to the matching equation can be easily determined. It is desired to work in the coordinates adapted to Γ , which exists due to Proposition 4.4.2. We call a coordinate chart (y^1, \dots, y^n) *adapted to* Γ if $y^{m+1} = \dots = y^n = 0$ defines Γ . In these coordinates, the

vertical and horizontal distribution and codistribution have the local representation:

$$\begin{aligned}\mathcal{V} &= \text{span} \left\{ \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^m} \right\}, & \mathcal{V}^* &= \text{span} \{ dy^1, \dots, dy^m \}, \\ \mathcal{H} &= \text{span} \left\{ \frac{\partial}{\partial y^{m+1}}, \dots, \frac{\partial}{\partial y^n} \right\}, & \mathcal{H}^* &= \text{span} \{ dy^{m+1}, \dots, dy^n \}.\end{aligned}$$

For the choice of F_d such that $g^\perp \bar{F}_d^\mathcal{V} = 0$, the homogeneous matching equation is

$$g^\perp \bar{F}_d^\mathcal{H} (H_d)_{\omega^\mathcal{H}} \omega^\mathcal{H} = 0,$$

and $g^\perp \bar{F}_d^\mathcal{H}$ is a local isomorphism between \mathcal{H}^* and \mathcal{H} .

Proposition 4.4.4. Let Σ be a control-affine system, Γ its equilibrium manifold at a regular admissible equilibrium p_* , and let $\omega = (\omega^\mathcal{V}, \omega^\mathcal{H})$ be a coframe adapted to Γ . Let F_d a structure tensor field such that $g^\perp \bar{F}_d^\mathcal{V} = 0$. Then, the coordinate functions y^1, \dots, y^m are linear independent solutions to the homogenous matching equation.

Proof: The adapted coframe is $\omega = dy = \frac{\partial y}{\partial x} dx$ and the coframe derivatives become the partial derivatives with respect to the adapted coordinates. The matching equation is

$$g^\perp \bar{F}_d^\mathcal{H} (H_d)_{\omega^\mathcal{H}} \omega^\mathcal{H} = g^\perp f$$

with $\bar{F}_d = F_d \frac{\partial y}{\partial x}^{-\top}$. The homogeneous PDE $g^\perp \bar{F}_d^\mathcal{H} (H_d)_{\omega^\mathcal{H}} \omega^\mathcal{H} = 0$ is equivalent to

$$\begin{aligned}(H_d)_{\omega^\mathcal{H}} \omega^\mathcal{H} &= \frac{\partial H_d}{\partial y^i} dy^i = 0 \\ \Rightarrow \frac{\partial H_d}{\partial y^i} &= 0 \quad i = m+1, \dots, n,\end{aligned}$$

since $g^\perp \bar{F}_d^\mathcal{H}$ is a local isomorphism. Furthermore, this implies that any function

$\Phi(y^1, \dots, y^m)$ of the coordinates y^1, \dots, y^m is a solution to the homogeneous matching equation. ■

4.4.3 Existence condition

Under the hypotheses of Proposition 4.4.4, we can establish sufficient conditions for the existence of a solution to the matching equation.

Proposition 4.4.5. Assume in addition to the hypotheses of Proposition 4.4.4 that the local expression of $g^\perp \bar{F}_d^{\mathcal{H}}$ does not depend on the coordinates y^1, \dots, y^m . Let $\mu = (g^\perp \bar{F}_d^{\mathcal{H}})^{-1} g^\perp f$. Then there exists a solution to the matching equation if

$$d\mu \equiv 0 \pmod{\mu}. \quad (4.12)$$

Proof: Since $g^\perp \bar{F}_d^{\mathcal{H}}$ is an isomorphism at every point in a neighborhood of p_* , we can write $g^\perp \bar{F}_d^{\mathcal{H}}(H_d)_{\omega^{\mathcal{H}}} \omega^{\mathcal{H}} = g^\perp f$ as

$$(H_d)_{\omega^{\mathcal{H}}} \omega^{\mathcal{H}} = \mu. \quad (4.13)$$

By assumption, $g^\perp \bar{F}_d^{\mathcal{H}}$ is independent of y^1, \dots, y^m and g^\perp can be chosen such that $g^\perp f = \sum_{i=m+1}^n y^i \frac{\partial}{\partial y^i}$. Hence, the right hand side of (4.13) does not depend on y^1, \dots, y^m , then it follows from (4.12) that there exists an integrating factor b such that $b\mu$ is exact and its integral is a solution to the matching equation. ■

A particular solution to the matching equation can be computed by integration themes, which are applicable to large scale systems recently presented in Yap [2009].

4.5 Application

In this section, we apply the results established in this chapter to a stabilization problem of a tube-shell heat exchanger model.

4.5.1 Heat exchanger control

We consider a simple model of a tube-shell heat exchanger presented in Hango et al. [2004]. Denote the inlet and outlet temperature of the cold stream by T_{c_i} and T_{c_o} , respectively. Similarly, denote the inlet and outlet temperature of the hot stream by T_{h_i} and T_{h_o} , respectively. The volumetric flow rates are denoted by f_c and f_h , respectively. The associated thermal capacities are denoted by C_c and C_h , whereas the heat transfer is modeled by a thermal conductance G_{hc} . The dynamics of the system is governed by the differential equations for T_{c_i} and T_{c_o} ,

$$\begin{aligned} C_c \dot{T}_{c_o} &= -G_{hc}(T_{c_o} - T_{h_o}) + \gamma_c(T_{c_o} - T_{c_i})f_c \\ C_h \dot{T}_{h_o} &= G_{hc}(T_{c_o} - T_{h_o}) + \gamma_h(T_{h_o} - T_{h_i})f_h, \end{aligned}$$

where γ_c and γ_h are constants which depend on the density and specific heat of the respective streams. We can write the system in control-affine form if we define

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} T_{c_o} \\ T_{h_o} \end{bmatrix}, \quad \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} T_{c_i} \\ T_{h_i} \end{bmatrix}, \quad \begin{bmatrix} u \\ \bar{u} \end{bmatrix} = \begin{bmatrix} \frac{\gamma_c f_c}{C_c} \\ \frac{\gamma_h f_h}{C_h} \end{bmatrix}.$$

We have then

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{G_{hc}}{C_c}(x_1 - x_2) \\ \frac{G_{hc}}{C_h}(x_1 - x_2) + (x_2 - \bar{x}_2)\bar{u} \end{bmatrix} + \begin{bmatrix} x_1 - \bar{x}_1 \\ 0 \end{bmatrix} u,$$

hence

$$f = \begin{bmatrix} -\frac{G_{hc}}{C_c}(x_1 - x_2) \\ \frac{G_{hc}}{C_h}(x_1 - x_2) + (x_2 - \bar{x}_2)\bar{u} \end{bmatrix}, \quad g = \begin{bmatrix} x_1 - \bar{x}_1 \\ 0 \end{bmatrix}$$

Let $\mathcal{X} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > \bar{x}_1\}$ define the state space. We determine the set of admissible equilibria by

$$\Gamma = \{(x_1, x_2) \in \mathcal{X} \mid \frac{G_{hc}}{C_h}(x_1 - x_2) + (x_2 - \bar{x}_2)\bar{u} = 0\}.$$

This set is represented by a single line as shown in Figure 4.2. Our control objective

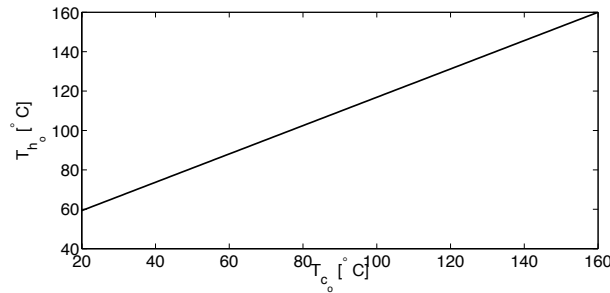


Figure 4.2: Equilibrium manifold Γ

is to achieve a desired temperature $T_{c_o}^* = x_1^*$ ($80^\circ C$ in the simulations) at the outlet on the cold temperature side. This implies a desired equilibrium at (x_1^*, x_2^*) , where x_2^* is determined by Γ and x_1^* . We use the framework established in the previous section to

design a passivity-based controller. The tangent space to Γ is spanned by the vector

field $X = \frac{G_{hc}}{c_h} \frac{\partial}{\partial x^1} + \left(-\frac{G_{hc}}{c_h} + \bar{u}\right) \frac{\partial}{\partial x^2}$, hence

$$\mathcal{V}^* = \text{span} \left\{ \frac{G_{hc}}{c_h} dx^1 + \left(-\frac{G_{hc}}{c_h} + \bar{u}\right) dx^2 \right\}$$

and $\mathcal{H}^* = \text{span} \left\{ -\frac{G_{hc}}{c_h} dx^2 + \left(-\frac{G_{hc}}{c_h} + \bar{u}\right) dx^1 \right\}$. The matrix that defines the adapted coframe is therefore a constant matrix given by

$$A = \begin{bmatrix} \frac{G_{hc}}{c_h} & -\frac{G_{hc}}{c_h} + \bar{u} \\ -\frac{G_{hc}}{c_h} + \bar{u} & -\frac{G_{hc}}{c_h} \end{bmatrix}.$$

Let

$$\bar{F}_d = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \nu_1 & \nu_2 \end{bmatrix}$$

with constant entries. Then the condition $g^\perp \bar{F}_d^\mathcal{V} = 0$ implies $\nu_1 = 0$. We also see that this condition establishes that $g^\perp \bar{F}_d^\mathcal{H} = \nu_2 \neq 0$. The coordinates induced by the equilibrium manifold are given by

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{G_{hc}}{c_h} x_1 \left(-\frac{G_{hc}}{c_h} + \bar{u}\right) x_2 \\ \frac{G_{hc}}{c_h} (x_1 - x_2) + (x_2 - \bar{x}_2) \bar{u} \end{bmatrix}.$$

In these coordinates, the homogeneous matching equation is given by $\frac{\partial H}{\partial y_2} = 0$. As a result, any function Φ , independent of y_2 , is an homogeneous solution. Furthermore, we require that $F_d + F_d^\top \leq 0$, this translates into $\nu_2 \neq 0$ and $4\alpha_1\nu_2 - \alpha_2^2 \leq 0$. Hence, we choose $\nu_2 = -\alpha_2$ and $\alpha_2 \neq 0$. Consequently, we compute a particular solution of

$\frac{\partial H}{\partial y_2} = (g^\perp \bar{F}_d^{\mathcal{H}})^{-1} g^\perp f = \nu_2 y_2$ by simple integration in the new coordinates. This yields $H = \frac{1}{2} \frac{1}{\nu_2} y_2^2$. The final solution is therefore given by

$$H_d = \frac{1}{2} \frac{1}{\nu_2} y_2^2 + \Phi(y_1).$$

To achieve stabilization of the system, we choose $\nu_2 = 10$ and $\Phi(y_1) = \frac{1}{2}(y_1 - y_1^*)^2$, where y_1^* is determined by (x_1^*, x_2^*) , such that H_d has a minimum at (y_1^*, y_2^*) . The influence of the two different terms in the solution becomes clear. The particular solution is such that the equilibrium manifold is stabilized and the homogeneous solution is chosen such that the desired equilibrium on the equilibrium manifold is stabilized. Applying the standard stabilizer feedback yields a locally asymptotically stable closed-loop system, a sample trajectory is shown in Figure 4.3. The values of the physical parameters of the system used in the simulation are those found in Çengel [2003], Example 11-4.

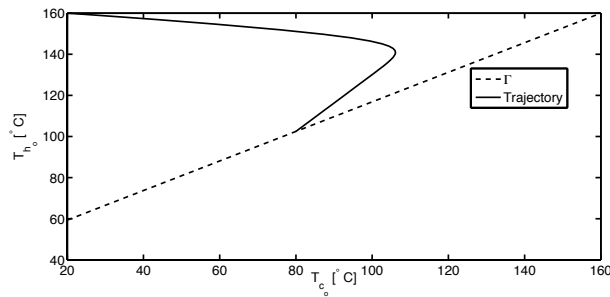


Figure 4.3: State space trajectories of the closed-loop

4.6 Conclusion

In this chapter, we exploited the implications of the assumption of a constant structure tensor on the matching problem. We established necessary and sufficient conditions for the existence of a solution to the matching equations based on exterior differential systems. A solution to an LMI is shown to be a candidate for a constant structure tensor that solves the matching problem. The concept of an equilibrium manifold introduces a decomposition of the structure tensor, which effectively reduces the degrees of freedom in the choice of the structure tensor while separating the problem of equilibrium assignment and existence of a solution.

4.7 Appendix: Exterior Differential Systems

Let M be an n -manifold. Denote by $\Omega^k(M)$ the space of differential k -forms on M , where $\Omega^0(M) = \mathcal{C}^\infty(M)$ is the space of functions on M . Then $\Omega(M) = \bigoplus_{k=0}^n \Omega^k(M)$ denotes the space of all differential forms on M . There are two operations defined on $\Omega^k(M)$ for all k , an associative but non-commutative product called the *wedge product* of two differential forms $\wedge : \Omega^k(M) \times \Omega^l(M) \rightarrow \Omega^{k+l}(M)$ and the map $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$ for all k called the *exterior derivative*. The space $\Omega(M)$ also forms a graded algebra under the wedge product. A subspace $\mathcal{I} \subset \Omega(M)$ is an *algebraic ideal* if it is a direct sum of homogeneous subspaces $\mathcal{I}^k \subset \Omega^k(M)$ and is closed under wedge product with arbitrary differential forms. An algebraic ideal is a *differential ideal* or *exterior differential system* if it is closed under exterior differentiation, $d\mathcal{I} \subseteq \mathcal{I}$. A Pfaffian system I is a submodule of one-forms over $\mathcal{C}^\infty(M)$ generated by a set of one-forms $\{\omega_1, \dots, \omega_k\}$, i.e., $I = \{\sum f_k \omega_k \mid f_k \in \mathcal{C}^\infty(M)\}$. A Pfaffian system I generates

a exterior differential system $\mathcal{I} = \langle \{dI, I\} \rangle$ of one-forms on M , where $\langle \cdot \rangle$ denotes the \mathbb{R} -linear span of elements in $\Omega^1(M)$. For exterior differential systems, the well-known Frobenius theorem takes the following form.

Proposition 4.7.1. A Pfaffian system I is completely integrable if and only if $d\alpha_i \equiv 0 \pmod{\mathcal{I}}, \forall \alpha_i \in \mathcal{I}$.

Let I be a Pfaffian system on M . A submanifold N of M is called an integral manifold of \mathcal{I} if $T_x N \subset \langle \mathcal{I}(x) \rangle^\perp$ for every $x \in N$. A curve $c : [a, b] \subset \mathbb{R} \rightarrow M$ is called an integral curve of \mathcal{I} if $c'(t) \in \text{ann } \langle \mathcal{I}(c(t)) \rangle$ for every $t \in [a, b]$. Let \mathcal{I} be an exterior differential system. An *independence condition* for \mathcal{I} is a decomposable p -form $\Omega = \omega^1 \wedge \cdots \wedge \omega^p$. The pair (\mathcal{I}, Ω) is called *exterior differential system with independence condition*. An integral manifold of (\mathcal{I}, Ω) is an integral manifold $i : N \rightarrow \mathcal{X}$ of \mathcal{I} such that $i^*(\Omega) \neq 0$. Given a Pfaffian system I , the smallest integrable subsystem contained in \mathcal{I} can be constructed as follows. Set $\mathcal{I}^{(0)} = \mathcal{I}$ and

$$\mathcal{I}^{(i)} = \{\omega \in \mathcal{I}^{(i-1)} \mid d\omega \equiv 0 \pmod{\mathcal{I}^{(i-1)}}\}.$$

This gives rise to a filtering

$$\mathcal{I}^{(0)} \supset \mathcal{I}^{(1)} \supset \cdots \supset \mathcal{I}^{(i)} \supset \cdots$$

called the *derived flag*. The filtration stabilizes for some integer N , *i.e.*, $\mathcal{I}^{(N)} = \mathcal{I}^{(N+1)}$ called the *derived length*, the system $\mathcal{I}^{(N)}$ is then called the *bottom derived system*. A derived system is called trivial if it only contains the null element. For a complete set of definitions and further information, the reader is referred to Bryant et al. [1990].

Chapter 5

The Matching Problem and Feedback Equivalence

The objective of this chapter is to connect the matching problem to a classical feedback equivalence approach. We show that feedback equivalence between control-affine systems can be used to define a family of matching problems for which an explicit solution is known. We illustrate our findings using a non-isothermal CSTR model.

This chapter is structured as follows. Cartan's method and its application to control theory is described in Section 5.1, and preliminary results on feedback equivalence are stated. The main results are presented in Section 5.3. We illustrate the results on a non-isothermal CSTR model in Section 5.4. A short conclusion and future directions of research are given in Section 5.5. We give further details on Cartan's method in Appendix 5.6.

5.1 Introduction

This section introduces Cartan’s method of equivalence, and describes its application to control theory. The method of equivalence offers a structured approach to specific geometric problems. An example is the equivalence problem of Riemannian metrics under conformal transformation [Gardner, 1989a]. In addition to the original papers by Cartan [Cartan, 1908], there are several books dedicated to this subject. One of the main references is Gardner [1989a], which gives a detailed description of Cartan’s method and gives applications to problems in mathematics and control theory. In Olver [1995], Cartan’s method is used in the context of invariants and symmetries of equivalence under transformation groups. In Ivey and Landsberg [2003], the authors consider the application to Pfaffian systems and partial differential equations. In the following, we give more details on the equivalence problem and focus on the application to control theory. Let $U, V \subset \mathbb{R}^n$ be open subsets, and let ω_U and Ω_V be two coframes on U and V , respectively. The study of equivalence involves three types of problems:

1. Determine whether Ω_V and ω_U are “equivalent”.
2. Determine the symmetries.
3. Determine the invariants of a symmetry group.

In the context of this thesis, the first problem translates into finding necessary and sufficient conditions for a control system to be feedback equivalent to a port-Hamiltonian system. The second problem is defined by determining the class of feedbacks that transforms port-Hamiltonian systems into port-Hamiltonian systems. For non-parametric IDA-PBC, the third problem translates into finding PDEs for the

unknown Hamiltonian function which are invariant functions. This will be further discussed in the following sections. We make the following definition.

Definition 5.1.1 (Gardner [1989a]). Let ω_U and Ω_V be n -vector of one-forms on open sets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$, respectively and let G be a Lie subgroup of $GL(n, \mathbb{R})$. Then ω_U is said to be G -equivalent to Ω_V if there exists a diffeomorphism $\Phi : U \rightarrow V$ and a function $\gamma : U \rightarrow G$ satisfying

$$\Phi^*(\Omega_V|_{\Phi(u)}) = \gamma(u)\omega_U|_u,$$

for all $u \in U$. Such a diffeomorphism, Φ , is called a G -equivalence of ω_U and Ω_V . The set of all G -equivalences of ω_U and Ω_V is denoted by $E_G(\omega_U, \Omega_V)$. If $G = GL(n, \mathbb{R})$, and there exists a G -equivalences between ω_U and Ω_V , then ω_U and Ω_V are said to be *equivalent*.

Note that this definition defines an equivalence relation on the set of control systems. With the notation, the equivalence problem can be stated as follows:

Equivalence Problem of É. Cartan: Let $\Omega_V = [\Omega_V^1, \dots, \Omega_V^n]^\top$ be a coframe on an open set $V \subset \mathbb{R}^n$, and let $\omega_U = [\omega_U^1, \dots, \omega_U^n]^\top$ be a coframe on $U \subset \mathbb{R}^n$, and let G be a Lie subgroup of $GL(n, \mathbb{R})$, then find a necessary and sufficient condition for the existence of a G -equivalence between ω_U and Ω_V .

5.1.1 Application to Control

It is possible to formulate the problem of feedback equivalence of two control systems as a problem of equivalence as described above. The method of equivalence has been

used to solve a number of feedback equivalence problems. The feedback equivalence problem of general control systems was treated in Gardner and Shadwick [1990a] and in Atkins and Shadwick [1993]. The presentation in this chapter is based on Atkins and Shadwick [1993], and Atkins [1992].

The study of invariants of a feedback equivalence is a possibility to characterize feedback equivalence classes. For general (not control-affine) control systems, it is difficult to find meaningful invariants. If further assumptions on the control system are made, then the method can reveal all invariants of that equivalence class. As a result one can consider the equivalence class of port-Hamiltonian systems and study ways to generate a complete set of invariants that describes the equivalence class. One special case, which has been well-treated in the literature, is the case of linear controllable systems. Several contributions by R. Gardner and W. Shadwick on this topic have been made (see Gardner and Shadwick [1992], Gardner [1989b] and references therein) including necessary and sufficient conditions for exact feedback linearization and an algorithm to compute the required output functions. The controllable linear case can be considered as a generic case, since it is defined by the vanishing of almost all invariant functions. The results in Gardner and Shadwick [1992] have been extended to orbital feedback linearization in Guay [1999]. The work presented in Gardner and Shadwick [1992] and Guay [1999] is based on equivalence of Pfaffian systems. It leads to a convenient procedure that can be implemented using symbolic computation software.

5.1.2 Preliminaries

A short outline of the application of the method of equivalence to feedback equivalences is presented in this section. We describe control-affine systems in terms of vectors of one-forms. To do so, let ω be a one-form on \mathcal{X} . Then for every vector field X , $\omega(X)$ is a function on \mathcal{X} . Let $\{\varepsilon^i\}_{i=1}^n$ be a local coframe on \mathcal{X} and let I be an open interval of \mathbb{R} including zero with dt , the coordinate one-form on I . Then we define for all $u = (u^1, \dots, u^m) \in \mathcal{U}$ the following one-forms:

$$\eta^j = \varepsilon^j - (\varepsilon^j(f) + u^i \varepsilon^j(g_i))dt, \quad j = 1, \dots, n, \quad (5.1)$$

which define the n -vector of one-forms $\eta = [\eta^1, \dots, \eta^n]^\top$. Based on this construction, we also denote a control-affine system by the triple $\Sigma = (\mathcal{X}, \eta, \mathcal{U})$. We define solutions of $\Sigma = (\mathcal{X}, \eta, \mathcal{U})$ as maps $\sigma : I \rightarrow I \times \mathcal{X} \times \mathcal{U}$ satisfying $\sigma^*(\eta^i) = 0$ for all $i = 1, \dots, n$ and $\sigma^*(dt) \neq 0$.

The equivalence of the two definitions of solutions of a control-affine system is established by the following proposition.

Proposition 5.1.2. Let (c, u) be a continues solution of $\Sigma = (\mathcal{X}, f, g, \mathcal{U})$, then the graph of (c, u) , $\sigma : s \mapsto (s, c(s), u(s))$ is a solution of $\Sigma = (\mathcal{X}, \eta, \mathcal{U})$. Conversely, let $\sigma : s \mapsto (s, \phi(s), \psi(s))$ be a solution to $\Sigma = (\mathcal{X}, \eta, \mathcal{U})$ then the curve $c : I \rightarrow \mathcal{X}$ and a control u defined by $c(s) = \phi(s)$ and $u(s) = \psi(s)$ is a solution to $\Sigma = (\mathcal{X}, f, g, \mathcal{U})$.

The proof of this proposition can be found in Lewis [1995], using the identification of η with an exterior differential system on $I \times \mathcal{X} \times \mathcal{U}$ described in Gardner and Shadwick [1990b]. The motivation for the alternative representation of a control-affine system is the following. We want to investigate the equivalence of control-affine

systems by studying the equivalence of coframes that are adapted to the control-affine systems as given in (5.1). The rationale for this investigation is that every trajectory $c(t)$ of Σ generated by some control $u(t)$ is in a one-to-one correspondence with an integral curve of the exterior differential system generated by $\{\eta^j\}_{j=1}^n$ (see Gardner and Shadwick [1990a] for further details).

Next we define feedback equivalence between two control-affine systems. Let $(x^1, \dots, x^n, u^1, \dots, u^m, t)$ be coordinates on $M = \mathcal{X} \times \mathcal{U} \times I$ and let $\pi_{\mathcal{X}} : M \rightarrow \mathcal{X}$ be the projection onto the first factor. Note that $\{\pi_{\mathcal{X}}^* \eta^j\}_{j=1}^n$ ¹ is a set of n linearly independent one-forms on M , which together with the set $\{dt, du^1, \dots, du^m\}$ defines a coframe $\omega_M = [\eta, du, dt]^\top$ on M , called the *zero-adapted coframe*. We use this construction of the coframe to define a feedback equivalence for control-affine systems. The following construction is needed for the definition of feedback equivalence. We need to define a *structure group*, which is a Lie subgroup of $GL(n+m+1, \mathbb{R})$ defined by

$$G_0 = \left\{ \left[\begin{array}{ccc} A & 0 & 0 \\ B & C & 0 \\ 0 & 0 & 1 \end{array} \right] \in GL(n+m+1, \mathbb{R}) \left| \begin{array}{l} A \in GL(n, \mathbb{R}) \\ B \in \mathbb{R}^{m \times n} \\ C \in GL(m, \mathbb{R}) \end{array} \right. \right\}.$$

Definition 5.1.3. Consider two control-affine systems $\Sigma_{\mathcal{X}} = (\mathcal{X}, \eta, \mathcal{U})$ and $\Sigma_{\mathcal{Y}} = (\mathcal{Y}, \mu, \mathcal{V})$. Let $\omega_M = [\eta, du, dt]^\top$ and $\Omega_N = [\mu, dv, dt]^\top$ be two zero-adapted coframes on $M = \mathcal{X} \times \mathcal{U} \times \mathbb{R}$ and $N = \mathcal{Y} \times \mathcal{V} \times \mathbb{R}$, respectively. We say that $\Sigma_{\mathcal{X}}$ is *feedback equivalent* to $\Sigma_{\mathcal{Y}}$ if ω_M is G_0 -equivalent to Ω_N , *i.e.*, there exists a G_0 -equivalence $\Phi : \mathcal{X} \times \mathcal{U} \times \mathbb{R} \rightarrow \mathcal{Y} \times \mathcal{V} \times \mathbb{R}$ of the form

$$\Phi(x, u, t) = (\phi(x), \psi(x, u), t)$$

¹We omit the pull-back of the projection if it is clear from the context.

for some local diffeomorphism $\phi : \mathcal{X} \rightarrow \mathcal{Y}$, called *coordinate transformation* and some map $\psi : \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{V}$, called *state feedback*, then Φ so defined is called a *feedback equivalence* between $\Sigma_{\mathcal{X}}$ and $\Sigma_{\mathcal{Y}}$.

This definition agrees with the idea that two systems are feedback equivalent if they generate the same trajectory under the same control input and initial conditions.

5.1.3 Matching Problem

Next, we describe the matching problem introduced in the IDA-PBC methodology as a feedback equivalence problem. We define the matching problem for non-parametric IDA-PBC:

Problem 5.1.4. Given a control-affine system Σ , and a structure tensor field F_d find a function H_d such that Σ is feedback equivalent to the port-Hamiltonian system $\Sigma_d = (\mathcal{X}, F_d, H_d, g, \mathcal{U})$.

For the purpose of this chapter, we are only concerned with the modification or generation of a Hamiltonian function, not with stabilization and equilibrium assignment.

5.2 Equivalence

In this section we outline a procedure to determine a feedback equivalence, this procedure was presented in Atkins [1995] and is based on the work of Élie Cartan [Cartan, 1908]. It will become clear that the matching equations appear naturally in this context.

5.2.1 Lifted Coframe

In the equivalence problem, as defined in the previous section, we are interested in finding a local diffeomorphism between $M = \mathcal{X} \times \mathcal{U} \times \mathbb{R}$ and $N = \mathcal{Y} \times \mathcal{V} \times \mathbb{R}$ such that its differential has a predetermined form given by G_0 . Proposition 5.2.1 below states that we can equivalently find a local diffeomorphism between $M \times G_0$ and $N \times G_0$ without any restrictions on its differential. Let $\pi : M \times G_0 \rightarrow M$ be the projection onto M and define the vector of one-forms $\omega = S\pi^*\omega_M$ on $M \times G_0$, where $S \in G_0$ and proceed similarly for Ω on $N \times G_0$. The lifted coframe ω on $M \times G_0$ is defined as

$$\omega = S\omega_M = \begin{bmatrix} \theta \\ \mu \\ dt \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ B & C & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \eta \\ du \\ dt \end{bmatrix}.$$

The following proposition allows us to lift the equivalence problem to the product $M \times G_0$.

Proposition 5.2.1 (Gardner [1989a]). There exists a local diffeomorphism $\Phi : M \rightarrow N$ satisfying

$$\Phi^*\Omega_N = \gamma_{NM}\omega_M \quad \text{with } \gamma_{NM} : M \rightarrow G_0$$

if and only if there exists a diffeomorphism $\Phi^1 : M \times G_0 \rightarrow N \times G_0$ such that $\Phi^{1*}\Omega = \omega$.

In order to solve the feedback equivalence problem we can equivalently determine if there exists a local diffeomorphism Φ^1 on $M \times G_0$ that solves $\Phi^{1*}\Omega = \omega$. We will use some terminology which is introduced in Appendix 5.6 and further explained in Gardner [1989a]. We study the structure equations of the feedback equivalence problem. Since we consider only time-invariant feedback, it is clear that t is invariant

for Φ and Φ^1 . Consequently, we omit this component in the structure equations. After Lie algebra compatible absorption equation (5.6) are

$$d \begin{bmatrix} \theta \\ \mu \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ \beta & \gamma \end{bmatrix} \wedge \begin{bmatrix} \theta \\ \mu \end{bmatrix} + \begin{bmatrix} dt \wedge K\mu \\ 0 \end{bmatrix}. \quad (5.2)$$

where α, β and γ are Lie algebra-valued one-forms. Furthermore, the term $dt \wedge K\mu$ is the only non-zero intrinsic torsion. The torsion coefficient K can be expressed parametrically as $K = AK_0C^{-1}$ with $K_0 = g$, *i.e.*, K_0 is the value of K for $S = e$ with e as the identity element of G_0 .

5.2.2 Reduction of the structure group

Next, we analyze how the intrinsic torsion changes under the action of G_0 . Instead of computing the orbit of the group action on the intrinsic torsion explicitly, we compute the infinitesimal group action to determine the reduced structure group and the adapted coframe as suggested by Gardner [1989a]. The infinitesimal group action on the intrinsic torsion is obtained by studying the integrability condition $d \circ d\theta = 0$. This yields the congruence

$$dK - \alpha K + K\gamma \equiv 0 \pmod{\{\theta, \mu, dt\}},$$

where two one-forms ω and η are said to be *congruent modulo* an algebraic ideal I if there exists $\alpha \in I$ such that $\omega = \eta + \alpha$, *i.e.*, $\omega \equiv \eta \pmod{I}$. We assumed that g has locally constant rank m around every point $p \in \mathcal{X}$, hence K also has locally constant rank m . The elements A and C are required to have full-rank since $G_0 \subset GL(n+m+1, \mathbb{R})$.

We choose to normalize K to the constant matrix

$$K_N = \begin{bmatrix} 0_{(n-m) \times m} \\ I_m \end{bmatrix}.$$

It follows that after this normalization $dK_N = 0$ and hence $\alpha K_N \equiv K_N \gamma \pmod{\theta, \mu, dt}$.

If we let

$$\alpha = \begin{bmatrix} \varepsilon & \alpha_{12} \\ \lambda & \alpha_{22} \end{bmatrix},$$

then we get the following congruences

$$\alpha_{12} \equiv 0 \pmod{\theta, \mu, dt}, \quad \alpha_{22} \equiv \gamma \pmod{\theta, \mu, dt} \quad (5.3)$$

which determine the first reduced structure group given by the isotropy group of K_N .

Remark 5.2.2. Parametric calculations show that the following equation

$$AK_N C^{-1} = K_N,$$

determines the first reduced structure group G_1 . Furthermore, elements of G_1 are then of the form

$$\begin{bmatrix} A_1 & 0 & 0 \\ A_2 & C & 0 \\ B_1 & B_2 & C \end{bmatrix}$$

where $A_1 \in GL(n-m, \mathbb{R})$, $A_2 \in \mathbb{R}^{n-m \times n}$ and $B = [B_1, B_2] \in \mathbb{R}^{n \times m}$.

In order to define the G_1 -adapted coframe we define the map

$$\beta_M : M \rightarrow G_0$$

$$(x, u, t) \mapsto \begin{bmatrix} \begin{bmatrix} g^\perp(x) \\ g^\dagger(x) \end{bmatrix} & 0 & 0 \\ 0 & g^\dagger(x)g(x) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where g^\perp is a full-rank left annihilator of g , *i.e.*, $g^\perp g = 0$ and g^\dagger is such that $g^\dagger g$ and $[g^\perp, g^\dagger]$ have full-rank. Note that one possible choice for g^\dagger is g^\top . The G_1 -adapted coframe is then defined by

$$\beta_M \omega_M = \begin{bmatrix} g^\perp dx & -g^\perp f dt \\ g^\dagger dx & -g^\dagger(f + gu) dt \\ (g^\dagger g) du \\ dt \end{bmatrix}.$$

Furthermore, define $\widehat{\theta} = [\theta^1, \dots, \theta^{n-m}]^\top$ and $\bar{\theta} = [\theta^{n-m+1}, \dots, \theta^n]^\top$ by

$$S_1 \beta_M \omega_M = \begin{bmatrix} \widehat{\theta} & \bar{\theta} & \mu & dt \end{bmatrix}^\top, \quad S_1 \in G_1.$$

We continue with the reduced equivalence problem with new structure equations for the G_1 -equivalence between the G_1 -adapted coframes $\beta_M \omega_M$ and $\beta_N \Omega_N$. The basis for only considering the reduced equivalence problem is given by Proposition 5.6.6. The new structure equations are obtained by substituting the congruences (5.3) into the structure equations (5.2), which introduces new torsion. The new structure equations

are

$$d \begin{bmatrix} \widehat{\theta} \\ \bar{\theta} \\ \mu \end{bmatrix} = \begin{bmatrix} \varepsilon & 0 & 0 \\ \lambda & \gamma & 0 \\ \beta_1 & \beta_2 & \gamma \end{bmatrix} \wedge \begin{bmatrix} \widehat{\theta} \\ \bar{\theta} \\ \mu \end{bmatrix} + \begin{bmatrix} \bar{\theta}^\top \wedge \mathcal{J}\bar{\theta} + dt \wedge \mathcal{H}\bar{\theta} \\ dt \wedge \mu \\ 0 \end{bmatrix}$$

where β_1 and β_2 are such that $\beta = [\beta_1, \beta_2]$. The term $\mathcal{H} = [\mathcal{H}_{ij}]$ is a matrix-valued function on $M \times G_1$ and $\mathcal{J} = [\mathcal{J}_{jk}^i] = -[\mathcal{J}_{kj}^i]$ are skew-symmetric matrix-valued functions on $M \times G_1$ for all $i = 1, \dots, n-m$. The terms \mathcal{H} and \mathcal{J} are the torsion coefficients for the new structure equations. The infinitesimal group action on the torsion coefficients \mathcal{J} and \mathcal{H} induced by the integrability condition $d \circ d\widehat{\theta} = 0$ is determined by

$$d\mathcal{J} - \varepsilon\mathcal{J} + \gamma^\top \mathcal{J} + \mathcal{J}\gamma \equiv 0 \pmod{\{\theta, dt\}}$$

$$d\mathcal{H} - \varepsilon\mathcal{H} + \mathcal{H}\gamma \equiv 0 \pmod{\{\theta, \mu, dt\}}.$$

Furthermore, we can determine the torsion coefficients parametrically by

$$\mathcal{J} = A_1 C^{-\top} \mathcal{J}_0 C^{-1} \quad \text{and} \quad \mathcal{H} = A_1 \mathcal{H}_0 C^{-1}$$

where \mathcal{J}_0 and \mathcal{H}_0 are the values of \mathcal{J} and \mathcal{H} at $e \in G_1$, respectively.

To determine the equivalence between the control-affine systems, we repeat the inductive process outlined above until the torsion does not explicitly depend on the group parameters. At this point there are two possibilities. Either there are no free group parameters left and we have identified a canonical coframe and this problem can be solved as an equivalence problem of coframes (see Olver [1995] for more details on equivalence of coframes), or there remain free group parameters and the involutivity

of the defining equations must be checked. This is an existence and uniqueness question for a set of partial differential equations and is answered by the Cartan-Kähler theorem, which is a geometric extension of the Cauchy-Kowalewski existence theorem (see Bryant et al. [1990]).

5.3 Port-Hamiltonian Systems

In this section, we first derive the matching equation associated to the matching problem and then use the transitivity property of the feedback equivalence to generate new interconnection and damping structures and associate a Hamiltonian function to them. Furthermore, we show that the choice of coordinates, together with an appropriate feedback law, can be used to define new matching problem for which an explicit solution is known.

5.3.1 Matching equations

We apply the procedure described in the previous section to the feedback equivalence of a control-affine system $\Sigma_{\mathcal{X}} = (\mathcal{X}, f, g, \mathcal{U})$ and a port-Hamiltonian system $\Sigma_d = (\mathcal{X}, F_d, H_d, g, \mathcal{U})$. We established that any feedback equivalence $\Phi = (\phi, \psi, \text{Id}_I)$ between $\Sigma_{\mathcal{X}}$ and Σ_d has to satisfy $\Phi^*\Omega_N = \omega_M$, where ω_M and Ω_N are the adapted coframes of $\Sigma_{\mathcal{X}}$ and Σ_d , respectively. Recognizing that the two control-affine systems have the same control vector fields, which is that the first torsion coefficients are the same. Hence, the first reduction of the structure group is identical for both control-affine systems, which implies that the first structure groups are the same. Based on

Proposition 5.6.6, the feedback equivalence also has to satisfy

$$\Phi^*(\beta_N\Omega_N) = \alpha_{NM}(\beta_M\omega_M),$$

where $\alpha_{NM} : M \rightarrow G_1$. Using the same notation as in the previous section. The $\widehat{\theta}$ -component is given by

$$-\phi^*(g^\perp F_d dH_d) = -A_1 g^\perp f \quad (5.4)$$

with $A_1 \in GL(n-m, \mathbb{R})$, which represents a subgroup of the structure group G_1 . We refer to equation (5.4) as the *matching equation* of the feedback equivalence problem. If we leave H_d free and fix Φ , it is a partial differential equation for the unknown H_d .

Remark 5.3.1. The matching equations can be understood as an equivalence condition after the first reduction of the structure group. The first reduction requires a constant torsion, which represents that the control vector fields of the given and desired system are related by a local diffeomorphism. It is interesting to understand what implication a constant torsion of the reduced structure equations has on the structure tensor field and the desired Hamiltonian function.

5.3.2 Transitivity

Next, a result that allows to define solutions to an equivalence problem by utilizing known solutions is presented. We first give a general result for arbitrary equivalence of coframes and then apply this result to the special case of the matching problem.

Proposition 5.3.2. Let $\omega_U, \Omega_V, \eta_W$ be coframes on $U, V, W \subset \mathbb{R}^n$, respectively, and let $G \subset GL(n, \mathbb{R})$ be a Lie group. Assume there exists a G -equivalence Φ for ω_U and

Ω_V and a G -equivalence Ψ for Ω_V and η_W , then there exists a G -equivalence for ω_U and η_W .

Proof: Since ω_U and Ω_V are G -equivalent, there exists $\gamma_{VU} : U \rightarrow G$ such that $\Phi^*(\Omega_V) = \gamma_{VU}\omega_U$ and there exists $\Gamma_{WV} : V \rightarrow G$ such that $\Psi^*(\eta_W) = \Gamma_{WV}\Omega_V$. Hence,

$$\begin{aligned} \Phi^*(\Omega_V) = \gamma_{VU}\omega_U &\Leftrightarrow \Phi^*(\Gamma_{WV}^{-1}\Psi^*(\eta_W)) = \gamma_{VU}\omega_U \\ &\Leftrightarrow \Gamma_{WV}^{-1} \circ \Phi(\Psi \circ \Phi)^*(\eta_W) = \gamma_{VU}\omega_U \\ &\Leftrightarrow (\Psi \circ \Phi)^*(\eta_W) = (\Gamma_{WV}^{-1} \circ \Phi)^{-1}\gamma_{VU}\omega_U, \end{aligned}$$

where $(\Gamma_{WV}^{-1} \circ \Psi)^{-1} : U \rightarrow G$ denotes the inverse of $\Gamma_{WV}^{-1} \circ \Phi(u) \in G$ for all $u \in U$, it follows that $\alpha_{WU} \triangleq (\Gamma_{WV}^{-1} \circ \Psi)^{-1}\gamma_{VU}$ defines a G -equivalence for ω_U and η_W . ■

Corollary 5.3.3. Let $\Sigma_{\mathcal{Y}} = (\mathcal{Y}, \mu, \mathcal{V})$ be a control-affine system and let F_d structure tensor on \mathcal{Y} , respectively. Assume there exists a solution to the matching problem define by $\Sigma_{\mathcal{Y}}$ and F_d . Let $\Sigma_{\mathcal{X}} = (\mathcal{X}, \eta, \mathcal{U})$ be a control-affine system, which is feedback equivalent to $\Sigma_{\mathcal{Y}}$ with feedback equivalence given by $\Phi : M \rightarrow N$, then there exists a feedback equivalence for $\Sigma_{\mathcal{X}}$ and Σ_d .

Proof: Let ω_M and Ω_N be the adapted coframe associated to $\Sigma_{\mathcal{X}}$ and $\Sigma_{\mathcal{Y}}$, respectively. Define the port-Hamiltonian system $\Sigma_d = (\mathcal{Y}, \mu_d, \mathcal{V}) = (\mathcal{Y}, J_d, R_d, H_d, g, \mathcal{V})$ and its adapted coframe ω_M^d . By assumption Σ_d is feedback equivalent to $\Sigma_{\mathcal{Y}}$, and there exists a G_0 -equivalence for $\Sigma_{\mathcal{Y}}$ and Σ_d . Hence, by Proposition 5.3.2, ω_M is G_0 -equivalent to ω_M^d . ■

The following proposition establishes that a feedback equivalence allows us to define a new structure tensor on \mathcal{Y} .

Proposition 5.3.4. Under the assumptions of Corollary 5.3.3, there exists a local diffeomorphism $\bar{\Phi} : M \rightarrow N$, $\bar{\Phi} = (\bar{\phi}, \bar{\psi}, \text{Id}_I)$ such that the matching problem defined by $\Sigma_{\mathcal{X}}$, $\bar{F}_d = \bar{\phi}_* F_d$ has a solution $\bar{H} = H \circ \bar{\phi}$.

Proof: Let $\Psi : N \rightarrow N$ be a the feedback equivalence for $\Sigma_{\mathcal{Y}}$ and Σ_d , then $\bar{\Phi} = \Psi \circ \Phi$ is a feedback equivalence for $\Sigma_{\mathcal{X}}$ and Σ_d such that

$$\bar{\Phi}^*(\omega_M^d) = S\omega_M \quad (5.5)$$

where $S : M \rightarrow G_0$. Note that (5.5) can be written in components as $\bar{\Phi}^*\mu_d^i = S_{ij}\eta^j$. Let us choose coordinate charts such that the local representative of $\bar{\phi}$ is given by $y^i = \bar{\phi}^i(x^1, \dots, x^n)$. Then $\bar{\phi}^*dy^i = S_{ij}(y)dx^j$ implies that $\frac{\partial(y^i \circ \bar{\phi})}{\partial x^j} = \frac{\partial y^i}{\partial x^j} = S_{ij}(x)$ for all $i, j = 1, \dots, n$. Define $A : M \rightarrow GL(n, \mathbb{R})$ such that $A_{ij} = S_{ij}$ for $i = 1, \dots, n$ and $j = 1, \dots, n$. We assumed that the feedback time-invariant. Hence $\bar{\Phi}^*(dt) = dt$ implies that the functions multiplying dt in $\bar{\Phi}^*(\mu_d)$ and $A_{ij}\eta^j$ must agree. Therefore,

$$\begin{aligned} & \bar{\Phi}^*(F_d(dy^i, dH_d) + dy^i(g_j)v^j) \\ &= A(x)(dx^i(f) + dx^i(g_j)w^j), \end{aligned}$$

where f is the drift vector field and g_1, \dots, g_m are the control vector fields of $\Sigma_{\mathcal{X}}$. Also, we have $\bar{\Phi}^*F_d(dy^i, dH_d) = \bar{\phi}^*F_d(dy^i, dH_d)$, since their component functions do not depend on v by assumption. Next, we show that these functions define the structure tensor field $\bar{F}_d = \bar{\phi}^*F_d$ on M . Using the definition of the pull-back, we get

$$\bar{\phi}^*(F_d(dy^i, dH_d)) = (F_d(dy^i, dH_d)) \circ \bar{\phi} = \left(F_d^{ik} \frac{\partial H_d}{\partial y^k} \right) \circ \bar{\phi}.$$

Furthermore, we have that

$$\bar{\phi}^* F_d = F_d^{kl} \circ \bar{\phi} \left(\bar{\phi}_*^{-1} \frac{\partial}{\partial y^k} \right) \otimes \left(\bar{\phi}_*^{-1} \frac{\partial}{\partial y^l} \right),$$

which we use to compute

$$\begin{aligned} & \bar{\phi}^* F_d(dx^i, \bar{\phi}^* dH_d) \\ &= F_d^{kl} \circ \bar{\phi} dx^i \left(\bar{\phi}_*^{-1} \frac{\partial}{\partial y^k} \right) \bar{\phi}^* dH_d \left(\bar{\phi}_*^{-1} \frac{\partial}{\partial y^l} \right) \\ &= F_d^{kl} \circ \bar{\phi} dx^i \left(\bar{\phi}_*^{-1} \frac{\partial}{\partial y^k} \right) dH_d \left(\bar{\phi}_* \bar{\phi}_*^{-1} \frac{\partial}{\partial y^l} \right) \\ &= F_d^{kl} \circ \bar{\phi} \frac{\partial y^i}{\partial x^k} \frac{\partial H_d}{\partial y^l} \circ \bar{\phi} \\ &= \left(\frac{\partial y^i}{\partial x^k} \right) \left(F_d^{kl} \frac{\partial H_d}{\partial y^l} \right) \circ \bar{\phi} \end{aligned}$$

for all $i = 1, \dots, n$. Furthermore, we have that

$$\bar{\phi}^* F_d(dx^i, \bar{\phi}^* dH_d) = \left(\frac{\partial y^i}{\partial x^k} \right) \bar{\phi}^* (F_d(dy^k, dH_d)).$$

As a result, we have defined a port-Hamiltonian system on \mathcal{X} with structure tensor field \bar{F}_d . The Hamiltonian function of this system is $\bar{H}_d = H_d \circ \bar{\phi}$. ■

Remark 5.3.5. Proposition 5.3.4 gives a general formulation to construct a structure tensor field on \mathcal{X} . In practice this is useful when the system under consideration has some known feedback invariant properties, *e.g.*, feedback linearizability. Then the matching problem is solved by finding any simple port-Hamiltonian system, which has the same feedback invariant properties.

5.4 Application

To illustrate the application of the results presented above, we apply the IDA-PBC methodology to a non-isothermal CSTR studied for example in Guay [2002]. Here we adopt the following perspective. First, we consider a matching problem that is inspired by a generic mechanical control system under full actuation with the objective to stabilize a generic equilibrium. Then, we establish a feedback equivalence between a generic simple mechanical control system and the non-isothermal CSTR. This equivalence is guaranteed by differential flatness of both systems. Using the transitivity property, the resulting matching problem for the non-isothermal CSTR system is solved easily. The dynamics of a non-isothermal CSTR are governed by the equations

$$\begin{aligned}\dot{C}_A &= r_A(C_A, C_B, T) + (C_{A,In} - C_A)u_1/V \\ \dot{C}_B &= r_B(C_A, C_B, T) + (C_{B,In} - C_B)u_1/V \\ \dot{T} &= h(C_A, C_B, T) + \alpha(T_J - T)/V + (T_{in} - T)u_1/V \\ \dot{T}_J &= \beta(T - T_J)/V_J + \lambda(T_{J_0} - T_J)u_2/V,\end{aligned}$$

where C_A and C_B are the concentrations of the components A and B , T is the reactor temperature, T_J is the jacket temperature, r_A and r_B are arbitrary chemical kinetic expressions, $h(C_A, C_B, T)$ is an arbitrary function of the heat of reaction, V is the reactor volume, V_J is the jacket volume, α , β and λ are constants (see also Figure 5.1). It is easy to see that the system has the required control-affine form. Let us denote this system by $\Sigma_{\mathcal{X}}$. The control objective is to stabilize a desired state $(C_A^*, C_B^*, T^*, T_J^*)$ using the inlet volumetric flow rate u_1 and the jacket volumetric

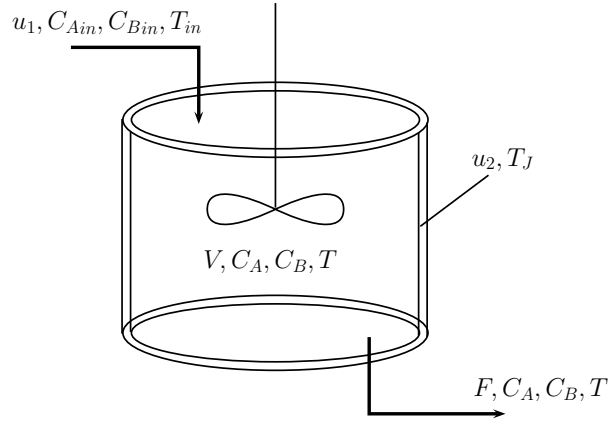


Figure 5.1: Non-isothermal continuous stirred-tank reactor

flow rate u_2 as the control input.

Matching equations

We consider a second control-affine control system, which is feedback equivalent to $\Sigma_{\mathcal{X}}$, with desired interconnection and damping structures such that the matching problem defined by these elements can be solved. For this purpose, we chose a simple mechanical control system with full actuation denoted by $\Sigma_{\mathcal{Y}}$. Let $Q \subset \mathbb{R}^2$ be the configuration space with coordinates $q = (q^1, q^2)$ and let $(q, p) = (q^1, q^2, p^1, p^2)$ be coordinates on T^*Q . The Euler–Lagrange equations are

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H^\top}{\partial q} \\ \frac{\partial H^\top}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} \tau,$$

where $B \in \mathbb{R}^{2 \times 2}$ is constant and non-singular and we consider the generalized torque τ as the control input. The total energy $H(q, p) = \frac{1}{2}p^\top M^{-1}p + V(q)$ is the sum of kinetic energy, with constant inertia matrix $M = M^\top > 0$, and potential energy function V ,

which is bounded from below with global minimum q^* . Since the system is fully actuated, we can shape the potential energy of the system without any constraints. In terms of our notation, this translates into a simple feedback equivalence $\Psi = (\text{id}_{\mathbb{R}}, \text{id}_{T^*Q}, KB^\top \frac{\partial H^\top}{\partial p} + u)$. The desired interconnection and damping structure is given by

$$J_d = \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix} \quad \text{and} \quad R_d = \begin{bmatrix} 0 & 0 \\ 0 & BKB^\top \end{bmatrix},$$

in coordinates, define the structure tensor as follows: $F_d = J_d - R_d$.

Transitivity

Next, we establish the feedback equivalence for $\Sigma_{\mathcal{X}}$ and $\Sigma_{\mathcal{Y}}$. It has been shown in Guay [2002] that the system $\Sigma_{\mathcal{X}}$ is state-feedback linearizable with linearizing output [Isidori, 1995]

$$h = \begin{bmatrix} \frac{C_B C_{A,In} - C_A C_{B,In}}{C_B - C_{B,In}} \\ \frac{T_{In} C_B - T C_{B,In}}{C_B - C_{B,In}} \end{bmatrix}.$$

Let us denote by Φ_1 the feedback equivalence that transforms $\Sigma_{\mathcal{X}}$ into Brunovsky normal form, *i.e.*, $\Phi_1 = (h, \mathcal{L}_f h, \mathcal{L}_f^2 h + \mathcal{L}_g \mathcal{L}_f h u)$. The simple mechanical control system $\Sigma_{\mathcal{Y}}$ is also feedback linearizable with linearizing output $y = q$. Let us denote by Φ_2 the feedback equivalence that transforms $\Sigma_{\mathcal{Y}}$ into Brunovsky normal form, *i.e.*, $\Phi_2(q, p, \tau) = (q, M^{-1}p, -\frac{\partial H}{\partial q} - BKB^\top \frac{\partial H^\top}{\partial p} + B\tau)$, where $KB^\top \frac{\partial H^\top}{\partial p}$ represents an additional damping injection feedback. Hence, one possible transformation that establishes the feedback equivalence for $\Sigma_{\mathcal{X}}$ and $\Sigma_{\mathcal{Y}}$ is given by $\Phi = \Phi_2^{-1} \circ \Phi_1$. The coordinate transformation on the state space is given by $\phi(C_A, C_B, T, T_J) = (h, M\mathcal{L}_f h)$. A new matching problem is defined by $\Sigma_{\mathcal{X}}$ and the desired structure tensor $\phi^* F_d$. Its solution is given

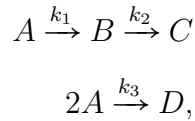
by

$$H \circ \phi = \frac{1}{2}(L_f h)^\top M \mathcal{L}_f h + \frac{1}{2}(h - q^*)^\top P(h - q^*),$$

if we chose a quadratic potential energy function with $P = P^\top > 0$. Note that this choice of interconnection and damping structure is not obvious in the natural coordinates of $\Sigma_{\mathcal{X}}$. Hence, we can stabilize the desired equilibrium by choosing q^* in $h^{-1}(C_A^*, C_B^*, T^*, T_J^*)$.

Simulation Example: van der Vusse Reactor

Next, we study how the generalized position of the simple mechanical control system $\Sigma_{\mathcal{Y}}$ can be understood in terms of the states of the non-isothermal CSTR. Consider the specific reaction mechanism



known as the van der Vusse reaction, as described for example in Chen et al. [1995] and Niemiec and Kravaris [2003]. Furthermore, we specify the reaction kinetics

$$\begin{aligned} r_A(C_A, C_B, T) &= -k_1(T)C_A - k_3(T)C_A^2, \\ r_B(C_A, C_B, T) &= k_1(T)C_A - k_2(T)C_B, \end{aligned}$$

where the rate coefficients k_i are dependent on the reactor temperature via the Arrhenius equation $k_i(T) = k_{i0}e^{-E_i/RT}$ for $i = 1, 2, 3$. From the discussion above, we know that the generalized positions q_1 and q_2 depend only on T, C_A and C_B .

The dependence of the states T , C_A and C_B on the generalized positions q_1 and q_2 , centered at the desired equilibrium q_* , are shown in Figure 5.2. It can be seen that the generalized coordinates are linearly dependent on the concentration C_A and C_B , and depend non-linearly on the temperature of the reactor. We also investigate the influence of the states T , C_A and C_B on the potential energy function, this is shown in Figure 5.3. The values of the physical parameters of the system used in the simulation are those found in Niemiec and Kravaris [2003].

5.5 Conclusion

It has been shown how feedback equivalence can be utilized to generate new matching problems for which an explicit solution is known. Furthermore, we have seen, in an example, that the transitivity property of the feedback equivalence can be used to define a simpler matching problem for feedback equivalent system. Future work will consider how the set of matching problems, which can be generated by an existing feedback equivalence, can be characterized.

To the best of our knowledge, there has not been any previous consideration of either using Cartan's method as a structured approach to the equivalence problem encoded in the IDA-PBC methodology nor to narrow down the specific class of Hamiltonian systems using Cartan's method.

5.6 Appendix: Cartan's Method

In this appendix some essential results of Cartan's method are stated (see Gardner [1989a], Olver [1995], Ivey and Landsberg [2003] for proofs and further details). We

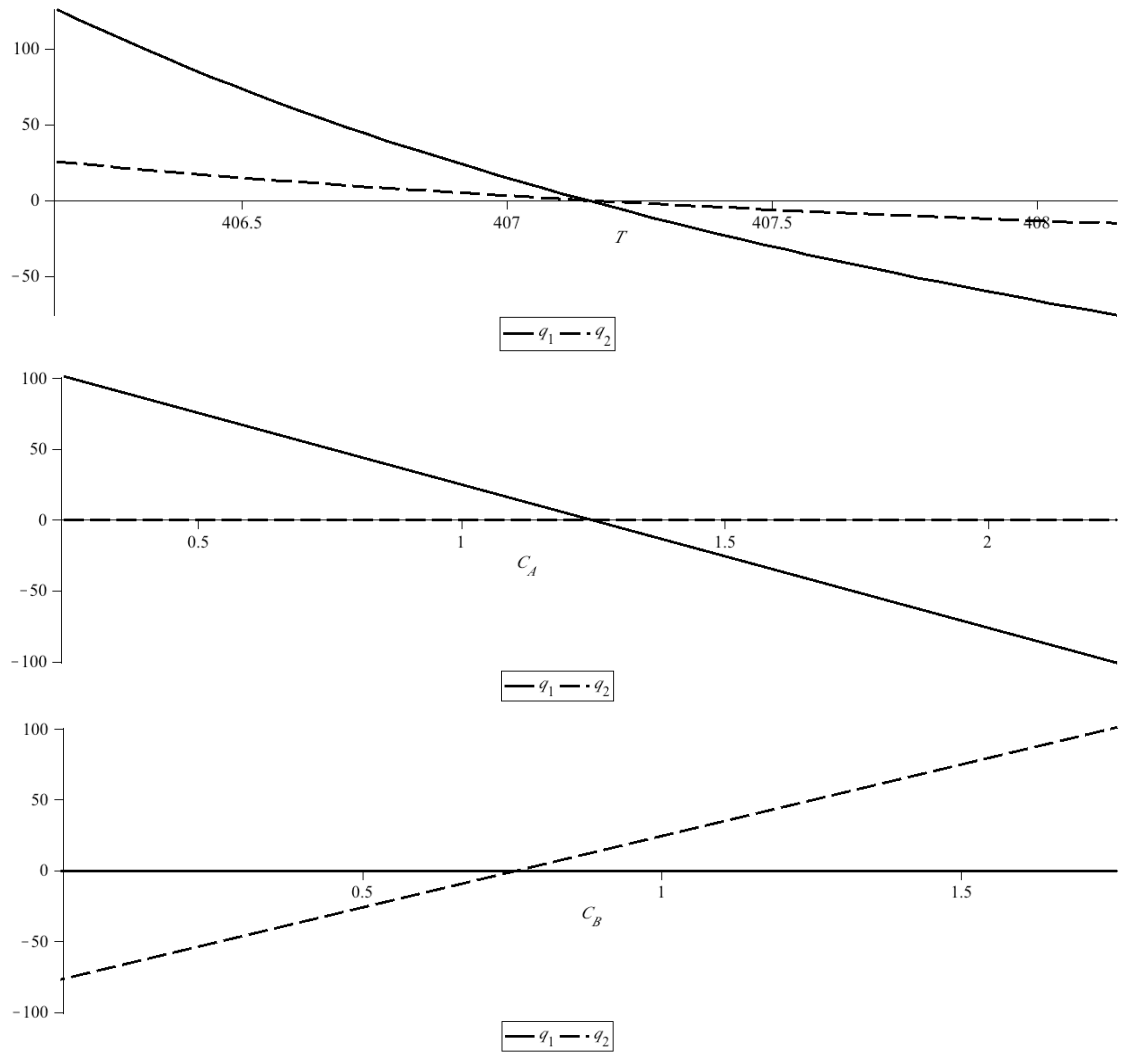
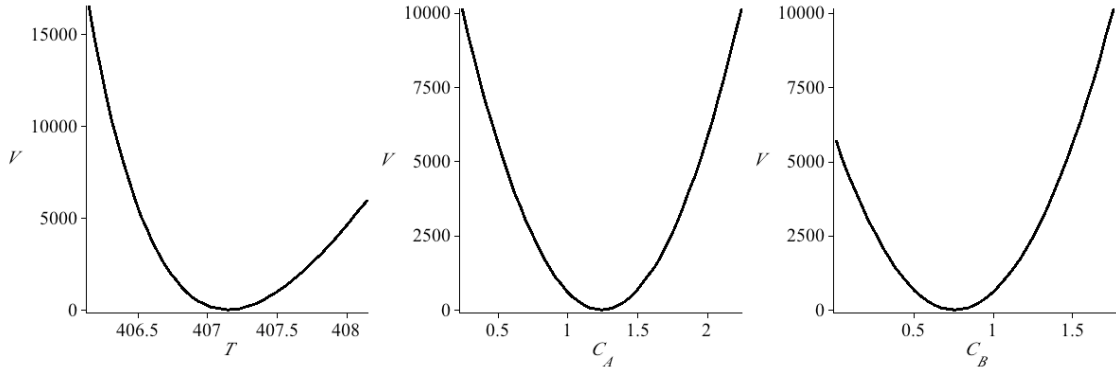


Figure 5.2: Dependency of (T, C_A, C_B) on (q_1, q_2)

define the structure equations and intrinsic torsion, and state results that allows us to reduce the equivalence problem to an equivalence problem with reduced structure group.

Let G be a Lie subgroup of $GL(n, \mathbb{R})$, $U \subset \mathbb{R}^n$, ω_U a coframe on U and $S : U \rightarrow G$ a smooth map. For our purpose it is beneficial to denote \mathbb{R}^n by V with basis $\{e_i\}_{i=1}^n$,


 Figure 5.3: Dependency of V on T , C_A and C_B

and denote by $\{f^i\}_{i=1}^n$ the basis of V^* dual to $\{e_i\}_{i=1}^n$. Furthermore, the duality pair between elements of V and V^* is denoted by $\langle \cdot, \cdot \rangle$. Hence, we write $\omega_U = \sum \omega_U^i e_i$. The lift of ω_U is then $\omega = S\pi^*\omega_U = S_j^i \omega_U^j e_i$. Let \mathfrak{g} denote the Lie algebra of G , suppose $\{\varepsilon_\alpha\}$ is a basis for \mathfrak{g} and $\{\pi^\beta\}$ right-invariant one-forms on G dual to $\{\varepsilon_\alpha\}$, then we have $\varepsilon_l = \sum a_{il}^j e_j \otimes f^i$.

Proposition 5.6.1. The exterior derivative of ω satisfies the *structure equations*

$$d\omega^i e_i = \sum a_{jl}^i \pi^l \wedge \omega^j + \sum \gamma_{jk}^i \omega^j \wedge \omega^k e_i \quad (5.6)$$

where the functions γ_{jk}^i are called the *torsion coefficients*

$$\gamma_{jk}^i = \sum (c_{jk}^i \circ \pi) \langle f^j, S^{-1} \cdot e_r \rangle \langle f^k, S^{-1} \cdot e_s \rangle \langle S^{-1} \cdot f^t, e_r \rangle$$

with c_{jk}^i given by $d\omega^i = \sum c_{jk}^i \omega^j \wedge \omega^k$.

Next we define the intrinsic torsion. Define the linear map $\mathbf{L} : \mathfrak{g} \otimes V^* \rightarrow \mathfrak{g} \otimes V^* \simeq V \otimes \wedge^2 V^*$ by $\nu = \sum \nu_k^l \varepsilon_l \otimes f^k \mapsto -\sum (a_{jl}^i \nu_k^l - a_{kl}^i \nu_j^l) e_i \otimes f^j \wedge f^k$. The *intrinsic torsion* is defined by $\tau_U(u, S) = \sum \gamma_{jk}^i(u, S) e_i \otimes f^j \wedge f^k + \text{Im } \mathbf{L}$. Furthermore, we define a

representation of G .

Definition 5.6.2. A *representation* of a group G is a group homomorphism $\rho : G \rightarrow \text{Aut}(V)$ from G to the space of invertible linear transformation of a vector space V . The dual of ρ is denoted by $\rho^\dagger : G \rightarrow \text{Aut}(V^*)$.

Lemma 5.6.3. Assume there exists an equivalence $\Phi^1 : U \times G \rightarrow V \times G$, then the intrinsic torsion is invariant, *i.e.*, $\tau_U = \tau_V \circ \Phi^1$.

Lemma 5.6.4. If the equivalence problem is of first-order constant type (Gardner [1989a]), then for each $u \in U$ there exists a $C(u) \in G$ such that $(\rho \otimes \wedge^2 \rho^\dagger)(C(u))\tau_U(u, e) = \tau_0$, where τ_0 is the normalized value of the intrinsic torsion and e denotes the identity element of G .

For a normalized value τ_0 of τ_U we can define the isotropy group $G_{\tau_0} \triangleq \{C \in G \mid \rho \otimes \wedge^2 \rho^\dagger(C)\tau_0 = \tau_0\}$. Further computations show that

$$\begin{aligned} \tau_U^{-1}(\tau_0) &\triangleq \{(u, s) \in U \times G \mid u \in U \text{ and } s \in G_{\tau_0}(C(u))\} \\ &= \{(u, G_{\tau_0}C(u)) \mid u \in U\} \end{aligned}$$

is a submersed submanifold of U .

Definition 5.6.5. A τ_0 -*modified coframe* is a section of $\tau_U^{-1}(\tau_0)$ which gives rise to a local section $\Gamma(p) = (p, \beta_M(p))$ of $U \times G$, where $\beta_U : U \rightarrow G$ satisfies $\tau_U(p, \beta_U(p)) = \tau_0$ ($\tau_U(\Gamma(p)) = \tau_0$).

Finally, we state one of the important results in Cartan's method.

Proposition 5.6.6. A map $\Phi : U \rightarrow V$ induces a G -equivalence if and only if Φ induces a G_{τ_0} -equivalence between τ_0 -modified coframes given by $\beta_U \omega_U$ and $\beta_V \Omega_V$.

Chapter 6

Conclusions

In this chapter, we summarize the contributions of this work and point out the strengths, as well as some of the weaknesses, of the results, and discuss future directions of research.

6.1 Summary

The success the IDA-PBC methodology is intrinsically connected to the ability of solving the matching problem. This makes it important to have a clear framework to describe the problem and develop new solutions. In this thesis, clarity is achieved by considering a geometric approach, hence working with an intrinsic definition of control systems, and in particular, port-Hamiltonian systems.

In Chapter 3, we established that linear port-Hamiltonian systems can be decomposed into reduced-order linear port-Hamiltonian systems, which are linear abstractions of the original system. The decomposition was defined intrinsically based on

the controllability matrices of the linear system. This is a first step towards a normal form for linear port-Hamiltonian control systems.

In Chapter 4, we considered control-affine system, and gave necessary and sufficient conditions for the existence of a solution to the matching equation, based on a coordinate-free description of the problem. Based on the assumption that the interconnection and damping structure is constant and additional assumptions on the control distribution, we derived conditions under which the matching problem can be solved. The assumptions have already been shown to be successful in specific applications [Johnsen and Allgöwer, 2007]. The procedure was, furthermore, adapted to yield a design with two degrees of freedom, where some degrees of freedom are separated into two directions, one influencing the dynamics tangential or horizontal to the equilibrium manifold, the other influencing the direction complementary or vertical to the equilibrium manifold. Our main result here are checkable conditions for a fixed, not necessarily constant structure tensor field, though we did not present a constructive procedure for finding a structure tensor. However, it is conceivable using the results presented in Chapter 4 that such a procedure could be developed. This is a topic for future research.

In Chapter 5, another approach to solving the matching problem was presented, based on the feedback equivalence of control systems. This allowed us to use the transitivity property of the feedback equivalence to define new structures tensor fields for which a solution to the matching equations exists. One benefit of the approaches presented is that a general adaption to a specific coframe simplifies the description

of the differential of the Hamiltonian function. A better understanding of the equivalence class of port-Hamiltonian systems and the associated symmetry group would strengthen the result.

6.2 Future Research Problems

In this section, we discuss some avenues for future work.

The results presented in Chapter 3 consider the decomposition of the interconnection structure of linear port-Hamiltonian systems. To extend the result to decomposition of damping structures, we have to satisfy both conditions in Proposition 3.2.14. This is analog to the construction of achievable Dirac structures with resistive element as discussed in Cervera et al. [2007]. Also, the concept of achievable Dirac structures is closely related to the control by interconnection methodology. Hence, the concept of achievable Casimir functions should appear naturally in the context of decomposition, reflecting additional degrees of freedom in the way a linear port-Hamiltonian system can be decomposed. The decomposition can also lead to a reduced order matching problem, since the controllability is preserved in the controllability abstractions and the existence of a solution to the matching problem depends on the stabilizability of the system. As already discussed in Chapter 3, the extension of the idea of decomposition is difficult to realize for control-affine systems. One possibility would be to consider partially feedback linearized systems such that the control-affine form is preserved for some controllability abstractions.

Based on the constant structure tensor assumption in Chapter 4, we propose a change of the structure tensor by means of $GL(n, \mathbb{R})$ acting on the space of $(0, 2)$ -tensor fields in the following way. Let F be a structure tensor field and define the action of $A \in G \subset GL(n, \mathbb{R})$ on F point-wise by $AF_p A^\top$ for all $p \in \mathcal{X}$, this action preserves the rank of the symmetrization. Furthermore, the effect of this action on the existence should be studied via the existence conditions presented in Section 4.2. A first step involves the characterization of the Lie subgroup for which the homogeneous solutions to the matching equations are preserved. Also, the symmetry group of the partial differential equation could be studied using the general framework presented in Olver [1993]. Furthermore, the results presented here may be useful for non-exact matching approaches, as presented in Section 2.2. Approximation techniques such as the homotopy operator approach, could be combined with the dynamic compensator proposed in Acosta and Astolfi [2009].

Following the method of equivalence, the equivalence class of systems for which the second order torsion terms are constant, and hence invariant, can be studied. Such a requirement imposes additional conditions on the first order derivatives of the structure tensor field and on the second order derivatives of the Hamiltonian function. It would be appropriate to begin with the constant structure tensor assumption discussed in Chapter 4.

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