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Localization Techniques, Yang-Mills Theory and Strings

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Abstract

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Equivariant localization techniques exploit symmetries of systems, represented by group actions on manifolds, and use them to evaluate certain partition functions exactly. In this master thesis we begin with the study of localization in finite dimensions. We then generalize this concept to infinite dimensions and study the partition function of two dimensional quantum Yang-Mills theory and its relation to string theory. The partition function can be written as a sum over the critical point set and be related to the topology of the moduli space of flat connections. Furthermore, for large N the partition function of the gauge groups SU(N) and U(N) can be interpreted as a string perturbation series. The coefficients of the expansion are given by a sum over maps from a two dimensional surface onto the two dimensional target space and thus the partition function is interpreted as a closed string theory. Also, a string theory action is discussed using topological field theory tools and localization techniques.

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POPULÄRVETENSKAPLIG SAMMANFATTNING

Lokaliseringstekniker använder symmetrier, som beskrivs som en gruppverkan på en mångfald, hos system för att beräkna vissa integraler exakt. Lokalisering betyder att integralen kan skrivas som en summa över en diskret mängd element. Eftersom dessa integraler, t.ex. vägintegraler för fysikaliska system, kan lösas exakt ger de en fullständig förståelse för fysiken/matematiken där. Vägintegraler är integraler som används i kvantfysiken och ersätter vägen som en partikel tar i den klassiska fysiken (en bana som minimerar energin) med att summera över *alla* möjliga vägar. Denna summa ger istället en sannolikhetesamplitud och beskriver hur ett system beter sig. Vägintegraler är oändligdimensionella men via lokalisering kan vissa av dessa vägintegraler reduceras till ändligdimensionella integraler, vilka är väldefinerade. Det är symmetrierna i den underliggande dynamiska teorin som säger om vägintegralerna kan reduceras till ändligdimensionella integral.

Det matematiska ramverket för att beskriva dessa symmetrier kallas ekvivariant kohomologi, vilket inkluderar gruppverkan i kohomologi beskrivningen. Kohomolgi är ett matematiskt verktyg för att studera topologin hos en mångfald. Det var på 1980-talet som det insågs att vissa integraler kunde skrivas exakt om det fanns vissa typer av symmetrier - det fundamentala lokaliseringsteoremet var fött.

I fysiken har symmetrier länge använts för att förenkla, beskriva och förstå olika fenomen i naturen. Till exempel bygger Standard Modellen, som beskriver elementarpartiklarna och deras interaktioner genom elektromagnetisk, stark och svag växelverkan, på symmetrier hos naturen. Standard Modellen bygger på en teori som heter Yang-Mills teori. Detta är en gauge teori. Dessa bygger på lokala symmetrier som i sin tur ger upphov till interaktionerna i teorin.

I denna uppsats kommer vi diskutera lokalisering för både ändligdimensionella integraler och oändligdimensionella integraler. Vi inför Cartans modell för ekvivariant kohomologi som liknar de Rham kohomologi med skillnaden att gruppverkan är inkluderad. Vi gör sedan beräkningar på supermångfalder genom att införa anti- kommutativa variabler. I det oändligdimensionella fallet kommer vi att studera tvådimensionell Yang-Mills teori med hjälp av lokaliseringsprincipen och även dess underliggande strängteori.

CONTENTS

1 INTRODUCTION

Localization techniques make use of symmetries, represented by group actions on a manifold, of systems and use these in evaluating certain integrals exactly. Specifically, localization means that the integral can be written as a sum over a discrete set of points. As these integrals, for example path integrals of physical systems, can be solved exactly they give a complete understanding of the physics/mathematics there.

There is a mathematical framework to describe these symmetries called *equivariant cohomology* (one includes the group action in the description of cohomology, the study of manifold topology). This gives us a general framework; *the equivariant cohomological framework*, which is a tool to develop geometric techniques for manipulating integrals and investigate the localization properties they possess.

This tool can be used to study Feynman path integrals, which was introduced in the 1940's as a new and original approach to quantum theory [1]. The path integral is understood as an integral over infinite dimensional functional space but without rigorous definition. It replaces the single trajectory of classical physics by integrating over all possible trajectories to calculate the quantum probability amplitude describing the behavior of the system. To calculate it one imitates how the evaluation is done in the finite dimensional case. The path integrals that can be solved exactly have some features in common. There is numerous (super-)symmetries in the underlying dynamical theory. This makes the integrals reduce (localize) to Gaussian finite-dimensional integrals where one can extract the physical/mathematical information (compare the Schrödinger equation; the O(4)-symmetry of the Coulomb problem in three dimensions makes the hydrogen atom exactly solvable [2]).

In the 1940's, there was only two examples that could be solved exactly; the harmonic oscillator and the free particle. The path integrals can then be calculated using

$$\int_{-\infty}^{\infty}\prod_{k=1}^{n} dx^k e^{\frac{i}{2}\sum_{ij}x^i M_{ij}x^j + i\sum_i \lambda_i x^i} = \frac{(2\pi e^{i\pi/2})^{\frac{n}{2}} e^{\frac{i}{2}\sum_{ij}\lambda_i (M^{-1})^{ij}\lambda_j}}{\sqrt{\det M}} \tag{1.1}$$

which is the functional analog of the Gaussian integration formula and M is a $n \times n$ non-singular symmetric matrix [3]. Using (1.1), the path integral can formally be evaluated for a field theory that is at most quadratic in the fields. If not, the arguments of the integrand can be expanded and the path integral can be approximated by (1.1). For a finite-dimensional integral this approximation is called the stationary phase approximation (or saddle-point or steepest-descent approximation) [4]. For path integrals, it is often denoted Wentzel-Kramers-Brillouin (WKB) approximation [2, 5]. As the solution of (1.1) is given by putting the global minimum of the quadratic form (the classical value) in the exponent and multiplying it by the second variation of the form (the fluctuation determinant) it is also denoted the semi-classical approximation (this is what gives the interpretation of quantum mechanics to be a sum over paths fluctuating about the classical trajectories). If the semi-classical approximation is exact, (1.1) can be seen as a localization of the complicated path integral onto the global minimum of the quadratic form.

A class of field theories that has path integrals that can be solved exactly in most cases is topological quantum field theories (have observables that are independent of the metric) and supersymmetric (spacetime symmetry that relates fermions to bosons) theories. Topological quantum field theories have some likeness to many interesting physical systems and physicists can use them for some insight to the structure of more complicated physical systems. In chapter 6 we will see that two dimensional quantum Yang-Mills theory can be studied from localization using a topological quantum field theory and we have supersymmetry such that the path integral can be evaluated exactly.

The application of localization techniques started in the beginning of the 1980s when Duistermaat and Heckman were studying symplectic geometry and understood that integrals of certain symmetry could be simplified to easier operations. In their paper [6], published in 1982, they showed the Duistermaat-Heckman theorem which states that the the semi-classical approximation for oscillatory integrals of finite dimension over compact manifolds is exact. This is the fundamental theorem of localization. The simplification was understood by Atiyah and Bott [7] to be a special case of a more general localization principle of equivariant cohomology. Thereafter, Berline and Vergne used this to prove the first general localization formula for Killing vector fields on compact Riemannian manifolds [8, 9]. We will prove this theorem in chapter 4.

In 1985 the Duistermaat-Heckman theorem was generalized to an infinite dimensional case by Atiyah and Witten [10]. In this work they studied the supersymmetric path integral for the Dirac operator index. They showed that the Duistermaat-Heckman theorem could be applied to the partition function of $\mathrm{N} = 1/2$ supersymmetric quantum mechanics on the loop space of a manifold (which in other words is the description of a supersymmetric spinning particle in a gravitational background [24]) and that it gave the Atiyah-Singer index theorem (which states that the analytical and topological index of the Dirac operator is the same. The topological index is given by an integral over characteristic classes and gives a measure of the curvature of a manifold).

In [11] Blau related supersymmetry and equivariant cohomology in the quantum mechanics of spin. From this work Blau, Keski-Vakkuri and Niemi [12] worked out a general supersymmetric, or equivariant cohomological, framework to study Duistermaat-Heckman localization formulas for path integrals of non-supersymmetric phase space and it is the foundation of equivariant localization theory. They showed that the partition function of quantum mechanics with circle actions on symplectic manifolds localizes and their work led to a lot of activity in this field. The proof uses Becchi-Rouet-Stora-Tyupin (BRST) quantization (see chapter 5). BRST cohomology is the fundamental structure in topological field theories and these BRST supersymmetries are the ones responsible for the localization.

The Duistermaat-Heckman theorem was first generalized to non-abelian group actions by Guilleman and Prato [13]. In 1992 Witten showed [14] that a more general non-abelian localization formula can be used to study the path integral of two dimensional quantum Yang-Mills theory. In that paper Witten showed that the path integral can be related to the topology of the moduli space of flat connections. This was of great importance and received a lot of interest as it showed that one can reduce the path integral (which is a very complicated integral over infinite dimensional functional space that one doesn't even know how to define properly) to intersection numbers, topological invariants, using the localization principle. In chapter 6 we will study the non-abelian localization formula and two dimensional quantum Yang-Mills theory.

Yang-Mills theory is a gauge theory (gauge means standard of calibration) that has been successfully used to explain the dynamics of the known elementary particles. The theory of elementary particle physics is put together in the Standard Model (however not a final theory of elementary particles). This is a non-abelian gauge theory, with symmetry group $U(1) \times SU(2) \times SU(3)$, which describes the elementary particles and the electro-weak and strong interactions. Gauge theories can be constructed from the following recipe. First one looks for a global symmetry of the physical system. Secondly one changes this symmetry to a local symmetry which destroys the invariance. To restore the invariance one has to add new fields. These fields gives the interactions of the theory. Finally it leaves us with a Lagrangian with local gauge invariance and interactions.

Gauge theories can be viewed more geometrically by the concept of fiber bundles. A fiber bundle is a manifold which locally is a direct product of two topological spaces (see figure 3.1 in chapter 3). This is the resulting structure constructed by attaching fibers to every point of the manifold. A fiber bundle can be written schematically as

$$\begin{array}{l} E \leftarrow G \\ \downarrow \\ B \end{array} \tag{1.2}$$

where E is the total space, B the base space and G is the fiber given by the symmetry group (can also be a vector field). For example, electrodynamics is described by a U(1) fibre bundle over the spacetime. The gauge field, which gives the photon field, is a connection on the bundle and the electromagnetic field strength is the curvature.

Yang-Mils theory has another expected interpretation in terms of string theory. The are many reasons for this. For example, one has seen that the strong interaction resembles strings. In the late 1960s string theory was actually found when people tried to guess a mathematical formula for the strong interaction scattering amplitudes that would agree with current experiments. However, viewing the strong interaction as a one dimensional string made a lot of contradictions with experimental results and in the middle of the 1970s this theory was abandoned for quantum electrodynamics (QCD). Later string theory has been used as a theory for trying to describe all the forces (including gravity) and matter in nature. In modern theoretical physics fundamental theories of nature are described by both geometry and symmetry; general relativity and gauge theory. It is believed that string theory can generalize general relativity and gauge theory to one final theory. And the hope of writing Yang-Mills theory as a string theory is still alive. However there are no experimental proofs for string theory.

1.1 AIM AND STRUCTURE OF THESIS

The aim of this master thesis is to study finite dimensional localization, two dimensional quantum Yang-Mills theory (using the concept of localization) and its relation to string theory using the localization principle. Two dimensional quantum Yang-Mills theory without matter is a gauge theory that has been studied a lot over the years and can be solved easily. Here we will re-examine it using a non-abelian localization formula to explain properties that can not be explained using standard methods. It is an interesting object of study as four dimensional quantum Yang-Mills theory is the basis of the Standard Model. We will use the localization principle to show that the partition function of two dimensional quantum Yang-Mills theory can be written as a sum over the critical point set of the action and that one can relate the partition function to the topology of the moduli space of flat connections. We will also write down general formulas for intersection pairings on moduli spaces of flat connections. To achieve this goal we will begin by proving localization formulas in a finite dimensional setting. This will be shown using supergeometry. We will then go on to study the infinite dimensional case of two dimensional quantum

Yang-Mills theory using a further generalization of the localization formula following [14]. Thereafter we will interpret the two dimensional Yang-Mills theory in terms of an equivalent string theory and write down a string action using topological field theory tools and localization techniques [15, 16, 17, 18].

The structure of the thesis is as follows. The thesis is divided into three blocks. The first block contains chapters 2 to 4, which discusses finite dimensional localization. The second block consists of chapter 5 and 6, were we study infinite dimensional localization for the case of two dimensional Yang-Mills theory. The third block contains chapter 7 and reviews the underlying string theory of the two dimensional quantum Yang-Mills theory. A short introduction/summary of each chapter can be found in the beginning of the chapters. To easily find references they are included there as well.

The chapters contain the following. In chapter 2 we will introduce basic notions of supergeometry. In particular we will look at Berezin integration, superalgebra and supermanifolds. In chapter 3 we will study manifolds acted on by a group. We will generalize concepts as cohomology and vector bundles to this case. This will be used in chapter 4 where we prove localization formulas in finite dimension. In particular we will prove the Berline-Vergne formula, the Duistermaat-Heckman formula and the localization formula of the degenerate case. Chapter 2 and 3 are reviews of necessary theory for doing the calculations in chapter 4, that may be skipped if familiar with the concepts. In chapter 5 we will discuss topological quantum field theory and gauge theory (Yang-Mills theory). We will look at Yang-Mills theory and compute its partition function in two dimensions. Chapter 5 contains the theory for understanding the calculations of chapter 6. In chapter 6 we study the localization of two dimensional quantum Yang-Mills theory. In chapter 7 we give an introduction to string theory, the symmetric group, Young tableaux and Riemann surfaces and use these concepts in interpreting the two dimensional Yang-Mills theory in terms of a string theory. Chapter 8 is devoted to the conclusions.

2

SUPERGEOMETRY

We will begin this master thesis by considering some basic concepts of supergeometry. Supergeometry extends classical geometry (commuting coordinates) by permitting odd coordinates which anticommute. These coordinates are realized through Grassmann variables. When gluing these new coordinate systems one gets supermanifolds. The notions of supermanifolds and integration over odd coordinates will be of importance in the work of the next chapters. To understand these we need to introduce Grassmann variables, Berezin integration and $\mathbb{Z}_2$-graded algebra (also called superalgebra). We will not discuss the sheaf and categorical notions of supergeometry, as this will not simplify the understanding of this work. Nevertheless it is important for the proper treatment of the subject (see [20, 21]). At the end of this chapter we will also introduce graded geometry, which is the generalization of supergeometry.

2.1 GRASSMANN VARIABLES

Grassmann variables (also called odd variables) are anticommuting variables satisfying

$$\theta_i\theta_j = -\theta_j\theta_i, \quad (\theta_i)^2 = 0. \tag{2.1}$$

These variables commute with ordinary numbers and allows for fermonic fields to have a path integral representation through Berezin integration (see below).

A general function of even variables x^j $(j = 1, \ldots, m)$ and odd variables θ^i $(i = 1, \ldots, n)$ can be written as

$$f = f_0(\bar{x}) + \sum_{k=1}^{n} \frac{1}{k!} f_{i_1 \ldots i_k}(\bar{x})\theta^{i_1} \cdots \theta^{i_k}. \tag{2.2}$$

Two homogeneous functions f and g with degree $|f|$, $|g|$ (for example $f(x)\theta$ is homogeneous, it only consist of a determined power of odd variables, of degree one) respectively satisfies

$$fg = (-1)^{|f||g|} gf. \tag{2.3}$$

This is known as the sign rule, which says that if two odd terms are interchanged a minus sign will appear.

Let us now introduce derivation of odd variables. The derivation is defined through

$$\left\{ \frac{\partial}{\partial \theta^i}, \theta^j \right\} = \delta_i^j. \tag{2.4}$$

Let f and g be homogeneous functions consisting of a specific number of odd variables. The generalized Leibniz rule is given by

$$\begin{aligned}
\frac{\partial}{\partial \theta_\beta} fg &= \frac{\partial}{\partial \theta_\beta} \sum_k \frac{1}{k!} f_{i_1 \dots i_k}(\bar{x}) \theta^{i_1} \cdots \theta^{i_k} \sum_l \frac{1}{l!} g_{j_1 \dots j_l}(\bar{x}) \theta^{j_1} \cdots \theta^{j_l} \\
&= \frac{\partial}{\partial \theta_\beta} \sum_k \sum_l \frac{1}{k!} \frac{1}{l!} f_{i_1 \dots i_k}(\bar{x}) g_{j_1 \dots j_l}(\bar{x}) \underbrace{\theta^{i_1} \cdots \theta^{i_k}}_{\theta_\beta \text{ in here}} \theta^{j_1} \cdots \theta^{j_l} \\
&+ \frac{\partial}{\partial \theta_\beta} \sum_k \sum_l \frac{1}{k!} \frac{1}{l!} f_{i_1 \dots i_k}(\bar{x}) g_{j_1 \dots j_l}(\bar{x}) \theta^{i_1} \cdots \theta^{i_k} \underbrace{\theta^{j_1} \cdots \theta^{j_l}}_{\theta_\beta \text{ in here}} \\
&= \frac{\partial}{\partial \theta_\beta} \sum_k \sum_l \frac{1}{k!} \frac{1}{l!} f_{i_1 \dots i_k}(\bar{x}) g_{j_1 \dots j_l}(\bar{x}) \theta^{i_1} \cdots \theta^{i_k} \theta^{j_1} \cdots \theta^{j_l} \\
&+ \frac{\partial}{\partial \theta_\beta} \sum_k \sum_l \frac{1}{k!} \frac{1}{l!} (-1)^{kl} f_{i_1 \dots i_k}(\bar{x}) g_{j_1 \dots j_l}(\bar{x}) \theta^{j_1} \cdots \theta^{j_l} \theta^{i_1} \cdots \theta^{i_k},
\end{aligned} \tag{2.5}$$

which for two homogenous functions f, g can be written as

$$\frac{\partial}{\partial \theta_\beta} fg = \left(\frac{\partial}{\partial \theta_\beta} f \right) g + (-1)^{|k||l|} \left(\frac{\partial}{\partial \theta_\beta} g \right) f. \tag{2.6}$$

Example 2.1. *Let us take the two functions $f_1(\theta_1, \theta_2) = \theta_1 \theta_2$ and $f_2(\theta_1, \theta_2) = \theta_2 \theta_1$. The derivative of f_1 and f_2 is $\frac{d}{\theta_1} \theta_1 \theta_2 = \theta_2$ and $\frac{d}{\theta_1} \theta_2 \theta_1 = -\theta_2$ respectively. This shows that one must look carefully at the order of the θ's when dealing with odd variables.*

2.2 BEREZIN INTEGRATION

We will now turn our attention to the integration of odd variables, denoted Berezin integration. The basic Berezin integration rules are

$$\int d\theta = 0, \int d\theta\theta = 1. \tag{2.7}$$

These rules are constructed in this way in order to satisfy the linearity condition and the partial integration formula:

$$\int [af(\theta) + bg(\theta)]d\theta = a\int f(\theta)d\theta + b\int g(\theta)d\theta, \tag{2.8}$$

$$\int \left[\frac{\partial}{\partial\theta} f(\theta)\right] d\theta = 0 \tag{2.9}$$

so that one can reproduce the path integral for a fermion field.

When integrating an even function $f(x)$ in one variable one can make a coordinate change by $x = cy$ and the measure is changed as

$$dx = cdy. \tag{2.10}$$

However, integrating a function $f = f_0 + f_1\theta$ (with one odd coordinate θ) using Berezin integration we get a different measure when changing coordinates. We like to change coordinates as $\theta = c\tilde{\theta}$. Looking back at (2.7) we have $\int d\theta\theta = 1$ and $\int d\tilde{\theta}\tilde{\theta} = 1$. Changing coordinates gives $\int d\theta c\tilde{\theta}$ with the measure

$$d\theta = \frac{1}{c}d\tilde{\theta}. \tag{2.11}$$

As a result we see that the odd measure transforms in the opposite way as for the even measure.

Next, we define the convention used in this work for integration over many θ's by

$$\int d\theta^n \cdots d\theta^1\theta^1 \cdots \theta^n = 1. \tag{2.12}$$

This says that the θ's must be put in this particular order to integrate out to one. Next we like to do the coordinate change

$$\theta^i = \sum_{j=1}^{m} A_{ij}\tilde{\theta}^j \tag{2.13}$$

This gives

$$\begin{aligned}\int d\theta^n \cdots d\theta^1 \theta^1 \cdots \theta^n &= \int d^n\theta \sum_{j=1}^{n} A_{1j}\tilde{\theta}^j \cdots \sum_{l=1}^{n} A_{nl}\tilde{\theta}^l \\ &= \det(A) \int d^n\theta \tilde{\theta}^1 \cdots \tilde{\theta}^n \\ &= \int d\tilde{\theta}^n \cdots d\tilde{\theta}^1 \tilde{\theta}^1 \cdots \tilde{\theta}^n.\end{aligned} \tag{2.14}$$

This implies that the measure is changed as

$$d^n\theta = \frac{1}{det(A)} d^n\tilde{\theta}. \tag{2.15}$$

One can also define an exponential function of θ's which will terminate after finitely many terms. An example, in two odd variables, is

$$e^{\theta_1\theta_2} = 1 + \theta_1\theta_2. \tag{2.16}$$

We will now discuss how to perform Gaussian integration with odd coordinates. First we recall how its done using even coordinates. Let A be a $n \times n$ symmetric, real matrix. A can be diagonalized by a matrix $B \in SO(n)$ and $D = B^T AB = \mathrm{diag}(\lambda_1 \ldots \lambda_n)$, where λ_i are the eigenvalues of A. We get

$$\begin{aligned}\int d^n x e^{-x^T Ax} &= \int d^n y e^{-y^T B^T ABy} \\ &= \int d^n y e^{-y^T Dy} \\ &= \prod_{i_1}^{n} \int d^n y_i e^{-\lambda_i y_i^2} \\ &= \frac{\pi^{n/2}}{\sqrt{\det(A)}}.\end{aligned} \tag{2.17}$$

Now we turn to the case of odd variables. Let B be a $2n \times 2n$ skew-symmetric matrix. We have

$$\begin{aligned}\int d\theta^{2m} \cdots d\theta^1 e^{-\sum_{i,j=1}^{2m} \theta^i B_{ij}\theta^j} &= \int d\theta^{2m} \cdots d\theta^1 \frac{(-\sum_{i,j=1}^{2m} \theta^i B_{ij}\theta^j)^n}{n!} \\ &= \frac{1}{n!}\epsilon_{i_1} \ldots \epsilon_{i_{2n}} B_{i_1 i_2} \cdots B_{i_{2n-1} i_{2n}} \\ &= (-2)^m Pf(B),\end{aligned} \tag{2.18}$$

where Pf is short for the pfaffian of B. The pfaffian of B is defined as

$$Pf(B) = \epsilon^{i_1 \ldots i_{2n}} B_{i_1 i_2} \ldots B_{i_{2n-1} i_{2n}} = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} sgn(\sigma) \prod_{1=1}^{n} b_{\sigma(2i-1),\sigma(2i)} \tag{2.19}$$

and is zero for $2n$ odd. The pfaffian is related to the determinant as $(Pf(B))^2 = \det(B)$.

2.3 SUPERALGEBRA

Now we will consider vector spaces constructed out of both even (ordinary) and odd (Grassman) variables [19]. These type of vector spaces are called *super vector spaces* or $\mathbb{Z}_2$-graded vector spaces. A super vector space V over a field $\mathbb{K}$ (usually $\mathbb{R}$ or $\mathbb{C}$) with $\mathbb{Z}_2$-grading is a vector space decomposed as

$$V = V_1 \bigoplus V_2, \tag{2.20}$$

where V_1 is called even and V_2 is called odd. If $\dim V_1 = m$ and $\dim V_2 = n$ then we write $V^{m|n}$ (compare $\mathbb{R}^n$), where the combination (m, n) is called the superdimension of V. The notions of $\mathbb{Z}_2$-grading can be generalized to any grading discussed at the end of this chapter.

An algebra is a vector space with bilinear multiplication. A *superalgebra* V is a $\mathbb{Z}_2$-graded vector space V with a product: $V \otimes V \to V$ that respects the grading.

A superspace with the Lie bracket $[a, b]$ that satisfies

$$\begin{aligned} [a, b] &= -(-1)^{|a||b|}[b, a], \\ [a, [b, c]] &= [[a, b], c] + (-1)^{|a||b|}[b, [a, c]], \end{aligned} \tag{2.21}$$

for $a, b \in V$ and $|a|$ the degree of a is called a Lie superalgebra. Note that if $[a, b] = ab - (-1)^{|a||b|}ba = 0$, i.e.

$$ab = (-1)^{|a||b|}ba, \tag{2.22}$$

the superalgebra is said to supercommutative. The exterior algebra (the algebra of differential forms with multiplication defined by the wedge product) is an important example of the supercommutative algebra.

Let's also introduce the parity reversion functor Π by $(\Pi V)_1 = V_2$ and $(\Pi V)_2 = V_1$ that changes the parity of the components of a superspace.

Example 2.2. *Take real vector space $\mathbb{R}^n$ and reverse the parity by $\Pi\mathbb{R}^n$. This gives the odd vector space $\mathbb{R}^{0|n}$. Now, pick a basis θ_i $(i = 1, \cdots, n)$ and define the multiplication as $\theta_i\theta_j = -\theta_j\theta_i$. The functions on $C^\infty(\mathbb{R})^{0|n}$ on $\mathbb{R}^{0|n}$ are*

$$f(\theta^1, \ldots, \theta^m) = \sum_{k=1}^{n} \frac{1}{k!}\theta^{i_1} \cdots \theta^{i_k}, \tag{2.23}$$

which corresponds to elements of the exterior algebra $\Lambda^\bullet(\mathbb{R}^n)$. The exterior algebra is a supervector space with the wedge product as the supercommutative multiplication. The multiplication of functions in $C^\infty(\mathbb{R})^{0|n}$ corresponds to the wedge product of the exterior algebra.

2.4 SUPERMANIFOLDS

We will now look at how to construct supermanifolds. This is done in a way analogously to the definition of ordinary manifolds (the resulting object when gluing together open subsets of $\mathbb{R}^n$ by smooth transformations) but using vector superspaces. A supermanifold M of dimension (n, m) has a local description of n even coordinates x^i $(i = 1, ..., n)$ and m odd coordinates θ^j $(j = 1, ..., m)$. We cover the supermanifold M by open sets U_α having coordinates $(x_\alpha, \theta_\alpha)$. At the intersection $U_\alpha \cap U_\beta$ we have the gluing rule

$$\begin{aligned} x^i_\alpha &= x^i_{\alpha\beta}(x_\beta, \theta_\beta), \\ \theta^j_\alpha &= \theta^j_{\alpha\beta}(x_\beta, \theta_\beta). \end{aligned} \tag{2.24}$$

The gluing map must have an inverse, be compatible with the gluing maps on triple intersections and preserve parity (the parity of the variables is 0 for even and 1 odd variables).

We will now turn to some examples of supermanifolds.

Example 2.3 (The odd tangent bundle)**.** *Let M be a smooth manifold. Then we can define a supermanifold called the odd tangent* *bundle* ΠTM *(or* $T[1]M$*) with coordinates* x^i *and* θ^i *by the rules of transformation*

$$\begin{cases} \tilde{x}^i = \tilde{x}^i(x), \\ \tilde{\theta}^i = \frac{\partial \tilde{x}^i}{\partial x^j}\theta^j, \end{cases} \tag{2.25}$$

with x *being local coordinates on* M *and the* θ*'s transforming as* dx^i*. Thus we have an identification* $\theta^i \sim dx^i$*.*

The functions on ΠTM *are given by*

$$f(x,\theta) = \sum_{k=1}^{n} \frac{1}{k!} f_{i_1 \dots i_k}(x)\theta^{i_1} \cdots \theta^{i_k}. \tag{2.26}$$

We can see that the functions on ΠTM *are identified naturally with differential forms on* M*, i.e.* $C^\infty(T[1]M) = \Lambda^\bullet(M)$*.*

Moreover, on ΠTM *we have a canonical way of defining integration. As we saw in section 2.2 the even and odd measure transform opposite canceling each other, i.e. the even part transforms as*

$$d^n\tilde{x} = det\left(\frac{\partial \tilde{x}}{\partial x}\right) d^n x \tag{2.27}$$

and the odd part as

$$d^n\tilde{\theta} = \frac{1}{det(\frac{\partial \tilde{x}}{\partial x})} d^n\theta. \tag{2.28}$$

Thus we have

$$\int d^n\tilde{x} d^n\tilde{\theta} = \int d^n x d^n\theta. \tag{2.29}$$

This result says that any top degree function can be integrated canonically.

Example 2.4 (The odd cotangent bundle)**.** *We can also define a supermanifold called the odd cotangent bundle* ΠT^*M *(or* $T^*[1]M$*) by the rules of transformation*

$$\begin{cases} \tilde{x}^i = \tilde{x}^i(x), \\ \tilde{\theta}_i = \frac{\partial x^j}{\partial \tilde{x}^i}\theta_j, \end{cases} \tag{2.30}$$

with x being local coordinates on M and the θ's transforming as ∂_i. The functions on ΠTM are given by

$$f(x,\theta) = \sum_{k=1}^{n} \frac{1}{k!} f^{i_1 \dots i_k}(x) \theta_{i_1} \cdots \theta_{i_k}. \tag{2.31}$$

We can see that the functions on ΠT^*M are identified with multi-vector fields, i.e. $C^\infty(\Pi T^*M) = \Gamma(\wedge^\bullet(TM))$.

On ΠT^*M there is no way of canonically defining integration. In this case the even and odd measure transform in the same way, i.e. the even part transforms as

$$d^n\tilde{x} = det\left(\frac{\partial\tilde{x}}{\partial x}\right) d^n x \tag{2.32}$$

and the odd part as

$$d^n\tilde{\theta} = det\left(\frac{\partial\tilde{x}}{\partial x}\right) d^n \theta. \tag{2.33}$$

To define integration we need a term transforming in the opposite way. If M is orientable we can pick a volume form $\rho(x)dx^1 \wedge \cdots \wedge dx^n$, where ρ transforms as

$$\tilde{\rho} = \frac{1}{det(\frac{\partial\tilde{x}}{\partial x})}\rho. \tag{2.34}$$

Using ρ we can define an invariant measure as follows

$$\int d^n\tilde{x} d^n\tilde{\theta} \tilde{\rho}^2 = \int d^n x d\theta \rho^2. \tag{2.35}$$

Example 2.5. *If we again look at the odd tangent bundle ΠTM we can write the de Rham operator d, the interior product i_V (the contraction of a differential form with a vector field) and the Lie derivative $\mathcal{L}_V$ as functions of x's and θ's. Let the vector field be $V = V^\mu \frac{\partial}{\partial x^\mu}$ then*

$$\begin{aligned} d &= \theta^\mu \frac{\partial}{\partial x^\mu}, \\ i_V &= V^\mu \frac{\partial}{\partial \theta^\mu}, \\ \mathcal{L}_V &= di_V + i_V d = (\theta^\mu \frac{\partial}{\partial x^\mu})(V^\mu \frac{\partial}{\partial \theta^\mu}) + (V^\mu \frac{\partial}{\partial \theta^\mu})(\theta^\mu \frac{\partial}{\partial x^\mu}) \\ &= \theta^\mu \frac{\partial}{\partial x^\mu} V^\nu \frac{\partial}{\partial \theta^\nu} + V^\mu \frac{\partial}{\partial x^\mu}. \end{aligned} \tag{2.36}$$

2.5 GRADED GEOMETRY - GENERALIZING SUPERGEOMETRY

We will end this chapter by a very brief review of the generalization of supergeometry (see [19] or [22, 23] for more details). Supergeometry (with $\mathbb{Z}_2$-grading) can be generalized to an $\mathbb{Z}$-grading called graded geometry. We will explain this concept in the following.

A vector space V with a $\mathbb{Z}$-grading is a vector space decomposed as

$$V = \bigoplus_{i \in \mathbb{Z}} V_i, \tag{2.37}$$

where v is a homogeneous element of V with degree $|v| = i$ if $v \in V_i$. The elements of V can be decomposed as homogeneous elements of a certain degree. The morphism between these graded vector spaces is defined as a grading preserving linear map. This is just a bookkeeping device to keep track of elements of certain degree.

V is a graded algebra if the graded vector space V has an associative product that respects the grading. The endomorphism of V is then a derivation D of degree $|D|$ satisfying (for $\mathbb{Z}_2$-grading)

$$D(ab) = (Da)b + (-1)^{|D||a|}a(Db). \tag{2.38}$$

The graded algebra V is called a graded commutative algebra if

$$vv' = (-1)^{|v||v'|}v'v, \tag{2.39}$$

for homogenous elements v and v'. We shall end by giving one important example of graded commutative algebra.

Example 2.6 (Graded symmetric space $S(V)$)**.** *We will now look at the graded symmetric algebra $S(V)$, which is a graded vector space V over $\mathbb{R}$ or $\mathbb{C}$ spanned by polynomial functions on V*

$$\sum_l f_{a_1 \dots a_l} v^{a_1} \dots v^{a_l} \tag{2.40}$$

with

$$v^a v^b = (-1)^{|v^a||v^b|} v^b v^a. \tag{2.41}$$

v^a and v^b are homogeneous elements of degree $|v^a|$ and $|v^b|$. $S(V)$ is a graded commutative algebra as the functions on V are graded and the multiplication is graded commutative.

3

THE EQUIVARIANT GROUP ACTION ON MANIFOLDS

In this chapter we will introduce Cartan's model of equivariant cohomology and the equivariant Euler class, which we will use in chapter 4 when we prove the localization formulas in finite dimension.

Let us start by a short reminder of the notions of ordinary vector bundles, de Rham cohomology and characteristic classes.

On a manifold we can introduce differential forms and define the de Rham cohomology as closed differential forms modulo exact forms. The failure of closed forms to be exact tell us something about the sort of topology ("holes") we have on the manifold. As an example, on the plane all closed forms are exact if there are no holes present and the de Rham cohomology gives a tool to measure this.

A manifold which locally is a direct product of two topological spaces is called a fiber bundle (see figure 3.1). This is the resulting structure constructed by attaching fibers to every point of the manifold. If the fiber is a vector space then the fiber bundle is called a vector bundle and if the fiber is a group then it is called a principle bundle.

To measure the twisting, or non-triviality, of a fiber bundle one introduces characteristic classes. This is a way to assign a global invariant (a cohomology class of the manifold) to the principle bundle, written as an integral using the fiber bundle curvature.

This said, we will now discuss the scenario when there is a group acting on the manifold, and in particular how this changes notations, following [24].

3.1 CARTAN'S MODEL OF EQUIVARIANT COHOMOLOGY

There are many problems in theoretical physics where one not only has a manifold but an action of a Lie group (a symmetry) on this manifold. In these cases we can introduce equivariant cohomology (the generalization of cohomology including the group

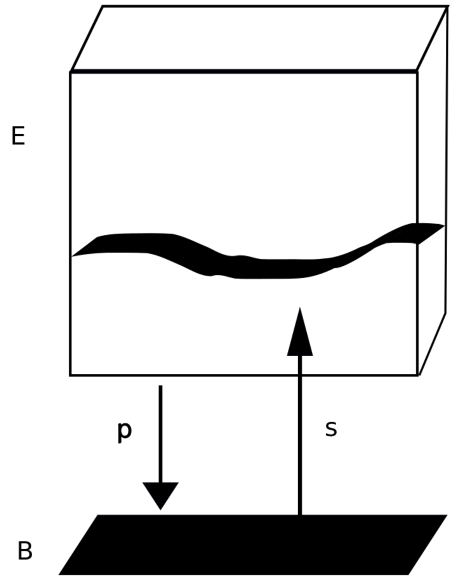

Figure 3.1: A fiber bundle $p : E \to B$ with base space B and total space E, that locally is a direct product of B and another topological spaces. A section s is a map from base space B to $s(B)$ of E.

action), which we will explain in what follows. There are different ways used to define equivariant cohomology but here we will use the Cartan model. The equivariant cohomology of M is then given by

$$H_G^*(X) = \ker D_{|\Lambda_G^k M} / \mathrm{im} D_{|\Lambda^{k-1}{}_G M} \quad , \tag{3.1}$$

which is the space of equivariantly closed forms ($D\alpha = 0$) modulo the space of equivariantly exact forms ($\alpha = D\beta$). We will explain this in the following.

When a differentiable manifold M is acted on by a group G it is denoted by

$$\begin{aligned} G \times M &\to M \\ (g, x) &\mapsto g \cdot x, \end{aligned} \tag{3.2}$$

with $e \cdot x = x \ \forall x \in M$ and $g_1 \cdot (g_2 \cdot x) = (g_1 \cdot g_2) \cdot x \ \forall g_1, g_2 \in G$. Often G is picked as the symmetry group of the physical problem. For example, in topological field theory (see chapter 5) it is common that the space of gauge connections is M (usually denoted $\mathcal{A}$), the group of gauge transformations is taken to be G

and M modulo G is called the moduli space (this will be used in chapter 5 and 6).

We assume that G is connected and has a smooth action on M. Given this action we denote the set of elements invariant of the group action by

$$M^G := \{x \in M | \forall g \in G, g \cdot x = x\}. \tag{3.3}$$

The power of G denotes the invariant part.

We now like to know the cohomology of M given the group action G. This is called the *equivariant cohomology* of M. We begin by the defining the space of orbits M/G (the orbit of an element $x \in X$ is the set of elements in X to which x can be moved by the elements of G given by $G.x = \{g.x \mid g \in G\}$). M/G is the set of equivalence classes (the equivalence class of an element a is the set $[a] = \{x \in X \mid a \sim x\}$), such that x and x' are equal iff $x' = g \cdot x$ for $g \in G$. If G acts freely on M ($g \cdot x = x$ iff g is the identity of G $\forall x \in M$) then M/G is a differential manifold and the equivariant cohomology is defined as

$$H^*_G(M) = H^*(M/G). \tag{3.4}$$

If there is a non-free action another way of defining the equivariant cohomology is needed. There are three different ways to do this. The three models are the Cartan model, the Weil model and the BRST model, which is a interpolation between the first two [25, 26]. These three ways of modeling $H^*_G(M)$ uses differential forms on M and polynomial functions and forms on the Lie algebra $\mathfrak{g}$ on G.

To write down the equivariant cohomology groups similar to the de Rham case we need to introduce equivariant differential forms. An equivariant differential form on M acted on by G is a polynomial map

$$\alpha \colon \mathfrak{g} \to \Lambda M, \tag{3.5}$$

from Lie algebra $\mathfrak{g}$ to the exterior algebra ΛM of differential forms on M. The equivariant differential forms are invariant under the G-action and thus the Lie derivative acting on the equivariant differential form is $\mathcal{L}_V \alpha = 0$. This is equivalent to say that α is an element of the G-invariant subalgebra

$$\Lambda_G M = (\mathrm{Sym}(\mathfrak{g}^*) \otimes \Lambda M)^G, \tag{3.6}$$

where G denotes the G-invariant part, $\mathfrak{g}^*$ is the dual vector space of $\mathfrak{g}$ and $\mathrm{Sym}(\mathfrak{g}^*)$ is the symmetric algebra over $\mathfrak{g}^*$.

Next we assign a $\mathbb{Z}$-grading (usually called ghost number in physical language) to the equivariant differential forms in (3.6). We can then define the *equivariant exterior derivative* D as the linear map

$$D\colon \Lambda^k_G M \to \Lambda^{k+1}_G M \tag{3.7}$$

on (3.6) by $D\phi^a = 0$ and $D\beta = (1 \otimes d - \phi^a \otimes i_{V^a})\beta$ for $\beta \in \Lambda M$. The degree of the equivariant form is two times its polynomial degree plus its form degree. The basis of $\mathfrak{g}^*$ is ϕ^a dual to T_a of g. The equivariant exterior derivative on an equivariant differential form is written as

$$(D\alpha)(X) = d(\alpha(X)) - i_V(\alpha(X)) \tag{3.8}$$

for $\alpha \in (\mathrm{Sym}(\mathfrak{g}^*) \otimes \Lambda M)^G$, V is the vector on M generated by Lie algebra element X, d is the exterior derivative and i_V is the interior product.

The square of the equivariant exterior derivative is given by the Lie derivative (see (2.36)). Thus $D^2\alpha = 0$ with $\alpha \in \Lambda_G M$. Finally we have reach the goal an can define the equivariant cohomology of M as

$$H^*_G(X) = \mathrm{ker} D_{|\Lambda^k_G M} / \mathrm{im} D_{|\Lambda^{k-1}{}_G M} \quad , \tag{3.9}$$

which is the space of equivariantly closed forms ($D\alpha = 0$) modulo the space of equivariantly exact forms ($\alpha = D\beta$).

Equivariant differential forms are of interest because when integrating over an equivariantly closed form one can evaluate the integral by summing over the fixed points of the action using localization techniques. This will be the object of the next chapter. Before going there we will conclude this chapter by introducing the equivariant Euler class. This class will be needed in the next chapter.

3.2 EQUIVARIANT VECTOR BUNDLES AND CHARACTERISTIC CLASSES

In this section we define equivariant vector bundles and generalize ordinary characteristic classes to the equivariant context (see for example [24]). These equivariant characteristic classes provide representatives of the equivariant cohomology and can be described by equivariant differential forms.

A fiber bundle $\pi : E \to M$ is called an *equivariant bundle* if there are group actions G on M and E such that π is an *equivariant map*, i.e.

$$g \cdot \pi(x) = \pi(g \cdot x) \quad \forall x \in E, \quad \forall g \in G. \tag{3.10}$$

The action of G on differential forms that has values in E is generated by the Lie derivatives $\mathcal{L}_V^a$ (see [24]).

As in the normal de Rham case we need to say how to connect the fibers if there are twists, which is done using a connection Γ. This object is defined over M and has values in E. The action of Γ on sections of the bundle gives the sections parallel transport along fibers. The covariant derivative, that generates the parallel transport, is given by

$$\nabla = d + \Gamma, \tag{3.11}$$

where d is the exterior derivative. This operator is a linear derivation and it associates to every section of the vector bundle a 1-form in $\Lambda^1(M, E)$. Let $x(t)$ be a path in M, then $(\nabla s)(\dot{x}(t)) = 0$ for $\dot{x}(t)$ a tangent vector along the path and s be the section. This gives the parallel transport along the path and let us connect different fibers of the bundle.

Let $E \to M$ be a equivariant vector bundle. Then we will assume that the covariant derivative is G-invariant, i.e.

$$[\nabla, \mathcal{L}_{V^a}] = 0. \tag{3.12}$$

Define the *equivariant covariant derivative* or equivariant connection as an operator on $\Lambda_G(M, E)$ (equivariant differential forms on M with values in E) by taking after (3.7) as

$$\nabla_{\mathfrak{g}} = 1 \otimes \nabla - \phi^a \otimes i_{V^a} \tag{3.13}$$

and define the *equivariant curvature* of the connection as

$$F_{\mathfrak{g}} = (\nabla_{\mathfrak{g}})^2 + \phi^a \otimes \mathcal{L}_{V^a}, \tag{3.14}$$

that satisfies $[\nabla_{\mathfrak{g}}, F_{\mathfrak{g}}] = 0$ and $F_{\mathfrak{g}}$ is an element of $\Lambda_G^2(M, E)$. These conditions reduce to the ordinary concepts for a vector bundle when G is trivial group.

Now we will introduce the equivariant characteristic classes. First recall that ordinary characteristic classes can be constructed using an invariant polynomial P on principal bundles with structure group H. For the equivariant case we can generalize this almost immediately. We now pick the curvature (3.14) invariant of the G-action as the argument of P. Then

$$DP(F_{\mathfrak{g}}) = rP(\nabla_{\mathfrak{g}} F_{\mathfrak{g}}) = 0, \tag{3.15}$$

where r is the degree of P. This implies that $P(F_{\mathfrak{g}})$ defines equivariant characteristic classes that are elements of the algebra $\Lambda_G M$. The equivariant cohomology class of $P_{\mathfrak{g}}(F)$ is connection independent.

3.2.1 *The Equivariant Euler Class*

In this subsection we define the equivariant Euler class which will be needed in the degenerate localization formula in section 4.3.

Let $E \to M$ be a real oriented equivariant vector bundle with a metric and a connection ∇ compatible with the metric, both invariant under the group action. Let $F_{\mathfrak{g}}$ be the equivariant extension of the curvature defined in (3.14). Then the equivariant Euler class is

$$e_{\mathfrak{g}}(F) = Pf(F_{\mathfrak{g}}) \tag{3.16}$$

which is an equivariantly closed form and its equivariant cohomology class depends only on the orientation of E.

4

LOCALIZATION IN FINITE DIMENSION

In this chapter we will first explain the localization principle using equivariant cohomology and then show the Berline-Vergne formula [8, 9], the Duistermaat-Heckman (DH) [6] formula and the localization formula for the degenerate case [24, 27].

4.1 LOCALIZATION PRINCIPLE

We will now start by explaining the localization principle. This is an application of equivariant cohomology (discussed in the previous chapter) which simplifies certain integrals as we will see in the following.

Assume that we want to integrate a closed equivariant differential form $\int \alpha$, $D\alpha = 0$, on a compact oriented manifold M without boundary with a G-action. α lies in the equivariant cohomology of M that we introduced in the previous chapter. Let $V = V^\mu \partial/\partial x^\mu$ be the vector field on M generated by the G-action that we will assume to be $G = U(1)$ for simplicity. The role of $\phi^a \in \mathrm{Sym}(\mathfrak{u}(1)^*)$ is not important here and we can "localize algebraically" by putting $\phi^a = -1$ in the equation for the equivariant exterior derivative (see chapter 3). The equivariant exterior derivative D then is

$$D = d + i_V = \theta^\mu \frac{\partial}{\partial x^\mu} + V^\mu \frac{\partial}{\partial \theta^\mu} \tag{4.1}$$

on

$$\Lambda_V M = \{\alpha \in \Lambda M : \mathcal{L}_V \alpha = 0\} \tag{4.2}$$

using (3.8) and (2.36) to get D.

It can be noticed (first shown by Atiyah and Bott [7] and Berline and Vergne [8, 9]) that the equivariant cohomology is determined by the fixed point set

$$M_V = \{x \in M | V(x) = 0\}. \tag{4.3}$$

This implies that as $\int_M \alpha$ depends only on the equivariant cohomology class of α (because $\int_M \alpha + D\lambda = \int_M \alpha + d\lambda + i_V\lambda = \int_M \alpha + \int_{\partial M} \lambda = \int_M \alpha$) it is determined by the fixed point set. This is the core of the localization theorems, both in finite dimension and in topological quantum field theory, and it is called *the equivariant localization principle*. We will now show the localization principle explicitly.

We start by picking an one form ω and a real positive number t such that

$$\int_{\Pi TM} d^n x d^n \theta \alpha = \int_{\Pi TM} d^n x d^n \theta \alpha e^{-tD\omega} \tag{4.4}$$

holds. This is so since

$$\begin{aligned} \frac{dZ(t)}{dt} &= -\int d^n x d^n \theta \alpha (D\omega) e^{-tD\omega} \\ &= -\int d^n x d^n \theta [D(\alpha \omega e^{-tD\omega}) - (D\alpha)\omega e^{-tD\omega} \\ &\qquad + \alpha (D^2 \omega) e^{-tD\omega}] = 0, \quad \text{if} \quad D^2\omega = 0. \end{aligned} \tag{4.5}$$

As $Z(t)$ is independent of t, given that we pick ω such that $D^2\omega = 0$, we can instead calculate

$$\lim_{t \to +\infty} \int d^n x d^n \theta \alpha(x, \theta) e^{-tD\omega(x,\theta)}. \tag{4.6}$$

Next we pick $\omega = g_{\mu\nu} \theta^\mu V^\nu(x)$, where $g = \frac{1}{2} g_{\mu\nu}(x) dx^\mu \otimes dx^\nu$ is the metric. Then

$$D\omega = \left(\theta^\mu \partial_\mu + V^\mu \frac{\partial}{\partial \theta^\mu}\right) g_{\alpha\beta} \theta^\alpha V^\beta = g_{\mu\beta} V^\mu V^\beta + \theta^\mu \left(\partial_\mu (g_{\alpha\beta}) V^\beta + g_{\alpha\beta} \partial_\mu V^\beta\right) \theta^\alpha. \tag{4.7}$$

The second derivative of ω gives

$$\begin{aligned} D^2\omega &= \left(\theta^\mu \partial_\mu V^\nu \frac{\partial}{\partial \theta^\nu} + V^\mu \partial_\mu\right) g_{\alpha\beta} \theta^\alpha V^\beta \\ &= \theta^\mu \partial_\mu V^\alpha g_{\alpha\beta} V^\beta + V^\mu \partial_\mu g_{\alpha\beta} \theta^\alpha V^\beta + V^\mu g_{\alpha\beta} \theta^\alpha \partial_\mu V^\beta \\ &= \theta^\alpha \partial_\alpha V^\mu g_{\mu\beta} V^\beta + V^\mu \partial_\mu g_{\alpha\beta} \theta^\alpha V^\beta + V^\beta g_{\alpha\mu} \theta^\alpha \partial \beta V^\mu \\ &= \left(\partial_\alpha V^\mu g_{\mu\beta} + V^\mu \partial_\mu g_{\alpha\beta} + g_{\alpha\mu} \partial_\beta V^\mu\right) \theta^\alpha V^\beta. \end{aligned} \tag{4.8}$$

For any compact manifold with a $U(1)$-action generated by V there exist an $U(1)$-invariant metric g satisfying the Killing equation (V is a Killing vector field of the metric g)

$$(\mathcal{L}_V g)_{\alpha\beta} = V^\mu \partial_\mu g_{\alpha\beta} + g_{\mu\beta} \partial_\alpha V^\mu + g_{\mu\alpha} \partial_\beta V^\mu = 0. \tag{4.9}$$

Comparing the two equations (4.8) and (4.9) we have

$$\mathcal{L}_V g = 0 \iff D^2\omega = 0. \qquad (4.10)$$

Equation (4.4) can be written out explicitly as

$$Z(t) = \int d^n x d^n \theta \alpha(x,\theta) e^{-t\left(g_{\mu\beta}V^\mu V^\beta + \theta^\mu[(\partial_\mu g_{\alpha\beta})V^\beta + g_{\alpha\beta}(\partial_\mu V^\beta)]\theta^\alpha\right)}. \qquad (4.11)$$

Then, as $t \to \infty$ only fixed points of the vector field, i.e. $V^\mu(x_i) = 0$, can contribute. This is the principle of localization. We will now continue by using this to prove the Berline-Vergne formula.

4.2 The Berline-Vergne Formula and the Symplectic Case

Theorem 4.1 (Berline-Vergne formula)**.** *Let M be a compact oriented boundary less even-dimensional manifold acted on by a $U(1)$-action. Let $V \in \Gamma(TM)$ be a vector field on M generated by the action and let $M_V = \{x \in M | V(x) = 0\}$ only consist of isolated points. Assume that α is a closed equivariant form then*

$$\int_M \alpha = \sum_{x_i \in M_V} (-2\pi)^{n/2} \frac{\alpha^{(0)}(x_i)}{Pf(\partial_\mu V^\nu(x_i))} \qquad (4.12)$$

Proof. To prove this formula we begin by expanding $V^\mu(x)$ and $g_{\alpha\beta}(x)$ around the fixed points.

In general,we have $T(x) = \sum_{\alpha|\geqslant 0} \frac{(x-x_i)^\alpha}{\alpha!}(\partial_\alpha f)(a)$. This gives

$$V^\mu(x) = V^\mu(x_i) + \partial_\nu V^\mu(x_i)(x-x_i)^\nu + ..., \qquad (4.13)$$

$$g_{\mu\nu}(x) = g_{\mu\nu}(x_i) + \partial_\alpha g_{\mu\nu}(x_i)(x-x_i)^\alpha + ... \qquad (4.14)$$

Expanding the terms we need to put in (4.11) gives

$$g_{\mu\nu}V^\mu V^\nu = g_{\mu\nu}(x_i)\partial_\alpha V^\nu(x_i)(x-x_i)^\alpha \partial_\beta V^\mu(x_i)(x-x_i)^\beta + ... \qquad (4.15)$$

$$\theta^\mu B_{\mu\alpha}\theta^\alpha = [(\partial_\mu g_{\alpha\beta}(x_i))V^\beta(x_i) + g_{\alpha\beta}(x_i)\partial_\mu V^\beta(x_i) + ...]\theta^\mu\theta^\alpha. \qquad (4.16)$$

Next we change the variables as

$$\tilde{x} = \sqrt{t}x \qquad (4.17)$$

$$\tilde{\theta} = \sqrt{t}\theta. \qquad (4.18)$$

Putting everything together we get the proof of the formula:

$$
\begin{aligned}
Z(t) = \lim_{t\to+\infty} \sum_{x_i\in M_v} &\int d^n\tilde{x}d^n\tilde{\theta}(\alpha^{(0)(x_i)} + ...)e^{-t[\frac{1}{t}g_{\mu\nu}(x_i)\partial_\alpha V^\nu(x_i)\partial_\beta V^\mu(x_i)(\tilde{x})^\alpha(\tilde{x})^\beta + \frac{1}{t}g_{\alpha\beta}(x_i)\partial_\mu V^\beta(x_i)\tilde{\theta^\mu}\tilde{\theta^\alpha} + ...]} \\
&= \sum_{x_i\in M_v} \alpha^{(0)}(x_i) \int d^n\tilde{x}e^{-g_{\mu\nu}(x_i)\partial_\alpha V^\nu(x_i)\partial_\beta V^\mu(x_i)(\tilde{x})^\alpha(\tilde{x})^\beta} \int d^n\tilde{\theta}e^{g_{\lambda\sigma}(x_i)\partial_\kappa V^\sigma(x_i)\tilde{\theta^\kappa}\tilde{\theta^\lambda}} \\
&= \sum_{x_i\in M_v} \alpha^{(0)}(x_i)\frac{\pi^{\frac{n}{2}}(-2)^{\frac{n}{2}}Pf(det(g_{\lambda\kappa}(x_i)\partial_\kappa V^\sigma(x_i)))}{\sqrt{det(g_{\mu\nu}(x_i)\partial_\alpha V^\nu(x_i)\partial_\beta V^\mu(x_i))}} \\
&= \sum_{x_i\in M_v} \alpha^{(0)}(x_i)\frac{(-2\pi)^{\frac{n}{2}}}{Pf(\partial_\mu V^\nu(x_i))},
\end{aligned}
\tag{4.19}
$$

where we in the third line used (2.18) for the Grassman coordinates and (2.17) for the even coordinates.

□

Example 4.1 (Area of S^2)**.** *As S^2 is a compact manifold with rotational symmetry around one axis we can use the the Berline-Vergne formula to calculate the area. By ordinary integration the area is calculated to be $\int_{S^2}\sin\phi d\phi d\varphi = 4\pi$. Now we instead want to find the equivariant extension of the volume form and make use of the Berline-Vergne formula to calculate the area.*

Let the U(1)-action rotate the sphere around its z-axis giving the vector field $V = \frac{\partial}{\partial\varphi}$. The equivariant extension can be written as a sum of a zero form and a two form, i.e. $\alpha = \alpha^{(2)} + \alpha^{(0)}$, where $D\alpha = 0$. We have

$$
\int_{\Pi S^2}\alpha^{(2)} + \alpha^{(0)} = \int_{\Pi S^2}\alpha^{(2)} = \int_{\Pi S^2}\sin\phi\theta^\varphi\theta^\phi = \int_{S^2}\sin\phi d\phi d\varphi,
\tag{4.20}
$$

as only the top form contributes in the Berezin integral.

Let us now find $\alpha^{(0)}$ using

$$
\begin{aligned}
0 = D\alpha &= (d + i_V)(\alpha^{(2)} + \alpha^{(0)}) &(4.21)\\
&= (\theta^\mu\partial_\mu + V^\mu\frac{\partial}{\partial\theta^\mu})(\sin\phi\theta^\varphi\theta^\phi + \alpha^{(0)}) &(4.22)\\
&= (\theta^\varphi\frac{\partial}{\partial\varphi} + \theta^\phi\frac{\partial}{\partial\phi} + \frac{\partial}{\partial\theta^\varphi})(\sin\phi\theta^\varphi\theta^\phi + \alpha^{(0)}) &(4.23)\\
&= \sin\phi\theta^\phi + d\alpha^{(0)}. &(4.24)
\end{aligned}
$$

This implies that $\alpha^{(0)} = \cos\phi$.

The sphere has two fixed points at $z = \pm 1$. *At these points the coordinate system is not well defined and we have to introduce local coordinates. Around* $z = 1$ *we have* $x = \cos\varphi$ *and* $y = \sin\varphi$. *This gives* $\frac{\partial}{\partial\varphi} = \frac{\partial x}{\partial\varphi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\varphi}\frac{\partial}{\partial y} = -\sin\varphi\frac{\partial}{\partial x} + \cos\varphi\frac{\partial}{\partial y} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. *Then*

$$\partial V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{4.25}$$

Around $z = -1$ *we have* $x = \cos\varphi$ *and* $y = -\sin\varphi$ *which in a similar way gives*

$$\partial V = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.26}$$

Now we can put everything in to calculate the area to be

$$\int_{\Pi S^2} \alpha = \sum_i (-2\pi)\frac{\alpha^{(0)}(x_i)}{Pf(\partial_\mu V^\nu(x_i))} = (-2\pi)(\frac{1}{-1} + \frac{-1}{1}) = 4\pi. \tag{4.27}$$

This might not be the most useful example but is shows how we can extend a form we want to integrate to a closed equivariant differential form and make use of the Berline-Vergne formula, which can be a great simplification.

We will now turn our attention to the partition function of classical statistical mechanics and how to simplify the calculations of these integrals by using the equivariant localization principle.

Assume that M is a compact symplectic manifold of dimension $2n$, with a symplectic form ω (for a review on symplectic geometry see [24]). Assume that M is acted symplectically on (i.e. the symplectic structure is preserved; $\mathcal{L}_V\omega = 0$) by a $U(1)$-action generated by a vector field V. If the action is Hamiltonian, i.e. there is a function H on M satisfying $DH = -i_V\omega$, then $D(H + \omega) = 0$. In local coordinates x^i if we write the symplectic form as

$$\omega = \frac{1}{2}\omega_{\mu\nu}dx^\mu \wedge dx^\nu \tag{4.28}$$

then

$$\partial_\mu H(x) = V^\nu(x)\omega_{\mu\nu}(x) \tag{4.29}$$

and

$$i_V\omega = V^i\omega_{ij}dx^j. \tag{4.30}$$

Equation (4.29) says that the critical point set M_V (where $V^\mu(x_i) = 0$) and the critical points of H coincide.

The volume form (or Liouville measure) is given by

$$\frac{\omega^n}{n!} = \sqrt{det\omega(x)}d^{2n}x. \tag{4.31}$$

The symplectic manifold is related to classical Hamiltonian mechanics through Darboux's theorem [28]. The theorem says that locally one can always find a coordinate system $(p_\mu, q^\mu)_{\mu=1}^n$ on M (called Darboux coordinates) such that

$$\omega = dp_\mu \wedge dq^\mu. \tag{4.32}$$

In classical statistical mechanics the partition function can then be written as

$$\int_M \frac{\omega^n}{n!}e^{-TH}. \tag{4.33}$$

Given this, we can write down the following formula which states that this integral can be calculated exactly as a sum over the critical points of H.

Theorem 4.2 (Duistermaat-Heckman theorem)**.** *Let M be a compact $2n$-dimensional symplectic manifold acted on symplectically by a $U(1)$-action generated by a vector field V. Assume that V generates a global Hamiltonian H given by (4.29) and that the critical points x_i of H are isolated and its Hessian matrix is non-degenerate. Then, assuming α is an closed equivariant form,*

$$\int \alpha = \sum_{x_i\in M_V} \left(\frac{-2\pi}{T}\right)^n \frac{e^{-TH(x_i)}}{Pf(\partial_\nu V(x_i))} \tag{4.34}$$

If the Hamiltonian is quadratic then

$$\int \alpha = \sum_{x_i\in M_V} \left(\frac{2\pi}{T}\right)^n \frac{\sqrt{det\omega(x_i)}}{\sqrt{det\partial^2 H(x_i)}}e^{-TH(x_i)}. \tag{4.35}$$

Proof. We begin by looking at the partition function we want to integrate

$$\int \alpha = \int_M d^{2n}x \sqrt{det\omega(x)} e^{-TH(x)} \tag{4.36}$$

$$= \sum_i \int d^{2n}x \sqrt{det\omega(x)} e^{-T(H(x_i)+\partial_\mu H(x_i)(x-x_i)^\mu + \frac{1}{2}\frac{\partial^2}{\partial x^\mu \partial x^\nu} H(x_i)(x-x_i)^\mu (x-x_i)^\nu + ...)} \tag{4.37}$$

$$\approx \sum_{x_i \in M_V} \left(\frac{2\pi}{T}\right)^n \frac{\sqrt{det\omega(x_i)}}{\sqrt{\det \partial^2 H(x_i)}} e^{-TH(x_i)}. \tag{4.38}$$

In the second line we have expanded the Hamiltonian around its fixed points and in the third line we used the saddle point approximation.

To be able to use the Berline-Vergne formula we have to find an equivariant extension of α that we call α', such that $D\alpha' = 0$. As we are working with a symplectic manifold we have $D(\omega + H) = 0$ and we can take $\alpha' = \frac{1}{T^n} e^{-T(w+H)}$.

Using the Berline-Vergne formula we directly get

$$\int \alpha' = \sum_i \left(\frac{-2\pi}{T}\right)^n \frac{e^{-TH(x_i)}}{Pf(\partial_\nu V(x_i))}. \tag{4.39}$$

If we have a quadratic Hamiltonian H we can write this slightly differently. The equivariant Darboux theorem [29] says that one can find local Darboux coordinates (p_μ, q^μ) such that $\omega = dp_\mu \wedge dq^\mu$ and that the origin can be located on the fixed point x_i. Then, the the action on M is locally given by

$$V = \sum_i \epsilon_i \left(-p_i \frac{\partial}{\partial q_i} + q_i \frac{\partial}{\partial p_i}\right), \tag{4.40}$$

with the weights ϵ_j. Then using (4.29) the quadratic Hamiltonian is written as $H = H(x_i) + \sum_i \epsilon_i \frac{p_i^2 + q_i^2}{2}$.

In the equivariant Darboux coordinates ω is the antidiagonal matrix

$$\omega = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & \ddots \end{pmatrix} \tag{4.41}$$

and the Hessian of H is given by

$$\partial_\nu V = \omega^{-1}\partial^2 H = \begin{pmatrix} 0 & \epsilon_i & \\ -\epsilon_i & 0 & \\ & & \ddots \end{pmatrix}. \tag{4.42}$$

Then we have $\partial_\nu V = (-1)^n \epsilon_1 ... \epsilon_n$. This implies that one can write

$$Z(T) = \sum_i \left(\frac{2\pi}{T}\right)^n \frac{\sqrt{det\omega(x_i)}}{\sqrt{\det\partial^2 H(x_i)}} e^{-TH(x_i)}. \tag{4.43}$$

□

4.3 DEGENERATE SYSTEMS

As a next generalization we will assume a degenerate vector field. In this case the set of fixed points become a submanifold $M_V \subset M$. M is again a compact manifold acted on by a $U(1)$-action. We still want to calculate

$$\int_M \alpha = \int_M \alpha e^{-tD\omega} = \int_M \alpha e^{-t(g_{AB}V^A V^B + (\Omega)_{AB}\theta^A\theta^B)} \tag{4.44}$$

with $D\alpha = 0$ and

$$(\Omega)_{AB} = \frac{1}{2}[\partial_A(g_{BC}V^C) - \partial_B(g_{AC}V^C)], \tag{4.45}$$

where $A, B, C = 1, \ldots, \dim M$. To simplify calculations we will have to introduce normal coordinates.

Before we go on with the theory we will give an example of where the critical points becomes a submanifold. This is the case of the height function on the torus, given that the torus is laying flat on the xy-plane in three dimensional space. This function has two extrema which are cirlcles (given a rotation around the z-axis). The set of fixed points consist of $S^1 \sqcup S^1$.

Having a degenerate system we can decompose the manifold M as $M = M_V \sqcup N_V$, where N_V is normal to M_V. Let $\dim M_V = m$ and $\dim M = n$. Locally on M we have coordinates

$$(x^1, \ldots, x^m, x^{m+1}, \ldots, x^n). \tag{4.46}$$

We will let the latin indices represent points on M_V, greek indices represent points on N_V and capital latin letters run from $1, \ldots, n$, i.e.

$$x = (x^A) = (x^i, x^\alpha), \quad x^\alpha = 0 \quad \text{if} \quad x \in M_V. \tag{4.47}$$

The tangent space $T_x M$ is decomposed as

$$T_x M = T_x M_V \oplus (T_x M_V)^\perp, \tag{4.48}$$

where $\{\partial/\partial x^i\}$ spans $T_x M_V$ and $\{\partial/\partial x^\alpha\}$ spans $(T_x M_V)^\perp$. The Grassman variables (generating the exterior algebra of M) can then be decomposed as

$$\theta = (\theta^A) = (\theta^i, \theta^\alpha). \tag{4.49}$$

The vector field $V = V^A(x^i, x^\alpha)\frac{\partial}{\partial x^A}$ satisfies

$$\begin{aligned} V^A(x^i, 0) &= 0, \\ \partial_i V^A(x^i, x^\alpha) &= 0. \end{aligned} \tag{4.50}$$

The tangent bundle, with the connection ∇, is an equivariant vector bundle. The Lie derivative acts non-trivially on the fibers as $\mathcal{L}_V = V^\mu \partial_\mu + dV^\mu i_V^\mu - dV$ and the moment map of the equivariant bundle is the Riemann moment map $\mu_V = \nabla V$ [24].

By (2.36), the moment map can be written in terms of Ω defined in (4.45) as $(\Omega)_{\mu\nu} = 2 g_{\mu\lambda} (\mu_V)^\lambda_\nu$. The equivariant curvature R_V of the bundle can be written as

$$R_V = R + \mu_V. \tag{4.51}$$

The 2-form Riemann curvature of the bundle is [24]

$$R^\mu_\nu = \frac{1}{2} R^\mu_{\nu\lambda\rho} \theta^\lambda \theta^\rho, \tag{4.52}$$

where $R^\mu_{\nu\lambda\rho}$ is the Riemann curvature tensor.

Let $p \in M_V$ and (x^i, x^α) be normal coordinates (see [31]) around p, then the metric g can be written as

$$g_{\mu\nu} = \delta_{\mu\nu} - \frac{1}{3} R_{\mu\rho\nu\sigma} x^\rho x^\sigma + O(x^3). \tag{4.53}$$

Using normal coordinates the metric and the curvature tensor satisfies [31]

$$\begin{aligned} \partial_C g_{AB}|_{(x^i,0)} &= 0, \\ R_{ABCD}(x^i,0) &= g_{AD,BC}(x^i,0) - g_{AC,BD}(x^i,0), \end{aligned} \tag{4.54}$$

where $g_{AD,BC} = \frac{\partial^2 g_{AB}}{\partial x^B \partial x^C}$.

With these notations we can write down a localization formula for the degenerate system.

Theorem 4.3 (The Degenerate Formula). *Let M be a compact manifold acted on by a $U(1)$-action. Assume that α is a closed equivariant differential form. Let r be the codimension of the critical point set M_V and $e_V(R)$ the equivariant Euler class of the normal bundle N_V. Let M_V have local coordinates x^i $(i = 1, \dots, m)$. Then*

$$\int_M \alpha = \int_{M_V \otimes \Lambda^1 M_V} dx^i d\theta^i \alpha(x^i, \theta^i) \frac{(-2\pi)^{r/2}}{e_V(R)}. \tag{4.55}$$

Proof. To prove this theorem we need to know the t-dependence of the terms in the exponent

$$e^{-tD\omega} = e^{-t(g_{AB}V^A V^B + (\Omega)_{AB}\theta^A\theta^B)}, \tag{4.56}$$

with $(\Omega)_{AB}$ given by (4.45).

First we expand $D\omega$ around $(x^i, 0)$ then do a change of variables as

$$\begin{aligned} x^\alpha &\to \sqrt{t} x^\alpha \\ \theta^\alpha &\to \sqrt{t}\theta^\alpha. \end{aligned} \tag{4.57}$$

The terms of concern are:

$$\begin{aligned} \text{Prop. to t} &: \theta^i (\Omega)_{ij} \theta^j \\ \text{Prop. to } \sqrt{t} &: \theta^i x^\alpha \partial_\alpha((\Omega)_{ij})\theta^j + \theta^\alpha (\Omega)_{\alpha i}\theta^i + \theta^i (\Omega)_{i\alpha}\theta^\alpha \\ \text{O}(1) &: \frac{1}{2}\theta^i\theta^j x^\alpha x^\beta \partial_\alpha \partial_\beta((\Omega)_{ij}) + \theta^\alpha V^\beta \partial_\beta((\Omega)_{\alpha i})\theta^i \\ &\quad + \theta^\alpha (\Omega)_{\alpha\beta}\theta^\beta + g_{\mu\nu} x^\alpha x^\beta \partial_\alpha V^\mu \partial_\beta V^\nu \end{aligned}$$

The term proportional to t is zero by (4.50). Next look at the first term proportional to $\sqrt{t}$

$$\begin{aligned}
\theta^i x^\alpha \partial_\alpha((\Omega)_{ij})\theta^j &= \frac{1}{2}\partial_\alpha[\partial_i(g_{jA}V^A) - \partial_j(g_{iA}V^A)]x^\alpha\theta^i\theta^j \\
&= \frac{1}{2}\partial_\alpha[\partial_i(g_{j\rho})V^\rho + g_{j\rho}\partial_i(V^\rho) - \partial_j(g_{i\rho})V^\rho - g_{i\rho}\partial_j(V^\rho)]x^\alpha\theta^i\theta^j \\
&= \frac{1}{2}[\partial_\alpha(\partial_i(g_{j\rho}))V^\rho + (\partial_i(g_{j\rho}))\partial_\alpha(V^\rho) + \partial_\alpha(g_{j\rho})\partial_i V^\rho \\
&+ g_{j\rho}\partial_\alpha\partial_i V^\rho - \partial_\alpha(\partial_j(g_{i\rho}))V^\rho - \partial_j(g_{i\rho})\partial_\alpha V^\rho \\
&- (\partial_\alpha g_{i\rho})\partial_j(V^\rho) - g_{i\rho})\partial_\alpha\partial_j(V^\rho)]x^\alpha\theta^i\theta^j \\
&= 0,
\end{aligned} \tag{4.58}$$

where we used (4.50). The other terms proportional to $\sqrt{t}$ are also zero by (4.50).

Of the terms of order one the second term is zero by inspection (using the calculations above). The first term is

$$\begin{aligned}
\frac{1}{2}\theta^i\theta^j x^\alpha x^\beta \partial_\alpha\partial_\beta((\Omega)_{ij}) &= \frac{1}{4}\theta^i\theta^j V^\alpha V^\beta \partial_\alpha\partial_\beta[\partial_i(g_{jA}V^A) - \partial_j(g_{iA}V^A)]x^\alpha\theta^i\theta^j \\
&= \frac{1}{4}\theta^i\theta^j x^\alpha x^\beta[(\partial_\beta(\partial_i g_{j\rho} - \partial_\beta\partial_j g_{i\rho})\partial_\alpha V^\rho \\
&+ (\partial_\alpha\partial_i g_{j\rho} - \partial_\alpha\partial_j g_{i\rho})\partial_\beta(V^\rho)] \\
&= \frac{1}{4}\theta^i\theta^j x^\alpha x^\beta[R_{\beta\rho ji}\partial_\alpha(V^\rho) + R_{\alpha\rho ji}\partial_\beta(V^\rho)] \\
&= \frac{1}{2}\theta^i\theta^j x^\alpha x^\beta R_{\beta\rho ji}\partial_\alpha(V^\rho) \\
&= \frac{1}{2}\theta^i\theta^j x^\alpha x^\beta R_{\rho\beta ij}g^{\rho\nu}g_{\nu\lambda}\partial_\alpha(V^\lambda) \\
&= \frac{1}{2}\theta^i\theta^j x^\alpha x^\beta R^\rho_{\beta ij}(\Omega)_{\alpha\rho},
\end{aligned} \tag{4.59}$$

where we in the second line used (4.50), in the third line we used (4.54), in the fourth line used the symmetry of the indices of the Riemann curvature tensor and in the last line we used (4.45).

Taking the limit $\lim_{t\to+\infty}$ of $\int_M \alpha e^{-tD\omega}$ we have:

$$\int_{M\otimes\Lambda^1 M=(M_v\otimes\Lambda^1 M_v)\coprod(N_\perp\otimes\Lambda^1 N^\perp)} dx^i d\theta^i dx^\alpha d\theta^\alpha \alpha(x^i,\theta^i) e^{-(\Omega)_{\alpha\nu}(\Omega)^\nu_\beta x^\alpha x^\beta+\frac{1}{2}\theta^i\theta^j x^\alpha x^\beta R^\rho_{\beta ij}(\Omega)_{\alpha\rho}+(\Omega)_{\alpha\beta}\theta^\alpha\theta^\beta} \tag{4.60}$$

$$= \int_{M_V\otimes\Lambda^1 M_V} dx^i d\theta^i \alpha(x^i,\theta^i)\frac{(-2\pi)^{r/2} Pf((\Omega)_{\alpha\beta})}{\sqrt{det((\Omega)_{\alpha\nu}+\frac{1}{2}R^\rho_{\beta ij}(\Omega)_{\alpha\rho}\theta^i\theta^j))}} \tag{4.61}$$

$$= \int dx^i d\theta^i \alpha(x^i,\theta^i)\frac{(-2\pi)^{r/2}}{Pf(\partial_\nu V^\mu+\frac{r}{2}R^\rho_{\beta ij}(\Omega)_{\alpha\rho}\theta^i\theta^j)} \tag{4.62}$$

$$= \int dx^i d\theta^i \alpha(x^i,\theta^i)\frac{(-2\pi)^{r/2}}{e_V(R)}, \tag{4.63}$$

where $r = codim(M)$ and $e_V(R)$ is the equivariant Euler class of the normal bundle defined in (3.16). In the first line we used (4.45) to rewrite the last term of order one in terms of Ω. To get second line we used (2.18) for the Grassman coordinates and (2.17) for the even ones. In the last line we used (4.52) and (4.51). □

5 TOPOLOGICAL QUANTUM FIELD THEORY AND GAUGE THEORY

We will now leave the finite dimensional setup and turn our attention to how the localization principle can be used in the infinite dimensional case (quantum field theory). To do so, further concepts have to be introduced that we will review in this chapter. We will discuss basic notions of topological field theory (TQFT) [32], non-abelian gauge theories [40] and how to compute the partition function of two dimensional Yang-Mills theory by using a lattice gauge regularization [33].

5.1 TOPOLOGICAL QUANTUM FIELD THEORY

In this section we review the general notions of TQFT following [32].

The study of TQFT started in the 1980s (the first TQFT was formulated in 1988 [35]) and was a new link between physics and mathematics. We will here study the TQFT from the viewpoint of the functional integral.

Let X be Riemannian manifold (real smooth manifold with inner product on the tangent space) endowed with a metric $g_{\mu\nu}$ and consider a quantum field theory defined over X. Let $\{\phi_i\}$ be a set of fields on X, with which the action $S(\phi_i)$ of the theory can be constructed. Let $\mathcal{O}_\alpha(\phi_i)$ be operators (arbitrary functionals of the fields). Then the vacuum expectation value of such operators is

$$\langle \mathcal{O}_{\alpha_1} \cdots \mathcal{O}_{\alpha_p} \rangle = \int [D\phi_i] \mathcal{O}_{\alpha_1}(\phi_i) \cdots \mathcal{O}_{\alpha_p}(\phi_i) e^{-S(\phi_i)}. \quad (5.1)$$

We can now define a TQFT as a quantum field theory with a set of operators, also known as topological observables, that has correlation functions independent of the metric, i.e,

$$\frac{\delta}{\delta g_{\mu\nu}} \langle \mathcal{O}_{\alpha_1} \cdots \mathcal{O}_{\alpha_p} \rangle = 0. \quad (5.2)$$

This condition can be realized in two ways. The first way is when both the action and the operators are independent of the metric. These are called TQFT's of Schwarz type [36], for example Chern-Simons gauge theory. In the second case the observables and the action can be dependent on the metric. However, the theory has a symmetry given by an operator Q (odd and nilpotent) that leaves the correlation functions independent of the metric. In this theory the observables are in the cohomology of Q. This second type is called cohomological TQFT's or TQFT's of Witten type [37]. One example is Donaldson-Witten theory [37].

Example 5.1 (TQFT of Schwarz type; Chern-Simons theory). The Chern-Simons theory is defined on a differentiable compact 3-manifold M with a simple compact gauge group G. The action is

$$S_{CS} = \int Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A), \tag{5.3}$$

where A is a gauge connection corresponding to G.

As it is a gauge invariant theory the operators have to be invariant under gauge transformations. Observables are written from Wilson loop

$$Tr_R(Hol_\gamma(A)) = Tr_R P \exp \int_\gamma A, \tag{5.4}$$

where the path-ordering operator is denoted by P, Tr_R is the trace of the holonomy (see figure 5.1) of A in representation R and γ is a 1-cycle.

5.1.1 *The Cohomological Type and the Mathai-Quillen Formalism*

In this subsection the second type of TQFT's mentioned above will be described in more detail.

To assure that (5.2) is satisfied in a cohomological TQFT one uses the symmetry of the theory (with infinitesimal form δ). Symmetry transformation of the fields are done so that the action and the operators are invariant under the symmetry, leaving the correlation functions independent of the metric. This is shown in what follows.

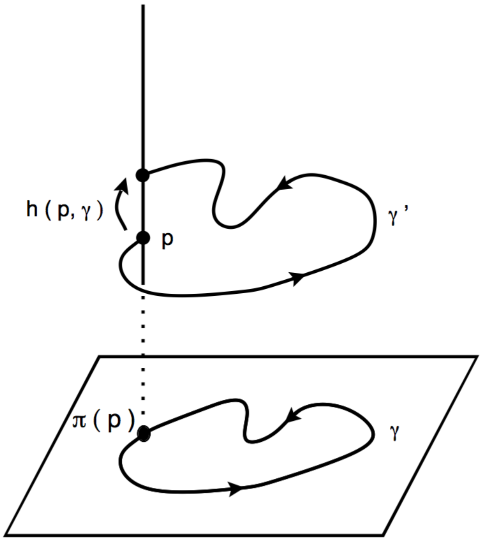

Figure 5.1: Holonomy [18]. If γ' isn't closed we have holonomy.

Assume δ satisfies

$$\begin{aligned} \delta \mathcal{O}_{\alpha_1}(\phi_i) &= 0, \\ T_{\mu\nu}(\phi_i) &= \delta G_{\mu\nu}(\phi_i), \end{aligned} \tag{5.5}$$

where $G_{\mu\nu}(\phi_i)$ is a tensor and $T_{\mu\nu}(\phi_i) = \frac{\delta}{\delta g^{\mu\nu}} S(\phi_i)$ is the energy-momentum tensor. These conditions gives the relation

$$\begin{aligned} \frac{\delta}{\delta g_{\mu\nu}} \langle \mathcal{O}_{\alpha_1} \cdots \mathcal{O}_{\alpha_p} \rangle &= - \int [D\phi_i] \mathcal{O}_{\alpha_1}(\phi_i) \cdots \mathcal{O}_{\alpha_p}(\phi_i) T_{\mu\nu} e^{-S(\phi_i)} \\ &= - \int [D\phi_i] \delta \left(\mathcal{O}_{\alpha_1}(\phi_i) \cdots \mathcal{O}_{\alpha_p}(\phi_i) G_{\mu\nu} e^{-S(\phi_i)} \right) \\ &= 0, \end{aligned} \tag{5.6}$$

assuming that the measure is invariant under δ and the observables are independent of the metric. This says that the theory is to be considered topological.

Cohomological TQFT's often satisfy

$$S(\phi_i) = \delta \Lambda(\phi_i), \tag{5.7}$$

$\Lambda(\phi_i)$ being some functional. This implies that the partition function and topological observables in general are independent of the coupling constant [32].

In a theory where (5.5) holds it is possible to construct correlation functions which correspond to topological invariants (invariant under deformations of the metric) by looking at operators that are invariant under the symmetry δ. Looking back at (5.6) we can see that if one of the operators can be written as $\delta\Gamma$ then it will make the correlation function vanish. Thus, we can identify operators differing by some $\delta\Gamma$ and having $\delta^2 = 0$ (and $\mathcal{O} \neq \delta\chi$). These operators, invariant under the symmetry, are the observables of the theory. It can be shown that δ^2 is a gauge transformation (see below), which means that the analysis is restricted to gauge invariant operators, a natural demand. Let Q be the operator giving this symmetry, then the observables of the theory are in the cohomology of Q (a state $|a> = Q|b>$ has zero norm as $<a|a> = <b|Q^2|b> = 0$).

The observables of the theory can be built out of gauge invariant polynomials in the fields $\phi(x)$ as $[Q, \phi(x)] = 0$. To construct other fields the topological descent equations

$$d\phi^{(n)} = i[Q, \phi^{(n+1)}] \tag{5.8}$$

can be used. From this the observables of the theory can be built;

$$W_\phi^{\gamma_n} = \int_{\gamma_n} \phi^{(n)}. \tag{5.9}$$

$W_\phi^{\gamma_n}$ is an observable as it is invariant under Q. If the cycle is the boundary of a surface then the observable is Q-exact and it correlation function will vanish. $W_\phi^{\gamma_n}$ depends only on the homology class of the cycle.

We now introduce shortly the Mathai-Quillen formalism, a more mathematical approach to TQFT introduced in [32, 41]. The cohomological TQFT's can be described by fields ϕ^i, equations $s(\phi)$ (s is a generic section) and symmetries. In general, the fields (elements of configuration space X and defined on a Riemann manifold) are acted on by a group G, for example a gauge symmetry group, and its natural to consider the quotient space of X modulo G. A subset of this space, called the *moduli space* (see figure 5.2), is

$$\mathcal{M} = Z(s)/G, \tag{5.10}$$

with $Z(s) = \{\phi^i \in X | s(\phi) = 0\}$. The symmetry δ gives a representation of the equivariant cohomology on the space of fields.

The moduli space is of great importance as the path integral localizes onto this subset, a fact that we will make use of in chapter 6 and 7 (compare (4.3) in chapter 4). We note that topological quantum field theories of this kind can be used to study the geometry of moduli spaces and from a mathematical viewpoint this is the study of intersection theory (intersection of two subspaces inside some space) on moduli spaces.

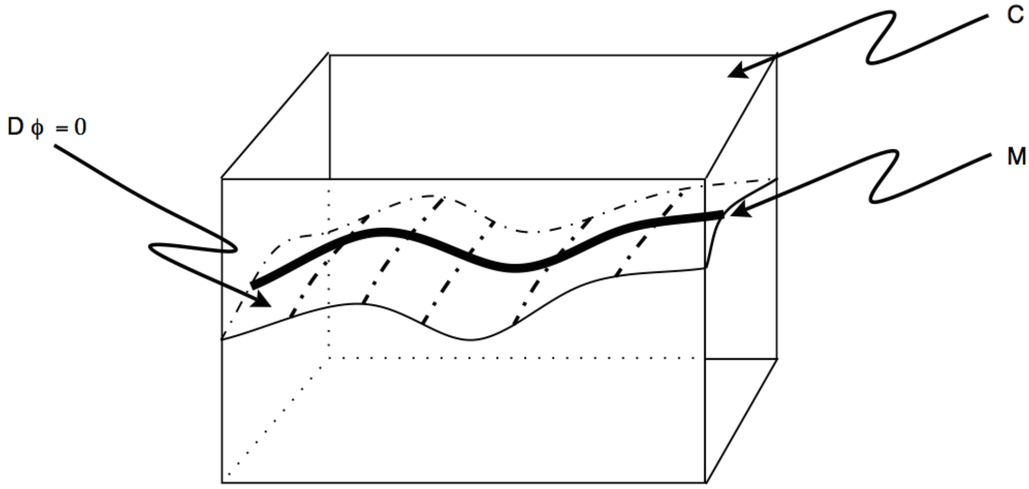

Figure 5.2: The moduli space where the space of fields is denoted C, $M = \{\phi \in C | D\phi = 0\}/G$ is the moduli space [18].

5.2 POINCARÉ DUALITY

In topological quantum field theories the correlation functions are related to topological invariants. For example, in three dimensions the correlation functions can be related to link invariants (a link is a union of knots). There are different ways to assign invariants to a knot. The linking number is computed from labeling each crossing as positive or negative and then adding the signs of the crossings. One can also assign a polynomial to a knot where the coefficients of the polynomial contain some properties of the knot.

In [38] Witten showed that in the topological quantum field theory Chern-Simons theory expectation values, constructed out of

Wilson loops, are given by knot polynomials (Jones polynomials of knot theory). Thus link invariants can be related to physical observables. The way of relating the integral of the wedge product of differential forms to homology cycles is done through the Poincaré duality.

Let M be a n-dimensional closed oriented smooth manifold. The Poincaré duality [39] relates the k-th cohomology group with the $(n-k)$-th homology group on the manifold as

$$H^k(M) \simeq H_{n-k}(M), \quad k \in \mathbb{Z}. \tag{5.11}$$

Thus, a differential form can be represented by a homology cycle and the wedge product of differential forms are the intersection numbers of homology cycles.

On the compact smooth manifold M of dimension 2n the intersection form is defined as a bilinear form

$$\lambda_M : H^n(M) \times H^n(M) \to \mathbb{R} \tag{5.12}$$

defined by

$$\lambda_M([\alpha],[\beta]) = \int_M \alpha \wedge \beta. \tag{5.13}$$

This is an important topological invariant. It can be thought of geometrically by Poincaré duality. Choose Poincaré duals of α and β to be submanifolds A and B then $\lambda_M(a,b)$ is the oriented intersection number of A and B.

We will see in chapter 6 how we can write general expressions for intersection numbers on moduli spaces of flat connections.

5.3 YANG-MILLS THEORY

In this section we will review Yang-Mills theory, which is a non-abelian gauge theory [40] with the gauge group given by $SU(N)$ or in general any compact, semi-simple Lie group (semi-simple means that it has a Lie algebra that is a semi-simple, i.e. a direct sum of simple Lie algebras. Simple non-abelian Lie algebras has only ideals that are $\{0\}$ and itself).

The two dimensional Yang-Mills theory (which is the object of interest for us) is a super-renormalizable quantum field theory that can be solved exactly and one can use it to study structures of more complicated non-abelian gauge theories, for example higher-dimensional cohomological field theories or physical models such as four dimensional quantum chromodynamics (QCD).

In Yang-Mills theory the gauge fields are the Yang-Mills fields $A^a_\mu(x)$. The symmetries are gauge symmetries and the action on a Riemann surface Σ (a complex one dimensional surface) of genus g is

$$S_{YM} = -\frac{1}{2e^2}\int_\Sigma Tr(F \wedge \star F), \tag{5.14}$$

where a gauge connection of a trivial principle bundle over Σ is denoted A, $F = dA + A \wedge A$ is the field strength, e^2 is the coupling constant and $\star$ is the Hodge star operator. The action is invariant under gauge transformations $A \to g^{-1}Ag + g^{-1}dg$, $g : \Sigma \to G$ for g in Lie algebra $\mathfrak{g}$ and Lie group G.

To quantize Yang-Mills theory one introduces the path integral

$$Z = \int_\mathcal{A} DAe^{-S_{YM}} \tag{5.15}$$

over the space of all connections. Because of the gauge invariance we like to divide the result by the gauge symmetries as we have a range of equivalent solutions that are related by gauge transformations. This is however ill-defined and instead one fixes a gauge and restrict integration to the space of all connections modulo gauge transformations [42]. This is done by adding some additional fields, called ghost fields or Faddeev-Popov ghosts, that break the gauge symmetry. These fields will not correspond to any physical particles.

The mentioned procedure has to do with integrals of the form $\int dxdye^{-S}$. Here the integral over y is redundant and we can define Z as

$$Z := \int dxe^{-S} = \int dxdy\delta ye^{-S}. \tag{5.16}$$

Without changing the result, the argument of the delta function can be shifted as $Z = \int dxdy\delta(y - f(x))e^{-S}$. Assume that we are told that $y = f(x)$ is the solution of $\mathcal{G}(x, y) = 0$ (for fixed x). Then we can substitute the delta-function in Z using the relation

$$\delta(\mathcal{G}(x, y)) = \frac{\delta(y - f(x))}{|\partial\mathcal{G}/\partial y|}. \tag{5.17}$$

Generalizing to integrate over $d^n x d^n y$, we get

$$Z = \int d^n x d^n y \det\left(\frac{\partial \mathcal{G}_i}{\partial y_i}\right) \Pi_i \delta(\mathcal{G}_i) e^{-S}, \tag{5.18}$$

assuming $\frac{\partial \mathcal{G}_i}{\partial y_i}$ is positive.

Now we can use the result to path integrals over non-abelian gauge fields. We let the redundant integration variable y be the set of all gauge transformations $\xi^a(x)$ and the integration variables x and y together is changed for the the gauge field $A^a_\mu(x)$. $\mathcal{G}$ is the gauge-fixing function (that one picks for ones purpose). Then (index a plays the role of the index i),

$$Z = \int DA \det\left(\frac{\partial \mathcal{G}}{\partial \xi}\right) \Pi_{x,a} \delta(\mathcal{G}) e^{-S_{YM}}. \tag{5.19}$$

To evaluate the functional derivative $\partial \mathcal{G}^a / \partial \xi^b$ one has to choose a particular gauge-fixing function. Its functional determinant can be written as a path integral over complex Grassmann variables. Let $c^a(x)$ and $\bar{c}^a(x)$ (the hermitian conjugate) be complex Grassmann fields. These are the Faddeev-Popov ghosts. We can write

$$\det\left(\frac{\partial \mathcal{G}^a}{\partial \xi^b}\right) \propto \int Dc D\bar{c} e^{-S_{gh}}. \tag{5.20}$$

We can also write $\delta(\mathcal{G})$ as an integral for the gauge fixing (gf), leaving

$$Z \propto \int DA Dc D\bar{c} e^{-S_{YM} - S_{gh} - S_{gf}}, \tag{5.21}$$

i.e. we can instead integrate also over the ghost fields to get rid of the gauge redundancy.

The gauge-fixed path integral can be re-derived in a different way because there is another symmetry of the Lagrangian in play, called the BRST symmetry [24, 42]. BRST quantization was first introduced in quantization of Yang-Mills theory used to prove the re-normalizability of four dimensional non-abelian gauge theories. It can also show which the physical particles of the theory are. We will now outline the BRST quantization procedure.

Let $A^a_\mu(x)$ be the gauge field in the non-abelian gauge theory and let $\phi_i(x)$ be a scalar or spinor field in representation R. The infinitesimal gauge transformations are

$$\begin{aligned} \delta A^a_\mu(x) &= -D^{ab}_\mu \xi^b(x), \\ \delta\phi_i(x) &= -ig\xi^a(x)(T^a_R)_{ij}\phi_j(x), \end{aligned} \tag{5.22}$$

with the gauge transformation parametrized by $\xi^a(x)$ and where D^{ab}_μ is the covariant derivative and T^a_R is the generator matrices of the representation. We can now introduce $c^a(x)$ being a scalar Grassman field in the adjoint representation. Then we can define a infinitesimal BRST transformation

$$\begin{aligned} \delta_B A^a_\mu(x) &= D^{ab}_\mu c^b(x), \\ \delta_B\phi_i(x) &= -igc^a(x)(T^a_R)_{ij}\phi_j(x), \end{aligned} \tag{5.23}$$

which is a infinitesimal gauge transformation with $-\xi^a(x) \to c^a(x)$. Then gauge invariance implies BRST invariance. In particular the Yang-Mills lagrangian is BRST invariant, i.e. $\delta_B L$. We will restrict the BRST transformation by requiring $\delta_B^2 = 0$, which determines the ghost field transformation to be $\delta_B c^c(x) = -\frac{1}{2}gf^{abc}c^a(x)c^b(x)$ (look at $\delta_B(\delta_B\phi_i) = 0$ to get the transformation. $\delta_B(\delta_B A^a_\mu)$ will is also be zero then) [42].

Now we introduce an antighost field that we call $\bar{c}^a(x)$ (independent of $c^a(x)$) with the BRST transformation $\delta_B\bar{c}^a(x) = B^a(x)$, with the commuting scalar field $B^a(x)$. Let us now add to our BRST invariant Yang-Mills lagrangian another term $\delta_B\mathcal{O}$, which corresponds to fixing a gauge (the gauge depends on our choice of $\mathcal{O}$), using our introduced fields. We then end up with the same terms L_{gh} and L_{gf} as in the procedure above. Ghost number conservation is one symmetry of the new lagrangian, which means that one assigns ghost number $+1$ to c^a and -1 to $\bar{c}^a$. All other fields have zero ghost number.

From the infinitesimal BRST transformation we can define the BRST charge (an operator giving the symmetry) as

$$Q_B = \int d^3x j^0_B(x), \tag{5.24}$$

where $j_B^\mu(x)$ is the associated Noether current. The BRST charge generates a BRST transformation

$$\begin{aligned} i[Q_B, A^a(x)] &= D_\mu^{ab} c^b(x), \\ i[Q_B, \phi_i(x)]_\pm &= igc^a(x)(T_R^a)_{ij}\phi_j(x), \\ i\{Q_B, c^a(x)\} &= -\frac{1}{2} g f^{abc} c^b(x) c^c(x), \\ i\{Q_B, \bar{c}^a(x)\} &= B^a(x), \\ i[Q_B, B^a(x)] &= 0, \end{aligned} \tag{5.25}$$

where f^{abc} are the structure coefficients. If ϕ_i is a scalar field $[]_\pm$ the commutator and if ϕ_i is a spinor field then $[]_\pm$ is the anticommutator (i.e. $[A, B] = AB - BA$). As $Q_B^2 = 0$ all physical states lie in the cohomology of Q_B.

We have now seen the Fadeev-Popov-BRST procedure to get the measure on $\mathcal{A}/G$, which introduces ghost field c^a transforming as (5.23). The gauge fixing term is $\delta_B \mathcal{O}$ for some $\mathcal{O}$.

For our purposes (see chapter 6) we will introduce $A_\mu^a(x)$ of ghost number zero, ghosts $\psi_\mu^a(x)$ of ghost number one with opposite statistics and fields $\phi^a(x)$ with ghost number two with quantum numbers of the generators of the symmetry group. Then, in the Yang-Mills theory, the anticommuting BRST symmetry Q has transformation rules given by

$$\begin{aligned} [Q, A_\mu^a] &= \psi_\mu^a(x), \\ \{Q, \psi_\mu^a\} &= -D_\mu \phi^a, \\ [Q, \phi^a] &= 0, \end{aligned} \tag{5.26}$$

where $Q^2 = 0$ on gauge invariant fields ($[Q^2, A_\mu^a] = \{Q, [Q, A_\mu^a(x)]\} = -D_\mu \phi^a$, which is an infinitesimal gauge transformation generated by gauge parameter ϕ^a of A_μ^a).

More abstractly the transformations in (5.26) can be written (with $\delta_\phi^{gauge}(field)$ being the infinitesimal gauge transformation generated by ϕ) as

$$\begin{aligned} [Q, field] &= ghost, \\ \{Q, ghost\} &= \delta_\phi^{gauge}(field), \\ [Q, \phi] &= 0. \end{aligned} \tag{5.27}$$

We can compare this with the Faddeev-Popov-BRST quantization (introduced in (5.23)) written in the abstract from

$$\begin{aligned} [Q_B, field] &= \delta_c^{gauge}(field), \\ \{Q_B, c\} &= \frac{1}{2}\delta_c^{gauge}(field). \end{aligned} \tag{5.28}$$

The ordinary BRST symmetry (5.23) can be related to the Lie algebra cohomology of the gauge group acting on the space of connections and (5.26) can be related to the *equivariant* cohomology of the gauge group acting on the space of connections.

The physical observables (gauge invariant operators) are in the cohomology of Q. As $[Q, \phi^a(x)] = 0$ and $\phi^a(x)$ is not on the right hand side of (5.26) we can construct a physical observable of gauge invariant polynomial in $\phi^a(x)$, as introduced in subsection 5.1.1 (we will use this in chapter 6). Take

$$\mathcal{O}_{k,0} = Tr\phi^k(x) \tag{5.29}$$

with ghost number $2k$. The derivative of $\mathcal{O}_{k,0}$ by x is Q-exact by (d is the exterior derivative)

$$d\mathcal{O}_{k,0}(x) = \{Q, \mathcal{O}_{k,1}\}, \tag{5.30}$$

which means that the correlation function and the BRST cohomology class of $\mathcal{O}_{k,0}$ are not dependent of x. This implies that we can construct a theory that is topologically invariant. In (5.30) the operator $\mathcal{O}_{k,1} = -kTr\phi^{k-1}\psi$. For a circle C we have

$$W_k(C) = \int_C \mathcal{O}_{k,1}, \tag{5.31}$$

which is BRST invariant that can be seen using (5.30). One can use the topological descent equations (5.8) to construct other operators, see that $W_k(C)$ depends only on the homology class of C and that

$$W_k(\Sigma) = \int_\Sigma \mathcal{O}_{k,2}(\Sigma) \tag{5.32}$$

is a BRST invariant term that can be added to the lagrangian.

We are now ready to apply the BRST gauge fixing more explicitly on our two dimensional Yang-Mills path integral $Z = \int_{\mathcal{A}} DAe^{-S_{YM}}$.

First of all, we will write the partition function in a simpler form of first order

$$Z = \int DAD\phi e^{-i\int_\Sigma Tr(i\phi F + \frac{e^2}{2}\phi \star \phi)}, \tag{5.33}$$

where $\phi = \star F$ is the Lie algebra valued scalar field. The gauge invariance of the action $S[\phi, A]$ is $S[g^{-1}\phi g, Ag] = S[\phi, A]$, where ϕ transforms under the adjoint representation of the gauge group.

Now we need to fix a gauge because of the gauge invariance and we will use the procedure of BRST gauge fixing. To do this procedure one introduces an anticommuting one form ψ^μ and writes (5.33) as

$$Z = \frac{1}{\mathrm{vol} C^\infty(\Sigma, \mathfrak{g})} \int DAD\psi D\phi e^{-i\int_\Sigma Tr(i\phi F - \frac{1}{2}\psi \wedge \psi) - i\frac{e^2}{2}\int_\Sigma Tr\phi \star \phi - i\int_\Sigma \star\{Q,\psi\}}. \tag{5.34}$$

In (5.34) the ordinary BRST and Faddeev-Popov gauge fixing terms are introduced. They are defined by the graded BRST commutator of a gauge fermion $\Psi = \psi^\mu \Pi_\mu(x)$ and Q (the BRST charge that satisfies $Q^2 = -i\delta_\phi$, where δ_ϕ is the generator of a gauge transformation with ϕ being the infinitesimal parameter).

On the physical states of the quantum field theory (states that are gauge invariant) Q is nilpotent. The (A, ψ, ϕ) field multiplet is the basic multiplet of *cohomological* Yang-Mills theory. The infinitesimal gauge invariance of (5.34) can be written as the infinitesimal BRST supersymmetry transformations

$$\begin{cases} \delta A_\mu = i\epsilon\psi_\mu, \\ \delta\psi_\mu = -\epsilon D_\mu\phi = -\epsilon(\partial_\mu\phi + [A_\mu, \phi]), \\ \delta\phi = 0, \end{cases} \tag{5.35}$$

where we have introduced the anticommuting parameter ϵ. These supersymmetric transformations are generated by the graded BRST commutator $\delta\Phi = -i\{Q, \Phi\}$ for every field Φ in (A, ψ, ϕ). The $\mathbb{Z}$-gradings (or ghost quantum numbers) of (A, ψ, ϕ) are $(0, 1, 2)$.

These results will be used in chapter 6 when we use the localization principle on two dimensional quantum Yang-Mills theory. The field strength tensor F corresponds to the Hamiltonian and the fields ϕ generate the G-equivariant cohomology.

5.3.1 *Computation of Partition Function*

A quite direct way of computing the partition function of two dimensional Yang-Mills theory is using a lattice gauge regularization [33]. In chapter 6, this result will be compared to the predictions of the localization principle for this type of integral.

We will start by putting everything in the right context. Assume that H is a compact, simple, connected Lie group with Lie algebra $\mathcal{H}$. Let $H = SU(N)$ and introduce a positive definite quadratic form $(,)$ on $\mathcal{H}$ by

$$(a,b) = -Tr\ ab \tag{5.36}$$

where the trace is in the N dimensional representation.

Let Σ be a two dimensional closed oriented Riemann surface of genus g. Take E to be an H bundle over Σ and take $\mathcal{A}$ to be the space of connections on E, whose tangent space is $\Lambda^1(\Sigma, ad(E))$ ($ad(E)$ is the adjoint vector bundle). Then, one can define a symplectic form on $\mathcal{A}$ by

$$\omega(a,b) = \frac{1}{4\pi^2}\int_\Sigma Tr(a \wedge b). \tag{5.37}$$

Let the group of gauge transformations on E be G and its Lie algebra $\mathfrak{g}$ be the space of $ad(E)$-valued zero-forms. G acts symplectically on $\mathcal{A}$, with a moment map given by

$$\mu(A) = -\frac{F}{4\pi^2}. \tag{5.38}$$

A is the connection and F the $ad(E)$-valued curvature two-form given by $F = dA + A \wedge A$. Therefore, $\mu^{-1}(0)$ consists of flat connections. $\mu^{-1}(0)/G$ is the moduli space $\mathcal{M}$ of flat connections on E up to gauge transformation.

Pick a measure μ on Σ with total area 1. This gives a metric (or quadratic form) $(a,a) = -\int_\Sigma d\mu Tr\ a^2$ on $\mathfrak{g}$ and thereby determines a quadratic form on the dual of $\mathfrak{g}$ as $(F,F) = -\int_\Sigma d\mu Tr\ f^2$, where $f = \star F$ (the Hodge star operator $\star$ depends only on a measure, not a metric, in two dimensions).

We can now write down the partition function of two dimensional quantum Yang-Mills theory on the surface Σ as

$$Z(\epsilon) = \frac{1}{vol(G)}\int_{\mathcal{A}} DA e^{-\frac{1}{2\epsilon}(F,F)}, \tag{5.39}$$

where DA is the symplectic measure on $\mathcal{A}$ and $\epsilon \in \mathbb{R}$.

The Yang-Mills theory can be written slightly different then we just have done, as mentioned above. If we let

$$L = -\frac{\epsilon^2}{2}\int_\Sigma d\mu Tr\phi^2 - i\int_\Sigma Tr\phi F \tag{5.40}$$

then we can integrate over ϕ and get back $\frac{1}{2\epsilon^2}\int_\Sigma Trf^2$. The good thing about (5.40) is that we can take $\epsilon \to 0$ which gives

$$L = -\frac{i}{4\pi^2}\int_\Sigma Tr\phi F. \tag{5.41}$$

This is a topological field theory and it is related to the Ray-Singer analytic torsion [43].

Depending on the regularization used to define the theory the partition functions can differ as

$$Z' = Ze^{t\int_\Sigma d\mu + v\int_\Sigma d\mu \frac{R}{2\pi}}, \tag{5.42}$$

where the two terms in the exponential are the volume and the Euler characteristic and t, v are constants. In chapter (6) the constants will be fixed such that t gives the right eigenvalues of the Hamiltonian and v to get agreement with the Ray Singer torsion.

We will now turn to compute the partition function of the Yang-Mills theory using a combinatorial approach. We can do a finite dimensional approximation using lattice gauge theory [44, 33]. In this approach one covers Σ with polygons and restricts E to the finite set S of vertices of the polygons. In this step we loose the topology of E and we will get a partition function summed over all kind of G bundle topology on Σ.

The lattice gauge transformations are a map $S \to G$. We pick a group element $g_x \in G$ to every $x \in S$ and only takes into account parallel transport along polygon edges.

Let the edge of a polygon between x and y be denoted γ and assign group element U_γ to the edge. U_γ is the parallel transport operator from $x \to y$, regarded as a map $E_x \to E_y$. It transforms as $U_\gamma \to g_y U_\gamma g_x^{-1}$ under gauge transformations.

It is possible (in two dimensional Yang-Mills theory) to write the lattice gauge theory such that Z is invariant under subdivision of the lattice [45, 33]. We will show this below.

Let us start by writing the integrand as

$$e^{-\int_\Sigma \mathcal{L}} = \Pi_i e^{-\int_{\omega_i} \mathcal{L}_i}, \tag{5.43}$$

were ω_i are the plaquettes (the interior of the polygons) dividing up Σ. We can see that the integrand is a product of local factors.

In the continuum limit we need to pick a measure, therefore we are interested in defining a lattice version of it. Let ρ_i be the area of each plaquette and by summing over them we get the total area ρ. From the connection elements assigned to each edge we can build the holonomy

$$\mathcal{U} = U_1 \dots U_n. \tag{5.44}$$

The conjugation class of the holonomy is gauge invariant and the local factors on the plaquettes are class functions of the holonomy. This then says that they are linear combination of group characters $\chi_\alpha(\mathcal{U})$, which constitutes the basis of the class functions. The character is the trace of $\mathcal{U}$ in the α-representation. The local factors proposed by [45] can be written as

$$\Gamma(\mathcal{U}, \rho_\omega) = \sum_\alpha \text{dim}\alpha \chi_\alpha(\mathcal{U}) e^{-\rho_\omega c_2(\alpha)/2}, \tag{5.45}$$

where c_2 is the quadratic Casimir operator associated with the quadratic form on $\mathcal{H}$. The quadratic Casimir is used because we have a Lagrangian density Trf^2 quadratic in f in the continuum limit. As $\rho \to 0$ then $\sum_\alpha \text{dim}\alpha \chi_\alpha(\mathcal{U}) = \delta(\mathcal{U} - 1)$.

As a reminder, the Casimir operator is a function of the generators on a semisimple n-dimensional Lie algebra, with basis $\{T_a\}_{i=1}^n$, that commutes with all the generators; $c_2 = \sum_{i=1}^n T_a T_a$, and there are as many Casimir operators as the rank of the group. For example, in $SU(2)$ there is one Casimir operator; $c_2 = {}^3/_4 I$. The constant proportional to the identity can be used to classify the representations and is often related to the mass or (iso)spin. Here, the constant is the "square" of total (iso)spin $T^2 = t(t+1)$ for $t = 1/2$.

Given this local factors we can write down a theory that is invariant under subdivision of the surface. First pick a polygon cover of the surface. Then the integral of the lattice approximation given the Haar measure dU_γ with vol$G = 1$ is

$$Z_{\Sigma,X}(\rho) = \int \Pi_\gamma dU_\gamma \Pi_i \Gamma(\mathcal{U}_i, \rho_i). \tag{5.46}$$

Imagine now a square with area ρ that is divided up in two triangles with areas ρ' and ρ'' with the extra edge V being the diagonal of the square. Then for the square

$$\Gamma = \sum_{\alpha} \dim\alpha \chi_\alpha(U_1U_2U_3U_4)e^{-\rho c_2(\alpha)/2} \tag{5.47}$$

and for the two triangles

$$\Gamma'\Gamma'' = \sum_{\alpha,\beta} \dim\alpha\dim\beta\chi_\alpha(U_1U_2V)\chi_\beta(V^{-1}U_3U_4)e^{-\rho' c_2(\alpha)/2-\rho'' c_2(\beta)/2}. \tag{5.48}$$

We now want to show that $\int dV\Gamma'\Gamma'' = \Gamma$ as this gives $Z_{\Sigma,X} = Z_{\Sigma,X'}$. This is shown by the use of the formula

$$\int dV\chi_\alpha(UV)\chi_\beta(V^{-1}W) = \delta_{\alpha\beta}\frac{1}{\dim\alpha}\cdot\chi_\alpha(UW). \tag{5.49}$$

Let us now try to compute the partition function $Z_\Sigma(\rho)$ with Σ being orientable of genus g. As we have just shown that it is invariant under subdivision we can divide the surface however we like. In particular it can by covered by one polygon of $4g$ number of sides (see figure 7.6). Then

$$Z_g(\rho) = \sum_\alpha \dim\alpha e^{-\rho c_2(\alpha)/2}\int dU_idV_j\chi_\alpha(U_1V_1U_1^{-1}V_1^{-1}\dots U_gV_gU_g^{-1}V_g^{-1}). \tag{5.50}$$

Using (5.49) and $\int dU\chi_\alpha(VUWU^{-1}) = \frac{1}{\dim\alpha}\chi_\alpha(V)\chi_\alpha(W)$ we get the Yang-Mills partition function on Σ (orientable surface of genus g) to be

$$Z_g(\rho) = \sum_\alpha \frac{e^{-\rho c_2(\sigma)/2}}{(\dim\alpha)^{2g-2}}. \tag{5.51}$$

This result can be rederived relating it to quantum field theory and Hilbert space. Let us instead cut the surface Σ through the circle C into Σ_L and Σ_R with areas ρ_L and ρ_R respectively [33] and cover them by polygons. We cover C with a single one-cell and attach the group element U to C (in general the holonomy around C). On Σ_L and Σ_R we assign group variables $U_{L,\gamma}$ and $U_{R,\delta}$ to their edges. Integrating over $U_{R,\delta}$ gives a function $\Psi_R(U)$ (for U fixed) given by

$$\Psi_R(U) = \int dU_{R,\delta}\Pi_{\omega_i\in\Sigma_R}\Gamma(\mathcal{U}_i,\rho_i). \tag{5.52}$$

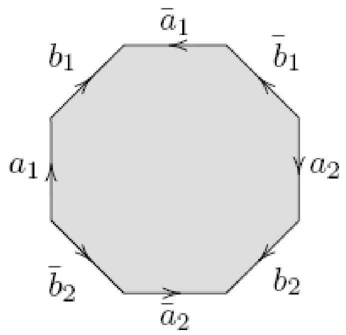

Figure 5.3: The orientable surface of genus g can be obtained by gluing the edges of a polygon with $4g$ sides, here shown for $g = 2$ [34].

Integrating over $U_{L,\gamma}$ gives

$$\Psi_L(U^{-1}) = \int dU_{L,\gamma} \Pi_{\omega_i \in \Sigma_L} \Gamma(\mathcal{U}_i, \rho_i). \tag{5.53}$$

Σ_L and Σ_R are class functions because they are gauge invariant; $\Sigma_L(AUA^{-1}) = \Sigma_L(U)$ and $\Sigma_R(AUA^{-1}) = \Sigma_R(U)$.

To get $Z_\Sigma(\rho)$ we integrate over U:

$$Z_\Sigma(\rho) = \int dU \Psi_L(U^{-1}) \Psi_R(U) = \int dU \Psi_L\bar{(}U) \Psi_R(U) \tag{5.54}$$

Introduce the Hilbert space of class functions on G with the inner product $(f, g) = \int dU f\bar{(}U) g(U)$ in which Ψ_L and Ψ_R are vectors. The partition function then is

$$Z_\Sigma(\rho) = (\Psi_L, \Psi_R). \tag{5.55}$$

Using that the characters constitute a orthogonal basis of the Hilbert space, a results from representation theory, we can write

$$(\Psi_L, \Psi_R) = \sum_\sigma (\Psi_L, \chi_\sigma) \cdot (\chi_\sigma, \Psi_R). \tag{5.56}$$

The partition function in a more general setting, with circles $C_1, \ldots, C_n$ making up the boundary with holonomies $U_1, \ldots U_n$ can then be written changing the first version by multiplication of a character giving

$$Z_\Sigma(\rho) = (\Psi_L, \Psi_R) = \sum_\sigma (\Psi_L, \chi_\sigma)(\chi_\sigma, \Psi_R). \tag{5.57}$$

Doing this rederivation of associating a Hilbert space with each circle gives the structure predicted from canonical quantization of $\int_\Sigma d\mu Tr\ f^2$ (described above), which predicts to give the Hilbert space of class functions that we have described [33].

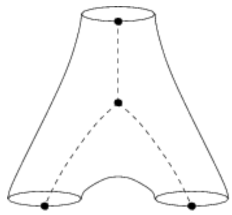

Figure 5.4: A three holed sphere

Next pick a surface Σ of genus greater than 1. This can be divided into $2g-2$ pieces of three holed spheres (see figure 5.4) cutting $3g-3$ circles. First we compute the partition function of a three holed sphere picking convenient covering using a similar method as above to be

$$Z_3(\rho;\alpha,\beta,\gamma) = \frac{e^{-\rho_0 c_2(\sigma)/2}}{\dim\alpha}\delta_{\alpha\beta\gamma} \tag{5.58}$$

Then, the partition function of an orientable surface of genus g is again

$$Z_g(\rho) = \sum_\alpha \frac{e^{-\rho c_2(\sigma)/2}}{(\dim\alpha)^{2g-2}}, \tag{5.59}$$

where every three holed sphere contributes a factor $1/\dim\alpha$.

6 TWO DIMENSIONAL GAUGE THEORY AND LOCALIZATION

In this chapter we will discuss how to apply the equivariant localization formalism to field theories. We will generalize the DH-formula of chapter 4 to an infinite dimensional setup in quantum field theory following the work by Witten [14]. More explicitly, we will study the partition function Z of two dimensional quantum Yang-Mills theory where we have a non-abelian action. We will define how to perform equivariant integration and use the localization principle of chapter 4 to show that the partition function can be written as a sum over the critical point set. We will map topological Yang-Mills theory to physical Yang-Mills theory, which can be seen as the localization in physical language. This can explain why in two dimensional quantum Yang-Mills theory one finds that the partition function $Z(\epsilon)$, expanded in powers of ϵ, has finitely many terms ($Z(\epsilon)$ is not a polynomial but contains exponentially small terms, explained as unstable classical solutions). It will be shown using localization that one can relate $Z(\epsilon)$ at $\epsilon \neq 0$ to the topology of moduli spaces of flat connections.

6.1 EQUIVARIANT INTEGRATION AND LOCALIZATION PRINCIPLE

Assume that we now have an action on a manifold X given by a compact, connected Lie group G, with Lie algebra $\mathcal{G}$, generating a vector field V_a on X. The Hamiltonian functions μ_a can be put together into $\mu : X \to \mathcal{G}^*$. In this case we consequently have a collection of $\mathcal{G}$-invariant functions so we need to modify the partition function in classical statistical mechanics (4.33) to

$$Z = \int_X \frac{\omega^n}{n!} e^{\frac{1}{2\epsilon}(\mu,\mu)}, \tag{6.1}$$

where $I = (\mu, \mu)$ is an invariant quadratic form on $\mathcal{G}$.

As in the abelian case, Z can be expressed as a sum over the critical points. To explain this we need to introduce a way to integrate equivariant differential forms. Usually one considers the pushforward $\Lambda^*_G(X) \to \Lambda^*_G(pt)$ by integrating over X, i.e. $\alpha \to \int_X \alpha$. However, we want the operation given by

$$\alpha \to \frac{1}{vol(G)} \int_{\mathcal{G}\times X} \frac{d\phi_1 \dots d\phi_s}{(2\pi)^s} \alpha, \tag{6.2}$$

which map $\Lambda^*_G(X) \to \mathbb{C}$ in analogy with normal de Rham integration. $\frac{1}{vol(H)} d\phi_1 \dots d\phi_s$ is a natural measure on $\mathcal{G}$, where $\phi_1 \dots \phi_s$ are Euclidean coordinates on $\mathcal{G}$ so that the measure on $\mathcal{G}$ agrees with the Haar measure chosen at the identity of G. The Haar measure is a G-invariant measure on G.

The integral in (6.2) does however not converge in general. We therefore need to multiply by a convergence factor $e^{-\frac{\epsilon}{2}(\phi,\phi)}$. This will give the definition of *equivariant* integration

$$\oint_X \alpha = \frac{1}{vol(G)} \int_{\mathcal{G}\times X} \frac{d\phi_1 \dots d\phi_s}{(2\pi)^s} \alpha e^{-\frac{\epsilon}{2}(\phi,\phi)}. \tag{6.3}$$

When mapping $\Lambda^*_G(X)$ to $\mathbb{C}$ most of the information is lost. To recover the information one can look at both $\oint \alpha$ and $\oint \alpha Q(\phi)$ with $Q \in \Lambda^*_G(pt)$, an arbitrary G invariant polynomial on $\mathcal{G}$.

Example 6.1 (Equivariant integration over of point). *As a first example we look at the G action on a point with $\alpha = 1$. This gives*

$$\oint_{pt} 1 = \frac{1}{vol(G)(2\pi\epsilon)^{s/2}}, \tag{6.4}$$

using (2.17). This illustrates how the singularities of the action and the singularities of the equivariant integral are related by functions of ϵ.

As we now have a notion of integration, we shall turn to the localization principle in this setting. In chapter 4 we found that we could calculate the integral $\int \alpha$ (α being an equivariantly closed form) by multiplying it with $e^{-tD\omega}$, where $D^2\omega = 0$ to ensure t-independence. By the same arguments we have

$$\oint \alpha = \oint \alpha e^{tD\lambda}. \tag{6.5}$$

We here call the G-invariant one-form λ and $D = d - i \cdot i_{V(\phi)}$.

In chapter 4 we explicitly calculated the integral by taking the limit $t \to \infty$ and the integral localized on the fixed points of the action. We will now explain the generalization to the infinite dimensional setting.

Let T_a be an orthonormal basis of the Lie algebra $\mathcal{G}$ and $V(\phi) = \sum_a \phi^a V_a$. Then

$$\begin{aligned}\oint \alpha &= \frac{1}{vol(G)} \int \frac{d\phi_1 \dots d\phi_s}{(2\pi)^s} \alpha e^{-td\lambda - it\phi^a \lambda(V_a) - \frac{\epsilon}{2}\sum_a (\phi^a)^2} \\ &= \frac{1}{vol(G)(2\pi\epsilon)^{s/2}} \int_X \alpha e^{-td\lambda - \frac{t^2}{2\epsilon}\sum_a (\lambda(V_a))^2},\end{aligned} \tag{6.6}$$

where in the line we performed the gaussian integral over ϕ assuming α is independent of ϕ and we have written $i_V(\lambda)$ as $\lambda(V)$.

Take $X' \subset X$ on which $\lambda(V_a) = 0$ $(a = 1, \cdots, s)$. We can write X' as a union of its connected components X_σ:

$$X' = \cup_{\sigma \in S} X_\sigma, \tag{6.7}$$

with S being the set of these components. Then take $W \subset X$ such that $W \cap X' = \emptyset$

As (6.6) is independent of t we can take the limit $t \to \infty$ leaving only contributions from X_σ introduced above as the integral over W vanishes as e^{-kt^2} with k a positive constant. Then the integral can be written as

$$\oint_X \alpha = \sum_{\sigma \in S} Z_\sigma, \tag{6.8}$$

where Z_σ are the integral over a tubular neighborhood of X_σ. We have found that Z can be express as a sum over the critical point set.

6.1.1 *Stationary Phase Arguments*

The argument just explained can be reformulated using the stationary phase method (see for example [46], chapter 7). The stationary phase method is used here to compute the integral in the first line of (6.6) in the large t limit (without integrating over ϕ) to show that Z_σ is a polynomial in ϵ. We will do this in the following.

Except from $e^{td\lambda}$, in the first line of (6.6), t only appears in e^{-itK} with $K = \phi^a \lambda(V_a)$ on $X \times \mathcal{G}$. This means that we have the conditions

$$\lambda(V_a) = 0, \tag{6.9}$$
$$\phi^a d(\lambda(V_a)) = 0, \tag{6.10}$$

for the condition of critical points $dK = 0$.

There is an invariance of the equations under scaling ϕ^a, as ϕ takes its values in a vector space that can be contracted to the origin. This means that we can set $\phi = 0$, i.e. the components of the critical point set of K is the same as X_σ introduced above. So, the components of the critical point set of K will make the same contribution of the first line of (6.6) as X_σ does to the second line of (6.6) when $t \to \infty$.

The scaling invariance leads to the fact that the critical point set of K is compact when that of X_σ is [14]. The compactness of the critical point set of K means that we don't need the convergence factor in (6.6) and thereby the integral has a limit for $\epsilon \to 0$. Stationary phase integration principles say that the integral will have *finitely* many terms and Z_σ will be a *polynomial* in ϵ. This is an important result we will use later on.

We will now explain this more explicitly. When G acts freely on X_σ then $H^*_G(X_\sigma) \simeq H^*_G(X_\sigma/G)$. Assume that Y is an equivariant tubular neighborhood of X_σ, so

$$H^*_G(Y) \simeq H^*_G(X_\sigma) \simeq H^*_G(X_\sigma/G). \tag{6.11}$$

Let $\pi : Y \to X_\sigma/G$ and $\pi^* : H^*_G(X_\sigma/G) \to H^*_G(Y)$ be the pullback. We can then write

$$\frac{-(\phi,\phi)}{2} = \pi^*(\theta) + D\omega, \tag{6.12}$$

with $\frac{-(\phi,\phi)}{2} \in H^4_G(X)$, $\omega \in \Lambda^3_G(Y)$ and $\theta \in H^4_G(X_\sigma/G)$. θ is a characteristic class of $\mu^{-1}(0)$.

As the components of the critical point set of K is equal to the components X_σ we can calculate the contribution of X_σ to $\oint \alpha$ as

$$\oint \alpha = \frac{1}{vol(G)} \int \frac{d\phi_1 \dots d\phi_s}{(2\pi)^s} \alpha e^{-td\lambda - it\phi^a \lambda(V_a) - \frac{\epsilon}{2}\sum_a (\phi^a)^2} \cdot u \tag{6.13}$$

using the stationary phase in which the integral of (6.6) is multiplied by u - a smooth function invariant of G zero outside of Y and 1 otherwise. The contribution Z_σ can be calculated with the substitution

$$e^{-\frac{\epsilon}{2}(\phi,\phi)} \to e^{\epsilon\theta} \tag{6.14}$$

using (6.12). We then see that Z_σ is a polynomial in ϵ at most of order $\frac{1}{4}\dim(X_\sigma/G)$. We can also make the substitution

$$\alpha \to \alpha' \tag{6.15}$$

for $\alpha \in H^*_G(X)$ and $\alpha' \in H^*(X_\sigma/G)$. That we can do these substitutions will be important in the calculations of next subsection.

6.1.2 *Contribution of Flat Connections to the Equivariant Integral on a Symplectic Manifold*

Let us now look at equivariant integration over a symplectic manifold X acted on by the group G with the moment map μ. As we will see, this integral can be written as a sum over the critical points of $I = (\mu, \mu)$. We want to calculate the contribution of $\mu^{-1}(0)$ (the minimum of $I = (\mu, \mu)$) to

$$\oint \alpha = \frac{1}{vol(G)} \int \frac{d\phi_1 \dots d\phi_s}{(2\pi)^s} \int_X \alpha e^{tD\lambda - \frac{\epsilon}{2}(\phi,\phi)} \tag{6.16}$$

on (X, ω), which gives the dominant contribution as $\epsilon \to 0$. Assume that $\mu^{-1}(0)$ is smooth and acted on freely by G then contribution of $\mu^{-1}(0)$ to $\oint_X \alpha$ is

$$Z(\mu^{-1}(0)) = \int_{\mu^{-1}(0)/G} \alpha' e^{\epsilon\theta} \tag{6.17}$$

The quotient space $\mu^{-1}(0)/G$ (introduced in chapter 5) is called the symplectic quotient of X and is a symplectic manifold. We will show how to get (6.17) in what follows.

We start by showing that the integral can be written as a sum over the critical points of $I = (\mu, \mu)$. As we have seen the integral (6.16) localizes on $\lambda(V_a) = 0$ by the argumentation after (6.6). Now we like to show that $\lambda(V_a) = 0$ implies $dI = 0$.

Let J be an almost complex structure on X and ω be positive (the metric $g(u,v) = \omega(u, J, v)$ is positive definite) and of type $(1,1)$. Let

$$\lambda = \frac{1}{2} J(dI). \tag{6.18}$$

On critical points of I, $\lambda = 0$ meaning that $\lambda(V_a) = 0$. We want the other way around to hold.

Assume $Y = \sum_a \mu_a V_a$ and use $d\mu_a = -i_{V_a}(\omega)$ to get $Y = \frac{1}{2}\omega^{-1} dI$. $\lambda(V_a) = 0$ gives $\lambda(Y) = 0$ or $\omega^{-1}(dI, JdI) = 0$. $\omega^{-1}(dI, JdI) = 0$ holds only for $dI = 0$ as the metric is positive definite. This means that $\lambda(V_a) = 0$ implies $dI = 0$, i.e. on symplectic manifolds the integral localizes to the the critical points of I using this particular λ.

We will now go on to calculate the contribution of $\mu^{-1}(0)$ to the integral (6.16) making use of (6.14) and (6.15). To see that the condition to use (6.14) holds see [p.23, [14]].

Assume that Y is an equivariant tubular neighborhood of $\mu^{-1}(0)$ and that the equivariant projection is $\pi : Y \to \mu^{-1}(0)/G$ composed of $Y \to \mu^{-1}(0)$ and $\psi : \mu^{-1}(0) \to \mu^{-1}(0)/G$.

To study the integral (6.16) we will integrate over the fibers of π to reduce it from an integral over Y to an integral over $\mu^{-1}(0)/G$. As $-(\phi,\phi)/2$ and $\alpha \in H^*_G(Y)$ are pullbacks by π of θ and α' we see that all terms in (6.16) is a pullback through π but $e^{tD\lambda}$. Performing the integration over the fibers of π means that we substitute $-(\phi,\phi)/2$ and α using (6.14) and (6.15). What is left to calculate is the integral

$$\int \frac{d\phi_1 \ldots d\phi_s}{(2\pi)^s} \int_{\pi^{-1}(pt)} e^{tD\lambda}. \tag{6.19}$$

This is 1 as $t \to \infty$ by the following arguments. $\pi^{-1}(pt)$ is fibered on $G \simeq \psi^{-1}(pt)$ and the G action on $\pi^{-1}(pt)$ can be modeled on a neighborhood of $G \subset T^*G$. T^*G can be seen as $\mathcal{G} \times G$. Written with $g \times \gamma \in \mathcal{G} \times G$ the symplectic form is $\omega = (d\gamma, dgg^{-1}) + (\gamma, dgg^{-1}dgg^{-1})$. In this setting $\lambda = \frac{1}{2}J(dI) = (\gamma, dgg^{-1})$. We can now write down $D\lambda$ and change variables as ϕ to $g^{-1}\phi g$ to get $-i(\gamma,\phi)\frac{(d\gamma, dgg^{-1})^n}{n!}$.

To calculate (6.19) first calculate the ϕ integral using $\int_{-\infty}^{\infty} due^{-iuv} = 2\pi\delta(v)$ and then calculate the γ integral over the delta functions. This gives 1. Thereby, we can write down the result as

$$Z(\mu^{-1}(0)) = \int_{\mu^{-1}(0)/G} \alpha' e^{\epsilon\theta}, \tag{6.20}$$

getting the result stated above. This will be important for later calculations.

6.2 LOCALIZATION IN QUANTUM FIELD THEORY LANGUAGE

So far, we have shown how the localization principle works. We will now continue by putting everything in the context of quantum field theory. Then we will show how we can map topological to physical Yang-Mills theory and how to interpret the partition function.

Let the local coordinates on the compact manifold X be x_i and anticommuting variables tangent to X be ψ^i. Let D be the equivariant exterior derivative (introduced in chapter 3) given by

$$D = \psi^i \frac{\partial}{\partial x^i} - i \sum \phi^a V_a^i \frac{\partial}{\partial \psi^i} \tag{6.21}$$

and write the equivariant integration as

$$\oint_X \alpha = \frac{1}{vol(G)} \int_X dx^i d\psi^i \int_{\mathcal{G}} \frac{d\phi_1 \dots d\phi_s}{(2\pi)^s} \alpha e^{-\frac{\epsilon}{2}(\phi,\phi)}. \tag{6.22}$$

Note that we now have a natural integration measure $dx^i d\psi^i$ on X because the Jacobians of the odd and even variables cancel when changing variables (as explained in chapter 2).

Now, choose $\lambda = \psi^i b_i(x)$. After integrating over ϕ and inserting $d\lambda$ we have

$$\oint_X \alpha = \frac{1}{vol(G)(2\pi\epsilon)^{s/2}} \int_X dx^i d\psi^i \alpha e^{t\sum_{i,j} \psi^i \psi^j \partial_i b_j - \frac{t^2}{2} \sum_a (V_a^i b_i)^2}. \tag{6.23}$$

As before, the integral localizes around solutions of $V_a^i b_i = 0$ $(a = 1, \dots, s)$ as $\to \infty$.

Let us now look at the symplectic manifold (X, ω) with $\omega = \frac{1}{2}\omega_{ij}\psi^i\psi^j$. The choice of λ implies that the integral localizes on the set of critical points, i.e. where $\partial I = 0$ with $I = \sum_a \mu_a^2$ (done similarly to what we saw in 6.1.2).

Let's now choose an α of the form:

$$\alpha = e^{\frac{1}{2}\omega_{ij}\psi^i\psi^j - i\phi^a\mu_a}\beta. \tag{6.24}$$

With the ω introduced above and with $\beta = 1$ we get the standard Liouville measure when integrating over ψ. This is done using (2.18) to get $\sqrt{\det\omega}$ normally written as $\frac{\omega^n}{n!}$.

Integrating over ϕ gives

$$\oint \alpha = \int_X \frac{\omega^n}{n!} e^{-\frac{1}{2\epsilon}(\mu,\mu)}. \tag{6.25}$$

In (6.25) we expect critical points c to contribute as $e^{\frac{-I(c)}{2\epsilon}}$ and we see that solutions $\mu = 0$ (the absolute minimum of I) give the dominant contribution.

6.3 GAUGE THEORY OF COHOMOLOGICAL TYPE

We will now look at cohomological Yang-Mills theory and find the relation of the contribution of solutions $\mu = 0$ of the partition function of two dimensional Yang-Mills theory with the cohomology of $\mathcal{M} = \mu^{-1}(0)/G$ by a map from cohomological gauge theory to physical field theory (compare result found in 6.1.2).

Will will now take the space X above to be the space of connections $\mathcal{A}$ on a vector bundle E over the two dimensional Riemann surface Σ. Assume that E has a compact structure group H and let G be the group of gauge transformations of E. Let A be the gauge field and replace the x's above with A. Let ψ be an one form in $ad(H)$ and ϕ a zero form on Σ.

Cohomological Yang-Mills theory have the multiplet (A, ψ, ϕ) introduced in 5. The BRST supersymmetry transformation laws are given by (5.35) but we repeat them again for convenience. Let ϵ is an anticommuting variable then

$$\begin{cases} \delta A = i\epsilon\psi \\ \delta\psi_i = -\epsilon D_i\phi = -\epsilon(\partial_i\phi + [A_i, \phi]) \\ \delta\phi = 0 \end{cases} \tag{6.26}$$

(compare $\delta x^\mu = \psi^\mu$, $\delta\psi_i = V^\mu(x)$ from the finite dimensional case in chapter 4). Using the exterior equivariant derivative we can write the relations as $\delta\Phi = iD\Phi$ or $\delta\Phi = -i\{Q,\Phi\}$, where Q is the BRST operator (so $Q = -D$ and $Q^2 = D^2 = 0$ up to gauge transformations)

As before (A,ϕ,ψ) have ghost numbers $(0,1,2)$. To write a Lagrangian with Q symmetry we additional multiplets. These can be written as pairs (u,v) with ghost numbers $(n,n+1)$ and opposite statistics transforming as

$$\begin{aligned}\delta u &= i\epsilon \nu \\ \delta \nu &= \epsilon[\phi,u].\end{aligned} \tag{6.27}$$

These are analogous to the antighost multiplets introduced in usual BRST quantization. There is a lot of freedom in the antighost choice, we just need to pick them such that a good lagrangian can be written. We will add (λ,η) and $(\chi,-iH)$ with ghost numbers $(-2,-1)$ and $(-1,0)$ respectively. λ is a commuting field and χ an anticommuting field.

Let us pick a Lagrangian L as

$$\begin{aligned}L = -i\{Q,V\} = \frac{1}{h^2}\int_\Sigma d\mu Tr(\frac{1}{2}(H-f)^2 - \frac{1}{2}f^2 - i\chi\star D\psi \\ +iD_i\eta\psi_i + D_i\lambda D^i\phi + \frac{i}{2}\chi[\chi,\phi] + i[\psi_i,\lambda]\psi^i),\end{aligned} \tag{6.28}$$

with

$$V = \frac{1}{h^2}\int_\Sigma d\mu Tr(\frac{1}{2}\chi(H-2\star F) + g^{ij}D_i\lambda\psi_j). \tag{6.29}$$

Here g is the metric on Σ, μ is the measure and F is the Yang-Mills field strength with $f = \star F = \frac{1}{2}\epsilon^{ij}F_{ij}$. This V is gauge invariant which make the lagrangian Q-invariant. The kinetic energy of all fields in V is nondegenarate. This choice of lagrangian is a topological field theory and independent of the coupling parameter h (see explanation in chapter 5).

We can delete $(H-f)^2$ from L through the equation of motion $H = f$. In the limit as $h \to 0$ we can expand the lagrangian around $F = 0$ to get the solution. The scalar energy is minimized by $D_i\phi = 0$ if $\lambda = \bar{\phi}$. When A is an irreducible solution of $F = 0$ then $\phi = 0$.

Let us denote the space of solutions to $F = 0$ and $D_i\phi = 0$ as $\mathcal{U}$. In the limit $h \to 0$ the integral localizes to an integral over $\mathcal{U}$. For H and E with only irreducible solutions of $F = 0$, $\mathcal{U}$ is equal to the moduli space $\mathcal{M}$ (the space of flat connections on E up to gauge transformations) and the correlation functions can be written, in more mathematical language, as intersection pairings on $\mathcal{M}$ [47]. If we have reducible connections then we can get zero modes of ϕ and $\lambda = \bar{\phi}$ [37]. Therefore we want to eliminate these fields as $\mathcal{U}$ and $\mathcal{M}$ might not be equal then.

We will now turn to the task of mapping the given theory with L as in (6.28) to a physical Yang-Mills theory. We will start this by letting $V \to tV'$. Then the new theory with $L(t) = -i\{Q, V + tV'\}$ is independent of t if $L(t)$ still has a non degenerate kinetic energy and that there are no new fixed points (solutions of $\delta\chi = \delta\psi = 0$). Here we will not fulfill the last condition. This means that the fixed point set will have one component $\mathcal{M}$ ($\mathcal{U}$ in general) and then the new extra component $\mathcal{M}_\alpha$. The path integral will then be a sum over $\mathcal{M}$ and $\mathcal{M}_\alpha$. We will find a way to disentangle the contributions of the two.

We will begin by eliminating λ from the theory. Pick

$$V' = -\frac{1}{h^2}\int_\Sigma d\mu Tr\chi\lambda. \tag{6.30}$$

The lagrangian is

$$\begin{aligned} L(t) &= -i\{Q, V + tV'\} \\ &= \frac{1}{h^2}\int_\Sigma d\mu Tr(\frac{1}{2}(H - \lambda t - f)^2 - \frac{1}{2}(\lambda t + f)^2 + i\chi \star D\psi \\ &+ iD_i\eta\psi_i - D_i\lambda D^i\phi + \frac{i}{2}\chi[\chi,\phi] + i[\psi_i,\lambda]\psi^i). \end{aligned} \tag{6.31}$$

We can integrate out H by $H - \lambda t - f = 0$. λ, χ, η can be integrated out too, with $\lambda = -\frac{f}{t}$, leaving a local lagrangian written as

$$\begin{aligned} L'(t) = \frac{1}{h^2}\int_\Sigma d\mu Tr(&\frac{1}{t}(D_i f D^i\phi + if[\psi_i,\psi^i] - iD_l\psi^l\epsilon^{ij}D_i\psi_j) \\ &+\frac{1}{t^2}(D_l\phi\psi^l[D_k\psi^k,\phi] + \frac{1}{2}(-D_kD^k\phi + i[\psi_k,\psi^k])^2)). \end{aligned} \tag{6.32}$$

In $L = -i\{Q, V\}$ given by (6.28) (the standard cohomological lagrangian) the BRST invariant operators have correlation functions given by the cohomology of $\mathcal{M}$. This can be perturbed to $L'(t)$ (given by (6.32)) that still has the BRST symmetry and is written in terms of (A, ψ, ϕ) only.

It is not sure that L in (6.28) and $L'(t)$ in (6.32) is equal as we can have the new components $\mathcal{M}_\alpha$. We like to study $L'(t)$ in (6.32) by putting $t = -iu$ which gives $L''(u)$. Considering the terms of $1/u$ we get

$$L''(t) = \frac{i}{h^2 u} \int_\Sigma d\mu Tr(D_i f D^i \phi + if[\psi_i, \psi^i] - iD_l \psi^l \epsilon^{ij} D_i \psi_j). \tag{6.33}$$

This gives the Feynman path integral

$$\frac{1}{vol(G)} \int DAD\psi D\phi e^{-L''(u)}. \tag{6.34}$$

Integrating (6.34) over ϕ we get

$$\int D\phi e^{\frac{i}{h^2 u} \int_\Sigma d\mu Tr \phi D_i D^i f} \sim \Pi_{x \in \Sigma} \delta(D_i D^i f) \tag{6.35}$$

The path integral therefore localizes on

$$0 = \int_\Sigma d\mu Tr f D_i D^i f = -\int_\Sigma Tr(D_i f)^2 \tag{6.36}$$

or equivalently

$$0 = D_i f, \tag{6.37}$$

which are the Yang-Mills equations. The solutions are $f = 0$ and $f \neq 0$ on higher critical points. The space of these solutions is constructed out of $\mathcal{M}$ (with $f = 0$) and $\mathcal{M}_\alpha$ with the higher critical points.

The correlation functions $\langle \mathcal{O} \rangle$ computed in (6.28) and $\langle \mathcal{O} \rangle'$ computed in (6.33) might not be equal because of $\mathcal{M}_\alpha$. We will now discuss a situation where they are equal.

First we introduce two BRST invariant operators

$$\begin{aligned} \omega &= \frac{1}{4\pi^2} \int_\Sigma Tr(i\phi F + \frac{1}{2}\psi \wedge \psi) \\ \theta &= \frac{1}{8\pi^2} \int_\Sigma d\mu Tr \phi^2. \end{aligned} \tag{6.38}$$

Then we compute their correlation function as

$$\left\langle \exp\omega + \epsilon\Theta \right\rangle' = \frac{1}{vol(G)} \int DAD\psi D\phi \beta e^{\frac{1}{h^2 u}\{Q, \int_\Sigma d\mu \psi^k D_k f\} + \frac{1}{4\pi^2}\int_\Sigma Tr(i\phi F + \frac{1}{2}\psi\wedge\psi) + \frac{\epsilon}{8\pi^2}\int_\Sigma d\mu Tr\phi^2}, \tag{6.39}$$

which is u-independent. Let us set $''u = \infty''$ and by this go to *physical* Yang-Mills theory as the first term of the exponent in (6.39) vanishes. We can here calculate the correlation functions using two dimensional Yang-Mills theory just found with the lagrangian

$$L(A, \psi, \phi) = \int_\Sigma Tr(-i\phi F - \frac{1}{2}\psi \wedge \psi) - \frac{\epsilon}{8\pi^2}\int_\Sigma d\mu Tr\phi^2. \tag{6.40}$$

Let us turn to the case where $\epsilon = 0$ (can extract all topological information there, see chapter 5). Put $\beta = 1$. Integrating over ϕ gives $\delta(F)$ and there are no higher critical points (even for $\beta \neq 1$) so the correlation functions are equal, i.e $\left\langle e^\omega \beta \right\rangle = \left\langle e^\omega \beta \right\rangle'$.

Let us now turn to the case where $\epsilon \neq 0$. We can integrate over ψ as before and then integrate over ϕ. This gives the path integral of two dimensional Yang-Mills theory:

$$\left\langle e^{\omega+\epsilon\Theta} \right\rangle' = \frac{1}{vol(G)} \int DAe^{-\frac{2\pi^2}{\epsilon}\int_\Sigma d\mu Trf^2}. \tag{6.41}$$

At $\epsilon \neq 0$ and $\beta = 1$ the correlation functions might not be the same, as we have $\mathcal{M}_\alpha$. These higher critical points contributes as $I = -\int_\Sigma d\mu Trf^2$ with $f \neq 0$ and we can write

$$\frac{1}{vol(G)} \int DAe^{-\frac{2\pi^2}{\epsilon}\int_\Sigma d\mu Trf^2} = \left\langle e^{\omega+\epsilon\Theta} \right\rangle + O(e^{-2\pi^2 c/\epsilon}), \tag{6.42}$$

where c is the lowest value of I of the higher critical points.

In chapter 5 we saw that in a cohomological field theory one builds operator out of BRST invariant polynomials. We can build an operator which is not Q-exact as $\mathcal{O}_T^{(0)}(P) = T(\phi(P))$, where T is an invariant polynomial on Lie algebra $\mathcal{H}$ and $P \in \Sigma$. We can then build the observables of the theory from these operators, with F^a being the components of the curvature F, as

$$\begin{aligned} T_{(0)} &= \mathcal{O}^{(0)}(P) \to \int_\Sigma d\mu T(\phi), \\ T_{(1)}(C) &= -\int_C \frac{\partial T}{\partial \phi^a}\psi^a, \\ T_{(2)}(\Sigma) &= \int_\Sigma \left(i\frac{\partial T}{\partial \phi^a}F^a + \frac{1}{2}\frac{\partial^2 T}{\partial \phi^a \partial \phi^b}\psi^a \wedge \psi^b \right). \end{aligned} \tag{6.43}$$

6.4 COMPARISON OF LOCALIZATION PRINCIPLES AND YANG-MILLS THEORY

Until now we have looked at cohomological gauge theory that we mapped to physical Yang-Mills theory. Now we like to compare localization principles from section 6.1 with two dimensional Yang-Mills theory.

As we saw in chapter 5 the partition function of of Yang-Mills theory at $\epsilon = 0$ can be written as a topological field theory with

$$L = -\frac{i}{4\pi^2}\int_\Sigma Tr\phi F, \tag{6.44}$$

which is related to the Ray Singer torsion. We also found that the Yang-Mills partition function on a trivial bundle to be given by (5.51). Thereto, we mentioned that the regularization used to define the theory can differ as (5.42) so we really should be writing

$$Z = v \cdot \sum_\alpha \frac{1}{(\dim\alpha)^{2g-2}} \tag{6.45}$$

without the Casimir operator for the topological field theory. v should now be adjusted such that the theory agrees with the Ray Singer torsion. This gives [14]

$$Z(\Sigma) = \left(\frac{\mathrm{vol}(H)}{(2\pi)^{\dim H}}\right)^{2g-2} \sum_\alpha \frac{1}{(\dim\alpha)^{2g-2}}. \tag{6.46}$$

Moreover, Z is related to $\mathrm{vol}(\mathcal{M})$ by

$$Z = {}^{\mathrm{vol}(\mathcal{M})}/_{\#Z(H)} \tag{6.47}$$

($\#Z(H)$ is the symmetry of the connection, the number of elements in center of H) [14, 33]. The volume of $\mathcal{M}$ can then be written directly as

$$\int_{\mathcal{M}} e^{\omega} = \int_{\mathcal{M}} \frac{\omega^n}{n!} = \#Z(H) \left(\frac{\mathrm{vol}(H)}{(2\pi)^{\mathrm{dim}H}} \right)^{2g-2} \sum_{\alpha} \frac{1}{(\mathrm{dim}\alpha)^{2g-2}}. \tag{6.48}$$

The topological interpretation is that each differential form represents the Poincaré dual of a homology cycle and the integral gives the intersection number of the homology cycles. Thus the integral gives the intersection pairings on the moduli space.

6.4.1 *Calculation of Twisted Partition Function*

We will now drop the assumption that the gauge group is simply connected and look at groups with non-trivial π_1. We still let E be the trivial H bundle.

Assume that $\Gamma \subset Z(H)$, where $Z(H)$ is the center of H, and that $H' = {}^{H}/_{\Gamma}$ is the gauge group. We let E' be the principle H' bundle over Σ. The possible bundles are defined by the choice of the monodromy $u \in \Gamma$ as we will see in the following.

The principal bundle E' is trivial restricted to $\Sigma - P$ (P is a point on Σ) and one can lift E' to the trivial H bundle E. We let A' be the connection on E' and we can lift A' to A on E when on $\Sigma - P$. A doesn't need to be smooth over P and the monodromy $u \in H$ of A around P projects to the identity element of H'. However, A' is smooth over P and $u \in \Gamma$. The monodromy u is a topological invariant of E'.

We now turn our attention to the non-singular cases. We will look at the space of flat connections $\mathcal{M}'(u)$ on E' being smooth and the gauge group acts freely on it. We have the Riemann surface Σ of genus 1 or higher, $H = SU(N)$, $H' = {}^{H}/_{\Gamma}$ (with Γ being the center of $SU(N)$) and u its generator.

We want to integrate the path integral for H'

$$\tilde{Z}(\Sigma) = \frac{1}{\mathrm{vol}(G')} \int DA' D\phi e^{-L} \tag{6.49}$$

by relating the A'-integral to the A-integral. We need the relation between the volumes of G' and G. We will make use of the following relations [14]

$$\begin{aligned} Vol(G) &= \#\Gamma^{1-2g}\mathrm{vol}(G'), \\ Vol(H) &= \#\Gamma \cdot \mathrm{vol}(H'), \\ \#Z(H) &= \#\Gamma \cdot \#Z(H'), \\ \#\pi_1(H') &= \#\Gamma. \end{aligned} \tag{6.50}$$

We also need to know how the singularity at P affects. Let us cut out disk D from Σ with $P \in D$. The monodromy about P given by $u \in \Gamma$ says that the monodromy U around C should be u and not 1 as before. We must therefore replace $\delta(U-1) = \sum_\alpha \dim(\alpha)\chi_\alpha(U)$ used to get the result of the partition function in (6.46) with

$$\delta(U-u) = \sum_\alpha \mathrm{dim}\alpha\chi_\alpha(U)\lambda_\alpha(u^{-1}). \tag{6.51}$$

We will now calculate the twisted partition function $\tilde{Z}(\Sigma, u)$. First we notice, by the arguments above, that $Z(\Sigma, u)$ on $\Sigma - P$ is given by multiplying every representation α by $\lambda_\alpha(u^{-1})$ in (6.46). Using this and (6.50) we get the result for the twisted partition function

$$\tilde{Z}(\Sigma, u) = \frac{1}{\#\pi_1(H')}\left(\frac{\mathrm{vol}(H')}{(2\pi)^{\mathrm{dim}H'}}\right)^{2g-2}\sum_\alpha \frac{\lambda_\alpha(u^{-1})}{(\mathrm{dim}\alpha)^{2g-2}}. \tag{6.52}$$

As before, the volume of $\mathcal{M}'(u)$ can be written down directly as

$$\int_{\mathcal{M}'(u)} e^\omega = \frac{\#Z(H')}{\#\pi_1(H')}\left(\frac{\mathrm{vol}(H')}{(2\pi)^{\mathrm{dim}H'}}\right)^{2g-2}\sum_\alpha \frac{\lambda_\alpha(u^{-1})}{(\mathrm{dim}\alpha)^{2g-2}}. \tag{6.53}$$

6.4.2 *The Interpretation of Yang-Mills theory using Localization*

We will now look at the physical Yang-Mills theory introduced earlier with the lagrangian

$$L = -\frac{i}{4\pi^2}\int_\Sigma Tr\phi F - \frac{\epsilon}{4\pi^2}\int_\Sigma d\mu Tr\phi^2, \tag{6.54}$$

where $\epsilon \in \mathbb{R}^+$. After integrating the path integral with the lagrangian in (6.54) we get the lagrangian

$$I = -\frac{1}{8\pi^2\epsilon}\int_\Sigma d\mu Tr f^2, \tag{6.55}$$

which defines the same theory.

The partition function, with $I \propto (F,F) = -\int_\Sigma d\mu Tr f^2$, is on the exact form (6.1) that is governed by the localization principle. One can interpret I as the square of the moment map. The critical points of I are the solutions to the Yang-Mills equations (6.37). The solutions of (6.37) are $f = 0$ and higher critical points $f \neq 0$. For the higher critical points f gives a reduced structure group H_0 that commutes with f and the solutions are flat connections twisted by constant curvature line bundles in the $U(1)$ subgroup generated by f.

The partition function of the twisted bundle $E'(u)$, with the Hamiltonian $H = \frac{\epsilon'}{2}c_2 + t\epsilon'$ ($\epsilon' = 4\pi^2\epsilon$) using canonical quantization, is

$$\tilde{Z}(\Sigma,\epsilon;u) = \frac{1}{\#\pi_1(H')}\left(\frac{\text{vol}(H')}{(2\pi)^{\text{dim}H}}\right)^{2g-2}\sum_\alpha \frac{\lambda_\alpha(u^{-1})e^{-\epsilon' C_2(\alpha)/2+\epsilon' t}}{(\text{dim}\alpha)^{2g-2}}. \tag{6.56}$$

t is the parameter introduced in chapter 5 that appears because of different ways of defining the path integral.

Now we turn our attention to comparing (6.56) with localization predictions in the beginning of this chapter. We will do this in the two following examples.

Example 6.2 ($H = SU(2)$)**.** *In this example we will start by writing down the partition function with gauge group H=$SU(2)$. Then we will rewrite the result as a sum over critical points to put it in the right context of the localization principle.*

To write down the partition function we need the value of the Casimir operator. As this group has one n-dimensional irreducible representation α_n for every n we can write the Casimir operator as $c_2(\alpha_n) = (n^2-1)/2$. Let $t = 1/4$ to get the eigenvalues of *the Hamiltonian to be* $\epsilon' n^2/4$*. Here, vol*$(SU(2)) = 2^{5/2}\pi^2$ *and dim*$SU(2) = 3$*. This gives the partition function*

$$\begin{aligned} Z(\Sigma,\epsilon) &= \left(\frac{2^5\pi^4}{(2\pi)^6}\right)^{2(g-1)} \sum_{n=1}^{\infty} \frac{1}{n^{2g-2}} e^{-\epsilon n^2/4} \\ &= \frac{1}{(2\pi^2)^{(g-1)}} \sum_{n=1}^{\infty} \frac{1}{n^{2g-2}} e^{\frac{-\epsilon n^2}{4}}. \end{aligned} \tag{6.57}$$

If $f \neq 0$ *in the Yang-Mills equations the structure group is reduced to* $U(1)$ *and from line bundle classifications we get the conjugacy class of* f *is*

$$f = 2\pi m \begin{pmatrix} i & 0 \\ 0 & -1 \end{pmatrix}, \tag{6.58}$$

were m *is an integer.* I *at these critical points gives*

$$I_m = \frac{(2\pi m)^2}{\epsilon'} \tag{6.59}$$

Now we want to write the partition function as a sum of over the critical points. Start by looking at

$$\frac{\partial^{g-1} Z}{\partial \epsilon^{g-1}} = \frac{(-1)^g}{(8\pi^2)^{g-1}} \sum_{n=1}^{\infty} e^{\frac{-\epsilon n^2}{4}} = \frac{(-1)^g}{2(8\pi^2)^{g-1}} \left(-1 + \sum_{n\in\mathbb{Z}} e^{\frac{-\epsilon n^2}{4}}\right). \tag{6.60}$$

We can write

$$\sum_{n=1}^{\infty} e^{\frac{-\epsilon n^2}{4}} = -\frac{1}{2} + \frac{1}{2} \sum_{n\in\mathbb{Z}} e^{\frac{-\epsilon n^2}{4}}. \tag{6.61}$$

Then we can use Poisson summation to get the Jacobi inversion formula

$$\begin{aligned} \sum_{n\in\mathbb{Z}} e^{\frac{-\epsilon n^2}{4}} &= \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} dn e^{2\pi i n m - \frac{\epsilon n^2}{4}} \\ &= \sqrt{\frac{4\pi}{\epsilon}} \sum_{m\in\mathbb{Z}} e^{-\frac{(2\pi m)^2}{\epsilon}} \end{aligned} \tag{6.62}$$

to get the result of the derivation as

$$\frac{\partial^{g-1} Z}{\partial \epsilon^{g-1}} = \frac{(-1)^g}{2(8\pi^2)^{g-1}} \left(-1 + \sqrt{\frac{4\pi}{\epsilon}} \sum_{m\in\mathbb{Z}} e^{-\frac{(2\pi m)^2}{\epsilon}}\right) \tag{6.63}$$

and we see that the exponent agrees with I_m. If we put $m = 0$ in (6.63) we see that the $(g-1)$th derivative of Z is proportional to $\epsilon^{-1/2}$. In general, the structure is

$$Z(\epsilon) = \sum_{k=0}^{g-2} a_k \epsilon^k + a_{g-3/2}\epsilon^{g-3/2} + \textit{exp. small terms.} \tag{6.64}$$

The terms of $Z(\epsilon)$ that doesn't exponentially vanish have to be interpreted as the $\mu^{-1}(0)$ contribution by the arguments of the beginning of this chapter. The non-analytic contribution of $\mu^{-1}(0)$ tell us that $\mu^{-1}(0)$ is singular.

Example 6.3 (Non-trivial $SO(3)$-bundle with $u = -1$)**.** *In this example we will start by writing down the twisted partition function for a non-trivial $SO(3)$-bundle and then rewrite the result as a sum over critical points. Thereafter we can interpret the result using the localization principles.*

The twisted partition function with $\lambda_n(u^{-1}) = (-1)^{n+1}$ (extracted from Verlinde's formula in [33]), $\#\pi_1(H') = 2$, vol$(SO(3)) = 2^{3/2}\pi^2$ is

$$\tilde{Z}(\Sigma, \epsilon, -1) = \frac{1}{2(8\pi^2)^{g-1}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{2g-2}} e^{-\epsilon' n^2/4}. \tag{6.65}$$

If $f \neq 0$ in the Yang-Mills equations the structure group is reduced to $U(1)$ and from line bundle classifications we get that the conjugacy class of f is

$$f = 2\pi\left(m + \frac{1}{2}\right)\begin{pmatrix} i & 0 \\ 0 & -1 \end{pmatrix}, \tag{6.66}$$

where m is an integer. At these critical points, I is

$$I_m = \frac{(2\pi(m+\frac{1}{2}))^2}{\epsilon'} \tag{6.67}$$

Next we want to write $\tilde{Z}$ as a sum of over the critical points. Start by looking at

$$\frac{\partial^{g-1}\tilde{Z}}{\partial \epsilon'^{g-1}} = \frac{(-1)^g}{2(32\pi^2)^{g-1}} \sum_{n=1}^{\infty} (-1)^n e^{\frac{-\epsilon' n^2}{4}}. \tag{6.68}$$

We can write

$$\sum_{n=1}^{\infty}(-1)^n e^{\frac{-\epsilon' n^2}{4}} = -\frac{1}{2} + \frac{1}{2}\sum_{n\in\mathbb{Z}}(-1)^n e^{\frac{-\epsilon' n^2}{4}}. \tag{6.69}$$

Then we can use Poisson summation to get the Jacobi inversion formula

$$\begin{aligned}\sum_{n\in\mathbb{Z}}(-1)^n e^{\frac{-\epsilon' n^2}{4}} &= \sum_{m\in\mathbb{Z}}\int_{-\infty}^{\infty} dn e^{2\pi i n m + i\pi n - \frac{\epsilon' n^2}{4}} \\ &= \sqrt{\frac{4\pi}{\epsilon'}}\sum_{m\in\mathbb{Z}} e^{-\frac{(2\pi(m+1/2))^2}{\epsilon'}}\end{aligned} \tag{6.70}$$

to get the result of the derivation as

$$\frac{\partial^{g-1}\tilde{Z}}{\partial\epsilon'^{g-1}} = \frac{(-1)^g}{4(32\pi^2)^{g-1}}\left(-1 + \sqrt{\frac{4\pi}{\epsilon'}}\sum_{m\in\mathbb{Z}} e^{-\frac{(2\pi(m+1/2))^2}{\epsilon'}}\right). \tag{6.71}$$

We are now at the point were we can use the localization principles explained in the beginning of this chapter and compare them with $\tilde{Z}$*. For this bundle* $\mu^{-1}(0)$ *is smooth and* G *acts freely on it and therefore* $\tilde{Z}(\epsilon)$ *has to be a sum of polynomials in* ϵ *(of degree dim* $\mathcal{M}/4$ *at highest) plus exponentially small terms of higher critical points (that are unstable) when* $\epsilon \to 0$*. We can see from (6.67) that these higher critical points must make the contribute of the exponential term in (6.71) for small* ϵ*. We also see in (6.71) that* $\tilde{Z}(\epsilon)$ *is a polynomial in* ϵ *of degree* $g-1$ *up to exponentially small terms.*

We can write $\tilde{Z}(\epsilon)$ *as*

$$\tilde{Z}(\epsilon) = \frac{1}{2(8\pi^2)^{g-1}}\sum_{k=0}^{g-2}\frac{(-\pi^2\epsilon)^k}{k!}(1-2^{3-2g+2k})\zeta(2g-2-2k) + O(\epsilon^{g-1}) \tag{6.72}$$

To get the ϵ^k*-terms expand (6.65) in* ϵ*.* $\zeta(2n) = \frac{(2\pi)^{2n}(-1)^{n+1}B_{2n}}{2(2n)!}$ *using Euler's formula and* B_{2n} *is the Bernoulli number.*

This implies that

$$\int_{\mathcal{M}'(-u)} e^{\omega+\epsilon\theta} = (-1)^{(g+1)}\sum_{k=0}^{g-1}\frac{\epsilon^k}{n!}\frac{(2^{g-2-2k})}{2^{3g-1}(2g-2-2k)!}B_{2g-2-2k}. \tag{6.73}$$

$\mathcal{M}'(-u)$ *is the moduli space of flat* $SO(3)$ *connections.*

6.4.3 *Cohomology of $SO(3)$ Moduli Space*

To calculate the cohomology of $SO(3)$ moduli space $\mathcal{M}'(-1)$ we need both classes θ and ω and some non-algebraic cycles [48, 49]. We already calculated the intersection pairings on $\mathcal{M}'(-1)$ of θ and ω in (6.73) which leaves us to discuss the non-algebraic cycles.

We will use the formula (introduced in section 6.3)

$$\left\langle \exp\left(\omega+\epsilon\Theta\right)\beta\right\rangle' = \frac{1}{Vol(G)}\int DAD\phi D\psi e^{\frac{i}{4\pi^2}\int_\Sigma Tr(i\phi F+\frac{1}{2}\psi\wedge\psi)+\frac{\epsilon}{8\pi^2}\int_\Sigma d\mu Tr\phi^2}\beta. \tag{6.74}$$

This correlation function is the same as the integral over moduli space up to exponentially small terms.

We now add the new cycles using the operator, with circle $C \in \Sigma$,

$$V_C = \frac{1}{4\pi^2}\int_C Tr\phi\psi, \tag{6.75}$$

a three dimensional class on moduli space (depending on the homology class of C).

We will need an even number of $V_C's$ because otherwise their intersection pairings will vanish as algebraic cycles are even dimensional. Let us first look at

$$\begin{aligned}
&\left\langle \exp\left(\omega+\epsilon\Theta\right)V_{C_1}V_{C_2}\beta\right\rangle' = \\
&\frac{1}{Vol(G)}\int DAD\phi D\psi e^{\frac{i}{4\pi^2}\int_\Sigma Tr(i\phi F+\frac{1}{2}\psi\wedge\psi)+\frac{\epsilon}{8\pi^2}\int_\Sigma d\mu Tr\phi^2}\frac{1}{4\pi^2}\int_{C_1}Tr\phi\psi\frac{1}{4\pi^2}\int_{C_2}Tr\phi\psi. \\
&= \frac{1}{Vol(G)}\int DAD\phi D\psi e^{\frac{i}{4\pi^2}\int_\Sigma Tr(i\phi F+\frac{1}{2}\psi\wedge\psi)+\frac{\epsilon}{8\pi^2}\int_\Sigma d\mu Tr\phi^2}\frac{1}{4\pi^2}\sum_{P\in C_1\cap C_2}\frac{-\sigma(P)}{4\pi^2}Tr\phi^2(P) \\
&= -2\#(C_1\cap C_2)\frac{\partial}{\partial\epsilon}\left\langle \exp\left(\omega+\epsilon\Theta\right)\beta\right\rangle',
\end{aligned} \tag{6.76}$$

where we in the third line performed the ψ-integral using

$$<\psi_i^a(x)\psi_j^b(y)>= -4\pi^2\epsilon_{ij}\delta^{ab}\delta^2(x-y) \tag{6.77}$$

and used that $\sigma(P) = \pm 1$ is the oriented intersection number summed over all the points P of intersection of C_1 and C_2. In the fourth line we used that the cohomology class of $Tr\phi^2(P)$

is not dependent on P and can be written as $\int_\Sigma Tr\phi^2$ and that $\sum_P \sigma(P) = \#(C_1 \cap C_2)$.

If we now interpret this integral to be the same as the integral over the moduli space $\mathcal{M}'(-1)$ up to exponentially small terms then we have

$$\int_{\mathcal{M}'(-1)} e^{\omega+\epsilon\theta} V_{C_1} V_{C_2} = -2\#(C_1 \cap C_2) \frac{\partial}{\partial \epsilon} \int_{\mathcal{M}'(-1)} e^{\omega+\epsilon\theta}, \tag{6.78}$$

where we know the answer for the integral of the right hand side from (6.73).

In general, this result can be written for arbitrary many $V's$ (using a similar method) as

$$\int_{\mathcal{M}'(-1)} e^{\omega+\epsilon\theta+\sum_{\sigma=1}^{2g}\eta_\sigma V_{C_\sigma}} = \int_{\mathcal{M}'(-1)} e^{\omega+\hat{\epsilon}\theta}, \tag{6.79}$$

where η_σ $(\sigma = 1 \ldots 2n)$ are anti commuting variables, $\hat{\epsilon} = \epsilon - 2\sum_{\sigma<\tau} \eta_\sigma \eta_\tau \gamma_{\sigma\tau}$ and $\gamma_{\sigma\tau} = \#(C_\sigma \cap C_\tau)$ is the intersection number matrix of circles C_σ $(\sigma = 1 \ldots 2g)$, which constitutes a basis of $H_1(\Sigma, \mathbb{Z})$.

6.5 INTERSECTION RING OF MODULI SPACES

So far, we have seen that the Yang-Mills path integral can be written as the intersection pairings on the moduli space of flat connections plus exponentially small terms that vanish when ϵ goes to zero. We have calculated two examples for the gauge groups $SO(3)$ and $SU(2)$. We will conclude this chapter by looking at the intersection pairings of arbitrary gauge group H' that we assume to be compact and connected. We are going to calculate the partition function in general and in the smooth cases we can interpret it as intersection pairings.

As before, we will use a H' bundle $E'(u)$ over Σ, oriented and closed, of genus g and our formula

$$\langle \exp(\omega + \epsilon\Theta)\beta \rangle' = \frac{1}{Vol(G)} \int DAD\phi D\psi e^{\frac{1}{4\pi^2}\int_\Sigma Tr(i\phi F + \frac{1}{2}\psi\wedge\psi) + \frac{\epsilon}{8\pi^2}\int_\Sigma d\mu Tr\phi^2} \beta. \tag{6.80}$$

We assume that β is an equivariant differential form of polynomial ϕ-dependence and we write $\beta = e^{\sum_i \delta_i \beta}$, as it will be easier to work with exponentials. δ_i is fermonic (then $\delta_i^2 = 0$) and bosonic variables. The possible β_i's are of the following form. Assume that Q is an invariant polynomial of degree r on $\mathcal{H}$ then the equivariant differential forms are

$$\begin{aligned} Q_{(0)} &= \int_\Sigma d\mu Q(\phi) \quad \text{(of deg 2r)}, \\ Q_{(1)} &= -\int_C \frac{\partial Q}{\partial \phi_a}\psi^a \quad \text{(of deg 2r - 1)}, \\ Q_{(2)} &= \int_\Sigma \left(i\frac{\partial Q}{\partial \phi_a}F^a + \frac{1}{2}\frac{\partial^2 Q}{\partial \phi_a \partial \phi^b}\psi^a \wedge \psi^b \right) \quad \text{(of deg 2r - 2)}. \end{aligned} \tag{6.81}$$

Pick $Q(\phi)$ and $T(\phi)$ (with deg$q_i(\phi) > 2$ and deg$t_i(\phi) > 2$) as

$$\begin{aligned} Q(\phi) &= \frac{1}{8\pi^2}Tr\phi^2 + \sum_i \delta_i q_i(\phi). \\ T(\phi) &= \frac{\epsilon}{8\pi^2}Tr\phi^2 + \sum_i \delta_i' t_i(\phi). \end{aligned} \tag{6.82}$$

Assume that we have the oriented circles $C_\rho \subset \Sigma$ $(\rho = 1, \cdots, 2g)$ generating $H_1(\Sigma, \mathbb{Z})$ with $\gamma_{\sigma\tau} = \#(C_\sigma \cap C_\tau)$ being the intersection pairings. Then pick an invariant polynomial for every ρ to be $S^\rho(\phi) = \sum_i \eta_i^\rho s_i^\rho$ with η_i^ρ being anticommuting and s_i^ρ being homogeneous invariant polynomials.

Assuming $\det\frac{\partial\hat{\phi}^a}{\partial\phi^b} = 1$ we can now compute

$$\begin{aligned} &\left\langle e^{Q_{(2)}+T_{(2)}+S^\rho_{(1)}(C_\rho)} \right\rangle' = \\ &= \frac{1}{vol(G')}\int DAD\phi D\psi e^{\int_\Sigma \left(i\frac{\partial Q}{\partial \phi^a}F^a + \frac{1}{2}\frac{\partial^2 Q}{\partial \phi^a \partial \phi^b}\psi^a \wedge \psi^b \right) - \sum_\sigma \oint_{C_\sigma} \frac{\partial S^\sigma}{\partial \phi^a}\psi^a + \int_\Sigma d\mu T(\phi)} \\ &= \frac{1}{vol(G')}\int DA'D\phi e^{\int_\Sigma i\frac{\partial Q}{\partial \phi^a}F^a + \int_\Sigma d\mu \hat{T}(\phi)} \\ &= \frac{1}{vol(G')}\int DA'D\hat{\phi} e^{\frac{i}{4\pi^2}\int_\Sigma Tr\hat{\phi}F + \int_\Sigma d\mu \hat{T}\circ W(\hat{\phi})}, \end{aligned} \tag{6.83}$$

where ψ is shifted to complete the square and integrated over in the third line giving the normal symplectic measure DA' on the space of connections (as we discussed in 6.2). $(\partial^2 Q)^{-1}$ is the inverse matrix of $\partial^2 Q/\partial\phi^a\partial\phi^b$ and $\hat{T(\phi)} = T(\phi) - \sum_{\sigma<\tau}\gamma_{\sigma\tau}\frac{\partial S^\sigma}{\partial \phi^a}\frac{\partial S^\tau}{\partial \phi^b}(\partial^2 Q)^{-1}_{ab}$.

In the fourth line ϕ is changed to $\hat{\phi}$ by $\hat{\phi}_a = 4\pi^2 \frac{\partial Q}{\partial \phi^a}$. The Jacobian is $\det \frac{\partial \hat{\phi}^a}{\partial \phi^b} = 1$ as we assumed above. The transformation is invertible, as δ_i^2 are nilpotent, and the inverse is $\phi^a = W^a(\hat{\phi})$.

To write the result of (6.83) we need to modify (6.56) by defining the Hamiltonian in this case (we need a prescription how to pick t introduced above). To do this we will consider the generalized path integral

$$\int DAD\phi e^{\frac{i}{4\pi^2}\int_\Sigma Tr\phi F + \int_\Sigma Q(\tilde{\phi})} \tag{6.84}$$

where $Q(\tilde{\phi})$ is an invariant polynomial on $\mathcal{H}$ of degree t and $\tilde{\phi} = \frac{\phi}{4\pi^2}$.

In general when one goes from classical mechanics to quantum mechanics the only ambiguity is renormalization. In two dimensional Yang-Mills theory we need to pick the quantum operator $\hat{Q}$ corresponding to $Q(\tilde{\phi})$ from the range $\hat{Q} = Q(-iT) +$ *Casimir operators of lower order*. The map $Q \rightarrow \hat{Q}$ have to be a ring homomorphism from invariant polynomials on the Lie algebra to quantum operators. Thereby it is enough to pick $\hat{Q}$ with Q belonging to the generators of the ring of invariant polynomials [14].

For example, we can look at $SU(2)$. The ring is a polynomial ring with one generator that we take to be $Q(\tilde{\phi}) = Tr\tilde{\phi}^2$. Then $\hat{Q} = -\sum_a TrT_a^2 = c_2$ and lower order Casimir operator must be a constant. This constant is the ambiguity of the normal-ordering for $SU(2)$, which has been called t above.

For compact Lie group H with r being the rank, the ring of invariant polynomials is a polynomial ring in r generators [50]. Given a H, we have a finite number of t analogs. For $Q(\tilde{\phi})$ the corresponding Casimir $Q(-iT)$ is $Q(-iT) = Q'(h+\delta) = Q(h+\delta) +$ *Casimir operators of lower order* on an irreducible representation α_h of highest weight h (for example we can take $SU(2)$ and then write states $|jm>$, then the top state $|jj>$ is the heighest weight). δ is the sum of the positive roots divided by two. Then we can pick $\hat{Q} = Q(h+\delta)$ on α_h [50] (In general treatment of Lie algebras generators are divided into set of generator that commute with each other $\{H\}$ and a set of generalized raising and lowering operators $\{E\}$. Lie groups can then be classified due to this dividing and of the commutation relations between the members of the two sets and between the raising and lowering operators. For example if $[H_i, E_\alpha] = (\alpha)_i E_\alpha$ then α are called the *roots* of the Lie algebra). The Hamiltonian is $H = -\hat{Q}$ so we can now write down (6.52) in its generalized form as

$$\tilde{Z}(\Sigma, Q, u) = \frac{1}{\#\pi_1(H')}\left(\frac{\mathrm{vol}(H')}{2\pi^{\mathrm{dim}H'}}\right)^{2g-2}\sum_h \frac{\lambda_{\alpha(h)}(u^{-1})e^{Q(h+\delta)}}{(\mathrm{dim}\alpha(h))^{2g-2}}. \tag{6.85}$$

$\sum_h$ are the dominant weights of H.

We are now ready to go back to the calculation of (6.83). In canonical quantization $\frac{\hat{\phi}^a}{4\pi^2}$ is the group generator $-iT^a$. Let $W(\hat{\phi}) = V(\frac{\hat{\phi}}{4\pi^2})$. $\hat{T}\circ W(\hat{\phi})$ is in quantum theory an operator written as $\hat{T}\circ V(h+\delta)$ on the highest weight h representation.

We can now write the result of (6.83) using (6.85) as

$$\frac{1}{\#\pi_1(H')}\left(\frac{\mathrm{vol}(H')}{2\pi^{\mathrm{dim}H'}}\right)^{2g-2}\sum_h \frac{\lambda_{\alpha(h)}(u^{-1})e^{\hat{T}\circ V(h+\delta)}}{(\mathrm{dim}\alpha(h))^{2g-2}}. \tag{6.86}$$

The calculation could have been done without the assumption of $\det\frac{\partial\hat{\phi}^a}{\partial\phi^b} = 1$. This will give a slightly different formula which is calculated in a similar way as we just did. In this general setting it can be shown [14] that the path integral is given by

$$\frac{1}{\#\pi_1(H')}\left(\frac{\mathrm{vol}(H')}{2\pi^{\mathrm{dim}H'}}\right)^{2g-2}\sum_h \frac{\lambda_{\alpha(h)}(u^{-1})e^{\tilde{T}\circ V(h+\delta)}}{(\mathrm{dim}\alpha(h))^{2g-2}}, \tag{6.87}$$

where $\tilde{T} = \hat{T} + (g-1)\ln\det(\frac{\partial^2 Q'}{\partial\phi^a\partial\phi^b})$.

We can now make some final conclusions. When $\epsilon > 0$ the path integral can be calculated as above without carrying about singularities of $\mathcal{M}'(u)$. If the moduli space of flat connections is smooth and acted on freely by the gauge group then (6.87) is a polynomial in ϵ up to exponentially small terms that vanish for $\epsilon \to 0$. The polynomial can be interpreted as the intersection numbers on the moduli space written as

$$\frac{\#Z(H')}{\#\pi_1(H')}\left(\frac{\mathrm{vol}(H')}{2\pi^{\mathrm{dim}H'}}\right)^{2g-2}\sum_h \frac{\lambda_{\alpha(h)}(u^{-1})e^{\tilde{T}\circ V(h+\delta)}}{(\mathrm{dim}\alpha(h))^{2g-2}} = \int_{\mathcal{M}'(u)} e^{Q_2+T_0+\sum_\sigma S^\sigma_{(1)}(C_\sigma)} + \text{exp. small terms.} \tag{6.88}$$

This is so, using the fact that in cohomological gauge theories, with $\mathcal{M}$ non-singular, we have [14, 47]

$$\langle \prod_{s=1}^{n} T_{j_s}(V_s) \rangle = \frac{1}{\#Z(H)} \int_{\mathcal{M}} \prod_{s=1}^{n} \hat{T}_{j_s}(V_s) \tag{6.89}$$

with $\hat{T}_j(V)$ in $H^{2r-j}(\mathcal{M}, \mathbb{R})$, the j-dimensional submanifold V of Σ and the homogeneous invariant polynomial T as above. This equality is true if we use (6.32) to calculate the left hand side. It also holds if we use (6.40) up to exponentially small terms for $\epsilon \to 0$.

7

STRING THEORY INTERPRETATION OF TWO DIMENSIONAL YANG-MILLS THEORY

The two dimensional Yang-Mills theory, or pure two dimensional QCD, can be solved exactly as we have seen. The result is stated quite formally as a sum over representations. We now like to write it explicitly for SU(N) and U(N) and interpret it in terms of strings. The parameter N can be viewed as a free parameter (taking N color states instead of three). In the large N limit the 1/N expansion can then be interpreted as a string perturbation series [16]. As we will see, it can be shown that the coefficients of the expansion are given by a sum over maps from a two dimensional surface onto the two dimensional target space. Thus we can interpret two dimensional QCD as a closed string theory.

In the first, second and third sections we will give a short introduction to string theory, the symmetric group and Young tableaux, and Riemann surfaces. Thereafter, we will move on to the interpretation of two dimensional Yang-Mills theory as a string theory.

7.1 SHORT INTRODUCTION TO STRING THEORY

We will now introduce some basic concepts of string theory (for an extensive introduction see [51, 52]). That the strong interaction might be represented by a string theory is an idea older than the QCD theory. In the late 1960s a string theory was found when people tried to guess a mathematical formula for the strong interaction scattering amplitudes that would agree with current experiments. A lot of the properties of hadrons can be understood if we look at the hadrons as string-like flux tubes. The picture is consistent with linear confinement (that quarks clump together and cannot be seen separately because the force between them grows as they move apart) and with the linear Regge trajectories [53] (linear correlation of mass squared and spin of families of mesons). But viewing the strong force as a one dimensional string made a lot of contradictions with experimental results and in the middle of the 1970s this theory was abandoned for QCD.

String theory has since developed a lot and what started as a theory for trying to describe the strong interaction has later been seen as a better fit for a theory of everything (describing all forces and matter in nature). However there are no experimental proofs for this and a lot remains to work out in the theory/theories.

To introduce strings we begin by considering a relativistic particle. It traces out a *world-line* when it moves through spacetime. The line can be parameterized using only one parameter. A string however traces out a *world-sheet* (see figure 7.1 for a comparison), a two dimensional surface, which can be parameterized by two parameters. The two dimensional surface lives in the space that is called the *target space* (see figure 7.2) [51].

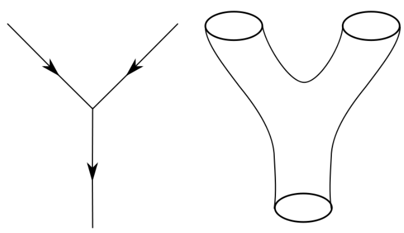

Figure 7.1: A world-line of a particle and a world-sheet of a string splitting into two.

The action of a relativistic point particle is proportional to the proper time elapsed on the point particle world-line. For strings the action is proportional to the proper area of the world-sheet. The Nambu-Goto action for a relativistic string is [51]

$$S = -\frac{T_0}{c} \int_{t_i}^{t_f} d\tau \int_0^{\sigma_1} \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 \cdot (X')^2}. \qquad (7.1)$$

This is a natural generalization of the relativistic point particle action. T_0 is the string tension, c the speed of light, $\dot{X}$ is the derivative of X (string coordinates) with respect to τ and X' the derivative with respect to σ. σ and τ are parameters of the world-sheet.

There is another action related to the Nambu-Goto action called the Polyakov action [52]

$$S = \frac{T}{2} \int d^2\sigma \sqrt{-h} h^{ab} g_{\mu\nu}(X) \partial_a X^\mu(\sigma) \partial_b X^\nu(\sigma). \tag{7.2}$$

T is the string tension, h_{ab} (with inverse h^{ab} and determinant h) is the metric of the world-sheet, $g_{\mu\nu}$ the metric of the target space. σ and τ are the parameters of the world-sheet.

The Polyakov action is more easily quantized because it is linear. The action is globally invariant under spacetime translations and infinitesimal Lorentz transformations. Locally it is invariant under world-sheet diffeomorphisms (coordinates transformations) and Weyl transformations (local rescaling of metric $g_{ab} \to e^{-2\omega(x)} g_{ab}$).

In string theory the proper area is invariant under the choice of parametrization. In other words, we can use many different grids (constant lines of the parameters) on the world-sheet to describe the same physical motion of the string. The reparameterization invariance (invariance under diffeomorphisms) is analogous to gauge invariance in electrodynamics.

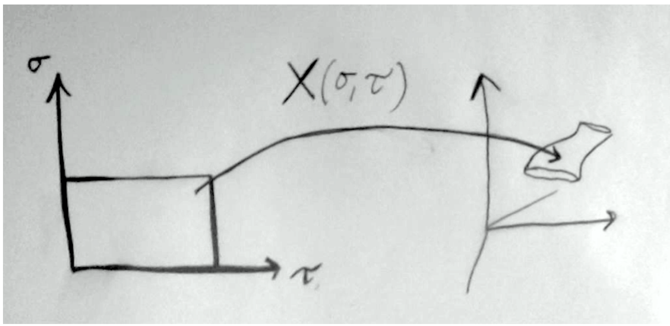

Figure 7.2: The map $X(\sigma, \tau)$ from the world-sheet parameterized by σ and τ to the target space.

7.2 THE SYMMETRIC GROUP AND YOUNG TABLEAUX

The symmetric group S_n (see [60]) is all permutations on n symbols, i.e the members of S_n are all permutations of a set $X = \{1, 2, \ldots, n\}$. For example take the set $\{1, 2, 3\}$, then we have the members $(1,2,3)$, $(1,3,2)$, $(2,1,3)$, $(2,3,1)$, $(3,1,2)$, and $(3,2,1)$.The number of elements, the order, of S_n is $n!$ as there are $n!$ permutations of n symbols and it has degree n. The group operation in S_n is function composition. For example, let f and g be two permutations given by

$$\begin{aligned} f = (1\ 3)(4\ 5) &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix} \\ g = (1\ 2\ 5)(3\ 4) &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 3 & 1 \end{pmatrix}. \end{aligned} \tag{7.3}$$

Then the composition is given by

$$fg = (1\ 2\ 4)(3\ 5) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix}. \tag{7.4}$$

The symmetric group is important in both mathematic and physics. For example, when one has a system of n identical particles the symmetric group will be in the symmetry group of the Hamiltonian.

If we like to know the dimension of the irreducible representation (irrep) of say S_4 corresponding to the partition $(2,1^2)$ (this means that we have one 2-cycle, two symbols permuting into each other, and two 1-cycles) we can use *Young tableaux* (see for example [55]). A Young tableau is a graph of a given shape, built up from square boxes. For $(2,1^2)$, or equally written as $(12)(3)(4)$, we can put the symbols into the Young tableau as

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array} \; . \tag{7.5}$$

The boxes and numbers are put out such that the first line has more or equal number of boxes as the second line etc. and that the numbers in the boxes increases to the right in a row and downwards in a column. The dimension is equal to the number of different diagrams, i.e. three in this example. The dimension can also be calculated using the Hook length formula

$$\dim \pi_\lambda = \frac{n!}{\prod_x \mathrm{hook}(x)}, \tag{7.6}$$

where hook(x) of a box x is the sum of the box x plus the boxes that are in the same row to the right of it plus the boxes in the same column below it.

The Young tableaux also classify the irreps of GL(N) on $V^{\otimes n}$. To construct irreps for SU(N) for example, we use vector spaces that carry the representation, building them by multiplying tensors that carry the fundamental [N] representation. The tensors can then be decomposed into irreps by (anti)symmetrization (divided into parts that don't transform into each other under group transformations). The Young tableaux can be used to represent the Clebsch-Gordan series (the decomposition of a product representation into irreducible components, i.e classifying the irreps).

Let us take SU(2) as an example. The basis state is represented by a box (which thus represents a particle). If the box is empty then it represents any state. If we put a number (1 or 2 here for the two basis states) in the box it represent a particular state

$$u_1 = \boxed{1} \quad u_2 = \boxed{2} \ . \tag{7.7}$$

Next we can take a direct product written as

$$\square \otimes \square = \square\square \oplus \begin{array}{c}\square\\ \square\end{array} \ . \tag{7.8}$$

If we use the rules introduced above (but with the change that the numbers in the boxes of a row must be nondecreasing from left to right) we can write this out as

$$\boxed{1}\boxed{1} \quad \boxed{1}\boxed{2} \quad \boxed{2}\boxed{2} \tag{7.9}$$

for the symmetric states and

$$\begin{array}{c}\boxed{1}\\ \boxed{2}\end{array} \tag{7.10}$$

for the antisymmetric state. We can do this procedure for $N > 2$ as well. The number of symmetric states for SU(N) are then $\frac{1}{2}N(N+1)$ and $\frac{1}{2}N(N-1)$ for antisymmetric states. An antiquark can be represented by $N-1$ boxes in a column denoted $\bar{N}$.

We can then move on and do a direct product of three states

$$\square \otimes \square \otimes \square = \square\square\square \oplus \begin{array}{l}\square\square\\ \square\end{array} \oplus \begin{array}{c}\square\\ \square\\ \square\end{array} \ . \tag{7.11}$$

The dimension of SU(N) can be calculated from a Young tableau by putting N in the top left corner and then increasing the number by 1 for each box to the left in the row but decreasing by 1 going down in a column. Continuing this procedure for all boxes one gets the dimension by taking the product of all numbers of the boxes and dividing by the hook lengths.

7.3 RIEMANN SURFACES

It is necessary to introduce the concept of Riemann surfaces (see for example [54]) a bit further before we can go on to the string interpretation of the Yang-Mills theory.

A Riemann surface is a one dimensional complex analytic connected manifold. It is a Hausdorff topological space with an atlas, thus for every point there is a neighborhood containing the point homeomorphic to the unit disk of the complex plane.

Riemann surfaces enters naturally when one tries to define $\sqrt{z}$. In real variable we learn that we can't take negative values of the variable but if we go to complex variable we can represent z by polar coordinates as $z = re^{i\theta}$. Then $\sqrt{z} = \sqrt{r}e^{i\theta/2}$. What goes wrong is that if we now smoothly vary θ from 0 to 2π, with $r = 1$ say, then wee see that $\sqrt{1}$ can be both -1 and 1. This is not a function (a function can't be multi-valued). What we really have to do is to take two complex planes and draw a branch cut (a line) on each from the origin (the branch point) to infinity (in which direction is not important) and when one crosses the branch cut one moves to the other complex plane. This is why $\sqrt{z}$ is really a complex function and it is required to be defined on a Riemann surface (the two complex planes (called sheets) fitted together along the branch cut), see figure 7.3. Thus we have resolved the ambiguities of a multi-valued function by changing its domain.

Let's give some more examples of Riemann surfaces. The complex plane $\mathbb{C}$ and the Riemann sphere (or the extended complex plane) $\mathbb{C} \cup \{\infty\}$ are both Riemann surfaces.

We will now turn our attention to some results concerning Riemann surfaces [54].

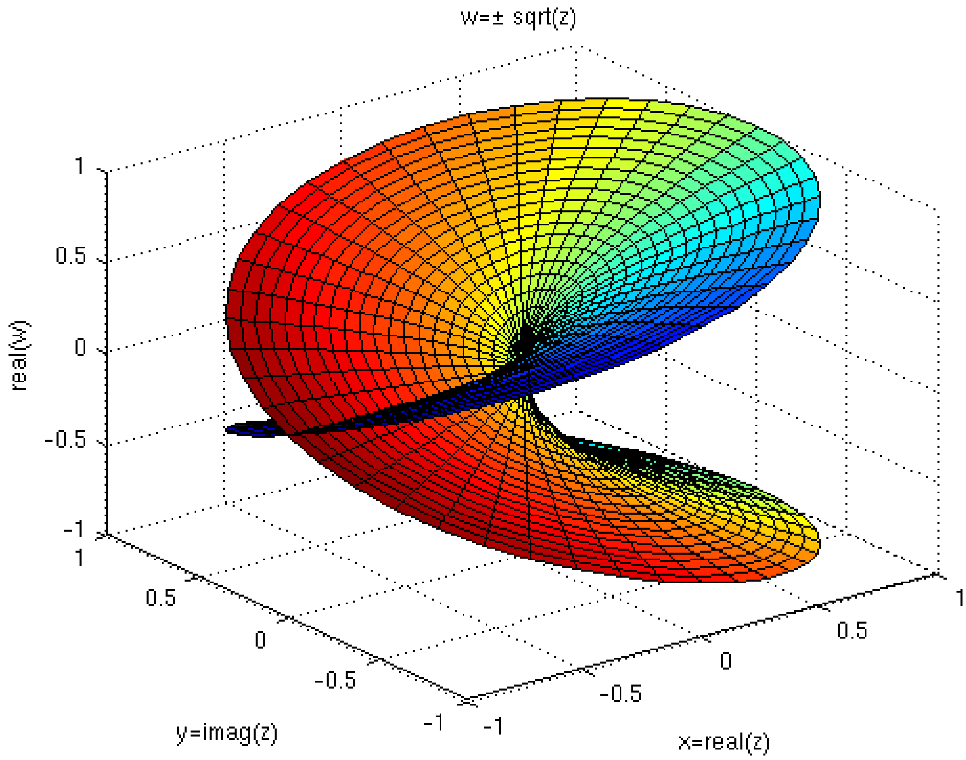

Figure 7.3: The Riemann surface for $f = \sqrt{z}$. The two horizontal axes represent the real and imaginary parts of z and the vertical axis represents the real part of $\sqrt{z}$.

Let $f : M \to N$ be a continuous map between Riemann surfaces. f is called *holomorphic* or *analytic* if for every local coordinate $\{U, z\}$ on M and $\{V\xi\}$ on N the map

$$\xi \circ f \circ z^{-1} : z(U \cap f^{-1}(V)) \to \xi(V) \tag{7.12}$$

is holomorphic (complex differentiable at every point in open set, essentially a complex function that doesn't depend on the complex conjugate of z).

Furthermore, let M be compact and f a holomorphic mapping. Then f is either constant or surjective (N is also compact in the latter case).

A continuous map $f : M \to N$ is called a *branched cover* (see figure 7.4) if we have a point $P \in N$ with neighborhood $U \in N$ so that the inverse $f^{-1}(U)$ is a union of disjoint open sets where f is topologically equivalent to $z \to z^n$ [18]. n is called the *ramification number* of f at P (or ramification index, winding number or that f takes the value of $f(P)$ n times at P) and $(n-1)$ is called the *branch number* of f at P denoted $b_f(P)$. Then there exists a positive integer m such that each $Q \in N$ is assumed precisely m times on M by f, thus

$$\sum_{P \in f^{-1}(Q)} (b_f(P) + 1) = m. \tag{7.13}$$

m is called the degree of f and one says that f is an m-sheeted cover of N by M (or f has m sheets).

Two branched covers f_1 and f_2 are *equivalent* if there is a homeomorphism $\phi : M \to N$ such that $f_1 \circ \phi = f_2$.

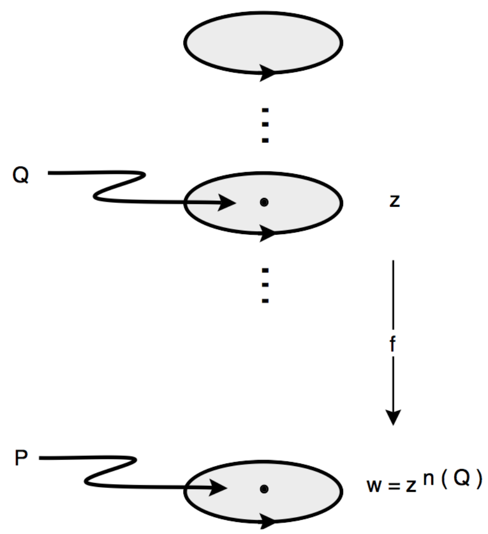

Figure 7.4: Branched covering. On the disk containing Q the map is $z \to w = z^{n(Q)}$ [18].

Now, pick a complex structure J on M. Then given a branched cover $f : M \to N$ there exist a unique complex structure on N making f holomorphic [18].

If f is a non-constant function on M then f has as many zeros as poles. And a single non-constant meromorhic (holomorphic except at isolated points) function completely determines the complex structure of the Riemann surface [54]. The local coordinate vanishing at P is given by

$$\begin{aligned} (f - f(P))^{1/n} \quad &\text{if} \quad f(P) \neq \infty, \\ f^{-1/n} \quad &\text{if} \quad f(P) = \infty. \end{aligned} \tag{7.14}$$

Next we turn our attention to the Riemann-Hurwitz theorem [54]. Let $f : M \to N$ be the non-constant holomorhic map between two compact Riemann surfaces. Let M be a surface of genus h and N of genus G and assume that the degree of f is n. The theorem states that

$$2h - 2 = n(2G - 2) + B, \tag{7.15}$$

where $B = \sum_{P \in M} (b_f(P))$ is the *total branching number* and 2-2h is the Euler character of M.

For smooth maps we have the Kneser's formula [62] given by

$$2(g - 1) \geqslant 2n(G - 1) + B. \tag{7.16}$$

Riemann surfaces can be defined by polynomial equations

$$P(z, w) = w^n + a_{n-1}(z)w^{n-1} + \cdots + a_1(z)w + a_0(z) = 0, \tag{7.17}$$

which can be compactified. This is given by a theorem that states that every compact Riemann surface is algebraic [54].

Let us give one example by looking at the equation

$$T = \{(z, w) \in \mathbb{C}^2 | w^2 = (z^2 - 1)(z^2 - k^2)\}, \quad k \neq \pm 1. \tag{7.18}$$

For each value of z there are two values of w, except $z = \pm 1$ and $z = \pm k$ where $w = 0$. Doing the analysis (see for example [56]) one can see that w behaves like the square root close to 1 and $\pm k$. We can draw two branch cuts between k and 1, and between 1 and k (this can be done in case of a surface). We then take two sheets of these with opposite signs. The matching edges of the cuts can then be aligned and stretched out around the cut to pull out tubes to connect the two sheets. The resulting surface is a torus and we can compactify it by adding two points at infinity, see figure 7.5. The projection to the z-axis has degree 2, and we have four branch points with total index 4.

A compact Riemann surface can be mapped to surfaces of lower genus but not to higher genus, except as constant maps. The reason for this is because that maps that are holomorphic or meromorphic behave like $z \mapsto z^n$ locally, thus non-constant maps are ramified covering maps. For compact Riemann surfaces they are

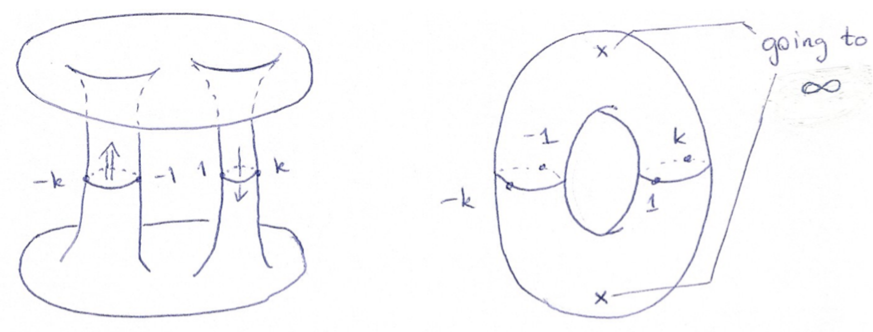

Figure 7.5: The torus given by (7.18) [56].

constrained by the Riemann-Hurwitz formula, that describes a relation between the Euler characteristics of two surfaces one being a ramified covering of the other.

In the next section we again turn our attention to the partition function of two dimensional Yang-Mills theory, expand it in a power series in $1/N$ and explain how it can be interpreted in terms of maps of two dimensional surfaces onto two dimensional surfaces.

7.4 YANG-MILLS THEORY AND STRINGS

Yang-Mills theory is the foundation for the formulation of QCD. QCD perturbation theory in the weak coupling expansion has proved to describe high-energy strong interaction scattering well. But as the energy gets lower, distances gets larger the coupling constant grows (no single quark has been seen in experiments) and the expansion is not valid. Because of this there is a necessity to find a theory for the infrared (IR) limit.

Lattice gauge theory has been able to give some insights to QCD in the IR limit [58]. It was seen that in the strong coupling expansion the free energy can be expressed as a sum over surfaces [57], which can be seen as an indication of a string equivalence of QCD. However it has shown to have very complicated weights of the surfaces.

It was shown in 1974 [57] that the expansion of the weak coupling perturbation theory of QCD can be interpreted as an expansion of an equivalent string theory, where the string coupling is given by 1/N. In the large N limit, with g^2N held fixed (g is the gauge coupling), planar diagrams dominate. They can be viewed as the world-sheet of a string. The Feynman diagrams is shown to be given by triangulations of a two dimensional surface. The main result was that 1/N can be used to pick the topology (genus). This is because a diagram corresponding to a Riemann surface of genus G is weighted by $(1/N)^{2G-2}$. In the expansion of the free energy the leading order in powers of 1/N is given by the planar diagrams and is proportional to N^2.

Thus, instead of doing the expansion in the gauge coupling, which is not good at low energies, one takes 1/N as expansion parameter. Then $N = 3$ has to be looked at as large and the limit $N \to \infty$, with $g^2 N =$ constant, is often used to give the right 1/N expansion.

Two dimensional QCD is a good model to test if there is an equivalence to a string theory for QCD. It has physically relevant features like confinement and a linear spectrum. We consider a theory without quarks that correspond to a string theory with only closed strings (quarks are attached to open string ends). A theory without quarks might look trivial because it has no physical degrees of freedom for the gluon in two dimensions. However, the free energy of the gluons will depend non-trivially on the manifold where they live.

The Yang-Mills theory on a two dimensional manifold M of genus G and area A is exactly solvable as shown in chapter 5 using lattice regularization. The partition function is given by [45]

$$Z[G, g^2 A, N] = \sum_R (\mathrm{dim} R)^{2-2G} e^{\frac{-\lambda A}{2N} c_2(R)}. \tag{7.19}$$

This depends only on the genus and the area of the manifold. The sum runs over all representations R of the gauge group (here U(N) or SU(N)), $c_2(R)$ is the quadratic Casimir and $\lambda = g^2 N$ with gauge coupling g.

In 1992 it was conjectured [15] that the free energy, $W = \ln Z$, is equal to some string theory partition function, where the string coupling is 1/N and string tension is $g^2 N$, i.e.

$$W[G, g^2 A, N] = Z^{\mathrm{string}}[\frac{1}{N}, g^2 N]. \tag{7.20}$$

For example, the free energy in the Polyakov string theory is written as a sum over all connected Riemann surfaces. Every term in the sum is weighted by $(g_{string})^{2h-2}$, with the string coupling g_{string} and h is the genus of the world-sheet. So, if two dimensional QCD is to have a corresponding string theory the free energy is expected to be given by even powers of $g_{string} = 1/N$ (at least perturbatively). The free energy, in this interpretation, is thus given by maps of the world-sheet of genus h into the target space of genus G. And we should expect that the $1/N$ expansion of the partition function to be of the form

$$Z \sim \exp \sum_{h \geqslant 0} \left(\frac{1}{N}\right)^{2h-2} Z_h. \tag{7.21}$$

By expanding the free energy W in powers of 1/N we write

$$W = \sum_{h=1}^{\infty} N^{2-2h} f_h^G(\lambda A), \tag{7.22}$$

Z_h and $f_h^G(\lambda A)$ should be interpreted as sums over maps from the word-sheet to the target space.

In 1993 [16] it was shown that two dimensional QCD is a string theory. The large N expansion of the partition function is described by two coupled parts, chiral sectors (one chiral partition function and an anti-chiral partition function coupled by an simple term) [16].

In the next section we will give the necessary formulas for the large N expansion. We then look at a single chiral sector and show that the leading terms are given by the sum of a symmetry factor over all branched covers of M. We then put everything together in a nonchiral sum giving the result for the partition function and free energy. At the end of this chapter we will write down a string theory action equivalent to two dimensional Yang-Mills theory, which was found through the tools of cohomological field theory [64].

7.5 THE LARGE N EXPANSION

The partition function in (7.19) is written as a sum over all representations. We now like to write it out explicitly for U(N) and SU(N).

The U(N) and SU(N) representations can be labeled by Young tableaux. The tableaux can be described by n boxes in rows with length n_i in row i and columns of length c_i [16]. Thus, there are different ways to describe a Young tableau:

$$\begin{aligned}
&\bullet \quad \{n_i\}, \quad \sum n_i = n \\
&\bullet \quad \{c_i\}, \quad \sum c_i = n \\
&\bullet \quad \{l_i\}, \quad \sum i l_i = n.
\end{aligned} \tag{7.23}$$

where $\{n_i\}$ are the number of boxes in each row, $\{c_i\}$, the number of boxes in each column and l_i are the number of columns of length i.

The quadratic Casimirs and dimensions are given by [15, 16, 59]

$$\begin{aligned}
c_2^{U(N)}(R) &= Nn + \tilde{c}(R), \\
c_2^{SU(N)}(R) &= Nn + \tilde{c}(R) - \frac{n^2}{N}, \\
\tilde{c}(R) = \sum_{i=1}^{m} n_i(n_i + 1 - 2i) &= \sum_{i=1}^{m} -c_i(c_i + 1 - 2i) = \sum_{i=1}^{m} n_i^2 - \sum_{i=1}^{m} c_i^2, \\
\dim(\mathrm{R}) &= \frac{\Delta(h)}{\Delta(h^0)}, \\
\Delta(h) &= \prod_{1 \leqslant i < j \leqslant N} (h_i - h_j), \\
h_i = N + n_i - i, &\quad h_i^0 = N - i.
\end{aligned} \tag{7.24}$$

The dimension d_R of the associated representation of the symmetric group S_n is given by [60]

$$d_R = n! \frac{\Delta(h)}{h_1! \ldots h_n!}. \tag{7.25}$$

It is related to $\dim(R)$ as [61]

$$\dim(R) = \frac{d_R}{n!} \prod_{i=1}^{r} \frac{\lambda_i!}{(N-i)!} \tag{7.26}$$

with r being the number of non-empty rows. Using $\frac{(N+n_i-i)!}{(N-i)!} = \prod_{k=1}^{n_i} (N + k - i) = N^{n_i} \prod_{k=1}^{n_i} \left(1 + \frac{k-i}{N}\right)$ it can be written as [15]

$$\dim(R) = \frac{d_R N^n}{n!} \prod_v (1 + \frac{\Delta_v}{N}) = \frac{d_R N^n}{n!} + O(N^{n-1}). \tag{7.27}$$

v runs over all boxes in the diagram, the difference of a column and row index of a box is denoted Δ_v.

7.5.1 *Chiral 1/N expansion*

We will now look at the single chiral sector and perform the asymptotic expansion of the partition function by expanding in 1/N. We put in the value of $c_2(R)$ from (7.24) for SU(N) (as the only difference compared to U(N) is the last term) and we have to interpret the four terms in the chiral partition function Z' given by

$$\begin{aligned} Z'(G,\lambda A,N) &= \sum_R \dim(R)^{2-2G} e^{-\frac{\lambda A c_2(R)}{2N}} \\ &= \sum_R \dim(R)^{2-2G} e^{-\frac{\lambda A n}{2}} e^{-\frac{\lambda A \tilde{c}(R)}{2N}} e^{-\frac{\lambda A n^2}{2N^2}}. \end{aligned} \tag{7.28}$$

We take $\lambda = g^2 N = \text{constant}$ when taking the limit $N \to \infty$ (motivated in [57]).

The first two exponentials is understood using branched coverings, where the first exponential looks like the area of a string which winds n times around the target space of area A. The last exponential give rise to the introduction of tubes and collapsed (infinitesimal) handles discussed further in 7.5.4.

Using (7.24) and (7.27) the chiral partition function (7.28) is written as (without writing the $e^{-\frac{\lambda A n^2}{2N^2}}$-term)

$$Z'[G,\lambda A,N] = \sum_{n=0}^{\infty} \sum_{R\in Y_n} (\dim(R))^{2-2G} e^{-\frac{\lambda A}{2N} c_2(R)}$$
$$= \sum_{n=0}^{\infty} \sum_{R\in Y_n} \left(\frac{n!}{d_R}\right)^{2G-2} e^{-\frac{n\lambda A}{2}} \times \sum_{i=0}^{\infty} \left[\frac{(-\lambda A \tilde{c}(R))^i}{2^i i!} N^{n(2-2G)-i} + O\left(N^{n(2-2G)-i-1}\right)\right]. \tag{7.29}$$

$O(N^{n(2-2G)-i-1})$ indicates the subleading contributions from the last term in $c_2(R)$ as well as contributions from the dimension (for $G \neq 1$). This expansion contains only half of the full theory because there is another set of representations that has quadratic Casimir of leading-order term nN, which we will see in the subsection of the nonchiral sum 7.5.3. First we will show that the leading terms in the chiral partition function (7.29) are given by the sum of maps from two dimensional surface to the two dimensional target space.

7.5.2 *The Symmetry Factor*

We will now look at the leading terms of the chiral partition function (7.29). Actually, we want a geometrical way of describing the coefficients in the 1/N expansion of the free energy in (7.22) but it will show easier to work with the chiral partition function [16] given by the 1/N expansion as

$$Z'(G, \lambda A, N) = \sum_{h=-\infty}^{\infty} \sum_{n} \sum_{i} \xi_{h,G}^{n,i} e^{-n\lambda A/2} (\lambda A)^i N^{2-2h}. \tag{7.30}$$

From (7.29) we see that when $2(h-1) = 2n(G-1) + i$ the coefficients $\xi_{h,G}^{n,i}$ are

$$\xi_{h,G}^{n,i} = \sum_{R} \frac{n!}{d_R} \frac{1}{i!} \left(\frac{\tilde{c}(R)}{2} \right)^i . \tag{7.31}$$

These coefficients has an interpretation in terms of covering maps. The covering maps should have a fixed winding number n and only be singular at points. We have to consider only singularities that come from branch points as the 2-fold cover $z \to z^2$ of the unit disk in $\mathbb{C}$.

Let $\Sigma(G, n, i)$ be the set of n-fold covers of the two dimensional Riemann surface M_G of genus G that has i branch points. Let $v : M_h \to M_G$ be such a cover map, then $2(h-1) = 2n(G-1) + i$. The covering spaces that are disconnected will also be included, such that h can be negative (the Euler characteristic χ of a disconnected surface is $2 - 2h = \chi$). To each cover v, we associate a symmetry factor $|S_v|$. $|S_v|$ is defined as the number of distinct homeomorphisms $\pi : M_h \to M_h$ such that $v\pi = v$. Then we can write [16]

$$i! \xi_{g,G}^{n,i} = \sum_{v \in \Sigma(G,n,i)} \frac{1}{|S_v|}. \tag{7.32}$$

To prove this we will count the number of branched covers of M_G.

Let's first look at $i = 0$, i.e no branch points. Let M be a surface of genus G and area A and chose a point $p \in M$. Take a set of generators for $\pi_a(M, p)$ (the fundamental group; determining when two paths that starts and ends at a fixed point can be continuously deformed into each other) $a_1, b_1, a_2, b_2, ...$, which can be identified with the $H_1(M)$ with the basis $a_1 \cdot b_1 = \delta_{ij}$, $a_i \cdot a_j = b_i \cdot b_j = 0$. Using these generators $\pi_1(M)$ can be defined through (when the space is homeomorphic to a simplicial complex)

$$a_1, b_1, a_1^{-1}, b_1^{-1}, \ldots, a_G, b_G, a_G^{-1}, b_G^{-1} = 1. \tag{7.33}$$

Fundamental groups π_1 can be studied using covering spaces. This is because a fundamental group is isomorphic the group of automorphism of the associated to the universal covering space (automorphism means isomorphism from the object to itself; an automorphism of a cover $p : C \to X$ is a homeomorphism $f : C \to C$ in the way that $p \circ f = p$ and the set of all automorphisms of p forms a group under composition, $\text{Aut}(p)$).

The universal covering space is a covering space that is simply connected. The name universal cover comes from the fact that if the map $q : D \to X$ is a universal cover of the space X and the map $p : C \to X$ is any cover of the space X, with the covering space C being connected, then there exists a covering map $f : D \to C$ such that $p \circ f = q$. We can say that any connected cover is covered by the universal cover. Each automorphism permutes the elements of each fiber, which defines the automorphism group action on each fiber.

Now, take v to be the n-fold unbranched cover of M, by choosing a labeling of the sheets of v over p with integers $I = \{1, \ldots, n\}$ then we can construct a map from $\pi_1(M)$ to S_n (the symmetric group defined above). We get the map by associating every element $t \in \pi_1(M)$ to I that comes from lifting t to the cover space and going around the paths transporting the labels on sheets.

This map defines a homomorphism $H_v : \pi_1(M) \to S_n$. If we look at a fixed covering of M then we have $n!$ possible labelings of the sheets over p.

If we have two labeling differing by $\rho \in S_n$, then we have homomorphisms H and $H' = \rho H \rho^{-1}$ that are related by conjugation by ρ. Every element of the symmetry group S_v produce a permutation ρ that leaves H_v invariant. The number of distinct homomorphisms $H : \pi_1(M) \to S_n$ related to a fixed cover v, that has symmetry factor $|S_v|$, is $n!/|S_v|$. To count every cover v with weight $1/|S_v|$, it is enough to sum distinct homomorphism $H : \pi_1(M) \to S_n$ with weight $1/n!$.

The weighted sum over unbranched covers is then given by

$$\sum_{v\in\Sigma(G,n,i)} \frac{1}{|S_v|} = \sum_{s_1,t_1,\ldots,s_G,t_G\in S_n} \left[\frac{1}{n!}\delta\left(\prod_{i=1}^{G} s_i t_i s_i^{-1} t_i^{-1}\right)\right] \tag{7.34}$$

where δ is a Kronecker delta function defined on S_n by $\delta(\rho) =$ identity if $\rho = 1$ and $\delta(\rho) = 0$ if $\rho \neq$ identity.

Now consider the case with branch points ($i \neq 0$). The branch points with branching number j are counted as j distinct branch points.

We can cut the surface M along curves $a_1, b_1, \ldots$ to construct a 4G-gon that has i branch points $q_1, \ldots, q_i$. We can choose the branch points in $A^i/i!$ ways. Furthermore we can take closed curves $c_1, ..., c_i$ on the 4G-gon and make c_j pass through p and go around q_j and not intersect any another curve, see figure 7.6. Then we can define $\pi_1(M\ \{q_1, \ldots q_i\})$ by

$$c_1 c_2 ... c_i a_1 b_1 a_1^{-1} b_1^{-1} ... a_G b_G a_G^{-1} b_G^{-1} = 1. \tag{7.35}$$

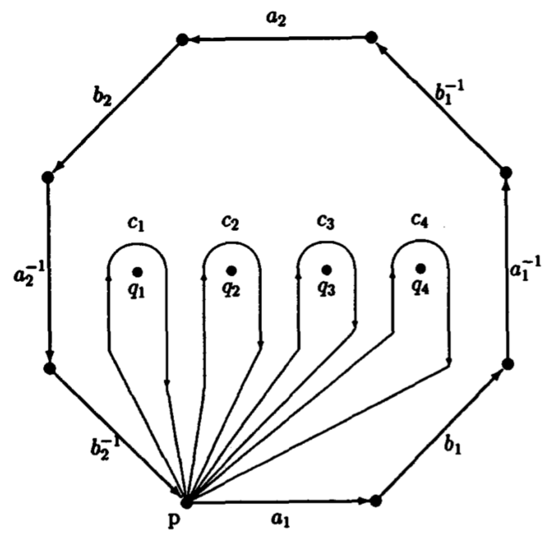

Figure 7.6: A two dimensional Riemann surface surface as a 4G-gon, here of genus 2 and four branch points [16].

Let v be a n-fold cover of M that has branch points $q_1, \ldots, q_i$ and we have a homomorphism $\pi_1(M\ \{q_1, \ldots q_i\}) \to S_n$ by labeling the sheets. There is a difference from the unbranched cover because as the branch points has branch number 1 then the permutations $p_1, \ldots, p_i$ related to the curves c_j is in the conjugacy class P_n (permutations that changes two elements). The formula (7.34) now reads

$$\sum_{v\in\Sigma(G,n,i)} \frac{1}{|S_v|} = \sum_{p_1,\ldots,p_i\in P_n} \sum_{s_1,t_1,\ldots,s_G,t_G\in S_n} \left[\frac{1}{n!}\delta\left(p_1\ldots p_i\prod_{j=1}^{h} s_j t_j s_j^{-1} t_j^{-1}\right)\right]. \tag{7.36}$$

To evaluate (7.36) we use the matrix $D_R(\rho)$ in representation R associated by element $\rho \in S_n$. The character is $\chi_R(\rho) = TrD_R(\rho)$. From group theory we have the formulas [16]

$$\begin{aligned} \delta(\rho) &= \frac{1}{n!}\sum_R d_R\chi_R(\rho), \\ \sum_{\rho\in S_n} \chi_R(\rho)D_R(\rho) &= \frac{n!}{d_R} I_R, \\ \sum_{\sigma\in S_n} D_R(\sigma\rho\sigma^{-1}) &= \frac{n!}{d_R}\chi_R(\rho) I_R. \end{aligned} \tag{7.37}$$

then

$$\sum_{\rho\in P_n} D_R(\rho) = \frac{n(n-1)}{2d_R}\chi_R(P) I_R.$$

I_R is the identity matrix and $\chi_R(P)$ is the character of any element of P_n in the representation R. Then using (7.37) we can write

$$\sum_{v\in\Sigma(G,n,i)} \frac{1}{|S_v|} = \sum_{p_1,\ldots,p_i\in P_n} \sum_{s_1,t_1,\ldots,s_G,t_G\in S_n} \left[\left(\frac{1}{n!}\right)^2 \sum_R d_R\chi_R\left(p_1\ldots p_i\prod_i s_i t_i s_i^{-1} t_i^{-1}\right)\right] \tag{7.38}$$

We have

$$\sum_{s,t\in S_n} D_R(sts^{-1}t^{-1}) = \sum_{s,t} D_R(sts^{-1})D_R(t^{-1}) = \sum_{s\in S_n} \frac{n!}{d_R}\chi_R(t)D_R(t^{-1}) I_R \tag{7.39}$$

and this gives

$$\sum_{v\in\Sigma(G,n,i)} \frac{1}{|S_v|} = \sum_R \left(\frac{n!}{d_R}\right)^{2G-2} \left(\frac{n(n-1)\chi_R(P)}{2d_R}\right)^i. \tag{7.40}$$

Furthermore it can be shown (see [16]) that

$$\tilde{c}(R) = \frac{n(n-1)\chi_R(P)}{d_R}. \tag{7.41}$$

We finally have

$$i!\xi^{n,i}_{h,G} = \sum_R \frac{n!}{d_R}\left(\frac{\tilde{c}(R)}{2}\right)^i = \sum_{v\in\Sigma(G,n,i)} \frac{1}{|S_v|}. \tag{7.42}$$

Thus the leading terms in the partition function are given by the sum of the symmetry factor over all n-fold covers (connected and homotopically distinct) of M_G. The factor $i!$ is there because of the $i = 2(h-1) - 2n(G-1)$ branch points when a surface of genus h covers a surface of genus G.

7.5.3 *Nonchiral Sum*

We now turn our attention to the other representations of interest. Let R and S be two representation, that has Young tableaux contain n and $\tilde{n}$ boxes respectively. Let the Young tableaux have columns of length c_i and $\tilde{c}_i$. We can then construct a new representation T, see figure 7.7, from the two old ones, with column lengths

$$\begin{aligned} N - \tilde{c}_{L+1-i}, \quad & i \leqslant L, \\ c_{i-L}, \quad & i > L, \end{aligned} \tag{7.43}$$

where L is the number of boxes in the first row of the Young tableau for S. We call the representation $T = \bar{S}R$ the *composite representation* of R and S. The composite representation contains L columns with $O(N)$ boxes.

The quadratic Casimir of the composite representation T is [16]

$$c_2(T) = c_2(R) + c_2(S) + \frac{2n\tilde{n}}{N}. \tag{7.44}$$

The dimension of the composite representation is [16]

$$\dim(T) = \dim(R)\dim(S)[1 + O(1/N^2)] \tag{7.45}$$

and for large N behaves as

$$\dim(T) = \frac{d_R d_S N^{n+\tilde{n}}}{n!\tilde{n}!}[1 + O(1/N^2)]. \tag{7.46}$$

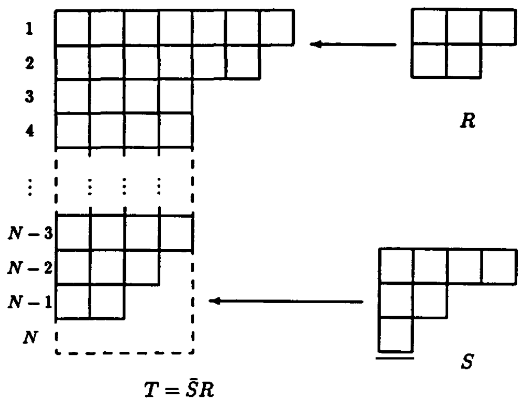

Figure 7.7: The composite representation [16].

The composite representation has a quadratic Casimir with leading order term $(n + \tilde{n})N$, which can be seen from (7.44). Therefore, it is necessary to use all composite representations to get all terms proportional to $e^{-n\lambda A/2}$. We can now write

$$Z[G, \lambda A, N] = \sum_{n} \sum_{\tilde{n}} \sum_{R \in Y_n} \sum_{S \in Y_{\tilde{n}}} (\dim(\bar{S}R))^{2-2G} e^{-\frac{\lambda A}{2N}[c_2(R)+c_2(S)+\frac{2n\tilde{n}}{N}]}. \tag{7.47}$$

We can now see from (7.45) that the partition function in (7.47) can be written as a product of two copies of the chiral partition function (7.29), omitting the $1/N^2$ corrections from the expansion of the dimensions and a coupling term $\exp\left(-\lambda A n\tilde{n}/N^2\right)$. As we have seen above the chiral partition function can be written as a sum over coverings of M with a fixed orientation. Thus the two factors of (7.29) coupled by the term $\exp\left(-\lambda A n\tilde{n}/N^2\right)$ can be interpreted as two chiral sectors correspond to orientation-preserving and orientation-reversing maps onto M. The coupling term can be seen as coupling an orientation-preserving cover with n sheets with an orientation-reversing cover with $\tilde{n}$ sheets. This term is exponentiated and can be described as infinitesimal orientation-reversing tubes that couples a sheet of the $\tilde{n}$-sheeted cover with a sheet of the n-sheeted cover. The arbitrary location of this infinitesimal tube gives the factor of λA.

7.5.4 *Tubes, Collapsed Handles and the Final Partition Function*

We can now interpret the last term of (7.28) [17]. We rewrite it as

$$\frac{\lambda A n^2}{2N^2} = \frac{n\lambda A}{2N^2} + \frac{n(n-1)}{2}\frac{\lambda A}{N^2}. \tag{7.48}$$

The first of the two terms can be seen as a handle connected to the covering space which is mapped to a single point in the target space. By doing this interpretation we can explain the factor of $1/N^2$ (as the genus increases by one for the handle), $n\lambda A$ (the choice of handle position, we have to integrate over the position because every position gives a different surface) and 1/2 (as the two ends of the handle are indistinguishable). The second term is interpreted as a tube (infinitesimal) connecting two sheets of the covering space over a point in M_G explaining λA. The genus will then be increased by one (because of the hole that the tube creates) explaining the $1/N^2$ factor and the $\frac{n(n-1)}{2}\lambda A$ factor is explained by the choice of position of the two ends of the tube on some pair of the n sheets. As the contributions are local they exponentiate. The tubes are similar to combining two branch points at a single point and therefore they are orientation-preserving (going through a tube preserves the orientation of the covering surface compared to the orientation of the target space). It is consistent with the interpretation of one chiral sector as corresponding to covering maps with a consistent relative orientation.

We can now write the partition function of two dimensional quantum Yang-Mills theory as a sum over the set of disconnected covering maps Σ_G given by

$$Z[G,\lambda A,N] = \sum_{v\in\Sigma_G} \frac{(-1)^{\tilde{t}}}{|S_v|} e^{-\frac{n\lambda A}{2}} \frac{(\lambda A)^{(i+t+\tilde{t}+h)}}{i!t!\tilde{t}!\tilde{h}!} N^{n(2-2G)-2(t+\tilde{t}+h)-i} \times [1+O(1/N)], \tag{7.49}$$

with n the winding number of v, the number of branch points i, the number of orientation-preserving (reversing) tubes $t(\tilde{t})$ and h is the number of handles that are mapped to points. The expansion is exact for the torus as there are no correction terms in that case.

Next we like to write down the free energy. The sum over connected Feynman diagrams determine the free energy. As a reminder the quantum partition function $Z[J]$ is

$$Z[J] \propto \sum_k D_k, \tag{7.50}$$

where D_k is some arbitrary Feynman diagram that consists of many connected components C_i. If we have n_i copies of a component C_i in D_k then we have a symmetry factor $n_i!$. Each Feynman diagram D_k contributes to the partition function as $\prod_i \frac{C_i^{n_i}}{n_i!}$, with i being the (infinite) number of connected Feynman diagrams. Therefore

$$Z[J] \propto \prod_i \sum_{n_i=0}^{\infty} \frac{C_i^{n_i}}{n_i!} = \exp \sum_i C_i \propto \exp W[J]. \tag{7.51}$$

Thus the free energy (the logarithm of the partition function (7.49) can be written as sums over the set of connected covering maps $\tilde{\Sigma}_G$ written as

$$W[G, \lambda A, N] = \sum_{v \in \tilde{\Sigma}_G} \frac{(-1)^{\tilde{t}}}{|S_v|} e^{-\frac{n\lambda A}{2}} \frac{(\lambda A)^{(i+t+\tilde{t}+h)}}{i!t!\tilde{t}!\tilde{h}!} N^{n(2-2G)-2(t+\tilde{t}+h)-i} \times \left[1 + O(1/N)\right]. \tag{7.52}$$

For a local discussion and rederivation of the results, done by calculating the partition function on a plaquette and then gluing the plaquettes together to form the space, see [16]. This result can be applied to manifolds with boundary.

In [63] the string theory equivalent theory of the $1/N$ expansion of two dimensional Yang-Mills theory is studied further. There is given a complete geometrical description of the partition function on an arbitrary manifold. It is stated in terms of maps from a orientable surface onto the target space and includes correction terms for surfaces of genus $G \neq 1$. This is described by extra "twist" points in the covering maps. See also [64]. For a discussion of two dimensional SO(N) and SP(N) Yang-Mills theories as closed string theories see [65].

7.6 A STRING THEORY ACTION

We will now discuss a path integral formulation for the equivalent string theory following [18, 64]. We will start with a recap of what we have done so far.

The main message up to now has been that the integral of an equivariant differential top form on a manifold with a G-action (which depends only on the equivariant cohomology class of the equivariant differential top form) is determined by, or localizes on, the fixed points of the action (as the equivariant cohomology is determined by the fixed point set of the action). In chapter 4 we applied equivariant cohomology by explaining the localization principle. We wanted to integrate a closed equivariant differential form $\int_M \alpha$ ($D\alpha = 0$ and α lies in the equivariant cohomology of M, a compact oriented manifold without boundary with a G-action and with the G-action generating a vector field $V = V^\mu \partial/\partial x^\mu$). We saw that the equivariant cohomology is determined by the fixed point set

$$M_V = \{x \in M | V(x) = 0\}. \tag{7.53}$$

and thus $\int_M \alpha$ is determined by the fixed point set. We performed a trick to see this explicitly by multiplying the integral by a factor $e^{-tD\Psi}$.

We then generalized this to the infinite dimensional case. In chapter 5 we discussed cohomological QFT's and that it can be described by fields ϕ^i, equations $s(\phi)$ and symmetries. By choosing the symmetry group G we study the G-equivariant cohomology. We had fields acted on by a group G and looked at the quotient space of the configuration space modulo G. A subset of this space, the moduli space, is

$$\mathcal{M} = Z(s)/G, \tag{7.54}$$

with

$$Z(s) = \{\phi^i \in X | s(\phi) = D\phi = 0\}. \tag{7.55}$$

on which the integral localizes.

We now would like to use the localization principle to work out our string theory path integral. The framework just described holds for topological string theory as well. Thus topological string theory can also be characterized by symmetries, fields and equations. The symmetries will now be a diffeomorphism group in our particular case. By choosing the group we study the G-equivariant cohomology and we must pick a model for the cohomology which is done by choosing the fields. The equations are as above with the section s of a vector bundle, the localization bundle, $s \in \Gamma[E_{\text{localization}} \to X]$.

The moduli space is a submanifold $Z \subset M$, where $M = P/G$ is seen as a principal bundle quotient. In topological string theory P is $\text{MAP}(M_h, M_G) \times \text{MET}(M_h)$ (the space of smooth metrics) and G is $\text{Diff}(M_h)$ and

$$Z \subset \frac{\text{MAP}(M_h, M_G) \times \text{MET}(M_h)}{\text{Diff}(M_h)}. \tag{7.56}$$

The moduli space Z is the space of holomorphic maps, which is defined by

$$Z = Z(s)/G = \{(f,h) \in \text{MAP} \times \text{MET} | R(h) = \pm 1, 0; s = df + Jdf\epsilon = 0\}/G, \tag{7.57}$$

where f is the map $f : M_h \to M_G$, R is the curvature, J is the complex structure on the target space M_G and ϵ is the complex structure on the world-sheet M_h. Z can be described in terms of the vanishing of a section of a vector bundle $V \to P$. The vector bundle and section are both G-equivariant. We can then define a bundle as

$$\begin{array}{ccc} V & \to & P \\ \downarrow & & \downarrow \\ E = V/G & \to & M = P/G \end{array} \tag{7.58}$$

The tools of cohomological field theory and the localization principle will be used to get the string action. The string theory must have the symmetries of two dimensional QCD, thus it has to be invariant under area-preserving diffeomorphisms. The Nambu-Goto and the Polyakov string action have this feature. As the free energy is an expansion in powers of $e^{-\lambda A}$ the string theory action must be proportional to the area of the map but folds should be suppressed (otherwise we would have terms in the sum corresponding to world-sheets that have an area that is not an integer multiple of the area A). The string reformulation of the Yang-Mills theory should have the property

$$\int D[f, h_{\alpha\beta,\ldots}] e^{-I[f, h_{\alpha\beta,\ldots}]} = \int_{\mathcal{H}} \chi[T\mathcal{H}(M_h, M_G) \to \mathcal{H}(M_h, M_G)], \tag{7.59}$$

as a conclusion from the 1/N expansion is that it generates Euler characters of moduli spaces of holomorphic maps, see [64] for a discussion of this. The map $f : M_h \to M_G$ is between the world-sheet surface and the target space, $h_{\alpha\beta}$ is a metric on M_h and $\mathcal{H}(M_h, M_G)$ is the space of holomorphic maps.

To make progress we will use the basic data of topological string theory. The original fieldspace of topological string theory is [18]

$$\tilde{M} = \{(f, g) | f \in C^\infty(M_h, M_G), g \in \mathrm{MET}(M_h)\}, \tag{7.60}$$

with $C^\infty(M_h, M_G)$ the space of smooth maps and $\mathrm{MET}(M_h)$ the space of smooth metrics. We then have fields $\mathbb{F} = (h, f) \in \tilde{M} = \mathrm{MAP} \times \mathrm{MET}$. Summarizing, to write the Lagrangian we will need [18]

$$\text{Fields: } \mathbb{F} = \begin{pmatrix} f^\mu \\ h_{\alpha\beta} \end{pmatrix}$$
$$\text{Ghosts: } G = \begin{pmatrix} \chi^\mu \\ \psi_{\alpha\beta} \end{pmatrix}$$
$$\text{Antighosts: } \rho^\alpha_\mu$$
$$\text{Lagrange multipliers: } \pi^\alpha_\mu.$$

Considering our string theory partition function we want to be able to localize to $\mathcal{H}(M_h, M_G)$. For this we can can use the basic fields $\mathbb{F}$ of topological string theory and the standard section

$$s(\mathbb{F}) = (df + Jdf\epsilon, R[h] + 1). \tag{7.61}$$

The density on the moduli space $\mathcal{H}(M_h, M_G)$ is given by $\chi(\ker\mathbb{O}/G)$ with the fermion kinetic operator $\mathbb{O}$ ($\ker\mathbb{O}_{f,h} \simeq T_{f,h}Z(s)$) [18]. $\ker\mathbb{O}$ is a $\mathrm{Diff}(M_h)$-equivariant bundle over

$$Z(s) = \{(h, f) \in \tilde{M}_{-1} | s(h, f) = df + Jdf\epsilon(h) = 0\}, \tag{7.62}$$

such that ker$\mathbb{O}/\text{Diff}(M_h) \simeq T_{[(f,h)]}\mathcal{H}$. We have denote the space satisfying $R[h] = -1$ (for genus greater than 1) $\tilde{M}_{-1}$.

To get the density of $\chi(T\mathcal{H} \to \mathcal{H})$ the fermion kinetic operator is $\mathbb{O} \oplus \mathbb{O}^\dagger$. We have to extend the space of fields compared to the standard topological string theory. The new fields, denoted "cofields" (written with a hat on them), are completely determined by the requirement that $\mathbb{O}^\dagger$ maps ghosts to antighosts, and by Q-symmetry. The ghosts are "hatted" version of the standard ghosts and takes values in the domain of $\mathbb{O}^\dagger$.

The ghosts are differential forms on fieldspace and $\tilde{M}$ has to be changed by the total space $\hat{E} \to \tilde{M}$ and fiber directions spanned by

$$\hat{\mathbb{F}} = \begin{pmatrix} \hat{f}^\alpha_\nu \\ \hat{h}^\alpha \end{pmatrix}, \tag{7.63}$$

with $\hat{\mathbb{F}} \in \Gamma\left(TM_h \oplus f^*(T^*M_G)\right)^+ \oplus \Gamma(TM_h)$.

The section chosen is

$$s : (\mathbb{F}, \hat{\mathbb{F}} \to (df + Jdf\epsilon, \mathbb{O}^\dagger\hat{\mathbb{F}}) = (s_1, s_2). \tag{7.64}$$

The Lagrangian for the two dimensional Yang-Mills equivalent string theory is a sum of a the topological string theory Lagrangian plus a Lagrangian for localizing to $\hat{\mathbb{F}} = 0$ (see [64]), i.e.

$$I_{YM_2} = I_{\text{topological string}} + I_{\text{"cofield"}}. \tag{7.65}$$

For the nonchiral case of the theory we have to localize on both the space of holomorphic and anti-holomorphic maps. We thus choose a section

$$\tilde{w}(f,h) \mapsto \dot{F} = [fd + Jdf\epsilon] \otimes [df - Jdf\epsilon] \tag{7.66}$$

that has $\dot{F}^{\mu\nu}_{\alpha\beta}$ written in indices. We have

$$\text{Fields: } \mathbb{F} = \begin{pmatrix} f^\mu \\ h_{\alpha\beta} \end{pmatrix}$$
$$\text{Ghosts: } \mathbb{G} = \begin{pmatrix} \chi^\mu \\ \psi_{\alpha\beta} \end{pmatrix}$$
$$\text{Antighosts: } A = \rho^{\mu\nu}_{\alpha\beta}$$
$$\text{Lagrange multipliers: } \Pi^{\mu\nu}_{\alpha\beta},$$

thus the anti-ghosts and Lagrange multipliers have changed compared to in the chiral theory.

We can write the action of the nonchiral theory as

$$I_{YM_2\text{nonchiral}} = I_{\text{topological string}} + I^{\text{nonchiral}}_{\text{topological sigma model}} + I^{\text{nonchiral}}_{\text{"cofield"}}, \tag{7.67}$$

where

$$I^{\text{nonchiral}}_{\text{topological sigma model}} = Q \int d^2z\sqrt{h}\{\rho^{\alpha\beta}_{\mu\nu}[i\dot{F}^{\mu\nu}_{\alpha\beta} - \Gamma^{\mu}_{\lambda\rho}\chi^{\lambda}\rho^{\mu\rho}_{\alpha\beta} + \frac{1}{2}\pi^{\mu\nu}_{\alpha\beta}]. \tag{7.68}$$

Expanding (7.68) and integrate out the Lagrange multiplier the bosonic term is given by

$$I^{\text{nonchiral}}_{t\sigma} = \int h^{z\bar{z}} G^2_{w\bar{w}} |\partial_w f^w|^2 |\partial_{\bar{z}} f^w|^2 + \dots \tag{7.69}$$

localizing on both holomorphic and anti-holomorphic maps.

We have now studied two dimensional Yang-Mills theory and its interpretation in terms of string theory. In 1997 the AdS/CFT correspondence (or gravity/gauge duality) was conjectured [66] giving a relation between string theory and quantum field theory. The string theory is defined on a product of anti de Sitter space (the Lorentzian analogue of hyperbolic space) and some closed manifold and the quantum field theory is conformal field theory (physics looks the same at all length scales) on the conformal boundary of the string theory space. For example $N = 4$ $SU(N)$ Yang-Mills theory in 3+1 dimensions on the boundary of AdS_5 is a string theory on $AdS_5 \times S^5$. There are a lot of nonrealistic theories with a lot of supersymmetry where the AdS/CFT correspondence applies and numerical tests, to hundreds of numbers, has shown agreement of calculations done on each side of the correspondence.

8 CONCLUSIONS

Localization formulas in finite dimensions (the Berline-Vergne formula, the Duistermaat-Heckman formula and the localization formula for the degenerate case) has been written down and proved.

Two dimensional quantum Yang-Mills theory on a compact Riemann surface is solved exactly. The partition function can be written as a sum over the critical point set and be related to the topology of the moduli space of flat connections. The $1/N$-expansion of the partition function can be written as a string theory and a string theory action is discussed. It is the mapping between the physical gauge theory and the cohomological quantum field theory that gives the basis for the localization of the partition function of two dimensional Yang-Mills theory. Thus the large supersymmetry of this theory explains its solvability properties.

The techniques that have been presented here motivate the approach to study physical problems by relating their properties to topological field theory properties. Gauge theories in low dimensions are useful playgrounds for the understanding of quantum field theory and string theory.

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