Al-Azhar University-Gaza

Deanship of Postgraduate Studies

Faculty of Science

Department of Mathematics

ON THE STABILITY AND CONTROL OF VIBRATIONS IN NONLINEAR DYNAMICAL SYSTEMS

BY

NOURA ABDUL RAHIM SALEM

(B.Sc., Mathematics, Faculty of Education, Islamic University-Gaza, 2011)

A THESIS

Submitted In Partial Fulfillment of The Requirements for The Master Degree In Mathematics

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DECLARATION SHEET

DATE: / /2015

I declare that this whole work submitted for the degree of Master is the result of my own work, except where otherwise acknowledged in the text, and that this work has not been submitted for another degree at any other university or institution.

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Signature: ………………..

Dedication

To the sake of Allah, my Creator and my Master,

To my great teacher and messenger, Mohammed (May Allah bless and grant him), who taught us the purpose of life,

To my homeland Palestine, the warmest womb,

To the great martyrs and prisoners, the symbol of sacrifice,

To my great parents, who never stop giving me their love, support and encouragement throughout my entire life,

To my beloved brothers and sisters,

To my friends who encourage and support me,

To All those whose names I forget to mention,

I dedicate this research.

Acknowledgement

All praise goes to the Almighty Allah, the one to whom all dignity, honor and glory are due. Peace and blessing of Allah be upon all the prophets and messengers. As prophet Mohammad, peace of Allah be upon him, said: "Who does not thank people, will not thank Allah".

First and foremost, I must acknowledge my limitless thanks to Allah, the Ever-Magnificent; the Ever-Thankful, for His help and bless. I am totally sure that this work would have never become truth, without His guidance.

I would like to acknowledge my sincere thanks and gratitude to my supervisor: Dr. Usama Hegazy for his time, efforts, advices and his encouragement. I am really grateful for his willingness to help in reviewing the study so that it might come out to light.

In addition, I am deeply grateful to Al-Azhar University of Gaza and its staff for all the facilitations, help and advice they offered.

I also would like to express my wholehearted thanks to my family for their generous support they provided me throughout my entire life and particularly through the process of pursuing the master degree. Because of their unconditional love and prayers, I have the chance to complete this thesis.

I would like to take this opportunity to extend warm thanks to all my beloved friends, who have been so supportive along the way of doing my thesis.

Finally, all appreciations are due to those whose kindness, patience and support were the candles that enlightened my way towards success.

Abstract

In this study, different controllers have been applied to investigate and suppress the vibrations of a second-order nonlinear dynamical system. Active controllers such as the position feedback (PF), negative velocity feedback (VF) and negative cubic velocity feedback controller are related directly to the considered system. While Passive controllers such as the nonlinear saturation (NS) and positive position feedback (PPF) controllers involve a second nonlinear oscillator coupled with the main system. The system under investigation is subjected to external and parametric excitation forces. The method of multiple scales as one of the perturbation techniques is used to reduce the second-order nonlinear differential equation into a set of two first-order differential equations that govern the time variation of the amplitude and phase of the response, and obtain the response equation near various resonance cases. The stability of the system is investigated by applying frequency response equations and phase-plane. The numerical solution and the effects of the parameters on the vibrating system are studied and reported. The simulation results are achieved using Maple13 software.

ملخص الرسالة

في هذه الرسالة تم دراسة أنواع مختلفة من أنظمة وطرق التحكم في اهتزازات نظام ديناميكي لا خطي من الرتبة الثانية , النوع الأول ويشمل التحكم المباشر في النظام باستخدام التغذية الراجعة في اتجاه الازاحة, السرعة الخطية "العكسية" و السرعة التكعيبية "العكسية", أما النوع الثاني فهو يتمثل في التحكم عبر نظام اهتزازي لا خطي مرتبط مع النظام الديناميكي قيد الدراسة فيتكون نظام من زوج من المعادلات التفاضلية غير الخطية وقد تم تطبيق نظام التشبع غير الخطي ونظام التغذية الراجعة للإزاحة .

لقد تم استخدام طريقة الأزمنة المضطربة لإيجاد الحل التقريبي للنظام الديناميكي غير الخطي تحت تأثير قوى خارجية وبارا مترية والمتمثل في معادلة تفاضلية غير خطية من الرتبة الثانية والتي تم تحويلها إلى معادلتين تفاضليتين من الرتبة الأولى تصفان حركة الإزاحة والطور بالنسبة للزمن ومن ثم تم الحصول على معادلات الاستجابة عند حالات رنين مختلفة .

ان استقرار النظام قد تم دراسته باستخدام طريقة معادلات الاستجابة ومستوى الطور وكذلك تم دراسة والتعرف على تأثير البارامترات المختلفة في المعادلات على حركة واهتزازات النظام .

ونشير إلى أنه تم استخدام برنامج Maple13 في ايجاد ورسم جميع الحلول في هذه الدراسة .

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Nomenclature

Symbol	***Description***
u , u' , u''	Displacement, velocity and acceleration
v , v' , v''	Displacement, velocity and acceleration
ω , ω_s , ω_c	Natural frequencies of the system
μ_1 , ξ	Damping coefficient
ε	Small dimensionless perturbation parameter
β_1, β_2, δ , τ , ρ	Nonlinear coefficient
f_1, f_2	Forcing amplitude
Ω	Excitation frequency
T	Control input
G	The gain
α	Equal 30°
t	Time
T_0	Fast time scale
T_1	Slow time scale
D_0 , D_1	Differential operators
$A(T_1)$, $B(T_1)$	Complex valued quantity
cc	Complex conjugate for preceding terms at the same equation
σ_1 , σ_2	Detuning parameter
a , a_1 , a_2	Steady-state amplitudes
p_1 , p_2 , p_3 , p_4	Real coefficients
θ , θ_1 , θ_2	Phase angles of the polar forms

Chapter1

Introduction and Literature review

1.1 Introduction

Important advances in mathematics, physics, biology, engineering and economics have shown the importance of the analysis of nonlinear vibrations, stabilities and dynamical behavior.

A nonlinear system refers to a set of nonlinear equations (algebraic, differential, integral, functional, or abstract operator equations, or a combination of some of these) used to describe a physical device or process that otherwise cannot be clearly defined by a set of linear equations of any kind. Dynamical system is used as a synonym of mathematical or physical system when the describing equations represent evolution of a solution with time and, sometimes, with control inputs and/or other varying parameters as well.

Vibration and dynamic chaos, occurring in most machines, vehicles, building, aircraft and structures are undesired phenomenon. Not only because of the resulting unpleasant motions. The dynamic stresses which may lead to fatigue and failure of the structure or machine. The energy losses and reduction in performance which accompany vibrations, but also because of the produced noise. Noise is an undesirable event. And since sound is produced by some source of motion or vibration causing pressure changes which propagate through the air or other transmitting medium. Vibration control is of vital importance to sound attenuation. Vibration analysis of machines and structures is often a necessary prerequisite for controlling vibration and noise. The theory and techniques of vibration suppression have been extensively studied for many years. Various types of controller are developed so as to channel the excess energy from excitation to the slave system in order that vibration in the primary system can be suppressed. The positive position feedback (PPF), velocity feedback (VF), acceleration feedback (AF) and nonlinear saturation (NS) controllers used extensively for vibration reduction for many linear and nonlinear dynamical systems, which show their feasibility and efficiency in practice.

In numerical analysis, the fourth order Runge-Kutta method can be used to solve differential equations. it is defined for any initial value problem of the following type:

$$y' = f(t, y), \qquad y(t_0) = y_0. \tag{1.1}$$

Where y is an unknown function (scalar or vector) of time t, y' the rate at which y changes.

The definition of the RK4 method for the initial value problem in equation (1.1) is shown in equation (1.2).

$$y_{n+1} = y_n + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right), \tag{1.2}$$

with h the time step, and the coefficients k_1, k_2, k_3 and k_4 are defined as follows:

$$\begin{aligned} k_1 &= f(t_n, y_n), \\ k_2 &= f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1), \\ k_3 &= f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2), \\ k_4 &= f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_3). \end{aligned} \tag{1.3}$$

These coefficients indicate the slope of the function at three points in the time interval, the beginning, the mid-point and the end. The slope at the mid-point is estimated twice, first using the value of k_1 to determine k_2 next using the value of k_2 to compute k_3.

Knowing the k-coefficients, the solution at the next time step can be computed by equation (1.2).

1.2 Literature Review

Vibrations are the cause of discomfort, disturbance, damage, and sometimes destruction of machines and structures. It must be reduced or controlled or eliminated. One of the most common methods of vibration control is the dynamic absorber. It has the advantages of low cost and simple operation at one model frequency. In the domain of many mechanical vibration systems the coupled non-linear vibration of such systems can be reduced to non-linear second order differential equations which are solved analytically and numerically.

Elhefnawy and Bassiouny [1], studied the nonlinear instability problem of two superposed dielectric fluids by using the method of multiple scales. Frequency response curves are presented graphically. The stability of the proposed solution is determined. Numerical solutions were presented graphically for the effects of the different parameters on the system stability, response and chaos. The method of multiple time scale perturbation technique is applied to solve the nonlinear differential equations up to and including the third order approximation [2,3,4]. Nayfeh and Mook [5], studied system having a single degree of freedom, which concerned with introducing basic concepts and analytic methods, then the concepts and methods are extended by them to systems having multi-degrees of freedom. All possible resonance cases were extracted at third approximation order and investigated numerically. The effects of the different parameters on system behavior are studied. The stability of the system is investigated using both frequency response functions and phase plane methods. The solutions of the frequency response functions regarding the stability of the system are shown graphically. Phase plane was shown for the steady state amplitudes as a criterion for system stability and chaos presence [6]. El Behady and El-Zahar [7], studied the effect of the nonlinear controller on the vibrating system. The approximate solutions up to the second order are derived using the method of multiple scale perturbation technique near the primary, principal parametric and internal resonance case. Moreover, they investigated the stability of the solution using both phase plane method and frequency response equations, and the effects of different parameters on the vibration of the system. Warminski et. al. [8], studied active suppression of nonlinear composite beam vibrations by selected control algorithms.The saturation phenomenon has been the subject of extensively theoretical and experimental research [9–10]. Eissa et. al. [11,12], investigated a single-degree-of-freedom non-linear oscillating systems subject to multi-parametric and/or external excitations. The multiple time scale perturbation technique is applied to obtain solution up to the third order approximation to extract and study the available resonance cases. They reported the occurrence of saturation phenomena at different parameters values. Kwak and Heo [13], presented effectiveness of the PPF algorithm applied for a model of a solar panel, where the first four modes of vibration have been considered. Siewe and Hegazy [14], applied different active controllers to suppress the vibration of the micromechanical resonator system. Moreover, a time-varying stiffness was introduced to control the chaotic motion of the considered system. Different techniques were applied to analyze the periodic and chaotic motions. Eissa and Amer [15] and Yaman and Sen [16] studied the vibration control of a cantilever beam subject to both external and parametric excitation but with different controllers. Sayed [17], studied the effects of different active controllers on simple and spring pendulum at the primary resonance via negative velocity feedback or its square or cubic. Golnaraghi [18] indicated that when the system is excited at a frequency near the high natural frequency, the structure responds at the frequency of the excitation and the amplitude of the response increases with the excitation amplitude. Oueini et al. [19], proposed a non-linear control law to suppress the vibrations of the first mode of a cantilever beam when subjected to a principal parametric excitation, which is based on cubic velocity feedback to suppress the vibration. The method of multiple scales was used to derive two first-order differential equations governing the time evolution of the amplitude and phase of the response. Then, a bifurcation analysis was conducted to examine the stability of the closed-loop system and investigate the performance of the control law. The theoretical and experimental findings indicate that the control law leads to effective vibration suppression and bifurcation control. El-Serafi et al. [20,21] showed how effective is the active control in vibration reduction at resonance at different modes of vibration. They demonstrated the advantages of active control over the passive one. Hegazy [22] studied the nonlinear dynamics and vibration control of an electromechanical seismograph system with time-varying stiffness. An active control method is applied to the system based on cubic velocity feedback. In [23], Hegazy investigated The problem of suppressing the vibrations of a hinged–hinged flexible beam that is subjected to primary and principal parametric excitations. Different control laws are proposed, and saturation phenomenon is investigated to suppress the vibrations of the system. El-Ganaini et. al. [24] applied positive position feedback active controller to suppress the vibration of a nonlinear system when subjected to external primary resonance excitation. The multiple scale perturbation method is applied to obtain a first-order approximate solution. The equilibrium curves for various controller parameters are plotted. The stability of the steady state solution is investigated using frequency-response equations. The approximate solution was numerically verified. They found that all predictions from analytical solutions are in good agreement with the numerical simulation.

1.3 Objective of The Work

The objective of this work is to study analytically and numerically techniques and to reduce the oscillations of a nonlinear dynamical system using different control (the position feedback (PF), negative velocity feedback (VF), a negative cubic velocity feedback and the nonlinear saturation (NS) controllers). Moreover we use the phase plane and frequency response method to investigate the systems stability. This study will include the following systems.

- Nonlinear differential equation with direct "active" controls
  - Position Feedback (PF) controller

$$u'' + \mu_1 u' + \omega^2 u + \beta_1 u^3 + \beta_2 u^5 - \delta\left(uu'^2 + u^2 u''\right) = f_1 \cos(\Omega t)\cos(\alpha) + uf_2 \cos(\Omega t)\sin(\alpha) + T \,. \tag{1.4}$$

Where T is a control input, that will expressed, separately, as Gu, Gu^3 and Gu^5 to give a linear, cubic, and quintic PF controllers, respectively. G is a positive constant called the gain.

  - Negative Velocity Feedback (VF) controller

$$u'' + \mu_1 u' + \omega^2 u + \beta_1 u^3 + \beta_2 u^5 - \delta\left(uu'^2 + u^2 u''\right) = f_1 \cos(\Omega t)\cos(\alpha) + uf_2 \cos(\Omega t)\sin(\alpha) - T \,. \tag{1.5}$$

Where T will expressed, separately, as Gu', Gu'^2 and Gu'^3 to give a negative linear, quadratic, and cubic VF controllers.

  - Negative Acceleration Feedback (AF) controller

$$u'' + \mu_1 u' + \omega^2 u + \beta_1 u^3 + \beta_2 u^5 - \delta\left(uu'^2 + u^2 u''\right) = f_1 \cos(\Omega t)\cos(\alpha) + uf_2 \cos(\Omega t)\sin(\alpha) - Gu''. \tag{1.6}$$

- Nonlinear differential equation with indirect "passive" controls
  - Positive Position Feedback (PPF) controller

$$u'' + \mu_1 u' + \omega_s^2 u + \beta_1 u^3 + \beta_2 u^5 - \delta\left(uu'^2 + u^2 u''\right) = f_1 \cos(\Omega t)\cos(\alpha) + uf_2 \cos(\Omega t)\sin(\alpha) + \tau v \,, \tag{1.7}$$

$$v'' + 2\xi\omega_c v' + \omega_c^2 v = \rho u. \tag{1.8}$$

- Nonlinear Saturation (NS) controller

$$u'' + \mu_1 u' + \omega_s^2 u + \beta_1 u^3 + \beta_2 u^5 - \delta\left(uu'^2 + u^2 u''\right) = f_1 \cos(\Omega t)\cos(\alpha) + u f_2 \cos(\Omega t)\sin(\alpha) + \tau v^2, \tag{1.9}$$

$$v'' + 2\xi\omega_c v' + \omega_c^2 v = \rho u v. \tag{1.10}$$

Chapter 2

Active Control of a Nonlinear Dynamical System

In this chapter we will consider a system of second-order nonlinear ordinary differential equation and apply a different active controllers to reduce the vibrations of the system and choose some of best active controllers. The nonlinear system with the chosen controllers is solved and studied using 4th order Rung-Kutta numerical method and the method of Multiple Scales perturbation technique. The stability of the controlled system is also conducted.

2.1 System model:

The considered equation is the modified non-linear ordinary differential equation describing the vibration of inclined beam which is given by [16] :

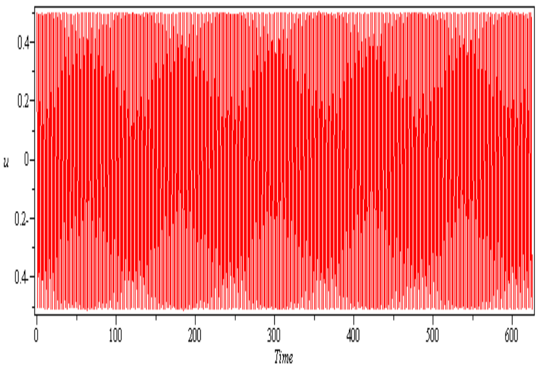
$$\begin{aligned} &u'' + \mu_1 u' + \omega^2 u + \beta_1 u^3 + \beta_2 u^5 - \delta\left(uu'^2 + u^2 u''\right) = f_1 \cos(\Omega t)\cos(\alpha) \\ &+ u f_2 \cos(\Omega t)\sin(\alpha) + T\,, \end{aligned} \tag{2.1}$$

where u, u' and u'' represent displacement, velocity and acceleration of the vibrating system, respectively, ω is the natural frequency, μ_1 is the damping coefficient , β_1, β_2 and δ are nonlinear coefficients, f_1 and f_2 are the forcing amplitude, Ω is the excitation frequency, $\alpha = 30^\circ$ and T is a control input.

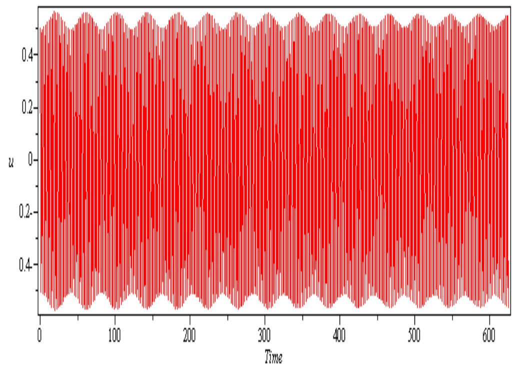
We will apply a different controllers and solve it by 4th order Rung-Kutta numerical method using Maple 13 then choose some of the best active controllers.

The different controllers are used to reduce the vibration of the considered system:

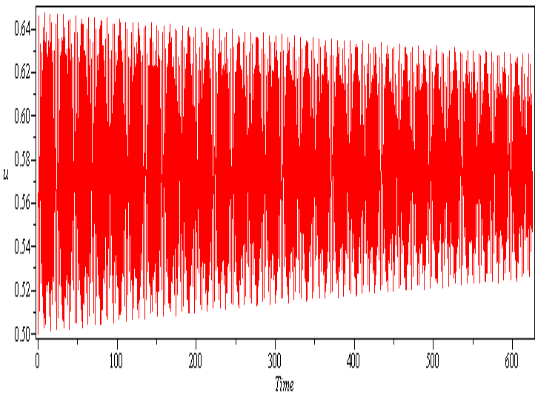
1. Position Feedback (PF) control $T = Gu$, this controller modifies the frequency of the system, where G is a positive constant called the gain.



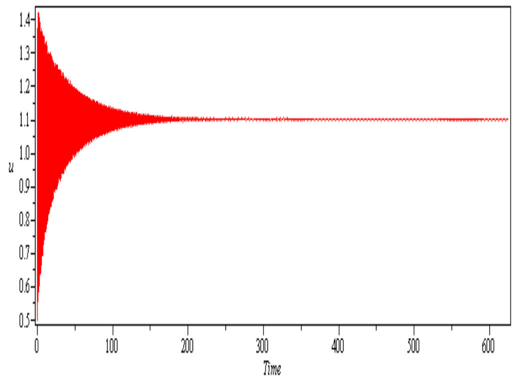
(a) $G = 0.05$



(b) $G = 0.5$



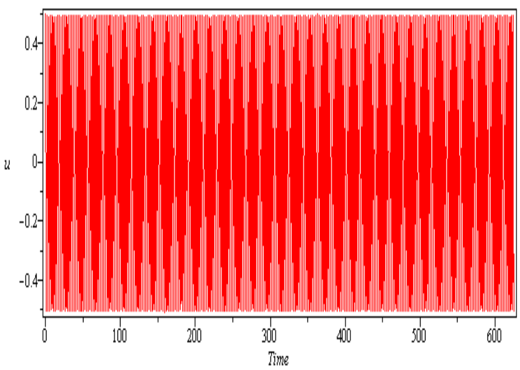
(c) $G = 10.0$



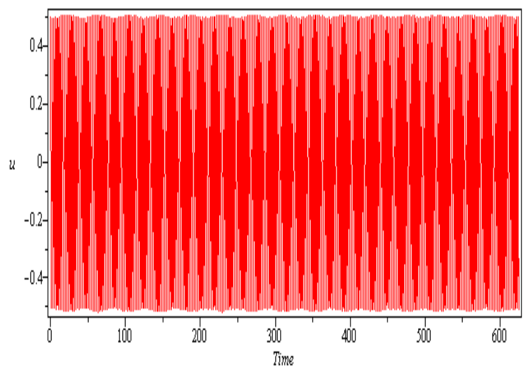
(d) $G = 30.0$

Fig. 2.1 Performance of (PF) controller for different values of the gain, ω=2.1, β_1=15.0, β_2=5.0, δ=0.03, μ_1=0.0005, Ω=2.7, f_1=0.4, f_2=0.2, α=30.0

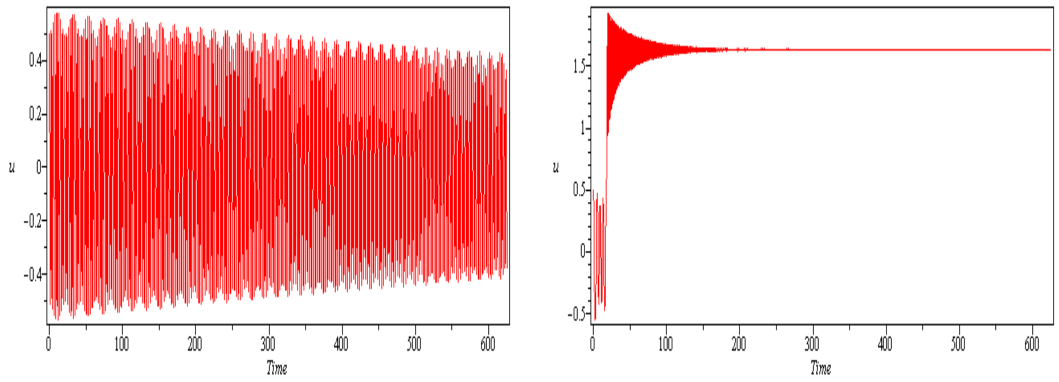
2. Cubic Position Feedback control $T = Gu^3$, this controller modifies $\beta_1 u^3$ due to non-linear curvature.



(a) $G = 0.05$



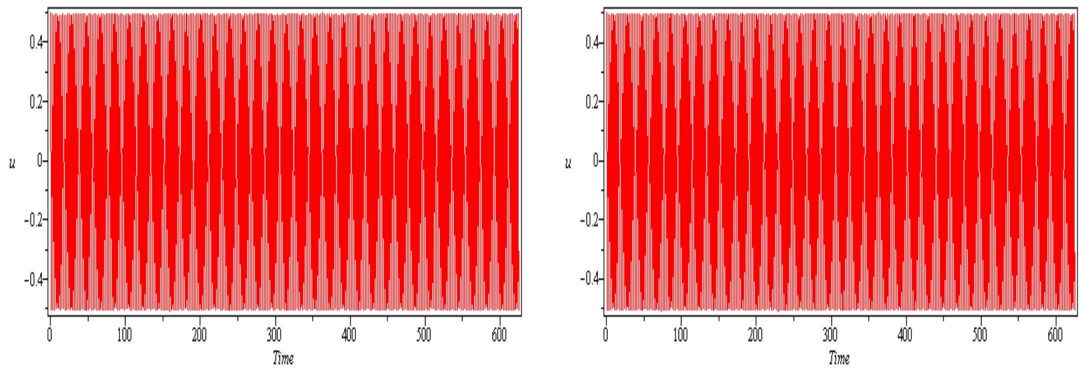
(b) $G = 0.5$



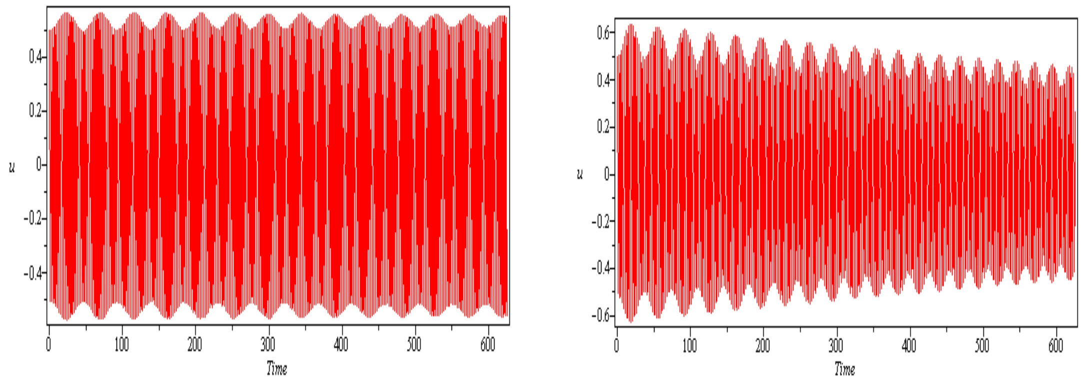
(c) $G = 10.0$ (d) $G = 30.0$

Fig. 2.2 Performance of cubic (PF) controller for different values of the gain, ω=2.1, β_1=15.0, β_2=5.0, δ=0.03, μ_1=0.0005, Ω=2.7, f_1=0.4, f_2=0.2, α=30.0

3. Quintic Position Feedback control $T = Gu^5$, this controller modifies $\beta_2 u^5$ due to non-linear curvature.



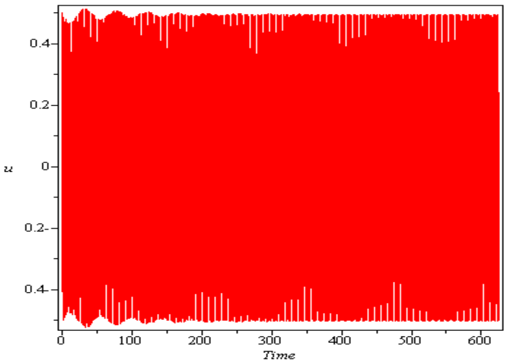
(a) $G = 0.05$ (b) $G = 0.5$



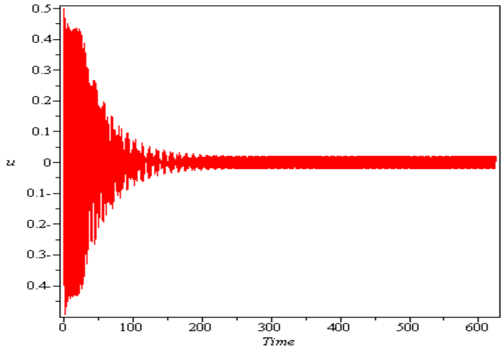
(c) $G = 10.0$ (d) $G = 30.0$

Fig. 2.3 Performance of quintic (PF) controller for different values of the gain, ω=2.1, β_1=15.0, β_2=5.0, δ=0.03, μ_1=0.0005, Ω=2.7, f_1=0.4, f_2=0.2, α=30.0

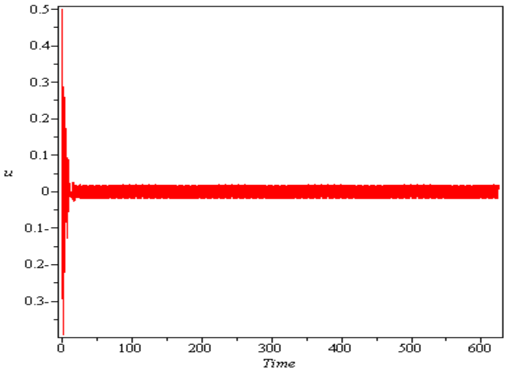
4. Negative Velocity Feedback (VF) control $T = -Gu'$, in this controller the damping of the system is modified.



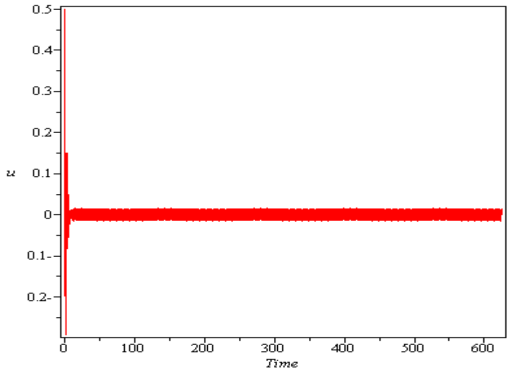
(a) $G = 0.02$



(b) $G = 0.05$



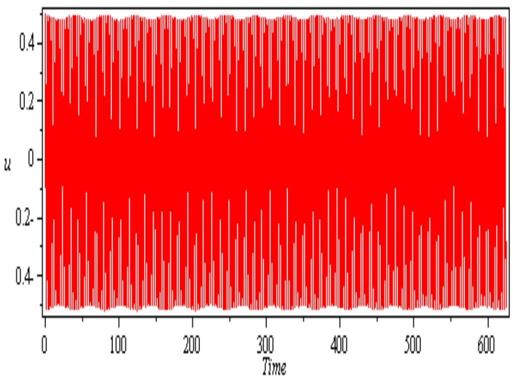
(c) $G = 0.5$



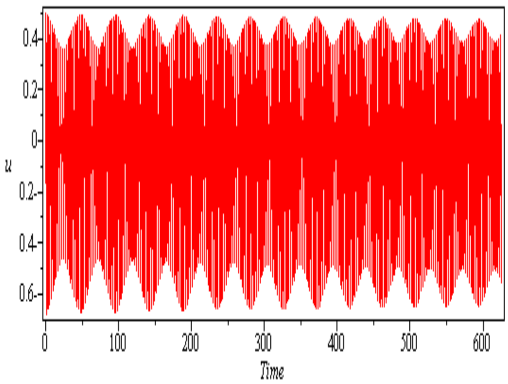
(d) $G = 1.0$

Fig. 2.4 Performance of negative (VF) controller for different values of the gain, ω=2.1, β_1=15.0, β_2=5.0, δ=0.03, μ_1=0.0005, Ω=2.7, f_1=0.4, f_2=0.2, α=30.0

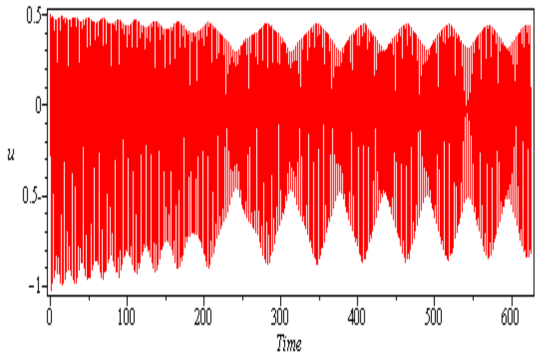
5. Negative Quadratic (VF) control $T = -Gu'^2$, in this controller the term $\delta uu'^2$ due to non-linear inertia of the system is modified.



(a) $G = 0.05$



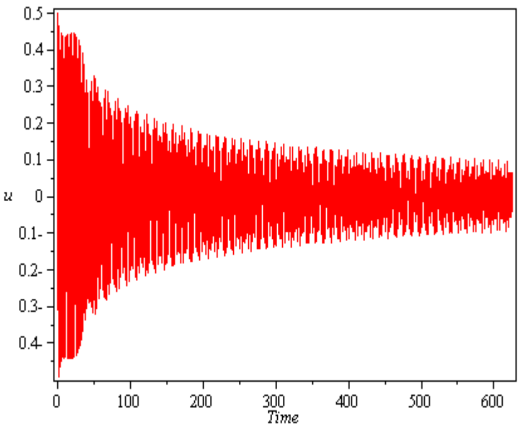
(b) $G = 0.5$



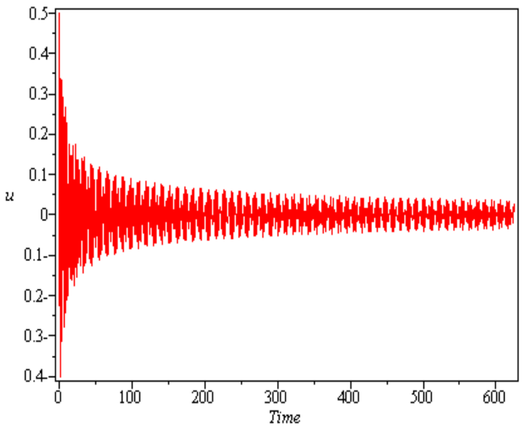
(c) $G = 1.0$

Fig. 2.5 Performance of negative quadratic (VF) controller for different values of the gain, ω=2.1, β_1=15.0, β_2=5.0, δ=0.03, μ_1=0.0005, Ω=2.7, f_1=0.4, f_2=0.2, α=30.0

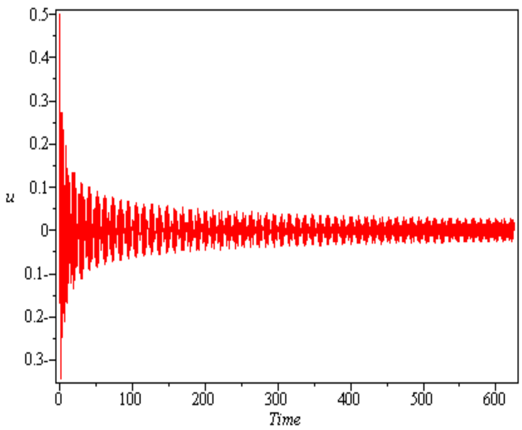
6. Negative cubic (VF) control $T = -Gu'^3$.



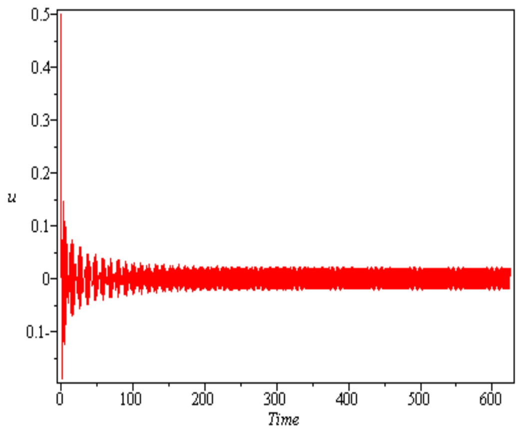
(a) $G = 0.05$



(b) $G = 0.5$



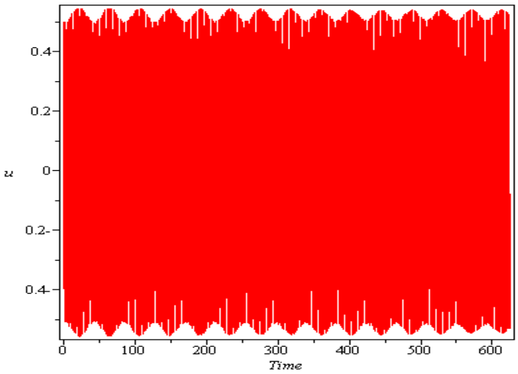
(c) $G = 1.0$



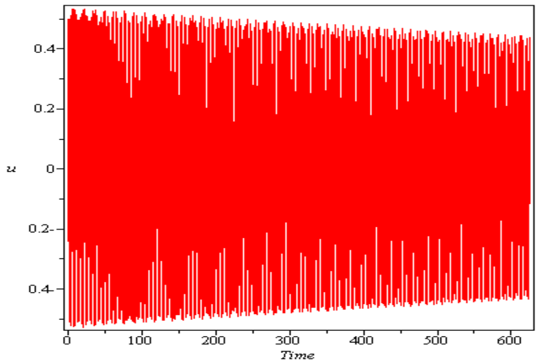
(d) $G = 5.0$

Fig. 2.6 Performance of negative cubic (VF) controller for different values of the gain, ω=2.1, β_1=15.0, β_2=5.0, δ=0.03, μ_1=0.0005, Ω=2.7, f_1=0.4, f_2=0.2, α=30.0

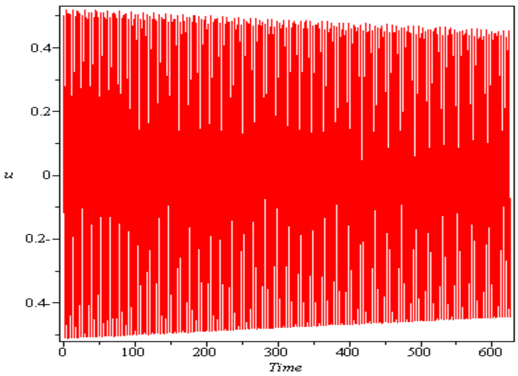
7. Negative Acceleration Feedback (AF) control $T = -Gu''$, which modifies the acceleration of the system.



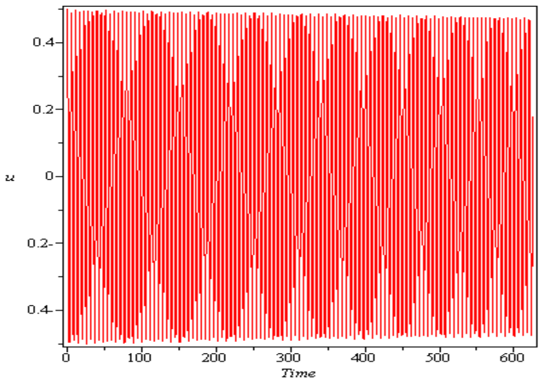
(a) $G = 0.05$



(b) $G = 0.5$



(c) $G = 1.0$



(d) $G = 5.0$

Fig. 2.7 Performance of negative (AF) controller for different values of the gain, ω=2.1, β_1=15.0, β_2=5.0, δ=0.03, μ_1=0.0005, Ω=2.7, f_1=0.4, f_2=0.2, α=30.0

The above figures show the effect of various active controllers for different values of the gain. In figure 2.1 more increase in G, for (PF) control lead to more decrease in the amplitude. In figure 2.2 cubic (PF) control is the same as (PF) control but with a few chaotic in the system. In figure 2.3 quintic (PF) control lead to small decrease in the amplitude. In figure 2.4 for negative (VF) control, it is clear that small values of the gain lead to significant decrease in the amplitude. In figure 2.5 increasing the gain, for negative quadratic (VF) control lead to chaotic behavior in the system . In figure 2.6 more increase in G, for negative cubic (VF) control would lead to more reduce in the amplitude. Figure 2.7 show that as the gain is increased the motion is changing to become stable but the amplitude is not.

So we will choose $T=-Gu'$ negative Velocity Feedback (VF), $T=-Gu'^3$ negative cubic (VF) and $T=Gu$ Position Feedback (PF) controllers as active internal controllers to investigate the behavior of the system analytically and numerically.

- $$u''+\mu_1 u'+\omega^2 u+\beta_1 u^3+\beta_2 u^5-\delta\left(uu'^2+u^2 u''\right)=f_1\cos(\Omega t)\cos(\alpha) +uf_2\cos(\Omega t)\sin(\alpha)+Gu. \tag{2.2}$$
- $$u''+\mu_1 u'+\omega^2 u+\beta_1 u^3+\beta_2 u^5-\delta\left(uu'^2+u^2 u''\right)=f_1\cos(\Omega t)\cos(\alpha) +uf_2\cos(\Omega t)\sin(\alpha)-Gu'. \tag{2.3}$$
- $$u''+\mu_1 u'+\omega^2 u+\beta_1 u^3+\beta_2 u^5-\delta\left(uu'^2+u^2 u''\right)=f_1\cos(\Omega t)\cos(\alpha) +uf_2\cos(\Omega t)\sin(\alpha)-Gu'^3. \tag{2.4}$$

2.2 Perturbation analysis for the nonlinear equation with (PF) control :

The nonlinear equation (2.2) with position feedback (PF) control is scaled using the perturbation parameter ε as follows

$$u''+\varepsilon\mu_1 u'+\omega^2 u+\varepsilon\beta_1 u^3+\varepsilon\beta_2 u^5-\varepsilon\delta\left(uu'^2+u^2 u''\right)=\varepsilon f_1\cos(\Omega t)\cos(\alpha) +\varepsilon uf_2\cos(\Omega t)\sin(\alpha)+\varepsilon Gu.$$

Applying the multiple scales method [2,3], we obtain first order approximate solutions for equation (2.2) by seeking the solutions in the form

$$u(t,\varepsilon)=u_0(T_0,T_1)+\varepsilon u_1(T_0,T_1), \tag{2.5}$$

where ε is a small dimensionless book keeping perturbation parameter, $T_0=t$ and $T_1=\varepsilon T_0=\varepsilon t$ are the fast and slow time scales, respectively, the time derivatives transform are recast in terms of the new time scales as

$$\frac{d}{dt}=D_0+\varepsilon D_1, \qquad \frac{d^2}{dt^2}=D_0^2+2\varepsilon D_0 D_1, \tag{2.6}$$

where $D_0=\dfrac{\partial}{\partial T_0}$, $D_1=\dfrac{\partial}{\partial T_1}$. (2.7)

Substituting u and time derivatives from equation (2.5) and (2.6)

$$u' = D_0 u_0 + \varepsilon D_0 u_1 + \varepsilon D_1 u_0 + \varepsilon^2 D_1 u_1,$$
$$u'' = D_0{}^2 u_0 + \varepsilon D_0{}^2 u_1 + 2\varepsilon D_0 D_1 u_0 + 2\varepsilon^2 D_0 D_1 u_1. \tag{2.8}$$

Substituting equation (2.8) into equation (2.2) we get,

$$\begin{aligned}
&D_0^2 u_0 + \varepsilon D_0^2 u_1 + 2\varepsilon D_0 D_1 u_0 + 2\varepsilon^2 D_0 D_1 u_1 + \omega^2 (u_0 + \varepsilon u_1) \\
&+\varepsilon \mu_1 (D_0 u_0 + \varepsilon D_0 u_1 + \varepsilon D_1 u_0 + \varepsilon^2 D_1 u_1) + \varepsilon \beta_1 (u_0 + \varepsilon u_1)^3 + \varepsilon \beta_2 (u_0 + \varepsilon u_1)^5 \\
&-\varepsilon \delta (u_0 + \varepsilon u_1)(D_0 u_0 + \varepsilon D_0 u_1 + \varepsilon D_1 u_0 + \varepsilon^2 D_1 u_1)^2 \\
&-\varepsilon \delta (u_0 + \varepsilon u_1)^2 (D_0^2 u_0 + \varepsilon D_0^2 u_1 + 2\varepsilon D_0 D_1 u_0 + 2\varepsilon^2 D_0 D_1 u_0) \\
&= \varepsilon f_1 \cos(\Omega t)\cos(\alpha) + \varepsilon (u_0 + \varepsilon u_1) f_2 \cos(\Omega t)\sin(\alpha) + \varepsilon G (u_0 + \varepsilon u_1).
\end{aligned} \tag{2.9}$$

Eliminating terms in which the powers of ε is more than or equal to 2 yields

$$\begin{aligned}
&D_0^2 u_0 + \varepsilon D_0^2 u_1 + 2\varepsilon D_0 D_1 u_0 + \varepsilon \mu_1 D_0 u_0 + \omega^2 u_0 + \varepsilon \omega^2 u_1 + \varepsilon \beta_1 u_0^3 + \varepsilon \beta_2 u_0^5 \\
&-2\varepsilon \delta D_0^2 u_0^3 - \varepsilon f_1 \cos(\Omega t)\cos(\alpha) - \varepsilon u_0 f_2 \cos(\Omega t)\sin(\alpha) - \varepsilon G u_0 = 0.
\end{aligned} \tag{2.10}$$

Equating the coefficient of same powers of ε in equation (2.10) gives

$$\begin{aligned}
&O(\varepsilon^0): D_0^2 u_0 + \omega^2 u_0 = 0, \\
&\Rightarrow \left(D_0^2 + \omega^2\right) u_0 = 0,
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
&O(\varepsilon^1): D_0^2 u_1 + 2 D_0 D_1 u_0 + \mu_1 D_0 u_0 + \omega^2 u_1 + \beta_1 u_0^3 + \beta_2 u_0^5 - 2\delta D_0^2 u_0^3 \\
&- f_1 \cos(\Omega t)\cos(\alpha) - u_0 f_2 \cos(\Omega t)\sin(\alpha) - G u_0 = 0.
\end{aligned} \tag{2.12}$$

Rearranging equation (2.12) to get,

$$\begin{aligned}
&\left(D_0^2 + \omega^2\right) u_1 = -2 D_0 D_1 u_0 - \mu_1 D_0 u_0 - \beta_1 u_0^3 - \beta_2 u_0^5 + 2\delta D_0^2 u_0^3 \\
&+ f_1 \cos(\Omega t)\cos(\alpha) + u_0 f_2 \cos(\Omega t)\sin(\alpha) + G u_0.
\end{aligned} \tag{2.13}$$

The general solution of (2.11) can be written in the form

$$u_0(T_0, T_1) = A(T_1) e^{i\omega T_0} + \bar{A}(T_1) e^{-i\omega T_0}, \tag{2.14}$$

where $A(T_1)$ is unknown function in T_1 .

In order to solve equation (2.12) for u_1, we substitute u_0 from equation (2.14) to get

$$\left(D_0^2+\omega^2\right)u_1=-2D_0D_1\left(Ae^{i\omega T_0}+\bar{A}e^{-i\omega T_0}\right)-\mu_1D_0\left(Ae^{i\omega T_0}+\bar{A}e^{-i\omega T_0}\right)$$
$$-\beta_1\left(Ae^{i\omega T_0}+\bar{A}e^{-i\omega T_0}\right)^3-\beta_2\left(Ae^{i\omega T_0}+\bar{A}e^{-i\omega T_0}\right)^5+2\delta D_0^2\left(Ae^{i\omega T_0}+\bar{A}e^{-i\omega T_0}\right)^3 \tag{2.15}$$
$$+f_1\cos(\Omega t)\cos(\alpha)+\left(Ae^{i\omega T_0}+\bar{A}e^{-i\omega T_0}\right)f_2\cos(\Omega t)\sin(\alpha)+G\left(Ae^{i\omega T_0}+\bar{A}e^{-i\omega T_0}\right),$$

which implies

$$\left(D_0^2+\omega^2\right)u_1=-2D_0D_1Ae^{i\omega T_0}-2D_0D_1\bar{A}e^{-i\omega T_0}-\mu_1D_0Ae^{i\omega T_0}-\mu_1D_0\bar{A}e^{-i\omega T_0}$$
$$-\beta_1A^3e^{3i\omega T_0}-3\beta_1A^2\bar{A}e^{i\omega T_0}-3\beta_1A\bar{A}^2e^{-i\omega T_0}-\beta_1\bar{A}^3e^{-3i\omega T_0}-\beta_2A^5e^{5i\omega T_0}$$
$$-5\beta_2A^4\bar{A}e^{3i\omega T_0}-10\beta_2A^3\bar{A}^2e^{i\omega T_0}-10\beta_2A^2\bar{A}^3e^{-i\omega T_0}-5\beta_2A\bar{A}^4e^{-3i\omega T_0}-\beta_2\bar{A}^5e^{-5i\omega T_0} \tag{2.16}$$
$$+2\delta D_0^2A^3e^{3i\omega T_0}+6\delta D_0^2A^2\bar{A}e^{i\omega T_0}+6\delta D_0^2A\bar{A}^2e^{-i\omega T_0}+2\delta D_0^2\bar{A}^3e^{-3i\omega T_0}$$
$$+f_1\cos(\Omega t)\cos(\alpha)+f_2Ae^{i\omega T_0}\cos(\Omega t)\sin(\alpha)+f_2\bar{A}e^{-i\omega T_0}\cos(\Omega t)\sin(\alpha)$$
$$+GAe^{i\omega T_0}+G\bar{A}e^{-i\omega T_0}.$$

Substituting equation (2.7) and using the form $\cos(\omega T_0)=\dfrac{e^{i\omega T_0}+e^{-i\omega T_0}}{2},\sin(\omega T_0)=\dfrac{e^{i\omega T_0}-e^{-i\omega T_0}}{2i}$ into equation (2.16), to get

$$\left(D_0^2+\omega^2\right)u_1=-2i\omega A'e^{i\omega T_0}+2i\omega\bar{A}'e^{-i\omega T_0}-\mu_1i\omega Ae^{i\omega T_0}+\mu_1i\omega\bar{A}e^{-i\omega T_0}-\beta_1A^3e^{3i\omega T_0}$$
$$-3\beta_1A^2\bar{A}e^{i\omega T_0}-3\beta_1A\bar{A}^2e^{-i\omega T_0}-\beta_1\bar{A}^3e^{-3i\omega T_0}-\beta_2A^5e^{5i\omega T_0}-5\beta_2A^4\bar{A}e^{3i\omega T_0}-10\beta_2A^3\bar{A}^2e^{i\omega T_0}$$
$$-10\beta_2A^2\bar{A}^3e^{-i\omega T_0}-5\beta_2A\bar{A}^4e^{-3i\omega T_0}-\beta_2\bar{A}^5e^{-5i\omega T_0}-18\omega^2\delta A^3e^{3i\omega T_0}-6\omega^2\delta A^2\bar{A}e^{i\omega T_0}$$
$$-6\omega^2\delta A\bar{A}^2e^{-i\omega T_0}-18\omega^2\delta\bar{A}^3e^{-3i\omega T_0}+\frac{1}{2}f_1e^{i\Omega T_0}\cos\left(\alpha\right)+\frac{1}{2}f_1e^{-i\Omega T_0}\cos\left(\alpha\right) \tag{2.17}$$
$$+\frac{1}{2}f_2Ae^{i\omega T_0+i\Omega T_0}\sin\left(\alpha\right)+\frac{1}{2}f_2Ae^{i\omega T_0-i\Omega T_0}\sin\left(\alpha\right)+\frac{1}{2}f_2\bar{A}e^{-i\omega T_0+i\Omega T_0}\sin\left(\alpha\right)$$
$$+\frac{1}{2}f_2\bar{A}e^{-i\omega T_0-i\Omega T_0}\sin\left(\alpha\right)+GAe^{i\omega T_0}+G\bar{A}e^{-i\omega T_0}.$$

Or simply,

$$\left(D_0^2+\omega^2\right)u_1=-2i\omega A'e^{i\omega T_0}-\mu_1i\omega Ae^{i\omega T_0}-\beta_1A^3e^{3i\omega T_0}-3\beta_1A^2\bar{A}e^{i\omega T_0}$$
$$-\beta_2A^5e^{5i\omega T_0}-5\beta_2A^4\bar{A}e^{3i\omega T_0}-10\beta_2A^3\bar{A}^2e^{i\omega T_0}-18\omega^2\delta A^3e^{3i\omega T_0}-6\omega^2\delta A^2\bar{A}e^{i\omega T_0} \tag{2.18}$$
$$+\frac{1}{2}f_1e^{i\Omega T_0}\cos\left(\alpha\right)+\frac{1}{2}f_2Ae^{i\omega T_0+i\Omega T_0}\sin\left(\alpha\right)+\frac{1}{2}f_2Ae^{i\omega T_0-i\Omega T_0}\sin\left(\alpha\right)+GAe^{i\omega T_0}+cc.$$

where cc denotes the complex conjugate terms.

Rearranging equation (2.18), to get

$$\left(D_0^2+\omega^2\right)u_1=\left(-2i\omega A'-\mu_1 i\omega A-3\beta_1 A^2\bar{A}-10\beta_2 A^3\bar{A}^2-6\omega^2\delta A^2\bar{A}+GA\right)e^{i\omega T_0}$$
$$+\left(-\beta_1 A^3-5\beta_2 A^4\bar{A}-18\omega^2\delta A^3\right)e^{3i\omega T_0}-\beta_2 A^5 e^{5i\omega T_0}+\frac{1}{2}f_1 e^{i\Omega T_0}\cos\left(\alpha\right) \tag{2.19}$$
$$+\frac{1}{2}f_2 Ae^{i(\omega+\Omega)T_0}\sin\left(\alpha\right)+\frac{1}{2}f_2 Ae^{i(\omega-\Omega)T_0}\sin\left(\alpha\right)+cc.$$

The particular solution of equation (2.19) can be written in the following form

$$u_1\left(T_0,T_1\right)=A_1\left(T_1\right)e^{i\omega T_0}-\frac{1}{8\omega^2}\left(-\beta_1 A^3-5\beta_2 A^4\bar{A}-18\omega^2\delta A^3\right)e^{3i\omega T_0}+\frac{1}{24\omega^2}\beta_2 A^5 e^{5i\omega T_0}$$
$$+\frac{1}{2\left(\omega-\Omega\right)\left(\omega+\Omega\right)}f_1\cos\left(\alpha\right)e^{i\Omega T_0}-\frac{1}{2\Omega\left(2\omega+\Omega\right)}f_2 Ae^{i(\omega+\Omega)T_0}\sin\left(\alpha\right) \tag{2.20}$$
$$+\frac{1}{2\Omega\left(2\omega-\Omega\right)}f_2 Ae^{i(\omega-\Omega)T_0}\sin\left(\alpha\right)+cc.$$

where A_1 is a function of T_1 to be determined in the next approximation .

From the equation (2.19), the reported resonance cases at this approximation order are

i. Primary resonance : $\Omega=\omega$
ii. Sub-harmonic resonance : $\Omega=2\omega$

2.3 Stability analysis

We will study the stability by considering the relation between the forcing frequency Ω and the natural frequency ω.

After studying perturbation analysis of the above system, we have two resonance cases,

2.3.1 Primary resonance $\Omega=\omega$:

In this case we introduce a detuning parameter σ_1 such that

$$\Omega=\omega+\varepsilon\sigma_1 \ , \tag{2.21}$$

Substituting equation (2.21) into equation (2.19), eliminating the terms that produce secular term and performing some algebraic manipulations, we obtain

$$-2i\omega A'-\mu_1 i\omega A-3\beta_1 A^2\bar{A}-10\beta_2 A^3\bar{A}^2-6\omega^2\delta A^2\bar{A}+GA+\frac{1}{2}f_1 e^{i\sigma_1 T_1}\cos\left(\alpha\right)=0, \tag{2.22}$$

Letting A as the polar form $A=\frac{1}{2}a(T_1)e^{i\theta(T_1)}$, where a and θ are the steady-state amplitude and the phases of the motions respectively, then we have

$$A=\frac{1}{2}ae^{i\theta}\quad,\quad A^2=\frac{1}{4}a^2e^{2i\theta}\quad,\quad A^3=\frac{1}{8}a^3e^{3i\theta},$$

$$\bar{A}=\frac{1}{2}ae^{-i\theta}\quad,\quad \bar{A}^2=\frac{1}{4}a^2e^{-2i\theta}\quad,\quad A'=\frac{1}{2}ai\,\theta' e^{i\theta}+\frac{1}{2}a'e^{i\theta}\quad.$$

Substituting $A,A^2,A^3,\bar{A},\bar{A}^2,A'$ in (2.22), we obtain

$$\begin{aligned}&\omega a\theta' e^{i\theta}-i\omega a' e^{i\theta}-\frac{1}{2}\mu_1 i\omega a e^{i\theta}-\frac{3}{8}\beta_1 a^3 e^{i\theta}-\frac{5}{16}\beta_2 a^5 e^{i\theta}-\frac{3}{4}\omega^2\delta a^3 e^{i\theta}\\&+\frac{1}{2}Gae^{i\theta}+\frac{1}{2}f_1 e^{i\sigma_1 T_1}\cos(\alpha)=0.\end{aligned}\tag{2.23}$$

Dividing equation (2.23) by $\omega e^{i\theta}$, we get

$$a\theta'-ia'-\frac{1}{2}\mu_1 ia-\frac{3}{8\omega}\beta_1 a^3-\frac{5}{16\omega}\beta_2 a^5-\frac{3}{4}\omega\delta a^3+\frac{1}{2\omega}Ga+\frac{1}{2\omega}f_1 e^{-i\theta+i\sigma_1 T_1}\cos(\alpha)=0.\tag{2.24}$$

Using the form $e^{ix}=\cos x+i\sin x$, to get

$$\begin{aligned}&a\theta'-ia'-\frac{1}{2}\mu_1 ia-\frac{3}{8\omega}\beta_1 a^3-\frac{5}{16\omega}\beta_2 a^5-\frac{3}{4}\omega\delta a^3+\frac{1}{2\omega}Ga\\&+\frac{1}{2\omega}f_1\cos(-\theta+\sigma_1 T_1)\cos(\alpha)+\frac{1}{2\omega}if_1\sin(-\theta+\sigma_1 T_1)\cos(\alpha)=0.\end{aligned}\tag{2.25}$$

Now equating the imaginary and real parts of equation (2.25) we obtain the following equations describing the modulation of amplitude and phase of the motions

$$a'=-\frac{1}{2}\mu_1 a+\frac{1}{2\omega}f_1\sin(-\theta+\sigma_1 T_1)\cos(\alpha),\tag{2.26}$$

And

$$a\theta'-\frac{3}{8\omega}\beta_1 a^3-\frac{5}{16\omega}\beta_2 a^5-\frac{3}{4}\omega\delta a^3+\frac{1}{2\omega}Ga+\frac{1}{2\omega}f_1\cos(-\theta+\sigma_1 T_1)\cos(\alpha)=0\,.\tag{2.27}$$

Sitting $\Lambda=\frac{1}{2\omega}f_1$, $\gamma=(-\theta+\sigma_1 T_1)$, equations (2.26) and (2.27), become as the following

$$a' = -\frac{1}{2}\mu_1 a + \Lambda \sin(\gamma)\cos(\alpha), \tag{2.28}$$

$$a\theta' - \frac{3}{8\omega}\beta_1 a^3 - \frac{5}{16\omega}\beta_2 a^5 - \frac{3}{4}\omega\delta a^3 + \frac{Ga}{2\omega} + \Lambda\cos(\gamma)\cos(\alpha) = 0\,. \tag{2.29}$$

since $\gamma' = -\theta' + \sigma_1$, (2.30)

then $a\theta' = a\sigma_1 - a\gamma'$. (2.31)

Substituting equation (2.31) into equation (2.29), gives

$$a\gamma' = \sigma_1 a - \frac{3}{8\omega}\beta_1 a^3 - \frac{5}{16\omega}\beta_2 a^5 - \frac{3}{4}\omega\delta a^3 + \frac{Ga}{2\omega} + \Lambda\cos(\gamma)\cos(\alpha),$$

(2.32)

For steady-state solutions, setting $a' = \gamma' = 0$, equation (2.28) becomes

$$\mu_1 a = 2\Lambda\sin(\gamma)\cos(\alpha), \tag{2.33}$$

and equation (2.32), becomes

$$2\sigma_1 a - \frac{3}{4\omega}\beta_1 a^3 - \frac{5}{8\omega}\beta_2 a^5 - \frac{3}{2}\omega\delta a^3 + \frac{Ga}{\omega} = -2\Lambda\cos(\gamma)\cos(\alpha)\ . \tag{2.34}$$

Squaring both sides of equations (2.33) , (3.34) and adding, we have

$$\mu_1^2 a^2 + \left(2a\sigma_1 - \frac{3}{4\omega}\beta_1 a^3 - \frac{5}{8\omega}\beta_2 a^5 - \frac{3}{2}\omega\delta a^3 + \frac{Ga}{\omega}\right)^2 = \left(2\Lambda\sin(\gamma)\cos(\alpha)\right)^2 + \left(-2\Lambda\cos(\gamma)\cos(\alpha)\right)^2, \tag{2.35}$$

$$\mu_1^2 a^2 + \left(2a\sigma_1 - \frac{3}{4\omega}\beta_1 a^3 - \frac{5}{8\omega}\beta_2 a^5 - \frac{3}{2}\omega\delta a^3 + \frac{Ga}{\omega}\right)^2 = 4\Lambda^2\sin^2(\gamma)\cos^2(\alpha) + 4\Lambda^2\cos^2(\gamma)\cos^2(\alpha), \tag{2.36}$$

$$\mu_1^2 a^2 + \left(2a\sigma_1 - \frac{3}{4\omega}\beta_1 a^3 - \frac{5}{8\omega}\beta_2 a^5 - \frac{3}{2}\omega\delta a^3 + \frac{Ga}{\omega}\right)^2 = 4\Lambda^2\cos^2(\alpha)\,. \tag{2.37}$$

Equation (2.37) is called the frequency response equation.

(a) Stability of trivial solution:

To determine the stability of the trivial solutions, we investigate the solutions of the linearized form of equation (2.22)

$$-2i\omega A' - \mu_1 i\omega A + GA = 0, \tag{2.38}$$

For stability analysis we expressed A in the Cartesian form

$$A = \frac{1}{2}\left(p_1 - ip_2\right)e^{i\phi T_1}, \tag{2.39}$$

where p_1 and p_2 are real.

Substituting A from equation (2.39) into equation (2.38), we get

$$\begin{aligned}&-2i\omega\left(\frac{1}{2}\left(p_1' - ip_2'\right)e^{i\phi T_1} + \frac{1}{2}i\phi\left(p_1 - ip_2\right)e^{i\phi T_1}\right) - \frac{1}{2}i\omega\mu_1\left(p_1 - ip_2\right)e^{i\phi T_1}\\ &+\frac{1}{2}G\left(p_1 - ip_2\right)e^{i\phi T_1} = 0.\end{aligned} \tag{2.40}$$

Dividing both sides of equation (2.40) by $\omega e^{i\phi T_1}$, to get

$$-ip_1' - p_2' + \phi p_1 - i\phi p_2 - \frac{1}{2}ip_1\mu_1 - \frac{1}{2}p_2\mu_1 + \frac{1}{2\omega}Gp_1 - \frac{1}{2\omega}iGp_2 = 0. \tag{2.41}$$

Separating real and imaginary parts, we get

$$p_1' = -\phi p_2 - \frac{1}{2}p_1\mu_1 - \frac{1}{2\omega}Gp_2, \tag{2.42}$$

And

$$p_2' = \phi p_1 - \frac{1}{2}p_2\mu_1 + \frac{1}{2\omega}Gp_1\ . \tag{2.43}$$

Rearranging equations (2.42), (2.43), gives

$$p_1' = \left(-\frac{1}{2}\mu_1\right)p_1 + \left(-\phi - \frac{1}{2\omega}G\right)p_2, \tag{2.44}$$

$$p_2' = \left(\phi + \frac{1}{2\omega}G\right)p_1 + \left(-\frac{1}{2}\mu_1\right)p_2. \tag{2.45}$$

The stability of the trivial solution is investigated by evaluating the eigenvalues of the Jacobian matrix of equations (2.44), (2.45)

$$\begin{bmatrix} p_1' \\ p_2' \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\mu_1 & -\phi - \frac{1}{2\omega}G \\ \phi + \frac{1}{2\omega}G & -\frac{1}{2}\mu_1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}.$$

The eigenvalues can be obtained by solving the determinant

$$\begin{vmatrix} -\frac{1}{2}\mu_1 - \lambda & -\phi - \frac{1}{2\omega}G \\ \phi + \frac{1}{2\omega}G & -\frac{1}{2}\mu_1 - \lambda \end{vmatrix} = 0 \quad ,$$

$$\lambda^2 + \mu_1\lambda + \frac{1}{4}\mu_1^2 + \left(\phi + \frac{1}{2\omega}G\right)^2 = 0 \ . \tag{2.46}$$

The solution of the equation (2.46) is

$$\lambda = -\frac{1}{2}\mu_1 \pm \sqrt{-\left(\phi + \frac{1}{2\omega}G\right)^2} \ . \tag{2.47}$$

The trivial solution is stable if $\lambda \le 0$, that is $\mu_1^2 \ge -4\left(\phi + \frac{1}{2\omega}G\right)^2$.

(b) Stability of non-trivial solution:

To determine the stability of the non-trivial solutions we let

$$a = a_0 + a_1(T_1), \text{ and } \gamma = \gamma_0 + \gamma_1(T_1). \tag{2.48}$$

Where a_0 and γ_0 correspond to a non-trivial solutions while a_1 and γ_1 are perturbation terms which are assumed to be small compared to a_0 and γ_0.

Substituting equation (2.48) into equations (2.28) and (2.32), using estimate $\sin\gamma_1 \approx \gamma_1$ and $\cos\gamma_1 \approx 1$, we get

$$a_0' + a_1' = -\frac{1}{2}\mu_1(a_0 + a_1) + \frac{1}{2\omega} f_1 \sin(\gamma_0 + \gamma_1)\cos(\alpha), \tag{2.49}$$

$$\begin{aligned}(a_0 + a_1)(\gamma_0' + \gamma_1') &= \sigma_1(a_0 + a_1) - \frac{3}{4\omega}\beta_1(a_0 + a_1)^3 - \frac{5}{8\omega}\beta_2(a_0 + a_1)^5 \\ &- \frac{3}{2}\omega\delta(a_0 + a_1)^3 + \frac{1}{2\omega}G(a_0 + a_1) + \frac{1}{2\omega} f_1 \cos(\gamma_0 + \gamma_1)\cos(\alpha).\end{aligned} \tag{2.50}$$

Simplifying equations (2.49), (2.50), to get

$$a_0' + a_1' = -\frac{1}{2}\mu_1 a_0 - \frac{1}{2}\mu_1 a_1 + \frac{1}{2\omega} f_1\left(\sin(\gamma_0) + \gamma_1\cos(\gamma_0)\right)\cos(\alpha), \tag{2.51}$$

$$\begin{aligned}a_0\gamma_0' + a_1\gamma_0' + a_0\gamma_1' + a_1\gamma_1' &= a_0\sigma_1 + a_1\sigma_1 - \frac{3}{4\omega}\beta_1\left(a_0^3 + 3a_0^2 a_1 + \ldots\right) \\ &- \frac{5}{8\omega}\beta_2\left(a_0^5 + 5a_0^4 a_1 + \ldots\right) - \frac{3}{2}\omega\delta\left(a_0^3 + 3a_0^2 a_1 + \ldots\right) + \frac{1}{2\omega}Ga_0 + \frac{1}{2\omega}Ga_1 \\ &+ \frac{1}{2\omega} f_1\left(\cos(\gamma_0) - \gamma_1\sin(\gamma_0)\right)\cos(\alpha).\end{aligned} \tag{2.52}$$

Since a_0 and g_0 are solutions of equation (2.28) and (2.32) then equation (2.51) and (2.52), becomes

$$a_1' = -\frac{1}{2}\mu_1 a_1 + \frac{1}{2\omega} f_1\gamma_1\cos(\gamma_0)\cos(\alpha), \tag{2.53}$$

$$\begin{aligned}a_1\gamma_0' + a_0\gamma_1' + a_1\gamma_1' &= a_1\sigma_1 - \frac{9}{4\omega}\beta_1 a_0^2 a_1 - \frac{25}{8\omega}\beta_2 a_0^4 a_1 - \frac{9}{2}\omega\delta a_0^2 a_1 + \frac{1}{2\omega}Ga_1 \\ &- \frac{1}{2\omega} f_1\gamma_1\sin(\gamma_0)\cos(\alpha).\end{aligned} \tag{2.54}$$

Substituting from equations (2.33) , (2.34) into equations (2.53) , (2.54), we get

$$a_1' = -\frac{1}{2}\mu_1 a_1 - \frac{1}{2}\gamma_1\left(2\sigma_1 a_0 - \frac{3}{4\omega}\beta_1 a_0^3 - \frac{5}{8\omega}\beta_2 a_0^5 - \frac{3}{2}\omega\delta a_0^3 + \frac{Ga_0}{\omega}\right), \tag{2.55}$$

$$a_1\gamma_0' + a_0\gamma_1' + a_1\gamma_1' = a_1\sigma_1 - \frac{9}{4\omega}\beta_1 a_0^2 a_1 - \frac{25}{8\omega}\beta_2 a_0^4 a_1 - \frac{9}{2}\omega\delta a_0^2 a_1 + \frac{1}{2\omega}Ga_1 - \frac{1}{2}\mu_1 a_0\gamma_1. \tag{2.56}$$

Simplifying equations (2.55) , (2.56), gives

$$a_1' = \left(-\frac{1}{2}\mu_1\right)a_1 + \left(-\sigma_1 a_0 + \frac{3}{8\omega}\beta_1 a_0^3 + \frac{5}{16\omega}\beta_2 a_0^5 + \frac{3}{4}\omega\delta a_0^3 - \frac{Ga_0}{2\omega}\right)\gamma_1\ , \tag{2.57}$$

$$a_0\gamma_1' + \left(\gamma_0' + \gamma_1'\right)a_1 = \left(\sigma_1 - \frac{9}{4\omega}\beta_1 a_0^2 - \frac{25}{8\omega}\beta_2 a_0^4 - \frac{9}{2}\omega\delta a_0^2 + \frac{1}{2\omega}G\right)a_1 + \left(-\frac{1}{2}\mu_1 a_0\right)\gamma_1 . \quad (2.58)$$

Dividing equation (2.58) by a_0 and using $\gamma_0' + \gamma_1' = \gamma' = 0$, we get

$$\gamma_1' = \left(\frac{\sigma_1}{a_0} - \frac{9}{4\omega}\beta_1 a_0 - \frac{25}{8\omega}\beta_2 a_0^2 - \frac{9}{2}\omega\delta a_0 + \frac{G}{2\omega a_0}\right)a_1 + \left(-\frac{1}{2}\mu_1\right)\gamma_1 \quad . \quad (2.59)$$

We can put equations (2.57) and (2.59) as the following form

$$a_1' = \Gamma_1 a_1 + \Gamma_2 \gamma_1 \ , \quad \gamma_1' = \Gamma_3 a_1 + \Gamma_1 \gamma_1 \quad (2.60)$$

Where $\Gamma_1 = -\frac{1}{2}\mu_1$, $\Gamma_2 = -\sigma_1 a_0 + \frac{3}{8\omega}\beta_1 a_0^3 + \frac{5}{16\omega}\beta_2 a_0^5 + \frac{3}{4}\omega\delta a_0^3 - \frac{G a_0}{2\omega}$,

$$\Gamma_3 = \frac{\sigma_1}{a_0} - \frac{9}{4\omega}\beta_1 a_0 - \frac{25}{8\omega}\beta_2 a_0^2 - \frac{9}{2}\omega\delta a_0 + \frac{G}{2\omega a_0} \ .$$

The eigenvalues can be obtained by solving the determinant of the Jacobian matrix of the equation (2.60)

$$\begin{bmatrix} a_1' \\ \gamma_1' \end{bmatrix} = \begin{bmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_3 & \Gamma_1 \end{bmatrix} \begin{bmatrix} a_1 \\ \gamma_1 \end{bmatrix}.$$

The eigenvalues can be obtained by solving the determinant

$$\begin{vmatrix} \Gamma_1 - \lambda & \Gamma_2 \\ \Gamma_3 & \Gamma_1 - \lambda \end{vmatrix} = 0 ,$$

$$\lambda^2 - \lambda\left(2\Gamma_1\right) + \Gamma_1^2 - \Gamma_2\Gamma_3 = 0 . \quad (2.61)$$

The eigenvalues of equation (2.61) are

$$\lambda = \Gamma_1 \pm \sqrt{\Gamma_2\Gamma_3} \ . \quad (2.62)$$

Therefore the steady-state solutions are stable if and only if $\Gamma_1^2 \le \Gamma_2\Gamma_3$.

2.3.2 Sub-harmonic resonance : $\Omega = 2\omega$

In this case we introduce a detuning parameter σ_2

$$\Omega = 2\omega + \varepsilon\sigma_2 \ , \tag{2.63}$$

Substituting equation (2.63) into equation (2.19), eliminating the terms that produce secular term and performing some algebraic manipulations, we obtain

$$-2i\omega A' - \mu_1 i\omega A - 3\beta_1 A^2\bar{A} - 10\beta_2 A^3\bar{A}^2 - 6\omega^2\delta A^2\bar{A} + GA + \frac{1}{2} f_2\bar{A}e^{i\sigma_2 T_1}\sin(\alpha) = 0 \, , \tag{2.64}$$

Letting A in the polar form $A = \frac{1}{2}a(T_1)e^{i\theta(T_1)}$, where a and θ are the steady-state amplitude and the phases of the motions respectively, then we have

$$\begin{aligned} &\omega a\theta' e^{i\theta} - i\omega a' e^{i\theta} - \frac{1}{2}\mu_1 i\omega a e^{i\theta} - \frac{3}{8}\beta_1 a^3 e^{i\theta} - \frac{5}{16}\beta_2 a^5 e^{i\theta} - \frac{3}{4}\omega^2\delta a^3 e^{i\theta} \\ &+ \frac{1}{2}Gae^{i\theta} + \frac{1}{4} f_2 a e^{-i\theta + i\sigma_2 T_1}\sin(\alpha) = 0. \end{aligned} \tag{2.65}$$

Dividing equation (2.65) by $\omega e^{i\theta}$, to get

$$\begin{aligned} &a\theta' - ia' - \frac{1}{2}\mu_1 ia - \frac{3}{8\omega}\beta_1 a^3 - \frac{5}{16\omega}\beta_2 a^5 - \frac{3}{4}\omega\delta a^3 + \frac{1}{2\omega}Ga \\ &+ \frac{1}{4\omega} f_2 a e^{-2i\theta + i\sigma_2 T_1}\sin(\alpha) = 0. \end{aligned} \tag{2.66}$$

Using the form $e^{ix} = \cos x + i\sin x$,we get

$$\begin{aligned} &a\theta' - ia' - \frac{1}{2}\mu_1 ia - \frac{3}{8\omega}\beta_1 a^3 - \frac{5}{16\omega}\beta_2 a^5 - \frac{3}{4}\omega\delta a^3 + \frac{1}{2\omega}Ga \\ &+ \frac{1}{4\omega} f_2 a\cos(-2\theta + \sigma_2 T_1)\sin(\alpha) + \frac{1}{4\omega} if_2 a\sin(-2\theta + \sigma_2 T_1)\sin(\alpha) = 0. \end{aligned} \tag{2.67}$$

Now equating the imaginary and real parts of equation (2.67), we obtain the following equations describing the modulation of amplitude and phase of the motions

$$-a' - \frac{1}{2}\mu_1 a + \frac{1}{4\omega} f_2 a\sin(-2\theta + \sigma_2 T_1)\sin(\alpha) = 0 \, , \tag{2.68}$$

And

$$a\theta' - \frac{3}{8\omega}\beta_1 a^3 - \frac{5}{16\omega}\beta_2 a^5 - \frac{3}{4}\omega\delta a^3 + \frac{1}{2\omega}Ga + \frac{1}{4\omega} f_2 a\cos\left(-2\theta + \sigma_2 T_1\right)\sin\left(\alpha\right) = 0 \ . \tag{2.69}$$

Sitting $\Lambda_1 = \frac{1}{4\omega} af_2$, $\gamma_2 = (-2\theta + \sigma_2 T_1)$, then equations (2.68) and (2.69) becomes

$$a' = -\frac{1}{2}\mu_1 a + \Lambda_1 \sin\left(\gamma_2\right)\sin\left(\alpha\right), \tag{2.70}$$

$$2a\theta' - \frac{3}{4\omega}\beta_1 a^3 - \frac{5}{8\omega}\beta_2 a^5 - \frac{3}{2}\omega\delta a^3 + \frac{Ga}{\omega} + 2\Lambda_1 \cos\left(\gamma_2\right)\sin\left(\alpha\right) = 0 \,. \tag{2.71}$$

Since $\gamma_2' = -2\theta' + \sigma_2$, then we have

$$2a\theta' = a\sigma_2 - a\gamma_2'. \tag{2.72}$$

Substituting equation (2.72) into equation (2.69), to get

$$a\gamma_2' = \sigma_2 a - \frac{3}{4\omega}\beta_1 a^3 - \frac{5}{8\omega}\beta_2 a^5 - \frac{3}{2}\omega\delta a^3 + \frac{Ga}{\omega} + 2\Lambda_1 \cos\left(\gamma_2\right)\sin\left(\alpha\right). \tag{2.73}$$

For steady-state solution, setting $a' = \gamma_2' = 0$ equation (2.70) and (2.73) becomes

$$\mu_1 a = 2\Lambda_1 \sin\left(\gamma_2\right)\sin\left(\alpha\right), \tag{2.74}$$

$$\sigma_2 a - \frac{3}{4\omega}\beta_1 a^3 - \frac{5}{8\omega}\beta_2 a^5 - \frac{3}{2}\omega\delta a^3 + \frac{Ga}{\omega} = -2\Lambda_1 \cos\left(\gamma_2\right)\sin\left(\alpha\right). \tag{2.75}$$

Squaring both sides of equation (2.74) and (2.75)and adding, we have

$$\mu_1^2 a^2 + \left(\sigma_2 a - \frac{3}{4\omega}\beta_1 a^3 - \frac{5}{8\omega}\beta_2 a^5 - \frac{3}{2}\omega\delta a^3 + \frac{Ga}{\omega}\right)^2 = \left(2\Lambda_1 \sin\left(\gamma_2\right)\sin\left(\alpha\right)\right)^2 + \left(-2\Lambda_1 \cos\left(\gamma_2\right)\sin\left(\alpha\right)\right)^2 . \tag{2.76}$$

More simply,

$$\mu_1^2 a^2 + \left(\sigma_2 a - \frac{3}{4\omega}\beta_1 a^3 - \frac{5}{8\omega}\beta_2 a^5 - \frac{3}{2}\omega\delta a^3 + \frac{Ga}{\omega}\right)^2 = 4\Lambda_1^2 \sin^2\left(\alpha\right). \tag{2.77}$$

Equation (2.77) is called the frequency response equation.

(a) Stability of trivial solution :

To determine the stability of the trivial solutions, we investigate the solutions of the linearized form of equation (2.64)

$$-2i\omega A' - \mu_1 i\omega A + GA + \frac{1}{2} f_2 \bar{A} e^{i\sigma_2 T_1} \sin(\alpha) = 0, \tag{2.78}$$

For stability analysis we expressed A in the Cartesian form

$$A = \frac{1}{2}(p_1 - ip_2) e^{i\phi T_1}, \tag{2.79}$$

where p_1 and p_2 are real.

Substituting A from equation (2.79) into equation (2.78), we get

$$\begin{aligned} &-2i\omega\left(\frac{1}{2}(p_1' - ip_2')e^{i\phi T_1} + \frac{1}{2}i\phi(p_1 - ip_2)e^{i\phi T_1}\right) - \frac{1}{2}i\omega\mu_1(p_1 - ip_2)e^{i\phi T_1} \\ &+ \frac{1}{2}G(p_1 - ip_2)e^{i\phi T_1} + \frac{1}{4} f_2 (p_1 + ip_2) e^{-i\phi T_1 + i\sigma_2 T_1} \sin(\alpha) = 0. \end{aligned} \tag{2.80}$$

Dividing both sides of equation (2.80) by $\omega e^{i\phi T_1}$ and simplifying , we get

$$\begin{aligned} &-ip_1' - p_2' + \phi p_1 - i\phi p_2 - \frac{1}{2}ip_1\mu_1 - \frac{1}{2}p_2\mu_1 + \frac{1}{2\omega}Gp_1 - \frac{1}{2\omega}iGp_2 \\ &+ \frac{1}{4\omega} f_2 p_1 e^{-2i\phi T_1 + i\sigma_2 T_1} \sin(\alpha) + \frac{1}{4\omega} if_2 p_2 e^{-2i\phi T_1 + i\sigma_2 T_1} \sin(\alpha) = 0. \end{aligned} \tag{2.81}$$

Using the form $e^{ix} = \cos x + i \sin x$, separating real and imaginary parts, we get

$$\begin{aligned} p_1' &= -\phi p_2 - \frac{1}{2}p_1\mu_1 - \frac{1}{2\omega}Gp_2 + \frac{1}{4\omega} f_2 p_1 \sin(-2\phi T_1 + \sigma_2 T_1)\sin(\alpha) \\ &+ \frac{1}{4\omega} f_2 p_2 \cos(-2\phi T_1 + \sigma_2 T_1)\sin(\alpha), \end{aligned} \tag{2.82}$$

And

$$\begin{aligned} p_2' &= \phi p_1 - \frac{1}{2}p_2\mu_1 + \frac{1}{2\omega}Gp_1 + \frac{1}{4\omega} f_2 p_1 \cos(-2\phi T_1 + \sigma_2 T_1)\sin(\alpha) \\ &- \frac{1}{4\omega} f_2 p_2 \sin(-2\phi T_1 + \sigma_2 T_1)\sin(\alpha). \end{aligned} \tag{2.83}$$

Sitting $\vartheta_1 = (-2\phi T_1 + \sigma_2 T_1)$, gives

$$p_1' = \left(-\frac{1}{2}\mu_1 + \frac{1}{4\omega} f_2 \sin(\vartheta_1)\sin(\alpha) \right) p_1 + \left(-\phi - \frac{1}{2\omega} G + \frac{1}{4\omega} f_2 \cos(\vartheta_1)\sin(\alpha) \right) p_2 , \qquad (2.84)$$

$$p_2' = \left(\phi + \frac{1}{2\omega} G + \frac{1}{4\omega} f_2 \cos(\vartheta_1)\sin(\alpha) \right) p_1 + \left(-\frac{1}{2}\mu_1 - \frac{1}{4\omega} f_2 \sin(\vartheta_1)\sin(\alpha) \right) p_2 . \qquad (2.85)$$

Sitting $\Gamma_4 = -\frac{1}{2}\mu_1 + \frac{1}{4\omega} f_2 \sin(\vartheta_1)\sin(\alpha)$, $\Gamma_5 = -\phi - \frac{1}{2\omega} G + \frac{1}{4\omega} f_2 \cos(\vartheta_1)\sin(\alpha)$,

$$\Gamma_6 = \phi + \frac{1}{2\omega} G + \frac{1}{4\omega} f_2 \cos(\vartheta_1)\sin(\alpha) \quad , \quad \Gamma_7 = -\frac{1}{2}\mu_1 + \frac{1}{4\omega} f_2 \sin(\vartheta_1)\sin(\alpha) .$$

The stability of the trivial solution is investigated by evaluating the eigenvalues of the Jacobian matrix of equations (2.84), (2.85) gives

$$\begin{bmatrix} p_1' \\ p_2' \end{bmatrix} = \begin{bmatrix} \Gamma_4 & \Gamma_5 \\ \Gamma_6 & \Gamma_7 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} .$$

The eigenvalues can be obtained by solving the determinant

$$\begin{vmatrix} \Gamma_4 - \lambda & \Gamma_5 \\ \Gamma_6 & \Gamma_7 - \lambda \end{vmatrix} = 0 ,$$

$$\lambda^2 - \lambda(\Gamma_4 + \Gamma_7) + \Gamma_4\Gamma_7 - \Gamma_5\Gamma_6 = 0 . \qquad (2.86)$$

The trivial solution is stable if $\Gamma_4\Gamma_7 - \Gamma_5\Gamma_6 \le 0$.

(b) Stability of non-trivial solution :

To determine the stability of the non-trivial solutions we let

$$a = a_0 + a_1(T_1) \text{, and } h = h_0 + h_1(T_1) . \qquad (2.87)$$

Where a_0 and h_0 correspond to a non-trivial solutions while a_1 and h_1 are perturbation terms which are assumed to be small compared to a_0 and h_0 .

Substituting equation (2.87) into equation (2.70), (2.73) where $h = \gamma_2$, using estimate $\sin h_1 \simeq h_1$ and $\cos h_1 \simeq 1$, to get

$$a_0' + a_1' = -\frac{1}{2}\mu_1\left(a_0 + a_1\right) + \frac{1}{4\omega} f_2\left(a_0 + a_1\right)\sin\left(h_0 + h_1\right)\sin\left(\alpha\right), \tag{2.88}$$

$$\begin{aligned}\left(a_0 + a_1\right)\left(h_0' + h_1'\right) &= \sigma_2\left(a_0 + a_1\right) - \frac{3}{4\omega}\beta_1\left(a_0 + a_1\right)^3 - \frac{5}{8\omega}\beta_2\left(a_0 + a_1\right)^5 - \frac{3}{2}\omega\delta\left(a_0 + a_1\right)^3 \\ &+ \frac{1}{\omega}G\left(a_0 + a_1\right) + \frac{1}{2\omega} f_2\left(a_0 + a_1\right)\cos\left(h_0 + h_1\right)\sin\left(\alpha\right).\end{aligned} \tag{2.89}$$

Simplifying the above equations, gives

$$\begin{aligned}a_0' + a_1' &= -\frac{1}{2}\mu_1 a_0 - \frac{1}{2}\mu_1 a_1 + \frac{1}{4\omega} f_2 a_0\left(\sin h_0 + h_1\cos h_0\right)\sin\left(\alpha\right) \\ &+ \frac{1}{4\omega} f_2 a_1\left(\sin h_0 + h_1\cos h_0\right)\sin\left(\alpha\right),\end{aligned} \tag{2.90}$$

$$\begin{aligned}a_0h_0' + a_1h_0' + a_0h_1' + a_1h_1' &= a_0\sigma_2 + a_1\sigma_2 - \frac{3}{4\omega}\beta_1\left(a_0^3 + 3a_0^2a_1 + \ldots\right) - \frac{5}{8\omega}\beta_2\left(a_0^5 + 5a_0^4a_1 + \ldots\right) \\ &- \frac{3}{2}\omega\delta\left(a_0^3 + 3a_0^2a_1 + \ldots\right) + \frac{Ga_0}{\omega} + \frac{Ga_1}{\omega} + \frac{1}{2\omega} f_2 a_0\left(\cos h_0 - h_1\sin h_0\right)\sin\left(\alpha\right) \\ &+ \frac{1}{2\omega} f_2 a_1\left(\cos h_0 - h_1\sin h_0\right)\sin\left(\alpha\right).\end{aligned} \tag{2.91}$$

Since a_0 and h_0 are solutions of equations (2.70) and (2.73) then equations (2.90) , (2.91) , become

$$\begin{aligned}a_1' &= -\frac{1}{2}\mu_1 a_1 + \frac{1}{4\omega} f_2 a_0 h_1\cos\left(h_0\right)\sin\left(\alpha\right) + \frac{1}{4\omega} f_2 a_1\sin\left(h_0\right)\sin\left(\alpha\right) \\ &+ \frac{1}{4\omega} f_2 a_1 h_1\cos\left(h_0\right)\sin\left(\alpha\right),\end{aligned} \tag{2.92}$$

$$\begin{aligned}a_0h_1' + a_1\left(h_1' + h_0'\right) &= a_1\sigma_2 - \frac{9}{4\omega}\beta_1 a_0^2 a_1 - \frac{25}{8\omega}\beta_2 a_0^4 a_1 - \frac{9}{2}\omega\delta a_0^2 a_1 + \frac{Ga_1}{\omega} \\ &- \frac{1}{2\omega} f_2 a_0 h_1\sin\left(h_0\right)\sin\left(\alpha\right) + \frac{1}{2\omega} f_2 a_1\cos\left(h_0\right)\sin\left(\alpha\right) - \frac{1}{2\omega} f_2 a_1 h_1\sin\left(h_0\right)\sin\left(\alpha\right).\end{aligned} \tag{2.93}$$

Now since a_1h_1 is a very small term and $h_0' + h_1' = h' = 0$ then they can be eliminated

Thus equations (2.92) , (2.93) can expressed as

$$a_1' = -\frac{1}{2}\mu_1 a_1 + \frac{1}{4\omega} f_2 a_0 h_1\cos\left(h_0\right)\sin\left(\alpha\right) + \frac{1}{4\omega} f_2 a_1\sin\left(h_0\right)\sin\left(\alpha\right), \tag{2.94}$$

$$a_0 h_1' = a_1\sigma_2 - \frac{9}{4\omega}\beta_1 a_0^2 a_1 - \frac{25}{8\omega}\beta_2 a_0^4 a_1 - \frac{9}{2}\omega\delta a_0^2 a_1 + \frac{G a_1}{\omega} - \frac{1}{2\omega} f_2 a_0 h_1 \sin(h_0)\sin(\alpha)$$
$$+\frac{1}{2\omega} f_2 a_1 \cos(h_0)\sin(\alpha). \tag{2.95}$$

Substituting from equations (2.74) , (2.75) into equations (2.94) , (2.95) and simplifying , to get

$$a_1' = \left(-\frac{1}{2}\sigma_2 a_0 + \frac{3}{8\omega}\beta_1 a_0^3 + \frac{5}{16\omega}\beta_2 a_0^5 + \frac{3}{4}\omega\delta a_0^3 - \frac{G a_0}{2\omega}\right) h_1 , \tag{2.96}$$

$$h_1' = \left(-\frac{3}{2\omega}\beta_1 a_0 - \frac{5}{2\omega}\beta_2 a_0^3 - 3\omega\delta a_0 - \frac{G}{2\omega a_0}\right) a_1 + (-\mu_1) h_1 . \tag{2.97}$$

We can put equations (2.96) and (2.97) as the following form

$$a_1' = h_1\Gamma_8 \ , \quad h_1' = a_1\Gamma_9 + h_1\Gamma_{10} \ . \tag{2.98}$$

Where $\Gamma_8 = -\frac{1}{2}\sigma_2 a_0 + \frac{3}{8\omega}\beta_1 a_0^3 + \frac{5}{16\omega}\beta_2 a_0^5 + \frac{3}{4}\omega\delta a_0^3 - \frac{G a_0}{2\omega}$,

, $\Gamma_9 = -\frac{3}{2\omega}\beta_1 a_0 - \frac{5}{2\omega}\beta_2 a_0^3 - 3\omega\delta a_0 - \frac{G}{2\omega a_0}$, $\Gamma_{10} = -\mu_1$.

The non-trivial solution is stable if and only if the real parts of equation (2.98) are less than or equal to zero using the Jacbian matrix mothed to solve the equation

$$\begin{bmatrix} a_1' \\ h_1' \end{bmatrix} = \begin{bmatrix} 0 & \Gamma_8 \\ \Gamma_9 & \Gamma_{10} \end{bmatrix} \begin{bmatrix} a_1 \\ h_1 \end{bmatrix}.$$

The eigenvalues can be obtained by solving the determinant

$$\begin{vmatrix} -\lambda & \Gamma_8 \\ \Gamma_9 & \Gamma_{10} - \lambda \end{vmatrix} = 0 ,$$

$$\lambda^2 - \lambda\Gamma_{10} - \Gamma_8\Gamma_9 = 0 . \tag{2.99}$$

The eigenvalues of equation (2.99) are

$$\lambda = \frac{\Gamma_{10} \pm \sqrt{\Gamma_{10}^2 + 4\Gamma_8\Gamma_9}}{2} \quad . \tag{2.100}$$

Therefore the steady-state solutions are stable if and only if $\Gamma_8\Gamma_9 \le 0$.

2.4 Perturbation analysis for the nonlinear equation with negative (VF) control :

The nonlinear equation (2.3) with negative velocity feedback (VF) control is scaled using the perturbation parameter ε as follows

$$u'' + \varepsilon\mu_1 u' + \omega^2 u + \varepsilon\beta_1 u^3 + \varepsilon\beta_2 u^5 - \varepsilon\delta\left(uu'^2 + u^2u''\right) = \varepsilon f_1 \cos(\Omega t)\cos(\alpha)$$
$$+\varepsilon u f_2 \cos(\Omega t)\sin(\alpha) - \varepsilon G u'$$

Applying the multiple scales mothed, similarly as in the perturbation analysis equations (2.5) – (2.9) , we have

$$\begin{aligned} &D_0^2 u_0 + \varepsilon D_0^2 u_1 + 2\varepsilon D_0 D_1 u_0 + \varepsilon\mu_1 D_0 u_0 \\ &+\omega^2 u_0 + \varepsilon\omega^2 u_1 + \varepsilon\beta_1 u_0^3 + \varepsilon\beta_2 u_0^5 - 2\varepsilon\delta D_0^2 u_0^3 \\ &-\varepsilon f_1 \cos(\Omega t)\cos(\alpha) - \varepsilon u_0 f_2 \cos(\Omega t)\sin(\alpha) + \varepsilon G D_0 u_0 = 0. \end{aligned} \tag{2.101}$$

Equating the coefficient of same powers of ε in equation (2.101), to get

$$O(\varepsilon^0) : \left(D_0^2 + \omega^2\right)u_0 = 0\,, \tag{2.102}$$

$$\begin{aligned} O(\varepsilon^1) : \left(D_0^2 + \omega^2\right)u_1 &= -2D_0D_1u_0 - \mu_1 D_0 u_0 - \beta_1 u_0^3 - \beta_2 u_0^5 + 2\delta D_0^2 u_0^3 \\ &+ f_1 \cos(\Omega t)\cos(\alpha) + u_0 f_2 \cos(\Omega t)\sin(\alpha) - G D_0 u_0. \end{aligned} \tag{2.103}$$

The general solution of (2.102) is given by

$$u_0 = A(T_1)e^{i\omega T_0} + \bar{A}(T_1)e^{-i\omega T_0}\,, \tag{2.104}$$

where $A(T_1)$ is unknown function in T_1 at this stage of the analysis.

Now to solve equation (2.103), substituting equation (2.104) into it then substituting equation (2.7), and using the form $\cos(\omega T_0) = \dfrac{e^{i\omega T_0} + e^{-i\omega T_0}}{2}$, $\sin(\omega T_0) = \dfrac{e^{i\omega T_0} - e^{-i\omega T_0}}{2i}$, to get this simplified equation,

$$\left(D_0^2+\omega^2\right)u_1=\left(-2i\omega A'-\mu_1 i\omega A-3\beta_1 A^2\bar{A}-10\beta_2 A^3\bar{A}^2-6\omega^2\delta A^2\bar{A}-Gi\omega A\right)e^{i\omega T_0}$$
$$+\left(-\beta_1 A^3-5\beta_2 A^4\bar{A}-18\omega^2\delta A^3\right)e^{3i\omega T_0}-\beta_2 A^5 e^{5i\omega T_0}+\frac{1}{2}f_1 e^{i\Omega T_0}\cos\left(\alpha\right) \tag{2.105}$$
$$+\frac{1}{2}f_2 A e^{i(\omega+\Omega)T_0}\sin\left(\alpha\right)+\frac{1}{2}f_2 A e^{i(\omega-\Omega)T_0}\sin\left(\alpha\right)+cc,$$

where cc denotes the complex conjugate terms.

The particular solution of equation (2.105) can be written in the following form

$$u_1\left(T_0,T_1\right)=A_1\left(T_1\right)e^{i\omega T_0}-\frac{1}{8\omega^2}\left(-\beta_1 A^3-5\beta_2 A^4\bar{A}-18\omega^2\delta A^3\right)e^{3i\omega T_0}+\frac{1}{24\omega^2}\beta_2 A^5 e^{5i\omega T_0}$$
$$+\frac{1}{2\left(\omega-\Omega\right)\left(\omega+\Omega\right)}f_1\cos\left(\alpha\right)e^{i\Omega T_0}-\frac{1}{2\Omega\left(2\omega+\Omega\right)}f_2 A e^{i(\omega+\Omega)T_0}\sin\left(\alpha\right) \tag{2.106}$$
$$+\frac{1}{2\Omega\left(2\omega-\Omega\right)}f_2 A e^{i(\omega-\Omega)T_0}\sin\left(\alpha\right)+cc.$$

From the equation (2.106), the reported resonance cases at this approximation order are

a. Primary resonance $\Omega=\omega$:
b. Sub-harmonic resonance $\Omega=2\omega$:

2.5 Stability analysis

2.5.1 Primary resonance $\Omega=\omega$:

To describe the nearness of excitation frequency Ω to frequency of the natural frequency ω introducing the detuning parameter σ_1 such that

$$\Omega=\omega+\varepsilon\sigma_1 \ , \tag{2.107}$$

Substituting equation (2.107) into equation (2.105), eliminating the terms that produce secular term and performing some algebraic manipulations, we obtain

$$-2i\omega A'-\mu_1 i\omega A-3\beta_1 A^2\bar{A}-10\beta_2 A^3\bar{A}^2-6\omega^2\delta A^2\bar{A}-Gi\omega A+\frac{1}{2}f_1 e^{i\sigma_1 T_1}\cos\left(\alpha\right)=0\ . \tag{2.108}$$

Substituting $A=\frac{1}{2}a(T_1)e^{i\theta(T_1)}$ similarly in equations (2.22) – (2.25) , we obtain the following equations describing the modulation of amplitude and phase of the motions

$$a' = -\frac{1}{2}\mu_1 a - \frac{1}{2}Ga + \Lambda\sin(\gamma)\cos(\alpha), \tag{2.109}$$

$$a\theta' - \frac{3}{8\omega}\beta_1 a^3 - \frac{5}{16\omega}\beta_2 a^5 - \frac{3}{4}\omega\delta a^3 + \Lambda\cos(\gamma)\cos(\alpha) = 0, \tag{2.110}$$

where $\Lambda = \frac{1}{2\omega} f_1$, $\gamma = (-\theta + \sigma_1 T_1)$.

since $\gamma' = -\theta' + \sigma_1$,

then $a\theta' = a\sigma_1 - a\gamma'$. (2.111)

Substituting equation (2.111) into equation (2.110), to get

$$a\gamma' = \sigma_1 a - \frac{3}{8\omega}\beta_1 a^3 - \frac{5}{16\omega}\beta_2 a^5 - \frac{3}{4}\omega\delta a^3 + \Lambda\cos(\gamma)\cos(\alpha). \tag{2.112}$$

For steady-state solutions, setting $a' = \gamma' = 0$, equation (2.109) and (2.112) become

$$\mu_1 a + Ga = 2\Lambda\sin(\gamma)\cos(\alpha) , \tag{2.113}$$

$$2\sigma_1 a - \frac{3}{4\omega}\beta_1 a^3 - \frac{5}{8\omega}\beta_2 a^5 - \frac{3}{2}\omega\delta a^3 = -2\Lambda\cos(\gamma)\cos(\alpha). \tag{2.114}$$

From equation (2.113) and (2.114), we have

$$(\mu_1 a + Ga)^2 + \left(2a\sigma_1 - \frac{3}{4\omega}\beta_1 a^3 - \frac{5}{8\omega}\beta_2 a^5 - \frac{3}{2}\omega\delta a^3\right)^2 = 4\Lambda^2\cos^2(\alpha). \tag{2.115}$$

Equation (2.115) is called the frequency response equation.

(a) Stability of trivial solution:

To determine the stability of the trivial solutions, we investigates the solutions of the linearized form of equation (2.108)

$$-2i\omega A' - \mu_1 i\omega A - Gi\omega A = 0. \tag{2.116}$$

For stability analysis we expressed A in the Cartesian form and substituting similarly as equation (2.38) – (2.43), we have

$$p_1' = \left(-\frac{1}{2}\mu_1 - \frac{1}{2}G\right)p_1 + (-\phi)\,p_2 \ , \tag{2.117}$$

$$p_2' = (\phi)\,p_1 + \left(-\frac{1}{2}\mu_1 - \frac{1}{2}G\right)p_2 . \tag{2.118}$$

The stability of the trivial solution is investigated by evaluating the eigenvalues of the Jacobian matrix of equations (2.117), (2.118) gives

$$\begin{vmatrix} -\frac{1}{2}\mu_1 - \frac{1}{2}G - \lambda & -\phi \\ \phi & -\frac{1}{2}\mu_1 - \frac{1}{2}G - \lambda \end{vmatrix} = 0 ,$$

$$\lambda^2 + (\mu_1 + G)\lambda + \frac{1}{4}\mu_1^2 + \frac{1}{4}G^2 + \frac{1}{2}\mu_1 G + \phi^2 = 0 . \tag{2.119}$$

The solution of the equation (2.119) is

$$\lambda = -\frac{1}{2}(\mu_1 + G) \pm \sqrt{\left(-\phi^2\right)} \ . \tag{2.120}$$

The trivial solution is stable if $\lambda \le 0$, that is $(\mu_1 + G)^2 \ge -4\phi^2$.

(b) Stability of non-trivial solution:

To determine the stability of the non-trivial solutions we let

$$a = a_0 + a_1(T_1) \text{ and } \gamma = \gamma_0 + \gamma_1(T_1) . \tag{2.121}$$

Substituting equation (2.121) into equations (2.109) , (2.112), and simplifying, similarly as in the above, we have

$$a_0' + a_1' = -\frac{1}{2}\mu_1 a_0 - \frac{1}{2}\mu_1 a_1 - \frac{1}{2}Ga_0 - \frac{1}{2}Ga_1 + \frac{1}{2\omega} f_1\left(\sin(\gamma_0) + \gamma_1\cos(\gamma_0)\right)\cos(\alpha) \ , \tag{2.122}$$

$$a_0\gamma_0' + a_1\gamma_0' + a_0\gamma_1' + a_1\gamma_1' = a_0\sigma_1 + a_1\sigma_1 - \frac{3}{4\omega}\beta_1\left(a_0^3 + 3a_0^2 a_1 + ...\right) - \frac{5}{8\omega}\beta_2\left(a_0^5 + 5a_0^4 a_1 + ...\right)$$
$$-\frac{3}{2}\omega\delta\left(a_0^3 + 3a_0^2 a_1 + ...\right) + \frac{1}{2\omega} f_1\left(\cos\left(\gamma_0\right) - \gamma_1 \sin\left(\gamma_0\right)\right)\cos\left(\alpha\right). \tag{2.123}$$

Since a_0 and γ_0 are solutions of equations (2.109),(2.112) and $\gamma_0' + \gamma_1' = \gamma' = 0$ then

$$a_1' = \left(-\frac{1}{2}\mu_1 - \frac{1}{2}G\right)a_1 + \left(-\sigma_1 a_0 + \frac{3}{8\omega}\beta_1 a_0^3 + \frac{5}{16\omega}\beta_2 a_0^5 + \frac{3}{4}\omega\delta a_0^3\right)\gamma_1, \tag{2.124}$$

$$\gamma_1' = \left(\frac{\sigma_1}{a_0} - \frac{9}{4\omega}\beta_1 a_0 - \frac{25}{8\omega}\beta_2 a_0^2 - \frac{9}{2}\omega\delta a_0\right)a_1 + \left(-\frac{1}{2}\mu_1 - \frac{1}{2}G\right)\gamma_1. \tag{2.125}$$

We can put equations (2.124) and (2.125) as the following form

$$a_1' = \Gamma_{11}a_1 + \Gamma_{12}\gamma_1 \ , \ \ \gamma_1' = \Gamma_{13}a_1 + \Gamma_{11}\gamma_1. \tag{2.126}$$

Where $\Gamma_{11} = -\frac{1}{2}\mu_1 - \frac{1}{2}G$, $\Gamma_{12} = -\sigma_1 a_0 + \frac{3}{8\omega}\beta_1 a_0^3 + \frac{5}{16\omega}\beta_2 a_0^5 + \frac{3}{4}\omega\delta a_0^3$,

$$\Gamma_{13} = \frac{\sigma_1}{a_0} - \frac{9}{4\omega}\beta_1 a_0 - \frac{25}{8\omega}\beta_2 a_0^2 - \frac{9}{2}\omega\delta a_0 \ .$$

The eigenvalues can be obtained by solving the determinant of the Jacobian matrix of the equation (2.126)

$$\begin{vmatrix} \Gamma_{11} - \lambda & \Gamma_{12} \\ \Gamma_{13} & \Gamma_{11} - \lambda \end{vmatrix} = 0,$$

$$\lambda^2 - \lambda\left(2\Gamma_{11}\right) + \Gamma_{11}^2 - \Gamma_{12}\Gamma_{13} = 0. \tag{2.127}$$

The eigenvalues of equation (2.127) are

$$\lambda = \Gamma_{11} \pm \sqrt{\Gamma_{12}\Gamma_{13}} \ . \tag{2.128}$$

Therefore the steady-state solutions are stable if and only if $\Gamma_{11}^2 \le \Gamma_{12}\Gamma_{13}$.

2.5.1 Sub-harmonic resonance : $\Omega = 2\omega$

In this case we introduce a detuning parameter σ_2

$$\Omega = 2\omega + \varepsilon\sigma_2 , \tag{2.129}$$

Substituting equation (2.129) into equation (2.105), eliminating the terms that produce secular term and performing some algebraic manipulations, we obtain

$$\begin{aligned} &-2i\omega A' - \mu_1 i\omega A - 3\beta_1 A^2 \bar{A} - 10\beta_2 A^3 \bar{A}^2 - 6\omega^2 \delta A^2 \bar{A} - Gi\omega A \\ &+ \frac{1}{2} f_2 \bar{A} e^{i\sigma_2 T_1} \sin(\alpha) = 0. \end{aligned} \tag{2.130}$$

Substituting $A = \frac{1}{2} a(T_1) e^{i\theta(T_1)}$ similarly in equations (2.65) – (2.67) , we obtain the following equations describing the modulation of amplitude and phase of the motions

$$-a' - \frac{1}{2}\mu_1 a - \frac{1}{2} G a + \frac{1}{4\omega} f_2 a \sin(-2\theta + \sigma_2 T_1) \sin(\alpha) = 0 , \tag{2.131}$$

And

$$a\theta' - \frac{3}{8\omega}\beta_1 a^3 - \frac{5}{16\omega}\beta_2 a^5 - \frac{3}{4}\omega\delta a^3 + \frac{1}{4\omega} f_2 a \cos(-2\theta + \sigma_2 T_1) \sin(\alpha) = 0 . \tag{2.132}$$

Sitting $\Lambda_1 = \frac{1}{4\omega} a f_2$, $\gamma_2 = (-2\theta + \sigma_2 T_1)$, then equations (2.163) , (2.164) becomes

$$a' = -\frac{1}{2}\mu_1 a - \frac{1}{2} G a + \Lambda_1 \sin(\gamma_2) \sin(\alpha), \tag{2.133}$$

$$a\gamma_2' = \sigma_2 a - \frac{3}{4\omega}\beta_1 a^3 - \frac{5}{8\omega}\beta_2 a^5 - \frac{3}{2}\omega\delta a^3 + 2\Lambda_1 \cos(\gamma_2) \sin(\alpha). \tag{2.134}$$

For steady-state solution, setting $a' = \gamma_2' = 0$ equation (2.133), (2.134) becomes

$$\mu_1 a + G a = 2\Lambda_1 \sin(\gamma_2) \sin(\alpha), \tag{2.135}$$

$$\sigma_2 a - \frac{3}{4\omega}\beta_1 a^3 - \frac{5}{8\omega}\beta_2 a^5 - \frac{3}{2}\omega\delta a^3 = -2\Lambda_1 \cos(\gamma_2) \sin(\alpha). \tag{2.136}$$

From (2.135) and (2.136) we have

$$\left(\mu_1 a+Ga\right)^2+\left(\sigma_2 a-\frac{3}{4\omega}\beta_1 a^3-\frac{5}{8\omega}\beta_2 a^5-\frac{3}{2}\omega\delta a^3\right)^2=4\Lambda_1^2\sin^2\left(\alpha\right). \tag{2.137}$$

Equation (2.137) is called the frequency response equation.

(a) **Stability of trivial solution**

To determine the stability of the trivial solutions, we investigates the solutions of the linearized form of equation (2.130)

$$-2i\omega A'-\mu_1 i\omega A-Gi\omega A+\frac{1}{2}f_2\bar{A}e^{i\sigma_2 T_1}\sin\left(\alpha\right)=0. \tag{2.138}$$

Substituting $A=\frac{1}{2}\left(p_1-ip_2\right)e^{i\phi T_1}$ into equation (2.138) and simplifying, then separating real and imaginary parts, we have

$$\begin{aligned}&-p_1'-\phi p_2-\frac{1}{2}p_1\mu_1-\frac{1}{2}Gp_1+\frac{1}{4\omega}f_2 p_1\sin\left(-2\phi T_1+\sigma_2 T_1\right)\sin\left(\alpha\right)\\&+\frac{1}{4\omega}f_2 p_2\cos\left(-2\phi T_1+\sigma_2 T_1\right)\sin\left(\alpha\right)=0,\end{aligned} \tag{2.139}$$

And

$$\begin{aligned}&-p_2'+\phi p_1-\frac{1}{2}p_2\mu_1-\frac{1}{2}Gp_2+\frac{1}{4\omega}f_2 p_1\cos\left(-2\phi T_1+\sigma_2 T_1\right)\sin\left(\alpha\right)\\&-\frac{1}{4\omega}f_2 p_2\sin\left(-2\phi T_1+\sigma_2 T_1\right)\sin\left(\alpha\right)=0.\end{aligned} \tag{2.140}$$

Sitting $\vartheta_1=\left(-2\phi T_1+\sigma_2 T_1\right)$, gives

$$p_1'=\left(-\frac{1}{2}\mu_1-\frac{1}{2}G+\frac{1}{4\omega}f_2\sin\left(\vartheta_1\right)\sin\left(\alpha\right)\right)p_1+\left(-\phi+\frac{1}{4\omega}f_2\cos\left(\vartheta_1\right)\sin\left(\alpha\right)\right)p_2, \tag{2.141}$$

$$p_2'=\left(\phi+\frac{1}{4\omega}f_2\cos\left(\vartheta_1\right)\sin\left(\alpha\right)\right)p_1+\left(-\frac{1}{2}\mu_1-\frac{1}{2}G-\frac{1}{4\omega}f_2\sin\left(\vartheta_1\right)\sin\left(\alpha\right)\right)p_2. \tag{2.142}$$

Sitting $\Gamma_{14}=-\frac{1}{2}\mu_1-\frac{1}{2}G+\frac{1}{4\omega}f_2\sin\left(\vartheta_1\right)\sin\left(\alpha\right)$, $\Gamma_{15}=-\phi+\frac{1}{4\omega}f_2\cos\left(\vartheta_1\right)\sin\left(\alpha\right)$,

$$\Gamma_{16} = \phi + \frac{1}{4\omega} f_2 \cos(\vartheta_1)\sin(\alpha) \quad , \quad \Gamma_{17} = -\frac{1}{2}\mu_1 - \frac{1}{2}G - \frac{1}{4\omega} f_2 \sin(\vartheta_1)\sin(\alpha).$$

The stability of the trivial solution is investigated by evaluating the eigenvalues of the Jacobian matrix of equations (2.141), (2.142)

$$\begin{vmatrix} \Gamma_{14} - \lambda & \Gamma_{15} \\ \Gamma_{16} & \Gamma_{17} - \lambda \end{vmatrix} = 0,$$

$$\lambda^2 - \lambda(\Gamma_{14} + \Gamma_{17}) + \Gamma_{14}\Gamma_{17} - \Gamma_{15}\Gamma_{16} = 0. \tag{2.143}$$

The trivial solution is stable if $\Gamma_{14}\Gamma_{17} - \Gamma_{15}\Gamma_{16} \le 0$.

(b) Stability of non-trivial solution

To determine the stability of the non-trivial solutions we let

$$a = a_0 + a_1(T_1) \text{ and } h = h_0 + h_1(T_1) . \tag{2.144}$$

Substituting equation (2.144) into equations (2.133) , (2.134), and simplifying, similarly as in the above, we have

$$a_0' + a_1' = -\frac{1}{2}\mu_1 a_0 - \frac{1}{2}\mu_1 a_1 - \frac{1}{2}G a_0 - \frac{1}{2}G a_1 + \frac{1}{4\omega} f_2 a_0 (\sin h_0 + h_1 \cos h_0)\sin(\alpha) + \frac{1}{4\omega} f_2 a_1 (\sin h_0 + h_1 \cos h_0)\sin(\alpha), \tag{2.145}$$

$$a_0 h_0' + a_1 h_0' + a_0 h_1' + a_1 h_1' = a_0\sigma_2 + a_1\sigma_2 - \frac{3}{4\omega}\beta_1\left(a_0^3 + 3a_0^2 a_1 + ...\right) - \frac{5}{8\omega}\beta_2\left(a_0^5 + 5a_0^4 a_1 + ...\right) - \frac{3}{2}\omega\delta\left(a_0^3 + 3a_0^2 a_1 + ...\right) + \frac{1}{2\omega} f_2 a_0 (\cos h_0 - h_1 \sin h_0)\sin(\alpha) + \frac{1}{2\omega} f_2 a_1 (\cos h_0 - h_1 \sin h_0)\sin(\alpha). \tag{2.146}$$

Since a_0 and h_0 are solutions of equations (2.133) , (2.134) , $a_1 h_1$ is a very small term and $h_0' + h_1' = h' = 0$ then they can be eliminated

Thus equations (2.145) , (2.146), becomes

$$a_1' = -\frac{1}{2}\mu_1 a_1 - \frac{1}{2}Ga_1 + \frac{1}{4\omega} f_2 a_0 h_1 \cos(h_0)\sin(\alpha) + \frac{1}{4\omega} f_2 a_1 \sin(h_0)\sin(\alpha), \tag{2.147}$$

$$a_0 h_1' = a_1\sigma_2 - \frac{9}{4\omega}\beta_1 a_0^2 a_1 - \frac{25}{8\omega}\beta_2 a_0^4 a_1 - \frac{9}{2}\omega\delta a_0^2 a_1 - \frac{1}{2\omega} f_2 a_0 h_1 \sin(h_0)\sin(\alpha) + \frac{1}{2\omega} f_2 a_1 \cos(h_0)\sin(\alpha). \tag{2.148}$$

Substituting from (2.147),(2.148) into equations (2.135),(2.136) and simplifying, we get

$$a_1' = h_1\left(-\frac{1}{2}a_0\sigma_2 + \frac{3}{8\omega}\beta_1 a_0^3 + \frac{5}{16\omega}\beta_2 a_0^5 + \frac{3}{4}\omega\delta a_0^3\right), \tag{2.149}$$

$$h_1' = \left(-\frac{3}{2\omega}\beta_1 a_0 - \frac{5}{2\omega}\beta_2 a_0^3 - 3\omega\delta a_0\right)a_1 + (-\mu_1 - G)h_1. \tag{2.150}$$

We can put equation (2.149) and (2.150) as the following form

$$a_1' = h_1\Gamma_{18}\ ,\ \ h_1' = a_1\Gamma_{19} + h_1\Gamma_{20}\ , \tag{2.151}$$

Where $\Gamma_{18} = -\frac{1}{2}a_0\sigma_2 + \frac{3}{8\omega}\beta_1 a_0^3 + \frac{5}{16\omega}\beta_2 a_0^5 + \frac{3}{4}\omega\delta a_0^3$,

, $\Gamma_{19} = -\frac{3}{2\omega}\beta_1 a_0 - \frac{5}{2\omega}\beta_2 a_0^3 - 3\omega\delta a_0$, $\Gamma_{20} = -\mu_1 - G$.

The non-trivial solution is stable if and only if the real parts of equation (2.193) are less than or equal to zero using the Jacbian matrix mothed to solve the equation

$$\begin{vmatrix} -\lambda & \Gamma_{18} \\ \Gamma_{19} & \Gamma_{20} - \lambda \end{vmatrix} = 0,$$

The eigenvalues are

$$\lambda = \frac{\Gamma_{20} \pm \sqrt{\Gamma_{20}^2 + 4\Gamma_{18}\Gamma_{19}}}{2}. \tag{2.152}$$

Therefore the steady-state solutions are stable if and only if $\Gamma_{18}\Gamma_{19} \le 0$.

2.6 perturbation analysis for the system with negative cubic (VF):

The nonlinear equation (2.4) with negative cubic velocity feedback (VF) control is scaled using the perturbation parameter ε as follows

$$u'' + \varepsilon\mu_1 u' + \omega^2 u + \varepsilon\beta_1 u^3 + \varepsilon\beta_2 u^5 - \varepsilon\delta\left(uu'^2 + u^2 u''\right) = \varepsilon f_1 \cos(\Omega t)\cos(\alpha)$$
$$+\varepsilon u f_2 \cos(\Omega t)\sin(\alpha) - \varepsilon G u'^3.$$

Applying the multiple scales mothed, similarly as in the perturbation analysis equations (2.5) – (2.9) , we have

$$\begin{aligned}&D_0^2 u_0 + \varepsilon D_0^2 u_1 + 2\varepsilon D_0 D_1 u_0 + \varepsilon\mu_1 D_0 u_0\\&+\omega_1^2 u_0 + \varepsilon\omega_1^2 u_1 + \varepsilon\beta_1 u_0^3 + \varepsilon\beta_2 u_0^5 - 2\varepsilon\delta D_0^2 u_0^3\\&-\varepsilon f_1 \cos(\Omega t)\cos(\alpha) - \varepsilon u_0 f_2 \cos(\Omega t)\sin(\alpha) + \varepsilon G D_0^3 u_0^3 = 0.\end{aligned} \quad (2.153)$$

Equating the coefficient of same powers of ε in equation (2.153), we have

$$O(\varepsilon^0):\left(D_0^2 + \omega_1^2\right)u_0 = 0\,, \quad (2.154)$$

$$\begin{aligned}O(\varepsilon^1):&\left(D_0^2 + \omega_1^2\right)u_1 = -2D_0 D_1 u_0 - \mu_1 D_0 u_0 - \beta_1 u_0^3 - \beta_2 u_0^5 + 2\delta D_0^2 u_0^3\\&+ f_1 \cos(\Omega t)\cos(\alpha) + u_0 f_2 \cos(\Omega t)\sin(\alpha) - G D_0^3 u_0^3.\end{aligned} \quad (2.155)$$

The general solution of (2.154) is given by

$$u_0 = A(T_1)e^{i\omega T_0} + \bar{A}(T_1)e^{-i\omega T_0} \quad , \quad (2.156)$$

where $A(T_1)$ is unknown function in T_1 .

To solve equation (2.155), substituting equation (2.156) into it then substituting equation (2.7), and using the form $\cos(\omega T_0) = \dfrac{e^{i\omega T_0} + e^{-i\omega T_0}}{2}$, $\sin(\omega T_0) = \dfrac{e^{i\omega T_0} - e^{-i\omega T_0}}{2i}$, to get this simplifying equation,

$$\begin{aligned}&\left(D_0^2 + \omega_1^2\right)u_1 = \left(-2i\omega A' - \mu_1 i\omega A - 3\beta_1 A^2\bar{A} - 10\beta_2 A^3\bar{A}^2 - 6\omega^2\delta A^2\bar{A} + 3i\omega^3 G A^2\bar{A}\right)e^{i\omega T_0}\\&+\left(-\beta_1 A^3 - 5\beta_2 A^4\bar{A} - 18\omega^2\delta A^3 + 18i\omega^3 G A^3\right)e^{3i\omega T_0} - \beta_2 A^5 e^{5i\omega T_0} + \frac{1}{2} f_1 e^{i\Omega T_0}\cos\left(\alpha\right)\\&+\frac{1}{2} f_2 A e^{i(\omega+\Omega)T_0}\sin\left(\alpha\right) + \frac{1}{2} f_2 A e^{i(\omega-\Omega)T_0}\sin\left(\alpha\right) + cc,\end{aligned} \quad (2.157)$$

where cc denotes the complex conjugate terms.

The particular solution of equation (2.157) can be written in the following form

$$u_0 = A_1(T_1)e^{i\omega T_0} - \frac{1}{8\omega^2}\left(-\beta_1 A^3 - 5\beta_2 A^4\bar{A} - 18\omega^2\delta A^3 + 18i\omega^3 G A^3\right)e^{3i\omega T_0} + \frac{1}{24\omega^2}\beta_2 A^5 e^{5i\omega T_0}$$
$$+\frac{1}{2(\omega-\Omega)(\omega+\Omega)} f_1\cos(\alpha)e^{i\Omega T_0} - \frac{1}{2\Omega(2\omega+\Omega)} f_2 A e^{i(\omega+\Omega)T_0}\sin(\alpha) \tag{2.158}$$
$$+\frac{1}{2\Omega(2\omega-\Omega)} f_2 A e^{i(\omega-\Omega)T_0}\sin(\alpha) + cc.$$

From the equation (2.157), the reported resonance cases at this approximation order are

a. Primary resonance : $\Omega = \omega$
b. Sub-harmonic resonance : $\Omega = 2\omega$

2.7 Stability analysis

2.7.1 Primary resonance $\Omega = \omega$

In this case we introduce a detuning parameter σ_1 such that

$$\Omega = \omega + \varepsilon\sigma_1, \tag{2.159}$$

Substituting equation (2.159) into (2.157), eliminating the terms that produce secular term and performing some algebraic manipulations, we obtain

$$-2i\omega A' - \mu_1 i\omega A - 3\beta_1 A^2\bar{A} - 10\beta_2 A^3\bar{A}^2 - 6\omega^2\delta A^2\bar{A} + 3i\omega^3 G A^2\bar{A}$$
$$+\frac{1}{2} f_1 e^{i\sigma_1 T_1}\cos(\alpha) = 0. \tag{2.160}$$

Substituting $A = \frac{1}{2}a(T_1)e^{i\theta(T_1)}$ and using the form $e^{ix} = \cos x + i\sin x$ and separating the imaginary and real parts, we obtain the following equations describing the modulation of amplitude and phase of the motions

$$a' = -\frac{1}{2}\mu_1 a + \frac{3}{8}a^3\omega^2 G + \frac{1}{2\omega} f_1 \sin(-\theta + \sigma_1 T_1)\cos(\alpha) = 0, \tag{2.161}$$

And

$$a\theta' - \frac{3}{8\omega}\beta_1 a^3 - \frac{5}{16\omega}\beta_2 a^5 - \frac{3}{4}\omega\delta a^3 + \frac{1}{2\omega} f_1\cos(-\theta + \sigma_1 T_1)\cos(\alpha) = 0. \tag{2.162}$$

Sitting $\Lambda = \frac{1}{2\omega} f_1$, $\gamma = (-\theta + \sigma_1 T_1)$.

Equation (2.161) and (2.162) become as the following

$$a' = -\frac{1}{2}\mu_1 a + \frac{3}{8}a^3\omega^2 G + \Lambda \sin(\gamma)\cos(\alpha), \tag{2.163}$$

$$a\gamma' = \sigma_1 a - \frac{3}{8\omega}\beta_1 a^3 - \frac{5}{16\omega}\beta_2 a^5 - \frac{3}{4}\omega\delta a^3 + \Lambda\cos(\gamma)\cos(\alpha). \tag{2.164}$$

For steady-state solutions, setting $a' = \gamma' = 0$ equations (2.163) , (2.164), becomes

$$\mu_1 a - \frac{3}{4}a^3\omega^2 G = 2\Lambda\sin(\gamma)\cos(\alpha), \tag{2.165}$$

$$2\sigma_1 a - \frac{3}{4\omega}\beta_1 a^3 - \frac{5}{8\omega}\beta_2 a^5 - \frac{3}{2}\omega\delta a^3 = -2\Lambda\cos(\gamma)\cos(\alpha). \tag{2.166}$$

From equation (2.165) and (2.166), we have

$$\left(\mu_1 a - \frac{3}{4}a^3\omega^2 G\right)^2 + \left(2a\sigma_1 - \frac{3}{4\omega}\beta_1 a^3 - \frac{5}{8\omega}\beta_2 a^5 - \frac{3}{2}\omega\delta a^3\right)^2 = 4\Lambda^2\cos^2(\alpha). \tag{2.167}$$

Equation (2.167) is called the frequency response equation.

(a) Stability of trivial solution:

To determine the stability of the trivial solutions, we investigates the solutions of the linearized form of equation (2.160)

$$-2i\omega A' - \mu_1 i\omega A = 0. \tag{2.168}$$

Substituting $A = \frac{1}{2}(p_1 - ip_2)e^{i\phi T_1}$ into equation (2.168) and simplifying, we get

$$-ip_1' - p_2' + \phi p_1 - i\phi p_2 - \frac{1}{2}ip_1\mu_1 - \frac{1}{2}p_2\mu_1 = 0. \tag{2.169}$$

Separating real and imaginary parts we get

$$p_1' = -\phi p_2 - \frac{1}{2} p_1 \mu_1, \tag{2.170}$$

$$p_2' = \phi p_1 - \frac{1}{2} p_2 \mu_1. \tag{2.171}$$

The stability of the trivial solution is investigated by evaluating the eigenvalues of the Jacobian matrix of equations (2.170), (2.171) gives

$$\begin{vmatrix} -\frac{1}{2}\mu_1 - \lambda & -\phi \\ \phi & -\frac{1}{2}\mu_1 - \lambda \end{vmatrix} = 0,$$

$$\lambda^2 + (\mu_1)\lambda + \frac{1}{4}\mu_1^2 + \phi^2 = 0. \tag{2.172}$$

The solution of the equation (2.172) is

$$\lambda = -\frac{1}{2}\mu_1 \pm \sqrt{(-\phi^2)}. \tag{2.173}$$

The trivial solution is stable if $\lambda \le 0$, that is $\mu_1^2 \ge -4\phi^2$.

(b) Stability of non-trivial solution:

To determine the stability of the non-trivial solutions we let

$$a = a_0 + a_1(T_1) \text{ and } \gamma = \gamma_0 + \gamma_1(T_1). \tag{2.174}$$

Substituting equation (2.174) into equations (2.163) , (2.164), and simplifying, we have

$$\begin{aligned} a_0' + a_1' &= -\frac{1}{2}\mu_1 a_0 - \frac{1}{2}\mu_1 a_1 + \frac{3}{8}\left(a_0^3 + 3a_0^2 a_1 + ...\right)\omega^2 G \\ &+ \frac{1}{2\omega} f_1\left(\sin(\gamma_0) + \gamma_1 \cos(\gamma_0)\right)\cos(\alpha), \end{aligned} \tag{2.175}$$

$$\begin{aligned} a_0\gamma_0' + a_1\gamma_0' + a_0\gamma_1' + a_1\gamma_1' &= a_0\sigma_1 + a_1\sigma_1 - \frac{3}{4\omega}\beta_1\left(a_0^3 + 3a_0^2 a_1 + ...\right) - \frac{5}{8\omega}\beta_2\left(a_0^5 + 5a_0^4 a_1 + ...\right) \\ &- \frac{3}{2}\omega\delta\left(a_0^3 + 3a_0^2 a_1 + ...\right) + \frac{1}{2\omega} f_1\left(\cos(\gamma_0) - \gamma_1 \sin(\gamma_0)\right)\cos(\alpha). \end{aligned} \tag{2.176}$$

Since a_0 and γ_0 are solutions of equations (2.163) , (2.164) and $\gamma_0' + \gamma_1' = \gamma' = 0$ then

$$a_1' = -\frac{1}{2}\mu_1 a_1 + \frac{9}{8}a_0^2 a_1 \omega^2 G + \frac{1}{2\omega} f_1 \gamma_1 \cos(\gamma_0)\cos(\alpha), \tag{2.177}$$

$$a_0 \gamma_1' = a_1 \sigma_1 - \frac{9}{4\omega}\beta_1 a_0^2 a_1 - \frac{25}{8\omega}\beta_2 a_0^4 a_1 - \frac{9}{2}\omega\delta a_0^2 a_1 - \frac{1}{2\omega} f_1 \gamma_1 \sin(\gamma_0)\cos(\alpha). \tag{2.178}$$

Substituting from equations (2.165) , (2.166) into equations (2.177) , (2.178), we get

$$a_1' = \left(-\frac{1}{2}\mu_1 + \frac{9}{8}a_0^2\omega^2 G\right)a_1 + \left(-\sigma_1 a_0 + \frac{3}{8\omega}\beta_1 a_0^3 + \frac{5}{16\omega}\beta_2 a_0^5 + \frac{3}{4}\omega\delta a_0^3\right)\gamma_1, \tag{2.179}$$

$$\gamma_1' = \left(\frac{\sigma_1}{a_0} - \frac{9}{4\omega}\beta_1 a_0 - \frac{25}{8\omega}\beta_2 a_0^2 - \frac{9}{2}\omega\delta a_0\right)a_1 + \left(\frac{3}{8}a_0^2\omega^2 G - \frac{1}{2}\mu_1\right)\gamma_1. \tag{2.180}$$

We can put equation (2.179) and (2.180) as the following form

$$a_1' = \Gamma_{21} a_1 + \Gamma_{22}\gamma_1 , \quad \gamma_1' = \Gamma_{23} a_1 + \Gamma_{24}\gamma_1 , \tag{2.181}$$

Where $\Gamma_{21} = -\frac{1}{2}\mu_1 + \frac{9}{8}a_0^2\omega^2 G$, $\Gamma_{22} = -\sigma_1 a_0 + \frac{3}{8\omega}\beta_1 a_0^3 + \frac{5}{16\omega}\beta_2 a_0^5 + \frac{3}{4}\omega\delta a_0^3$,

$$\Gamma_{23} = \frac{\sigma_1}{a_0} - \frac{9}{4\omega}\beta_1 a_0 - \frac{25}{8\omega}\beta_2 a_0^2 - \frac{9}{2}\omega\delta a_0 \quad , \quad \Gamma_{24} = \frac{3}{8}a_0^2\omega^2 G - \frac{1}{2}\mu_1 .$$

The eigenvalues can be obtained by solving the determinant of the Jacobian matrix of the equation (2.181)

$$\begin{vmatrix} \Gamma_{21} - \lambda & \Gamma_{22} \\ \Gamma_{23} & \Gamma_{24} - \lambda \end{vmatrix} = 0 ,$$

$$\lambda^2 - \lambda(\Gamma_{21} + \Gamma_{24}) + \Gamma_{21}\Gamma_{24} - \Gamma_{22}\Gamma_{23} = 0 . \tag{2.182}$$

Therefore the steady-state solutions are stable if and only if $\Gamma_{21}\Gamma_{24} - \Gamma_{22}\Gamma_{23} \le 0$.

2.7.2 Sub-harmonic resonance : $\Omega = 2\omega$

In this case we introduce a detuning parameter σ_2

$$\Omega = 2\omega + \varepsilon\sigma_2 \ , \tag{2.183}$$

Substituting equation (2.183) into equation (2.157), eliminating the terms that produce secular term and performing some algebraic manipulations, we obtain

$$\begin{aligned}&-2i\omega A' - \mu_1 i\omega A - 3\beta_1 A^2\bar{A} - 10\beta_2 A^3\bar{A}^2 - 6\omega^2\delta A^2\bar{A} + 3i\omega^3 GA^2\bar{A}\\&+\frac{1}{2} f_2\bar{A}e^{i\sigma_2 T_1}\sin(\alpha) = 0.\end{aligned} \tag{2.184}$$

Substituting $A = \frac{1}{2}a(T_1)e^{i\theta(T_1)}$ and using the form $e^{ix} = \cos x + i\sin x$ and separating the imaginary and real parts, we obtain the following equations describing the modulation of amplitude and phase of the motions

$$-a' - \frac{1}{2}\mu_1 a + \frac{3}{8}a^3\omega^2 G + \frac{1}{4\omega} f_2 a\sin(-2\theta + \sigma_2 T_1)\sin(\alpha) = 0\,, \tag{2.185}$$

And

$$a\theta' - \frac{3}{8\omega}\beta_1 a^3 - \frac{5}{16\omega}\beta_2 a^5 - \frac{3}{4}\omega\delta a^3 + \frac{1}{4\omega} f_2 a\cos(-2\theta + \sigma_2 T_1)\sin(\alpha) = 0\,. \tag{2.186}$$

Sitting $\Lambda_1 = \frac{1}{4\omega}af_2$, $\gamma_2 = (-2\theta + \sigma_2 T_1)$, then

$$a' = -\frac{1}{2}\mu_1 a + \frac{3}{8}a^3\omega^2 G + \Lambda_1\sin(\gamma_2)\sin(\alpha)\,, \tag{2.187}$$

$$a\gamma_2' = \sigma_2 a - \frac{3}{4\omega}\beta_1 a^3 - \frac{5}{8\omega}\beta_2 a^5 - \frac{3}{2}\omega\delta a^3 + 2\Lambda_1\cos(\gamma_2)\sin(\alpha)\,. \tag{2.188}$$

For steady-state solution, setting $a' = \gamma_2' = 0$ equation (2.187) , (2.188) becomes

$$\mu_1 a - \frac{3}{4}a^3\omega^2 G = 2\Lambda_1\sin(\gamma_2)\sin(\alpha)\,, \tag{2.189}$$

$$\sigma_2 a - \frac{3}{4\omega}\beta_1 a^3 - \frac{5}{8\omega}\beta_2 a^5 - \frac{3}{2}\omega\delta a^3 = -2\Lambda_1\cos(\gamma_2)\sin(\alpha)\,. \tag{2.190}$$

From (2.189) and (2.190) we have

$$\left(\mu_1 a-\frac{3}{4}a^3\omega^2 G\right)^2+\left(\sigma_2 a-\frac{3}{4\omega}\beta_1 a^3-\frac{5}{8\omega}\beta_2 a^5-\frac{3}{2}\omega\delta a^3\right)^2=4\Lambda_1^2\sin^2(\alpha). \tag{2.191}$$

Equation (2.191) is called the frequency response equation.

(a) Stability of trivial solution

To determine the stability of the trivial solutions, we investigates the solutions of the linearized form of equation (2.184)

$$-2i\omega A'-\mu_1 i\omega A+\frac{1}{2}f_2\bar{A}e^{i\sigma_2 T_1}\sin(\alpha)=0. \tag{2.192}$$

Substituting $A=\frac{1}{2}(p_1-ip_2)e^{i\phi T_1}$ and simplifying, we get

$$\begin{aligned}&-ip_1'-p_2'+\phi p_1-i\phi p_2-\frac{1}{2}ip_1\mu_1-\frac{1}{2}p_2\mu_1+\frac{1}{4\omega}f_2 p_1 e^{-2i\phi T_1+i\sigma_2 T_1}\sin(\alpha)\\&+\frac{1}{4\omega}if_2 p_2 e^{-2i\phi T_1+i\sigma_2 T_1}\sin(\alpha)=0.\end{aligned} \tag{2.193}$$

Using the form $e^{ix}=\cos x+i\sin x$ and separating real and imaginary parts we get

$$\begin{aligned}&-p_1'-\phi p_2-\frac{1}{2}p_1\mu_1+\frac{1}{4\omega}f_2 p_1\sin(-2\phi T_1+\sigma_2 T_1)\sin(\alpha)\\&+\frac{1}{4\omega}f_2 p_2\cos(-2\phi T_1+\sigma_2 T_1)\sin(\alpha)=0,\end{aligned} \tag{2.194}$$

$$\begin{aligned}&-p_2'+\phi p_1-\frac{1}{2}p_2\mu_1+\frac{1}{4\omega}f_2 p_1\cos(-2\phi T_1+\sigma_2 T_1)\sin(\alpha)\\&-\frac{1}{4\omega}f_2 p_2\sin(-2\phi T_1+\sigma_2 T_1)\sin(\alpha)=0.\end{aligned} \tag{2.195}$$

Sitting $\vartheta_1=(-2\phi T_1+\sigma_2 T_1)$, gives

$$p_1'=\left(-\frac{1}{2}\mu_1+\frac{1}{4\omega}f_2\sin(\vartheta_1)\sin(\alpha)\right)p_1+\left(-\phi+\frac{1}{4\omega}f_2\cos(\vartheta_1)\sin(\alpha)\right)p_2, \tag{2.196}$$

$$p_2' = \left(\phi + \frac{1}{4\omega} f_2 \cos(\vartheta_1)\sin(\alpha)\right) p_1 + \left(-\frac{1}{2}\mu_1 - \frac{1}{4\omega} f_2 \sin(\vartheta_1)\sin(\alpha)\right) p_2 . \qquad (2.197)$$

Sitting $\Gamma_{25} = -\frac{1}{2}\mu_1 + \frac{1}{4\omega} f_2 \sin(\vartheta_1)\sin(\alpha)$, $\Gamma_{26} = -\phi + \frac{1}{4\omega} f_2 \cos(\vartheta_1)\sin(\alpha)$,

$$\Gamma_{27} = \phi + \frac{1}{4\omega} f_2 \cos(\vartheta_1)\sin(\alpha) \quad , \quad \Gamma_{28} = -\frac{1}{2}\mu_1 - \frac{1}{4\omega} f_2 \sin(\vartheta_1)\sin(\alpha).$$

The stability of the trivial solution is investigated by evaluating the eigenvalues of the Jacobian matrix of equations (2.196), (2.197) gives

$$\begin{vmatrix} \Gamma_{25} - \lambda & \Gamma_{26} \\ \Gamma_{27} & \Gamma_{28} - \lambda \end{vmatrix} = 0 ,$$

$$\lambda^2 - \lambda(\Gamma_{25} + \Gamma_{28}) + \Gamma_{25}\Gamma_{28} - \Gamma_{26}\Gamma_{27} = 0 . \qquad (2.198)$$

The trivial solution is stable if $\Gamma_{25}\Gamma_{28} - \Gamma_{26}\Gamma_{27} \le 0$.

(b) Stability of non-trivial solution

To determine the stability of the non-trivial solutions we let

$$a = a_0 + a_1(T_1) \text{ and } h = h_0 + h_1(T_1) . \qquad (2.199)$$

Substituting equation (2.199) into equations (2.187), (2.188) and simplifying, we get

$$a_0' + a_1' = -\frac{1}{2}\mu_1 a_0 - \frac{1}{2}\mu_1 a_1 + \frac{3}{8}\left(a_0^3 + 3a_0^2 a_1 + ...\right)\omega^2 G + \frac{1}{4\omega} f_2 a_0 (\sin h_0 + h_1 \cos h_0)\sin(\alpha)$$
$$+ \frac{1}{4\omega} f_2 a_1 (\sin h_0 + h_1 \cos h_0)\sin(\alpha), \qquad (2.200)$$

$$a_0 h_0' + a_1 h_0' + a_0 h_1' + a_1 h_1' = a_0 \sigma_2 + a_1 \sigma_2 - \frac{3}{4\omega}\beta_1\left(a_0^3 + 3a_0^2 a_1 + ...\right) - \frac{5}{8\omega}\beta_2\left(a_0^5 + 5a_0^4 a_1 + ...\right)$$
$$-\frac{3}{2}\omega\delta\left(a_0^3 + 3a_0^2 a_1 + ...\right) + \frac{1}{2\omega} f_2 a_0 (\cos h_0 - h_1 \sin h_0)\sin(\alpha) \qquad (2.201)$$
$$+ \frac{1}{2\omega} f_2 a_1 (\cos h_0 - h_1 \sin h_0)\sin(\alpha).$$

Since a_0 and h_0 are solutions of equations (2.187), (2.188), $a_1 h_1$ is a very small term and $h_0' + h_1' = h' = 0$ then they can be eliminated, we have

$$a_1' = -\frac{1}{2}\mu_1 a_1 + \frac{9}{8}a_0^2 a_1 \omega^2 G + \frac{1}{4\omega} f_2 a_0 h_1 \cos(h_0)\sin(\alpha) + \frac{1}{4\omega} f_2 a_1 \sin(h_0)\sin(\alpha), \tag{2.202}$$

$$a_0 h_1' = a_1\sigma_2 - \frac{9}{4\omega}\beta_1 a_0^2 a_1 - \frac{25}{8\omega}\beta_2 a_0^4 a_1 - \frac{9}{2}\omega\delta a_0^2 a_1 - \frac{1}{2\omega} f_2 a_0 h_1 \sin(h_0)\sin(\alpha) + \frac{1}{2\omega} f_2 a_1 \cos(h_0)\sin(\alpha). \tag{2.203}$$

Substituting from equations (2.189) , (2.190) into equations (2.202) , (2.203) and simplifying, we have

$$a_1' = \left(\frac{9}{8}a_0^2\omega^2 G - \frac{3}{8}a_1^2\omega^2 G\right)a_1 + \left(-\frac{1}{2}a_0\sigma_1 + \frac{3}{8\omega}\beta_1 a_0^3 + \frac{5}{16\omega}\beta_2 a_0^5 + \frac{3}{4}\omega\delta a_0^3\right)h_1, \tag{2.204}$$

$$h_1' = \left(-\frac{3}{2\omega}\beta_1 a_0 - \frac{5}{2\omega}\beta_2 a_0^3 - 3\omega\delta a_0\right)a_1 + \left(\frac{3}{4}a_0^2\omega^2 G - \mu_1\right)h_1. \tag{2.205}$$

We can put equations (2.204) and (2.205) as the following form

$$a_1' = \Gamma_{29} a_1 + h_1\Gamma_{30}, \quad h_1' = a_1\Gamma_{31} + h_1\Gamma_{32}, \tag{2.206}$$

Where $\Gamma_{29} = \frac{9}{8}a_0^2\omega^2 G - \frac{3}{8}a_1^2\omega^2 G$, $\Gamma_{30} = -\frac{1}{2}a_0\sigma_1 + \frac{3}{8\omega}\beta_1 a_0^3 + \frac{5}{16\omega}\beta_2 a_0^5 + \frac{3}{4}\omega\delta a_0^3$,

$\Gamma_{31} = -\frac{3}{2\omega}\beta_1 a_0 - \frac{5}{2\omega}\beta_2 a_0^3 - 3\omega\delta a_0$, $\Gamma_{32} = \frac{3}{4}a_0^2\omega^2 G - \mu_1$.

The non-trivial solution is stable if and only if the real parts of (2.206) are less than or equal to zero, using the Jacbian matrix mothed to solve the equation

$$\begin{vmatrix} \Gamma_{29} - \lambda & \Gamma_{30} \\ \Gamma_{31} & \Gamma_{32} - \lambda \end{vmatrix} = 0,$$

The eigenvalues $\lambda = \dfrac{(\Gamma_{29} + \Gamma_{32}) \pm \sqrt{(\Gamma_{29} + \Gamma_{32})^2 - 4\Gamma_{29}\Gamma_{32} + 4\Gamma_{30}\Gamma_{31}}}{2}$. (2.207)

Therefore the steady-state solutions are stable if and only if $\Gamma_{29}\Gamma_{32} - \Gamma_{30}\Gamma_{31} \le 0$.

2.8 Numerical results and discussions

In this section the steady state response of the nonlinear dynamical system is investigated for various system parameters under primary and sub-harmonic resonance conditions when the negative linear velocity feedback is considered. The stability of the numerical solution is studied using the frequency response function and the phase plane method.

2.8.1 Time response solution

The time history and stability of the dynamical system (inclined beam) subject to both harmonic and parametric excitations are obtained under position feedback, linear negative velocity feedback and cubic negative velocity feedback controllers at nonresonance, as shown in Figs. (2.8a,9a,10a), and at primary resonance case, as shown in Figs. (2.8b,9b,10b), and sub-harmonic resonance case, as shown in Figs. (2.8c,9c,10c). Comparing these figures, we may notice the followings:

Control Type	The response at primary resonance ($\Omega = \omega$)	The response at sub-harmonic resonance ($\Omega = 2\omega$).
Position Feedback	Chaotic with multi limit cycles.	May reach steady state at t >>> 600s.
Negative cubic velocity Feedback	Modulated with multi limit cycles.	May reach steady state at t > 600s.
Negative linear velocity Feedback	Modulated then stable after t=500s, with multi limit cycles.	Reaches steady state at t = 400s.

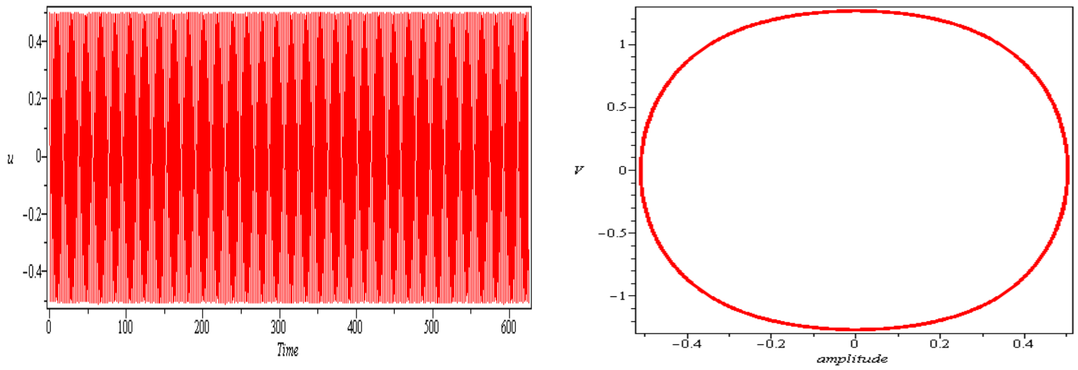
Based on the above comparison, we may conclude that the best performance among the three active controllers is the negative velocity feedback one as it suppresses the vibration to the minimum steady state amplitude at a shorter time when the system is at principal parametric resonance case.

2.8.2 Theoretical frequency response solution

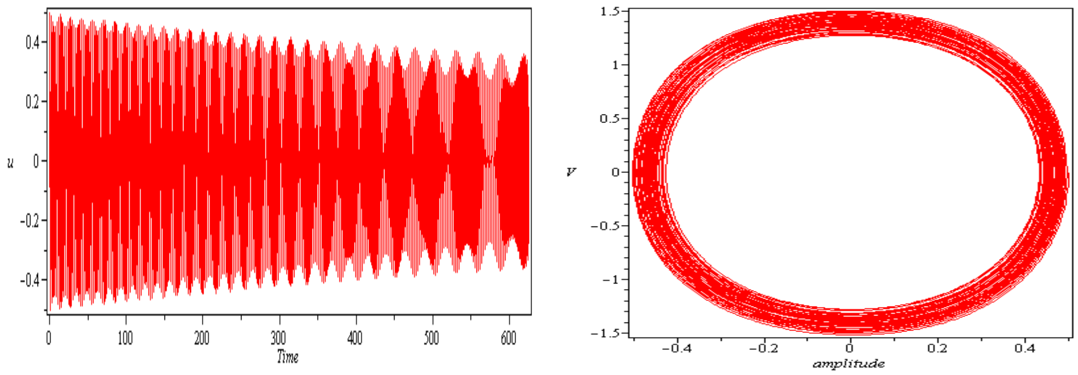
The frequeny response equations (2.115) and (2.137) under primary and subharmonic resonance conditions with positive position feedback controller, is solved and the stabillity of the steady state response is obtained from the eigenvalues of the corresponding Jacobian matrix. The results are shown in Figs. (2.11) and (2.12), respectively, as the steady state amplitude against the detuning parameter σ for different values of the system parameters.

Considering Fig. (2.11a) as a basic case for comparison. It is noted that the frequeny response curve consists of two branches that are bent to right showing that the system posseses hardenning nonlinearity charateristic. It can be seen from Fig. (2.11b), (2.11c), (2.11d) and (2.11g) that the steady state amplitude increases as each of the natural frequency ω, the linear damping coefficient μ_1 and the nonlinear coefficients β_1 and δ decrease. Figure (2.11f) shows that as the excitation force amplitude f increases, the branches of the response curves diverge away and the amplitude increases. The effect of the gain of the position feedback control is illustrated in Fig. (2.11h).

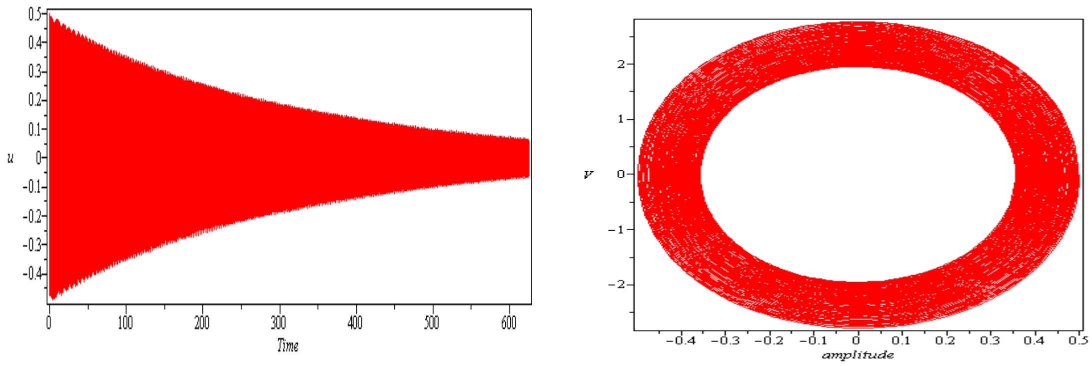
Fig.(2.12) represents the solution of the the sub-harmonic resonant frequency response equation (2.137) under the positive position feedback controller, which shows a different kind of frequeny curves but result in same effects of the system parameters that discussed in Fig.(2.11). It should be mentiond that the resonant frequency response curves under the other studied controllers (negative linear and cubic velocity feedback) have not been included since they do not represent significant change in the behavior or the shape of the curves discussed in Figs.(2.11) and (2.12).



(a) Nonresonant time series

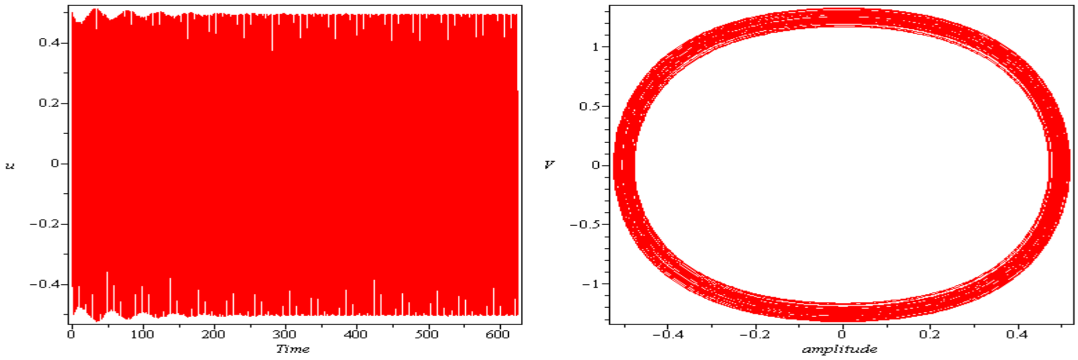


(b) Resonant time series when $\Omega = \omega$

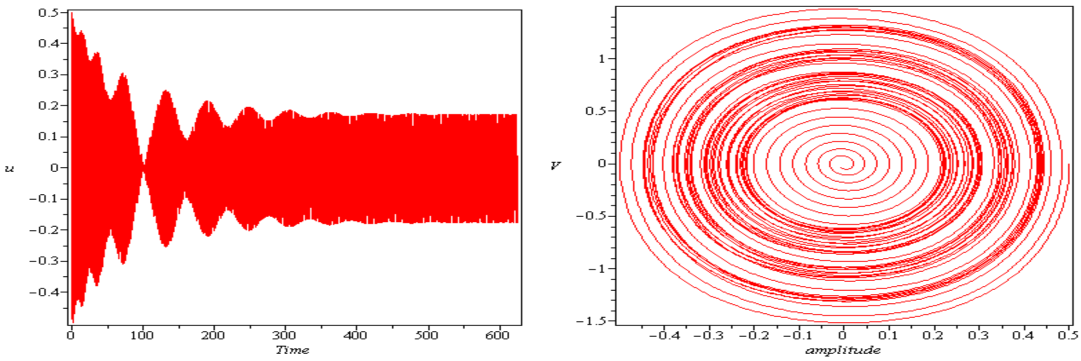


(c) Resonant time series when $\Omega = 2\omega$

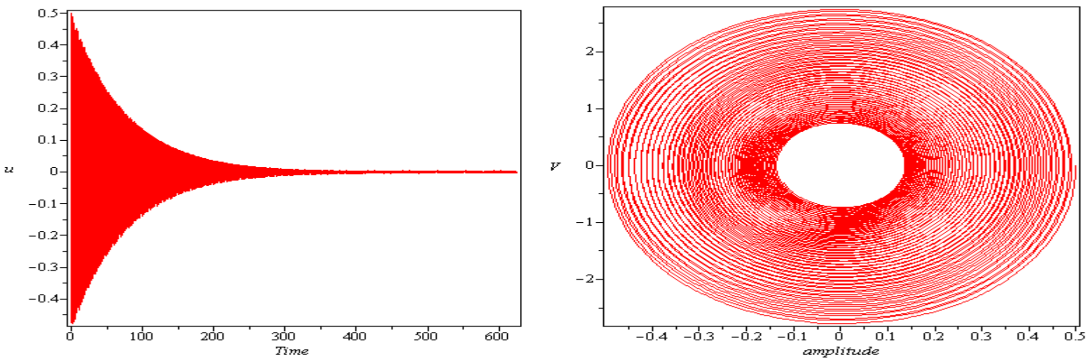
Fig 2.8 Resonant time history solution of the system with (PF) control when: $\omega = 2.1$, $\beta_1 = 15.0$, $\delta = 0.03$, $\mu_1 = 0.0005$, $\Omega = 2.7$, $\beta_2 = 5.0$, $f_1 = 0.4$, $f_2 = 0.2$, $\alpha = 30$, $G = 0.05$



(a) Nonresonant time series

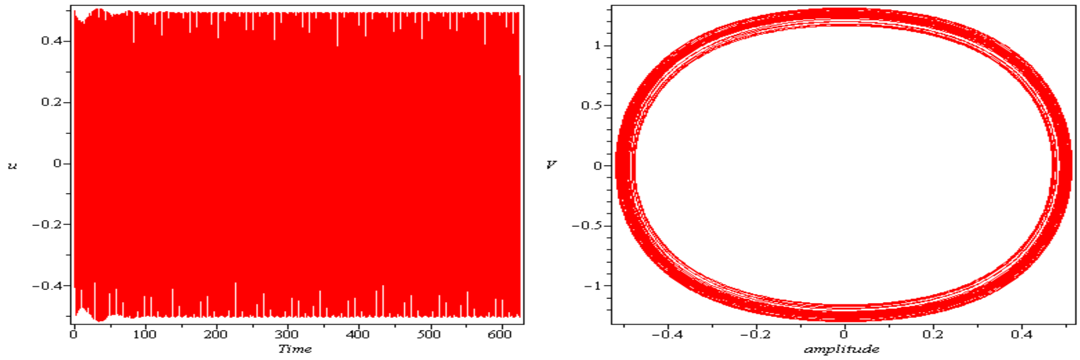


(b) Resonant time series when $\Omega = \omega$

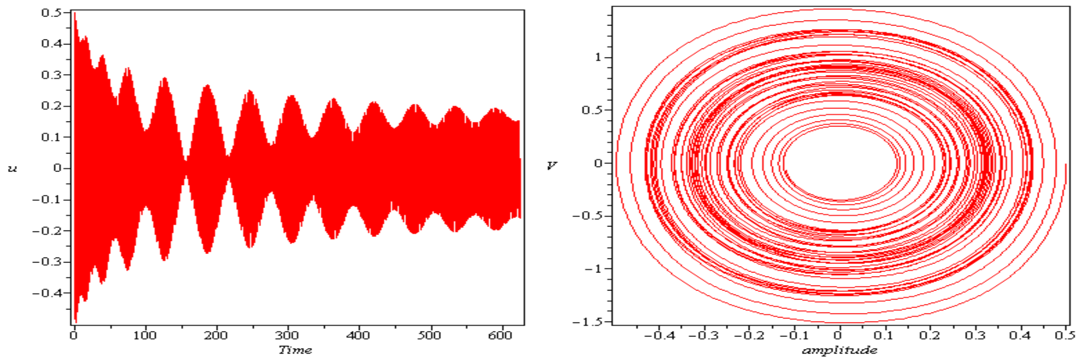


(c) Resonant time series when $\Omega = 2\omega$

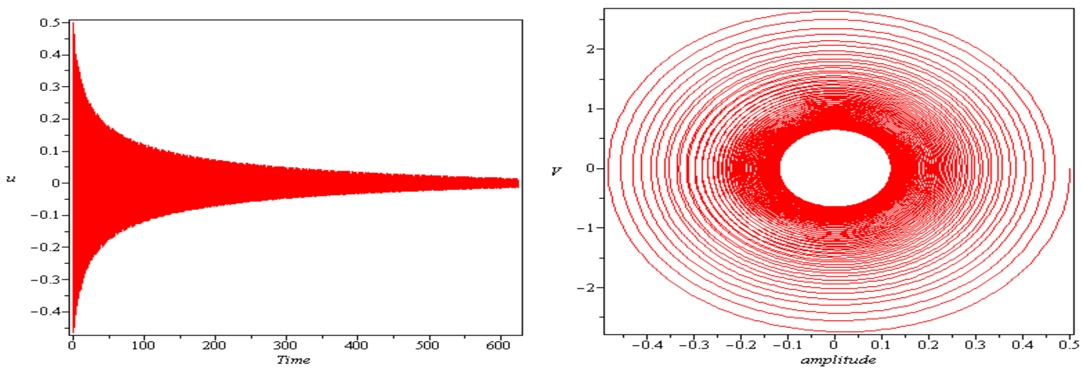
Fig 2.9 Resonant time history solution of the system with negative (VF) control when: $\omega = 2.1$, $\beta_1 = 15.0$, $\delta = 0.03$, $\mu_1 = 0.0005$, $\Omega = 2.7$, $\beta_2 = 5.0$, $f_1 = 0.4$, $f_2 = 0.2$, $\alpha = 30$, $G = 0.05$



(a) Nonresonant time series

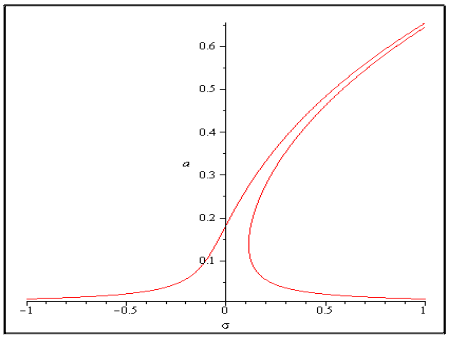


(b) Resonant time series when $\Omega = \omega$

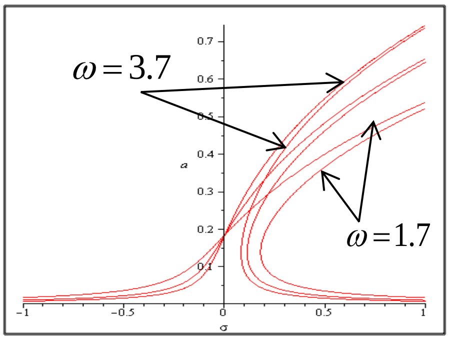


(c) Resonant time series when $\Omega = 2\omega$

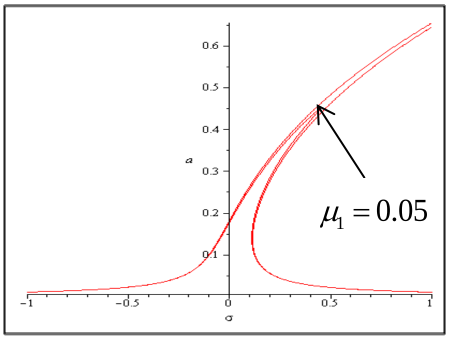
Fig 2.10 Resonant time history solution of the system with negative cubic (VF) control when: ω= 2.1, β_1= 15.0, δ= 0.03, μ_1 = 0.0005, Ω= 2.7, β_2= 5.0, f_1= 0.4, f_2 = 0.2, α = 30, G = 0.05



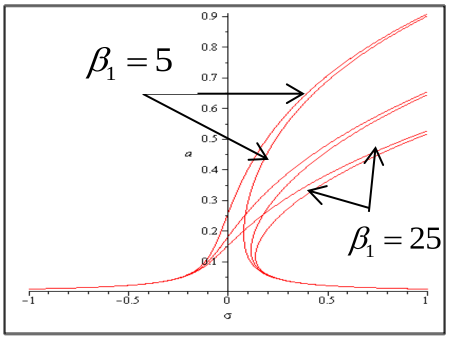
(a) Basic case



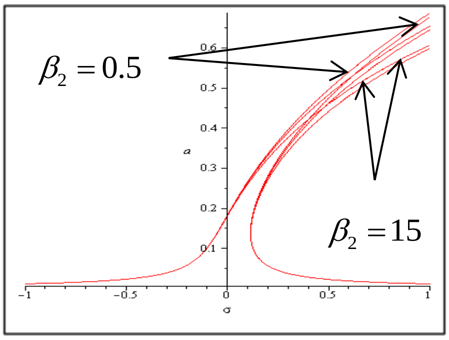
(b) Natural frequency



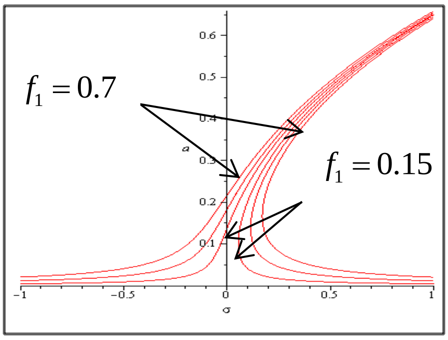
(c) The damping coefficient



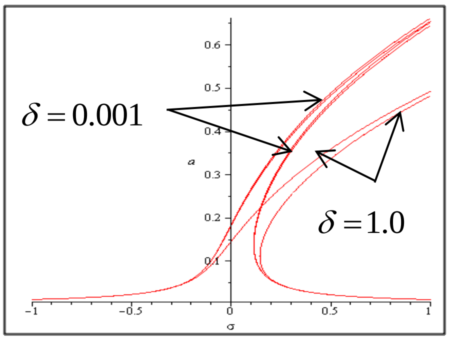
(d) Nonlinear coefficient



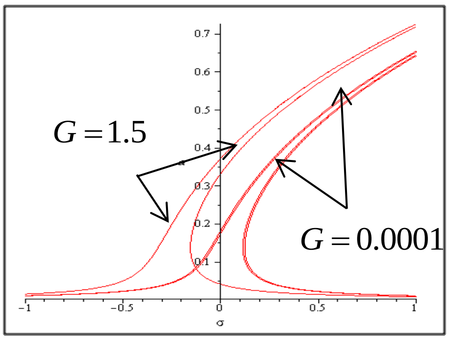
(e) Nonlinear coefficient



(f) The forcing amplitude

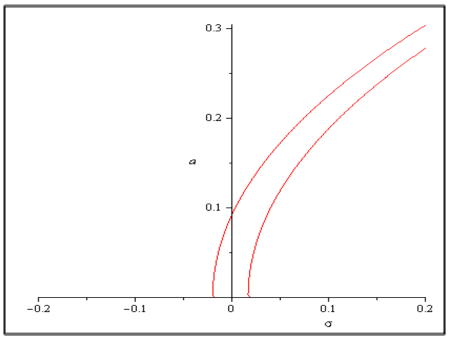


(g) Nonlinear coefficient

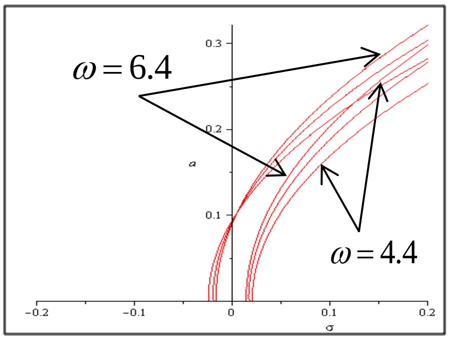


(h) The gain

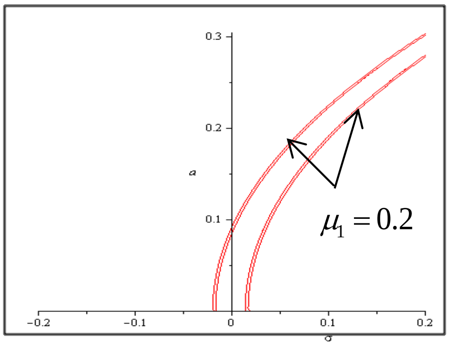
Fig 2.11 Theoretical frequency response curves to primary resonance case for (PF) control $\omega = 2.7$, $\beta_1 = 15.0$, $\delta = 0.03$, $\mu_1 = 0.0005$, $\Omega = 2.7$, $\beta_2 = 5.0$, $f_1 = 0.4$, $\alpha = 30$, $G = 0.05$.



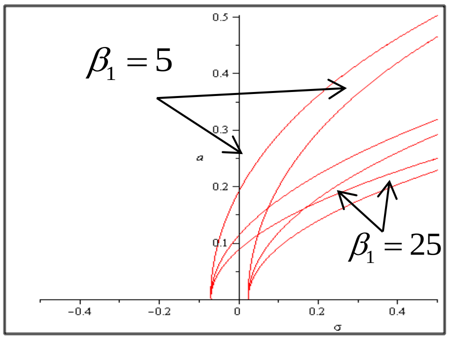
(a) Basic case



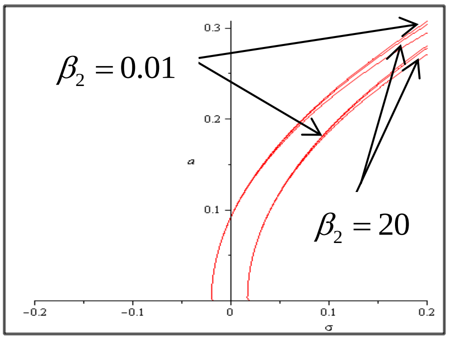
(b) Natural frequency



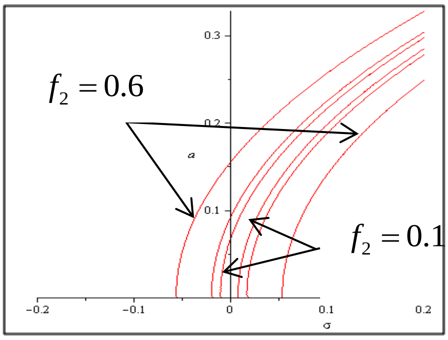
(c) The damping coefficient



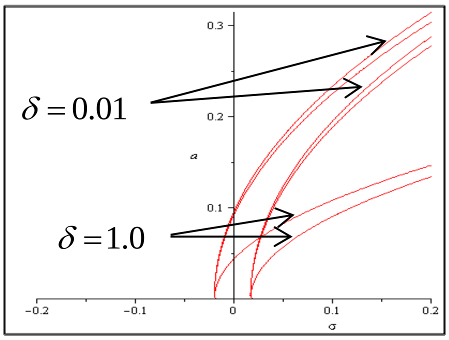
(d) Nonlinear coefficient



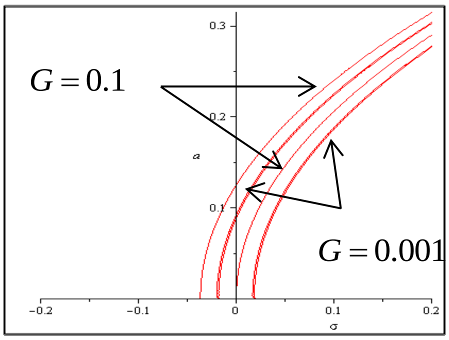
(e) Nonlinear coefficient



(f) The forcing amplitude

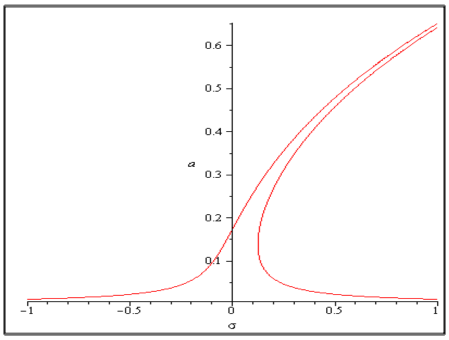


(g) Nonlinear coefficient

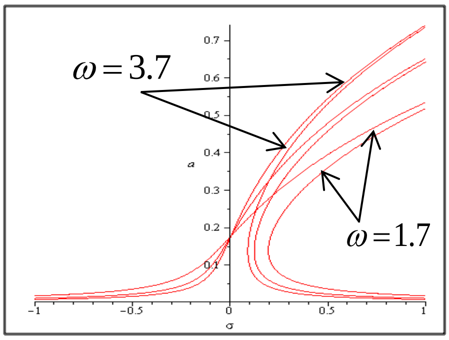


(h) The gain

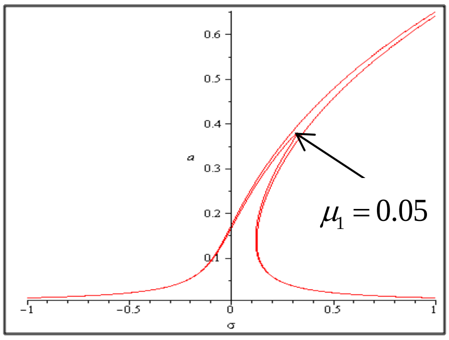
Fig 2.12 Theoretical frequency response curves to sub-harmonic resonance case for (PF) control ω= 5.4 , β_1= 15.0 , δ= 0.03, μ_1 = 0.0005 , Ω= 2.7 , β_2= 5.0 , $f_2 = 0.2$, $\alpha = 30$, $G = 0.01$.



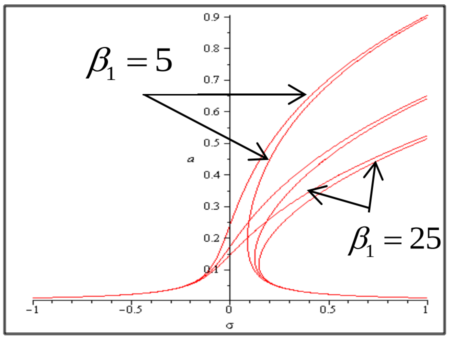
(a) Basic case



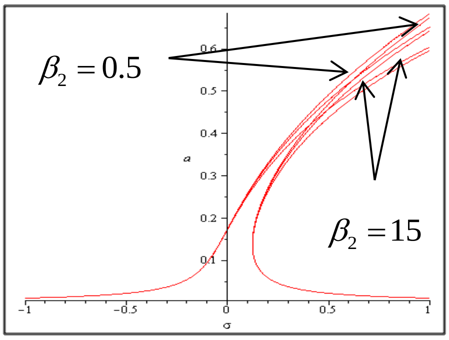
(b) Natural frequency



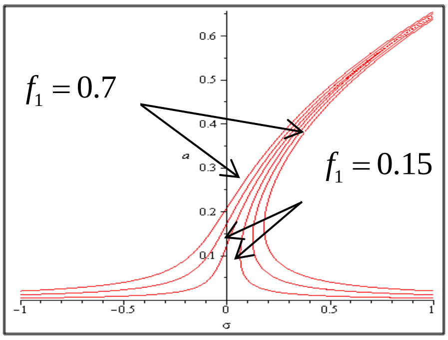
(c) The damping coefficient



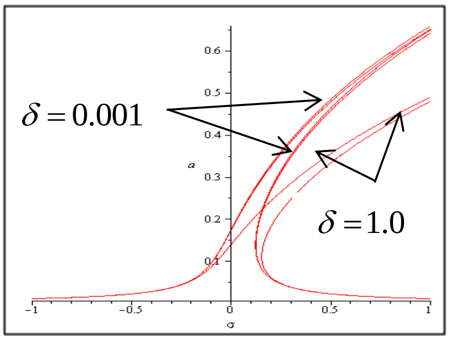
(d) Nonlinear coefficient



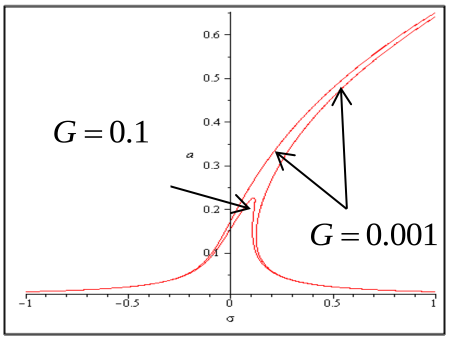
(e) Nonlinear coefficient



(f) The forcing amplitude

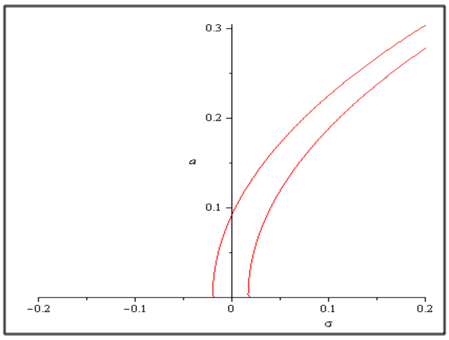


(g) Nonlinear coefficient

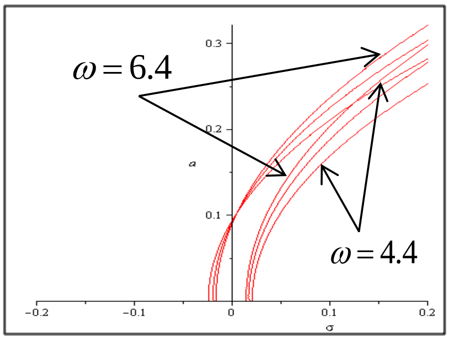


(h) The gain

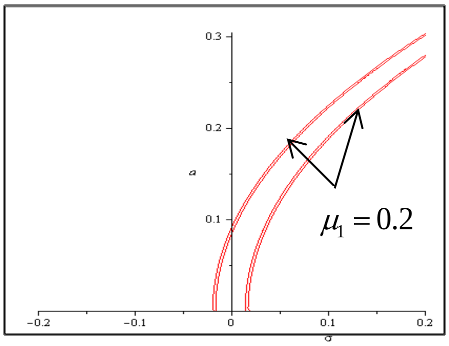
Fig 2.13 Theoretical frequency response curves to primary resonance case for negative (VF) control ω= 2.7, β_1= 15, δ= 0.03, μ_1 = 0.0005, Ω= 2.7, β_2= 5.0, f_1= 0.4, $\alpha = 30$, $G = 0.01$.



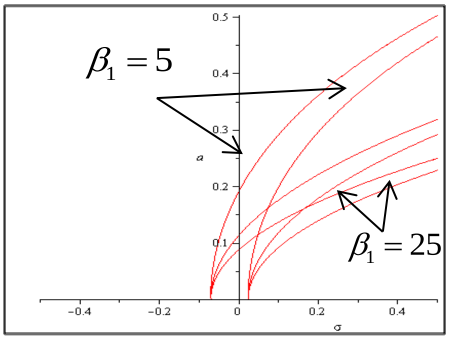
(a) Basic case



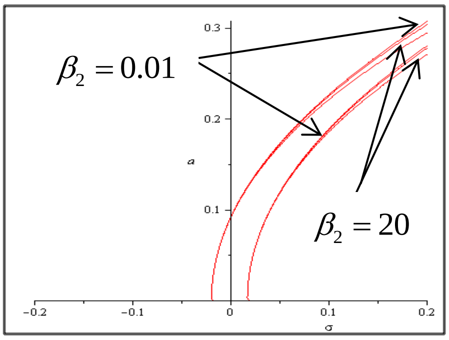
(b) Natural frequency



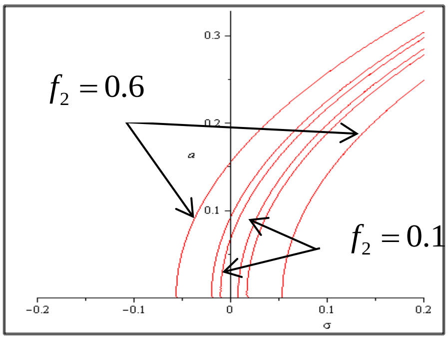
(c) The damping coefficient



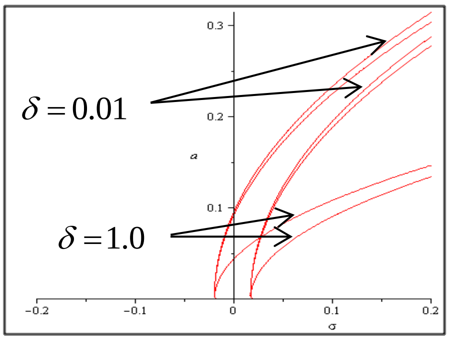
(d) Nonlinear coefficient



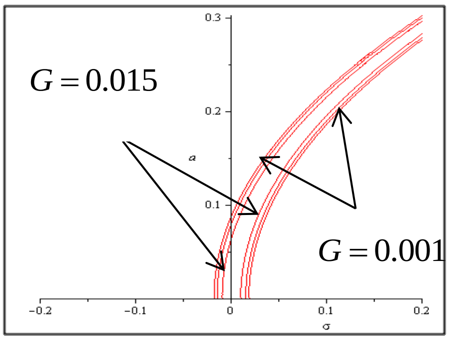
(e) Nonlinear coefficient



(f) The forcing amplitude

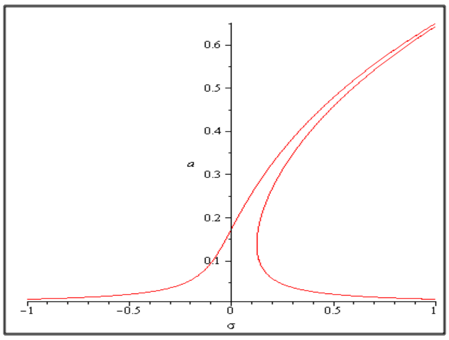


(g) Nonlinear coefficient

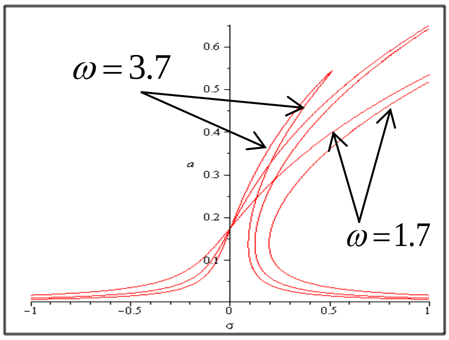


(h) The gain

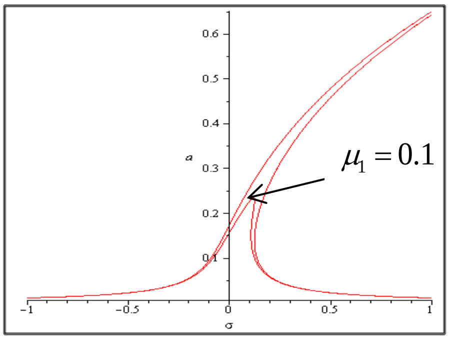
Fig 2.14 Theoretical frequency response curves to sub-harmonic resonance case for negtive (VF) control $\omega = 5.4$, $\beta_1 = 15.0$, $\delta = 0.03$, $\mu_1 = 0.0005$, $\Omega = 2.7$, $\beta_2 = 5.0$, $f_2 = 0.2$, $\alpha = 30$, $G = 0.01$.



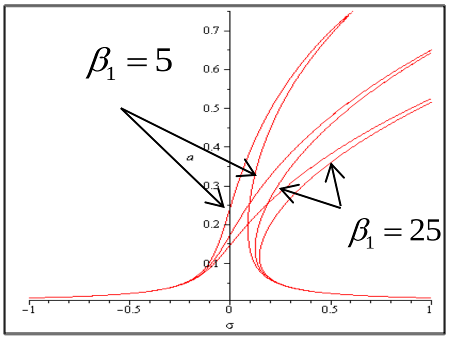
(b) Basic case



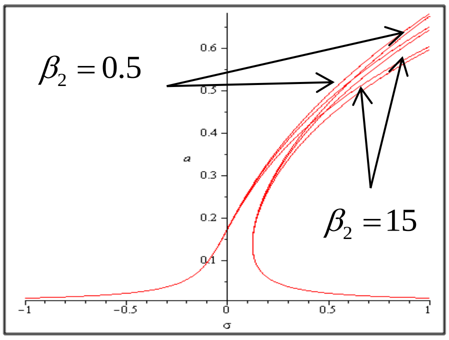
(b) Natural frequency



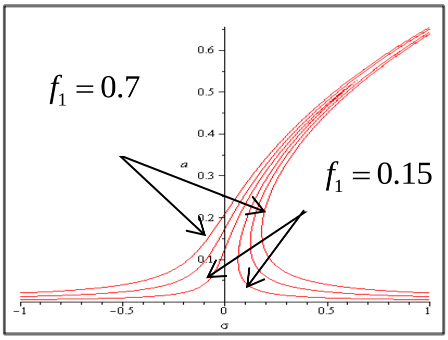
(c) The damping coefficient



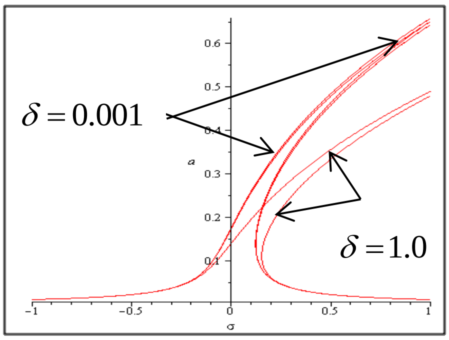
(d) Nonlinear coefficient



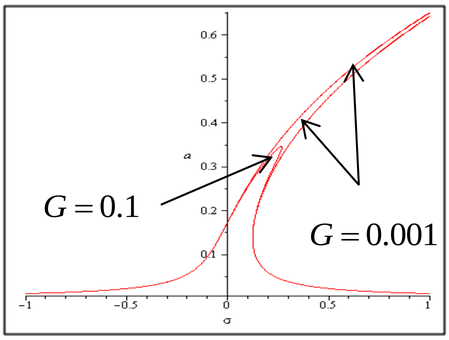
(e) Nonlinear coefficient



(f) The forcing amplitude

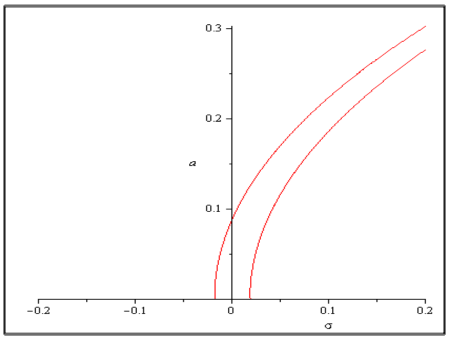


(g) Nonlinear coefficient

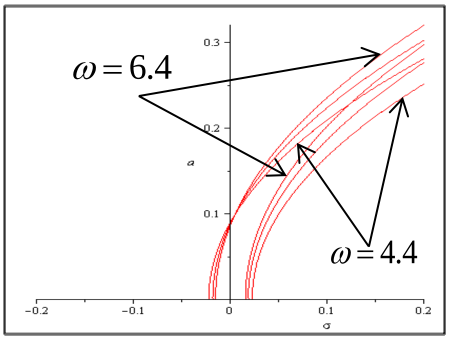


(h) The gain

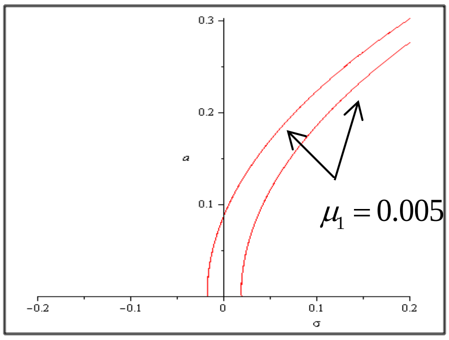
Fig 2.15 Theoretical frequency response curves to primary resonance case for negative cubic (VF) control $\omega = 2.7, \beta_1 = 15$, $\delta = 0.03$, $\mu_1 = 0.0005$, $\Omega = 2.7$, $\beta_2 = 5.0$, $f_1 = 0.4$, $\alpha = 30, G = 0.01$.



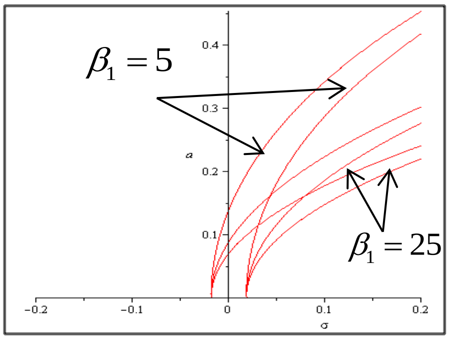
(b) Basic case



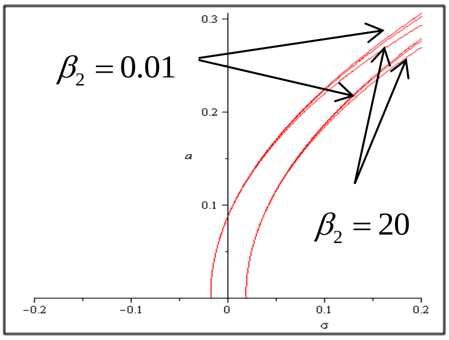
(b) Natural frequency



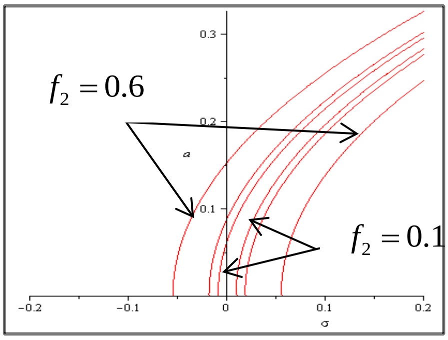
(c) The damping coefficient



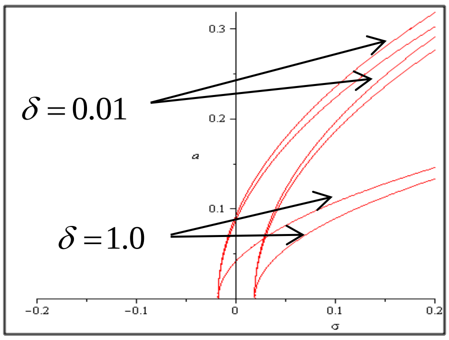
(d) Nonlinear coefficient



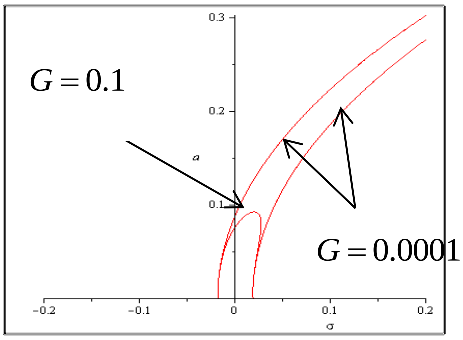
(e) Nonlinear coefficient



(f) The forcing amplitude



(g) Nonlinear coefficient



(h) The gain

Fig 2.16 Theoretical frequency response curves to sub-harmonic resonance case for negtive cubic (VF) control $\omega = 5.4$, $\beta_1 = 15.0$, $\delta = 0.03$, $\mu_1 = 0.0005$, $\Omega = 2.7$, $\beta_2 = 5.0$, $f_2 = 0.2$, $\alpha = 30$, $G = 0.001$.

Chapter 3

Passive Control of a Nonlinear Dynamical System

In this chapter, we present the perturbation and numerical solutions of two-dimensional nonlinear differential equations with two different controller, Positive Position Feedback (PPF) control and Nonlinear Saturation (NS) control . The multiple scale analytical method and Rung-Kutta fourth order numerical methods are used to investigate the system behavior and its stability. All possible resonance cases will be extracted and effect of different parameters on system behavior at resonance cases were studied.

3.1 System model

The modified second-order nonlinear ordinary differential equation that describes the dynamical behavior is given as [10,24]

$$\begin{aligned} &u'' + \mu_1 u' + \omega_s^2 u + \beta_1 u^3 + \beta_2 u^5 - \delta\left(uu'^2 + u^2 u''\right) = f_1 \cos(\Omega t)\cos(\alpha) \\ &+ u f_2 \cos(\Omega t)\sin(\alpha) + \tau F_c(t). \end{aligned} \tag{3.1}$$

We introduce two a second-order non-linear controllers, which are coupled to the main system through a control law. Then, the equation governing the dynamics of the controllers is suggested as $v'' + 2\xi\omega_c v' + \omega_c^2 v = \rho F_f(t)$. (3.2)

We choose the control signal $F_c = v$, and feedback signal $F_f = u$,for (PPF) control, and $F_c = v^2$, $F_f = uv$, for (NS) control.

So the closed loop system equations to the both controllers are

- Positive Position Feedback (PPF) control

$$\begin{aligned} &u'' + \mu_1 u' + \omega_s^2 u + \beta_1 u^3 + \beta_2 u^5 - \delta\left(uu'^2 + u^2 u''\right) = f_1 \cos(\Omega t)\cos(\alpha) \\ &+ u f_2 \cos(\Omega t)\sin(\alpha) + \tau v, \end{aligned} \tag{3.3}$$

$$v'' + 2\xi\omega_c v' + \omega_c^2 v = \rho u, \tag{3.4}$$

- Nonlinear Saturation (NS) control

$$u'' + \mu_1 u' + \omega_s^2 u + \beta_1 u^3 + \beta_2 u^5 - \delta\left(uu'^2 + u^2 u''\right) = f_1 \cos(\Omega t)\cos(\alpha) + u f_2 \cos(\Omega t)\sin(\alpha) + \tau v^2, \tag{3.5}$$

$$v'' + 2\xi\omega_c v' + \omega_c^2 v = \rho u v\ , \tag{3.6}$$

where v, v' and v'' represent displacement, velocity and acceleration of the controller, ω_c is the natural frequency of the controller, ξ is the damping coefficient of the controller , τ , ρ is nonlinear coefficients of the controller, the main system parameter is shown in chapter 2.

3.2 Perturbation analysis for the main system with indirect (PPF) control

The nonlinear differential equation (3.3) with PPF control (3.4) is scaled using the perturbation parameter ε as follows

$$u'' + \varepsilon\mu_1 u' + \omega_s^2 u + \varepsilon\beta_1 u^3 + \varepsilon\beta_2 u^5 - \varepsilon\delta\left(uu'^2 + u^2 u''\right) = \varepsilon f_1 \cos(\Omega t)\cos(\alpha) + \varepsilon u f_2 \cos(\Omega t)\sin(\alpha) + \varepsilon\tau v,$$

$$v'' + 2\xi\varepsilon\omega_c v' + \omega_c^2 v = \varepsilon\rho u\ .$$

Applying the multiple scales method, we obtain first order approximate solutions for equation (3.3) and (3.4) by seeking the solutions in the form

$$\begin{aligned} u(T_0,T_1) &= u_0(T_0,T_1) + \varepsilon u_1(T_0,T_1), \\ v(T_0,T_1) &= v_0(T_0,T_1) + \varepsilon v_1(T_0,T_1), \end{aligned} \tag{3.7}$$

where ε is a small dimensionless book keeping perturbation parameter, $T_0 = t$ and $T_1 = \varepsilon T_0 = \varepsilon t$ are the fast and slow time scales, respectively. The time derivatives transform are recast in terms of the new time scales as

$$\begin{aligned} \frac{d}{dt} &= D_0 + \varepsilon D_1, \\ \frac{d^2}{dt^2} &= D_0^2 + 2\varepsilon D_0 D_1, \end{aligned} \tag{3.8}$$

where $D_0 = \dfrac{\partial}{\partial T_0}$, $D_1 = \dfrac{\partial}{\partial T_1}$. (3.9)

Substituting u and time derivatives from equations (3.7) and (3.8), we get

$$
\begin{aligned}
u &= u_0 + \varepsilon u_1 \\
u' &= D_0 u_0 + \varepsilon D_0 u_1 + \varepsilon D_1 u_0 + \varepsilon^2 D_1 u_1 \\
u'' &= D_0{}^2 u_0 + \varepsilon D_0{}^2 u_1 + 2\varepsilon D_0 D_1 u_0 + 2\varepsilon^2 D_0 D_1 u_1,
\end{aligned} \tag{3.10}
$$

and

$$
\begin{aligned}
v &= v_0 + \varepsilon v_1 \\
v' &= D_0 v_0 + \varepsilon D_0 v_1 + \varepsilon D_1 v_0 + \varepsilon^2 D_1 v_1 \\
v'' &= D_0{}^2 v_0 + \varepsilon D_0{}^2 v_1 + 2\varepsilon D_0 D_1 v_0 + 2\varepsilon^2 D_0 D_1 v_1.
\end{aligned} \tag{3.11}
$$

Substituting equations (3.10) and (3.11) into equations (3.3) and (3.4) we get,

$$
\begin{aligned}
& D_0^2 u_0 + \varepsilon D_0^2 u_1 + 2\varepsilon D_0 D_1 u_0 + 2\varepsilon^2 D_0 D_1 u_1 + \varepsilon \mu_1 (D_0 u_0 + \varepsilon D_0 u_1 + \varepsilon D_1 u_0 + \varepsilon^2 D_1 u_1) \\
& + \omega_s^2 (u_0 + \varepsilon u_1) + \varepsilon \beta_1 (u_0 + \varepsilon u_1)^3 + \varepsilon \beta_2 (u_0 + \varepsilon u_1)^5 \\
& - \varepsilon \delta (u_0 + \varepsilon u_1)(D_0 u_0 + \varepsilon D_0 u_1 + \varepsilon D_1 u_0 + \varepsilon^2 D_1 u_1)^2 \\
& - \varepsilon \delta (u_0 + \varepsilon u_1)^2 (D_0^2 u_0 + \varepsilon D_0^2 u_1 + 2\varepsilon D_0 D_1 u_0 + 2\varepsilon^2 D_0 D_1 u_0 \\
& = \varepsilon f_1 \cos(\Omega t) \cos(\alpha) + \varepsilon (u_0 + \varepsilon u_1) f_2 \cos(\Omega t) \sin(\alpha) + \varepsilon \tau (v_0 + \varepsilon v_1),
\end{aligned} \tag{3.12}
$$

and

$$
\begin{aligned}
& D_0^2 v_0 + \varepsilon D_0^2 v_1 + 2\varepsilon D_0 D_1 v_0 + 2\varepsilon^2 D_0 D_1 v_1 + 2\varepsilon \xi \omega_c (D_0 v_0 + \varepsilon D_0 v_1 + \varepsilon D_1 v_0 + \varepsilon^2 D_1 v_1) \\
& + \omega_c^2 (v_0 + \varepsilon v_1) = \varepsilon \rho (u_0 + \varepsilon u_1).
\end{aligned} \tag{3.13}
$$

Eliminating terms containing the power of $\varepsilon \geq 2$, equation (3.12) and (3.13) become

$$
\begin{aligned}
& D_0^2 u_0 + \varepsilon D_0^2 u_1 + 2\varepsilon D_0 D_1 u_0 + \varepsilon \mu_1 D_0 u_0 + \omega_s^2 u_0 + \varepsilon \omega_s^2 u_1 + \varepsilon \beta_1 u_0^3 + \varepsilon \beta_2 u_0^5 - 2\varepsilon \delta D_0^2 u_0^3 \\
& - \varepsilon f_1 \cos(\Omega t) \cos(\alpha) - \varepsilon u_0 f_2 \cos(\Omega t) \sin(\alpha) - \varepsilon \tau v_0 = 0,
\end{aligned} \tag{3.14}
$$

and

$$
D_0^2 v_0 + \varepsilon D_0^2 v_1 + 2\varepsilon D_0 D_1 v_0 + 2\varepsilon \xi \omega_c D_0 v_0 + \omega_c^2 v_0 + \varepsilon \omega_c^2 v_1 - \varepsilon \rho u_0 = 0. \tag{3.15}
$$

Equating the coefficient of same powers of ε in equation (3.14) and (3.15), gives

$O(\varepsilon^0)$:

$$
\left(D_0^2 + \omega_s^2 \right) u_0 = 0\,, \tag{3.16}
$$

and

$$\left(D_0^2+\omega_c^2\right)v_0=0. \tag{3.17}$$

$O(\varepsilon^1)$:

$$\left(D_0^2+\omega_s^2\right)u_1=-2D_0D_1u_0-\mu_1D_0u_0-\beta_1u_0^3-\beta_2u_0^5+2\delta D_0^2u_0^3+f_1\cos(\Omega t)\cos(\alpha)$$
$$+u_0f_2\cos(\Omega t)\sin(\alpha)+\tau v_0, \tag{3.18}$$

and

$$\left(D_0^2+\omega_c^2\right)v_1=-2D_0D_1v_0-2\varepsilon\xi\omega_cD_0v_0+\rho u_0. \tag{3.19}$$

The general solution of equation (3.16) and (3.17) is given by

$$u_0(T_0,T_1)=A(T_1)e^{i\omega_sT_0}+\bar{A}(T_1)e^{-i\omega_sT_0}, \tag{3.20}$$

and

$$v_0(T_0,T_1)=B(T_1)e^{i\omega_cT_0}+\bar{B}(T_1)e^{-i\omega_cT_0}. \tag{3.21}$$

Where the quantities $A(T_1)$ and $B(T_1)$ are unknown function in T_1 at this stage of the analysis.

Substituting equation (3.20) and (3.21) into equation (3.18) and (3.19), we get

$$\left(D_0^2+\omega_s^2\right)u_1=-2D_0D_1\left(Ae^{i\omega_sT_0}+\bar{A}e^{-i\omega_sT_0}\right)-\mu_1D_0\left(Ae^{i\omega_sT_0}+\bar{A}e^{-i\omega_sT_0}\right)$$
$$-\beta_1\left(Ae^{i\omega_sT_0}+\bar{A}e^{-i\omega_sT_0}\right)^3-\beta_2\left(Ae^{i\omega_sT_0}+\bar{A}e^{-i\omega_sT_0}\right)^5+2\delta D_0^2\left(Ae^{i\omega_sT_0}+\bar{A}e^{-i\omega_sT_0}\right)^3$$
$$+f_1\cos(\Omega t)\cos(\alpha)+\left(Ae^{i\omega_sT_0}+\bar{A}e^{-i\omega_sT_0}\right)f_2\cos(\Omega t)\sin(\alpha)+\tau\left(Be^{i\omega_cT_0}+\bar{B}e^{-i\omega_cT_0}\right), \tag{3.22}$$

and

$$\left(D_0^2+\omega_c^2\right)v_1=-2D_0D_1\left(Be^{i\omega_cT_0}+\bar{B}e^{-i\omega_cT_0}\right)-2\xi\omega_cD_0\left(Be^{i\omega_cT_0}+\bar{B}e^{-i\omega_cT_0}\right)$$
$$+\rho\left(Ae^{i\omega_sT_0}+\bar{A}e^{-i\omega_sT_0}\right). \tag{3.23}$$

Expanding and simplifying equation (3.22) and (3.23), we get

$$\left(D_0^2+\omega_s^2\right)u_1=-2D_0D_1Ae^{i\omega_sT_0}-2D_0D_1\bar{A}e^{-i\omega_sT_0}-\mu_1D_0Ae^{i\omega_sT_0}-\mu_1D_0\bar{A}e^{-i\omega_sT_0}$$
$$-\beta_1A^3e^{3i\omega_sT_0}-3\beta_1A^2\bar{A}e^{i\omega_sT_0}-3\beta_1A\bar{A}^2e^{-i\omega_sT_0}-\beta_1\bar{A}^3e^{-3i\omega_sT_0}-\beta_2A^5e^{5i\omega_sT_0}$$
$$-5\beta_2A^4\bar{A}e^{3i\omega_sT_0}-10\beta_2A^3\bar{A}^2e^{i\omega_sT_0}-10\beta_2A^2\bar{A}^3e^{-i\omega_sT_0}-5\beta_2A\bar{A}^4e^{-3i\omega_sT_0} \tag{3.24}$$
$$-\beta_2\bar{A}^5e^{-5i\omega_sT_0}+2\delta D_0^2A^3e^{3i\omega_sT_0}+6\delta D_0^2A^2\bar{A}e^{i\omega_sT_0}+6\delta D_0^2A\bar{A}^2e^{-i\omega_sT_0}$$
$$+2\delta D_0^2\bar{A}^3e^{-3i\omega_sT_0}+f_1\cos(\Omega t)\cos(\alpha)+f_2Ae^{i\omega_sT_0}\cos(\Omega t)\sin(\alpha)$$
$$+f_2\bar{A}e^{-i\omega_sT_0}\cos(\Omega t)\sin(\alpha)+\tau Be^{i\omega_cT_0}+\tau\bar{B}e^{-i\omega_cT_0},$$

and

$$\left(D_0^2+\omega_c^2\right)v_1=-2D_0D_1Be^{i\omega_cT_0}-2D_0D_1\bar{B}e^{-i\omega_cT_0}-2\xi\omega_cD_0Be^{i\omega_cT_0}$$
$$-2\xi\omega_cD_0\bar{B}e^{-i\omega_cT_0}+\rho Ae^{i\omega_sT_0}+\rho\bar{A}e^{-i\omega_sT_0}. \tag{3.25}$$

Substituting equations (3.9) and Using the form $\cos(\omega T_0)=\dfrac{e^{i\omega T_0}+e^{-i\omega T_0}}{2}$, $\sin(\omega T_0)=\dfrac{e^{i\omega T_0}-e^{-i\omega T_0}}{2i}$ into equation (3.24) and (3.25), to get

$$\left(D_0^2+\omega_s^2\right)u_1=-2i\omega_sA'e^{i\omega_sT_0}+2i\omega_s\bar{A}'e^{-i\omega_sT_0}-\mu_1i\omega_sAe^{i\omega_sT_0}+\mu_1i\omega_s\bar{A}e^{-i\omega_sT_0}$$
$$-\beta_1A^3e^{3i\omega_sT_0}-3\beta_1A^2\bar{A}e^{i\omega_sT_0}-3\beta_1A\bar{A}^2e^{-i\omega_sT_0}-\beta_1\bar{A}^3e^{-3i\omega_sT_0}-\beta_2A^5e^{5i\omega_sT_0}$$
$$-5\beta_2A^4\bar{A}e^{3i\omega_sT_0}-10\beta_2A^3\bar{A}^2e^{i\omega_sT_0}-10\beta_2A^2\bar{A}^3e^{-i\omega_sT_0}-5\beta_2A\bar{A}^4e^{-3i\omega_sT_0}-\beta_2\bar{A}^5e^{-5i\omega_sT_0}$$
$$-18\omega_s^2\delta A^3e^{3i\omega_sT_0}-6\omega_s^2\delta A^2\bar{A}e^{i\omega_sT_0}-6\omega_s^2\delta A\bar{A}^2e^{-i\omega_sT_0}-18\omega_s^2\delta\bar{A}^3e^{-3i\omega T_0}+\frac{1}{2}f_1e^{i\Omega T_0}\cos\left(\alpha\right) \tag{3.26}$$
$$+\frac{1}{2}f_1e^{-i\Omega T_0}\cos\left(\alpha\right)+\frac{1}{2}f_2Ae^{i\omega T_0+i\Omega T_0}\sin\left(\alpha\right)+\frac{1}{2}f_2Ae^{i\omega T_0-i\Omega T_0}\sin\left(\alpha\right)$$
$$+\frac{1}{2}f_2\bar{A}e^{-i\omega T_0+i\Omega T_0}\sin\left(\alpha\right)+\frac{1}{2}f_2\bar{A}e^{-i\omega T_0-i\Omega T_0}\sin\left(\alpha\right)+\tau Be^{i\omega_cT_0}+\tau\bar{B}e^{-i\omega_cT_0},$$

and

$$\left(D_0^2+\omega_c^2\right)v_1=-2i\omega_cB'e^{i\omega_cT_0}+2i\omega_c\bar{B}'e^{-i\omega_cT_0}-2i\xi\omega_c^2Be^{i\omega_cT_0}$$
$$+2i\xi\omega_c^2D_0\bar{B}e^{-i\omega_cT_0}+\rho Ae^{i\omega_sT_0}+\rho\bar{A}e^{-i\omega_sT_0}. \tag{3.27}$$

Simplifying equation (3.26) and (3.27) we get

$$\left(D_0^2+\omega_s^2\right)u_1=\left(-2i\omega_s A'-\mu_1 i\omega_s A-3\beta_1 A^2\bar{A}-10\beta_2 A^3\bar{A}^2-6\omega_s^2\delta A^2\bar{A}\right)e^{i\omega_s T_0}$$
$$+\left(-\beta_1 A^3-5\beta_2 A^4\bar{A}-18\omega_s^2\delta A^3\right)e^{3i\omega_s T_0}-\beta_2 A^5 e^{5i\omega_s T_0}+\frac{1}{2}f_1 e^{i\Omega T_0}\cos(\alpha) \tag{3.28}$$
$$+\frac{1}{2}f_2 A e^{i(\omega+\Omega)T_0}\sin(\alpha)+\frac{1}{2}f_2 A e^{i(\omega-\Omega)T_0}\sin(\alpha)+\tau B e^{i\omega_c T_0}+cc,$$

and

$$\left(D_0^2+\omega_c^2\right)v_1=\left(-2i\omega_c B'-2i\xi\omega_c^2 B\right)e^{i\omega_c T_0}+\rho A e^{i\omega_s T_0}+cc. \tag{3.29}$$

where cc denotes the complex conjugate terms.

The particular solution of equation (3.28) and (3.29) can be written in the following form

$$u_1(T_0,T_1)=A_1(T_1)e^{i\omega T_0}-\frac{1}{8\omega_s^2}\left(-\beta_1 A^3-5\beta_2 A^4\bar{A}-18\omega_s^2\delta A^3\right)e^{3i\omega_s T_0}+\frac{1}{24\omega_s^2}\beta_2 A^5 e^{5i\omega_s T_0}$$
$$+\frac{1}{2(\omega-\Omega)(\omega+\Omega)}f_1\cos(\alpha)e^{i\Omega T_0}-\frac{1}{2\Omega(2\omega+\Omega)}f_2 A e^{i(\omega+\Omega)T_0}\sin(\alpha) \tag{3.30}$$
$$+\frac{1}{2\Omega(2\omega-\Omega)}f_2 A e^{i(\omega-\Omega)T_0}\sin(\alpha)+\frac{1}{(-\omega_c+\omega_s)(\omega_c+\omega_s)}\tau B e^{i\omega_c T_0}+cc,$$

and

$$v_1(T_0,T_1)=B_1(T_1)e^{i\omega_c T_0}+\frac{1}{(-\omega_s+\omega_c)(\omega_s+\omega_c)}\rho A e^{i\omega_s T_0}+cc\,. \tag{3.31}$$

From the equation (3.28) and (3.29) the reported resonance cases at this approximation order is simultaneous resonance $\Omega=\omega_s$ and $\omega_s=\omega_c$

3.3 Stability analysis

3.3.1 Simultaneous primary resonance $\Omega=\omega_s$ and $\omega_s=\omega_c$

In this case we introduce a detuning parameters σ_1 and σ_2 such that

$$\Omega=\omega_s+\varepsilon\sigma_1 \quad , \quad \omega_c=\omega_s+\varepsilon\sigma_2\,. \tag{3.32}$$

Substituting equation (3.32) into equation (3.28) and (3.29), eliminating the terms that produce secular term and performing some algebraic manipulations, we obtain

$$-2i\omega_s A' - \mu_1 i\omega_s A - 3\beta_1 A^2\bar{A} - 10\beta_2 A^3\bar{A}^2 - 6\omega_s^2\delta A^2\bar{A} + \frac{1}{2} f_1 e^{i\sigma_1 T_1}\cos(\alpha) + \tau B e^{i\sigma_2 T_1} = 0, \qquad (3.33)$$

and

$$-2i\omega_c B' - 2i\xi\omega_c^2 B + \rho A e^{-i\sigma_2 T_1} = 0. \qquad (3.34)$$

Now we use polar forms

$$A = \frac{1}{2}a_1 e^{i\theta_1} \quad , \quad B = \frac{1}{2}a_2 e^{i\theta_2} \ , \qquad (3.35)$$

where $a_1, a_2, \theta_1, \theta_2$ are functions in T_1.

Substituting A, B from equation (3.35) into equation (3.33) and (3.34), we get

$$\begin{aligned}&\omega_s a_1\theta_1' e^{i\theta_1} - i\omega_s a_1' e^{i\theta_1} - \frac{1}{2}\mu_1 i\omega_s a_1 e^{i\theta_1} - \frac{3}{8}\beta_1 a_1^3 e^{i\theta_1} - \frac{5}{16}\beta_2 a_1^5 e^{i\theta_1}\\ &-\frac{3}{4}\omega_s^2\delta a_1^3 e^{i\theta_1} + \frac{1}{2} f_1 e^{i\sigma_1 T_1}\cos(\alpha) + \frac{1}{2}\tau a_2 e^{i\theta_2 + i\sigma_2 T_1} = 0,\end{aligned} \qquad (3.36)$$

and

$$\omega_c a_2\theta_2' e^{i\theta_2} - i\omega_c a_2' e^{i\theta_2} - i\xi\omega_c^2 a_2 e^{i\theta_2} + \frac{1}{2}\rho a_1 e^{i\theta_1 - i\sigma_2 T_1} = 0\,. \qquad (3.37)$$

Dividing equation (3.36) by $\omega_s e^{i\theta_1}$ and dividing equation (3.37) by $\omega_c e^{i\theta_2}$, we obtain

$$\begin{aligned}&a_1\theta_1' - ia_1' - \frac{1}{2}\mu_1 i a_1 - \frac{3}{8\omega_s}\beta_1 a_1^3 - \frac{5}{16\omega_s}\beta_2 a_1^5 - \frac{3}{4}\omega_s\delta a_1^3\\ &+\frac{1}{2\omega_s} f_1 e^{-i\theta_1 + i\sigma_1 T_1}\cos(\alpha) + \frac{1}{2\omega_s}\tau a_2 e^{-i\theta_1 + i\theta_2 + i\sigma_2 T_1} = 0,\end{aligned} \qquad (3.38)$$

and

$$a_2\theta_2' - ia_2' - i\xi\omega_c a_2 + \frac{1}{2\omega_c}\rho a_1 e^{-i\theta_2 + i\theta_1 - i\sigma_2 T_1} = 0. \qquad (3.39)$$

Using the form $e^{ix} = \cos x + i\sin x$,we get

$$a_1\theta_1' - ia_1' - \frac{1}{2}\mu_1 ia_1 - \frac{3}{8\omega_s}\beta_1 a_1^3 - \frac{5}{16\omega_s}\beta_2 a_1^5 - \frac{3}{4}\omega_s \delta a_1^3$$
$$+\frac{1}{2\omega_s} f_1 \cos\left(-\theta_1 + \sigma_1 T_1\right)\cos\left(\alpha\right) + \frac{1}{2\omega_s} if_1 \sin\left(-\theta_1 + \sigma_1 T_1\right)\cos\left(\alpha\right) \tag{3.40}$$
$$+\frac{1}{2\omega_s}\tau a_2 \cos(-\theta_1 + \theta_2 + \sigma_2 T_1) + \frac{1}{2\omega_s} i\tau a_2 \sin(-\theta_1 + \theta_2 + \sigma_2 T_1) = 0,$$

and

$$a_2\theta_2' - ia_2' - i\xi\omega_c a_2 + \frac{1}{2\omega_c}\rho a_1 \cos\left(-\theta_2 + \theta_1 - \sigma_2 T_1\right) + \frac{1}{2\omega_c} i\rho a_1 \sin\left(-\theta_2 + \theta_1 - \sigma_2 T_1\right) = 0\,. \tag{3.41}$$

Separating imaginary and real parts of equations (3.40) and (3.41), we get

$$2a_1' = -\mu_1 a_1 + \frac{1}{\omega_s} f_1 \sin\left(-\theta_1 + \sigma_1 T_1\right)\cos\left(\alpha\right) + \frac{1}{\omega_s}\tau a_2 \sin(-\theta_1 + \theta_2 + \sigma_2 T_1)\,, \tag{3.42}$$

$$2a_1\theta_1' - \frac{3}{4\omega_s}\beta_1 a_1^3 - \frac{5}{8\omega_s}\beta_2 a_1^5 - \frac{3}{2}\omega_s \delta a_1^3 + \frac{1}{\omega_s} f_1 \cos\left(-\theta_1 + \sigma_1 T_1\right)\cos\left(\alpha\right)$$
$$+\frac{1}{\omega_s}\tau a_2 \cos(-\theta_1 + \theta_2 + \sigma_2 T_1) = 0, \tag{3.43}$$

$$a_2' = -\xi\omega_c a_2 - \frac{1}{2\omega_c}\rho a_1 \sin\left(-\theta_1 + \theta_2 + \sigma_2 T_1\right), \tag{3.44}$$

and

$$a_2\theta_2' + \frac{1}{2\omega_c}\rho a_1 \cos\left(-\theta_1 + \theta_2 + \sigma_2 T_1\right) = 0. \tag{3.45}$$

Letting $\Lambda_1 = \frac{1}{\omega_s} f_1, \Lambda_2 = \frac{1}{\omega_s}\tau a_2, \Lambda_3 = \frac{1}{2\omega_c}\rho a_1, \gamma_1 = \left(-\theta_1 + \sigma_1 T_1\right)$ and $\gamma_2 = \left(-\theta_1 + \theta_2 + \sigma_2 T_1\right)$

.

Then, equations (3.42) - (3.45) become

$$2a_1' = -\mu_1 a_1 + \Lambda_1 \sin\left(\gamma_1\right)\cos\left(\alpha\right) + \Lambda_2 \sin\left(\gamma_2\right), \tag{3.46}$$

$$2a_1\gamma_1' = 2\sigma_1 a_1 - \frac{3}{4\omega_s}\beta_1 a_1^3 - \frac{5}{8\omega_s}\beta_2 a_1^5 - \frac{3}{2}\omega_s \delta a_1^3 + \Lambda_1 \cos\left(\gamma_1\right)\cos\left(\alpha\right) + \Lambda_2 \cos\left(\gamma_2\right), \tag{3.47}$$

$$a_2' = -\xi\omega_c a_2 - \Lambda_3 \sin\gamma_2 , \tag{3.48}$$

and

$$a_2(\gamma_1' - \gamma_2') = a_2(\sigma_1 - \sigma_2) + \Lambda_3 \cos\gamma_2 . \tag{3.49}$$

The steady state solutions correspond to constant $a_1, a_2, \gamma_1, \gamma_2$ that is $a_1' = a_2' = \gamma_1' = \gamma_2' = 0$

$$\mu_1 a_1 = \Lambda_1 \sin(\gamma_1)\cos(\alpha) + \Lambda_2 \sin(\gamma_2) , \tag{3.50}$$

$$-2\sigma_1 a_1 + \frac{3}{4\omega_s}\beta_1 a_1^3 + \frac{5}{8\omega_s}\beta_2 a_1^5 + \frac{3}{2}\omega_s \delta a_1^3 = \Lambda_1 \cos(\gamma_1)\cos(\alpha) + \Lambda_2 \cos(\gamma_2), \tag{3.51}$$

$$\xi\omega_c a_2 = -\Lambda_3 \sin\gamma_2 , \tag{3.52}$$

and

$$-a_2(\sigma_1 - \sigma_2) = \Lambda_3 \cos\gamma_2 . \tag{3.53}$$

Squaring both sides of equations (3.50), (3.51) and adding, to gives

$$\begin{aligned}(\mu_1 a_1)^2 + \left(-2\sigma_1 a_1 + \frac{3}{4\omega_s}\beta_1 a_1^3 + \frac{5}{8\omega_s}\beta_2 a_1^5 + \frac{3}{2}\omega_s \delta a_1^3\right)^2 &= \left(\Lambda_1 \sin(\gamma_1)\cos(\alpha) + \Lambda_2 \sin(\gamma_2)\right)^2 \\ &+ \left(\Lambda_1 \cos(\gamma_1)\cos(\alpha) + \Lambda_2 \cos(\gamma_2)\right)^2 ,\end{aligned} \tag{3.54}$$

and squaring both sides of equations (3.52), (3.53) and adding, to gives

$$\left(\xi\omega_c a_2\right)^2 + \left(-a_2(\sigma_1 - \sigma_2)\right)^2 = \left(-\Lambda_3 \sin\gamma_2\right)^2 + \left(\Lambda_3 \cos\gamma_2\right)^2 . \tag{3.55}$$

Simplifying (3.54) and (3.55), we get

$$(\mu_1 a_1)^2 + \left(-2\sigma_1 a_1 + \frac{3}{4\omega_s}\beta_1 a_1^3 + \frac{5}{8\omega_s}\beta_2 a_1^5 + \frac{3}{2}\omega_s \delta a_1^3\right)^2 = \Lambda_1^2 \cos^2(\alpha) + \Lambda_2^2 + 2\Lambda_1\Lambda_2 \cos(\alpha) , \tag{3.56}$$

and

$$\left(\xi\omega_c a_2\right)^2 + \left(a_2(\sigma_1 - \sigma_2)\right)^2 = \Lambda_3^2 . \tag{3.57}$$

Equation (3.56) and (3.57) are called frequency response equations

(a) Trivial solution :

To determine the stability of the trivial solutions, we investigate the solutions of the linearized form equations (3.33) and (3.34)

$$-2i\omega_s A' - \mu_1 i\omega_s A + \tau B e^{i\sigma_2 T_1} = 0, \tag{3.58}$$

and

$$-2i\omega_c B' - 2i\xi\omega_c^2 B + \rho A e^{-i\sigma_2 T_1} = 0. \tag{3.59}$$

A and B are expressed in cartesian form as

$$A = \frac{1}{2}\left(p_1 - ip_2\right)e^{i\phi_1 T_1} \text{ and } B = \frac{1}{2}\left(p_3 - ip_4\right)e^{i\phi_2 T_1},$$

where p_1, p_2, p_3, p_4 are real.

Substituting in equations (3.58) and (3.59), we get

$$\begin{aligned}&-2i\omega_s\left(\frac{1}{2}\left(p_1' - ip_2'\right)e^{i\phi_1 T_1} + \frac{1}{2}i\phi_1\left(p_1 - ip_2\right)e^{i\phi_1 T_1}\right) - \frac{1}{2}i\omega_s\mu_1\left(p_1 - ip_2\right)e^{i\phi_1 T_1} \\ &+ \frac{1}{2}\tau\left(p_3 - ip_4\right)e^{i\phi_2 T_1 + i\sigma_2 T_1} = 0,\end{aligned} \tag{3.60}$$

and

$$\begin{aligned}&-2i\omega_c\left(\frac{1}{2}\left(p_3' - ip_4'\right)e^{i\phi_2 T_1} + \frac{1}{2}i\phi_2\left(p_3 - ip_4\right)e^{i\phi_2 T_1}\right) - i\xi\omega_c^2\left(p_3 - ip_4\right)e^{i\phi_2 T_1} \\ &+ \frac{1}{2}\rho\left(p_1 - ip_2\right)e^{i\phi_1 T_1 - i\sigma_2 T_1} = 0.\end{aligned} \tag{3.61}$$

Dividing both sides of equation (3.60) by $\omega_s e^{i\phi_1 T_1}$ and both sides of equation (3.61) by $\omega_c e^{i\phi_2 T_1}$ and using the form $e^{ix} = \cos x + i\sin x$, to get

$$
\begin{aligned}
&-ip_1' - p_2' + \phi_1 p_1 - i\phi_1 p_2 - \frac{1}{2} ip_1\mu_1 - \frac{1}{2} p_2\mu_1 + \frac{1}{2\omega_s}\tau p_3 \cos\left(-\phi_1 T_1 + \phi_2 T_1 + \sigma_2 T_1\right) \\
&+\frac{1}{2\omega_s} i\tau p_3 \sin\left(-\phi_1 T_1 + \phi_2 T_1 + \sigma_2 T_1\right) - \frac{1}{2\omega_s} i\tau p_4 \cos\left(-\phi_1 T_1 + \phi_2 T_1 + \sigma_2 T_1\right) \\
&+\frac{1}{2\omega_s}\tau p_4 \sin\left(-\phi_1 T_1 + \phi_2 T_1 + \sigma_2 T_1\right) = 0,
\end{aligned}
\tag{3.62}
$$

and

$$
\begin{aligned}
&-ip_3' - p_4' + \phi_2 p_3 - i\phi_2 p_4 - i\omega_c p_3\xi - \omega_c p_4\xi + \frac{1}{2\omega_c}\rho p_1 \cos\left(-\phi_2 T_1 + \phi_1 T_1 - \sigma_2 T_1\right) \\
&-\frac{1}{2\omega_c} i\rho p_2 \cos\left(-\phi_2 T_1 + \phi_1 T_1 - \sigma_2 T_1\right) + \frac{1}{2\omega_c} i\rho p_1 \sin\left(-\phi_2 T_1 + \phi_1 T_1 - \sigma_2 T_1\right) \\
&+\frac{1}{2\omega_c}\rho p_2 \sin\left(-\phi_2 T_1 + \phi_1 T_1 - \sigma_2 T_1\right) = 0.
\end{aligned}
\tag{3.63}
$$

Separating real and imaginary parts in equations (3.62) and (3.63), we get

$$
\begin{aligned}
&-p_1' - \phi_1 p_2 - \frac{1}{2} p_1\mu_1 + \frac{1}{2\omega_s}\tau p_3 \sin\left(-\phi_1 T_1 + \phi_2 T_1 + \sigma_2 T_1\right) \\
&-\frac{1}{2\omega_s}\tau p_4 \cos\left(-\phi_1 T_1 + \phi_2 T_1 + \sigma_2 T_1\right) = 0,
\end{aligned}
\tag{3.64}
$$

$$
\begin{aligned}
&-p_2' + \phi_1 p_1 - \frac{1}{2} p_2\mu_1 + \frac{1}{2\omega_s}\tau p_3 \cos\left(-\phi_1 T_1 + \phi_2 T_1 + \sigma_2 T_1\right) \\
&+\frac{1}{2\omega_s}\tau p_4 \sin\left(-\phi_1 T_1 + \phi_2 T_1 + \sigma_2 T_1\right) = 0,
\end{aligned}
\tag{3.65}
$$

$$
\begin{aligned}
&-p_3' - \phi_2 p_4 - \omega_c p_3\xi - \frac{1}{2\omega_c}\rho p_2 \cos\left(-\phi_2 T_1 + \phi_1 T_1 - \sigma_2 T_1\right) \\
&+\frac{1}{2\omega_c}\rho p_1 \sin\left(-\phi_2 T_1 + \phi_1 T_1 - \sigma_2 T_1\right) = 0,
\end{aligned}
\tag{3.66}
$$

and

$$
\begin{aligned}
&-p_4' + \phi_2 p_3 - \omega_c p_4\xi + \frac{1}{2\omega_c}\rho p_1 \cos\left(-\phi_2 T_1 + \phi_1 T_1 - \sigma_2 T_1\right) \\
&+\frac{1}{2\omega_c}\rho p_2 \sin\left(-\phi_2 T_1 + \phi_1 T_1 - \sigma_2 T_1\right) = 0.
\end{aligned}
\tag{3.67}
$$

Setting $\vartheta_1 = \left(-\phi_1 T_1 + \phi_2 T_1 + \sigma_2 T_1\right)$, $\vartheta_2 = \left(\phi_1 T_1 - \phi_2 T_1 - \sigma_2 T_1\right)$ and rearranging the above equations, we get

$$p_1' = \left(-\frac{1}{2}\mu_1\right) p_1 + \left(-\phi_1\right) p_2 + \left(\frac{1}{2\omega_s}\tau \sin \vartheta_1\right) p_3 + \left(-\frac{1}{2\omega_s}\tau \cos \vartheta_1\right) p_4, \tag{3.68}$$

$$p_2' = \left(\phi_1\right) p_1 + \left(-\frac{1}{2}\mu_1\right) p_2 + \left(\frac{1}{2\omega_s}\tau \cos \vartheta_1\right) p_3 + \left(\frac{1}{2\omega_s}\tau \sin \vartheta_1\right) p_4, \tag{3.69}$$

$$p_3' = \left(\frac{1}{2\omega_c}\rho \sin \vartheta_2\right) p_1 + \left(-\frac{1}{2\omega_c}\rho \cos \vartheta_2\right) p_2 + \left(-\omega_c \xi\right) p_3 + \left(-\phi_2\right) p_4, \tag{3.70}$$

and

$$p_4' = \left(\frac{1}{2\omega_c}\rho \cos \vartheta_2\right) p_1 + \left(\frac{1}{2\omega_c}\rho \sin \vartheta_2\right) p_2 + \left(\phi_2\right) p_3 + \left(-\omega_c \xi\right) p_4. \tag{3.71}$$

Setting $J_{11} = -\frac{1}{2}\mu_1$, $J_{12} = -\phi_1$, $J_{13} = \frac{1}{2\omega_s}\tau \sin \vartheta_1$, $J_{14} = -\frac{1}{2\omega_s}\tau \cos \vartheta_1$, $J_{31} = \frac{1}{2\omega_c}\rho \sin \vartheta_2$, $J_{32} = -\frac{1}{2\omega_c}\rho \cos \vartheta_2$, $J_{33} = -\omega_c \xi$ and $J_{34} = -\phi_2$.

The stability of the trivial solution is investigated by evaluating the eigenvalues of the Jacobian matrix of equations (3.68) - (3.71)

$$\begin{vmatrix} J_{11} - \lambda & J_{12} & J_{13} & J_{14} \\ -J_{12} & J_{11} - \lambda & -J_{14} & J_{13} \\ J_{31} & J_{32} & J_{33} - \lambda & J_{34} \\ -J_{32} & J_{31} & -J_{34} & J_{33} - \lambda \end{vmatrix} = 0,$$

$$\lambda^4 + \eta_1 \lambda^3 + \eta_2 \lambda^2 + \eta_3 \lambda + \eta_4 = 0, \tag{3.72}$$

where

$$\eta_1 = -2J_{11} - 2J_{33},$$

$$\eta_2 = J_{11}^2 + 4J_{11}J_{33} - 2J_{31}J_{13} + J_{33}^2 + J_{34}^2 + J_{12}^2,$$

$$\eta_3 = -2J_{11}^2 J_{33} + 2J_{11}J_{31}J_{13} + 2J_{32}J_{34}J_{13} + 2J_{31}J_{13}J_{33} + 2J_{12}J_{32}J_{13} \\ -2J_{12}^2 J_{33} - 2J_{34}^2 J_{11} - 2J_{33}^2 J_{11},$$

and

$$\eta_4 = -2J_{11}J_{32}J_{34}J_{13} + J_{11}J_{31}J_{14}J_{34} - 2J_{11}J_{31}J_{13}J_{33} - 2J_{12}J_{32}J_{13}J_{33} + 2J_{12}J_{31}J_{34}J_{13} \\ -J_{31}J_{11}J_{14}J_{33} + J_{11}^2 J_{33}^2 + J_{11}^2 J_{34}^2 + J_{12}^2 J_{33}^2 + J_{12}^2 J_{34}^2 + J_{31}^2 J_{13}^2 - J_{31}^2 J_{14}^2 + J_{32}^2 J_{13}^2 - J_{32}^2 J_{14}^2,$$

The trivial solution is stable if $\eta_1 > 0$, $\eta_1\eta_2 - \eta_3 > 0$, $\eta_3(\eta_1\eta_2 - \eta_3) - \eta_1^2\eta_4 > 0$, $\eta_4 > 0$.

(a) Non-trivial solution:

To determine the stability of the non-trivial solutions

We let $a_1 = b_0 + b_1(T_1)$, $a_2 = c_0 + c_1(T_1)$ and $\varphi = \varphi_0 + \varphi_1(T_1)$, $\psi = \psi_0 + \psi_1(T_1)$, (3.73)

where $b_0, c_0, \varphi_0, \psi_0$ correspond to a non-trivial solution, while $b_1, c_1, \varphi_1, \psi_1$ are perturbation terms which are assumed to be small compared to $b_0, c_0, \varphi_0, \psi_0$

Substituting equation (3.73) into equations (3.46) , (3.47) and (3.48) , (3.49) , where $\varphi = \gamma_1$, $\psi = \gamma_2$, using estimate $\sin\varphi_1 \approx \varphi_1$, $\cos\varphi_1 \approx 1$, $\sin\psi_1 \approx \psi_1$, and $\cos\psi_1 \approx 1$

$$2(b_0' + b_1') = -\mu_1(b_0 + b_1) + \frac{1}{\omega_s} f_1 \sin(\varphi_0 + \varphi_1)\cos(\alpha) + \frac{1}{\omega_s}\tau(c_0 + c_1)\sin(\psi_0 + \psi_1) , \quad (3.74)$$

$$2(b_0 + b_1)(\varphi_0' + \varphi_1') = 2\sigma_1(b_0 + b_1) - \frac{3}{4\omega_s}\beta_1(b_0 + b_1)^3 - \frac{5}{8\omega_s}\beta_2(b_0 + b_1)^5 \\ -\frac{3}{2}\omega_s\delta(b_0 + b_1)^3 + \frac{1}{\omega_s} f_1 \cos(\varphi_0 + \varphi_1)\cos(\alpha) + \frac{1}{\omega_s}\tau(c_0 + c_1)\cos(\psi_0 + \psi_1), \quad (3.75)$$

$$(c_0' + c_1') = -\xi\omega_c(c_0 + c_1) - \frac{1}{2\omega_c}\rho(b_0 + b_1)\sin(\psi_0 + \psi_1), \quad (3.76)$$

and

$$(c_0 + c_1)\left((\varphi_0' + \varphi_1') - (\psi_0' + \psi_1')\right) = (c_0 + c_1)(\sigma_1 - \sigma_2) + \frac{1}{2\omega_c}\rho(b_0 + b_1)\cos(\psi_0 + \psi_1). \quad (3.77)$$

Simplifying equations (3.74) - (3.77), we get

$$2b_0' + 2b_1' = -\mu_1 b_0 - \mu_1 b_1 + \frac{1}{\omega_s} f_1 \left(\sin\varphi_0 + \varphi_1 \cos\varphi_0\right)\cos\left(\alpha\right)$$
$$+\frac{1}{\omega_s}\tau c_0\left(\sin\psi_0 + \psi_1\cos\psi_0\right) + \frac{1}{\omega_s}\tau c_1\left(\sin\psi_0 + \psi_1\cos\psi_0\right), \tag{3.78}$$

$$2b_0\varphi_0' + 2b_1\varphi_0' + 2b_0\varphi_1' + 2b_1\varphi_1' = 2\sigma_1 b_0 + 2\sigma_1 b_1 - \frac{3}{4\omega_s}\beta_1\left(b_0^3 + 3b_0^2 b_1 + ...\right)$$
$$-\frac{5}{8\omega_s}\beta_2\left(b_0^5 + 5b_0^4 b_1 + ...\right) - \frac{3}{2}\omega_s\delta\left(b_0^3 + 3b_0^2 b_1 + ...\right) + \frac{1}{\omega_s} f_1\left(\cos\varphi_0 - \varphi_1\sin\varphi_0\right)\cos\left(\alpha\right) \tag{3.79}$$
$$+\frac{1}{\omega_s}\tau c_0\left(\cos\psi_0 - \psi_1\sin\psi_0\right) + \frac{1}{\omega_s}\tau c_1\left(\cos\psi_0 - \psi_1\sin\psi_0\right),$$

$$c_0' + c_1' = -\xi\omega_c c_0 - \xi\omega_c c_1 - \frac{1}{2\omega_c}\rho b_0\left(\sin\psi_0 + \psi_1\cos\psi_0\right) - \frac{1}{2\omega_c}\rho b_1\left(\sin\psi_0 + \psi_1\cos\psi_0\right), \tag{3.80}$$

and

$$c_0\varphi_0' + c_0\varphi_1' + c_1\varphi_0' + c_1\varphi_1' - c_0\psi_0' - c_0\psi_1' - c_1\psi_0' - c_1\psi_1' = c_0\left(\sigma_1 - \sigma_2\right) + c_1\left(\sigma_1 - \sigma_2\right)$$
$$+\frac{1}{2\omega_c}\rho b_0\left(\cos\psi_0 - \psi_1\sin\psi_0\right) + \frac{1}{2\omega_c}\rho b_1\left(\cos\psi_0 - \psi_1\sin\psi_0\right). \tag{3.81}$$

Since $b_0, c_0, \varphi_0, \psi_0$ are solution of equation (3.46), (3.47) and (3.48), (3.49) then

$$2b_1' = -\mu_1 b_1 + \frac{1}{\omega_s} f_1\varphi_1\cos\varphi_0\cos\left(\alpha\right) + \frac{1}{\omega_s}\tau c_0\psi_1\cos\psi_0 + \frac{1}{\omega_s}\tau c_1\sin\psi_0 + \frac{1}{\omega_s}\tau c_1\psi_1\cos\psi_0, \tag{3.82}$$

$$2b_1\varphi_0' + 2b_0\varphi_1' + 2b_1\varphi_1' = 2\sigma_1 b_1 - \frac{9}{4\omega_s}\beta_1 b_0^2 b_1 - \frac{25}{8\omega_s}\beta_2 b_0^4 b_1 - \frac{9}{2}\omega_s\delta b_0^2 b_1$$
$$-\frac{1}{\omega_s} f_1\varphi_1\sin\varphi_0\cos\left(\alpha\right) - \frac{1}{\omega_s}\tau c_0\psi_1\sin\psi_0 + \frac{1}{\omega_s}\tau c_1\cos\psi_0 - \frac{1}{\omega_s}\tau c_1\psi_1\sin\psi_0, \tag{3.83}$$

$$c_1' = -\xi\omega_c c_1 - \frac{1}{2\omega_c}\rho b_0\psi_1\cos\psi_0 - \frac{1}{2\omega_c}\rho b_1\sin\psi_0 - \frac{1}{2\omega_c}\rho b_1\psi_1\cos\psi_0, \tag{3.84}$$

and

$$c_0\varphi_1' + c_1\varphi_0' + c_1\varphi_1' - c_0\psi_1' - c_1\psi_0' - c_1\psi_1' = c_1\left(\sigma_1 - \sigma_2\right) - \frac{1}{2\omega_c}\rho b_0\psi_1 \sin\psi_0$$
$$+\frac{1}{2\omega_c}\rho b_1 \cos\psi_0 - \frac{1}{2\omega_c}\rho b_1\psi_1 \sin\psi_0. \tag{3.85}$$

Now since $b_1\varphi_1$ and $c_1\psi_1$ are a very small term and $\varphi_0' + \varphi_1' = \varphi' = 0$, $\psi_0' + \psi_1' = \psi' = 0$ then they can be eliminated,

$$2b_1' = -\mu_1 b_1 + \frac{1}{\omega_s} f_1\varphi_1 \cos\varphi_0 \cos\left(\alpha\right) + \frac{1}{\omega_s}\tau c_0\psi_1 \cos\psi_0 + \frac{1}{\omega_s}\tau c_1 \sin\psi_0, \tag{3.86}$$

$$2b_0\varphi_1' = 2\sigma_1 b_1 - \frac{9}{4\omega_s}\beta_1 b_0^2 b_1 - \frac{25}{8\omega_s}\beta_2 b_0^4 b_1 - \frac{9}{2}\omega_s\delta b_0^2 b_1$$
$$-\frac{1}{\omega_s} f_1\varphi_1 \sin\varphi_0 \cos\left(\alpha\right) - \frac{1}{\omega_s}\tau c_0\psi_1 \sin\psi_0 + \frac{1}{\omega_s}\tau c_1 \cos\psi_0, \tag{3.87}$$

$$c_1' = -\xi\omega_c c_1 - \frac{1}{2\omega_c}\rho b_0\psi_1 \cos\psi_0 - \frac{1}{2\omega_c}\rho b_1 \sin\psi_0, \tag{3.88}$$

and

$$c_0\varphi_1' - c_0\psi_1' = c_1\left(\sigma_1 - \sigma_2\right) - \frac{1}{2\omega_c}\rho b_0\psi_1 \sin\psi_0 + \frac{1}{2\omega_c}\rho b_1 \cos\psi_0. \tag{3.89}$$

Rearranging equations (3.86) - (3.89), to gives

$$b_1' = \left(-\frac{1}{2}\mu_1\right)b_1 + \left(\frac{1}{2\omega_s} f_1 \cos\varphi_0 \cos\left(\alpha\right)\right)\varphi_1 + \left(\frac{1}{2\omega_s}\tau \sin\psi_0\right)c_1 + \left(\frac{1}{2\omega_s}\tau c_0 \cos\psi_0\right)\psi_1, \tag{3.90}$$

$$\varphi_1' = \left(\frac{\sigma_1}{b_0} - \frac{9}{8\omega_s}\beta_1 b_0 - \frac{25}{16\omega_s}\beta_2 b_0^3 - \frac{9}{4}\omega_s\delta b_0\right)b_1$$
$$+\left(-\frac{1}{2\omega_s b_0} f_1 \sin\varphi_0 \cos\left(\alpha\right)\right)\varphi_1 + \left(\frac{1}{2\omega_s b_0}\tau \cos\psi_0\right)c_1 + \left(-\frac{1}{2\omega_s b_0}\tau c_0 \sin\psi_0\right)\psi_1, \tag{3.91}$$

$$c_1' = \left(\frac{1}{2\omega_c}\rho \sin\psi_0\right)b_1 + \left(-\xi\omega_c\right)c_1 + \left(\frac{1}{2\omega_c}\rho b_0 \cos\psi_0\right)\psi_1, \tag{3.92}$$

and

$$\psi_1' = \left(-\frac{\sigma_1}{b_0} + \frac{9}{8\omega_s}\beta_1 b_0 + \frac{25}{16\omega_s b_0}\beta_2 b_0^3 + \frac{9}{4b_0}\omega_s \delta b_0 - \frac{1}{2c_0\omega_c}\rho\cos\psi_0 \right) b_1$$
$$+ \left(\frac{1}{2\omega_s b_0} f_1 \sin\varphi_0 \cos(\alpha) \right)\varphi_1 + \left(-\frac{1}{c_0}(\sigma_1 - \sigma_2) - \frac{1}{2\omega_s b_0}\tau\cos\psi_0 \right) c_1 \tag{3.93}$$
$$+ \left(\frac{1}{2\omega_s b_0}\tau c_0 \sin\psi_0 + \frac{1}{2c_0\omega_c}\rho b_0 \sin\psi_0 \right)\psi_1.$$

Letting $J_{11} = -\frac{1}{2}\mu_1$, $J_{12} = \frac{1}{2\omega_s} f_1 \cos\varphi_0 \cos(\alpha)$, $J_{13} = \frac{1}{2\omega_s}\tau\sin\psi_0$, $J_{14} = \frac{1}{2\omega_s}\tau c_0 \cos\psi_0$,

$J_{21} = \frac{\sigma_1}{b_0} - \frac{9}{8\omega_s}\beta_1 b_0 - \frac{25}{16\omega_s}\beta_2 b_0^3 - \frac{9}{4}\omega_s \delta b_0$, $J_{22} = -\frac{1}{2\omega_s b_0} f_1 \sin\varphi_0 \cos(\alpha)$,

$J_{23} = \frac{1}{2\omega_s b_0}\tau\cos\psi_0$, $J_{24} = -\frac{1}{2\omega_s b_0}\tau c_0 \sin\psi_0$, $J_{31} = \frac{1}{2\omega_c}\rho\sin\psi_0$, $J_{33} = -\xi\omega_c$,

$J_{34} = \frac{1}{2\omega_c}\rho b_0 \cos\psi_0$, $J_{41} = -\frac{\sigma_1}{b_0} + \frac{9}{8\omega_s}\beta_1 b_0 + \frac{25}{16\omega_s b_0}\beta_2 b_0^3 + \frac{9}{4b_0}\omega_s \delta b_0 - \frac{1}{2c_0\omega_c}\rho\cos\psi_0$,

$J_{43} = -\frac{1}{c_0}(\sigma_1 - \sigma_2) - \frac{1}{2\omega_s b_0}\tau\cos\psi_0$ and $J_{44} = \frac{1}{2\omega_s b_0}\tau c_0 \sin\psi_0 + \frac{1}{2c_0\omega_c}\rho b_0 \sin\psi_0$.

The stability of the non-trivial solution is investigated by evaluating the eigenvalues of the Jacobian matrix of equations (3.90) - (3.93)

$$\begin{vmatrix} J_{11}-\lambda & J_{12} & J_{13} & J_{14} \\ J_{21} & J_{22}-\lambda & J_{23} & J_{24} \\ J_{31} & 0 & J_{33}-\lambda & J_{34} \\ J_{41} & -J_{22} & J_{43} & J_{44}-\lambda \end{vmatrix} = 0,$$

$$\lambda^4 + \eta_1\lambda^3 + \eta_2\lambda^2 + \eta_3\lambda + \eta_4 = 0 \ . \tag{3.94}$$

The non-trivial solution is stable if $\eta_1 > 0, \eta_1\eta_2 - \eta_3 > 0, \eta_3(\eta_1\eta_2 - \eta_3) - \eta_1^2\eta_4 > 0, \eta_4 > 0$.

3.4 Perturbation analysis for the main system with indirect (NS) controls

The nonlinear differential equation (3.5) with NS control (3.6) is scaled using the perturbation parameter ε as follows

$$u'' + \varepsilon\mu_1 u' + \omega_s^2 u + \varepsilon\beta_1 u^3 + \varepsilon\beta_2 u^5 - \varepsilon\delta\left(uu'^2 + u^2 u''\right) = \varepsilon f_1 \cos(\Omega t)\cos(\alpha)$$
$$+\varepsilon u f_2 \cos(\Omega t)\sin(\alpha) + \varepsilon\tau v^2,$$

$$v'' + 2\xi\varepsilon\omega_c v' + \omega_c^2 v = \varepsilon\rho u v\,.$$

Applying the multiple scales method,

Similarly as in equations (3.7) - (3.13) , we have

$$D_0^2 u_0 + \varepsilon D_0^2 u_1 + 2\varepsilon D_0 D_1 u_0 + \varepsilon\mu_1 D_0 u_0 + \omega_s^2 u_0 + \varepsilon\omega_s^2 u_1 + \varepsilon\beta_1 u_0^3 + \varepsilon\beta_2 u_0^5 - 2\varepsilon\delta D_0^2 u_0^3$$
$$-\varepsilon f_1 \cos(\Omega t)\cos(\alpha) - \varepsilon u_0 f_2 \cos(\Omega t)\sin(\alpha) - \varepsilon\tau v_0^2 = 0, \qquad (3.95)$$

and

$$D_0^2 v_0 + \varepsilon D_0^2 v_1 + 2\varepsilon D_0 D_1 v_0 + 2\varepsilon\xi\omega_c D_0 v_0 + \omega_c^2 v_0 + \varepsilon\omega_c^2 v_1 - \varepsilon\rho u_0 v_0 = 0\,. \qquad (3.96)$$

Equating the coefficient of same powers of ε in equation (3.95) and (3.96), gives

$O(\varepsilon^0)$:

$$\left(D_0^2 + \omega_s^2\right) u_0 = 0\,, \qquad (3.97)$$

and

$$\left(D_0^2 + \omega_c^2\right) v_0 = 0\,. \qquad (3.98)$$

$O(\varepsilon^1)$:

$$\left(D_0^2 + \omega_s^2\right) u_1 = -2D_0 D_1 u_0 - \mu_1 D_0 u_0 - \beta_1 u_0^3 - \beta_2 u_0^5 + 2\delta D_0^2 u_0^3 + f_1 \cos(\Omega t)\cos(\alpha)$$
$$+u_0 f_2 \cos(\Omega t)\sin(\alpha) + \tau v_0^2, \qquad (3.99)$$

and

$$\left(D_0^2 + \omega_c^2\right) v_1 = -2D_0 D_1 v_0 - 2\varepsilon\xi\omega_c D_0 v_0 + \rho u_0 v_0\,. \qquad (3.100)$$

The general solution of equation (4.14) and (4.15) is given by

$$u_0 = A(T_1)e^{i\omega_s T_0} + \bar{A}(T_1)e^{-i\omega_s T_0}\,, \qquad (3.101)$$

and

$$v_0 = B(T_1)e^{i\omega_c T_0} + \bar{B}(T_1)e^{-i\omega_c T_0}\,. \qquad (3.102)$$

where the quantities $A(T_1)$ and $B(T_1)$ are unknown function in T_1

Now to solve equation (3.99) and (3.100) substituting equations (3.101) and (3.102) into it, then Substituting equation (3.9) and Using the form $\cos(\omega T_0)=\dfrac{e^{i\omega T_0}+e^{-i\omega T_0}}{2}$, $\sin(\omega T_0)=\dfrac{e^{i\omega T_0}-e^{-i\omega T_0}}{2i}$, similarly as equations (3.22) – (3.27), we have

$$\left(D_0^2+\omega_s^2\right)u_1=\left(-2i\omega_s A'-\mu_1 i\omega_s A-3\beta_1 A^2\bar{A}-10\beta_2 A^3\bar{A}^2-6\omega_s^2\delta A^2\bar{A}\right)e^{i\omega_s T_0}$$
$$+\left(-\beta_1 A^3-5\beta_2 A^4\bar{A}-18\omega_s^2\delta A^3\right)e^{3i\omega_s T_0}-\beta_2 A^5 e^{5i\omega_s T_0}+\frac{1}{2}f_1 e^{i\Omega T_0}\cos(\alpha) \quad (3.103)$$
$$+\frac{1}{2}f_2 A e^{i(\omega+\Omega)T_0}\sin(\alpha)+\frac{1}{2}f_2 A e^{i(\omega-\Omega)T_0}\sin(\alpha)+\tau B^2 e^{2i\omega_c T_0}+\tau B\bar{B}+cc,$$

and

$$\left(D_0^2+\omega_c^2\right)v_1=\left(-2i\omega_c B'-2i\xi\omega_c^2 B\right)e^{i\omega_c T_0}+\rho BAe^{i(\omega_s+\omega_c)T_0}+\rho\bar{A}Be^{i(\omega_c-\omega_s)T_0}+cc\,. \quad (3.104)$$

where cc denotes the complex conjugate terms.

The particular solution of equation (3.103) and (3.104) can be written in the following form

$$u_1(T_0,T_1)=A_1 e^{i\omega_s T_0}-\frac{1}{8\omega_s^2}\left(-\beta_1 A^3-5\beta_2 A^4\bar{A}-18\omega_s^2\delta A^3\right)e^{3i\omega_s T_0}+\frac{1}{24\omega_s^2}\beta_2 A^5 e^{5i\omega_s T_0}$$
$$+\frac{1}{2(\omega-\Omega)(\omega+\Omega)}f_1\cos(\alpha)e^{i\Omega T_0}-\frac{1}{2\Omega(2\omega+\Omega)}f_2 A e^{i(\omega+\Omega)T_0}\sin(\alpha) \quad (3.105)$$
$$+\frac{1}{2\Omega(2\omega-\Omega)}f_2 A e^{i(\omega-\Omega)T_0}\sin(\alpha)+\frac{1}{(-2\omega_c+\omega_s)(2\omega_c+\omega_s)}\tau B^2 e^{2i\omega_c T_0}+\tau B\bar{B}+cc,$$

and

$$v_1(T_0,T_1)=B_1 e^{i\omega_c T_0}-\frac{1}{\omega_s(\omega_s+2\omega_c)}\rho BAe^{i(\omega_s+\omega_c)T_0}-\frac{1}{\omega_s(\omega_s-2\omega_c)}\rho\bar{A}Be^{i(\omega_c-\omega_s)T_0}+cc. \quad (3.106)$$

From the equation (3.105) and (3.106) the reported resonance cases at this approximation order are

(a) simultaneous resonance $\Omega=\omega_s$ and $\omega_c=\frac{1}{2}\omega_s$

(b) simultaneous resonance $\Omega = 2\omega_s$ and $\omega_c = \frac{1}{2}\omega_s$

3.5 Stability analysis

3.5.1 simultaneous resonance $\Omega = \omega_s$ and $\omega_c = \frac{1}{2}\omega_s$

In this case we introduce a detuning parameter σ_1 and σ_2 such that

$$\Omega = \omega_s + \varepsilon\sigma_1 \quad , \quad \omega_c = \frac{1}{2}\omega_s + \varepsilon\sigma_2 \ . \tag{3.107}$$

Substituting equation (3.107) into equation (3.103) and (3.104), eliminating the terms that produce secular term and performing some algebraic manipulations, we obtain

$$\begin{aligned} &-2i\omega_s A' - \mu_1 i\omega_s A - 3\beta_1 A^2\bar{A} - 10\beta_2 A^3\bar{A}^2 - 6\omega_s^2\delta A^2\bar{A} + \frac{1}{2} f_1 e^{i\sigma_1 T_1}\cos(\alpha) \\ &+\tau B^2 e^{i2\sigma_2 T_1} = 0, \end{aligned} \tag{3.108}$$

and

$$\left(-2i\omega_c B' - 2i\xi\omega_c^2 B\right) + \rho A\bar{B}e^{-2i\sigma_2 T_1} = 0 \, . \tag{3.109}$$

Substituting $A = \frac{1}{2}a_1 e^{i\theta_1}, B = \frac{1}{2}a_2 e^{i\theta_2}$, similarly in equations (3.36) – (3.41) , we obtain the following equations describing the modulation of amplitude and phase of the motions

$$2a_1' = -\mu_1 a_1 + \frac{1}{\omega_s} f_1 \sin(-\theta_1 + \sigma_1 T_1)\cos(\alpha) + \frac{1}{2\omega_s}\tau a_2^2 \sin(-\theta_1 + 2\theta_2 + 2\sigma_2 T_1), \tag{3.110}$$

$$\begin{aligned} &2a_1\theta_1' - \frac{3}{4\omega_s}\beta_1 a_1^3 - \frac{5}{8\omega_s}\beta_2 a_1^5 - \frac{3}{2}\omega_s\delta a_1^3 + \frac{1}{\omega_s} f_1 \cos(-\theta_1 + \sigma_1 T_1)\cos(\alpha) \\ &+\frac{1}{2\omega_s}\tau a_2^2 \cos(-\theta_1 + 2\theta_2 + \sigma_2 T_1) = 0. \end{aligned} \tag{3.111}$$

$$a_2' = -\xi\omega_c a_2 - \frac{1}{4\omega_c}\rho a_1 a_2 \sin(-\theta_1 + 2\theta_2 + 2\sigma_2 T_1), \tag{3.112}$$

and

$$a_2\theta_2' + \frac{1}{4\omega_c}\rho a_1 a_2 \cos\left(-\theta_1 + 2\theta_2 + 2\sigma_2 T_1\right) = 0\,. \tag{3.113}$$

Letting $\Lambda_1 = \frac{1}{\omega_s} f_1$, $\Lambda_2 = \frac{1}{2\omega_s}\tau a_2^2$, $\Lambda_3 = \frac{1}{4\omega_c}\rho a_1 a_2$, $\gamma_1 = \left(-\theta_1 + \sigma_1 T_1\right)$ and $\gamma_2 = (-\theta_1 + 2\theta_2 + 2\sigma_2 T_1)$

Then, equations (3.110) - (3.113) becomes

$$2a_1' = -\mu_1 a_1 + \Lambda_1 \sin\left(\gamma_1\right)\cos\left(\alpha\right) + \Lambda_2 \sin\left(\gamma_2\right), \tag{3.114}$$

$$2a_1\gamma_1' = 2\sigma_1 a_1 - \frac{3}{4\omega_s}\beta_1 a_1^3 - \frac{5}{8\omega_s}\beta_2 a_1^5 - \frac{3}{2}\omega_s \delta a_1^3 + \Lambda_1 \cos\left(\gamma_1\right)\cos\left(\alpha\right) + \Lambda_2 \cos\left(\gamma_2\right), \tag{3.115}$$

$$a_2' = -\xi\omega_c a_2 - \Lambda_3 \sin\left(\gamma_2\right), \tag{3.116}$$

and

$$\frac{1}{2}a_2(\gamma_1' - \gamma_2') = \frac{1}{2}a_2(\sigma_1 - 2\sigma_2) + \Lambda_3 \cos\left(\gamma_2\right). \tag{3.117}$$

The steady state solutions correspond to constant $a_1, a_2, \gamma_1, \gamma_2$ that is $a_1' = a_2' = \gamma_1' = \gamma_2' = 0$

$$\mu_1 a_1 = \Lambda_1 \sin\left(\gamma_1\right)\cos\left(\alpha\right) + \Lambda_2 \sin\left(\gamma_2\right), \tag{3.118}$$

$$-2\sigma_1 a_1 + \frac{3}{4\omega_s}\beta_1 a_1^3 + \frac{5}{8\omega_s}\beta_2 a_1^5 + \frac{3}{2}\omega_s \delta a_1^3 = \Lambda_1 \cos\left(\gamma_1\right)\cos\left(\alpha\right) + \Lambda_2 \cos\left(\gamma_2\right), \tag{3.119}$$

$$\xi\omega_c a_2 = -\Lambda_3 \sin\left(\gamma_2\right), \tag{3.120}$$

and

$$-\frac{1}{2}a_2(\sigma_1 - 2\sigma_2) = \Lambda_3 \cos\left(\gamma_2\right). \tag{3.121}$$

From equations (3.118) - (3.121), we have

$$(\mu_1 a_1)^2 + \left(-2\sigma_1 a_1 + \frac{3}{4\omega_s}\beta_1 a_1^3 + \frac{5}{8\omega_s}\beta_2 a_1^5 + \frac{3}{2}\omega_s \delta a_1^3\right)^2 = \Lambda_1^2 \cos^2\left(\alpha\right) + \Lambda_2^2 + 2\Lambda_1\Lambda_2 \cos\left(\alpha\right), \tag{3.122}$$

$$\left(\xi\omega_c a_2\right)^2 + \left(\frac{1}{2}a_2(\sigma_1 - 2\sigma_2)\right)^2 = \Lambda_3^2. \tag{3.123}$$

Equation (3.122) and (3.123) are called frequency response equations.

(a) Trivial solution :

To determine the stability of the trivial solutions, we investigate the solutions of the linearized form equation (3.108) and (3.109)

$$-2i\omega_s A' - \mu_1 i\omega_s A = 0\,, \tag{3.124}$$

and

$$-2i\omega_c B' - 2i\xi\omega_c^2 B = 0\,. \tag{3.125}$$

We expressed A and B in Cartesian form

$$A = \frac{1}{2}\left(p_1 - ip_2\right)e^{i\phi_1 T_1} \quad B = \frac{1}{2}\left(p_3 - ip_4\right)e^{i\phi_2 T_1}\,,$$

where p_1, p_2, p_3, p_4 are real

$$-2i\omega_s\left(\frac{1}{2}\left(p_1' - ip_2'\right)e^{i\phi_1 T_1} + \frac{1}{2}i\phi_1\left(p_1 - ip_2\right)e^{i\phi_1 T_1}\right) - \frac{1}{2}i\omega_s\mu_1\left(p_1 - ip_2\right)e^{i\phi_1 T_1} = 0\,, \tag{3.126}$$

and

$$-2i\omega_c\left(\frac{1}{2}\left(p_3' - ip_4'\right)e^{i\phi_2 T_1} + \frac{1}{2}i\phi_2\left(p_3 - ip_4\right)e^{i\phi_2 T_1}\right) - i\xi\omega_c^2\left(p_3 - ip_4\right)e^{i\phi_2 T_1} = 0\,. \tag{3.127}$$

Dividing both sides of equation (3.126) by $\omega_s e^{i\phi_1 T_1}$ and both of sides of equation (3.127) by $\omega_c e^{i\phi_2 T_1}$

$$-ip_1' - p_2' + \phi_1 p_1 - i\phi_1 p_2 - \frac{1}{2}ip_1\mu_1 - \frac{1}{2}p_2\mu_1 = 0\,, \tag{3.128}$$

and

$$-ip_3' - p_4' + \phi_2 p_3 - i\phi_2 p_4 - i\omega_c p_3\xi - \omega_c p_4\xi = 0\,. \tag{3.129}$$

Separating real and imaginary parts in equation (3.128) and (3.129) to get

$$p_1' = \left(-\frac{1}{2}\mu_1 \right) p_1 + \left(-\phi_1 \right) p_2 , \qquad (3.130)$$

$$p_2' = \left(\phi_1 \right) p_1 + \left(-\frac{1}{2}\mu_1 \right) p_2 , \qquad (3.131)$$

$$p_3' = \left(-\omega_c \xi \right) p_3 + \left(-\phi_2 \right) p_4 , \qquad (3.132)$$

and

$$p_4' = \left(\phi_2 \right) p_3 + \left(-\omega_c \xi \right) p_4 . \qquad (3.133)$$

Sitting $J_{11} = -\frac{1}{2}\mu_1$, $J_{12} = -\phi_1$, $J_{33} = -\omega_c \xi$, $J_{34} = -\phi_2$.

The stability of the trivial solution is investigated by evaluating the eigenvalues of the Jacobian matrix of equations (3.130) - (3.133)

$$\begin{vmatrix} J_{11} - \lambda & J_{12} & 0 & 0 \\ -J_{12} & J_{11} - \lambda & 0 & 0 \\ 0 & 0 & J_{33} - \lambda & J_{34} \\ 0 & 0 & -J_{34} & J_{33} - \lambda \end{vmatrix} = 0 ,$$

$$\lambda^4 + \eta_1 \lambda^3 + \eta_2 \lambda^2 + \eta_3 \lambda + \eta_4 = 0 . \qquad (3.134)$$

The trivial solution is stable if $\eta_1 > 0 , \eta_1 \eta_2 - \eta_3 > 0 , \eta_3 \left(\eta_1 \eta_2 - \eta_3 \right) - \eta_1^2 \eta_4 > 0 , \eta_4 > 0$.

(a) Non-trivial solution:

To determine the stability of the non-trivial solutions

We let $a_1 = b_0 + b_1(T_1)$, $a_2 = c_0 + c_1(T_1)$ and $\varphi = \varphi_0 + \varphi_1(T_1)$, $\psi = \psi_0 + \psi_1(T_1)$. (3.135)

Substituting equation (3.135) into equations (3.114) - (3.117) similarly as in above, we have

$$2b_0' + 2b_1' = -\mu_1 b_0 - \mu_1 b_1 + \frac{1}{\omega_s} f_1\left(\sin\varphi_0 + \varphi_1\cos\varphi_0\right)\cos(\alpha)$$
$$+\frac{1}{2\omega_s}\tau c_0^2\left(\sin\psi_0 + \psi_1\cos\psi_0\right) + \frac{1}{2\omega_s}\tau c_0 c_1\left(\sin\psi_0 + \psi_1\cos\psi_0\right), \tag{3.136}$$

$$2b_0\varphi_0' + 2b_1\varphi_0' + 2b_0\varphi_1' + 2b_1\varphi_1' = 2\sigma_1 b_0 + 2\sigma_1 b_1 - \frac{3}{4\omega_s}\beta_1\left(b_0^3 + 3b_0^2 b_1 + \ldots\right)$$
$$-\frac{5}{8\omega_s}\beta_2\left(b_0^5 + 5b_0^4 b_1 + \ldots\right) - \frac{3}{2}\omega_s\delta\left(b_0^3 + 3b_0^2 b_1 + \ldots\right) + \frac{1}{\omega_s} f_1\left(\cos\varphi_0 - \varphi_1\sin\varphi_0\right)\cos(\alpha) \tag{3.137}$$
$$+\frac{1}{2\omega_s}\tau c_0^2\left(\cos\psi_0 - \psi_1\sin\psi_0\right) + \frac{1}{2\omega_s}\tau c_0 c_1\left(\cos\psi_0 - \psi_1\sin\psi_0\right),$$

$$c_0' + c_1' = -\xi\omega_c c_0 - \xi\omega_c c_1 - \frac{1}{4\omega_c}\rho b_0 c_0\left(\sin\psi_0 + \psi_1\cos\psi_0\right) - \frac{1}{4\omega_c}\rho b_1 c_0\left(\sin\psi_0 + \psi_1\cos\psi_0\right)$$
$$-\frac{1}{4\omega_c}\rho b_0 c_1\left(\sin\psi_0 + \psi_1\cos\psi_0\right), \tag{3.138}$$

and

$$\frac{1}{2}\left(c_0\varphi_0' + c_0\varphi_1' + c_1\varphi_0' + c_1\varphi_1' - c_0\psi_0' - c_0\psi_1' - c_1\psi_0' - c_1\psi_1'\right) = \frac{1}{2}c_0\left(\sigma_1 - \sigma_2\right) + \frac{1}{2}c_1\left(\sigma_1 - \sigma_2\right)$$
$$+\frac{1}{4\omega_c}\rho b_0 c_0\left(\cos\psi_0 - \psi_1\sin\psi_0\right) + \frac{1}{4\omega_c}\rho b_1 c_0\left(\cos\psi_0 - \psi_1\sin\psi_0\right) \tag{3.139}$$
$$+\frac{1}{4\omega_c}\rho b_0 c_1\left(\cos\psi_0 - \psi_1\sin\psi_0\right).$$

Since $b_0, c_0, \varphi_0, \psi_0$ are solutions of equations (3.114) - (3.117), $b_1\varphi_1$, $c_1\psi_1$ are a very small term and $\varphi_0' + \varphi_1' = \varphi' = 0$, $\psi_0' + \psi_1' = \psi' = 0$ then they can be eliminated, we have

$$b_1' = \left(-\frac{1}{2}\mu_1\right)b_1 + \left(\frac{1}{2\omega_s} f_1\cos\varphi_0\cos(\alpha)\right)\varphi_1 + \left(\frac{1}{4\omega_s}\tau c_0\sin\psi_0\right)c_1$$
$$+\left(\frac{1}{4\omega_s}\tau c_0^2\cos\psi_0\right)\psi_1, \tag{3.140}$$

$$\varphi_1' = \left(\frac{\sigma_1}{b_0} - \frac{9}{8\omega_s}\beta_1 b_0 - \frac{25}{16\omega_s}\beta_2 b_0^3 - \frac{9}{4}\omega_s\delta b_0\right)b_1 + \left(-\frac{1}{2\omega_s b_0} f_1\sin\varphi_0\cos(\alpha)\right)\varphi_1$$
$$+\left(\frac{1}{2b_0\omega_s}\tau c_0\cos\psi_0\right)c_1 + \left(-\frac{1}{4b_0\omega_s}\tau c_0^2\sin\psi_0\right)\psi_1, \tag{3.141}$$

$$c_1' = \left(-\frac{1}{4\omega_c} \rho c_0 \sin\psi_0 \right) b_1 + \left(-\xi\omega_c - \frac{1}{4\omega_c} \rho b_0 \sin\psi_0 \right) c_1 + \left(-\frac{1}{4\omega_c} \rho b_0 c_0 \cos\psi_0 \right) \psi_1 \ , \qquad (3.142)$$

and

$$\psi_1' = \left(-\frac{\sigma_1}{b_0} + \frac{9}{8\omega_s} \beta_1 b_0 + \frac{25}{16\omega_s} \beta_2 b_0^3 + \frac{9}{4} \omega_s \delta b_0 - \frac{1}{2\omega_c} \rho \cos\psi_0 \right) b_1$$
$$+ \left(\frac{1}{2\omega_s b_0} f_1 \sin\varphi_0 \cos(\alpha) \right) \varphi_1 + \left(-\frac{1}{c_0} (\sigma_1 - \sigma_2) - \frac{1}{2b_0\omega_s} \tau c_0 \cos\psi_0 - \frac{1}{2c_0\omega_c} \rho b_0 \cos\psi_0 \right) c_1 \qquad (3.143)$$
$$+ \left(\frac{1}{2\omega_c} \rho b_0 \sin\psi_0 + \frac{1}{4b_0\omega_s} \tau c_0^2 \sin\psi_0 \right) \psi_1.$$

Letting $J_{11} = -\frac{1}{2}\mu_1$, $J_{12} = \frac{1}{2\omega_s} f_1 \cos\varphi_0 \cos(\alpha)$, $J_{13} = \frac{1}{4\omega_s} \tau c_0 \sin\psi_0$, $J_{14} = \frac{1}{4\omega_s} \tau c_0^2 \cos\psi_0$,

$$J_{21} = \frac{\sigma_1}{b_0} - \frac{9}{8\omega_s} \beta_1 b_0 - \frac{25}{16\omega_s} \beta_2 b_0^3 - \frac{9}{4} \omega_s \delta b_0 , \quad J_{22} = -\frac{1}{2\omega_s b_0} f_1 \sin\varphi_0 \cos(\alpha),$$

$$J_{23} = \frac{1}{2\omega_s b_0} \tau c_0 \cos\psi_0 , \quad J_{24} = -\frac{1}{4\omega_s b_0} \tau c_0^2 \sin\psi_0 , \quad J_{31} = -\frac{1}{4\omega_c} \rho c_0 \sin\psi_0 ,$$

$$J_{33} = -\xi\omega_c - \frac{1}{4\omega_c} \rho b_0 \sin\psi_0 , \quad J_{34} = -\frac{1}{4\omega_c} \rho b_0 c_0 \cos\psi_0 ,$$

$$J_{41} = -\frac{\sigma_1}{b_0} + \frac{9}{8\omega_s} \beta_1 b_0 + \frac{25}{16\omega_s} \beta_2 b_0^3 + \frac{9}{4} \omega_s \delta b_0 - \frac{1}{2\omega_c} \rho \cos\psi_0 ,$$

$$J_{43} = -\frac{1}{c_0} (\sigma_1 - \sigma_2) - \frac{1}{2b_0\omega_s} \tau c_0 \cos\psi_0 - \frac{1}{2c_0\omega_c} \rho b_0 \cos\psi_0 ,$$

$$J_{44} = \frac{1}{2\omega_c} \rho b_0 \sin\psi_0 + \frac{1}{4b_0\omega_s} \tau c_0^2 \sin\psi_0 .$$

The stability of the non-trivial solution is investigated by evaluating the eigenvalues of the Jacobian matrix of equations (3.140) - (3.143)

$$\begin{vmatrix} J_{11} - \lambda & J_{12} & J_{13} & J_{14} \\ J_{21} & J_{22} - \lambda & J_{23} & J_{24} \\ J_{31} & 0 & J_{33} - \lambda & J_{34} \\ J_{41} & -J_{22} & J_{43} & J_{44} - \lambda \end{vmatrix} = 0 ,$$

$$\lambda^4+\eta_1\lambda^3+\eta_2\lambda^2+\eta_3\lambda+\eta_4=0. \tag{3.144}$$

The non-trivial solution is stable if $\eta_1>0\,,\eta_1\eta_2-\eta_3>0\,,\eta_3\left(\eta_1\eta_2-\eta_3\right)-\eta_1^2\eta_4>0\,,\eta_4>0$.

3.5.2 simultaneous resonance $\Omega=2\omega_s$ and $\omega_c=\frac{1}{2}\omega_s$

In this case we introduce a detuning parameter σ_1 and σ_2 such that

$$\Omega=2\omega_s+\varepsilon\sigma_1 \quad , \quad \omega_c=\frac{1}{2}\omega_s+\varepsilon\sigma_2 . \tag{3.145}$$

Substituting equation (3.145) into equation (3.103) and (3.104), similarly as above resonance, we have

$$\begin{aligned}&\left(-2i\omega_s A'-\mu_1 i\omega_s A-3\beta_1 A^2\bar{A}-10\beta_2 A^3\bar{A}^2-6\omega_s^2\delta A^2\bar{A}\right)+\frac{1}{2}f_2\bar{A}e^{i\sigma_1 T_1}\sin\left(\alpha\right)\\&+\tau B^2 e^{i2\sigma_2 T_1}=0,\end{aligned} \tag{3.146}$$

and

$$\left(-2i\omega_c B'-2i\xi\omega_c^2 B\right)+\rho A\bar{B}e^{-2i\sigma_2 T_1}=0 . \tag{3.147}$$

Substituting $A=\frac{1}{2}a_1e^{i\theta_1}, B=\frac{1}{2}a_2e^{i\theta_2}$, similarly in equations (3.36) – (3.41) , we obtain the following equations describing the modulation of amplitude and phase of the motions

$$2a_1'=-\mu_1 a_1+\frac{1}{2\omega_s}f_2 a_1\sin\left(-2\theta_1+\sigma_1 T_1\right)\sin\left(\alpha\right)+\frac{1}{2\omega_s}\tau a_2^2\sin\left(-\theta_1+2\theta_2+2\sigma_2 T_1\right), \tag{3.148}$$

$$\begin{aligned}&2a_1\theta_1'-\frac{3}{4\omega_s}\beta_1 a_1^3-\frac{5}{8\omega_s}\beta_2 a_1^5-\frac{3}{2}\omega_s\delta a_1^3+\frac{1}{2\omega_s}f_2 a_1\cos\left(-2\theta_1+\sigma_1 T_1\right)\sin\left(\alpha\right)\\&+\frac{1}{2\omega_s}\tau a_2^2\cos\left(-\theta_1+2\theta_2+2\sigma_2 T_1\right)=0,\end{aligned} \tag{3.149}$$

$$a_2'=-\xi\omega_c a_2-\frac{1}{4\omega_c}\rho a_1 a_2\sin\left(-\theta_1+2\theta_2+2\sigma_2 T_1\right), \tag{3.150}$$

and

$$a_2\theta_2' + \frac{1}{4\omega_c}\rho a_1 a_2 \cos\left(-\theta_1 + 2\theta_2 + 2\sigma_2 T_1\right) = 0\,. \tag{3.151}$$

Letting $\mathrm{E}_1 = \frac{1}{2\omega_s} a_1 f_2$, $\mathrm{E}_2 = \frac{1}{2\omega_s}\tau a_2^2$, $\mathrm{E}_3 = \frac{1}{4\omega_c}\rho a_1 a_2$, $\phi_1 = \left(-2\theta_1 + \sigma_1 T_1\right)$ and $\phi_2 = \left(-\theta_1 + 2\theta_2 + 2\sigma_2 T_1\right)$

Then, equations (3.148) - (3.151) becomes

$$2a_1' = -\mu_1 a_1 + \mathrm{E}_1 \sin\left(\phi_1\right)\sin\left(\alpha\right) + \mathrm{E}_2 \sin\left(\phi_2\right), \tag{3.152}$$

$$a_1\phi_1' = \sigma_1 a_1 - \frac{3}{4\omega_s}\beta_1 a_1^3 - \frac{5}{8\omega_s}\beta_2 a_1^5 - \frac{3}{2}\omega_s \delta a_1^3 + \mathrm{E}_1\cos\left(\phi_1\right)\sin\left(\alpha\right) + \mathrm{E}_2\cos\left(\phi_2\right), \tag{3.153}$$

$$a_2' = -\xi\omega_c a_2 - \mathrm{E}_3 \sin\left(\phi_2\right), \tag{3.154}$$

and

$$\frac{1}{4}a_2(\phi_1' - 2\phi_2') = \frac{1}{4}a_2\sigma_1 - a_2\sigma_2 + \mathrm{E}_3\cos\left(\phi_2\right). \tag{3.155}$$

The steady state solutions correspond to constant a_1, a_2, ϕ_1, ϕ_2 that is $a_1' = a_2' = \phi_1' = \phi_2' = 0$

$$\mu_1 a_1 = \mathrm{E}_1 \sin\left(\phi_1\right)\sin\left(\alpha\right) + \mathrm{E}_2 \sin\left(\phi_2\right) \tag{3.156}$$

$$-\sigma_1 a_1 + \frac{3}{4\omega_s}\beta_1 a_1^3 + \frac{5}{8\omega_s}\beta_2 a_1^5 + \frac{3}{2}\omega_s\delta a_1^3 = \mathrm{E}_1\cos\left(\phi_1\right)\sin\left(\alpha\right) + \mathrm{E}_2\cos\left(\phi_2\right) \tag{3.157}$$

And

$$\xi\omega_c a_2 = -\mathrm{E}_3 \sin\left(\phi_2\right) \tag{3.158}$$

$$a_2\sigma_2 - \frac{1}{4}a_2\sigma_1 = \mathrm{E}_3\cos\left(\phi_2\right) \tag{3.159}$$

From equations (3.156) - (3.159), we have

$$(\mu_1 a_1)^2 + \left(-\sigma_1 a_1 + \frac{3}{4\omega_s}\beta_1 a_1^3 + \frac{5}{8\omega_s}\beta_2 a_1^5 + \frac{3}{2}\omega_s\delta a_1^3\right)^2 = \mathrm{E}_1^2\sin^2\left(\alpha\right) + \mathrm{E}_2^2 + 2\mathrm{E}_1\mathrm{E}_2\sin\left(\alpha\right), \tag{3.160}$$

$$\left(\xi\omega_c a_2\right)^2 + \left(a_2\sigma_2 - \frac{1}{4}a_2\sigma_1\right)^2 = \mathrm{E}_3^2. \tag{3.161}$$

Equations (3.160) and (3.161) are called frequency response equations.

3.6 Numerical Results and Discussions

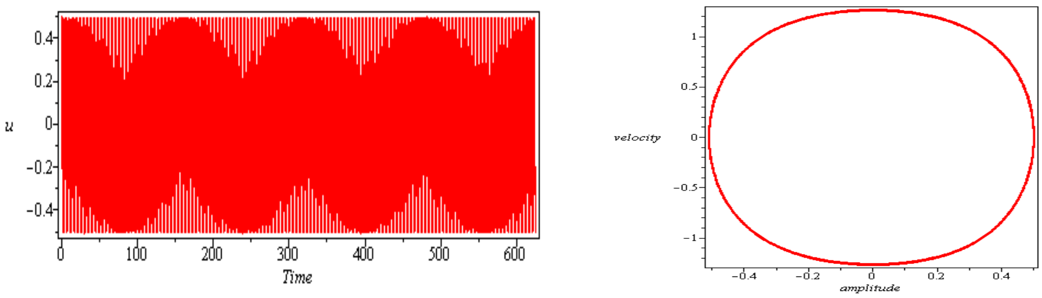
The numerical study of the response and the stability of two nonlinear systems, are conducted. Each system is represented by two (the plant and the absorber) coupled second order nonlinear differential equations. The plant (oriented beam) has quadratic, cubic and quintic nonlinearities and is subjected to external and parametric excitations. The coupling terms are either produce the positive position absorber or nonlinear sink absorber. All possible resonance cases were extracted and effects of different parameters and controllers on the plant were discussed and reported.

3.6.1 Time-response solution

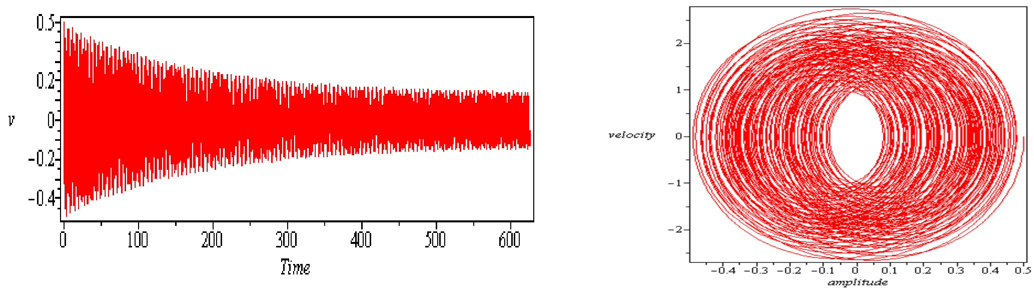
The time response of the nonlinear systems (3.3), (3.4) and (3.5), (3.6) has been investigated applying fourth order Runge-Kutta numerical method and the results are shown in Figs. (3.1) and (3.2), respectively. The phase plane method is used to give an indication about the stability of the system. Figs. (3.1a) and (3.1b) show the non-resonant behavior of the main system and the PPF absorber, respectively, with fine limit cycle for the plant. Whereas, a chaotic behaviour is illustrated in Figs. (3.1c) and (3.1d) for both the plant and the absorber at the simultaneous primary resonance case. The responses of the plant and the NS absorber at non-resonance and at two resonance cases are shown in Fig.3.2. It is clear that the response of the plant with the NS absorber is much better than of PPF absorber. The NS might be more effective in controlling behavior of the main system at resonance, which resulted in a slight chaotic response, Fig. (3.2c) or a modulated amplitude, (Fig.3.2e).

3.6.2 Theoretical Frequency Response solution

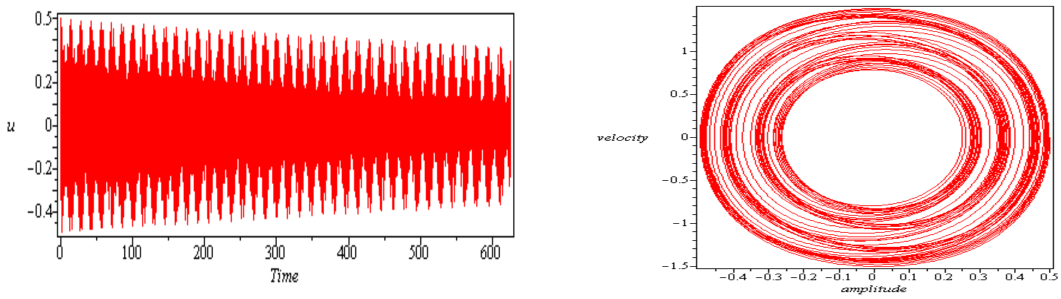
The resonant frequency response equations of the main system (3.56, 57), with PPF controller, and (3.122,123), with NS controller are solved numerically. The results are shown in Figs. (3.3,4) and (3.5,6) which represents the variation of the steady state amplitudes $a_{1,2}$ against the detuning parameter $\sigma_{1,2}$, respectively, for different values of other parameters. Fig. 3.3 shows the theoretical frequency response curves of the main system to simultaneous primary resonance case. It can be noted from Fig. (3.3b,c,d,g) that steady state amplitude increases as each of the natural frequency ω, the linear damping coefficient μ_1 and the nonlinear coefficients β_1 and δ decrease. Figure (3.3f) indicates that as the excitation force amplitude f increases, the branches of the response curves diverge away and the amplitude increases. The effect of the gain is shown in Fig. (3.3h). Fig. 3.4 iluustrates the resonant frequency response curves of the PPF control for various parameters. Each figure consists of two curves that either diverge away when the gain ρ, and the steady state ampltiude of the plant increase, Fig.(3.4b,f). Or they converge to each others as the natural frequency ω_c, and the linear damping ζ are decreased as shown in Fig.(3.4c,d). The curves in Fig. (3.4e) shifts to the right as the detuning parameter σ increases. Fig. 3.5 shows similar effects of the parameters of the system that were explained and disussed previously in Figs. (3.3) and (3.4).



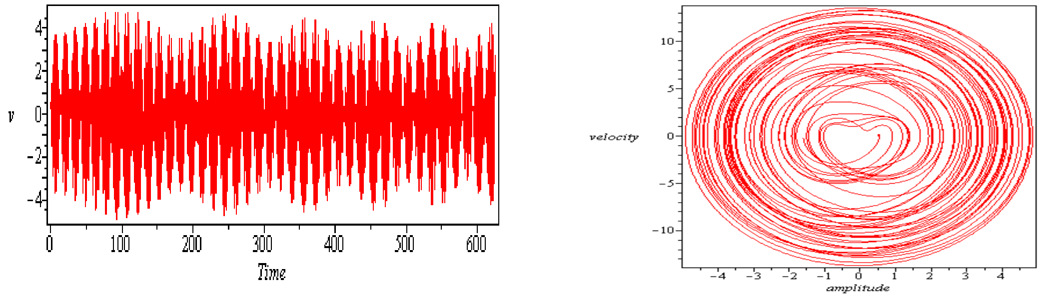
(a) Non-resonance time series of the main system



(b) Non-resonance time series of the controller system

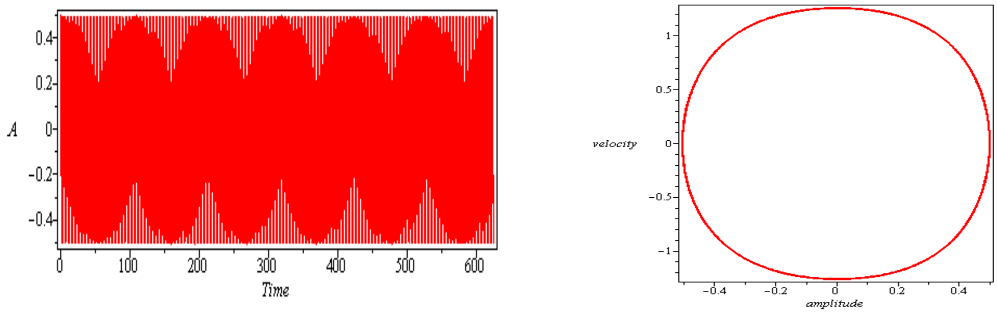


(c) Resonant time series of the main system when $\Omega = \omega_s$ and $\omega_s = \omega_c$

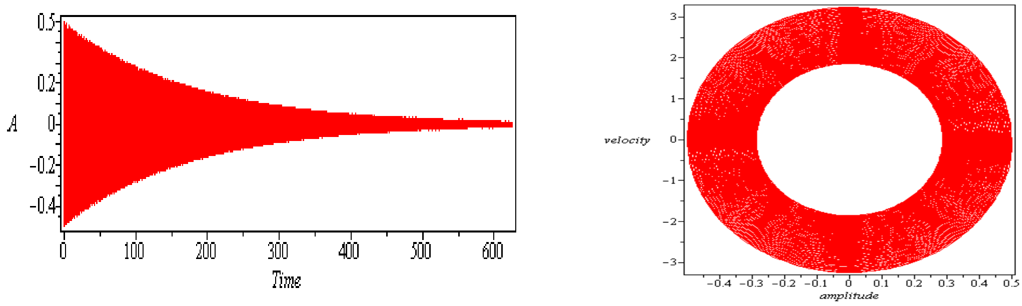


(d) Resonant time series of the controller system when $\Omega = \omega_s$ and $\omega_s = \omega_c$

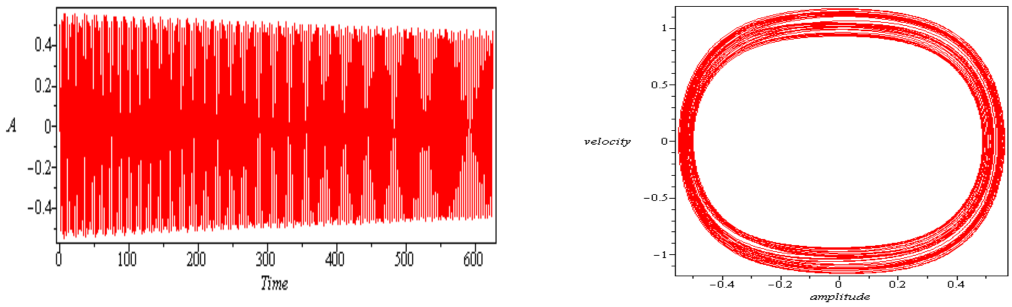
Fig 3.1 Non-resonance and resonant time history solution of the main system and (PPF) controller system when: $\omega_s = 2.1$, $\beta_1 = 15.0$, $\delta = 0.03$, $\mu_1 = 0.0005$, $\Omega = 2.7$, $\beta_2 = 5.0$, $f_1 = 0.4$, $f_2 = 0.2$, $\alpha = 30$, $\tau = 0.1$, $\xi = 0.0001$, $\rho = 10.0$, $\omega_c = 6.5$.



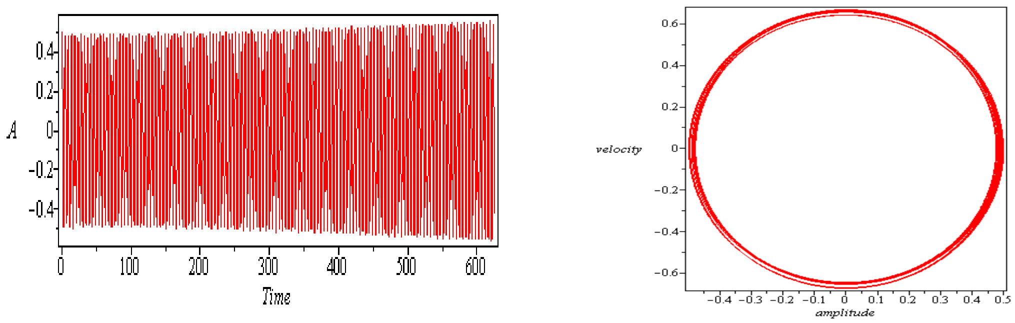
(a) Non-resonance time series of the main system



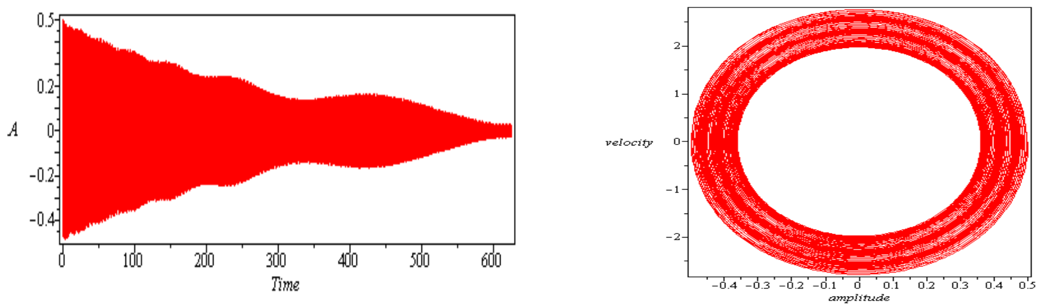
(b) Non-resonance time series of the controller system



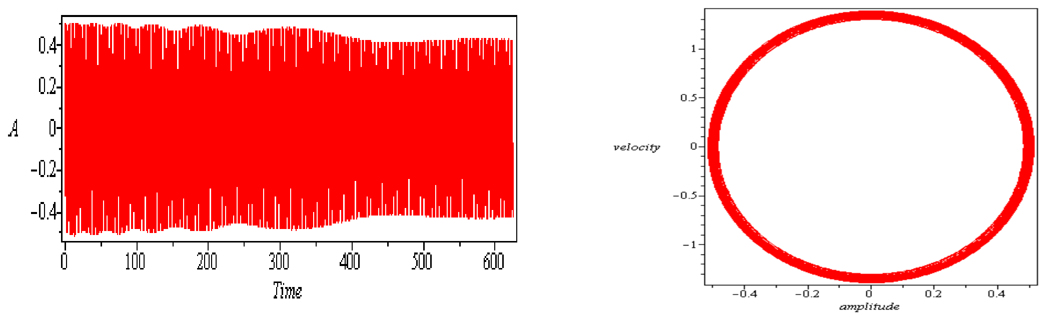
(c) Resonance time series of the main system when $\Omega = \omega_s$ and $\omega_c = \frac{1}{2}\omega_s$



(d) Resonance time series of the controller system when $\Omega = \omega_s$ and $\omega_c = \frac{1}{2}\omega_s$

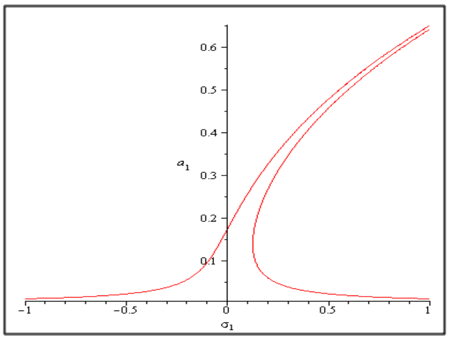


(e) Resonance time series of the main system when $\Omega = 2\omega_s$ and $\omega_c = \frac{1}{2}\omega_s$

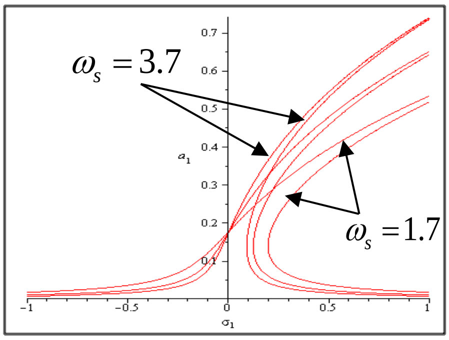


(f) Resonance time series of the controller system when $\Omega = 2\omega_s$ and $\omega_c = \frac{1}{2}\omega_s$

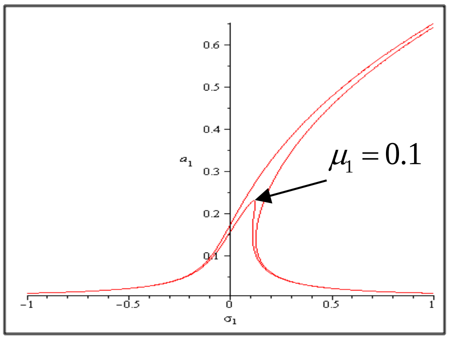
Fig 3.2 Non-resonance and resonant time history solution of the main system and (NS) controller system when: $\omega_s = 2.1$, $\beta_1 = 15.0$, $\delta = 0.03$, $\mu_1 = 0.0005$, $\Omega = 2.7$, $\beta_2 = 5.0$, $f_1 = 0.4$, $f_2 = 0.2$, $\alpha = 30$, $\tau = 0.1$, $\xi = 0.0001$, $\rho = 0.1$, $\omega_c = 6.5$.



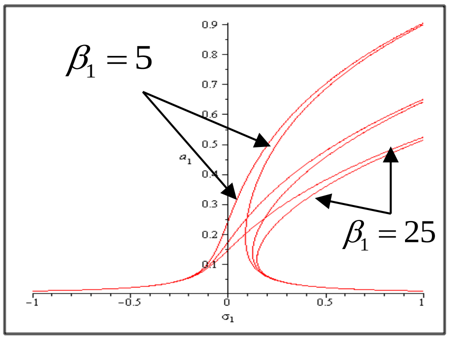
(a) Basic case



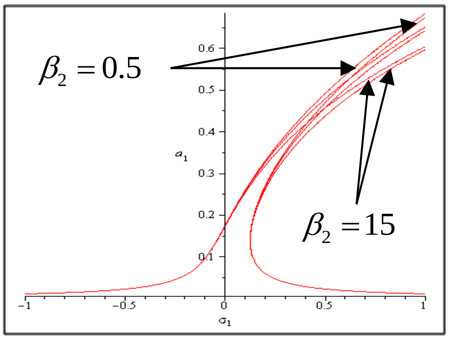
(b) Natural frequency



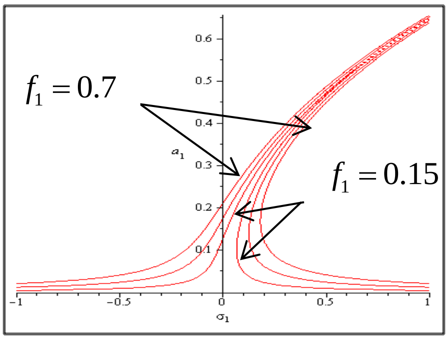
(c) The damping coefficient



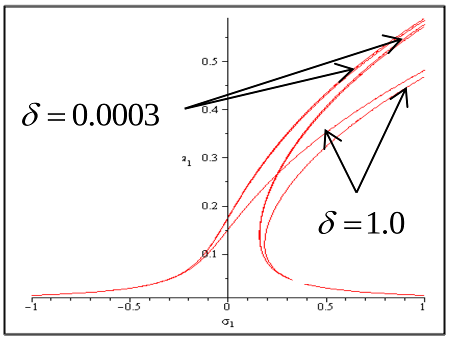
(d) Nonlinear coefficient



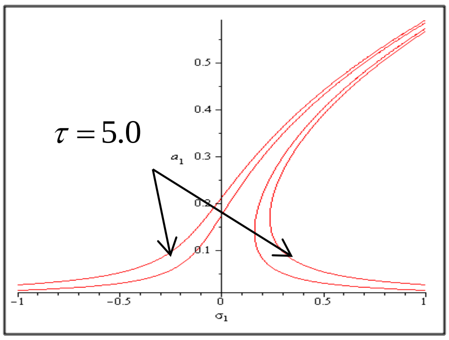
(e) Nonlinear coefficient



(f) The forcing amplitude

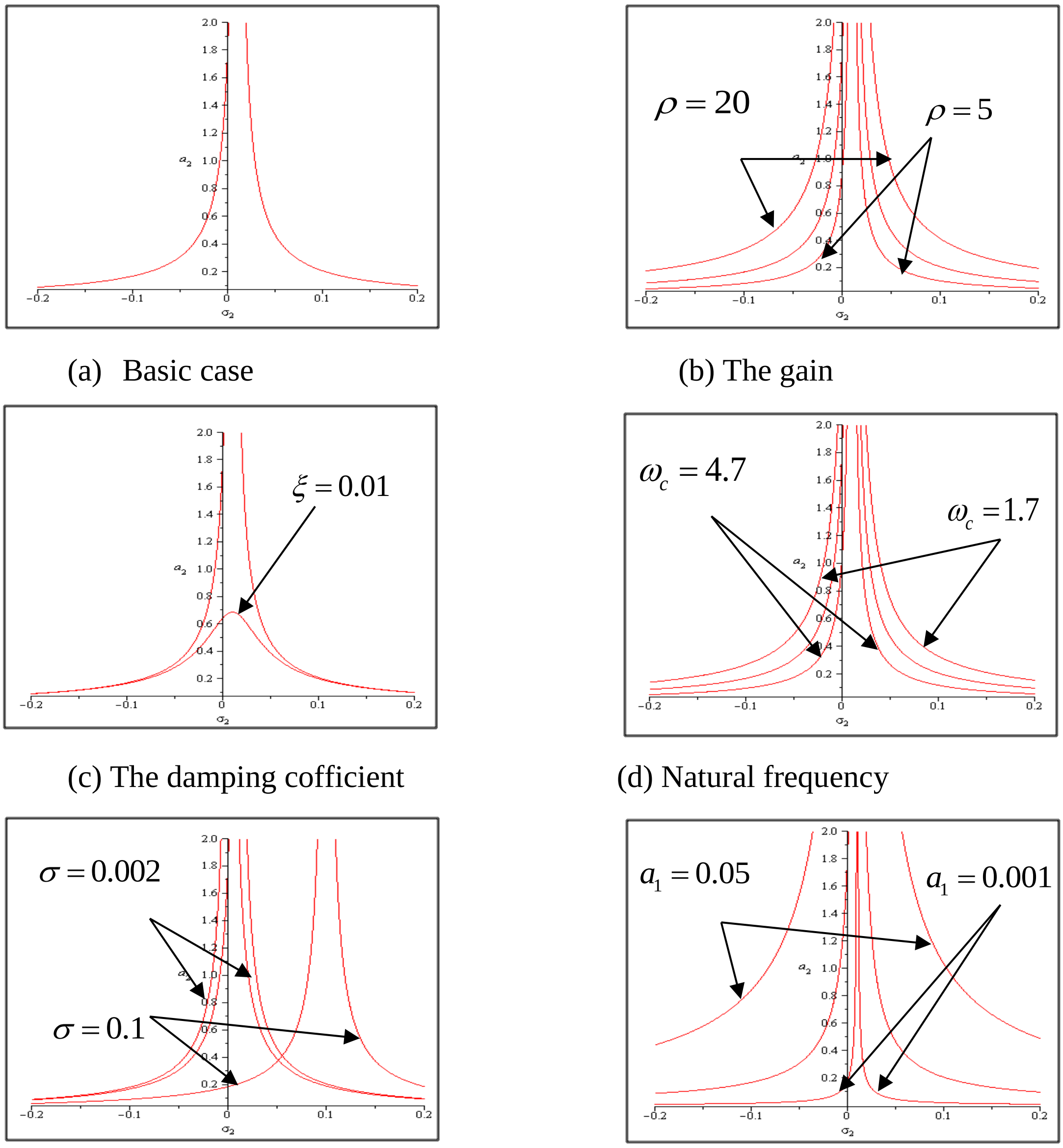


(g) Nonlinear coefficient



(h) The gain

Fig 3.3 Theoretical frequency response curves to simultaneous primary resonance case in the main system $\omega_s = 2.7$, $\beta_1 = 15.0$, $\delta = 0.03$, $\mu_1 = 0.0005$, $\beta_2 = 5.0$, $f_1 = 0.4$, $\alpha = 30$, $\tau = 0.1$.



(a) Basic case

(b) The gain

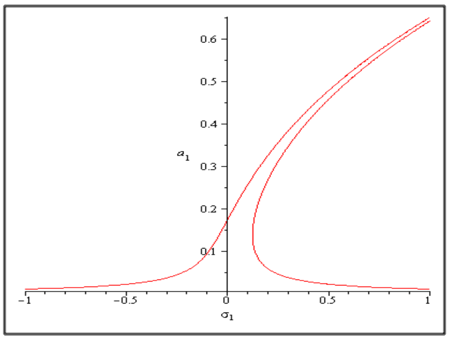
(c) The damping cofficient

(d) Natural frequency

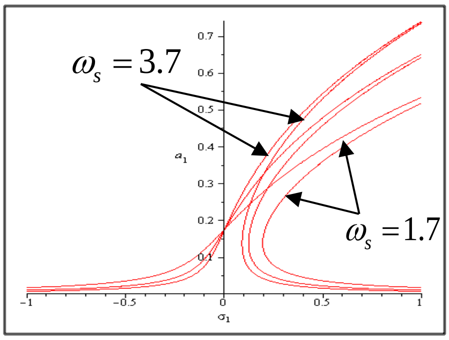
(e) Nonlinear coefficient

(f) The steady state amplitude

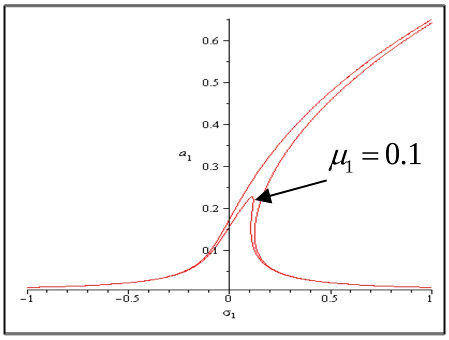
Fig 3.4 Theoretical frequency response curves to simultaneous primary resonance case in the (PPF) controller system $\omega_c = 2.7$, $\xi = 0.0001$, $\sigma = 0.01$, $a_1 = 0.01$, $\rho = 10.0$.



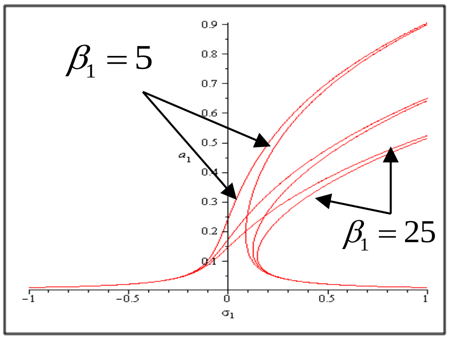
(a) Basic case



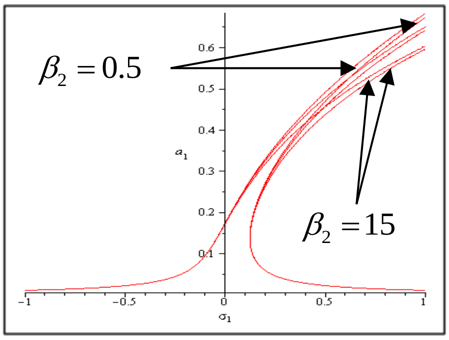
(b) Natural frequency



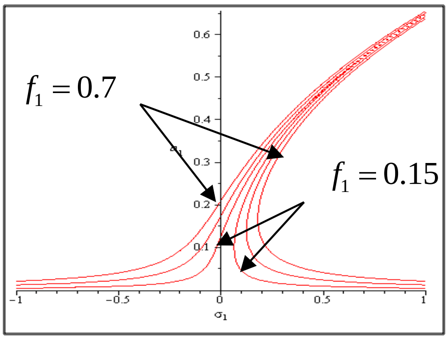
(c) The damping coefficient



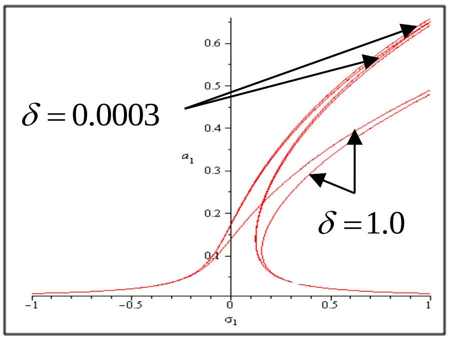
(d) Nonlinear coefficient



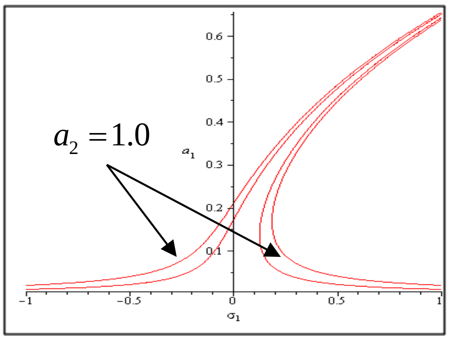
(e) Nonlinear coefficient



(f) The forcing amplitude



(g) Nonlinear coefficient



(h) The steady state amplitude

Fig 3.5 Theoretical frequency response curves to simultaneous resonance case in the main system $\omega_s = 2.7$, $\beta_1 = 15.0$, $\delta = 0.03$, $\mu_1 = 0.0005$, $\beta_2 = 5.0$, $f_1 = 0.4$, $\alpha = 30$, $\tau = 0.1$

Chapter 4

Conclusions

4.1 Summary

The control and stability of a non-linear differential equation representing the non-linear dynamical one-degree-of-freedom inclined beam are studied. The inclined beam has cubic and quintic nonlinearities subjected to external and parametric excitation forces. Various active and passive control techniques have been applied. The investigation includes the solutions applying both Runge-Kutta numerical method and the perturbation technique. The stability of the system under the applied control techniques is investigated applying both the phase plane and the frequency response equation. The phase-plane is a good criterion for the presence of dynamic chaos. From the study it is concluded that the negative velocity active controller is very effective tool in vibration reduction at many different resonance cases. Passive controllers are very helpful in suppressing the undesired vibration of the nonlinear dynamical system but more expensive than active ones.

4.2 Future Work

There are many directions of future research in which the present work can be extended, such as

1. investigate the non-linear vibration of the inclined beam to multi-excitations (harmonic and parametric), tuned forces, mixed excitation forces.
2. apply other different tools of control such as time delay control technique, if applicable.
3. validate the theoretical and numerical obtained results of the nonlinear dynamical system experimentally.

References

[1] A. R. F. Elhefnawy and A. F. El-Bassiouny "Nonlinear stability and chaos in electrodynamics" Chaos, Solitons & Fractals, Vol 23, pp. 289-312, 2005.

[2] A. H. Nayfeh and D. T. Mook "Perturbation Methods" John Wiley & Sons Inc., 1973.

[3] A. H. Nayfeh "Introduction to Perturbation Techniques" John Wiley & Sons Inc., 1981.

[4] A. H. Nayfeh "Nonlinear Interaction and Analytical, Computational, and Experimental Methods" John Wiley & Sons, Inc., 2000.

[5] A.H. Nayfeh, D. Mook "Nonlinear Oscillations" Wiley, New York, 1995.

[6] D. W. Jordan and P. Smith "Nonlinear Ordinary Differential Equations" Oxford University Press, 1985.

[7] E. E. El Behady, E. R. El-Zahar "Vibration reduction and stability study of a dynamical system under multi-excitation forces via active absorber" International Journal of Physical Sciences Vol. 48, pp. 6203-6209, December 2012.

[8] J. Warminski, M. Bochenski, W. Jarzyna, P. Filipek, M. Augustyniak, "Active suppression of nonlinear composite beam vibrations by selected control algorithms" Commun Nonlinear Sci Numer Simulat (Elsevier), pp. 2237-2248, 2011.

[9] L. Jun, H. Hongxing, S. Rongying "Saturation-based active absorber for a non-linear plant to principal external excitation" Mech Syst Signal Process, Vol 21, pp.98-1489, 2007.

[10] L. Jun, L. Xiaobin, H. Hongxing " Active nonlinear saturation-based control for suppressing the free vibration of a self-excited plant" Commun Nonlinear Sci Numer Simulat (Elsevier), Vol. 15, pp. 1071-1079, 2010.

[11] M. Eissa, W.A.A. El-Ganaini, Y. S. Hamed "On the saturation phenomena and resonance of non-linear differential equations" Menufiya Journal of Electronic Engineering Research (MJEER), Vol 15, pp. 73-84, January 2005.

[12] M. Eissa, W.A.A. El-Ganaini, Y. S. Hamed "Saturation and stability resonance of non-linear systems" Physica A, Vol 356, pp. 341-358, 2005.

[13] M.K. Kwak, S. Heo "Active vibration control of smart grid structure by multi-input and multi-output positive position feedback controller" J Sound Vibration , Vol 304, pp. 45-230, 2007.

[14] M. Siewe Siewe and Usama H. Hegazy, " Homoclinic bifurcation and chaos control in MEMS Resonators " Applied Mathematical Modelling (Elsevier), 35(12), pp. 5533-5552, 2011.

[15] M. Eissa, Y.A. Amer "Vibration control of a cantilever beam subject to both exernal and parametric exciation" Applied mathematics and computation, Vol 152, pp.611-619, 2004.

[16] M. Yaman, S. Sen " Vibration control of a cantilever beam of varying orientation" International Journal of Solids and Structures, Vol 44, pp.1210-1220, 2007.

[17] M. Sayed "Improving the Mathematical Solutions of Nonlinear Differential Equations Using Different Control Methods" Ph.D. Thesis, Menofia University, Shebin El-Koom, 2006.

[18] M.F. Golnaraghi "Regulation of flexible structures via non-linear coupling" Jouurnal of Dynamics and Control ,pp. 405-428, 1991.

[19] S.S. Oueni, C.M. Chin, A.H. Nayfeh "Dynamics of a cubic nonlinear vibration absorber" Nonlinear Dynam (Springer), Vol 20, pp.283-295, 1999.

[20] S. El-Serafi, M. Eissa, H. El-Sherbiny, T.H. El-Ghareeb "On passive and Active control of vibrating system" International Journal of Applied Mathematics, Vo1 18, pp. 515, 2005.

[21] S.A. El-Serafi, M.H. Eissa, H.M. El-Sherbiny, T.H. El-Ghareeb " Comparison between passive and active control of a non-linear dynamical system" Japan Journal of Industrial and Applied Mathematics, Vol. 23 , pp. 139-161, 2006.

[22] U. H. Hegazy, "Dynamics and Control of a Self-sustained Electromechanical Seismographs With Time-Varying Stiffness" Meccanica (Springer) , Vol. 44, pp. 355-368, 2009.

[23] U. H. Hegazy, "Single-Mode Response and Control of a Hinged-Hinged Flexible Beam" Archive of Applied Mechanics (Springer), Vol. 79, pp. 335-345, 2009.

[24] W.A.A. El-Ganaini, N.A. Saeed, M. Eissa "Positive position feedback (PPF) controller for suppression of nonlinear system vibration" Nonlinear Dynam (Springer), pp.517-537, 2013.