2010

Aircraft Leasing with Contracts

Qiong ZUO
Singapore Management University, qiong.zuo.2008@smu.edu.sg

Thesis Proposal Title:

Aircraft Leasing With Contracts

Student Name: ZUO Qiong

Student ID: G0755180R

Master of Science in Operations Management

Thesis chair: Prof. DING Qing

Committee member:

Prof. Brian RODRIGUES   Prof. LIM Yun Fong

Abstract:

We study a problem of rental rate pricing and rental contract designing in aircraft leasing industry. In a framework of Stackelberg game, the system is composed of an airline company (carrier) and an aircraft leasing company (lessor). Acting as the leader, the lessor announces daily rental rates and/or provides long-term contracts on a finite horizon with multiple periods. For each period, the carrier determines the aircraft leasing number to adjust the flight capacity, and applies a dynamic pricing policy for air-tickets based on a seasonally stochastic demand and some economic factor, such as oil price. We find the optimal policies for both lessor and carrier through a dynamic program approach.

Then, we consider a "forward-like" long-term contract in this paper. The lessor provides an identical rental rate if the carrier promises to rent a pre-determined number of aircraft on the whole horizon. Applying an appropriate long-term contract, the lessor can make more money from a large required leasing number. The carrier can improve performance from providing additional flights. Meanwhile, the customers enjoy more flight service. We are able to obtain the optimal contract design and the associated optimal policies for the entire system.

In the future research, we will study more flexible contracts for the carrier and lessor to improve the profit and share the risk.

Key words:

Aircraft leasing, batch size, contracts, oil price

1. Introduction

1.1 The emergence of aircraft leasing industry

Over the years, air travel has come to play a more prominent role in economic development as air transportation usage increased worldwide. In fact, historical analysis of transportation trends has showed that the role of air travel as a transportation mode will keep increasing in the future since people tend to shift to faster and more expensive transportation modes, such as air transportation, as their income increases.

Commercial airlines use their fleets of aircraft to transport passengers and cargo around the world. The aircraft are a significant and essential component of their business and great care is taken in matters related to these aircraft. Periodically it is necessary for an airline to acquire or rent new aircraft in order to expand their business or to replace older units which may no longer be safe or economically viable to operate. The decision making process for determining what type and what quantity of aircraft to obtain is often complex. Each different aircraft type has its own capabilities and characteristics which may or may not make it suitable for a given airline or route. Especially a new aircraft will likely be in service for 20 or more years and the purchase price for aircraft and the operating expenses associated with them are very high. Then the industry begins to consider the aircraft leasing topics to cut the expensive cost for obtain a new aircraft.

First used in the mid-1970s by ILFC, an aircraft lessor, the aircraft leasing became popular in the mid-1980s after airline deregulation in the United States and Europe. In the initial stages, operating lessors were mainly buying surplus second-hand aircraft from carriers and leasing them to other carriers, particularly those with poor access to debt and equity markets. In the mid-1980s, lessors started to acquire new aircraft directly from manufacturers and also entered into sale-leaseback transactions with carriers. As a result, today, almost all airlines use operating leases as a component of their capital structure, and most of all operating lessors' purchases of new aircraft have a designated lessee at the time of the order.

Till now, about one third of the aircraft currently operated by major carriers are under an operating lease: a rental contract between a lessor and an airline for use of the aircraft for a short period (mainly 4-5 years). GECAS—the largest lessor, a unit of General Electric Company —today owns approximately 1200 aircraft. As a means of comparison, the largest carrier in the world, American Airlines, operates around 800 aircraft.

1.2 back ground knowledge of aircraft leasing

Before we approach to the research about aircraft leasing contract, it is necessary for us to add some basic knowledge about aircraft leasing here.

1.2.1 The concept of leasing

Leasing is defined as a contract between a lessor and a lessee where the lessor provides the lessee with the right to use assets, property owned by the lessor. The contract is usually for a specified period of time, referred to as the 'the lease term'-for which the lessee is obliged to pay a stream of rental payments as agreed to between the lessor and the lessee. Generally, a lease contract may not be cancelled by either party unless certain terms and conditions specified in the contract trigger its termination (e.g., non-payment, bankruptcy). A lease contract may however grant an early termination option on a specific date with or without penalty for various predetermined reasons. At the expiry of the lease term, the lessee is usually required to return the asset to the lessor, unless the lessor provides an option to the lessee to purchase of the asset. A purchase option price is usually formula based and may be a variant on fair market value or a nominal amount depending on the type of lease.

1.2.2 Capital leasing and operating leasing

The two basic types of lease—the operating lease and the capital lease—have important differences. Generally speaking, if ownership of the leased asset is transferred to the lessee at the end of the lease term following payments that represent the full value of the asset, it is a capital lease; other wise, it is an operating lease. The precise classification changes slightly for legal, taxation, and accounting purposes, but the main idea is that the greater the extent to which the lessee acquires control and residual claims on the asset, the more likely it is that the lease is classified as a capital lease.

In the United States, conventional criteria used for lease classification include:

(1) Transfer of ownership:

If a lease contract has the provision of ownership transfer of the asset, it is classified as capital lease. It refers to the legal ownership in the lease contract which can be easily implemented in practice.

(2) Bargain purchase option:

A bargain purchase option is a provision allowing the lessee to purchase the leased property for a price that is significantly lower than the property's expected fair value at the date the option becomes exercisable. At the inception of the lease, the difference between the option price and the expected fair market value must be large enough to make exercise of the option reasonably assured. If there is a bargain purchase option in the lease contract, it is recognized as capital lease.

(3) Economic life:

If the lease period equals or exceeds 75% of the assets economic life, most of the risks and rewards of ownership are transferred to the lessee, and the capitalization is therefore appropriate. The lease term is generally considered to be fixed, non-cancelable term of the lease.

(4) Recovery of investment:

If the present value of the minimum lease payments equal or exceeds 90% of the fair market value of the asset, then the leased asset should be capitalized. The rationale for this test is that if the present value of the minimum lease payments is reasonably close to the market price of the asset, the asset is effectively being purchased.

In this paper, we will focus on the operating leasing. The reasons include operating perspective, contract design period, practice in industry, etc. More detailed explanations will be offered in next section.

1.3 The academic research of aircraft leasing

The airline industry is one of the most successful examples of applying operations research methods and tools for the planning and scheduling of resources. Optimization-based decision support systems have proven to be efficient and cost-saving for the scheduling of aircraft and crew, not to mention the short term rescheduling problems, where modifications to the initial plans are required before the final schedules can be executed.

On the other hand, leasing contracts are extensively used in capital-equipment markets, and leasing is one of the major sources of financing for firms. In fact, 80% of U.S. companies lease capital equipment. Of the $668 billion spent by American businesses on productive assets in 2003, $208 billion was acquired through leasing, according to the Equipment Lease Foundation. Graham, Lemmon, and Schallheim (1998) report that operating leases, capital leases, and debt are42%, 6%, and52% of fixed claims, respectively, in the1981-1992 data. Eisfeldt and Rampini (2009) document that leasing is the largest source of external finance for small firms. In a typical lease contract, the owner of the asset (the lessor) grants to another party (the lessee) the exclusive right to use the asset for an agreed period of time, in return for periodic payments. Hence, the lessee takes the risks and returns from the use of the asset, and the lessor takes the risks and returns from ownership of the asset. As in any financial contract, the risk of default by the lessee is a primary element in the risk of ownership, with the liquidation value of the asset playing a key role if the lessee defaults. Especially, lease of aircraft has become an increasingly important tool for the airline industry.

While in academic area, there is little research on leasing contract design, especially for some certain cases. The benefits of lease were traditionally viewed as financial. Gritta et al. (1994) examined the role of lease as sources of off-balance-sheet financing. As operating lease is not capitalized, carriers can substantially lower their debt/equity ratio on their balance sheet if they finance their aircraft fleet by leasing rather than by traditional debt. Another well-known financial benefit is that leasing separates the ownership of an aircraft from the aircraft's user. Therefore, it is the lessors who own the aircraft while the carrier operate the aircraft. This separation of ownership enables valuable depreciation allowances to be used more effectively by the lessors for tax purposes. Indeed, in certain international leasing arrangements, when the lessors and the airlines belong to different tax regimes, it was reported that depreciation allowances were claimed by both parties in the leasing contract, a practice commonly referred to as "double dip".

It may be argued that the effect of off-balance-sheet financing is largely cosmetic because financial analysts would not be fooled when it is publicly known that an airline has taken up a substantial lease obligation. Indeed, Marston and Harris (1988) demonstrated, using a large sample of US firms, that lease and debt are substitutes as it would under efficient financial markets. Results from a survey study by Bayliss and Diltz (1986) also showed that bank loan officers reduce their willingness to lend when a firm takes up lease obligations. Therefore, lease as source of off-balance-sheet financing does not appear to be able to significantly increase firms' debt capacity. Furthermore, with increasingly stringent accounting and tax rules, the tax effects of lease are also limited. Now, the major attractions of operating lease of aircraft are viewed as more operational than financial in nature. First, while the aircraft manufacturers currently have substantial order backlogs, major aircraft leasing companies have inventories for immediate delivery. Hence, airlines desiring a quick expansion need not wait for the production backlogs. Second, short-term operating lease provides the flexibility to the airlines so that they can manage fleet size and composition as closely as possible, expanding and contracting to match demand.

While significant use of operating lease affords the airlines the flexibility to change aircraft fleet size as demand for air transport changes, it created a burden to the leasing companies to maintain efficient utilization of their inventory of aircraft. In a recession, when demand for aircraft is low, the leasing companies will also suffer from excess capacity. Indeed, the last recession was devastating to dozens of leasing companies when demand and aircraft values dropped. In essence, through the flexibility of operating leases, the airlines shifted part of their business risks to the leasing companies. However, although short-term operating lease reduces the risks of excess capacity for the airlines, it does not eliminate uncertainties in the financial costs. During recession, when costs of short-term leasing are low, airlines have little incentive to expand their fleet. On the other hand, during the booming period, when the airlines need the capacity most, the costs of leasing will also be highest. Thus the operating lease provides a vehicle which enables the airlines and the leasing companies to share the risks of uncertain demand. For the airline industry which faces a cyclical demand, this risk-sharing aspect of operating lease is highly desirable. Needless to say, the aircraft leasing companies are in the business for profits. They purchase aircrafts from the manufacturers by means of long-term financing, and then lease the aircraft to the airlines. For short-term operating lease, it would take at least two or more lease transactions on an aircraft for the leasing companies to recover the costs. Therefore, the expected revenues from operating lease must not only cover the long-term financing costs of the aircraft, but also provide the leasing company with a profit (premium) adequate to compensate for the risks involved with aircraft release and residual value.

To the carriers, optimal use of operating lease then presents the problem of a trade-off between operational flexibility and higher financial costs inherent in the short-term lease. The historical trend has been an ever-increasing use of operating lease, in tandem with the development of an active aircraft leasing market. Now, with the market becoming mature, whether airlines should continue to increase reliance on operating leases has become a strategic question to the fleet management of the airlines.

1.4 The research problem and paper's structure

Operating lease of the aircraft gives the airlines flexibility in capacity management. However, airlines pay a risk premium to the leasing companies for bearing part of the risks. Therefore, the airlines face a trade-off between flexibility of capacity and higher costs.

More details, we can list the three main incentives for aircraft leasing here, and which are the key points we want to study in this paper.

(1) Risk transfer

The airline industry is traditionally cyclical, with large swings, and the waves have greater swings recently. The figure1 and 2 show the trends of airlines' net profit in these years. Hence, both airline profits and aircraft values carry substantial financial risk, and they are almost perfectly correlated. Leasing allows carriers to transfer some of the aircraft-ownership risk to operating lessors. And lessors are better suited to assuming the risk of aircraft ownership through their specific knowledge of the markets, their economies of scale, and their broader diversification of aircraft types and lessees operating in different geographic regions. Moreover, the largest lessors (GECAS and ILFC) belong to large financial conglomerates, which allow them to diversify the aggregate risk of aircraft ownership and to have a lower cost of funds, thanks to a higher credit rating.

(2) Flexibility

As an example, GECAS cites the following benefits of an operating lease: "Fleet flexibility to introduce new routes or aircraft types" and "Flexibility to increase or reduce capacity quickly." In exchange for operational flexibility, lessors require a substantial return. The airlines that use operating leases consider that the flexibility such leases provide makes up for the fact that the cash costs of the leases can be greater than the cost of acquiring the same aircraft through ownership.

(3) Price discounts

In most cases, lessors' scale enables them to extract price discounts from the manufacturers by purchasing a large number of aircraft, which means they can pass part of these savings to lessees, and smaller carriers find that particularly attractive.

Here in our model, we study a problem of rental rate pricing and rental contract designing in aircraft leasing industry. In a framework of Stackelberg game, the system is composed of an airline company (carrier) and an aircraft leasing company (lessor). Acting as the leader, the lessor announces daily rental rates and/or provides long-term contracts on a finite horizon with multiple periods. For each period, the carrier determines the aircraft leasing number to adjust the flight capacity, and applies a dynamic pricing policy for air-tickets based on a seasonally stochastic demand and some economic factor, such as oil price. We find the optimal policies for both lessor and carrier through a dynamic program approach.

Then, we consider a "forward-like" long-term contract in this paper. The lessor provides an identical rental rate if the carrier promises to rent a pre-determined number of aircraft on the whole horizon. Applying an appropriate long-term contract, the lessor can make more money from a large required leasing number. The carrier can improve performance from providing additional flights. Meanwhile, the customers enjoy more flight service. We are able to obtain the optimal contract design and the associated optimal policies for the entire system. In the future research, we will study more flexible contracts for the carrier and lessor to improve the profit and share the risk. We want to fill some gap about aircraft leasing applied in OM area by investing this contracts design problem.

The structure of the paper is as below:

The first section is the introduction about the aircraft leasing problem, together with a brief description of the problem we want to investigate. In section 2, we list the literatures about this topic and specially three most relative papers to our study; the section 3 explain the main assumptions and settings we will use in this paper, including the choice of pricing policies, demand characters, and parameter's setting, for example, the identical size of aircrafts. The base model is formulated in section 4, and we will list all the notations and obtain some basic but important characters about the information of optimal solution. The key part of this paper, section 5, will propose a forward-liking contract which is used between carrier and lessor in order to reach the win-win. We will provide both of the advantage and disadvantage of this contract which will be produced in application, in addition to the proof process. A sensitive analysis is provided in section 6 and we will test our result's robustness given different basic parameter's value. As the final part, in section 7, there is the conclusion of this paper and expectations for future research.

2. Literature review

2.1 Airline revenue management

Airline revenue management is the first relative literature area to our paper. To increase total revenue and in an effort to better match the demand for each flight with its capacity, airlines offer a variety of fare products at different price levels for the same flight (differential pricing). Much of the rationale for price differentiation lies not in discrimination but rather in the different costs of serving passengers associated with different requirements. A leisure traveler who is willing to book well ahead when seats are more readily available is less expensive to accommodate than a business traveler who demands flexibility. Business passengers will often need to book a flight shortly before the departure, in the event of an expected business meeting, for example. Therefore airlines are obliged to have seats available at the last minute for a number of business travelers. This result in lower load factors on days when the actual demand is lower than expected and must be accounted for when pricing different products. Fares thus must reflect the cost of providing different degrees of flexibility, even in economy class. Offering different fare products is also inevitably derived from the nature of airline market segmentation.

Revenue management is therefore the practice of determining the number of seats to be made available at each fare level, limiting low-fare seats and protecting higher-fare seats for late-booking passengers. The airline market has several niches, each with specific customer features. As a consequence, the price structure on a competitive route is not very transparent. The evolution of RM systems has traced the development of RM practices; the first of such systems appeared in the 1980s and mainly relied on overbooking (as described in Thompson (1961) and Littlewood (1972)) so as to increase revenue. A complete literature review dealing with revenue management and related problems can be found in Barnhart, Belobaba and Odoni(2003) and McGill and van Ryzin(1999).

2.2 Rental contract

The second stream of related research focuses on the analysis of contract design. Fisher and Raman (1996) show how demand forecasts can be improved by utilizing up-to-date demand information. Our results show how spot trading can help improve the supply chain performance while aggregating different pieces of private information that are dispersed among competing participants. Anupindi and Bassok (1999) and Mahajan and van Ryzin (1999) show that in a setting where competing retailers' market shares are determined by their inventory levels, the retailers overstock and supply chain coordination suffers. Kouvelis and Lariviere (2000) examine the role of an internal market in coordinating the supply chain. They find that when a central agent whose goal is maximizing supply chain profits can set (near) linear transfer prices; the supply chain can be coordinated. This coordination is based on the intervention of a benevolent agent and does not result from independent market forces. Corbett et al. (2004) investigate various two-part tariff contracts under a one-supplier; one-buyer setting where the buyer's cost is private information. Their study focuses on supplier-initiated contracts and evaluates the value of information for the supplier. In their article, the supplier does not know the buyer's internal variable cost; however, they do not consider supplier or buyer investment decisions to influence the demand through higher quality or selling efforts, respectively. In line with their article, we observe that as the information gap about retailer type increases, the flexibility provided by using transfer fees is reduced. However, we also note that the manufacturer may prefer the fixed-fee contract over the general franchise menu contract (two part tariff contract) under certain conditions. In a paper with retailer selling effort, Krishnan et al. (2004) consider the effect of retailer promotions on demand in a stochastic model. They discuss that a buy-back contract alone cannot coordinate the channel as buy-backs reduce the incentive for retailer's promotional efforts. As such, they suggest using buy-backs coupled with promotional cost-sharing agreements when effort cost is observed in order to coordinate the channel. Other coordinating mechanisms for the case of observable (but not verifiable) demand, and for the case of verifiable demand are also discussed.

Especially for rental industry, Bayiz and Tang (2004) develop an integrated planning system for a dosimeter service company that is based on deterministic demand and return processes. This planning system is designed to help the firm to manage inventory in an effective manner by taking the return process into consideration. Other work that relates to the rental industry tends to concentrate on the analysis of the revenue sharing scheme. Dana and Spier (2001) show that the revenue sharing scheme is valuable in vertically separated industries in which demand is either uncertain or variable. They also show that revenue sharing enables the supply chain to achieve the first best outcome by softening retail price competition without distorting the retailer's stocking decisions. Utilizing the panel data collected at 6137 video rental stores in the US between 1998 and 2000, Mortimer (2008) compares the stocking levels, rental prices, etc., across different stores for the same title as well as across different titles within the same store. In addition, she conducts regression analysis to examine the effect of revenue sharing scheme on the retailer's profit. Her analysis shows that the revenue sharing scheme has a small positive effect on the retailer's profit for popular titles, and a small negative effect for less popular titles.

To explore the notion of revenue sharing contracts further, Cachon and Lariviere (2005) provide an analytical comparison between revenue sharing contracts and other supply chain contracts such as buy-back contracts, price-discount contracts, quantity- flexibility contracts, sales-rebate contracts, etc. They show that revenue sharing contracts are equivalent to buy back contracts in the newsvendor case and are equivalent to price discount contracts in the price-setting newsvendor case. In contrast to Cachon's approach, Øystein Foros et al. (2009) discuss a price-dependent profit-sharing rule and show how an upstream firm, by using this rule, can prevent destructive competition between downstream firms that produce relatively close substitutes. With this rule, the upstream firm induces the retailers to behave as if demand has become less price elastic. As a result, competing downstream firms will maximize aggregate total channel profit. When downstream firms are better informed about demand conditions than the upstream firm, the same outcome cannot be achieved by vertical restraints such as resale price maintenance. Price-dependent profit sharing may also ensure that the downstream firms undertake efficient market expanding investments. The model is consistent with observations from the market for content commodities distributed by mobile networks.

2.3 Three most relative papers

Our departure point will be a prototypical single-product revenue management problem first introduced and formalized by Gallego and van Ryzin (1994). This formulation models realized demand as a Poisson process whose intensity at each point in time is determined by a price set by the decision maker. Given an initial inventory, the objective is to dynamically price the product so as to maximize expected revenues over a finite selling horizon. In the dynamic optimization problem considered in their paper, the decision maker knows the demand function prior to the start of the selling season and designs optimal policies based on this information.

The second is a discussion of the demand for operating leasing of aircrafts. Tae Hoon Oum et al. (2000) develop a model for the airlines to determine their optimal mix of leased and owned capacity, taking into consideration that the demand for air transportation is uncertain and cyclical. Empirical results based on the model suggested that the optimal demand by 23 major airlines in the world would range between 40% and 60% of their total fleet, for the reasonable range of premiums of operating lease. For the leasing companies, this indicates huge potential of the market given strong forecast for the growth of air transportation in the next decade.

The third inspire our contracts design problem. It is the Cachon and Lariviere (2005)'s paper motioned in last section. We get inspiration from the proof process and the beautiful results of this paper. Since we have listed it in the previous part, we just ignore its details here.

3. Assumptions and basic settings

3.1 Pricing policy

When we determine what pricing policy the airline company will take, we have to the balance between cost and flexibility. It is obviously the dynamic pricing policy costs more than fixed prices list, but gain more revenues.

In many settings, dynamic pricing (DyP) can augment or replace traditional capacity-control RM in which multiple product "classes" are offered at different posted prices, and revenues are controlled by allocating capacity to the different price classes over time.

There is an extensive literature on DyP. For surveys, see Bitran and Caldentey (2003) and Elmaghraby and Keskinocak (2003). Research on coordinated pricing and inventory decisions is surveyed by Chan et al. (2004) and Yano and Gilbert (2003). Broad discussions of RM and pricing can be found in Talluri and van Ryzin (2004).

From the consideration of the cost of changing price, Levy et al. (1997) analyze the data of pricing systems of supermarket chains, derived from five different costs of price changes. They found that if the costs of price adjustment made the adjustments unprofitable they are not implemented. Levy et al. (1998) extend this analysis by describing the price adjustment process and involved decisions in more detail. Bergen et al. (2003) emphasize that implementing price changes is not costless and offer recommendations for a price change strategy at the managerial level. Zbaracki et al. (2004) report on the empirical data of costs for price adjustments. These empirical studies support the argument that the costs of price changes should be considered when applying dynamic pricing policies.

Considering the price changing costs, Netessine (2006) analyzes the dynamic pricing problem of a single product when only a limited number of price changes are allowed. He considers the timing of the price changes in a deterministic environment in cases of constraint and unconstraint capacity. This work focuses on an analytic solution for maximum expected revenue, whereas our paper looks at the question which price changing costs are acceptable under downside risk.

There are several papers which analyze dynamic and static pricing policies. Gallego and van Ryzin (1994, 1997) discuss the numerical results of static pricing in comparison to the optimal price. They conclude that for large sized problems with known demand functions and no constraint on price setting, there were no great benefits when using dynamic pricing. Cooper (2002) and Maglaras and Meissner (2006) present results of contrasting static and resolving pricing policies. Cooper (2002) provides an example showing that resolving policies does not necessarily lead to a better result, whereas the numerical results of Maglaras and Meissner (2006) show that its expected revenue is superior to static pricing in non-asymptotically settings. However, the policies generally assume a long-term perspective when studying expected revenue and a short-term view when risk sensitivity might be more important.

On the other side, most classical DyP models assume that consumer behavior is myopic-a consumer makes a purchase as soon as the price is below his/her valuation for the product. At its core, any DyP system is based on a model of demand; that is, a model of how demand responds to price changes. A typical one is to formulate demand at each point in time as only a function of the price charged at that time, the implicit assumption being that customers do not anticipate future prices; that is, they are myopic and buy if the current price is less than their reservation price. This model is pervasive; most commercial pricing software uses it, and it is common in the research literature too. For example, see Gallego and van Ryzin (1994), Feng and Gallego (1995), Federgruen and Heching (1999), and Chen and Simchi-Levi (2003). The myopic customer assumption is reasonable when customers make impulse purchases and for consumable goods (food, beverages, etc.). In addition, it has the considerable practical advantage of leading to mathematically tractable models.

Monopoly models with strategic consumers are more complex, however, and are often considered in the deterministic form. For example, Besanko and Winston (1990) present a general deterministic DyP model. They show that the subgame-perfect equilibrium policy for the firm is to lower prices over time in a manner similar to price skimming. Su (2007) considers a deterministic demand model with consumers partitioned into four segments according to their valuation level and whether they are strategic or myopic. Consumers arrive continuously with fixed rates, and the seller looks for the optimal price and capacity rationing schedule. The article shows when the seller should use markdowns or markups. Aviv and Pazgal (2008) study the optimal pricing of fashion-like seasonal goods in the presence of forward-looking consumers who arrive according to a Poisson process with constant rate and have declining valuations for the product over the course of the season. This work considers the Nash equilibrium between a seller and strategic consumers for the cases of inventory-contingent pricing strategies and announced fixed discount strategies. Levin et al. (2010) study optimal DyP of perishable items by a monopolist facing strategic consumers. This has become the pop trends of principal when we consider the use of DyP.

3.2 Demand

3.2.1 Demand forecast

In order to schedule the fleet size and frequency, together with regarding to the development of infrastructure facilities and to reduce the airport risk, it is important for the carriers to evaluate and to forecast the volume of air passenger demand in the future. Peak demand in passenger flows at the airports, typically determined by seasonal and cyclical patterns. Therefore, it is essential to provide more capacity or choose better pricing policy to cover demand during the planning horizon and maximum the revenues.

Poore (1993) has developed a study to test the hypothesis that forecasts of the future demand for air transportation offered by aircraft manufacturers and aviation regulators are reasonable and representative of the trends implicit in actual experience. He compared forecasts issued by Boeing, Airbus Industry and the International Civil Aviation Organization (ICAO) which has actual experience and the results of a baseline model for revenue passenger kilometers (RPKs) demand. Svrcek (1994) has analyzed three fundamental measures of capacity, including static capacity that is used to describe the storage capability of a holding facility or area, dynamic capacity which refers to the maximum processing rate or flow rate of pedestrians and sustained capacity that is used to describe the overall capacity of a subsystem to accommodate traffic demand over a sustained period. Inzerilli and Sergioc (1994) have developed an analytical model to analyze optimal price capacity adjustments in air transportation. From this study, they used numerical examples to analyze the behavior of the policy variables (and the resulting load factor) under different degrees of uncertainty. Matthews (1995) has done measurement and forecasting of peak passenger flow at several airports in the United Kingdom. According to his research, annual passenger traffic demand can be seen as the fundamental starting point, driven by economic factors and forecasting. While forecasts of hourly flows are needed for long-term planning related with infrastructure requirements. Hourly forecasts are almost always based on forecasts of annual flows. Bafail, Abed, and Jasimuddin (2000) have developed a model for forecasting the long-term demand for domestic air travel in Saudi Arabia. They utilized several explanatory variables such as total expenditures and population to generate model formulation. Yamaguchi et al. (2001) have analyzed the economic impact analysis of deregulation for airport capacity expansion in Japanese domestic aviation market. According to their research, deregulation and airport capacity expansion play significant roles in realizing full benefit of aviation market growth. In line of deregulation policy, airport capacity expansion was accelerated to meet the growth demand. Swan (2002) has analyzed airline demand distributions model. The model explains when the Gamma shape will dominate and when the Normal will determine the shape. From his study, he found that Gamma shapes are probably better for revenue management and Normal for spill modeling. Another study for air travel demand forecasting has done by Grosche, Rothlauf, and Heinzl (2007). According to their research, there are some variables that can affect the air travel demand, including population, GDP and buying power index. He considered GDP as a representative variable for the level of economic activity.

3.2.2 Demand uncertainty

Economists have used a variety of demand models which have been applied in the revenue management literature. The problem of demand uncertainty has motivated a significant amount of literature in the field of revenue management and pricing. A number of different approaches have been introduced to model this uncertainty. Zabel (1970)considers two models of uncertain demand: a multiplicative model $\left(d_t = \eta_t u\left(p_t\right)\right)$ and additive model $\left(d_t = u\left(p_t\right) + \eta_t\right)$, where d_t is the demand at time t, p_t is the price at time t, $u\left(p\right) = a - \frac{a}{b} p$,i.e. a downward sloping linear demand curve, and η_t is assumed to be either exponentially or uniformly distributed with $E\left[\eta_t\right] > 0$. Young (1978) and Federgruen and Heching (1999) generalize the demand model to be of the form $d_t = \gamma_t\left(p\right)\varepsilon_t + \delta_t\left(p\right)$, where γ and δ have first derivatives non positive and ε_t is a random term with a finite mean. Gallego and van Ryzin (1994, 1997) as well as Bitran and Mondschein(1997)assume that demand follows a Poison process with a deterministic intensity that depends on price and time. Raman and Chatterjee (1997) model the stochastic characters of the demand by introducing an additive model where the random noise is a continuous time Wiener process.

The operations research literature treats the presence of data uncertainty in optimization problem in several ways. The problem is sometimes solved assuming all parameters are deterministic; subsequently sensitivity analysis is performed to study the stability of the nominal solution with respect to small perturbations of the data. Stochastic programming is used when a probability distribution of the underlying uncertain parameters is available, and seeks a solution that performs well and has low probability of constraint violation. Robust optimization is also an alternative way to seek an optimal solution of a problem when its data is uncertain.

3.3 Identical leased aircraft size and fixed fleet frequency

The recognition and study of the impact of aircraft size and frequency on airline demand started with the introduction of the concept of “schedule delay”, first introduced by Douglas and Miller (1974), and subsequently applied by Viton (1986). “Schedule delay” has two components.

The first is frequency delay, which represents the elapsed time between an individual traveler’s preferred time and the time of a scheduled flight. The second component is stochastic delay, which represents the additional elapsed time when preferred flights are fully booked. Douglas and Miller estimated empirical frequency and stochastic delay functions by using regression and simulation methods. Frequency delay decreases with frequency, while stochastic delay decreases with frequency and aircraft size, and increases with demand in the market. For the same service frequency provided by the airlines, the larger the aircraft, the higher the probability that a passenger can get a seat on a preferred flight and therefore enjoy a more convenient service. The concept of "schedule delay" was used in a linear regression model by Abrahams (1983) to estimate total air travel demand in a single market. In order to specify "schedule delay", Abrahams used the frequency delay function introduced by Eriksen (1977), and the stochastic delay function introduced by Swan (1979). These two functions have the same form as those proposed by Douglas and Miller (1974), but the parameter values are different. Thus these models capture effects of both frequency and aircraft size.

Instead of using the negative term" schedule delay", Eriksen (1977) and Russon and Hollingshead (1989) used the terms "level of service" or "quality of service"-which are functions of service frequency and aircraft size in a format similar to "schedule delay"-in their models of air passenger travel demand. Hansen (1990) used service frequency, fare and flight distance to specify a passenger's utility function, and built a logic model for demand analysis. Norman and Strandens (1990) directly related service frequency to the waiting time and cost of passengers, and built a probabilistic air travel demand model under the assumption of uniform distribution for desired departure times over a time interval.

More recently, Coldren et al. (2003) built an itinerary level market share model using aggregate multinomial logic methodology. Aircraft size and type, together with such variables as fares, time of day, carrier market presence, itinerary level-of-service (non-stop, direct, single-connect, or double-connect) and connecting quality, are taken as independent variables in the model to measure various itinerary characteristics. Proussaloglou and Koppelman (1995) and Nako (1992) both applied the logic model to study airlines' demand using the survey data from individual passenger. They both investigated the effectiveness of the frequent flyer programs, but did not take aircraft size or type as a factor influencing passengers' choice of airlines. Wenbin Wei and Mark Hansen (2005) focus on the analysis of the role of aircraft size on airlines' demand and market share in a duopoly competitive environment at the market level, with one major airport in origin and one major airport in destination. They study the roles of aircraft size both in an individual airline's market share and in total air travel demand in the market. Not only aircraft size but also the proportion of aircraft capacity available to local passengers will be taken into consideration in their research.

3.4 Other assumptions

Need to be fulfilled.

Comment:

Should add more reasonable point to explain why choose these assumptions and our setting in the model.

4. Basic model

4.1 Notations

The system is composed of a carrier and a lessor, and the whole time horizon is divided into N days (in fact, it should be N days but not limited as days, and here we use the days only for simplification).That is, the carrier and the lessor repeat their decision process N times in our model. In each decision day, assumed as day n, the sequence is as this way: acting as the leader, the lessor announces daily rental rates from the day n the end of horizon-day N , here we denote the leasing rates vector as L_n, where the n is the decision making day. It is obvious that $L_n = \left[l_{nn}, l_{n,n+1}, l_{n,n+2}, ...l_{nN} \right]^T$.

After observed the leasing rates vector, the carrier will determine the leasing number vector for this and following days, denoted as $X_n = \left[x_{nn}, x_{n,n+1}, x_{n,n+2}, ...x_{nN} \right]^T$, where the n is still the decision making day. Once the leasing number is determined for day n, the carrier's total capacity for this day is fixed. Here we use vector Y_n to denote the carrier's aircraft capacity, or we call it inventory, at the beginning of day n, where n is the decision making day, $Y_n = \left[y_{nn}, y_{n,n+1}, y_{n,n+2}, ...y_{nN} \right]^T$. We denote y_0 the number of carrier owned aircrafts. It is sure that at the beginning of the first day, the inventory for each day is the same as equal to the carrier's owned capacity, that is $y_{11} = y_{12} = y_{13} = ... = y_{1N} = y_0$;for the days $n > 1$, the carrier's inventory for day i at the beginning of day n should be the sum of all the leasing number rented for the day i till day $n-1$, that is $y_{ni} = x_{1i} + x_{2i} + ... + x_{n-1,i} = y_{n-1,i} + x_{n-1,i}$,for each i, $i \ge n$.

Now we consider what happens in day n after the capacity is fixed. The carrier applies a dynamic pricing policy for air-tickets based on a seasonally stochastic demand and some economic factor, such as oil price. We denote the carrier's ticket pricing policy at day n as P_n, which belongs to a set P. P_n is a function of demand, $P_n = P_n(\lambda_n)$, λ_n is the demand at day n ,it is also a function of P_n,which means the demand can also be expressed as $\lambda_n = \lambda_n(P_n)$. Also, we have to take the notation t to denote the ticket booking point in the ticket sale process, and T_n the carrier's ticket booking horizon for day n, $t \in T_n$.

As for the parameters, here in the model we pay attention to the oil prices and the seats number in aircrafts. The oil price for day n is O_n, and it is stochastic. In corresponding to this, we use o_n to denote the observed oil price in day n. We only consider the identical aircrafts and the number of seats in each aircraft is q .based on these notations, in the day n we can calculate the total seats available for day $n, n+1, n+2 ... N$, and it is a vector $Z_n = \left[z_{nn}, z_{n,n+1}, z_{n,n+2} ... z_{nN} \right]^T$, $Z_n = q\left(X_n + Y_n \right)$.

The above notations can be listed in the following table:

Notation list:

N	The number of decision days.
L_n	The leasing rate vector at day n, n is the decision making day, $L_n = \left[l_{nn}, l_{n,n+1}, l_{n,n+2}, ... l_{nN} \right]^T$
X_n	The leasing number vector at day n, n is the decision making day, $X_n = \left[x_{nn}, x_{n,n+1}, x_{n,n+2}, ... x_{nN} \right]^T$
P_n	Carrier's ticket pricing policy at day n, belongs to a set P, $P_n = P_n\left(\lambda_n \right)$
λ_n	The demand at day n, $\lambda_n = \lambda_n\left(P_n \right)$
t	Ticket booking point
T_n	Carrier's ticket booking horizon for day n, $t \in T_n$
Y_n	The carrier's aircraft inventory at the beginning of day n, n is the decision making day, $Y_n = \left[y_{nn}, y_{n,n+1}, y_{n,n+2}, ... y_{nN} \right]^T$
O_n	The oil price in day n
o_n	The observed oil price in day n
q	The number of seats in each aircraft
Z_n	The total seats available for day $n, n+1, ... N$, it is a vector $Z_n = \left[z_{nn}, z_{n,n+1}, z_{n,n+2} ... z_{nN} \right]^T$

In order to describe the problem more clearly, we take the method of back-ward analysis in order to get some straightforward conclusions, and then, test them in multiple consecutive days' environment. We first demonstrate the carrier's and lessor's problem in the last day N , and then go to the case of last two days', for day N and day $N-1$.

4.2 Carrier's profit function:

Let's consider the carrier's problem. We need to add some new notations here to describe the carrier's decision process. As the objective is to maximize the carrier's profit, we use Π_N^C to denote the carrier's optimal profit at day N ;and if given the observed oil price o_N , leasing rate vector L_N , and the inventory Y_N at the beginning of day N , the decision variables relative to the profit are the leasing number X_N and pricing policy P_N . So the expression of the optimal profit can be written as $\Pi_N^C\left(X_N, P_N \middle| o_N, L_N, Y_N\right)$. If we consider the profit at day N , we use $\pi_N^C\left(X_N, P_N \middle| o_N, L_N, Y_N\right)$ to denote this, and still X_N, P_N is the decision variables.

Now we discuss how the profit is created in carrier's perspective. First, $R_N\left(P_N \middle| Z_N\right)$ is used as the ticket sales revenue at day N , given the aircraft seats capacity Z_N . And from lots of literatures we know that once the capacity is fixed, the revenue is determined by the adjustment of tickets price, so the pricing policy is the variable here. Then we consider the cost part, which includes the cost for leasing aircrafts, and operating costs for daily use, especially, the oil payment. Here we use ${L_N}^T X_N$ to denote the leasing payment at day N , and $o_N I^T X_N$ is the oil payment at day N .

So, the decision-making process should be acted this way: once the carrier knows about the realized oil price o_N at day N , the announced leasing rate L_N and the aircraft inventory at the beginning of day N , it chooses X_N, P_N to calculate the maximum of its profit to reach the optimal solution for day N .In day N , the carrier's profit function should be as below:

$$\pi_N^C\left(X_N, P_N \left|o_N, L_N, Y_N\right.\right) = R_N\left(P_N \left|Z_N\right.\right) - L_N{}^T X_N - o_N I^T X_N$$

And the objective of the carrier at day N should be:

$$\Pi_N^C\left(X_N, P_N \left|o_N, L_N, Y_N\right.\right) = \max_{X_N, P_N} \underset{\lambda_N}{E}\left[\pi_N^C\left(X_N, P_N \left|o_N, L_N, Y_N\right.\right)\right]$$

This is only for the last day's problem, and we can observe from the equation that it is a single period's problem and the decision is not affected by previous decisions. The logic we want to use to in this model is that after set up the formulation for last day, we will go back to study the problem of two days, day N and $N-1$, and to observe some characters for consecutive two days. Then we use back-ward analysis to extend the conclusion to multiple days' case and to check the system's characters.

In this way, if we consider any day n 's problem, $n = 1,2,3...N-1$, now in the day n ,the carrier's profit function should be:

$$\pi_n^C\left(X_n, P_n \left|o_n, L_n, Y_n\right.\right) = R_n\left(P_n \left|Z_n\right.\right) - L_n{}^T X_n - o_n I^T X_n + \underset{O_{n+1}}{E}\left[\Pi_{n+1}^C\left(X_{n+1}, P_{n+1} \left|O_{n+1}, L_{n+1}, Y_{n+1}\right.\right)\right]$$

Compared with the last day's case, the new function has one more item, which is the expected forward profit that will be produced in the future. And the objective of the carrier at day n should be:

$$\Pi_n^C\left(X_n, P_n \left|O_n, L_n, Y_n\right.\right) = \max_{X_n, P_n} \underset{\lambda_n}{E}\left[\pi_n^C\left(X_n, P_n \left|O_n, L_n, Y_n\right.\right)\right]$$

Where

$\Pi_n^C\left(X_n, P_n \left|O_n, L_n, Y_n\right.\right)$ is the optimal profit at day n ;

$\pi_n^C\left(X_n, P_n \left|o_n, L_n, Y_n\right.\right)$ is the profit at day n , X_n, P_n is the decision variables;

$R_n\left(P_n \left| Z_n\right.\right)$ is the ticket sales revenue at day n;

$L_n^{\ T} X_n$ is the leasing payment at day n;

$O_n I^T X_n$ is the oil payment at day n.

$\underset{O_{n+1}}{E}\left[\Pi_{n+1}^C\left(X_{n+1}, P_{n+1} \left| O_{n+1}, L_{n+1}, Y_{n+1}\right.\right)\right]$: The expected optimal profit from day $n+1$ to N. Because the oil prices in the future are stochastic, we have to take expectation operation here.

Especially, we now have to pay attention to the terminate case and conditions. When $n = N-1$, the profit function will become:

$$\pi_n^C\left(X_n, P_n \left| o_n, L_n, Y_n\right.\right) = R_n\left(P_n \left| Z_n\right.\right) - L_n^{\ T} X_n - o_n I^T X_n + \underset{O_N}{E}\left[\Pi_N^C\left(X_N, P_N \left| O_N, L_N, Y_N\right.\right)\right]$$

The last item of this equation is just the expectation of the last day's optimal profit, and referring to what we have discussed, it is in fact a single period's problem, and the forward process will be terminated here. This is the reason why we can use back-ward analysis to obtain the desired results.

4.3 Lessor's profit function:

Compared with the carrier's problem, the lessor's profit function is much simpler. Because we don't consider any operating cost generated in the leasing process for lessor, so the leasing revenue will be the profit of the lessor. If we only study the last day's case, then

$$\pi_N^L\left(L_N\right) = L_N^{\ T} X_N$$

And still the optimal profit should be

$$\Pi_N^L\left(L_N\right) = \max_{L_N} \underset{\lambda_N}{E}\left[\pi_N^L\left(L_N\right)\right]$$

As for multiple days, the profit should also be the profit produced in the current day plus the profit to go in the future, that is, if we consider any day n 's problem, $n = 1, 2, 3 \ldots N-1$, now in the day n ,the lessor's profit function should be:

$$\pi_n^L(L_n) = L_n^{\ T} X_n + \underset{O_{n+1}}{E}\left[\Pi_{n+1}^L(L_{n+1})\right]$$

And the objective function is:

$$\Pi_n^L(L_n) = \max_{L_n} \underset{\lambda_n}{E}\left[\pi_n^L(L_n)\right]$$

Where

$\Pi_n^L(L_n)$ is the optimal profit at day n;

$\pi_n^L(L_n)$ is the profit at day n, L_n is the decision variables;

$L_n^{\ T} X_n$ is the leasing revenue at day n;

$\underset{O_{n+1}}{E}\left[\Pi_{n+1}^L(L_{n+1})\right]$: The expected optimal profit from day $n+1$ to N .

4.4 Two day's problem:

In this section, we will study the problem of two days' case. That is, if we assume $T = N$, now we only consider what will happen when the system's whole time horizon is only composed of the day N and day $N-1$. We want to use two day's problem to explore how to solve the whole multiple days' repeating problem. And based on the two parties' decision sequence, we still first consider the carrier's problem.

For day N , we have already obtained the following equation for the carrier:

$$\Pi_N^C(X_N, P_N | o_N, L_N, Y_N) = \max_{X_N, P_N} \underset{\lambda_N}{E}\left[\pi_N^C(X_N, P_N | o_N, L_N, Y_N)\right]$$

Where $\pi_N^C(X_N, P_N | o_N, L_N, Y_N) = R_N(P_N | Z_N) - L_N^{\ T} X_N - o_N I^T X_N$

Now, we should check the characters of the left three items.

4.4.1 The characters of $\frac{\Delta R_N\left(P_N\left|z_{NN}\right.\right)}{\Delta x_{NN}}$

To make the expression more clear, we list the parameters and variables here, that are: $X_N=\left[x_{NN}\right]^T, Y_N=\left[y_{NN}\right]^T, Z_N=\left[z_{NN}\right]^T, L_N=\left[l_{NN}\right]^T$.

In order to calculate $\frac{\Delta\pi_N^C}{\Delta x_{NN}}$,

Where

$$\frac{\Delta\pi_N^C\left(X_N,P_N\left|o_N,L_N,Y_N\right.\right)}{\Delta x_{NN}}=\frac{\Delta\left[R_N\left(P_N\left|Z_N\right.\right)-L_N{}^T X_N-o_N I^T X_N\right]}{\Delta x_{NN}}=\frac{\Delta R_N\left(P_N\left|z_{NN}\right.\right)}{\Delta x_{NN}}-l_{NN}-o_N$$

In Gallego's theorem 1, we obtain the conclusion that $R_N\left(P_N\left|z_{NN}\right.\right)$ is concave on z_{NN}, which means that $\frac{\partial R_N\left(P_N\left|z_{NN}\right.\right)}{\partial z_{NN}}$ is decreasing on z_{NN}. Referring to his conclusion, here we only need to test the characters of $\frac{\Delta R_N\left(P_N\left|z_{NN}\right.\right)}{\Delta x_{NN}}$ to show $\frac{\Delta\pi_N^C}{\Delta x_{NN}}$'s trends.

We omit the proof process in the paper, and the result is $\frac{\Delta R_N\left(P_N\left|z_{NN}\right.\right)}{\Delta x_{NN}}$ is decreasing on x_{NN}. Detailed proof please refers to appendix I.

4.4.2 The character of $\frac{\Delta\Pi_N^C}{\Delta y_{NN}}$

Because the objective is to obtain the optimal profit, only characters of $\frac{\Delta\pi_N^C}{\Delta x_{NN}}$ is not enough. And as we take the expectation operation on π_N^C to calculate Π_N^C, here we need to check the character of $\frac{\Delta\Pi_N^C}{\Delta y_{NN}}$.

As already pointed out,

$$\Pi_N^C\left(X_N, P_N \left| o_N, L_N, Y_N\right.\right) = \max_{X_N, P_N} \underset{\lambda_N}{E}\left[\pi_N^C\left(X_N, P_N \left| o_N, L_N, Y_N\right.\right)\right],$$

Then

$$\frac{\Delta\Pi_N^C}{\Delta y_{NN}} = \frac{\Delta\left\{\max\limits_{X_N, P_N} \underset{\lambda_N}{E}\left[\pi_N^C\left(X_N, P_N \left| o_N, L_N, Y_N\right.\right)\right]\right\}}{\Delta y_{NN}} = \frac{\underset{\lambda_N}{E}\left\{\Delta\left[\max\limits_{X_N, P_N} \pi_N^C\left(X_N, P_N \left| o_N, L_N, Y_N\right.\right)\right]\right\}}{\Delta y_{NN}}$$

Because $\pi_N^C\left(X_N, P_N \left| o_N, L_N, Y_N\right.\right) = R_N\left(P_N \left| Z_N\right.\right) - L_N{}^T X_N - o_N I^T X_N$,

$X_N = \left[x_{NN}\right]^T$, $Y_N = \left[y_{NN}\right]^T$, $Z_N = \left[z_{NN}\right]^T$, $L_N = \left[l_{NN}\right]^T$ here, we also only need to calculate $\frac{\Delta\left(\max \pi_N^C\right)}{\Delta y_{NN}}$,

Where

$$\begin{aligned}
&\frac{\Delta\left[\max\limits_{X_N, P_N} \pi_N^C\left(X_N, P_N \left| o_N, L_N, Y_N\right.\right)\right]}{\Delta y_{NN}} \\
&= \frac{\Delta\left[\max\limits_{X_N, P_N} R_N\left(P_N \left| Z_N\right.\right) - L_N{}^T X_N - o_N I^T X_N\right]}{\Delta y_{NN}} \\
&= \frac{\Delta\left[\max\limits_{X_N, P_N} R_N\left(P_N \left| z_{NN}\right.\right)\right]}{\Delta y_{NN}}
\end{aligned}$$

Based on the definition of

$$\Pi_N^C\left(X_N, P_N \left| o_N, L_N, Y_N\right.\right) = \max_{X_N, P_N} \mathop{E}_{\lambda_N}\left[\pi_N^C\left(X_N, P_N \left| o_N, L_N, Y_N\right.\right)\right],$$

$$\max_{X_N, P_N} R_N\left(P_N \left| z_{NN}\right.\right) = \max_{X_N, P_N} R_N\left(P_N \left| q\left(x_{NN} + y_{NN}\right)\right.\right),$$

That is

$$\max_{X_N, P_N} R_N\left(P_N \left| q\left(x_{NN}\left(y_{NN}\right) + y_{NN}\right)\right.\right) = R_N\left[P_N^* \left| q\left(x_{NN}^*\left(y_{NN}\right) + y_{NN}\right)\right.\right]$$

$$\max_{X_N, P_N} R_N\left[P_N \left| q\left(x_{NN}\left(y_{NN}+1\right) + \left(y_{NN}+1\right)\right)\right.\right] = R_N\left[P_N^* \left| q\left(x_{NN}^*\left(y_{NN}+1\right) + \left(y_{NN}+1\right)\right)\right.\right]$$

$$\max_{X_N, P_N} R_N\left[P_N \left| q\left(x_{NN}\left(y_{NN}+2\right) + \left(y_{NN}+2\right)\right)\right.\right] = R_N\left[P_N^* \left| q\left(x_{NN}^*\left(y_{NN}+2\right) + \left(y_{NN}+2\right)\right)\right.\right]$$

We use these three equation to obtain the result is that $\frac{\Delta\Pi_N^C}{\Delta y_{NN}}$ is decreasing on y_{NN} ; detailed proof please refers to appendix II.

5. Following work

The following work includes these items:

Section 3: add explanations to the model assumptions;

Section 4: conclude the characters we obtained in the basic model;

Section 5: propose the forward-liking contract, formulate and prove it;

Section 6: sensitive analysis and data test;

Section 7: conclusion and future research.

List of references

[1]. Abrahams, M., 1983. A service quality model of air travel demand: an empirical study. Transportation Research A 17, 385-393.

[2]. Anupindi, R., Y. Bassok, 1999. Centralization of stocks: Retailers vs. manufacturer. Management Science 45, 178-191.

[3]. Aviv, Y., A. Pazgal, 2008. Optimal pricing of seasonal products in the presence of forward-looking consumers. Manufacturing and Service Operations Management 10, 339-359.

[4]. Bafail, A. O., Abed, S. Y., & Jasimuddin, S. M., 2000. The determinants of domestic air travel demand in the Kingdom of Saudi Arabia. Journal of Air Transportation World Wide 5, 72-86.

[5]. Bayiz, M., Tang, C., 2004. An integrated system for managing the refurbishment of thermo luminescent badges. Interfaces 24, 383-393.

[6]. Bayliss, M., Diltz, J., 1986. An empirical study of the debt displacement effects of leasing. Financial Management 15, 53-60.

[7]. Bergen, M., Ritson, M., Dutta, S., Levy, D., Zbaracki, M.J., 2003. Shattering the myth of costless price changes. European Management Journal 21, 663-669.

[8]. Besanko, D., W. L. Winston, 1990. Optimal price skimming by a monopolist facing rational consumers. Management Science 36, 555-567.

[9]. Bitran, G., R. Caldentey. 2003. An overview of pricing models for revenue management. Manufacturing Service Operation Management 5, 203-229.

[10]. Cachon, G., Lariviere, M., 2005. Supply chain coordination with revenue sharing contracts: Strengths and limitations. Management Science 51, 30-44.

[11]. Chan, L., Z. Shen, D. Simchi-Levi, J. Swann, 2004. Coordination of pricing and inventory decisions: A survey and classification. D. Simchi-Levi, D. Wu, Z. Shen, eds. Handbook of Quantitative Supply Chain Analysis- Modeling in the E-Business Era. Springer, New York, 335-392.

[12]. Chen, X., D. Simchi-Levi, 2003. Coordinating inventory control and pricing strategies with random demand and fixed ordering costs: The finite horizon case. Ph.D. thesis, Massachusetts Institute of Technology, Cambridge.

[13]. C.J. Corbett, D. Zhou, and C.S. Tang, 2004.Designing supply contracts: contract type and information asymmetry, Management Science 50, 550-559.

[14]. Coldren, G.M., Koppelman, F., Kasturirangan, K., Mukherjee, A., 2003. Air travel itinerary share prediction: Logic Model Development at Major US Airline. In: Transportation Research Board annual meeting CD ROM.

[15]. Cooper, W., 2002. Asymptotic behavior of an allocation policy for revenue management. Operations Research 50, 720-727.

[16]. Cynthia Barnhart, Peter Belobaba, and Amadeo Odoni, 2003. Applications of operations research in the air transport industry. Transportation Science 37, 368-391.

[17]. Dana, J., Spier, K., 2001. Revenue sharing and vertical control in the video rental industry. Journal of Industrial Economics 59, 223-245.

[18]. Douglas, G.W., Miller III, J.C., 1974. Economic Regulation of Domestic Air Transport: Theory and Policy.

[19]. Eisfeldt, A., Rampini, A., 2009.Leasing, ability to repossess, and debt capacity. Review of Financial Studies 22, 1621-1657.

[20]. Elmaghraby, W., P. Keskinocak, 2003. Dynamic pricing in the presence of inventory considerations: Research overview, current practices and future directions. Management Science 49, 1287-1309.

[21]. Eriksen, S.E., 1977. Policy Oriented Multi-Equation Models of US domestic Air Passenger Markets. Ph.D. Dissertation, Massachusetts Institute of Technology.

[22]. E.Zable, 1970. Monopoly and uncertainty. The Review of Economic Studies 37, 205-219.

[23]. Federgruen, A., A. Heching. 1999. Combined pricing and inventory control under uncertainty. Operations Research 47, 454-475.

[24]. Feng, Y., G. Gallego. 1995. Optimal starting times for end-of season sales and optimal stopping times for promotional fares. Management Science 41, 1371-1391.

[25]. Fisher, M., A. Raman, 1996. Reducing the cost of demand uncertainty through accurate response to early sales. Operations Research 44, 87-99.

[26]. G.Bitran and S.Monstein, 1997. Periodic pricing of seasonal product in retailing. Management Science 43, 427-443.

[27]. Gallego, G., G. van Ryzin, 1994. Optimal dynamic pricing of inventories with stochastic demand over finite horizons. Management Science 50, 999-1020.

[28]. Gallego, G., G. van Ryzin, 1997. A multiproduct dynamic pricing problem and its applications to network yield management. Operations Research 45, 24-41.

[29]. Graham,J., Lemmon,M., Schallheim,J., 1998. Debt, leases, taxes, and the endogenously of corporate tax status. Journal of Finance 53, 131-162.

[30]. Gritta, R.D., Lippman, E., Chow, G., 1994. The impact of the capitalization of leases on airline financial analysis: An issue revisited. Logistics and Transportation Review 30, 189-202.

[31]. Grosche, T., Rothlauf, F., Heinzl, A., 2007. Gravity models for airline passenger volume estimation. Journal of Air Transport Management, 13, 175-183.

[32]. Hansen, M., 1990. Airline's competition in a hub-dominated environment: an application of non cooperative game theory. Transportation Research 24B (1).

[33]. H. Krishnan, R. Kapuscinski, D.A. Butz, Coordinating contracts for decentralized supply chains with retailer promotional effort, Management Science 50 (2004), 48-63.

[34]. H.R.Thompson, 1961. Statistical problem in airline reservations control. Operations Research Quarterly 12, 167-185.

[35]. Inzerilli, F., Sergioc, R., 1994. Uncertain demand, modal competition and optimal price-capacity adjustments in air transportation. Transportation 21, 91-101.

[36]. J.I. McGill, G.J.van Ryzin, 1999. Revenue Management: research overview and prospects. Transportation Science 33, 233-256.

[37]. K.Littlewood, 1972. Forecasting and control of passenger bookings. 12th AGIFORS Annual Symposium Proceedings, Nathanya, Israel, 95-117.

[38]. Kouvelis, P., M. Lariviere. 2000. Decentralizing cross-functional decisions: Coordination through internal markets. Management Science 46, 1049-1058.

[39]. K.Raman, R.Chatterjee, 1997. Optimal monopolist pricing and ordering decisions by a monopolist. Management Science 3, 219-245.

[40]. Levin Yuri, Jeff McGill, Mikhail Nediak, 2010. Optimal dynamic pricing of perishable items by a monopolist facing strategic consumers. Production and Operations Management 19, 40-60.

[41]. Levy, D., Bergen, M., Dutta, S., Venable, R., 1997. The magnitude of menu costs: Direct evidence from large US supermarket chains. The Quarterly Journal of Economics 112, 791-825.

[42]. Levy, D., Dutta, S., Bergen, M., Venable, R., 1998. Price adjustment at multiproduct retailers. Managerial and Decision Economics 19, 81-120.

[43]. L.Young, 1978. Price, inventory and the structure of uncertain demand. New Zealand Operations Research 6, 157-177.

[44]. Maglaras, C., Meissner, J., 2006. Dynamic pricing strategies for multiproduct revenue management problems. Manufacturing and Service Operations Management 8, 136-148.

[45]. Mahajan, S., G. van Ryzin. 1999. Retail inventories and consumer choice. S. Tayur, R. Ganeshan, M. Magazine, eds. Quantitative Models for Supply Chain Management. Kluwer, Boston, MA, 491-551.

[46]. Marston, F., Harris, R.S., 1988. Substitutability of leases and debt in corporate capital structures. Journal of Accounting, Auditing and Finance 3, 147-164.

[47]. Matthews, L., 1995. Forecasting peak passenger flows at airports. Transportation 22, 55-72.

[48]. Mortimer, J., 2008. Vertical contracts in the video rental industry. Review of Economic Studies 75, 165-199.

[49]. Nako, S.M., 1992. Frequent flyer programs and business travelers: an empirical investigation. The Logistics and Transportation Review 28 (4).

[50]. Netessine, S., 2006. Dynamic pricing of inventory/capacity with infrequent price changes. European Journal of Operational Research 174, 553-580.

[51]. Norman, V.D., Strandens, S.P., 1990. Deregulation of Scandinavian Airlines: A Case Study of Oslo-Stockholm Route O'Connor, W.E., 1995. An Introduction to Airline Economics, fifth Ed.

[52]. Øystein Foros, Kåre P. Hagen, Hans Jarle Kind, 2009. Price-dependent profit sharing as a channel coordination device. Management Science 55, 1280-1291.

[53]. Poore, J. W., 1993. Forecasting the demand for air transportation services. Journal of Transportation Engineering 19, 22-34.

[54]. Proussaloglou, K., Koppelman, F.S., 1995. Air carrier demand: an analysis of market share determinants. Transportation 22.

[55]. Russon, M., Hollingshead, C.A., 1989. Aircraft size as a quality of service variable in a short-haul market. International Journal of Transport Economics XVI (3).

[56]. Su, X., 2007. Inter-temporal pricing with strategic customer behavior. Management Science 53, 726-741.

[57]. Svrcek, T., 1994. Planning level decision support for the selection of robust configurations of airport passenger buildings. Ph.D. Thesis, Department of Aeronautics and Astronautics, Flight Transportation Laboratory Report R94-6, Massachusetts Institute of Technology, Cambridge, MA, USA.

[58]. Swan, W. M., 2002. Airline demand distributions: Passenger revenue management and spill. Transportation Research Part E 38, 253-263.

[59]. Tae Hoon Oum, Anming Zhang, Yimin Zhang, 2000. Optimal demand for operating lease of aircraft. Transportation Research Part B 34, 17-29.

[60]. Talluri, K., G. van Ryzin, 2004. The Theory and Practice of Revenue Management. Kluwer, Norwell, MA.

[61]. Viton, P., 1986. Air deregulation revisited: choice of aircraft, load factors, and marginal-cost fares for domestic air travel. Transportation Research, Part A 20A (5), 361-371.

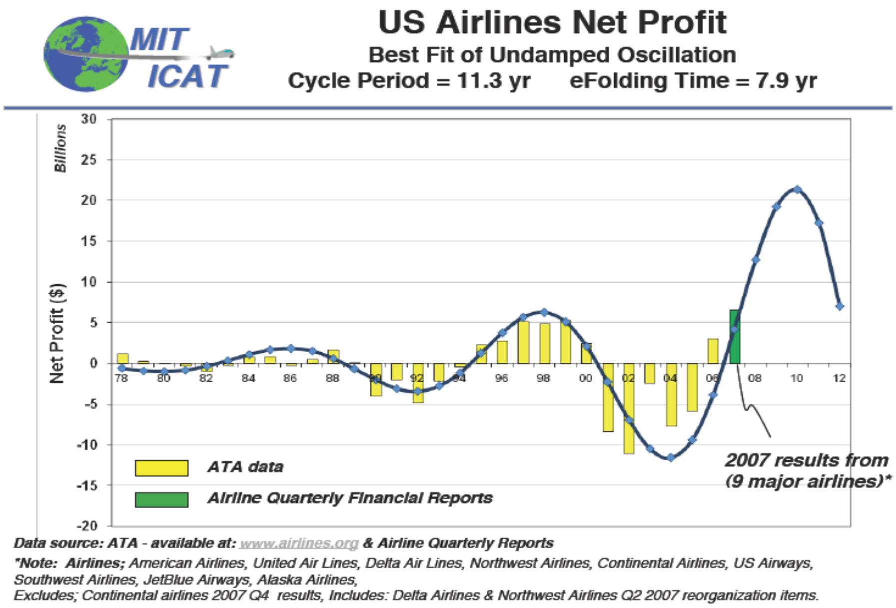
[62]. Wenbin Wei, Mark Hansen, 2005.Impact of aircraft size and seat availability on airlines' demand and market share in duopoly markets. Transportation Research Part E 41, 315-327.

[63]. Yamaguchi, K., Ueda, T., Ohashi, T., Takuma, F., Tsuchiya, K., Hikada, T., 2001.Economic impact analysis of deregulation and airport capacity expansion in Japanese domestic aviation market.

[64]. Yano, C., S. Gilbert, 2003. Coordinated pricing and production procurement decisions: A review. A. Chakravarty, J. Eliashberg, eds. Managing Business Interfaces: Marketing, Engineering, and Manufacturing Perspectives. Kluwer, Dordrecht, the Netherlands, 65-103.

[65]. Zbaracki, M.J., Ritson, M., Levy, D., Dutta, S., Bergen, M., 2004. Managerial and customer costs of price adjustment: Direct evidence from industrial markets. The Review of Economics and Statistics 86, 514-533.

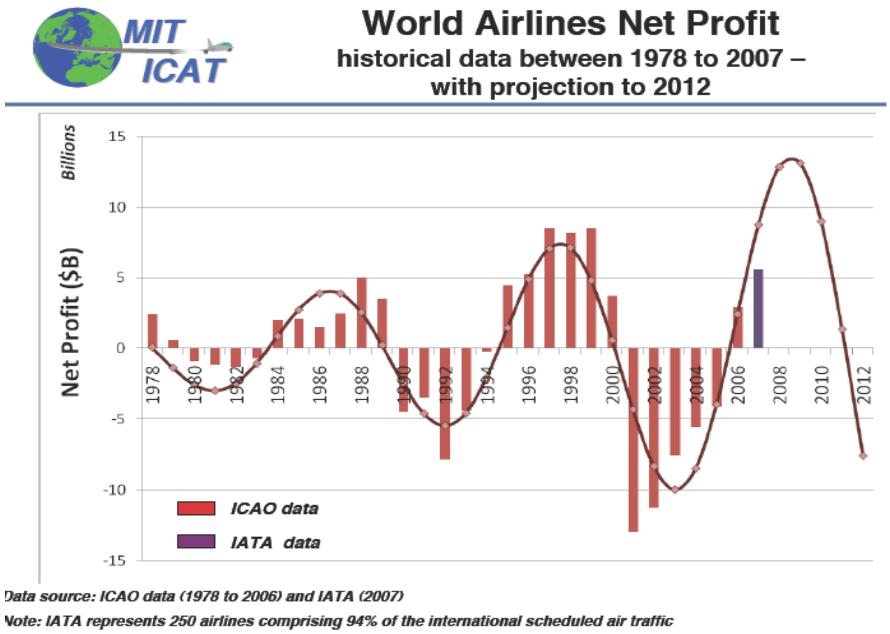
Figure 1: US airlines net profit



Source:

Overview of recent forces & trends in the airline industry, MIT ICAT, R.John Hansman

Figure 2: World airlines net profit



Source:

Overview of recent forces & trends in the airline industry, MIT ICAT, R.John Hansman

Appendix I: Characters of $\frac{\Delta R_N\left(P_N \mid z_{NN}\right)}{\Delta x_{NN}}$ (1)

There are two ways to check the characters.

Proof 1:

$$\frac{\Delta R_N\left(P_N \mid z_{NN}\right)}{\Delta x_{NN}} = \frac{\Delta R_N\left(P_N \mid z_{NN}\right)}{\Delta z_{NN}} \frac{\Delta z_{NN}}{\Delta x_{NN}} = \frac{\Delta R_N\left(P_N \mid z_{NN}\right)}{\Delta z_{NN}} \frac{\Delta\left[q\left(x_{NN}+y_{NN}\right)\right]}{\Delta x_{NN}} = q\frac{\Delta R_N\left(P_N \mid z_{NN}\right)}{\Delta z_{NN}}$$

Because

(1) $z_{NN} = x_{NN} + y_{NN}$, z_{NN} is increasing on x_{NN}

(2) $\frac{\partial R_N\left(P_N \mid z_{NN}\right)}{\partial z_{NN}}$ is decreasing on z_{NN}, $\frac{\Delta R_N\left(P_N \mid z_{NN}\right)}{\Delta z_{NN}}$ is also decreasing on z_{NN}

From the above equation we can get the conclusion that $\frac{\Delta R_N\left(P_N \mid z_{NN}\right)}{\Delta x_{NN}}$ is decreasing on x_{NN}.

Appendix I: Characters of $\frac{\Delta R_N\left(P_N \left| z_{NN}\right.\right)}{\Delta x_{NN}}$ (2)

Proof 2:

Because $R_N\left(P_N \left| z_{NN}\right.\right)$ is concave on z_{NN}, that is

$$R_N\left(P_N \left| z_{NN}+2\right.\right)-R_N\left(P_N \left| z_{NN}+1\right.\right) \le R_N\left(P_N \left| z_{NN}+1\right.\right)-R_N\left(P_N \left| z_{NN}\right.\right) \qquad (1)$$

Sequentially we know that

$$R_N\left(P_N \left| z_{NN}+3\right.\right)-R_N\left(P_N \left| z_{NN}+2\right.\right) \le R_N\left(P_N \left| z_{NN}+2\right.\right)-R_N\left(P_N \left| z_{NN}+1\right.\right) \qquad (2)\ \ldots$$

$$R_N\left(P_N \left| z_{NN}+q\right.\right)-R_N\left(P_N \left| z_{NN}+q-1\right.\right) \le R_N\left(P_N \left| z_{NN}+q-1\right.\right)-R_N\left(P_N \left| z_{NN}+q-2\right.\right) \qquad (q-1)$$

$$R_N\left(P_N \left| z_{NN}+q+1\right.\right)-R_N\left(P_N \left| z_{NN}+q\right.\right) \le R_N\left(P_N \left| z_{NN}+q\right.\right)-R_N\left(P_N \left| z_{NN}+q-1\right.\right) \qquad (q)$$

$$R_N\left(P_N \left| z_{NN}+q+2\right.\right)-R_N\left(P_N \left| z_{NN}+q+1\right.\right) \le R_N\left(P_N \left| z_{NN}+q+1\right.\right)-R_N\left(P_N \left| z_{NN}+q\right.\right) \qquad (q+1)\ldots$$

$$R_N\left(P_N \left| z_{NN}+2q-1\right.\right)-R_N\left(P_N \left| z_{NN}+2q-2\right.\right) \le R_N\left(P_N \left| z_{NN}+2q-2\right.\right)-R_N\left(P_N \left| z_{NN}+2q-3\right.\right) \qquad (2q-2)$$

$$R_N\left(P_N \left| z_{NN}+2q\right.\right)-R_N\left(P_N \left| z_{NN}+2q-1\right.\right) \le R_N\left(P_N \left| z_{NN}+2q-1\right.\right)-R_N\left(P_N \left| z_{NN}+2q-2\right.\right) \qquad (2q-1)$$

Take the partial sum of these inequalities,

From (q) to $(2q-1)$, a total of q inequalities

$$R_N\left(P_N \left| z_{NN}+2q\right.\right)-R_N\left(P_N \left| z_{NN}+q\right.\right) \le R_N\left(P_N \left| z_{NN}+2q-1\right.\right)-R_N\left(P_N \left| z_{NN}+q-1\right.\right)$$

From $(q-1)$ to $(2q-2)$, a total of q inequalities

$$R_N\left(P_N \left| z_{NN}+2q-1\right.\right)-R_N\left(P_N \left| z_{NN}+q-1\right.\right) \le R_N\left(P_N \left| z_{NN}+2q-2\right.\right)-R_N\left(P_N \left| z_{NN}+q-2\right.\right)$$

….

From (2) to $(q+1)$, a total of q inequalities

$$R_N\left(P_N\left|z_{NN}+q+2\right.\right)-R_N\left(P_N\left|z_{NN}+2\right.\right)\le R_N\left(P_N\left|z_{NN}+q+1\right.\right)-R_N\left(P_N\left|z_{NN}+1\right.\right)$$

From (1) to (q), a total of q inequalities

$$R_N\left(P_N\left|z_{NN}+q+1\right.\right)-R_N\left(P_N\left|z_{NN}+1\right.\right)\le R_N\left(P_N\left|z_{NN}+q\right.\right)-R_N\left(P_N\left|z_{NN}\right.\right)$$

From these inequalities, we can obtain the result that

$$R_N\left(P_N\left|z_{NN}+2q\right.\right)-R_N\left(P_N\left|z_{NN}+q\right.\right)\le R_N\left(P_N\left|z_{NN}+q\right.\right)-R_N\left(P_N\left|z_{NN}\right.\right)$$

Because

$$\begin{aligned}&R_N\left(P_N\left|q\left[\left(x_{NN}+2\right)+y_{NN}\right]\right.\right)-R_N\left(P_N\left|q\left[\left(x_{NN}+1\right)+y_{NN}\right]\right.\right)\\&\le R_N\left(P_N\left|q\left[\left(x_{NN}+1\right)+y_{NN}\right]\right.\right)-R_N\left(P_N\left|q\left(x_{NN}+y_{NN}\right)\right.\right)\end{aligned}$$

$$\begin{aligned}&R_N\left(P_N\left|q\left(x_{NN}+y_{NN}\right)+2q\right.\right)-R_N\left(P_N\left|q\left(x_{NN}+y_{NN}\right)+q\right.\right)\\&\le R_N\left(P_N\left|q\left(x_{NN}+y_{NN}\right)+q\right.\right)-R_N\left(P_N\left|q\left(x_{NN}+y_{NN}\right)\right.\right)\end{aligned}$$

$$R_N\left(P_N\left|z_{NN}+2q\right.\right)-R_N\left(P_N\left|z_{NN}+q\right.\right)\le R_N\left(P_N\left|z_{NN}+q\right.\right)-R_N\left(P_N\left|z_{NN}\right.\right)$$

So we have proved that $\dfrac{\Delta R_N\left(P_N\left|z_{NN}\right.\right)}{\Delta x_{NN}}$ is decreasing on x_{NN}.

Appendix II: Characters of $\frac{\Delta\Pi_N^C}{\Delta y_{NN}}$

Proof:

(1) Because $R_N\left(P_N\left|z_{NN}\right.\right)$ is increasing with z_{NN} ,we know that

$$\left(x_{NN}^*\left(y_{NN}\right)+2\right)+y_{NN}\geq\left(x_{NN}^*\left(y_{NN}\right)+1\right)+y_{NN}$$
$$=>R_N\left(P_N\left|q\left[\left(x_{NN}^*\left(y_{NN}\right)+2\right)+y_{NN}\right]\right.\right)-R_N\left(P_N\left|q\left[\left(x_{NN}^*\left(y_{NN}\right)+1\right)+y_{NN}\right]\right.\right)\geq 0$$

Since $\frac{\Delta R_N\left(P_N\left|z_{NN}\right.\right)}{\Delta x_{NN}}$ is decreasing on x_{NN} (or accurately speaking, $\frac{\Delta R_N\left(P_N\left|z_{NN}\right.\right)}{\Delta z_{NN}}$ is decreasing on z_{NN}),we obtain

$$R_N\left(P_N\left|q\left[\left(x_{NN}^*\left(y_{NN}\right)+2\right)+y_{NN}\right]\right.\right)-R_N\left(P_N\left|q\left[\left(x_{NN}^*\left(y_{NN}\right)+1\right)+y_{NN}\right]\right.\right)$$
$$\leq R_N\left(P_N\left|q\left[\left(x_{NN}^*\left(y_{NN}\right)+1\right)+y_{NN}\right]\right.\right)-R_N\left(P_N\left|q\left[x_{NN}^*\left(y_{NN}\right)+y_{NN}\right]\right.\right)$$

Based on the definition of “max”, we know

$$R_N\left(P_N\left|q\left[\left(x_{NN}^*\left(y_{NN}\right)+1\right)+y_{NN}\right]\right.\right)\leq R_N\left[P_N^*\left|q\left(x_{NN}^*\left(y_{NN}+1\right)+\left(y_{NN}+1\right)\right)\right.\right]$$
$$=>$$
$$R_N\left(P_N\left|q\left[\left(x_{NN}^*\left(y_{NN}\right)+1\right)+y_{NN}\right]\right.\right)-R_N\left(P_N\left|q\left[x_{NN}^*\left(y_{NN}\right)+y_{NN}\right]\right.\right)$$
$$\leq R_N\left[P_N^*\left|q\left(x_{NN}^*\left(y_{NN}+1\right)+\left(y_{NN}+1\right)\right)\right.\right]-R_N\left[P_N^*\left|q\left(x_{NN}^*\left(y_{NN}\right)+y_{NN}\right)\right.\right]$$

So, we get our first conclusion that

$$R_N\left[P_N^*\left|q\left(x_{NN}^*\left(y_{NN}+1\right)+\left(y_{NN}+1\right)\right)\right.\right]-R_N\left[P_N^*\left|q\left(x_{NN}^*\left(y_{NN}\right)+y_{NN}\right)\right.\right]\geq 0$$

Similarly,

$$R_N\left[P_N^*\left|q\left(x_{NN}^*\left(y_{NN}+2\right)+\left(y_{NN}+2\right)\right)\right.\right]-R_N\left[P_N^*\left|q\left(x_{NN}^*\left(y_{NN}+1\right)+\left(y_{NN}+1\right)\right)\right.\right]\geq 0$$

(2) Also because $R_N\left(P_N \left| z_{NN}\right.\right)$ is increasing with z_{NN} ,

$$
\begin{aligned}
& R_N\left[P_N^* \left| q\left(x_{NN}^*\left(y_{NN}+2\right)+\left(y_{NN}+2\right)\right)\right.\right]-R_N\left[P_N^* \left| q\left(x_{NN}^*\left(y_{NN}+1\right)+\left(y_{NN}+1\right)\right)\right.\right] \geq 0 \\
& => \left(x_{NN}^*\left(y_{NN}+2\right)+\left(y_{NN}+2\right)\right) \geq \left(x_{NN}^*\left(y_{NN}+1\right)+\left(y_{NN}+1\right)\right) \\
& R_N\left[P_N^* \left| q\left(x_{NN}^*\left(y_{NN}+1\right)+\left(y_{NN}+1\right)\right)\right.\right]-R_N\left[P_N^* \left| q\left(x_{NN}^*\left(y_{NN}\right)+y_{NN}\right)\right.\right] \geq 0 \\
& => \left(x_{NN}^*\left(y_{NN}+1\right)+\left(y_{NN}+1\right)\right) \geq \left(x_{NN}^*\left(y_{NN}\right)+y_{NN}\right)
\end{aligned}
$$

We get our second conclusion that

$$
\left(x_{NN}^*\left(y_{NN}+2\right)+\left(y_{NN}+2\right)\right) \geq \left(x_{NN}^*\left(y_{NN}+1\right)+\left(y_{NN}+1\right)\right) \geq \left(x_{NN}^*\left(y_{NN}\right)+y_{NN}\right)
$$

This can be expressed as below:

$$
x_{NN}^*\left(y_{NN}+2\right)+2 \geq x_{NN}^*\left(y_{NN}+1\right)+1 \geq x_{NN}^*\left(y_{NN}\right)
$$

Note:

The above inequality can be expressed as following:

$$
\begin{aligned}
& 0 \leq x_{NN}^*\left(y_{NN}\right)-x_{NN}^*\left(y_{NN}+1\right) \leq 1 \\
& 0 \leq x_{NN}^*\left(y_{NN}+1\right)-x_{NN}^*\left(y_{NN}+2\right) \leq 1
\end{aligned}
$$

Because x, y can only be integer in our problem, this means for any $x_{NN}^*\left(y_{NN}\right)$ and $x_{NN}^*\left(y_{NN}+1\right)$, there are only two cases:

$$
\begin{aligned}
& (1)\ x_{NN}^*\left(y_{NN}\right)=x_{NN}^*\left(y_{NN}+1\right); \\
& (2)\ x_{NN}^*\left(y_{NN}\right)=x_{NN}^*\left(y_{NN}+1\right)+1
\end{aligned}
$$

Guess: $x_{NN}^*\left(y_{NN}\right)-x_{NN}^*\left(y_{NN}+1\right) \geq x_{NN}^*\left(y_{NN}+1\right)-x_{NN}^*\left(y_{NN}+2\right)$

Remark:

When y_{NN} grows, that is the inventory at the beginning of day N grows, the corresponding leased aircrafts in the day will keep unchanged.

(3) Because $\frac{\Delta R_N\left(P_N\left|z_{NN}\right.\right)}{\Delta z_{NN}}$ is decreasing on z_{NN} ,together with our second conclusion

$$\left(x_{NN}^*\left(y_{NN}+2\right)+\left(y_{NN}+2\right)\right)\geq\left(x_{NN}^*\left(y_{NN}+1\right)+\left(y_{NN}+1\right)\right)\geq\left(x_{NN}^*\left(y_{NN}\right)+y_{NN}\right)$$

We get our third conclusion that

$$\frac{R_N\left[P_N^*\left|q\left(\left(x_{NN}^*\left(y_{NN}+2\right)+2\right)+y_{NN}\right)\right.\right]-R_N\left[P_N^*\left|q\left(\left(x_{NN}^*\left(y_{NN}+1\right)+1\right)+y_{NN}\right)\right.\right]}{\left(x_{NN}^*\left(y_{NN}+2\right)+2\right)-\left(x_{NN}^*\left(y_{NN}+1\right)+1\right)}$$
$$\leq\frac{R_N\left[P_N^*\left|q\left(\left(x_{NN}^*\left(y_{NN}+1\right)+1\right)+y_{NN}\right)\right.\right]-R_N\left[P_N^*\left|q\left(x_{NN}^*\left(y_{NN}\right)+y_{NN}\right)\right.\right]}{\left(x_{NN}^*\left(y_{NN}+1\right)+1\right)-x_{NN}^*\left(y_{NN}\right)}$$

=>

$$R_N\left[P_N^*\left|q\left(x_{NN}^*\left(y_{NN}+2\right)+\left(y_{NN}+2\right)\right)\right.\right]-R_N\left[P_N^*\left|q\left(x_{NN}^*\left(y_{NN}+1\right)+\left(y_{NN}+1\right)\right)\right.\right]$$
$$\leq R_N\left[P_N^*\left|q\left(x_{NN}^*\left(y_{NN}+1\right)+\left(y_{NN}+1\right)\right)\right.\right]-R_N\left[P_N^*\left|q\left(x_{NN}^*\left(y_{NN}\right)+y_{NN}\right)\right.\right]$$

This is,

$$\max_{X_N,P_N}R_N\left[P_N\left|q\left(x_{NN}\left(y_{NN}+2\right)+\left(y_{NN}+2\right)\right)\right.\right]-\max_{X_N,P_N}R_N\left[P_N\left|q\left(x_{NN}\left(y_{NN}+1\right)+\left(y_{NN}+1\right)\right)\right.\right]$$
$$\leq\max_{X_N,P_N}R_N\left[P_N\left|q\left(x_{NN}\left(y_{NN}+1\right)+\left(y_{NN}+1\right)\right)\right.\right]-\max_{X_N,P_N}R_N\left(P_N\left|q\left(x_{NN}\left(y_{NN}\right)+y_{NN}\right)\right.\right)$$

So, $\frac{\Delta\Pi_N^C}{\Delta y_{NN}}$ is decreasing on y_{NN} .