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Supersymmetric Quantum Mechanics

By

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Abstract

This Master thesis considers certain aspects of Supersymmetric Quantum Mechanics in the context of Path integral approach. First we state all the basic mathematical structure involved, and carry out some basic Gaussian integrals for both commutative and non-commutative variables. Later in the thesis these simple results obtained are generalized to study the Supersymmetric sigma models on flat and curved space. And we will recover the beautiful relationship between the supersymmetric sigma model and the geometry of the target manifold in the form of topological invariants of the manifold, for the models on curved space.

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Contents

0 Introduction

Supersymmetry is a quantum mechanical space-time symmetry which relates bosons and fermions. The understanding of consequences of supersymmetry has proven daunting and it has been difficult to develop theories which could account for symmetry breaking, meaning the inability to observe the superpartner. To do some progress on these problems physicists developed supersymmetric quantum mechanics which is an application of supersymmetry superalgebra to quantum mechanics instead of quantum field theory and was hoped that studying the results of supersymmetry in this simpler settings would allow new and better understanding of the subject, supersymmetric quantum mechanics has very important mathematical and physical consequences, for example it has been used to demonstrate dynamical supersymmetry breaking,to prove the Atiyah-Singer index theorem[1], and remarkably these efforts opened the door for new research areas in quantum mechanics itself.

In this master thesis we will illustrate some aspects of supersymmetric quantum mechanics in the context of path integrals. It is as well the purpose of this work to achieve enough formality to make the reader easily understand the mathematical framework involved. We will start by introduction to the basic concepts used in path integral formulation of quantum mechanics[2] by beginning from reviewing the very basic concepts about probability and probability amplitude, later in section 1 we will focus our attention towards the mathematical formulation used in path integrals and further develop the concept by applying it to a simple example of a free particle.[3]

In section 2 we will introduce the calculus of Grassmann variables, study some basic definitions and properties of the Grassmann variables, and their differentiation and integration rules. Then in section 3 we will discuss about the Gaussian Integrals by taking into account their algebraic properties and we will present some of the main results about them for the n-dimensional Gaussian integral.[6] First we will consider the Gaussian integrals with real and commuting variables, then generalize the idea to the complex variable case and in section 3.2 first we will discuss the Gaussian integral for the case of anticommuting n-Grassmann variables and develop the concept with computing an n=2 example, later in section 3.2 we will consider the same integral for n-complex Grassmann variables and elaborate the general result by presenting simple example.[4]

Then in section 4.1 we will consider the supersymmetric lagrangian for the simplest case of supersymmetric sigma model in the context of path integrals and use our ideas developed in section 3 to solve this model when the target space is flat. In section 4.2 we will consider another supersymmetric sigma model on a Riemannian manifold[5] with supersymmetric lagrangian containing a bosonic kinetic term, two different fermions and a curvature term and we will elaborate the results obtained. In section

4.3 we will focus on another supersymmetric sigma model on a Riemannian manifold[6],[7] with two identical fermions and bosonic kinetic term in the lagrangian, In the last two models the target space will be curved. The consideration of path integrals will be central in this thesis.

1 Path Integrals in Quantum Mechanics

1.1 Probability Concepts

From Quantum Mechanics we study a quantity called *Probability amplitude* which is associated with every event which occurs in the nature. For example suppose an electron has to come from a source S and pass through a slit either slit A of slit B, to reach a detector D in figure (1.1), it has one amplitude for completing this course while passing through the slit A and reaching the detector, and another amplitude in passing through the slit B in completing its course.

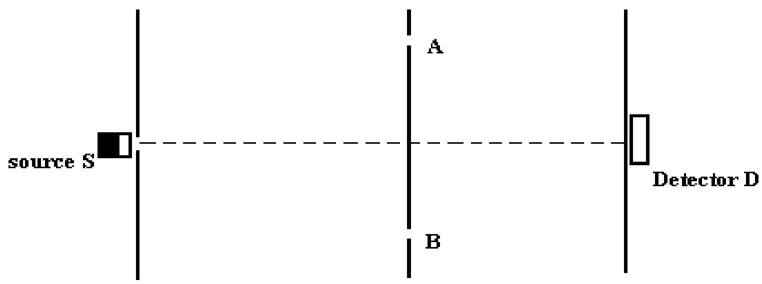

Figure 1.1

In this way we can associate an amplitude with the overall event by adding together the amplitudes of each alternative method for the completion of that event. Thus the overall amplitude for arrival at the detector D in the above discussed case will be given by,

$$\varphi = \varphi_1 + \varphi_2$$

Where φ_1 is the amplitude for the electron to pass through slit A and reach the detector and φ_2 is the amplitude for the electron to pass through slit B and reach the detector and φ is the total amplitude for the event.

We regard the absolute square of the total amplitude as the probability that the event will occur, in the above mentioned case the probability that the electron reaches the detector D is given by,

$$P = |\varphi_1 + \varphi_2|^2$$

If we try to investigate the course of the event by observation or some sort of measurement on the state of involved particle, we destroy the construction of total amplitude, If a system can be in more than one possible states and if we observe the system to be in one particular state, we destroy the possibility for it to be in any other state and thus the amplitude associated with the excluded state can no longer be added into the total amplitude. Considering our above discussed example if we determine with some sort of measuring device that the electron passes through slit A , the amplitude for arrival at detector D is just φ_1.

Furthermore it does not matter if we actually record the outcome of the measurement or not, as long as the measuring equipment is in place, it is enough to disturb the system and its probability amplitude.

1.2 The Structure of Amplitude

The amplitude of an event is the sum of amplitudes of various alternative ways in which the event can occur. This allows us to analyze the amplitude in various different ways depending upon the different classes into which the alternatives can be divided. The most detailed analysis results from considering that a particle going from A to B , for example, in a given time interval can be considered to have done this by going in a certain motion or path. We can therefore associate an amplitude with each possible motion. The total amplitude will be the sum of a contribution from each of the paths.

This scheme can be made more clear by considering the above example with two slits. Suppose we put a couple of more screens between the source and the detector, and in each of them we drill a few holes (see Fig 1.2).

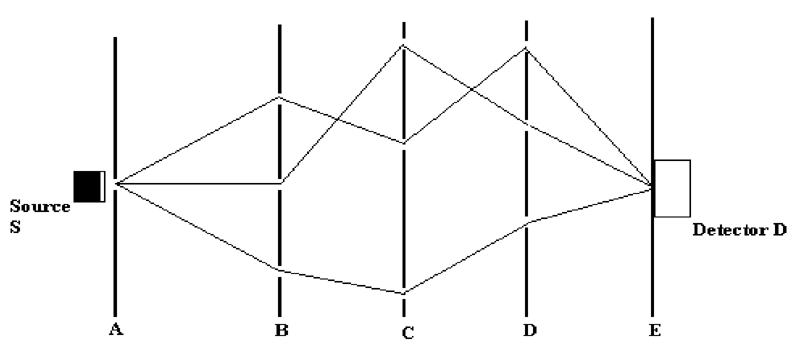

Figure 1.2

It is clear from the above figure that the electron can take many different paths on its way from the source S to the detector D, and each of these paths has its own amplitude. The complete amplitude is the sum of all of these amplitudes.

We can continue this process of putting more and more screens and drilling more and more holes, until a time when actually there is no screen left, we apply the same process and we will recover the integral over all paths of the amplitude for each path, which for the obvious reason is called a path integral.

1.3 Mathematical formulation and General Discussion

We consider a particle moving in one dimension, the Hamiltonian is given by

$$H = \frac{P^2}{2m} + V(q)$$

The basic question in this formulation is that: If the particle is at a position q at time $t = 0$, what is the probability amplitude that the it will be at some other position q' at a later time $t = T$?

To get a formal expression for this amplitude in Schrödinger formulation of quantum mechanics, we introduce the eigenstates of position operator $\hat{\boldsymbol{q}}$ which form a complete orthonormal set:

$$\hat{q}|q\rangle = q|q\rangle\,, \quad \langle q'|q\rangle = \delta(q'-q)\,, \quad \int dq|q\rangle\langle q| = 1$$

The initial state is $|\psi(0)\rangle = |q\rangle$. If we let this state evolve in time, and projecting on the state $|q'\rangle$, we get for the amplitude A, the following

$$A = \langle\, q'|\psi(T)\,\rangle = K(q',T;q,0) = \langle q'|e^{-iHT}|q\rangle$$

This object is known as propagator from initial point $(q,0)$ to final point (q',T) and it is independent of the origin of time, i-e $K(q',T+t;q,t) = K(q',T;q,0)$.

We can derive an expression for this amplitude in the form of a summation over all possible paths between initial and final points, and in this way we derive the Path integral from quantum mechanics.

As a first step we separate the time evolution in the above amplitude into two smaller time evolutions by writing $e^{-iHT} = e^{-iH(T-t_1)}e^{-iHt_1}$, the amplitude now becomes

$$A = \langle q'|e^{-iH(T-t_1)}e^{-iHt_1}|q\rangle$$

Next we insert a factor 1 in the form of a sum over position eigenstates, it gives

$$A = \langle q'|e^{-iH(T-t_1)} \int dq_1 |q_1\rangle\langle q_1|\, e^{-iHt_1}|q\rangle$$

$$= \int dq_1 \langle q'|e^{-iH(T-t_1)}|q_1\rangle\langle q_1|e^{-iHt_1}|q\rangle$$

$$= \int dq_1\, K(q', T; q_1, t_1)\, K(q_1, t_1; q, 0) \tag{1.1}$$

This is the expression for quantum mechanical rule for combination of amplitudes; if a process can occur in a number of ways, the amplitude for each of these ways add together. In propagating from q to q' the particle must be somewhere at an intermediate time t_1; we label the corresponding position q_1, and compute the amplitude for propagating via the point q_1 [As in (1.1)] and integrate over all possible intermediate positions, this result is a remembrance of Young's double slit experiment, where amplitudes passing through each of the two slits interfere.

We can repeat the division of time interval T ; we divide it into a large number N of time intervals each of duration $\delta = \frac{T}{N}$, then we can write for the propagator

$$A = \langle q'|\left(e^{-iH\delta}\right)^N|q\rangle = \langle q'|e^{-iH\delta}e^{-iH\delta} \dots . e^{-iH\delta}|q\rangle$$

We can insert a complete set of states between each exponential,

$$A = \langle q'|e^{-iH\delta} \int dq_{N-1}\, |q_{n-1}\rangle\langle q_{n-1}|e^{-iH\delta} \int dq_{N-2}\, |q_{n-2}\rangle\langle q_{n-2}| \dots ..$$

$$\dots . . \int dq_2\, |q_2\rangle\langle q_2|e^{-iH\delta} \int dq_1 |q_1\rangle\langle q_1|e^{-iH\delta}|q\rangle$$

$$= \int dq_1 \dots . . dq_{N-1} \langle q'|e^{-iH\delta}|q_{N-1}\rangle\, \langle q_{N-1}|e^{-iH\delta}|q_{N-2}\rangle \dots \dots$$

$$\dots . \langle q_1|e^{-iH\delta}|q\rangle$$

$$= \int dq_1 \dots \dots dq_{N-1}\, K_{qN,qN-1}\, K_{qN-1,qN-2} \dots . . K_{q2,q1}\, K_{q1,q0}$$

We have $q_0 = q$, $q_N = q'$ and we don't integrate over these initial and final positions, this discussion means that the amplitude is the integral of amplitude of all N paths.

It is clearly the sum over all possible paths of the amplitude for each path:

$$A = \sum_{paths} A_{path}$$

Where

$$\sum_{paths} = \int dq_1 \ldots \ldots dq_{N-1}, \quad A_{path} = K_{qN,qN-1}\, K_{qN-1,qN-2} \ldots \ldots K_{q2,q1}\, K_{q1,q0}$$

Lets look at it a bit in detail.

Propagator for one subinterval is $K_{q_{j+1},q_j} = \langle q_{j+1}| e^{-iH\delta} |q_j\rangle$ since δ is small, we can expand the exponential:

$$\begin{aligned} K_{q_{j+1},q_j} &= \langle q_{j+1}| \left(1 - iH\delta - \frac{1}{2}H^2\delta^2 + \cdots\right) |q_j\rangle \\ &= \langle q_{j+1}|q_j\rangle - i\delta\langle q_{j+1}|H|q_j\rangle + O(\delta^2). \end{aligned} \quad (1.2)$$

First term is delta function,

$$\langle q_{j+1}|q_j\rangle = \delta\left(q_{j+1} - q_j\right) = \int \frac{dp_j}{2\pi} e^{\,ip_j(q_{j+1}-q_j)} \quad (1.3)$$

In the second term of (1.2), we insert a factor 1 in the form of an integral over momentum eigenstates between H and $|q_j\rangle$;

$$-i\delta\, \langle q_{j+1}| \left(\frac{\hat{p}^2}{2m} + V(\hat{q})\right) \int \frac{dp_j}{2\pi} |p_j\rangle\langle p_j|q_j\rangle$$

$$= -i\delta \int \frac{dp_j}{2\pi} \left(\frac{{p_j}^2}{2m} + V(q_{j+1})\right) \langle q_{j+1}|p_j\rangle\langle p_j|q_j\rangle$$

$$= -i\delta \int \frac{dp_j}{2\pi} \left(\frac{{p_j}^2}{2m} + V\left(q_{j+1}\right)\right) e^{ip_j(q_{j+1}-q_j)} \quad (1.4)$$

If we had put H towards the left side of second term in (1.2), we would have obtained $V(q_j)$ in (1.4). In what follows we will simply write $V(\overline{q_J})$ where $\overline{q_J} = \frac{1}{2}(q_j + q_{j+1})$. Combining (1.3) and (1.4), we get the subinterval propagator

$$K_{q_{j+1},q_j} = \int \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} \left(1 - i\delta\left(\frac{{p_j}^2}{2m} + V(\overline{q_J})\right) + O(\delta^2)\right)$$

$$= \int \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} e^{-i\delta H(p_j,\overline{q_J})} \left(1 + O(\delta^2)\right) \qquad (1.5)$$

There are N such factors in the amplitude, combining them and writing $\dot{q_J} = \frac{(q_{j+1}-q_j)}{\delta}$, we get

$$A_{path} = \int \prod_{j=0}^{N-1} \frac{dp_j}{2\pi} exp\, i\delta \sum_{j=0}^{N-1} \left(p_j\dot{q_J} - H\left(p_j, \overline{q_J}\right)\right) \qquad (1.6)$$

Then the propagator becomes

$$K = \int dq_1 \dots \dots dq_{N-1}\, A_{path}$$

$$= \int \prod_{j=1}^{N-1} dq_j \int \prod_{j=0}^{N-1} \frac{dp_j}{2\pi} exp\, i\delta \sum_{j=0}^{N-1} \left(p_j\dot{q_J} - H\left(p_j, \overline{q_J}\right)\right) \qquad (1.7)$$

There is one momentum integral for each interval (N total) , while there is one position integral for each intermediate position (N - 1 total).

If $N \to \infty$ this gives an integral over all functions $p(t)$, $q(t)$. We write as following

$$K \equiv \int \mathcal{D}p(t)\mathcal{D}q(t) \exp i \int_0^T dt\left(p\dot{q} - H(p,q)\right) \qquad (1.8)$$

This integral is viewed as over all the functions $p(t)$, $q(t)$ where $q(0) = q$, $q(T) = q'$, eq(1.8) is just a short hand for a more complicated integral (1.7).

If the Hamiltonian is of the standard form, namely $H = \frac{p^2}{2m} + V(q)$, we can carry out momentum integrals in (1.7), as

$$K = \int \prod_{j=1}^{N-1} dq_j \, exp - i\delta \sum_{j=0}^{N-1} V(\bar{q}_J) \int \prod_{j=0}^{N-1} \frac{dp_j}{2\pi} exp \, i\delta \sum_{j=0}^{N-1} \left(p_j \dot{q}_J - \frac{{p_j}^2}{2m} \right)$$

The p integrals are all Gaussian, one such integral is

$$\int \frac{dp}{2\pi} e^{i\delta\left(p\dot{q} - \frac{p^2}{2m}\right)} = \sqrt{\frac{m}{2\pi i\delta}} \, e^{i\delta m \dot{q}^2}/_2$$

The propagator becomes

$$K = \int \prod_{j=1}^{N-1} dq_j \, exp - i\delta \sum_{j=0}^{N-1} V(\bar{q}_J) \prod_{j=0}^{N-1} \left(\sqrt{\frac{m}{2\pi i\delta}} \exp i\delta \frac{m{\dot{q}_J}^2}{2} \right)$$

$$= \left(\frac{m}{2\pi i\delta}\right)^{N/_2} \int \prod_{j=1}^{N-1} dq_j \exp i\delta \sum_{j=0}^{N-1} \left(\frac{m{\dot{q}_J}^2}{2} - V(\bar{q}_J) \right) \quad (1.9)$$

The argument of exponential is the action passing through discrete points. As above we can write this is more compact form as

$$K = \int \mathcal{D}q(t) \, e^{iS[q(t)]} \quad (1.10)$$

It is known as configuration space path integral. It must be viewed as a notation for a more precise expression (1.9).

1.4 Free Particle

Lets calculate the propagator for a free particle. Its Hamiltonian is given by

$$H = \frac{p^2}{2m}$$

We have

$$K = \langle q' | e^{-iHT} | q \rangle$$

$$= \langle q'| e^{-iT\frac{p^2}{2m}} \int \frac{dp}{2\pi}\, |p\rangle\langle p|q\rangle$$

$$= \int \frac{dp}{2\pi} e^{-iT\frac{p^2}{2m}} \langle q'|p\rangle\langle p|q\rangle$$

$$= \int \frac{dp}{2\pi} e^{-iT\frac{p^2}{2m}+i(q'-q)p} \tag{1.11}$$

It is a Gaussian integral, giving

$$K = \left(\frac{m}{2\pi iT}\right)^{1/2} \exp\frac{im(q'-q)^2}{2T} \tag{1.12}$$

Now we try with the Path Integrals, which is given by (1.10)

$$K = \lim_{N\to\infty}\left(\frac{m}{2\pi i\delta}\right)^{N/2} \int \prod_{j=1}^{N-1} dq_j \exp i\frac{m\delta}{2}\sum_{j=0}^{N-1}\left(\frac{q_{j+1}-q_j}{\delta}\right)^2$$

$$= \lim_{N\to\infty}\left(\frac{m}{2\pi i\delta}\right)^{N/2} \int \prod_{j=1}^{N-1} dq_j \exp i\frac{m}{2\delta}[(q_N-q_{N-1})^2+(q_{N-1}-q_{N-2})^2+\cdots$$
$$+(q_2-q_1)^2+(q_1-q_0)^2]$$

These integrals are Gaussian, and can be evaluated exactly, the result is

$$K = \lim_{N\to\infty}\left(\frac{m}{2\pi i\delta}\right)^{N/2} \frac{1}{\sqrt{N}}\left(\frac{2\pi i\delta}{m}\right)^{(N-1)/2} e^{im(q'-q)^2/2N\delta}$$

$$= \lim_{N\to\infty}\left(\frac{m}{2\pi iN\delta}\right)^{1/2} e^{im(q'-q)^2/2N\delta}$$

$N\delta$ is the total time interval T, therefore

$$K = \left(\frac{m}{2\pi i T}\right)^{1/2} e^{im(q'-q)^2/2T}$$

Which is the same as (1.12).

2 Grassmannian Variables

The Grassmann variables facilitate a mathematical construction which allows us to give a path integral representation for the fermionic fields. Grassmann variables are also important in the superspace formalism, where they serve as ‘’anti-commuting coordinates’’. A collection of Grassmann variables $\{\theta_i\}$ are independent elements of the algebra which anti-commute with each other but commute with the ordinary numbers.

So the Grassmann numbers are anti-commuting numbers. While the ordinary real and complex numbers are the commuting numbers. Let *n* generators $\{\theta_1, \theta_2, \dots, \theta_n\}$ satisfy the following anti-commutation relations,

$$\{\theta_i, \theta_j\} = 0$$

$$\Rightarrow \quad \theta_i\theta_j + \theta_j\theta_i = 0 \tag{2.1}$$

The above relation implies that $\theta_i{}^2 = 0$.

Then the set of linear combinations of $\{\theta_i\}$ with the commuting number coefficients is called the Grassmann variable, and with the commuting numbers, $\{\theta_i\}$ satisfy the following relation

$$\theta_j\, X^A = X^A\, \theta_j \tag{2.2}$$

Where X^A are ordinary commuting numbers.

First we should look at, how a general function of this kind would look like. Let $F(\theta_1, \theta_2, \dots, \theta_n)$ be a function of Grassmann numbers, since these numbers satisfy (2.1), each θ_i should appear at most to the power of one. So we can write a general function of this kind in the following way

$$F = f_0 + f_i\theta_i + f_{ij}\theta_i\theta_j + \dots + f_{12\dots n}\theta_1\theta_2 \dots \theta_n \tag{2.2}$$

Where *f* are real coefficients.

A function of this kind would be called even when it only has an even number of $\theta_i$variables in each factor of its expansion (2.2), and would be called odd if it has odd number of $\theta_i$variables in each of its factors.

These numbers also satisfy the following relations,

$$\theta_i{}^2 = 0$$

$$\theta_{i_1}\theta_{i_2}\ldots\theta_{i_n} = \epsilon_{i_1 i_2 \ldots . i_n}\theta_1\theta_2\ldots\theta_n$$

$$\theta_{i_1}\theta_{i_2}\ldots\theta_{i_m} = 0 \qquad if\ m > n$$

where

$$\epsilon_{i_1 i_2 \ldots . i_n} = \begin{cases} +1\ \ if\ i_1 i_2 \ldots . i_n\ is\ an\ even\ permutation\ of\ 1 \ldots n \\ -1\ if\ i_1 i_2 \ldots . i_n\ is\ an\ odd\ permutation\ of\ 1 \ldots n \\ 0\ otherwise \end{cases}$$

2.1 Differentiation

The right derivative $\frac{\vec{\partial}}{\partial\theta_i}$ satisfies the following properties.

$$\frac{\vec{\partial}\theta_i}{\partial\theta_j} = \delta_{ij} \qquad (2.3)$$

$$\frac{\vec{\partial}(f.g)}{\partial\theta_i} = (-1)^{|f|}f\ \frac{\vec{\partial}g}{\partial\theta_i} + \frac{\vec{\partial}f}{\partial\theta_i}g \qquad (2.4)$$

$$\frac{\vec{\partial}(af + bg)}{\partial\theta_i}\ =\ a\frac{\vec{\partial}f}{\partial\theta_i} + b\frac{\vec{\partial}g}{\partial\theta_i} \qquad (2.5)$$

Where f and g are functions, a and b are real (complex) numbers and $|f|$ is 1 when f is an odd function of θ_i and is 0 when f is an even function of θ_i.

Lets apply the above properties to a general function of Grassmann variables, we will get

$$\frac{\vec{\partial}}{\partial\theta_i}F = f_i + \delta_{ij}f_{jk}\theta_k - \delta_{ij}f_{kj}\theta_k + \delta_{ij}f_{jkl}\theta_k\theta_l - \delta_{ij}f_{kjl}\theta_k\theta_l + \delta_{ij}f_{klj}\theta_k\theta_l + \cdots \qquad (2.6)$$

Einstein summation convention is used.

We can define the left derivative according to the following formula

$$\frac{\vec{\partial}}{\partial\theta_j}\theta_i = \theta_i\frac{\overleftarrow{\partial}}{\partial\theta_j}$$

So choosing the left or right derivative is just a matter of convenience, the result obtained by both the left and right derivatives must be equivalent, but we must choose one convention for a problem and stick to it while going through the whole solution.

2.2 Integration

Integration can be defined over the functions of Grassmann variables, and these integrals satisfy the following rules

$$\int (af(\theta_i) + bg(\theta_i))\, d\theta_i = a\int f(\theta_i)\, d\theta_i + b\int g(\theta_i)\, d\theta_i$$

$$\int d\theta_i = 0 \tag{2.7}$$

$$\int \theta_i\, d\theta_j = \delta_{ij}$$

And

$$\int F_1(\theta_1)F_2(\theta_2) \dots F_n(\theta_n)\, d\theta_1 d\theta_2 \dots d\theta_n = \int F_1(\theta_1)d\theta_1 \int F_2(\theta_2)d\theta_2 \dots \int F_n(\theta_n)d\theta_n \tag{2.8}$$

From the above properties we come to know that integration for Grassmann variables is the same as the differentiation.

Now we check the general function of Grassmann variables (2.2) under the above mentioned properties as under.

$$\int F\, d\theta_1 d\theta_2 \dots d\theta_n = \int \left(f_0 + f_i\theta_i + f_{ij}\theta_i\theta_j + \cdots + f_{12\dots n}\theta_1\theta_2 \dots \theta_n\right) d\theta_1 d\theta_2 \dots d\theta_n$$

$$= f_{12\dots n}$$

This factor will only survive from all the terms in the general function because this is the only term which contains all the θ_i variables, and rest of the terms will become zero.

The equivalence of differentiation and integration leads to an odd behavior of integration under the change of variables of integration. Lets consider the case of n=1 first. Under the change of variable

$\theta' = a\theta$, we obtain the following

$$\int d\theta\, f(\theta) = \frac{\partial f(\theta)}{\partial \theta}$$

$$= \frac{\partial f(\theta'/a)}{\partial\, \theta'/a}$$

$$= a \int d\theta'\, f(\theta'/a)$$

This leads to $d\theta' = (1/a)\, d\theta$

This can be extended to the case of *n* variables. Suppose $\theta_i \rightarrow \theta'_i = a_{ij}\theta_j$, then we have

$$\int d\theta_1 \ldots \theta_n\, f(\theta) = \frac{\partial}{\partial \theta_1} \cdots \frac{\partial}{\partial \theta_n} f(\theta)$$

$$= \sum_{i_i=1}^{n} \frac{\partial \theta'_{i_1}}{\partial \theta_1} \cdots \frac{\partial \theta'_{i_n}}{\partial \theta_n} \frac{\partial}{\partial \theta'_{i_1}} \cdots \frac{\partial}{\partial \theta'_{i_n}}\, f\left(\frac{\theta'}{a}\right)$$

$$= \sum_{i_i=1}^{n} \varepsilon_{i_1 i_2 \ldots . i_n} a_{i_1 1} \ldots a_{i_n\, n}\, \frac{\partial}{\partial \theta'_{i_1}} \cdots \frac{\partial}{\partial \theta'_{i_n}}\, f\left(\frac{\theta'}{a}\right)$$

$$= \det a \int d\theta'_1 \ldots \theta'_n\, f\left(\frac{\theta'}{a}\right)$$

3 Gaussian Integral

Gaussian integrals play an important role in Quantum mechanics, Quantum field theory, in statistical physics etc. They are closely linked to the Path integrals as well, so its worthwhile to recall here some of the algebraic properties of Gaussian integrals.

We start our analysis by introducing the simple one dimensional Gaussian integral which is quite familiar to us as follows, for $a > 0$

$$\int_{-\infty}^{+\infty} dx\, e^{-ax^2/2} = \sqrt{\frac{2\pi}{a}} \tag{A}$$

In the next subsections we take up the case of higher dimensional Gaussian integrals, the technique is easy but a bit more tricky, we take up the case for commuting variables separately from the non-commuting variables.

3.1 For commuting variables

For the n-dimensional Gaussian integral for the commuting variables, we have the integral in the following form,

$$G(\mathbf{A}) = \int d^n x \;\; exp\left(- \sum_{i,j=1}^{n} \frac{1}{2} x_i \, A_{ij} x_j \right) \tag{3.1}$$

The above integral converges if the matrix **A** with the elements A_{ij} is a *symmetric positive definite* matrix. We give a definition of a *symmetric positive definite* matrix below.

3.1.1 Symmetric positive definite matrix (Real and Complex)

A symmetric matrix M, is a matrix which satisfies the following relation,

$$\mathrm{M} = \mathrm{M}^{\mathrm{T}}$$

Where M^{T} is the transpose of M.

Now an nxn real symmetric matrix M, is called positive definite if,

$$\mathrm{z}^{\mathrm{T}}\mathrm{M}\,\mathrm{z} > 0$$

Where z is a non-zero vector with real entries and z^T denotes the transpose of z. For the case of complex matrices, before taking up this definition, we will look into the definition of a *Hermitian matrix*.

A Hermitian matrix M, is a square matrix with complex entries which is equal to its own conjugate transpose, which means that the element in the ith row and jth column is equal to the complex conjugate of the element in jth row and ith column,

$$M_{ij} = \overline{M_{ji}}$$

Hermitian matrices are considered to be the complex counterparts of symmetric matrices.

Now a Hermitian matrix M, is called positive definite if,

$$z^\dagger M z > 0$$

Where $z^\dagger$ is conjugate transpose of z. The important thing to note is that all the eigenvalues of symmetric (Hermitian) positive definite matrix are positive.

Now we introduce another class of matrices which will be necessary for our later considerations.

3.1.2 Normal Matrices

A complex square matrix A is a normal matrix if,

$$A^\dagger A = AA^\dagger$$

Where $A^\dagger$ is the conjugate transpose of A. So it means that for complex square matrices, a matrix is normal if it commutes with its conjugate transpose. For the case of real matrices, a real square matrix A is a normal matrix if,

$$A^T A = AA^T$$

Where A^T is the transpose of A.

Normal matrices provide convenience for diagonalizability which we will see later, and it will justify our need for introducing normal matrices here. Among complex matrices, all hermitian matrices are normal matrices and among the real matrices, all symmetric matrices are normal matrices.

3.1.3 Unitary and Orthogonal Matrices

A unitary matrix is a complex nxn matrix U which satisfies the following condition,

$$\mathrm{U}^{\dagger}\mathrm{U} = \mathrm{U}\mathrm{U}^{\dagger} = \mathrm{I}$$

$$\Rightarrow \quad \mathrm{U}^{-1} = \mathrm{U}^{\dagger}$$

Moreover we have $\det \mathrm{U} = e^{i\varphi} \quad \Rightarrow \; |\det \mathrm{U}| = 1.$

Where I is an identity matrix and $\mathrm{U}^{\dagger}$ is the conjugate transpose of U.

Orthogonal matrices are the real counter part of the Unitary matrices, An orthogonal matrix O is a real nxn matrix which satisfies the following conditions,

$$\mathrm{O}^{\mathrm{T}}\mathrm{O} = \mathrm{O}\mathrm{O}^{\mathrm{T}} = \mathrm{I}$$

$$\Rightarrow \quad \mathrm{O}^{-1} = \mathrm{O}^{\mathrm{T}}$$

Now we can look at the integral (3.1), different methods give us the following result

$$G(\mathbf{A}) = (2\pi)^{n/2}\,(\det \mathbf{A})^{-1/2} \tag{3.2}$$

When the matrix is complex, the meaning of square root and determinant require some special care.

We derive below the result (3.2) for real positive matrices, and later in this subsection we will derive the result for the complex matrices, and the variables x are the commuting real variables.

In general, any normal matrix can be diagonalized by an orthogonal transformation and matrix **A**

in (3.1) can thus be written as following,

$$\mathbf{A} = \mathbf{O}\mathbf{D}\mathbf{O}^{\mathbf{T}} \tag{3.3}$$

Where the matrix **O** is orthogonal and matrix **D** with elements D_{ij} is diagonal,

$$\mathbf{O}^{\mathbf{T}}\mathbf{O} = \mathbf{1}\,, \qquad D_{ij} = a_i\delta_{ij}$$

Then we change the variables, **x** → **y** in the integral (3.1)

$$x_i = \sum_{j=1}^{n} O_{ij}y_j \;\; \Rightarrow \; \sum_{i,j} x_iA_{ij}x_j \; = \; \sum_{i,j} x_iO_{ik}a_kO_{jk}x_j$$

$$= \sum_{i} a_i{y_i}^2$$

The corresponding jacobian is given as follows

$$J = |\det \mathbf{O}| = 1$$

Then the integral (3.1) becomes

$$G(\mathbf{A}) = \prod_{i=1}^{n} \int dy_i \; e^{-a_i \, y_i{}^2/2}$$

The matrix **A** is positive, which means that all eigenvalues a_i are thus positive and each integral converges, From the result (A) we can then get the following result,

$$G(\mathbf{A}) = (2\pi)^{n/2} (a_1 a_2 \ldots\ldots a_n)^{-1/2}$$

$$= (2\pi)^{n/2} \, (\det \mathbf{A})^{-1/2} \qquad (3.4)$$

Which is the desired result.

We extend the above discussion to the case of complex matrices. Above proof based on diagonalization, used for real matrices has a complex generalization. Any hermitian (Normal) matrix **A** has a decomposition of the following form,

$$\mathbf{A} = \mathbf{U} \, \mathbf{D} \, \mathbf{U}^{\dagger}$$

Where **U** is a unitary matrix and **D** is a diagonal matrix. Then in the integral (3.1) we change the variables $\mathbf{x} \to \mathbf{y}$

$$x_i = \sum_{i=j}^{n} U_{ij} y_j$$

The above change of variables is a complex generalization of the orthogonal transformation (3.3).

The integral (3.1) then factorizes and the result is a product of the integral and a non-trivial jacobian of the change of variables, therefore

$$G(\mathbf{A}) = (2\pi)^{n/2} \; \frac{(\det \mathbf{D})^{-1/2}}{\det \mathbf{U}} \qquad (3.5)$$

Since we have the following identity

$$\det \mathbf{A} = \det \mathbf{D} \, (\det \mathbf{U})^2$$

Plugging it in (3.5), we get the following desired result

$$G(\mathbf{A}) = (2\pi)^{n/2} \, (\det \mathbf{A})^{-1/2} \qquad (3.6)$$

We have to keep in mind that the variables x in both the above mentioned integrals (with real and complex matrices) are commuting real ones.

For the following case,

$$G(\mathbf{A}) = \int dz_1 d\overline{z_1} \dots \dots dz_n d\overline{z_n} \ exp\left(-\sum_{i,j=1}^{n} \frac{1}{2} z_i A_{ij} \overline{z_j}\right) \tag{3.7}$$

We have here the commuting complex variable z and the matrix is *nxn* Hermite and positive definite one.

By the similar arguments used in the above examples, we get the following answer for the integral in (3.7)

$$G(\mathbf{A}) = (2\pi)^n \, (\det \mathbf{A})^{-1} \tag{3.8}$$

The disappearance of the square root sign from (3.8) is not surprising because the integration measure now runs two times *n*.

3.2 For anticommuting (grassmann) variables

We now consider a Gaussian integral over *n* grassmann variables, as follows

$$G(\mathbf{A}) = \int d\theta_1 \dots d\theta_{2n} \ \exp\left(\frac{1}{2}\theta_i A_{ij} \, \theta_j\right) \tag{3.9}$$

Where A is a real antisymmetric matrix and θ is a column vector with components $(\theta_1, \theta_2, \dots, \theta_{2n})$.

We must not take A to be a symmetric matrix here, as we did in the previous integrals, because we are dealing with grassmann variables here and if we do so, the property (2.1) for the grassmann variables will immediately imply that the integral (3.9) is zero.

Each nonzero term in the expansion of exponential in (3.9) contains an even number of factors of θ_i which must all be different because of the property (2.1) of grassmann variables.

On the other hand we take the integration measure to 2*n* instead of *n* to make sure that it remains even, because if we take it as *n* and if *n* is odd, there is an odd number of factors $d\theta_i$. So there must be at least one factor $\int d\theta_i$ where the integrand is 1. Thus using the property (2.7), we can say that (3.9) is zero for odd *n*.

When *n* is even, the only term which needs to be retained in the expansion of exponential of (3.9) is the term which involves *n* factors of θ.Terms with more than *n* factors of θ are immediately zero because of

(2.1) and terms with less than *n* factors of θ give zero upon integration because there is at least one factor $\int d\theta_i$ where the integrand is 1. Thus for 2*n*,

$$G(\mathbf{A}) = \frac{1}{2^n n!}\int d\theta_1 \dots d\theta_{2n} \left(\sum_{i,j=1}^{2n} \theta_i A_{ij}\, \theta_j\right)^n$$

In the expansion of the product, only the terms containing a permutation of $\theta_1, \theta_2, \dots, \theta_{2n}$ do not vanish, so we get

$$G(\mathbf{A}) = \frac{1}{2^n n!} \sum_{\substack{permutations \\ of\ i_1 \dots i_{2n}}} \epsilon_{i_1 \dots i_{2n}}\, A_{i_1 i_2}\, A_{i_3 i_4} \dots A_{i_{2n-1} i_{2n}} \tag{3.10}$$

Where $\epsilon = \pm 1$ is the signature of the permutation, and the quantity on the right hand side of (3.10) is known as *pfaffian* of the anti symmetric matrix A. So we have the following,

$$G(\mathbf{A}) = \mathrm{Pf}(\mathbf{A}) \tag{3.11}$$

And $$\mathrm{Pf}^2(\mathbf{A}) = \det \mathbf{A}$$

In order to understand the result (3.10) we consider the integral (3.9) for a simpler case of *n*=2 , and then we will see that it can be generalized to any even *n*.

We consider the integral

$$G(\mathbf{A}) = \int d\theta_1 d\theta_2\ \exp(\theta^T A\, \theta) \tag{3.12}$$

Now we expand the exponential,

$$= \int d\theta_1 d\theta_2 \left[\left\{1 + \theta_i A_{ij}\, \theta_j + \frac{(\theta_i A_{ij}\, \theta_j)^2}{2!} + \dots + \frac{(\theta_i A_{ij}\, \theta_j)^n}{n!} + \cdots\right\}\right] \tag{3.13}$$

Note that only the term $\theta_i A_{ij}\, \theta_j$ will survive from the above expansion in (3.13), because this is the only term which can saturate the number of variables in the integral measure, all the higher and lower order terms than this term will vanish upon integration, also note that there is summation going on in each pair of brackets in the expansion, so we will have the following,

$$G(\mathbf{A}) = \int d\theta_1 d\theta_2 \left(\theta_i A_{ij}\, \theta_j\right)$$

$$\theta_i A_{ij}\, \theta_j = \theta_1 A_{11}\, \theta_1 + \theta_1 A_{12}\, \theta_2 + \theta_2 A_{21}\, \theta_1 + \theta_2 A_{22}\, \theta_2$$

When we integrate the above expression, the terms that will survive will be the ones which have both $\theta_1$and $\theta_2$and rest of the two terms will go to zero.

$$G(\mathbf{A}) = \int d\theta_1 d\theta_2 (\theta_1 A_{12}\, \theta_2 + \theta_2 A_{21}\, \theta_1)$$

$$= A_{12} \int d\theta_1 d\theta_2 \theta_1 \theta_2 + A_{21} \int d\theta_1 d\theta_2\, \theta_2 \theta_1$$

$$= -A_{12} + A_{21}$$

$$= 2A_{21}$$

$$\Rightarrow \qquad G(\mathbf{A}) \sim \sqrt{\det A}$$

We consider another similar integral for *n* complex grassmann variables,

$$\int \prod d\bar{\theta}_i d\theta_i\, e^{-\bar{\theta}_i A_{ij} \theta_j} \tag{3.14}$$

Where A is a hermitian matrix $\bar{A}_{ij} = A_{ji}$ and $\bar{\theta}_i$ are the complex conjugates of θ_i.

Expanding the exponential and remembering that there is only one term which is nonzero, we get

$$\int d\overline{\theta_1}\, d\theta_1 \dots d\overline{\theta_n} d\theta_n \frac{1}{n!} [-\overline{\theta_{i_1}} A_{i_1 j_1} \theta_{j_1}][-\overline{\theta_{i_2}} A_{i_2 j_2} \theta_{j_2}] \dots [-\overline{\theta_{i_n}} A_{i_n j_n} \theta_{j_n}]$$

$$= \frac{1}{n!} \int d\overline{\theta_1} \dots .\, d\overline{\theta_n}\, \overline{\theta_{i_1}} \dots \overline{\theta_{i_n}} \int d\theta_1 \dots d\theta_n\, \theta_{j_1} \dots \theta_{j_n}\, A_{i_1 j_1} \dots A_{i_n j_n}$$

$$= \frac{1}{n!}\, \epsilon_{i_1 \dots i_n} \epsilon_{j_1 \dots j_n}\, A_{i_1 j_1} \dots A_{i_n j_n}$$

$$= \det A$$

On the second line we have the factor $(-1)^{\frac{1}{2} n(n-1) + \frac{1}{2} n(n-1) + n^2 + n} = (-1)^{2n^2} = +1$

In order to better understand the result of integral (3.14) in a more simple way, we consider the case for *n*=2.

$$G(\mathbf{A}) = \int d\overline{\theta_1}\, d\theta_1\, d\overline{\theta_2} d\theta_2\, e^{-\bar{\theta}_i A_{ij} \theta_j} \tag{3.15}$$

$$\Rightarrow \qquad G(\mathbf{A}) = \int d\overline{\theta_1}\, d\theta_1\, d\overline{\theta_2} d\theta_2 \left\{-\left(1 + \theta_i{}^* A_{ij}\theta_j + \frac{\left(\overline{\theta_\iota} A_{ij}\theta_j\right)^2}{2!} + \cdots\right)\right\}$$

The quadratic term will be the only contribution from above integral, because only this term can saturate the number of grassmann variables in the integral measure.

Keeping in mind the relation (2.1) for grassmann variables and also noting that there is a summation going on in each pair of brackets in the above expansion and then expanding the quadratic term, we get

$$\frac{\left(\overline{\theta_\iota} A_{ij}\theta_j\right)^2}{2!} = \frac{1}{2}\{(\overline{\theta_1}A_{11}\theta_1)(\overline{\theta_2}A_{22}\theta_2) + (\overline{\theta_1}A_{12}\theta_2)(\overline{\theta_2}A_{21}\theta_1) + (\overline{\theta_2}A_{21}\theta_1)(\overline{\theta_1}A_{12}\theta_2) + (\overline{\theta_2}A_{22}\theta_2)(\overline{\theta_1}A_{11}\theta_1)\}$$

Integration of the above term gives the following result,

$$\int d\overline{\theta_1}\, d\theta_1\, d\overline{\theta_2} d\theta_2 \left\{-\frac{\left(\overline{\theta_\iota} A_{ij}\theta_j\right)^2}{2!}\right\} = \frac{1}{2!}\{2(A_{11}A_{22} - A_{12}A_{21})\}$$

$$= \det A$$

Hence the result.

4 Supersymmetric sigma models

Any model which has both the bosonic and fermionic fields in it is called a supersymmetric model. We will see a relation between the topology of the target manifold and the structure of the supersymmetric sigma model. In this section we will consider some supersymmetric sigma models, and we will use the results of the integrals obtained in previous section in solving these models and path integrals will be used.

4.1 Supersymmetric sigma model on a flat space

We consider the simplest model first where the target space in flat. We can construct the supersymmetric Lagrangian for this case for bosonic and fermionic fields as follows,

$$L = \dot{x}^2 + \bar{\psi}\, \partial_t\, \psi$$

We have periodic bosonic and fermionic fields, that is

$$x(t + 2\pi\beta) = x(t)$$

$$\psi(t + 2\pi\beta) = \psi(t)$$

$$\bar{\psi}(t + 2\pi\beta) = \bar{\psi}(t)$$

And the action S is given by

$$S = \int_0^\beta L\, dt$$

Now we use the path integral formulation to construct this model as follows,

$$\int \mathcal{D}x\, \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, \exp\left\{-\int (\dot{x}^2 + \bar{\psi}\, \partial_t\, \psi) dt\right\} \tag{4.1}$$

Here the variable *x* represents the bosonic fields, and the variable ψ and $\bar{\psi}$ is representing the fermionic fields.

Using integration by parts, we can write the following,

$$\int \dot{x}^2 = -\int x\ {\partial_t}^2\, x$$

Using it in (4.1), we get

$$\int \mathcal{D}x\, \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\ \exp\left\{-\int \left(x\ \partial_t{}^2\ x\ +\ \bar{\psi}\, \partial_t\, \psi\right) dt\right\}$$

$$= \int \mathcal{D}x\ \ \exp\left\{-\int \left(x\ \ \partial_t{}^2\ x\ \right) dt\right\} . \int \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\ \ \exp\left\{-\int (\bar{\psi}\, \partial_t\, \psi\ ) dt\right\}$$

Now we use the result of Gaussian integrals for bosonic and fermionic variables and get the following,

$$= \frac{1}{\sqrt{\det\ \ \partial_t{}^2}} . \det\ \partial_t \tag{4.2}$$

We can define the determinant of an operator by properly regularized infinite product of its eigenvalues as $\det\ \partial_t\ = \prod_{n=1}^{\infty} \lambda_n$, where λ_n are the eigenvalues of the operator ∂_t , So we have

$\det\ \ \partial_t{}^2 = \prod_{n=1}^{\infty} \lambda_n{}^2$, and so formally we can write $\sqrt{\det\ \ \partial_t{}^2} = \det\ \partial_t$

$$\Rightarrow \qquad \int \mathcal{D}\bar{\psi}\, \mathcal{D}x\, \mathcal{D}\psi\ \exp\left\{-\int (\dot{x}^2\ +\ \bar{\psi}\, \partial_t\, \psi) dt\right\} = 1$$

4.2 Supersymmetric sigma model on a curved space

We now move to supersymmetric systems with more interesting target manifolds. Now we consider another supersymmetric sigma model on a curved space, we consider the model on a Riemannian manifold *M* of dimension *d* with the metric g. The theory involves bosonic variables which are defined by the map $x: S^1 \to M$ and the fermions ψ and $\bar{\psi}$ which are the complex conjugates of each other, are the odd counter-parts of the bosonic variables. The supersymmetric Lagrangian for the bosonic and fermionic fields is given by,

$$L = \partial_t x^\mu g_{\mu\nu} \partial_t x^\nu + \bar{\psi}^\mu D_t \psi^\nu g_{\mu\nu} + R_{\mu\nu\rho\sigma}(x) \psi^\mu \psi^\nu \bar{\psi}^\rho \bar{\psi}^\sigma \tag{4.3}$$

Note that in (4.3) we have two different fermionic fields ψ and $\bar{\psi}$, and D_t is the covariant derivative and is given by,

$$D_t\psi^\nu = \frac{\partial\psi^\nu}{\partial t} + \dot{x}^\lambda\, \Gamma^\nu{}_{\lambda K}(x)\, \psi^K$$

Where $\Gamma^\nu{}_{\lambda K}$ is the Christoffel symbol of the Levi-Civita connection, Levi-Civita connection is used because we are working in a curved space and ordinary derivative does not suffice here. Levi-Civita connection is a unique connection which preserves the metric of the manifold and has a vanishing torsion.

Further in (4.3) $g_{\mu\nu}$ is the metric associated with the manifold M, and $R_{\mu\nu\rho\sigma}$ is the Riemann curvature tensor.

From (4.3), we have

$$S = \int_0^\beta \left(\partial_t x^\mu g_{\mu\nu}\partial_t x^\nu + \bar{\psi}^\mu D_t\psi^\nu g_{\mu\nu} + R_{\mu\nu\rho\sigma}(x)\psi^\mu\psi^\nu\bar{\psi}^\rho\bar{\psi}^\sigma\right)\, dt \qquad (4.4)$$

This action is invariant under the following supersymmetry transformation,

$$\delta x^\nu = \epsilon\bar{\psi}^\nu - \bar{\epsilon}\psi^\nu$$

$$\delta\psi^\nu = \epsilon\left(i\dot{x}^\nu - \Gamma^\nu{}_{\lambda K}\bar{\psi}^\lambda\psi^K\right)$$

$$\delta\bar{\psi}^\nu = \bar{\epsilon}\left(-i\dot{x}^\nu - \Gamma^\nu{}_{\lambda K}\bar{\psi}^\lambda\psi^K\right)$$

Where ϵ and $\bar{\epsilon}$ are the infinitesimal real and complex Grassmann constants.

We can expand the metric $g_{\mu\nu}$ in the Taylor expansion as follows,

$$g_{\mu\nu}(x_0 + \delta x) = g_{\mu\nu}(x_0) + \partial_\alpha g_{\mu\nu}(x_0)\delta x^\alpha + \tfrac{1}{2}\left\{\partial_\alpha\partial_\beta g_{\mu\nu}(x_0)\delta x^\alpha\delta x^\beta\right\} + \cdots \qquad (4.5)$$

Using this expansion in the integral (4.4), The integral (4.4) now becomes,

$$S = \int_0^\beta \left[\partial_t x^\mu\partial_t x^\nu\left(g_{\mu\nu}(x_0) + \partial_\alpha g_{\mu\nu}(x_0)\delta x^\alpha + \cdots\right) + \right.$$

$$\left. + \bar{\psi}^\mu D_t\psi^\nu\left(g_{\mu\nu}(x_0) + \partial_\alpha g_{\mu\nu}(x_0)\delta x^\alpha + \cdots\right) + R_{\mu\nu\rho\sigma}(x)\psi^\mu\psi^\nu\bar{\psi}^\rho\bar{\psi}^\sigma\right] dt \qquad (4.6)$$

Now we expand the bosonic and fermionic fields in the integral (4.4) in the relevant fourier modes, as

$$x = x_0 + \sum_{n\neq 0} \sqrt{\beta}\, x_n e^{2\pi i n t/\beta}$$

$$\bar{\psi} = \bar{\psi}_0 + \sum_{n\neq 0} \sqrt{\beta}\, \bar{\psi}_n e^{2\pi i n t/\beta}$$

$$\psi = \psi_0 + \sum_{n\neq 0} \sqrt{\beta}\, \psi_n e^{2\pi i n t/\beta}$$

Using the above modes and expansion (4.5) into (4.6), and also rescaling the time as $t \to \beta t$, in the integral (4.6) ,we have

$$S = \int_0^1 \beta \left[\frac{1}{\beta^2} \left\{ \beta \sum_{n\neq 0} \sum_{m\neq 0} (2\pi i n)\,(2\pi i m)\, x_n{}^{\mu}\, x_m{}^{\nu} e^{2\pi i (n+m) t} \right\} \left\{ g_{\mu\nu}(x_0) \quad + \partial_\alpha g_{\mu\nu}(x_0) \sqrt{\beta} \sum_{l\neq 0} x_l{}^{\alpha} e^{2\pi i l t} + \right. \right.$$

$$\left. + \cdots \right\} + \left\{ \bar{\psi}_0{}^{\mu} + \sum_{n\neq 0} \sqrt{\beta} \bar{\psi}_n{}^{\mu} e^{2\pi i n t} \right\} \times$$

$$\times \left\{ \frac{\sqrt{\beta} \sum_{p\neq 0} (2\pi i p) \psi_p{}^{\nu}\, e^{2\pi i p t}}{\beta} + \frac{\sqrt{\beta}}{\beta} \sum_{r\neq 0} (2\pi i r) x_r{}^{\lambda}\ \Gamma^{\nu}{}_{\lambda K}(x)\, \psi^{K} \right\} \left\{ g_{\mu\nu}(x_0) \right.$$

$$\left. \left. + \partial_\alpha g_{\mu\nu}(x_0) \sqrt{\beta} \sum_{l\neq 0} x_l{}^{\alpha} e^{2\pi i l t} + \cdots \right\} + R_{\mu\nu\rho\sigma}(x_0) \psi_0{}^{\mu} \psi_0{}^{\nu} \bar{\psi}_0{}^{\rho} \bar{\psi}_0{}^{\sigma} + \cdots \right] dt$$

Now we can check the order of β in the above integral, and when we consider the small β limit, we throw away the terms of the order of $\beta^{1/2}$ and higher order, and hence we are left with the following,

$$\Rightarrow \quad S = \sum_{n\neq 0} (2\pi i n)^2\ x_n{}^{\mu} g_{\mu\nu}(x_0)\, x_n{}^{\nu} + O\left(\beta^{1/2}\right) + \sum_{n\neq 0} (2\pi i n) \bar{\psi}_n{}^{\mu} g_{\mu\nu}(x_0)\, \psi_n{}^{\nu} + O\left(\beta^{1/2}\right)$$
$$+ R_{\mu\nu\rho\sigma}(x_0) \psi_0{}^{\mu} \psi_0{}^{\nu} \bar{\psi}_0{}^{\rho} \bar{\psi}_0{}^{\sigma} \qquad (4.7)$$

And our main integral reads as follows,

$$I = \int \mathcal{D}x\, \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\;\; e^{-S} \tag{4.8}$$

$$\text{where}\;\; \mathcal{D}x = \prod_n d^d x_n\;, \mathcal{D}\psi = \prod_n d^d\,\psi_n \;\text{ and }\; \mathcal{D}\bar{\psi} = \prod_n d^d\,\bar{\psi}_n$$

where d is the dimension of the manifold.

We divide above integral (4.8) in three steps to make the calculation easily understandable.

$$I_1 = \int d^d x_n\;\; exp\left\{\sum_{n\neq 0} (2\pi in)^2\;\, x_n{}^{\mu} g_{\mu\nu}(x_0)\, x_n{}^{\nu}\right\}$$

Put $\qquad 2\pi in\, x = y$

$$\Rightarrow 2\pi in\; dx = dy$$

$$I_1 = \int \frac{d^d y_n}{(2\pi in)^d}\;\;\; exp\{-\, y_n{}^{\mu} g_{\mu\nu}(x_0)\, y_n{}^{\nu}\}$$

$$\Rightarrow\; I_1 = \frac{1}{(2\pi in)^d}\, \frac{1}{\sqrt{\det g}} \tag{4.9}$$

$$I_2 = \int d^d\psi_n\;\, d^d\bar{\psi}_n\;\; exp\left\{\sum_{n\neq 0} (2\pi in)\bar{\psi}_n{}^{\mu} g_{\mu\nu}(x_0)\, \psi_n{}^{\nu}\right\}$$

Putting $\qquad \xi = \sqrt{2\pi in}\;\psi \quad \Rightarrow d\xi = \frac{1}{\sqrt{2\pi in}}\, d\psi$

and $\qquad \bar{\xi} = \sqrt{2\pi in}\;\bar{\psi} \quad \Rightarrow \quad d\bar{\xi} = \frac{1}{\sqrt{2\pi in}}\; d\bar{\psi}$

$$\Rightarrow\;\; I_2 = (2\pi in)^d \int e^{\bar{\xi}_n\, g_{\mu\nu}\, \xi_n}\;\; d^d\xi_n\, d^d\bar{\xi}_n$$

We cannot use the grassmannian integration technique to solve I_2 yet, because $g_{\mu\nu}$ is not antisymmetric, to make it antisymmetric, we go as follows,

Let $\lambda = \begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix}$ and $G_{\mu\nu} = \begin{bmatrix} 0 & g_{\mu\nu} \\ -g_{\mu\nu}{}^{T} & 0 \end{bmatrix}$, We can rewrite I_2 as,

$$I_2 = (2\pi in)^d \int d^{2d}\lambda\;\; exp\{\lambda\, G_{\mu\nu}\lambda\}$$

$$= (2\pi in)^d \int d^{2d}\lambda\;\; \left\{1 + \left(\lambda G_{\mu\nu}\lambda\right)^2 + \cdots + \frac{\left(\lambda G_{\mu\nu}\lambda\right)^{2d}}{2d!}\right\}$$

$$= \frac{(2\pi i n)^d}{2d!} \int d^{2d}\lambda \left(\lambda G_{\mu\nu}\lambda\right)^{2d}$$

$$= \frac{(2\pi i n)^d}{2d!} \int d^2\lambda\, d^4\lambda \dots d^{2d}\lambda \left(\lambda_{i_2} G_{\mu_2\nu_2} \lambda_{j_2}\right) \dots \left(\lambda_{i_{2d}} G_{\mu_{2d}\nu_{2d}} \lambda_{j_{2d}}\right)$$

$$= \frac{(2\pi i n)^d}{2d!} \epsilon_{i_2} \dots \epsilon_{i_{2d}} \epsilon_{j_2} \dots \epsilon_{j_{2d}}\, G_{\mu_2\nu_2} \dots G_{\mu_{2d}\nu_{2d}}$$

$$\sim \frac{(2\pi i n)^d}{2d!} \sqrt{\det G}$$

$$= \frac{(2\pi i n)^d}{2d!} \left[-\det\{g(-g^T)\}\right]^{1/2}$$

$$= \frac{(2\pi i n)^d}{2d!} \sqrt{(\det g)^2}$$

$$\Rightarrow\ I_2 = \frac{(2\pi i n)^d}{2d!} \det g \tag{4.10}$$

Now for the curvature term, it only involves fermionic zero modes, thus we have the following integral,

$$I_3 = \int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\ \ exp\left\{R_{\mu\nu\rho\sigma}(x_0) {\psi_0}^\mu {\psi_0}^\nu {\bar{\psi}_0}^\rho {\bar{\psi}_0}^\sigma\right\}$$

$$= \frac{1}{(d/2)!} \int d^d\psi_0\ d^d\bar{\psi}_0\ \left\{R_{\mu\nu\rho\sigma}(x_0) {\psi_0}^\mu {\psi_0}^\nu {\bar{\psi}_0}^\rho {\bar{\psi}_0}^\sigma\right\}^{d/2}$$

$$= \frac{1}{(d/2)!} \left\{R_{\mu\nu\rho\sigma}(x_0)\right\}^{d/2} \int d^d\psi_0\ ({\psi_0}^\mu {\psi_0}^\nu)^{d/2} \int d^d\bar{\psi}_0\ \left({\bar{\psi}_0}^\rho {\bar{\psi}_0}^\sigma\right)^{d/2}$$

$$= \frac{1}{(d/2)!} \left\{R_{\mu\nu\rho\sigma}(x_0)\right\}^{d/2} \epsilon^{\mu_1\nu_2\dots\mu_{d-1}\nu_d}\, \epsilon^{\rho_1\sigma_2\dots\rho_{d-1}\sigma_d}$$

$$\Rightarrow \quad I_3 = \frac{1}{(d/2)!} R_{\mu_1\nu_2\rho_1\sigma_2} \dots\dots R_{\mu_{d-1}\nu_d\,\rho_{d-1}\sigma_d}\ \epsilon^{\mu_1\nu_2\dots\mu_{d-1}\nu_d}\, \epsilon^{\rho_1\sigma_2\dots\rho_{d-1}\sigma_d} \tag{4.11}$$

Combining all three results (4.9), (4.10), and (4.11), and noting that we have carried out the integral over the fermionic zero modes, and thus we are left with the integral over bosonic zero modes only, so we get the result for our model,

$$I = \int d^d x_0\ \frac{1}{(2\pi i n)^d} \frac{1}{\sqrt{\det g}} \times \frac{(2\pi i n)^d}{2d!} \det g\ \times$$
$$\times \frac{1}{(d/2)!} R_{\mu_1\nu_2\rho_1\sigma_2} \dots\dots R_{\mu_{d-1}\nu_d\,\rho_{d-1}\sigma_d}\ \epsilon_{\mu_1\nu_2\dots\mu_{d-1}\nu_d}\, \epsilon_{\rho_1\sigma_2\dots\rho_{d-1}\sigma_d}$$

$$\Rightarrow I = \int d^d x_0 \sqrt{\det g}\ \frac{1}{2d!}\ \frac{1}{(d/2)!}\ R_{\mu_1 \nu_2 \rho_1 \sigma_2} \ldots .. R_{\mu_{d-1} \nu_d\, \rho_{d-1} \sigma_d}\ \epsilon^{\mu_1 \nu_2 \ldots \mu_{d-1} \nu_d}\ \epsilon^{\rho_1 \sigma_2 \ldots \rho_{d-1} \sigma_d}$$

This is an important characteristic class called the Euler characteristic or the Euler class of the target manifold M. Euler class $e(M)$ is defined in terms of the curvature of a d dimensional manifold as follows,

$$e(M) = Pf(R/2\pi)$$

$$= \frac{(-1)^{d/2}}{(4\pi)^{d/2} \left(\frac{d}{2}\right)!} R_{\mu_1 \nu_2 \rho_1 \sigma_2} \ldots .. R_{\mu_{d-1} \nu_d\, \rho_{d-1} \sigma_d}\ \epsilon^{\mu_1 \nu_2 \ldots \mu_{d-1} \nu_d}\ \epsilon^{\rho_1 \sigma_2 \ldots \rho_{d-1} \sigma_d}$$

Which is almost identical to our result upto some numerical factor. Hence we can say that in our case, we get the following integral,

$$I \sim \int d^d x_0 \sqrt{\det g}\, e(M) = \chi(M)$$

Where $\chi(M)$ is called the Euler number of the manifold M, and it is a topological invariant of the manifold.

4.3 Another Supersymmetric sigma model on a curved space

In this section we consider another supersymmetric sigma model on a curved space, the supersymmetric lagrangian for this model is given as follows,

$$L = \dot{x}^2 + \psi D \psi$$

$$\Rightarrow L = \dot{x}^\mu g_{\mu\nu}\, \dot{x}^\nu + g_{\mu\nu}\, \psi^\mu \frac{D\psi^\nu}{Dt} \tag{4.12}$$

Note that in (4.12) we have the same fermionic fields, as compared to the previous model where we had different fermionic fields. The metric components are continuous and differentiable, We have the following expansion of the metric in Taylor series around a point x_0 as,

$$g_{\mu\nu}(x_0 + x) = g_{\mu\nu}(x_0) + g_{\mu\nu,\alpha}(x_0)\, x^\alpha + \frac{1}{2!} g_{\mu\nu,\alpha\beta}(x_0)\, x^\alpha x^\beta + \cdots \tag{4.13}$$

Where $g_{\mu\nu}$ is evaluated at the point x_0 and in general the symbol $g_{\mu\nu,\alpha\beta\ldots}$ stands for the partial derivatives of $g_{\mu\nu}$ with respect to $x^\alpha x^\beta$ ... at the point x_0 . Thus we have,

$$g_{\mu\nu,\alpha} = \left(\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right)_{x_0} \quad , \ g_{\mu\nu,\alpha\beta} = \left(\frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha \partial x^\beta}\right)_{x_0} \quad , \ldots ..$$

In studying this model (4.12), we will use the Riemannian normal coordinates in order to make our calculations easier. At the origin of Riemann normal coordinates, the expansion of the metric has the following properties,

$$g_{\mu\nu}(x_0) = \delta_{\mu\nu} \ , \qquad g_{\mu\nu,\alpha}(x_0) = 0$$

So at the origin of Riemann normal coordinates, (4.13) becomes

$$g_{\mu\nu}(x_0 + x) = \delta_{\mu\nu} + \frac{1}{2!} g_{\mu\nu,\alpha\beta}(x_0)\, x^\alpha x^\beta + \cdots \tag{4.14}$$

And the covariant derivative in (4.12) is given by,

$$D_t \psi^\nu = \frac{\partial \psi^\nu}{\partial t} + \dot{x}^\lambda\, \Gamma^\nu{}_{\lambda K}(x)\, \psi^K$$

Where $\Gamma^\nu{}_{\lambda K}$ is the Christoffel symbol of the Levi-Civita connection.

Using the expansion (4.14) and above covariant derivative in the supersymmetric lagrangian in (4.12), we get the following,

$$L = \left(\delta_{\mu\nu} + \frac{1}{2!} g_{\mu\nu,\rho\sigma}(x_0)\, x^\rho x^\sigma + \cdots\right) \dot{x}^\mu\, \dot{x}^\nu + g_{\mu\nu}\, \psi^\mu \left(\frac{\partial \psi^\nu}{\partial t} + \dot{x}^\lambda\, \Gamma^\nu{}_{\lambda K}(x)\, \psi^K\right) \tag{4.15}$$

We can also expand the christoffel connection term in (4.15) in Taylor series around the point x_0 as follows,

$$\Gamma^\nu{}_{\lambda K}(x_0 + x) = \Gamma^\nu{}_{\lambda K}(x_0) + \Gamma^\nu{}_{\lambda K,\alpha}(x_0) + \cdots \tag{4.16}$$

Where we have,

$$\Gamma^\nu{}_{\lambda K} = \frac{1}{2} g^{\delta\nu}\left(g_{\delta\lambda,K} + g_{\delta K,\lambda} - g_{\lambda K,\delta}\right) \tag{4.17}$$

Keeping (4.17) in mind and also the properties of the metric at the origin of Riemann normal coordinates, we can safely say that the zeroth order expansion term in (4.16) is zero, because it contains the first derivatives of the metric, so if we want to work at the origin of Riemann normal coordinates, this term must be zero. Then we are left with the first and higher order derivatives of the christoffel connection in the expansion (4.16). Also note that in the bosonic part of the lagrangian, we have,

$$\left(\delta_{\mu\nu} + \frac{1}{2!} g_{\mu\nu,\rho\sigma}(x_0)\, x^\rho x^\sigma + \cdots\right) \dot{x}^\mu\, \dot{x}^\nu$$

From here, we will consider only the first term of the above expansion, because the second term is fourth order, and the contribution from this part is negligible in small β limit, which we will consider later in this discussion and our this argument will be justified.

Using these arguments, and putting (4.16) back into (4.15), we get

$$L = \dot{x}^\mu \dot{x}^\mu + g_{\mu\nu}\, \psi^\mu \left\{\frac{\partial \psi^\nu}{\partial t} + \dot{x}^\lambda \psi^K \Gamma^\nu{}_{\lambda K,\alpha}(x_0) x^\alpha\right\} \tag{4.18}$$

Since we are working at the origin of Riemann normal coordinates centered at x_0 ,we can write the second derivative of the metric in terms of Riemann curvature tensor, and we can make use of the following useful identity,

$$\partial_\alpha \partial_\beta g_{\mu\nu}(x_0) = \{R_{\mu\alpha\nu\beta}(x_0)\} \tag{4.19}$$

And the second term in the expansion (4.16) can be written as the following,

$$\Gamma^\nu{}_{\lambda K,\alpha} = \frac{1}{2} g^{\delta\nu}\left(g_{\delta\lambda,K\alpha} + g_{\delta K,\lambda\alpha} - g_{\lambda K,\delta\alpha}\right)$$

Using the identity (4.19) in the above expression, we get

$$g_{\delta\nu}\Gamma^\nu{}_{\lambda K,\alpha} = \frac{1}{2}(R_{\delta\alpha\lambda K} + R_{\delta\alpha K\lambda} - R_{\lambda\alpha K\delta}) \tag{4.20}$$

Where the Riemann curvature tensor has the following symmetries,

$$R_{\delta\alpha\lambda K} = -R_{\alpha\delta\lambda K}$$

$$R_{\delta\alpha\lambda K} = -R_{\delta\alpha K\lambda}$$

$$R_{\delta\alpha\lambda K} = R_{\lambda K\delta\alpha}$$

And $$R_{\delta\alpha\lambda K} + R_{\delta\lambda K\alpha} + R_{\delta K\alpha\lambda} = 0$$

Above symmetry properties imply that the Riemann curvature tensor is symmetric under the exchange of first and second pair of indices, and it is antisymmetric under the exchange of two elements in first and second pair. The last identity is well known and is called the first Bianchi identity.

Now we make use of these symmetry properties, and we can get (4.20) in the following form,

$$g_{\delta\nu}\Gamma^{\nu}{}_{\lambda K,\alpha} = -\frac{1}{2}R_{\lambda\alpha K\delta}$$

$$\Rightarrow \quad \Gamma^{\nu}{}_{\lambda K,\alpha} = -g^{\delta\nu}\frac{1}{2}R_{\lambda\alpha K\delta}$$

$$\Rightarrow \quad \Gamma^{\nu}{}_{\lambda K,\alpha} = -\tfrac{1}{2}R_{\lambda\alpha K}{}^{\nu} \tag{4.21}$$

Using the result (4.21) in the equation (4.18), we get

$$L = \dot{x}^{\mu}\dot{x}^{\mu} + g_{\mu\nu}\,\psi^{\mu}\left\{\frac{\partial\psi^{\nu}}{\partial t} + \dot{x}^{\lambda}\psi^{K}x^{\alpha}\left(-\frac{1}{2}R_{\lambda\alpha K}{}^{\nu}(x_0)\right)\right\}$$

$$\Rightarrow \quad L = \dot{x}^{\mu}\dot{x}^{\mu} + \psi^{\mu}\left\{g_{\mu\nu}\frac{\partial\psi^{\nu}}{\partial t} - \frac{1}{2}\dot{x}^{\lambda}\psi^{K}x^{\alpha}R_{\lambda\alpha K\mu}(x_0)\right\} \tag{4.22}$$

In (4.22), the term $g_{\mu\nu}\frac{\partial\psi^{\nu}}{\partial t}$ can be simplified if we expand the metric in Taylor series like we did before, and we will see that only the first term in the expansion of the metric should be considered because all other terms will be of higher order which can be considered to be negligible in small β limit. So keeping these considerations in mind, we get

$$L = \dot{x}^{\mu}\dot{x}^{\mu} + \psi^{\mu}\frac{\partial\psi^{\mu}}{\partial t} - \frac{1}{2}\dot{x}^{\lambda}\psi^{K}x^{\alpha}\psi^{\mu}\,R_{\lambda\alpha K\mu}(x_0) \tag{4.23}$$

The action is given by,

$$S = \int_{0}^{\beta} L\,dt$$

Taking the periodic boundary conditions of x and ψ into account as follows,

$$x(t + 2\pi\beta) = x(t)$$

$$\psi(t + 2\pi\beta) = \psi(t)$$

We can expand the fields in their Fourier expansions as follows,

$$x^{\mu}(t) = x_0{}^{\mu} + \beta \sum_{n\neq 0} \xi_n{}^{\mu}\, e^{2\pi i n t/\beta}$$

$$\psi^{\mu}(t) = \psi_0{}^{\mu} + \sqrt{\beta} \sum_{n\neq 0} \eta_n{}^{\mu}\, e^{2\pi i n t/\beta}$$

When we consider the complete Lagrangian (4.15), we get

$$L = \left(\dot{x}^{\mu}\dot{x}^{\mu} + \frac{1}{2!} g_{\mu\nu,\rho\sigma}(x_0)\, x^{\rho}x^{\sigma}\dot{x}^{\mu}\,\dot{x}^{\nu} + \cdots\right) + \left(\psi^{\mu}\frac{\partial\psi^{\mu}}{\partial t} + \frac{1}{2!} g_{\mu\nu,\rho\sigma}(x_0)\, x^{\rho}x^{\sigma}\psi^{\mu}\frac{\partial\psi^{\mu}}{\partial t} + \cdots\right) - \frac{1}{2}\dot{x}^{\lambda}\psi^{K}x^{\alpha}\psi^{\mu}\, R_{\lambda\alpha K\mu}(x_0)$$

Considering only the order of β from the above expansions and If we rescale the time as $t \to \beta t$, we get

$$S = \int_0^1 \beta\left[\frac{1}{\beta^2}\{\beta^2\dot{x}^{\mu}\dot{x}^{\mu}\} + \left\{\frac{1}{2!} g_{\mu\nu,\rho\sigma}(x_0)\left(\frac{1}{\beta^2}\beta^4 x^{\rho}x^{\sigma}\dot{x}^{\mu}\,\dot{x}^{\nu}\right)\right\} + \cdots + \right.$$

$$\left. + \left\{\frac{\beta}{\beta}\psi^{\mu}\frac{\partial\psi^{\mu}}{\partial t} + \frac{\beta^3}{\beta}\frac{1}{2!} g_{\mu\nu,\rho\sigma}(x_0)x^{\rho}x^{\sigma}\psi^{\mu}\frac{\partial\psi^{\mu}}{\partial t} + \cdots\right\} - \frac{\beta^2}{\beta}\frac{1}{2}\, R_{\lambda\alpha K\mu}(x_0)\dot{x}^{\lambda}\psi_0{}^{K}x^{\alpha}\psi_0{}^{\mu}\right] dt$$

In the above action we did not yet expand the fields into their Fourier components except for the fermions in the curvature term and we take the zero modes only in order to make the curvature term survive in the small β limit, so we check the order of β in the above process. We see that in the small β limit, we neglect the terms of the order of β^3 and then we are left with the following integral,

$$S = \int_0^1 \left[\{\beta\dot{x}^{\mu}\dot{x}^{\mu} + O(\beta^3)\} + \left\{\beta\psi^{\mu}\frac{\partial\psi^{\mu}}{\partial t} + O(\beta^3)\right\} - \beta^2\frac{1}{2}\, R_{\lambda\alpha K\mu}(x_0)\dot{x}^{\lambda}\psi_0{}^{K}x^{\alpha}\psi_0{}^{\mu}\right] dt$$

After making these arguments and droping out higher order terms in β, we can say that our Lagrangian (4.23) can now be written as follows,

$$L = \dot{x}^{\mu}\dot{x}^{\mu} + \psi^{\mu}\frac{\partial\psi^{\mu}}{\partial t} - \frac{1}{2}R_{\lambda\alpha K\mu}(x_0)\,\psi_0{}^{\mu}\psi_0{}^{K}x^{\alpha}\dot{x}^{\lambda} \qquad (4.24)$$

If we define fluctuations in the coordinate system as follows,

$$x^{\mu}(t) = x_0{}^{\mu} + \xi^{\mu}(t)$$

$$\psi^{\mu}(t) = \psi_0{}^{\mu} + \eta^{\mu}(t)$$

And noting that $dx^{\mu} = d\xi^{\mu}$, $d\psi^{\mu} = d\eta^{\mu}$, we can cast the action into the following form,

$$S = \int_0^{\beta} \left[\dot{\xi}^{\mu}\dot{\xi}^{\mu} + \eta^{\mu}\dot{\eta}^{\mu} - \frac{1}{2} R_{\lambda\alpha K\mu}(x_0)\psi_0{}^{\mu}\psi_0{}^{K}\xi^{\alpha}\dot{\xi}^{\lambda} \right] dt \tag{4.25}$$

In (4.25) the operator which is associated with the ξ field is given as,

$$-\delta_{\mu\nu}\frac{d^2}{dt^2} + \widetilde{R_{\lambda\alpha}}\frac{d}{dt} \tag{4.26}$$

Where we have $\widetilde{R_{\lambda\alpha}}(x_0) = -\frac{1}{2} R_{\lambda\alpha K\mu}(x_0)\psi_0{}^{\mu}\psi_0{}^{K}\xi^{\alpha}\dot{\xi}^{\lambda}$

And the operator associated with the η field is given by,

$$\delta_{\mu\nu}\frac{d}{dt} \tag{4.27}$$

We have to consider the following path integral,

$$I = \int \mathcal{D}\xi\, \mathcal{D}\eta\, e^{-S} \tag{4.28}$$

In this integral, we will consider the modes $\xi_0{}^{\mu}$ and $\eta_0{}^{\mu}$ for which n= 0 which are called the zero modes, separately in the following integral, Considering the non-zero modes, we see that the operator (4.26) is associated with the bosinic fields whereas the operator (4.27) is associated with the fermionic fields, so we can make use of the results of Gaussian integral results for bosonic fields and fermionic fields here, taking into account these considerations, we get

$$I = \int \prod_{\mu=1}^{d} d\,\xi_0{}^{\mu} d\eta_0{}^{\mu} \frac{\sqrt{\mathrm{Det}\left(\delta_{\mu\nu}\frac{d}{dt}\right)}}{\sqrt{\mathrm{Det}\left(-\delta_{\mu\nu}\frac{d^2}{dt^2} + \widetilde{R_{\lambda\alpha}}(x_0)\frac{d}{dt}\right)}}$$

$$\Rightarrow \quad I = \int \prod_{\mu=1}^{d} d\,\xi_0{}^{\mu} d\eta_0{}^{\mu} \frac{1}{\sqrt{\mathrm{Det}\left(-\delta_{\mu\nu}\frac{d}{dt} + \widetilde{R_{\lambda\alpha}}(x_0)\right)}} \tag{4.29}$$

Where *d* is the dimension of the manifold, and We need to remember that the determinants obtained in the above integral are the result of integration over non-zero modes only, they do not include the zero modes.

Now we compute the functional determinant in (4.29) and in (4.29) the fermionic variables are only contained in $\widetilde{R_{\lambda\alpha}}(x_0)$ we suppose for the time being that this part is a commuting number, and we know that Riemann tensor is anti-symmetric which implies that $\widetilde{R_{\lambda\alpha}}(x_0)$ satisfies the following,

$$\widetilde{R_{\lambda\alpha}} = -\widetilde{R_{\alpha\lambda}}$$

Since real skew-symmetric matrices are normal matrices, it is possible to bring every skew-symmetric matrix into a block diagonal form by an orthogonal transformation, so in an even dimensional manifold it is possible to block diagonalize $\widetilde{R_{\lambda\alpha}}$ in the following form,

$$\widetilde{R_{\lambda\alpha}} = \begin{pmatrix} \begin{matrix} 0 & \alpha_1 \\ -\alpha_1 & 0 \end{matrix} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \begin{matrix} 0 & \alpha_n \\ -\alpha_n & 0 \end{matrix} \end{pmatrix}$$

Let us focus on the first block, the operator $-\delta_{\mu\nu}\frac{d}{dt} + \widetilde{R_{\lambda\alpha}}(x_0)$ is real and hence the eigenvalues are made of complex conjugate pairs, when this operator is applied to this block, and the determinant of this block can be computed as follows,

$$\det\begin{pmatrix} -\frac{d}{dt} & \alpha_1 \\ -\alpha_1 & -\frac{d}{dt} \end{pmatrix} = \mathrm{Det}\left(\frac{d^2}{dt^2} + \alpha_1{}^2\right)$$

$$= \prod_{n\neq 0}\left(\alpha_1{}^2 - \left(\frac{2n\pi}{\beta}\right)^2\right)$$

$$= \left[\prod_{n>1} \left(\frac{2n\pi}{\beta} \right)^2 \prod_{n>1} \left\{ 1 - \left(\frac{\alpha_1 \beta}{2n\pi} \right)^2 \right\} \right]^2$$

$$= \left(\frac{sin\beta\alpha_1/2}{\alpha_1/2} \right)^2 \tag{4.30}$$

If we consider all the n blocks, we will get the following result for the integral (4.29)

$$I = \int \prod_{\mu=1}^{2n} d\xi_0{}^{\mu} d\eta_0{}^{\mu} \prod_{j=1}^{n} \frac{\alpha_j/2}{sin\beta\alpha_j/2}$$

The integration over ξ_0 is equivalent as that of integration over x_0 and integration over η_0 is equivalent to that of over ψ_0 and also the product *j* in this integral can be written in terms of the Riemann curvature tensor as,

$$I = \int \prod_{\mu=1}^{2n} dx_0{}^{\mu} d\psi_0{}^{\mu} \frac{1}{\beta^{d/2}} \det \left(\frac{\beta\tilde{R}/2}{sin\,\beta\tilde{R}/2} \right)^{1/2}$$

We can make the following change of variables to remove the β dependence from above integral,

$$\psi_0{}^{\mu} = \frac{\chi_0{}^{\mu}}{\sqrt{\beta}}, \qquad d\psi_0{}^{\mu} = \sqrt{\beta}\, d\chi_0{}^{\mu}$$

And we substitute

$$\beta\tilde{R} = -\frac{1}{2} R_{\lambda\alpha K\mu}(x_0)\psi_0{}^{\mu}\psi_0{}^{K}$$

$$I = \int \prod_{\mu=1}^{2n} dx_0{}^{\mu} d\psi_0{}^{\mu} \; det \left(\frac{\frac{1}{2}\frac{1}{2} R_{\lambda\alpha K\mu}(x_0)\psi_0{}^{\mu}\psi_0{}^{K}}{sin\frac{1}{2}\frac{1}{2} R_{\lambda\alpha K\mu}(x_0)\psi_0{}^{\mu}\psi_0{}^{K}} \right)^{1/2} \tag{4.31}$$

We have to note that we have dropped out infinitely many terms in going through this model, when we were doing different expansions, and the result (4.31) is a close approximation to the actual result. If we take into account this fact and also take care of the factors $i/2\pi$ arising from the Feynman measure in the above integral, after carrying out integral over zero modes, we would get something like

$$I = \int det\left(\frac{\frac{1}{2}\frac{1}{2\pi}\ R}{\sinh\frac{1}{2}\frac{1}{2\pi}\ R}\right)^{1/2}$$

And if we again consider the anti-symmetry of R, we can again diagonalize R like we did before, and after doing this, we can recover the following result,

$$I = \int \prod_{j=1}^{n} \frac{x_j/2}{\sinh x_j/2} \tag{4.32}$$

This result (4.32) is a topological invariant of the manifold, It is the index of Dirac operator on a spin manifold. It is another important characteristic class which is often called as Dirac A-genus.

5 Conclusion

In this Master thesis we have presented some basic aspects of Supersymmetric Quantum Mechanics, in the context of Path integrals, and we investigated how we can solve supersymmetric sigma models on curved space through very simple and straightforward calculations.

We introduced very basic concepts about probability and the amplitude in Quantum Mechanics, and used this basic idea to describe the mathematical framework involved in path integrals, and then studied the example of a free particle to establish our idea of Path integrals.

Through the introduction of Grassmann variables we constructed basic ideas about how to handle these kind of variables, and how we define rules for differentiation and integration for such variables, then we introduced Gaussian integrals over normal variables and later generalized those Gaussian integrals to incorporate the idea of Gaussian integrals with Grassmann variables and studied some of very simple examples to carry out these integrals.

Finally we started our analysis of supersymmetric sigma models where the target space was flat, then we studied the supersymmetric sigma model with a curvature term on a circle, where the target space was curved, and we carried out the computation using path integrals as our central tool and used our ideas of Gaussian integrals for commutating and non-commutating variables developed in the previous sections, and in the end recovered the Euler characteristic of the manifold and hence the Euler number, which is a topological invariant of the manifold and is a connection between the geometry of the target manifold and the physical model of the system. Lastly we considered another supersymmetric sigma model on a curved space and carried out a similar computation as in the previous model using Path integrals as central tool, we recovered another important characteristic class for the manifold known as Dirac A-Genus, which is a topological invariant of the target manifold, hence again we recovered a connection between the physical model and the geometry of the target space.

This Master thesis went through simple and basic but important aspects of supersymmetric quantum mechanics, by showing calculations and proofs that will allow the reader to further understand the properties of supersymmetric quantum mechanics that help the simplification of calculations needed by physicists in supersymmetric field theories.

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