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Index Theorems and Supersymmetry

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Abstract

The Atiyah-Singer index theorem, the Euler number, and the Hirzebruch signature are derived via the supersymmetric path integral. Concisely, the supersymmetric path integral is a combination of a bosonic and a femionic path integral. The action in the supersymmetric path integral includes here bosonic, fermionic- and isospin fields (background fields), where the cross terms in the Lagrangian are nicely eliminated due to scaling of the fields and using techniques from spontaneous breaking of supersymmetry (that give rise to a mechanism, analogous to the Higgs-mechanism, but here regarding the so called superparticles instead). Thus, the supersymmetric path integral is a product of three path integrals over the three given fields, respectively, that can be evaluated exactly by means of Gaussian integrals. The closely related Witten index is a measure of the failure of spontaneous breaking of supersymmetry. In addition, the basic concepts of supersymmetry breaking are reviewed.

Contents

1 Introduction

In this thesis we derive index theorems by using techniques from mathematical physics and quantum mechanics. We will use here mainly the *supersymmetric path integral* in the derivations below.

The path integral describes the time-evolution of a quantum mechanical system given an initial- and a final position in space-time. There are two kinds of path integrals; the bosonic and the fermionic path integral, where in the former kind we use commutative variables and periodic boundary conditions, while in the latter kind we implement instead anti-commutative variables and anti-periodic boundary conditions.

Supersymmetry, on the other hand, treats bosons and fermions on an equal footing, thus the supersymmetric path integral includes both commutative- and anti-commutative variables and the boundary conditions, implemented over both variables, are periodic.

Index theorems relates analysis to topology by means of the solutions of a differential equation to a topological invariant, i.e. a topological number. In this thesis we are only concerned with the topological number called the *Euler number*, $\chi(M)$, where M is some manifold. Given a manifold that admits the spin structure, the index of the *Dirac operator* leads to the *Atiyah-Singer index theorem* and it is to be considered here as one of the main derivations using the supersymmetrical path integral.

The Atiyah-Singer index theorem originates from the early 1960s and can be considered as a vast generalization of earlier versions of index theorems such as the *Hirzebruch signature theorem*, also derived here using supersymmetry. In the early 1980s, physicists realized that the well known results in mathematical index theory could be derived by using relatively simple techniques from supersymmetric quantum mechanics and thereby, *possibly*, relate mathematical theory to physics. (Notice that there is not yet, as of this writing, any experimental verification of *supersymmetric* quantum mechanics.) All the path integrals in the derivations below can be solved exactly by using *Gaussian integrals*, thus neither Feynman diagrams, nor Feynman rules, are needed to yield the solutions.

The *Witten index* determines whether it is *not* possible to spontaneously break the supersymmetry in a supersymmetric model. The index of the Dirac operator is closely related to the Witten index; the Atiyah-Singer index theorem is equal to the Witten index and thus relates index theorems to supersymmetry. A broken supersymmetry implies that there is a mechanism that gives mass to supersymmetric particles (i.e. fermions with integer spin, or bosons with half-integer spin), analogous to the Higgs-mechanism[1] in the Standard Model.

The aim of this thesis is to present the most necessary preliminaries and to derive index theorems using the supersymmetric path integral.

Outline of the Thesis

The thesis is organized as follows: In **chapter 2** we introduce the index theorems from a non-supersymmetric point of view. Mathematical concepts and terminology is briefly reviewed. Elliptic differential operators, such as the Dirac operator in Euclidean metric, and common characteristic classes used in the index theorems are presented.

In **chapter 3** we review the theory of path integrals. Various standard techniques used in evaluating path integrals, e.g., Gaussian integrals, are introduced. The similarities and

[1] The author apologizes for leaving out Brout, Englert, Guralnik, Kibble and possibly other names in the •-mechanism.

differences in construction of the bosonic- and the fermionic path integral are emphasized. The final topic of the chapter is the supersymmetric path integral.

In **chapter 4** we review the concept of spontaneous breaking of supersymmetry in contrast to symmetry breaking in quantum field theory. The famous Wess-Zumino model serves as an example of whether supersymmetry is broken, and hence describes nature.

In the final chapter, **chapter 5**, we use the results from the consecutive chapters to derive the aforementioned index theorems. Two extensive examples; the Gauss-Bonnet theorem, and the winding number, serves as an in depth review on the geometrical and topological meaning of the Euler number and its relation to physics. This chapter can be considered as the main chapter while the previous chapters are preliminaries.

Four appendices follow the chapters described above: In **appendix A**, we show that the supersymmetric Lagrangian fulfills the principle of least action, by using the supersymmetry transformations and the Bianchi identities for the field strength- and the Riemann curvature tensors.

In **appendix B**, we derive an important formula used in the path integrals that are implemented in the derivation of the index theorems.

In **appendix C**, we derive the Riemann curvature tensor and the field strength curvature tensor explicitly. The similarities in construction of the two curvature tensors are emphasized.

Finally in **appendix D**, a gauge choice, heuristically introduced in the derivation of the Atiyah-Singer index theorem in chapter 5, is here calculated explicitly.

2 Index Theorems

In this chapter, elementary concepts and terminology of the theory of index theorems are presented. In chapter 5 below the same expressions of the index theorems presented here are derived using the supersymmetric path integral. Here, however, we follow closely the seminal articles [1]. The aim of this chapter is to state the major results of index theory in a rather non-technical review. For a more mathematical exposure, we refer to the aforementioned reference. Complementary references to the review given here include [3, 7, 9, 10, 12]. The mathematical preliminaries are, more or less, omitted here and we refer instead to the review article [3] for a more comprehensive exposure.

The hallmark of index theorems is that they give information about differential equations, provided that we understand the topology of the fiber bundles upon which the differential operators are defined. We outline several examples below, illustrating the connection between the index of an operator and the related topological numbers.

2.1 Elliptic Operators

In this section we review the theory of elliptic operators. Elliptic operators on compact manifolds are important in defining index theorems, since the dimension of the kernel of the operator is finite, thus the *analytical index* is well defined. Consider the eigenvalue problem of the generic operator $\mathbf{O}_p$ acting on some differential form $\omega \in \Lambda^p(M)$ of order p; $\mathbf{O}_p\omega = \lambda_n\omega$, where $\Lambda^p(M)$ is the space of p-forms. The constants λ_n, for $n = 0, 1, \ldots$, are the eigenvalues and the kernel of $\mathbf{O}_p$ is defined as the set of differential forms

$$\ker \mathbf{O}_p = \{\omega; \mathbf{O}_p\omega = 0\}.$$

As an example of an elliptic operator we take the *Laplacian*, Δ_p, which act on p-forms and is defined on compact Riemannian manifolds M of dimension n. The Laplacian requires a metric $g_{\mu\nu}(x)$ for its definition, hence we have a link between analysis and geometry. The *Hodge-de Rham theorem* yields topological information of the Laplacian

$$\dim \ker \Delta_p = \dim H^p_{\mathrm{dR}}(M; \mathbb{R}),$$

where $H^p_{\mathrm{dR}}(M; \mathbb{R})$ is the de Rham cohomology group. Next, we define the *Fourier transformation* $\mathcal{F}\{f(x)\}$ of a function $f(x)$ by the formula

$$\mathcal{F}\{f(x)\} = \frac{1}{(2\pi)^n} \int d^n x \exp(\mathrm{i}\xi x) f(x) =: \hat{f}(\xi).$$

The Laplacian (in Cartesian coordinates) is defined as

$$\Delta = -\frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_n^2},$$

and with Δ acting on $f(x)$ under the inverse Fourier transform yields the equation

$$\Delta f(x) = \frac{1}{(2\pi)^n} \int d^n \xi [\xi_1^2 + \cdots + \xi_n^2] \hat{f}(\xi) \exp(-\mathrm{i}\xi x).$$

The leading *symbol*, denoted by $\sigma_{\mathrm{L}}(\Delta)$, of the differential operator is the highest order part of its Fourier transform:

$$\sigma_{\rm L}(\Delta) = \xi_1^2 + \cdots + \xi_n^2,$$

and for $\sigma_{\rm L}(\Delta)$ equal to a constant we obtain the equation of a sphere. We can generalize the Laplacian by a change of scale a_i in the coordinates x_i, accordingly,

$$L = -\sum_i a_i \frac{\partial^2}{\partial x_i^2},$$

then the symbol of L set equal to a constant c is given by

$$a_1\xi_1^2 + \cdots + a_n\xi_n^2 = c,$$

which is the equation of an ellipsoid in $\mathbb{R}^n$, hence the name *elliptic operator*. A more formal definition of ellipticity is formulated as follows; if the leading symbol $\sigma_{\rm L}(x,\xi)$ is always non-zero for all x in $\mathbb{R}^n$, then the associated differential operator is called elliptic. As a counter example of an elliptic operator, consider the Bessel's equation of order λ given by the differential equation

$$x^2\frac{d^2u(x)}{dx^2} + x\frac{du(x)}{dx} + (x^2-\lambda^2)u(x) = 0; \qquad \lambda \in \mathbb{R},$$

which have the leading symbol

$$\sigma_{\rm L}(x,\xi) = x^2\xi^2,$$

that vanish at the at the origin $x = 0$.

It is common in the literature to use *multi-index* notation. Let L be a linear differential operator, defined in $\mathbb{R}^n$, of order m

$$L = \sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha.$$

The n-tuple $\alpha = (\alpha_1,\ldots,\alpha_n)$, where $\alpha_i \geq 0$, is called a multi-index and $|\alpha| = \sum\alpha_i$ is its length. Furthermore, we have $p^\alpha = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_n^{\alpha_n}$ and $D^\alpha = (-\mathrm{i})^{|\alpha|}(\partial/\partial x_1)^{\alpha_1}\ldots(\partial/\partial x_n)^{\alpha_n}$, thus the linear differential operator is given by

$$L = \sum_{|\alpha|\leq m} a_{(\alpha_1,\ldots\alpha_n)}(x)(-\mathrm{i})^{|\alpha|}\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}.$$

Using the Fourier transform, we get the symbol $\sigma_m(x,\xi)$:

$$\begin{aligned} Lu(x) = \sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha u(x) &= \sum_{|\alpha|\leq m} a_\alpha(x)\int_{\mathbb{R}^n} d^n\xi\xi^\alpha \exp(-\mathrm{i}\xi x)\hat{u}(\xi) \\ &= \int_{\mathbb{R}^n} d^n\xi[\sigma_m(x,\xi)]\exp(-\mathrm{i}\xi x)\hat{u}(\xi), \end{aligned}$$

hence,

$$\sigma_m(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha.$$

The leading symbol is then equal to

$$\sigma_{\mathrm{L}}(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha.$$

We are here mainly interested in the cases $m = 1$ (Dirac operator) and $m = 2$ (the Laplacian).

Elliptic operators on compact manifolds are called *Fredholm operators*, and we assume from now on that all differential operators are Fredholm, unless it is stated as non-Fredholm in a certain case.

2.2 Characteristic Classes

A fiber bundle is a manifold that locally looks like a direct product of two topological spaces. As an example, a direct product of a circle S^1 and some non-zero interval $I = [a,b]$, is a cylinder denoted by $S^1 \times I$. The manifold $M = S^1$ is called the base space and $F = I$ the fiber. A collection of all the fibers is called a fiber bundle. Since the cylinder can be expressed as a direct product, locally as well as globally, it is a so called trivial bundle. A Möbius strip, on the other hand, cannot be a direct product as in the case for a cylinder, since it is twisted globally (if wee zoom in and merely look at a small segment of its surface, it is indeed a direct product that looks like $\mathbb{R}^2$). Characteristic classes measure the non-triviality, or twisting, of a bundle. The measure of the twisting is equal to an integer, a topological constant, expressed as an integral involving the curvature of the fiber bundle.

In this section we present the most important characteristic classes that appear in the index theorems in the subsequent sections and in chapter 5. Several examples of integrals over characteristic classes are given in the next section, used in the evaluated index theorems.

2.2.1 The Chern Character

Let E be a complex vector bundle, whose fiber is $\mathbb{C}^k$. Given a gauge potential $A_\mu(x)$ and a field strength curvature two-form, $\mathscr{F} = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$, we define the total *Chern class* by

$$c(\mathscr{F}) = \det\left(I + \frac{\mathrm{i}\mathscr{F}}{2\pi}\right) = 1 + c_1(\mathscr{F}) + c_2(\mathscr{F}) + \ldots,$$

where $c_j(\mathscr{F})$ is the jth Chern class and I is a unit matrix. In an m-dimensional base space M, the Chern class $c_j(\mathscr{F})$ with $2j > m$ vanish, thus the series terminates at $c_k(\mathscr{F}) = \det(\mathrm{i}\mathscr{F}/2\pi)$ and $c_j(\mathscr{F}) = 0$ for $j > k$. The Chern classes are given, explicitly, by

$$
\begin{aligned}
c_0(\mathscr{F}) &= 1\\
c_1(\mathscr{F}) &= \frac{\mathrm{i}}{2\pi}\operatorname{Tr}\mathscr{F}\\
c_2(\mathscr{F}) &= \frac{1}{2}\left(\frac{\mathrm{i}}{2\pi}\right)^2[\operatorname{Tr}\mathscr{F}\wedge\operatorname{Tr}\mathscr{F}-\operatorname{Tr}(\mathscr{F}\wedge\mathscr{F})]\\
&\vdots\\
c_k(\mathscr{F}) &= \left(\frac{\mathrm{i}}{2\pi}\right)^k\det\mathscr{F}.
\end{aligned}
$$

If we now let E be a real vector bundle with rank $\dim_{\mathbb{R}} E = k$, we define the total *Pontrjagin class* by

$$p(\mathscr{F}) = \det\left(I+\frac{\mathscr{F}}{2\pi}\right) = 1 + p_1(\mathscr{F}) + p_2(\mathscr{F}) + \dots.$$

The relation between the Pontrjagin classes and the Chern classes is given by

$$p_j(E) = (-\mathrm{i})^j c_{2j}(E_{\mathbb{C}}),$$

where $E_{\mathbb{C}}$ denotes the complexification of the real vector bundle E, i.e., $E\otimes_{\mathbb{R}}\mathbb{C} = E_{\mathbb{C}}$. Finally, the total *Chern character* is defined by

$$ch(\mathscr{F}) = \operatorname{Tr}\exp\left(\frac{\mathrm{i}\mathscr{F}}{2\pi}\right) = k + c_1(\mathscr{F}) + \frac{1}{2}[c_2(\mathscr{F})^2 - 2c_2(\mathscr{F})] + \dots.$$

2.2.2 The Todd Class

Let E now be a complex vector bundle of rank k, i.e. $\dim_{\mathbb{R}} E = k$. We define the total *Todd class* of E by

$$
\begin{aligned}
td(E) = \prod_{j=1}^{k}\frac{x_j}{1-e^{-x_j}} =& 1 + \frac{1}{2}c_1(E) + \frac{1}{12}[c_1(E)^2 + c_2(E)] + \dots\\
=& 1 - \frac{1}{12}p_1(E) + \frac{1}{720}[3p_1(E)^2 - p_2(E)] + \dots,
\end{aligned}
$$

where the x_j's comes from the *splitting principle*; the bundle E can be written as a Whitney sum of n complex line bundles,

$$E = L_1\oplus L_2\oplus\cdots\oplus L_n.$$

The Whitney sum of the Chern class is, given a direct sum $E = E_1\otimes E_2$, equal to $c(E_1\oplus E_1) = c(E_1)\wedge c(E_1)$. The Chern class $c_i(E) = 0$ for $k_1+1\le i\le k_1+k_2$, where $k_1 = \dim_{\mathbb{R}} E_1$ and $k_2 = \dim_{\mathbb{R}} E_2$. For the sum of n complex line bundles L defined above, we get the wedge product

$$c(E) = c(L_1)\wedge c(L_2)\wedge\cdots\wedge c(L_n).$$

The rth Chern class $c_r(L) = 0$ for $r \geq 2$ since $\dim_{\mathbb{R}} L_i = 1$, thus we write the Chern class of L_i as

$$c(L_i) = 1 + c_1(L_i) \equiv 1 + x_i,$$

and the total Chern class is now expressed as

$$c(E) = \prod_{i=1}^{n}(1 + x_i).$$

The Chern character behaves well under Whitney sums; $ch(E \otimes F) = ch(E) \wedge ch(F)$ and $ch(E \oplus F) = ch(E) \oplus ch(F)$, and they are an important property in evaluating the index theorems as will be demonstrated below.

2.2.3 The Euler Class

Let the base space M be a $2l$-dimensional orientable Riemannian manifold. The real tangent bundle $TM = \bigcup_{p \in M}(T_pM)$ of M is the collection of all the tangent spaces T_pM of M. We define the *Euler class* as the square root of the highest Pontrjagin class:

$$p_{k/2}(E) = e^2(E),$$

where $k = 2l$ is the rank of the real vector bundle $E = TM$. For a complex vector bundle $E_{\mathbb{C}}$ the Euler class is equal to the top Chern class:

$$c_k(E_{\mathbb{C}}) = e(E_{\mathbb{C}}).$$

If the rank k is even, $k = 2l$ say, the Euler class can be associated to the *Pfaffian*:

$$Pf(\mathbf{A}) = \sqrt{\det(\mathbf{A})},$$

where $\mathbf{A}$ is an even dimensional, skew-symmetric matrix of the form

$$\mathbf{A} = \begin{pmatrix} 0 & x_1 & \cdots & & \\ -x_1 & 0 & \cdots & & \\ \vdots & \vdots & \ddots & & \\ & & & 0 & x_k \\ & & & -x_k & 0 \end{pmatrix}.$$

The Pfaffian is defined only for matrices of even order. For an odd-dimensional skew-symmetric matrix, the Pfaffian vanishes, thus the Euler class for an odd-dimensional manifold M is equal to zero. In chapter 3 we define the Pfaffian in terms of a Gaussian integral and, in chapter 3 and 5, Gaussian integrals are used in evaluating path integrals.

2.2.4 The $\hat{A}$-genus

The $\hat{A}$-genus (called A-roof genus or, common in physics literature, the Dirac genus) is defined by

$$\hat{A}(\mathscr{F}) = \prod_{j=1}^{k} \frac{x_j/2}{\sinh(x_j/2)} = 1 - \frac{1}{24}p_1 + \frac{1}{5760}(7p_1^2 - 4p_2) + \ldots,$$

where the x_j's are the eigenvalues of the field strength curvature two form, put in block diagonal form similar to **A** above. The index of the Dirac operator is the Atiyah-Singer index theorem and it is equal to an integral of $\hat{A}(TM)$ over a manifold M. The manifold M must admit a spin structure and the *Stiefel-Whitney classes* singles out all such manifolds. For a real bundle E, we define the total Stiefel-Whitney class by

$$w(E) = 1 + w_1(E) + w_2(E) + \ldots,$$

where only the first two classes are important in order to determine whether a manifold allows spin structure. If the base space is orientable, the first Stiefel-Whitney class $w_1(TM)$ is zero. The manifold is a spin-manifold if the second Stiefel-Whitney class $w_2(TM)$ is also zero, this means that parallel transport of spinors can be globally defined on $E = TM$ if and only if $w_1(TM) = w_2(TM) = 0$.

We give here two examples of spin-manifolds; (i) the complex projective spaces of odd dimension, denoted $\mathbb{C}P^1$, $\mathbb{C}P^3, \ldots$, and (ii) any sphere S^n.

2.2.5 The Hirzebruch L-polynomial

Let $k = \dim_{\mathbb{R}} E$ be the rank of a real bundle E over an n-dimensional manifold M. The *Hirzebruch L-polynomial* is defined by

$$L(x) = \prod_{j=1}^{k} \frac{x_j}{\tanh x_j} = 1 + \frac{1}{3}p_1 + \frac{1}{45}(-p_1^2 + 7p_2) + \ldots.$$

An alternative definition of the L-polynomial can be found in the literature:

$$L(x) = 2^k \prod_{j=1}^{k} \frac{x_j/2}{\tanh(x_j/2)}.$$

In the *Hirzebruch signature theorem*, only the highest order term is evaluated and both terms are equal, as can be realized by expanding the former definition up to order k. Hence either definition can be used in the signature theorem. The lower order terms, on the other hand, are sensitive to which definition is used.

2.3 Index Theorems and Classical Complexes

First we state a general index theorem formula, expressed in terms of the characteristic classes outlined in the previous section. We then apply the index theorem on *complexes*, a finite sequence of elliptic differential operators acting on fiber bundles. The order of the operators in a complex is important so that we get a certain chain of operators (in contrast to a partial derivative where the order can be chosen arbitrary). The index theorem of the de Rham complex yields the *Gauss-Bonnet theorem*. The Dolbeault complex can be considered as the complex variable analogue to the de Rham complex and leads to the *Riemann-Roch theorem*. The *Hirzebruch signature theorem* is derived in the context of the signature complex and, finally, from the spin complex we get the *Atiyah-Singer index theorem*.

2.3.1 A General Formula for Index Theorems

We are already familiar with the concept of fiber bundles from the previous section. Defined more formally, we have the base space M, the fiber F and the total space E, where E is a collection of all fibers, i.e., a fiber bundle. A map $f : A \to B$ that maps every element in the domain A to every element in the target B (not necessary one-to-one) is a surjective map, or a surjection. The surjection $\pi : E \to M$ is called the projection and its inverse $\pi^{-1}(p) = F_p$ is the fiber at $p \in M$ and it is one-to-one and onto to F, hence an isomorphism denoted by $F_p \cong F$. A (cross) section $s : M \to E$ satisfies $\pi \circ s = id_M$, the identity map $id_M : M \to M$. A section of our trivial bundle $S^1 \times I$ introduced above is just a fraction of the circle $M = S^1$, or the entire circle depending on how many fibers one chooses to take the cross section of.

A generic differential operator $\mathbf{D}$ can now be defined in terms of fiber bundles $E \xrightarrow{\pi} M$ and sections. Let $\Gamma(M, E)$ denote the set of sections on M, thus we define $\mathbf{D}$, and is dual $\mathbf{D}^\dagger$, by

$$\begin{aligned} \mathbf{D} &: \Gamma(M, E^0) \to \Gamma(M, E^1), \\ \mathbf{D}^\dagger &: \Gamma(M, E^1) \to \Gamma(M, E^0), \end{aligned}$$

where E^0 and E^1 are vector bundles over M. The kernels of $\mathbf{D}$ and $\mathbf{D}^\dagger$ are given by

$$\begin{aligned} \ker \mathbf{D} &\equiv \{s \in \Gamma(M, E^0); \mathbf{D}s = 0\}, \\ \ker \mathbf{D}^\dagger &\equiv \{s \in \Gamma(M, E^1); \mathbf{D}^\dagger s = 0\}. \end{aligned}$$

The operator $\mathbf{D}$ carries analytical information, from the solutions of the differential equation $\mathbf{D}s = 0$, hence the analytical index is defined by

$$index(\mathbf{D}) = \dim \ker \mathbf{D} - \dim \ker \mathbf{D}^\dagger.$$

A finite sequence of operators $\mathbf{D}_i$ is given by

$$0 \longrightarrow \Gamma(M, E^0) \xrightarrow{\mathbf{D}_0} \Gamma(M, E^1) \xrightarrow{\mathbf{D}_1} \cdots \xrightarrow{\mathbf{D}_n} \Gamma(M, E^{n+1}) \longrightarrow 0$$

and is called an *elliptic complex* if the composition $\mathbf{D}_i \circ \mathbf{D}_{i-1} = 0$ for any i.

A generalization of the definition of $index(\mathbf{D})$ above, given in terms of characteristic classes, is given by the formula

$$index(\mathbf{D}) = (-1)^n \{ch(\sigma_L(\mathbf{D}))td(TM_{\mathbb{C}})\}[TM]$$

where $TM_{\mathbb{C}}$ is the complexification of the tangent bundle TM, i.e., $TM_{\mathbb{C}} = M \otimes_{\mathbb{R}} \mathbb{C}$. The expression $[TM]$ is an abbreviation of taking the integral of the characteristic classes over TM. The right hand side can be generalized even further by rewriting the Chern character of the leading symbol as a fraction of the Chern character of an alternating sum of fiber bundles and the Euler class:

$$index(\mathbf{D}) = (-1)^{n(n+1)/2} \frac{ch(\sum_i (-1)^i E^i) td(TM_{\mathbb{C}})}{e(TM)}[M]. \tag{2.1}$$

The latter index formula defined above is valid only for even dimensional and orientable manifolds M. The Euler class vanishes for odd dimensions and consequently the index is defined to be equal to zero in the case when the dimension is odd.

Next, we apply the generalized index formula (2.1) over four different complexes.

2.3.2 The de Rham Complex

The (complexified) de Rham complex is defined by

$$\cdots \xrightarrow{d_{r-2}} \Lambda^{r-1}(M)_{\mathbb{C}} \xrightarrow{d_{r-1}} \Lambda^{r}(M)_{\mathbb{C}} \xrightarrow{d_{r}} \Lambda^{r+1}(M)_{\mathbb{C}} \xrightarrow{d_{r+1}} \cdots$$

where $\Lambda^p(M)_{\mathbb{C}} = \Gamma(M, \wedge^p T^* M_{\mathbb{C}})$ is the vector space of p-forms, d is the exterior derivative and $T^*M_{\mathbb{C}}$ is the complexified *co*tangent bundle (which is dual to $TM_{\mathbb{C}}$). For M an even dimensional manifold, $n = 2l$ and $l \geq 0$, we write the right hand side of the generalized index formula (2.1) as

$$(-1)^{l(2l+1)} ch\left(\sum_{r=0}^{n}(-1)^r E^r\right)\frac{td(TM_{\mathbb{C}})}{e(TM)}[M].$$

The Chern character in the index formula can be written as an alternating sum of Chern characters of vector bundles:

$$ch\left(\sum_{r=0}^{n}(-1)^r E^r\right) = \sum_{r=0}^{n}(-1)^r ch(E^r)$$

with $E^r = \wedge^r T^* M_{\mathbb{C}}$. For a line bundle L_i we have $ch(L_i) = \exp(x_i)$, where $x_i = c_1(L_i)$, and using the splitting principle we get the characteristic classes

$$\begin{aligned} ch\left(\sum_{r=0}^{n}(-1)^r \wedge^r T^* M_{\mathbb{C}}\right) &= \prod_{i=1}^{n}(1 - e^{-x_i})(TM_{\mathbb{C}}), \\ td(TM_{\mathbb{C}}) &= \prod_{i=1}^{n}\frac{x_i}{1 - e^{-x_i}}(TM_{\mathbb{C}}), \\ e(TM) &= \prod_{i=1}^{l} x_i(TM_{\mathbb{C}}). \end{aligned}$$

Substituting the Chern character, the Todd class, and the Euler class into the index formula we arrive at the *topological index* (given by the integral in the far right hand side)

$$index(d) = \int_M (-1)^{l(2l+1)}(-1)^l\left(\prod_{i=1}^{l} x_i(TM_{\mathbb{C}})\right) = \int_M e(TM),$$

where in the first integral we used the following relation between the Euler class and the top Chern class $c_n(TM_{\mathbb{C}}) = x_1 x_2 \ldots x_n$:

$$c_n(TM_{\mathbb{C}}) = (-1)^{n/2} e(TM \oplus TM) = (-1)^{n/2} e^2(TM).$$

The exterior derivative $d : \Lambda^r(M) \to \Lambda^{r+1}(M)$ is not Fredholm in the space $\Lambda^\bullet(M)$, thus we have to define d in the de Rham cohomology group $H^r_{\mathrm{dR}}(M)$ instead. Hence the *analytical index* is (given by the expressions in the first and second equality)

$$\begin{aligned}
index(d) &= \sum_{r=0}^{n} (-1)^r \dim H^r_{\mathrm{dR}}(M;\mathbb{C}) \\
&= \sum_{r=0}^{n} (-1)^r \dim H^r_{\mathrm{dR}}(M;\mathbb{R}) = \chi(M)
\end{aligned}$$

where the second equality follows from the de Rham's theorem and the third equality from the Euler-Poincaré theorem, via Hodge's theorem. The topological constant $\chi(M)$ is the *Euler number.*

The *Gauss-Bonnet theorem* is the index of the de Rham operator d:

$$\int_M e(TM) = \chi(M).$$

2.3.3 The Dolbeault Complex

Without going into too many details[2], the Dolbeault complex is analogous to the de Rham complex, using instead complex variables of the form $z^\mu = x^\mu + \mathrm{i}y^\mu$ and its complex conjugate $\bar{z}^\mu = x^\mu - \mathrm{i}y^\mu$. The manifold M is now a complex manifold of complex dimension $n/2$. The exterior derivative is defined as $d = \partial + \bar{\partial}$, where the Dolbeault operator ∂, and its dual $\bar{\partial}$, is given by

$$\partial = dz^\mu \wedge \partial/\partial z^\mu; \qquad \bar{\partial} = d\bar{z}^\mu \wedge \bar{\partial}/\partial \bar{z}^\mu.$$

The complex analogue of the de Rham sequence is

$$\begin{aligned}
\cdots \xrightarrow{\bar{\partial}} \Lambda^{p,q}(M) \xrightarrow{\bar{\partial}} \Lambda^{p,q+1}(M) \xrightarrow{\bar{\partial}} \ldots, \\
\cdots \xrightarrow{\partial} \Lambda^{p,q}(M) \xrightarrow{\partial} \Lambda^{p+1,q}(M) \xrightarrow{\partial} \ldots.
\end{aligned}$$

The Dolbeault complex is obtained with $p = 0$:

$$\cdots \xrightarrow{\bar{\partial}} \Lambda^{0,q}(M) \xrightarrow{\bar{\partial}} \Lambda^{0,q+1}(M) \xrightarrow{\bar{\partial}} \ldots.$$

Using similar arguments as in the de Rham case above, we have the characteristic classes

$$\begin{aligned}
c_{n/2}(\overline{TM}) &= (-1)^{n/2} c_{n/2}(TM) = (-1)^{n/2} e(TM), \\
td(TM_{\mathbb{C}}) &= td(TM \oplus \overline{TM}) = td(TM)td(\overline{TM}), \\
ch(\sigma_{\mathrm{L}}) &= \sum_{q=0}^{n/2} ch(\wedge^q TM) = \frac{c_{n/2}(\overline{TM})}{td(\overline{TM})}.
\end{aligned}$$

The index formula reduces to

$$index(\bar{\partial}) = (-1)^{l(2l+1)} \frac{(-1)^l e(TM)}{e(TM)td(\overline{TM})} td(TM)td(\overline{TM})[M] = td(TM)[M].$$

[2]See for instance *Kähler Geometry* in [7], or *Complex Manifolds* in [3] or in [10].

There is a relation between the classical *Betti numbers* $b_q = \dim H^q_{\mathrm{dR}}(M;\mathbb{R})$ and the *Hodge numbers* $h_{p,q}$:

$$\chi(M) = \sum_q (-1)^q b_q = \sum_{p,q} (-1)^{p+q} h_{p,q}.$$

The Hodge numbers can be regarded as a refinement of the Betti numbers. If we denote the Dolbeault complex by $\mathcal{E}$ we get the topological index

$$index(\bar{\partial}) = \sum_q (-1)^q h_{0,q} = \chi(\mathcal{E})$$

Finally, the *Riemann-Roch theorem* is given by

$$\int_M td(TM) = \chi(\mathcal{E}),$$

where $\chi(\mathcal{E})$ is called the *arithmetic genus* of the complex manifold M.

2.3.4 The Signature Complex

Let M be an oriented manifold of even dimension, $n = 2l$. We define a bilinear form $B : H^l(M;\mathbb{R}) \times H^l(M;\mathbb{R}) \to \mathbb{R}$ by

$$B(\alpha, \beta) \equiv \int_M \alpha \wedge \beta,$$

where $\alpha, \beta \in H^l(M;\mathbb{R})$, which is the middle cohomology group. The form $B(\alpha,\beta)$ is a $b^l \times b^l$ symmetric matrix if l is even, where $b^l = \dim H^l(M;\mathbb{R})$ is the Betti number. If $l = 2k$ (so n is divisible by four) the symmetric form $B(\alpha,\beta)$ has real eigenvalues where the number of positive (negative) eigenvalues is denoted by b^+ (b^-). The *Hirzebruch signature* of M is defined by

$$signature(M) := b^+ - b^-.$$

For l odd, $signature(M)$ is defined to vanish.

The Hodge star operator $*$ is a duality transformation; $* : \Lambda^r \to \Lambda^{n-r}$, and it satisfies $*^2 = 1$ when acting on a $2k$-form in a $4k$-dimensional manifold, hence $*$ has eigenvalues ± 1. We define an operator $\mathcal{D}$ by the sum

$$\mathcal{D} = d + d^\dagger,$$

which is the square root of the Laplacian $\Delta = dd^\dagger + d^\dagger d = \mathcal{D}^2$ (since $d^2 = (d^\dagger)^2 = 0$). Let $Harm^{2k}(M) = \{\omega \in \Lambda^{2k}(M); \mathcal{D}\omega = 0\}$ be the set of harmonic $2k$-forms on M, which is isomorphic to the cohomology groups of order $2k$, i.e., $Harm^{2k}(M) \cong H^{2k}(M;\mathbb{R})$. Due to the ± 1 eigenvalues of the operator $*$, the set of harmonic forms $Harm^{2k}(M)$ can be decomposed, accordingly,

$$Harm^{2k}(M) = Harm^{2k}_+(M) \oplus Harm^{2k}_-(M).$$

The Betti numbers are $b^\pm = \dim Harm^{2k}_\pm(M)$ and the Hirzebruch signature is given by

$$signature(M) = \dim Harm^{2k}_+(M) - \dim Harm^{2k}_-(M).$$

When dealing with elliptical complexes we can split the space of forms $\Lambda^\bullet(M)$ in a similar way as for $Harm^{2k}(M)$. We define an operator τ, that acts on r-forms ω, accordingly,

$$\tau := \mathrm{i}^{r(r-1)+l} * : \Lambda^r(M) \to \Lambda^{n-r}(M),$$

which satisfies $\tau^2 = 1$ and $\tau\mathcal{D} + \mathcal{D}\tau = 0$. The exterior algebra $\Lambda^\bullet(M)$ is decomposed as

$$\Lambda^\bullet(M) = \bigoplus_r \Lambda^r(M) = \Lambda^+ \oplus \Lambda^-.$$

The anti-commutativity $\tau\mathcal{D} = -\mathcal{D}\tau$ implies that we can define a restriction $\mathcal{D}^+$, and its dual $\mathcal{D}^-$, of the operator $\mathcal{D}$ given by

$$\begin{aligned}\mathcal{D}^+ :&\Lambda^+(M) \to \Lambda^-(M),\\ \mathcal{D}^- :&\Lambda^-(M) \to \Lambda^+(M).\end{aligned}$$

On the exterior algebra Λ^{2k} for dimension $n = 4k$ we have that $\pi = *$, and the index of the signature complex reduces to the Hirzebruch signature:

$$index(\mathcal{D}^+) = \dim\ker\mathcal{D}^+ - \dim\ker\mathcal{D}^- = signature(M).$$

The topological index is given by the formula

$$(-1)^l\{ch(\wedge^+T^*M \otimes_\mathbb{R} \mathbb{C}) - ch(\wedge^-T^*M \otimes_\mathbb{R} \mathbb{C})\}\frac{td(TM \otimes_\mathbb{R} \mathbb{C})}{e(TM)}[M].$$

From the splitting principle of the characteristic classes, we get

$$\begin{aligned}ch(\wedge^+T^*M \otimes_\mathbb{R} \mathbb{C}) - ch(\wedge^-T^*M \otimes_\mathbb{R} \mathbb{C}) &= \prod_{i=1}^{n/2}(e^{-x_i} - e^{x_i}),\\ td(TM \otimes_\mathbb{R} \mathbb{C}) &= \frac{x_i}{1-e^{x_i}}\frac{-x_i}{1-e^{-x_i}},\\ e(TM) &= x_1x_2\dots x_{n/2}.\end{aligned}$$

Hence, substituting the characteristic classes into the index formula yields:

$$\begin{aligned}index(\mathcal{D}^+) &= (-1)^{n/2}\left\{\prod_{i=1}^{n/2}\left(\frac{e^{-x_i} - e^{x_i}}{x_i}\frac{x_i}{1-e^{x_i}}\frac{-x_i}{1-e^{-x_i}}\right)\right\}[M]\\ &= \prod_{i=1}^{n/2}\frac{x_i(e^{x_i}+1)}{e^{x_i}-1}[M]\\ &= 2^{n/2}\prod_{i=1}^{n/2}\frac{x_i/2}{\tanh(x_i/2)}[M]\\ &= \prod_{i=1}^{n/2}\frac{x_i}{\tanh x_i}[M].\end{aligned}$$

As discussed above, the last equality can be realized by expansion of $\prod x_i/\tanh x_i$ up to order $n/2$. The $n/2$-order term coincides with the expression in the penultimate equality since it is only the highest term that is evaluated in the index.

The *Hirzebruch signature theorem* states that, for a compact oriented manifold of dimension n, where n is divisible by 4, the signature of M is given by

$$\mathcal{L}(x) = signature(M).$$

The integer

$$\mathcal{L}(x) = \int_M \prod_{i=1}^{n/2} \frac{x_i}{\tanh x_i}$$

is called the *L-genus* of M.

The Hirzebruch signature can be used in order to determine whether a manifold M admits a complex structure. In $\dim_{\mathbb{R}} M = 4$ we have the following relations

$$index(\bar{\partial}) = (\chi(M) + \tau(M))/4.$$

Example: If $M = S^4$ is the four-sphere then $\chi(S^4) = 2$ and $\tau(S^4) = 0$, hence the arithmetic genus is given by $index(\bar{\partial}) = 1/2$ which is not an integer and it means that S^4 is not complex. We can draw the same conclusion for the complex projective space, with the orientation $-\mathbb{C}P^2$, since $index(\bar{\partial}) = (3-1)/4 = 1/2$. For the opposite orientation, $+\mathbb{C}P^2$, it is complex; $index(\bar{\partial}) = (3+1)/4 = 1$.

2.3.5 The Spin Complex

Let $TM \xrightarrow{\pi} M$ be a tangent bundle, where $\dim M = n = 2l$ even and M orientable. A spin structure can be defined on, e.g., $M = S^2$ as discussed above. We define the double covering by the map

$$\rho : Spin(n) \to SO(n).$$

The $Spin(2)$ group is the double covering of S^2. Geometrically it is visualized as the splitting of the sphere into two half-spheres that are covering the upper- and lower hemispheres, respectively. The super orthogonal Lie-group $SO(2)$, that we can regard as a differentiable manifold, describe rotations in $\mathbb{R}^3$, hence $\rho : S^2 \to S^2$. The two-sphere can also be defined as the complex projective space $\mathbb{C}P^1 = S^2$, with transition functions $t_{ij} = -\exp(-\mathrm{i}2\theta)$, where θ is an angle describing the rotation, i.e., the double covering $\rho : \theta \mapsto 2\theta$. Topologically $Spin(2)$ is a latitudinal circle describing spin states on the double cover of S^2. The set of transition functions defines a spin bundle SM, and the set of sections of SM is denoted by $\Delta(M) = \Gamma(M, SM)$. The spin-group is generated by n numbers of *Dirac matrices*, $\{\gamma^\mu\}$, which satisfy the following conditions

$$\begin{aligned} \gamma^{\mu\dagger} &= \gamma^\mu, \\ \{\gamma^\mu, \gamma^\nu\} &= \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}. \end{aligned}$$

We define the gamma matrix of dimension $n+1$ as

$$\gamma^{n+1} \equiv (\mathrm{i})^{n/2}\gamma^1\gamma^2\ldots\gamma^n = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix},$$
$$(\gamma^{n+1})^2 = I,$$

where I is a $2^{n/2} \times 2^{n/2}$ unit matrix. For $n = 2$ we yield the Pauli matrices $\sigma_{1,2,3}$, and they are related to the rotations of a spin-1/2 particle on S^2 in the x-, y- and z-direction, respectively,

$$\gamma^0 = \sigma_2, \qquad \gamma^1 = \sigma_1, \qquad \gamma^2 = \mathrm{i}\gamma^0\gamma^1 = \sigma_3.$$

Since the eigenvalues of γ^{n+1}, called the *chirality*, are equal to ± 1, the set of sections of the spin bundle $\Delta(M)$ is decomposed into two eigenspaces, accordingly,

$$\Delta(M) = \Delta^+(M) \oplus \Delta^-(M).$$

The spin complex is defined in terms of the *Dirac operator* $\mathbf{D}$, and its dual $\mathbf{D}^\dagger$, by

$$\mathbf{D} : \Delta^+(M) \to \Delta^-(M),$$
$$\mathbf{D}^\dagger : \Delta^-(M) \to \Delta^+(M).$$

The analytical index of the spin complex is

$$\mathit{index}(\mathbf{D}) = \dim\ker\mathbf{D} - \dim\ker\mathbf{D}^\dagger = n_+ - n_-,$$

where n_+ (n_-) is the number of zero-energy modes of chirality $+$ ($-$). The Dirac operator is elliptic only in Euclidean metric[3] ,i.e., $g^{\mu\nu} = \delta^{\mu\nu}$, which is the ordinary Kronecker delta; a diagonal matrix of the form $\delta^{\mu\nu}$ =diag(+1,+1,+1,+1). Thus, on the Riemann sphere $M = S^2$ we assume that the metric is *locally flat*; $g_{\mu\nu}(x_0) = \delta_{\mu\nu}$ and $\partial_\lambda g_{\mu\nu}(x_0) = 0$, $x_0 \in M$. This choise of coordinates is called the *Riemann normal coordinates* (see appendix D for further details).

The index theorem for the spin complex is given by the index formula

$$(-1)^{n/2}\{ch(\Delta^+(M) - \Delta^-(M))\}\frac{td(TM_{\mathbb{C}})}{e(TM)}[M].$$

From the splitting principle we have

$$(-1)^{n/2}\{ch(\Delta^+(M)) - ch(\Delta^-(M))\} = \prod_{i=1}^{n/2}(e^{x_i/2} - e^{-x_i/2}),$$

Thus the topological index is equal to

[3] In relativistic quantum mechanics the Dirac operator $\mathbf{D}$ is defined in the Lorentzian metric given by $\eta^{\mu\nu}$ =diag(-1,1,1,1). The index of $\mathbf{D}$ is related to spontaneous breaking of supersymmetry (chapter four), where we are only interested of the physics in the ground state, i.e., the zero energy state. The total energy is $E \geq |P|$, thus in the ground state the momentum is $P = 0$.

$$\begin{aligned}
index(\mathbf{D}) &= \prod_{i=1}^{n/2} \left(\frac{e^{x_i/2} - e^{-x_i/2}}{x_i} \frac{x_i}{1 - e^{-x_i}} \frac{-x_i}{1 - e^{x_i}} \right) [M] \\
&= \prod_{i=1}^{n/2} \frac{x_i}{e^{x_i/2} - e^{-x_i/2}} [M] = \prod_{i=1}^{n/2} \frac{x_i/2}{\sinh(x_i/2)} [M] = \hat{A}(TM)[M].
\end{aligned}$$

The *Atiyah-Singer index theorem* is given by

$$index(\mathbf{D}) = \int_M \hat{A}(TM),$$

where the $\hat{A}$-genus contains only $4i$-forms, hence the index, as presented above, vanishes unless the dimension of M is a multiple of four.

Furthermore, The Dirac operator $\mathbf{D}$ can be "twisted" if the spin bundle SM is replaced by the tensor product $SM \otimes V$, where V is a vector bundle. Using the multiplicativity property of the Chern character, the index theorem applied to the twisted spin complex $\mathbf{D}_V : \Delta^+(M) \otimes V \to \Delta^-(M) \otimes V$ is then equal to

$$index(\mathbf{D}_V) = \int_M \hat{A}(TM) \wedge ch(V).$$

For $\dim M = 2$, we have

$$n_+ - n_- = \int_M ch_1(V) = \frac{\mathrm{i}}{2\pi} \int_M \mathrm{Tr}(V)$$

where $\mathrm{Tr}(V)$ is associated to the trace of the field strength curvature two-form $\mathscr{F}$, i.e., a background field that causes the twisting of the operator $\mathbf{D}$.

The Atiyah-Singer index theorem of the twisted Dirac operator is derived in the context of supersymmetry, in chapter 5 below.

3 Path Integrals

In this chapter we review the theory of path integrals and anti-commuting algebra, also called Grassmann algebra. We arrive in the end of this chapter at the path integral for fermions and, finally, the supersymmetric path integral. The fermionic and supersymmetric path integral play a crucial role in the proofs of the index theorems, presented in chapter 5 below.

3.1 General Formalism of Path Integrals

3.1.1 The Bosonic Path Integral

The dynamics of a quantum mechanical system can be described by a path integral, which is a sum of all field configurations[4] between a given initial point and a final point in space-time. We first consider the case of a system with one degree of freedom, and later generalize to a system with several degrees of freedom. In this section we deal with the bosonic case, hence the variables are commutative, in contrast to anti-commutative in the fermionic case. A picture of the quantum process in space-time is given in figure (1) below.

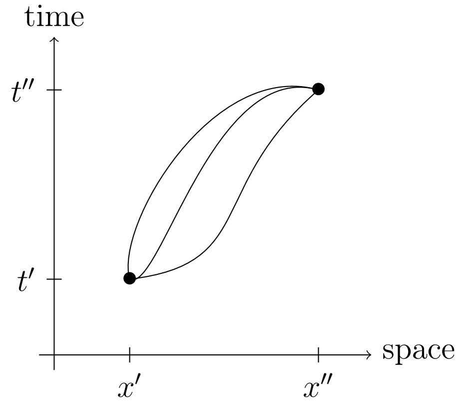

Figure 1: A path integral is a sum over all field configurations in space-time, where the paths in the figure describes a dynamical quantum process evolving from an initial point to a final point. The initial position is denoted by x' at the initial time t', and the evolution to the final position x'' is taking place at time t''.

The derivation of the path integral starts with the classical Lagrangian L of the form

$$L = L(x, \dot{x}) = \frac{m}{2}\dot{x}^2 - V(x),$$

where $K = (m/2)\dot{x}^2$ is the kinetic energy of a particle of mass m under the influence of the time independent force $F(x) = -dV(x)/dx$, and $V(x)$ is the potential energy for the classical trajectory $x = x(t)$. The Hamiltonian H is the sum of the kinetic and potential energy

[4]The terminology *sum of all paths*, or *sum of all histories*, can also be found in the literature. Since paths are not well defined in quantum mechanics, due to the Heisenberg uncertainty principle given by $\Delta x \Delta p \geq \hbar/2$, sum over all histories attempts to avoid such terminology. See the discussion on the validity of the path integral, further below in this section.

$$H = H(p, x) := p\dot{x} - L = \frac{p^2}{2m} + V(x),$$

where $p = m\dot{x}$ is the (generalized) momentum. Replacing the variables (x, p) by the time independent operators $\hat{x}$ and $\hat{p} = -id/dx$ in the Hamiltonian above we get the *quantum Hamiltonian* $\hat{H}$

$$\hat{H} := H(\hat{x}, \hat{p}) = \frac{\hat{p}^2}{2m} + V(\hat{x}).$$

The time dependent state vector $|\Psi(t)\rangle$ describes the physical state of a quantum mechanical system at a given time t, and the time-evolution of the states is governed by the *Schrödinger equation*

$$i\hbar\frac{d}{dt}|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle.$$

If we know the state at some initial time t', we then want to compute $|\Psi(t)\rangle$ for a final time $t'' > t'$. Solving the Schrödinger equation

$$\frac{d}{dt}|\psi(t)\rangle - \frac{i}{\hbar}\hat{H}|\psi(t)\rangle = 0; \quad t' < t < t'',$$

we find, from the general solution of the differential equation, the time-evolution operator

$$\hat{U}(t'', t') = \exp\left(-\frac{i}{\hbar}\hat{H}(t'' - t')\right),$$

i.e., the final state vector is of the form $|\Psi(t'')\rangle = \hat{U}(t'', t')|\Psi(t')\rangle$. The time-evolution operator$\hat{U}$ fulfills the Schrödinger equation as well and for, e.g., $t' < t_1 < t_2 < t''$ we have the composition law of $\hat{U}$; $\hat{U}(t'', t') = \hat{U}(t'', t_2)\hat{U}(t_2, t_1)\hat{U}(t_1, t')$. Since $\hat{H}$ depends on $\hat{x}$ and $\hat{p}$ we work in the *x-representation* and *p-representation*, respectively. Instead of $|\Psi(t)\rangle$ we use the state vectors $|x\rangle$ and $|p\rangle$, having the following properties

$$\begin{aligned}
\hat{x}|x\rangle = x|x\rangle; \quad \langle x'|x\rangle = \delta(x' - x); \quad \int_{\mathbb{R}} dx|x\rangle\langle x| = \mathbb{1},\\
\hat{p}|p\rangle = p|p\rangle; \quad \langle p'|p\rangle = \delta(p' - p); \quad \int_{\mathbb{R}} dp|p\rangle\langle p| = \mathbb{1}.
\end{aligned}$$

These properties are the *eigenvalue equation*; the *orthogonality of states*; and the *completeness relation* for x- and p-representation, respectively.

The path integral describes the evolution of the initial state $|x(t')\rangle = |x'\rangle$ at time t', evolving to the final state $|x(t'')\rangle = |x''\rangle$, at time t''. Hence, we shall calculate the *Feynman Kernel* $K(x'', x'; t'', t')$

$$\begin{aligned}
\langle x''|\hat{U}(t'', t')|x'\rangle &= \left\langle x'' \middle| \exp\left(-\frac{i}{\hbar}\hat{H}T\right) \middle| x' \right\rangle \\
&:= K(x'', x'; t'', t'); \qquad T = t'' - t', \quad t' < t < t''.
\end{aligned}$$

The transformation function, in the coordinate to momentum representation, is given by the plane wave

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}.$$

With the transformation function defined above, we compute the matrix element $\langle x|\hat{H}|p\rangle$, expressed in the classical Hamiltonian $H(p,x)$:

$$\langle x|\hat{H}|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar} H(p,x).$$

For small $T = t'' - t'$ we expand the time-evolution operator up to first order in T

$$\exp\left(-\frac{i}{\hbar}\hat{H}(t''-t')\right) \cong 1 - \frac{i}{\hbar}\hat{H}(t''-t'),$$

and the matrix element $\langle p|\hat{U}(t'',t')|x\rangle$ is equal to

$$\begin{aligned}\langle p|\hat{U}(t'',t')|x\rangle &\cong \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar}\left(1 - \frac{i}{\hbar}H(p,x)(t''-t')\right) \\ &\cong \frac{1}{\sqrt{2\pi\hbar}} \exp\left(-\frac{i}{\hbar}px - \frac{i}{\hbar}H(p,x)(t''-t')\right).\end{aligned}$$

Inserting the completeness relation, $\int dp|p\rangle\langle p| = \mathbb{1}$, inside the Feynman kernel gives

$$\begin{aligned}\langle x''|\mathbb{1}\hat{U}(t'',t')|x'\rangle &= \int_{\mathbb{R}} dp\langle x''|p\rangle\langle p|\hat{U}(t'',t')|x'\rangle \\ &= \frac{1}{2\pi\hbar}\int_{\mathbb{R}} dp \exp\left(\frac{i}{\hbar}p(x''-x') - \frac{i}{\hbar}H(p,x')(t''-t')\right). \qquad (3.1)\end{aligned}$$

The time-evolution operator fulfills the composition law as mentioned above, hence in the right hand side of the kernel we use the composition $\hat{U}(t'',t') = \hat{U}(t'',t_{N-1})\dots\hat{U}(t_1,t')$; a factorization into N factors. We divide the time interval $t''-t'$ into N steps:

$$\Delta t = \frac{t''-t'}{N} \ll 1,$$

hence we can carry out the integration of the term dependent on the Hamiltonian in (3.1). The time-evolution operator $\hat{U}(t'',t')$ is now a product, written as

$$\hat{U}(t'',t') \cong \left(1 - \frac{i}{\hbar}\hat{H}\frac{(t''-t')}{N}\right)^N = \left(\exp\left(-\frac{i}{\hbar}\hat{H}\Delta t\right)\right)^N.$$

Inserting the completeness relation, $\int dx|x\rangle\langle x| = \mathbb{1}$, $N-1$ times to the right of every factor, except the ultimate one, of $\hat{U}(t'',t')$ gives

$$
\begin{aligned}
\langle x''|\hat{U}(t'',t')|x'\rangle &= \int_{\mathbb{R}} dp \langle x''|p\rangle\langle p|\hat{U}(t'',t_{N-1})\mathbb{1}\dots\hat{U}(t_2,t_1)\mathbb{1}\hat{U}(t_1,t')|x'\rangle \\
&= \int_{\mathbb{R}} \prod_{i=1}^{N}\frac{dp_i}{2\pi}\prod_{j=1}^{N-1}dx_j \langle x_N|p_N\rangle\langle p_N|\hat{U}(t_n,t_{N-1})|x_{N-1}\rangle\dots\langle p_1|\hat{U}(t_1,t_0)|x_0\rangle \\
&= \int_{\mathbb{R}} \prod_{i=1}^{N}\frac{dp_i}{2\pi}\prod_{j=1}^{N-1}dx_j \exp\Big[\frac{i}{\hbar}(p_N(x_N-x_{N-1})+\dots+p_1(x_1-x_0)) \\
&\qquad -\frac{i}{\hbar}(H(p_N,x_{N-1})+\dots+H(p_1,x_0))\Delta t\Big],
\end{aligned}
$$

where $x_N = x''$ and $x_0 = x'$. In the limits $N\to\infty$ and $\Delta t \to dt$, we integrate over $p_N \to p(t)$ and $(x_N - x_{N-1})/\Delta t \to \dot{x}(t)$ for $t' < t < t''$. The boundary terms of the coordinates are $x(t') = x'$ and $x(t'') = x''$, hence the argument of the exponential transforms into the classical action

$$S = \int_{t'}^{t''} dt[p(t)\dot{x}(t) - H(p(t),x(t))] = \int_{t'}^{t''} dt L(x,\dot{x}).$$

The measure is a product of Liouville measures; they are all classical quantities,

$$\frac{dp''}{2\pi}\prod_{i=1}^{N-1}\frac{dp_i(t)dx_i(t)}{2\pi} := \mathscr{D}p(t)\mathscr{D}x(t).$$

In summary, the path integral is given by

$$K(x'',x';t'',t') = \int \mathscr{D}p(t)\mathscr{D}x(t)e^{iS/\hbar}. \tag{3.2}$$

Since both the measure $\mathscr{D}p(t)\mathscr{D}x(t)$ and the Lagrangian $L(x,\dot{x})$ are classical quantities, it might seem to be a contradiction that quantum mechanics can be expressed in terms of classical mechanics. The path integral expressed in the right hand side of (3.2) is written out *symbolically*; which means that it is to be considered as a limiting process, valid in the framework of perturbation theory in quantum mechanics. For a comprehensive review on path integrals, we refer to [4, 8].

3.1.2 Gaussian Integrals

We often use the Gaussian integral when evaluating path integrals. The Gaussian integral is defined as

$$\mathcal{F}(z,w) = \int_{\mathbb{R}} dx e^{-zx^2+wx} = \sqrt{\frac{\pi}{z}}\exp\left(\frac{w^2}{4z}\right); \quad z,w\in\mathbb{R}, \quad z\neq 0.$$

The one-dimensional Gaussian integral $\mathcal{F}(z,0)$ can be generalized to d-dimensions

$$\mathcal{F}_d(\mathbf{M}) := \int_{\mathbb{R}^d} dx^1\dots dx^d \exp\left(-\sum_{i,j=1}^{d} x^i M_{ij} x^j\right) \equiv \int_{\mathbb{R}^d} d\mathbf{x} e^{-\mathbf{x}^t\mathbf{M}\mathbf{x}},$$

where $\mathbf{M}$ is a real symmetric $d \times d$ matrix, $\mathbf{x}$ is a column vector and $\mathbf{x}^t$ its transpose. We can diagonalize the matrix $\mathbf{M}$ accordingly $\mathbf{M} = \mathbf{N}^t\mathbf{M}_D\mathbf{N}$, where $\mathbf{N}$ is an orthogonal matrix; $\mathbf{N}^t = \mathbf{N}^{-1}$ and $\det \mathbf{N} = 1$. The matrix $\mathbf{M}_D$ is diagonal with real, assuming all non-zero, eigenvalues $\lambda_1, \ldots, \lambda_d$. Hence, for a change of variable $\mathbf{y} = \mathbf{N}\mathbf{x}$, the Gaussian integral is written as

$$\begin{aligned}\mathcal{F}_d(\mathbf{M}) &= \det \mathbf{N} \int_{\mathbb{R}^d} d\mathbf{y} e^{-\mathbf{y}^t\mathbf{M}_D\mathbf{y}} = \prod_{k=1}^{d} \int_{\mathbb{R}} dy_k e^{-\lambda_k (y_k)^2} = \pi^{d/2}(\lambda_1\lambda_2\ldots\lambda_d)^{-1/2} \\ &= \pi^{d/2}(\det \mathbf{M}_D)^{-1/2} = \pi^{d/2}(\det \mathbf{M})^{-1/2}.\end{aligned}$$

A more general Gaussian integral is given by

$$\mathcal{F}(\mathbf{M}, \mathbf{u}) = \int_{\mathbb{R}^d} d\mathbf{x} e^{-\mathbf{x}^t\mathbf{M}\mathbf{x}+\mathbf{u}^t\mathbf{x}+\mathbf{x}^t\mathbf{u}} = \pi^{d/2}(\det \mathbf{M})^{-1/2} e^{\mathbf{u}\mathbf{M}^{-1}\mathbf{u}}.$$

3.1.3 Zeta Function Regularization

When evaluating path integrals via the Gaussian integral we need to solve functional determinants, e.g. $\det(d^2/dt^2)$, via an eigenvalue problem. Imposing Dirichlet (or periodic) boundary conditions on the path integral, we solve eigenvalue equations of the form

$$-\frac{d^2}{dt^2}x_n(t) = \lambda_n x_n(t); \qquad 0 \le t \le T; \quad x_n(0) = x_n(T) = 0.$$

The eigenfunctions x_n are, due to the boundary values, proportional to $\sin(n\pi t/T)$ and the eigenvalues are $\lambda_n = (n\pi/T)^2$, $n \ge 1$. Hence, the functional determinant is equal to

$$\det\left(-\frac{d^2}{dt^2}\right) = \prod_{n=1}^{\infty} \lambda_n = \prod_{n=1}^{\infty} \left(\frac{n\pi}{T}\right)^2 < \infty.$$

Let $\hat{O}$ be a generic operator whose eigenvalues are positive definite, i.e. $\det \hat{O} = \lambda_1\lambda_2\ldots\lambda_n > 0$, and from the formula $\det \hat{O} = \exp[\operatorname{Tr} \log \hat{O}]$ we have

$$\log \det \hat{O} = \operatorname{Tr} \log \hat{O} = \sum_{n=1}^{\infty} \log \lambda_n.$$

We define the *MP zeta function*[5], associated to $\hat{O}$, as

$$\zeta_{\hat{O}}(s) := \operatorname{Tr} \hat{O}^{-s} = \sum_{n=1}^{\infty} \frac{1}{{\lambda_n}^s}; \qquad s \in \mathbb{C},$$

where the sum converges for sufficiently large $\Re(s)$. Notice

$$\frac{d}{dt}({\lambda_n}^{-s}) = -\log \lambda_n \exp(-s \log \lambda_n)$$

and

[5]The zeta function of Minakshisundaram and Pleijel. There are several zeta functions; the Riemann zeta function is also referred to in this thesis.

$$\zeta'_{\hat{O}}(0) = \left.\frac{d\zeta_{\hat{O}}(s)}{ds}\right|_{s=0} = -\sum_{n=1}^{\infty} \log \lambda_n,$$

hence we arrive at

$$\det \hat{O} = \exp\left(-\zeta'_{\hat{O}}(s)\right).$$

Thus with the operator $\hat{O} = -d^2/dt^2$ mentioned as an example above, we find

$$\zeta_{-d^2/dt^2}(s) = \sum_{n=1}^{\infty} \left(\frac{n\pi}{T}\right)^{-2s} = \left(\frac{T}{\pi}\right)^{2s} \zeta(2s),$$

where $\zeta(2s) = \sum_{n=1}^{\infty} n^{-2s}$ is the *Riemann zeta function*, with well-defined $\zeta(0) = -1/2$, and $\zeta'(0) = -\log(2\pi)/2$. Finally, we get the derivative of the zeta function at $s = 0$ equal to

$$\zeta'_{-d^2/dt^2}(0) = 2\log\left(\frac{T}{\pi}\right)\zeta(0) + 2\zeta(2s) = -\log(2T).$$

The final result of the functional determinant is

$$\det\left(-\frac{d^2}{dt^2}\right) = 2T.$$

We give an example below on how to evaluate a path integral using the zeta function regularization and Fourier series.

3.1.4 Fourier Series and Path Integrals

Previously, we divided the time period T into N steps, i.e. $\Delta t = (t'' - t')/N = T/N$. Instead of discretizing the time interval we can evaluate the path integral using a Fourier series

$$x(t) = \sum_{n=1}^{\infty} a_n f_n(t); \qquad t' < t < t'',$$

where a_n are the Fourier coefficients and f_n are trigonometric functions. Hence, we discretize the trajectory x by the finite series

$$x^N(t) = \sum_{n=1}^{N} a_n f_n(t); \qquad t' < t < t''.$$

The approximate paths $x^N(t)$ are functions of the Fourier coefficients $\{a_n\}$, thus the measure is $\mathscr{D}^{(N)}a \sim \prod_{n=1}^{N} da_n$. We can choose here $t' = 0$, $t'' = T$ and the boundary conditions $x(0) = x(T) = 0$. Due to the boundary conditions we must use the sine-Fourier series:

$$x^N = \sum_{n=1}^{N} a_n \sin\left(\frac{n\pi t}{T}\right).$$

The path integral, here denoted $F(T)$, is then equal to

$$F(T) = \int_{x(0)=0}^{x(T)=0} \mathscr{D}x(t) \exp\left(\frac{i}{\hbar}S[x(t)]\right) = \int_{PBCs} \mathscr{D}a \exp\left\{\frac{i}{\hbar}S\left[\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi t}{T}\right)\right]\right\}$$
$$:= \lim_{N\to\infty} \left[\left(\frac{\pi}{\sqrt{2}}\right)^N N! \left(\frac{m}{2\pi i\hbar T}\right)^{(N+1)/2}\right] \prod_{n=1}^{N} \int_{\mathbb{R}} da_n \exp\left(\frac{i}{\hbar}S[x^N(t)]\right) \qquad (3.3)$$

The prefactor inside the square brackets in (3.3) is chosen so that in the end we get the result of the free field case (for which $V(x) \equiv 0$) multiplied by the trigonometric term $(\omega T)^{1/2}(\sin\omega T)^{-1/2}$. As an example we compute the path integral of the *one-dimensional harmonic oscillator* whose Lagrangian is given by

$$L_{\text{osc}}(x,\dot{x}) = \frac{m}{2}\dot{x}^2 - \frac{m}{2}\omega^2 x^2,$$

where ω is the oscillation frequency. Here we use $t' = 0$, $t'' = T$ and denote the path integral by $K_{\text{osc}}(x'', x'; T)$:

$$K_{\text{osc}}(x'', x'; T) = \int_{x(0)=x'}^{x(T)=x''} \mathscr{D}x(t) e^{iS_{\text{osc}}[x(t)]/\hbar},$$

where the action is equal to

$$S_{\text{osc}}[x(t)] = \frac{m}{2}\int_0^T dt(\dot{x}^2 - \omega^2 x^2) = \frac{m}{2}\int_0^T dt x(t)\left(-\frac{d^2}{dt^2} - \omega^2\right)x(t).$$

Expanding the variable $x(t)$ as

$$x(t) = x_{\text{cl}}(t) + q(t); \quad x_{\text{cl}}(0) = x', \quad x_{\text{cl}}(T) = x'', \quad \text{and} \quad q(0) = q(T) = 0,$$

where x_{cl} is the classical trajectory and $q(t)$ is the closed quantum fluctuation. The exponent of the action is factorized into a classic factor, where the equations of motion is given by the Euler-Lagrange equation, and a quantum factor F_{osc} which is the path integral of the quantum fluctuations:

$$K_{\text{osc}}(x'', x'; T) = \exp(iS_{\text{osc}}[x_{\text{cl}}]/\hbar)F_{\text{osc}}(T).$$

For closed quantum fluctuations $q(t)$ we use a sine-Fourier series and obtain the path integral $F_{\text{osc}}(T)$

$$F_{\text{osc}}(T) := K_{\text{osc}}(0,0;T) = \int_{q(0)=0}^{q(T)=0} \mathscr{D}q(t) e^{iS_{\text{osc}}[q(t)]/\hbar}.$$

The finite series approximation $q^N(t)$ of the action $S_{\text{osc}}[q^N(t)]$ is equal to

$$
\begin{aligned}
S_{\mathrm{osc}}[q^N(t)] &= \frac{m}{2}\int_0^T dt(\dot{q}^2 - \omega^2 q^2) \\
&= \frac{m}{2}\sum_{n=1}^{N} a_n^2 \int_0^T dt \left[\left(\frac{n\pi}{T}\right)^2 \cos^2\left(\frac{n\pi t}{T}\right) - \omega^2 \sin^2\left(\frac{n\pi t}{T}\right)\right] \\
&= \frac{mT}{4}\sum_{n=1}^{N} a_n^2 \left[\left(\frac{n\pi}{T}\right)^2 - \omega^2\right]
\end{aligned}
$$

Now we can motivate, more explicitly, the choice of the prefactor in (3.3) by evaluating $F_{\mathrm{osc}}^N(T)$:

$$
\begin{aligned}
F_{\mathrm{osc}}^N(T) &= \prod_{n=1}^{N}\int da_n \exp\left\{-\frac{mT}{4i\pi}\sum_{n=1}^{N}\left[\left(\frac{n\pi}{T}\right)^2 - \omega^2\right] a_n^2\right\} \\
&= \pi^{N/2}\left(\frac{mT}{4i\hbar}\right)^{-N/2}\prod_{n=1}^{N}\left(\frac{T}{n\pi}\right)\left\{\prod_{n=1}^{N}\left[1-\left(\frac{\omega T}{n\pi}\right)^2\right]\right\}^{-1/2} \\
&= \left(\frac{\pi}{\sqrt{2}}\right)^{-N}\left(\frac{m}{2\pi \mathrm{i}\hbar T}\right)^{-N/2}\frac{1}{N!}\left\{\prod_{n=1}^{N}\left[1-\left(\frac{\omega T}{n\pi}\right)^2\right]\right\}^{-1/2},
\end{aligned}
$$

where in the second equality the Gaussian integral was used in evaluating the path integral, and the two products come from factorizing $(\prod[(n\pi/T)^2 - \omega^2])^{-1/2}$. Substituting $F_{\mathrm{osc}}^N(T)$ in (3.3,) and taking the limit $N \to \infty$, yields

$$
F_{\mathrm{osc}}^N(T) = \sqrt{\frac{m}{2\pi i\hbar T}}\left\{\prod_{n=1}^{N}\left[1-\left(\frac{\omega T}{n\pi}\right)^2\right]\right\}^{-1/2} = \sqrt{\frac{m}{2\pi i\hbar T}}\sqrt{\frac{\omega T}{\sin\omega T}} \quad \text{for } T > 0,
$$

where the first square root is the free field result of the path integral. As an example of evaluating a path integral using the result of the zeta function regularization above, we consider the Lagrangian $L = (1/2)m\dot{q}^2$ and check the free field case $F_0(T)$:

$$
\begin{aligned}
F_0(T) &= \int_{q(0)=0}^{q(t)=0} \mathscr{D}q(t) \exp\left[-\frac{1}{2}\left(\frac{m}{i\hbar}\right)\int_0^T dt q(t)\left(-\frac{d^2}{dt^2}\right)q(t)\right] \\
&= \sqrt{\frac{m}{\pi i\hbar}}\left[\det\left(-\frac{d^2}{dt^2}\right)\right]^{-1/2} = \sqrt{\frac{m}{2\pi i\hbar T}}.
\end{aligned}
$$

3.1.5 Coherent States

Previously we introduced the classical Lagrangian $L = (1/2)m\dot{x}^2 - (1/2)m\omega^2 x^2$ for the simple one dimensional harmonic oscillator. The (quantum) Hamiltonian is then equal to

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2}.$$

We define the annihilation and creation operator, respectively, as

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right); \qquad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right). \tag{3.4}$$

With the commutation relation $[\hat{x},\hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar$, we get the Hamiltonian in terms of the number operator $N = a^\dagger a$:

$$a^\dagger a = \frac{m\omega}{2\hbar}\left(\hat{x}^2 + \frac{\hat{p}^2}{m\omega^2}\right) + \left(\frac{i}{2\hbar}\right)[\hat{x},\hat{p}] = \frac{\hat{H}}{\hbar\omega} - \frac{1}{2},$$

or

$$\hat{H} = \hbar\omega\left(N + \frac{1}{2}\right). \tag{3.5}$$

The eigenvalue equation of N, acting on the energy eigenkets $|n\rangle$, is equal to

$$N|n\rangle = n|n\rangle.$$

The eigenvalues n are positive integers, and the annihilation (creation) operator acting on $|n\rangle$ decreases (increases) the energy state by one unit, accordingly

$$a|n\rangle = \sqrt{n}|n-1\rangle; \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle.$$

One can show [13], by using the Heisenberg equations of motion, that the time evolution of a and $a^\dagger$ are

$$a(t) = a(0)\exp(-i\omega t); \quad a^\dagger(t) = a^\dagger(0)\exp(i\omega t) \tag{3.6}$$

Expressing $\hat{x}$ and $\hat{p}$ in terms of a and $a^\dagger$, by rewriting (3.4), we get $\hat{x}(t)$ and $\hat{p}(t)$ from (3.6):

$$\hat{x}(t) = \hat{x}(0)\cos\omega t + \left[\frac{\hat{p}(0)}{m\omega}\right]\sin\omega t,$$
$$\hat{p}(t) = -m\omega\hat{x}(0)\sin\omega t + \hat{p}(0)\cos\omega t.$$

The variables of $\hat{x}(t)$ and $\hat{p}(t)$ seem to oscillate, analogous to the case in classical mechanics. Notice, however, that $\hat{x}(0) \sim a + a^\dagger$ and $\hat{p} \sim -a + a^\dagger$; computing the expectation values $\langle n|\hat{x}(t)|n\rangle$ and $\langle n|\hat{p}(t)|n\rangle$ gives zero in both cases due to the orthogonality $\langle n|n\pm 1\rangle = 0$.

In order to observe oscillations of $\hat{x}(t)$ and $\hat{p}(t)$ we must use instead a superposition of energy eigenstates, e.g. using $|0\rangle$ and $|1\rangle$,

$$|\alpha\rangle = c_0|0\rangle + c_1|1\rangle; \quad c_0, c_1 \in \mathbb{C}.$$

A *coherent state* is defined by the following eigenvalue equation:

$$a|\lambda\rangle = \lambda|\lambda\rangle; \quad \lambda \in \mathbb{C},$$

where the eigenket $|\lambda\rangle$ is a superposition of $|n\rangle$:

$$|\lambda\rangle = \sum_{n=0}^{\infty} f_n(n)|n\rangle.$$

The distribution of $|f_n(n)|^2$ is of *Poisson type*; $|f_n(n)|^2 = (\bar{n}^n/n!)\exp(-\bar{n})$, where $\bar{n}$ is a mean of n measurements. For large values of n, the Poisson distribution approaches a bell-shaped *Gauss distribution* [2].

In summary; a coherent state is an oscillator ground state (a Gauss distribution) that can bounce back and forth by some finite distance in space. The shape of a wave-package translated in space remains in an oscillator ground state, for all time intervals Δt, without spreading in shape.

We use coherent states in the derivation of the fermionic path integral below.

3.2 Grassmann Algebra

The Pauli exclusion principle states that no two electrons with identical quantum numbers can occupy the same quantum state. Consider, e.g., an electron with, say, spin up and is in a state $|n\rangle$, if another electron is in the same state, then the latter electron must have a spin down.

In the next section the path integral for fermions will be derived. Instead of commuting numbers, as used in the construction of the bosonic path integral, anti-commuting Grassmann numbers are thus imposed in the Lagrangian and the measures.

3.2.1 Grassmann Algebra

Let $\{\theta_1, \dots, \theta_n\}$ be a set of Grassmann variables, satisfying the anti-commutation relation

$$\{\theta_i, \theta_j\} = \theta_i\theta_j + \theta_j\theta_i = 0 \quad \forall i, j.$$

A set of linear combinations of $\{\theta_i\}$, with coefficients that are complex numbers, is called a *Grassmann number*, e.g. for $n = 2$,

$$f(\theta) = f_0 + f_1\theta_1 + f_2\theta_2 + f_{12}\theta_1\theta_2; \quad f_0, f_1, f_2, f_{12} \in \mathbb{C}.$$

From the anti-commutative relation above, we have that $(\theta_i)^2 = 0$. We define a *function of Grassmann numbers* as a Taylor expansion. E.g., for $n = 1$ and θ a Grassmann variable, a Grassmann function $\exp(\theta)$ is equal to

$$e^\theta = 1 + \theta.$$

The exponential of one, or several, Grassmann numbers is a Grassmann function we encounter frequently when evaluating integrals in the following sections of this chapter, and in chapter 5 below.

3.2.2 Differentiation

The differential operator $\partial/\partial\theta_i$ act acts on a function from the left, in a similar way as the ordinary differential operator:

$$\frac{\partial}{\partial\theta_i}\theta_j = \delta_{ij}.$$

E.g., taking the derivative of $f(\theta)$ defined above with respect to the operators $\frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2}$ yields

$$\frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2}f(\theta) = -f_{12}.$$

Notice the order of the differential operators and that $\theta_1\theta_2 = -\theta_2\theta_1$ in the fourth term of $f(\theta)$.

3.2.3 Integration

integration with respect to a Grassmann variable θ is equivalent to differentiation. We introduce the *Berezin integrals*

$$\int d\theta\theta = 1, \quad \text{and} \quad \int d\theta 1 = 0.$$

For a general function $f(\theta)$ we have

$$\int d\theta f(\theta) = \frac{\partial f(\theta)}{\partial\theta},$$

i.e.,

$$\int d\theta_1 d\theta_2 \ldots d\theta_n f(\theta_1, \theta_2, \ldots, \theta_n) = \frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2}\cdots\frac{\partial}{\partial\theta_n}f(\theta_1, \theta_2, \ldots, \theta_n).$$

Since the order of the differentials $\partial/\partial\theta_i$ is the same as for $d\theta_1 \ldots d\theta_n$, one must use the anti-commutation rule to, if necessary, arrange the Grassmann variables of $f(\theta)$ in a descending order with respect to the differentials.

Integration under a change of variable $\theta^{'} = a\theta$, $a \in \mathbb{C}$, transform as

$$\int d\theta f(\theta) = \frac{\partial f(\theta)}{\partial\theta} = \frac{\partial f(\theta^{'}/a)}{\partial(\theta^{'}/a)} = a\int d\theta^{'} f(\theta^{'}/a),$$

i.e., $d\theta^{'} = (1/a)d\theta$. Extending to the case of n variables; $\theta_i \to \theta_i = a_{ij}\theta_j$, gives the transformation

$$\int d\theta_1 \ldots d\theta_n f(\vec{\theta}) = \det \mathbf{a} \int d\theta_1^{'} \ldots d\theta_n^{'} f(\mathbf{a}^{-1}\vec{\theta}^{'}),$$

where $\vec{\theta} = (\theta_1, \ldots, \theta_n)$ is a column vector and $\mathbf{a} = [a_{ij}]$ a matrix. We use the n-case change of variables when computing path integrals and Gaussian integrals in a following chapter. The Gaussian integral using Grassmann variables is defined below.

3.2.4 Gaussian Integral of Grassmann Variables

The Gaussian integral is given by

$$I = \int d\theta_1^* d\theta_1 \ldots d\theta_n^* d\theta_n \, e^{-\sum_{i,j}\theta_i^* M_{ij}\theta_j}.$$

The matrix $\mathbf{M} = [M_{ij}]$ is skew-symmetric, i.e. $M_{ij} = -M_{ji}$, since $\theta_i \theta_i^* = -\theta_i^* \theta_i$ and the sets $\{\theta_i\}$ and $\{\theta_i^*\}$ are independent. Under a change of variables $\theta_i^{'} = \sum_j M_{ij}\theta_j$ the integral is evaluated as

$$
\begin{aligned}
I &= \det \mathbf{M} \int d\theta_1^* d\theta_1^{'} \dots d\theta_n^* d\theta_n^{'} \, e^{-\sum_i \theta_i^* \theta_i^{'}} \\
&= \det \mathbf{M} \int d\theta_1^* d\theta_1^{'} e^{-\theta_1^* \theta_1^{'}} \dots d\theta_n^* d\theta_n^{'} e^{-\theta_n^* \theta_n^{'}} \\
&= \det \mathbf{M} \left[\int d\theta^* d\theta^{'} (1 + \theta^{'} \theta^*) \right]^n \\
&= \det \mathbf{M}.
\end{aligned}
$$

Notice the lack of square-root when integrating over two independents sets of variables. The determinant is in the nominator, rather than in the denominator as in the bosonic case, when implementing Grassmann variables into the Gaussian Integral.

If the Grassmann variable θ is complex, then θ^* is the complex conjugate of θ. (In a later chapter we introduce an anti-commutative field η, called the isospin field which dual to the spin field ψ, and the complex conjugate of η is then denoted by $\bar{\eta}$.)

We can show that the Gaussian integral vanishes if we have an odd number of factors in the measure. We define[10] the *Pfaffian* of the anti-symmetric matrix $\mathbf{A} = [A_{ij}]$ of order $2n$ as

$$
\mathrm{Pf}(\mathbf{A}) = \frac{1}{2^n n!} \sum_{\substack{\text{Permutations of} \\ \{i_1, \dots, i_{2n}\}}} \mathrm{sgn}(P) a_{i_1 i_2} a_{i_3 i_4} \dots a_{i_{2n-1} i_{2n}}.
$$

where $\mathrm{sgn}(P)$ is the signature of the permutation P. Recall the definition[2] of a determinant D_n of order n:

$$
D_n = \sum_{i,j,k,\dots} \varepsilon_{ijk\dots} a_i b_j c_k \dots
$$

where $\varepsilon_{ijk\dots}$ is the n-dimensional analogue to the Levi-Civita symbol. As an example, we consider the familiar case $n = 3$:

$$
D_3 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = +a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1.
$$

Now we can clearly see the meaning of $\mathrm{sgn}(P)$; an even (odd) permutation P yields $\mathrm{sgn}(P) = +1$ ($\mathrm{sgn}(P) = -1$). In the example above, we see that a simple transposition of a subscript of the matrix elements, with respect to the linear sequence (123), gives a minus, hence an odd permutation. For instance; $(123) \to -(213) \to +(231) \to -(321)$, hence $\mathrm{sgn}(P) = -1$ in the last term of D_3. Notice that there are $3! = 6$ terms in the sum of D_3. From a combinatorical point of view, there are 2^n ways of swapping the indices, e.g.,

$$
a_{i_1 i_2} a_{i_3 i_4} \dots a_{i_{2n-1} i_{2n}} \longrightarrow a_{i_2 i_1} a_{i_3 i_4} \dots a_{i_{2n-1} i_{2n}}.
$$

There are $n!$ permutations of the pairs of indices, e.g.,

$$a_{i_1 i_2} a_{i_3 i_4} \dots a_{i_{2n-1} i_{2n}} \longrightarrow a_{i_3 i_4} a_{i_1 i_2} \dots a_{i_{2n-1} i_{2n}}.$$

In order to avoid double counting of the terms, there is a fraction $1/(2^n n!)$ in front of the sum in the definition of the Pfaffian. The matrix $\mathbf{A}$ can be block diagonalized, accordingly,

$$\mathbf{N}^t \mathbf{A} \mathbf{N} = \mathbf{A}_D = \begin{pmatrix} 0 & \lambda_1 & \dots & & \\ -\lambda_1 & 0 & \dots & & \\ \vdots & \vdots & \ddots & & \\ & & & 0 & \lambda_{2n} \\ & & & -\lambda_{2n} & 0 \end{pmatrix},$$

and the determinant of $\mathbf{A}$ is equal to

$$\det(\mathbf{A}) = \det(\mathbf{A}_D) = \prod_{i=1}^{2n} \lambda_i^2.$$

The Pfaffian of a block diagonalized matrix is given by

$$Pf(\mathbf{A}_D) = a_{i_1 i_2} a_{i_3 i_4} \dots a_{i_{2n-1} i_{2n}} = \prod_{i=1}^{2n} \lambda_i,$$

which yields the relation between the Pfaffian and the determinant:

$$\det(\mathbf{A}) = [\mathrm{Pf}(\mathbf{A})]^2.$$

The Gaussian integral can be expressed in terms of the Pfaffian:

$$I = \int d\theta_{2n} \dots d\theta_1 \exp\left[\frac{1}{2} \sum_{i,j} \theta_i A_{ij} \theta_j\right] = \frac{1}{2^n n!} \int d\theta_{2n} \dots d\theta_1 \left(\sum_{i,j} \theta_i A_{ij} \theta_j\right)^n = \mathrm{Pf}(\mathbf{A}).$$

Notice here the factor $1/2$ in the argument of the exponential in the absence of pairs of $d\theta$'s. In the second equality, the exponential is expanded and the only term that saturates the measure $d\theta_{2n} \dots d\theta_1$ is of the order n since there are two Grassmann variables θ_i and θ_j in the sum. The Pfaffian vanishes for odd order matrices. As we shall see in chapter 5, the order of the matrix is associated to the dimension of a manifold and the non-vanishing of the analytical index.

3.3 Fermionic Path Integral

The fermionic path integral is constructed analogous to the bosonic case. We use instead Grassmann variables and arrive at a path integral identical, except for the boundary conditions, to the bosonic path integral. The boundary conditions are now anti-periodic rather than periodic.

3.3.1 Fermionic Harmonic Oscillator

Here we consider a quantum system with a single spin-1/2 particle described by the *Pauli matrices* σ_{x}, σ_{y} and σ_{z}. With $\sigma_\pm = (\sigma_{\mathrm{x}} \pm \mathrm{i}\sigma_{\mathrm{y}})/2$ we define the fermionic annihilation and creation operators, respectively,

$$c = \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \quad c^\dagger = \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The operators c and $c^\dagger$ satisfy the anti-commutation relations

$$\{c, c^\dagger\} = cc^\dagger + c^\dagger c = \mathbb{1}, \quad \{c, c\} = \{c^\dagger, c^\dagger\} = 0.$$

Hence $c^2 = (c^\dagger)^2 = 0$. The fermionic harmonic oscillator is described by the Hamiltonian[6]

$$\hat{H} = \frac{1}{2}(c^\dagger c - c^\dagger c)\omega = \frac{1}{2}[c^\dagger c - (\mathbb{1} - c^\dagger c)]\omega = \omega\left(N - \frac{1}{2}\right)$$

where $N = c^\dagger c$ is the number operator (cf. the bosonic case $H = \omega(N + 1/2)$). The eigenvalue of N is either zero or one; $N^2 = c^\dagger c c^\dagger c = c^\dagger(1 - c^\dagger c)c = N$, or $N(N-1) = 0$. Let the energy state $|n\rangle$, $n = 0$ or 1, be an eigenvector of $\hat{H}$:

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

then $c^\dagger|0\rangle = |1\rangle$, $c|0\rangle = c^\dagger|1\rangle = 0$, and $c|1\rangle = |0\rangle$. Hence the eigenvalues of the Hamiltonian are given by the eigenvalue equations

$$\hat{H}|0\rangle = -\frac{\omega}{2}|0\rangle, \quad \hat{H}|1\rangle = \frac{\omega}{2}|1\rangle.$$

3.3.2 Fermionic Coherent States

The number operator N has eigenvectors $|0\rangle$ and $|1\rangle$, hence an arbitrary vector $|f\rangle$ can be written as $|f\rangle = \sum f_n|n\rangle = f_0|0\rangle + f_1|1\rangle$. In the fermionic coherent state representation we have two basis functions $f_0 = 1$ and $f_1 = \theta$, θ a Grassmann variable, hence the fermionic coherent state $|\theta\rangle$, and its dual $\langle\theta|$, are equal to

$$|\theta\rangle = |0\rangle + |1\rangle\theta, \quad \langle\theta| = \langle 0| + \theta^*\langle\theta|.$$

The coherent states are eigenstates of c and $c^\dagger$, respectively:

$$\begin{aligned} c|\theta\rangle &= |0\rangle\theta = |0\rangle\theta + 0 = |0\rangle\theta + |1\rangle\theta^2 = |\theta\rangle\theta, \\ \langle\theta|c^\dagger &= \theta^*\langle\theta|. \end{aligned}$$

[6] In this section we set $\hbar = m = 1$, since it is more convenient to introduce this notation here, in agreement with the notation in chapter 5 below.

3.3.3 Fermionic Partition Function

We introduce the partition function $\mathcal{Z}(\beta)$, derived from the path integral by the replacement $t = -\mathrm{i}\tau$, i.e. the imaginary time (or Euclidean time). The partition function is identical to the path integral, except for the absence of a factor $\sqrt{-1} = \mathrm{i}$ in the exponent; e.g. $\langle x_\mathrm{f}|\exp(-\mathrm{i}\hat{H}T)|x_\mathrm{i}\rangle$ is identical to $\mathcal{Z}(\beta) = \langle x_\mathrm{f}|\exp(-\beta\hat{H})|x_\mathrm{i}\rangle$, where $\beta = \mathrm{i}T$.

The reason we introduce the partition function here is due to the notation used in the index theorems, presented in chapter 5 below.

First, we define and compute the partition function of a fermionic harmonic oscillator:

$$\mathcal{Z}(\beta) = \mathrm{Tr}\, e^{-\beta\hat{H}} = \sum_{n=0}^{1}\langle n|e^{-\beta\hat{H}}|n\rangle = e^{\beta\omega/2} + e^{-\beta\omega/2} = 2\cosh(\beta\omega/2) \tag{3.7}$$

This partition function is of great importance in proving the *Hirzebruch signature*, as will be verified in chapter 5. Using the completeness relation for fermionic coherent states:

$$\int d\theta^* d\theta |\theta\rangle\langle\theta| e^{-\theta^*\theta} = \mathbb{1},$$

one can show [10] that the partition function is related to the integral over Grassmann variables, accordingly

$$\mathrm{Tr}\, e^{-\beta\hat{H}} = \int d\theta^* d\theta \langle -\theta|e^{-\beta\hat{H}}|\theta\rangle e^{-\theta^*\theta}.$$

We emphasize here the anti-periodic boundary conditions (APBCs) over $[0,\beta]$ in the trace $\mathrm{Tr}(\bullet)$ above. The initial state is $|\theta\rangle$, evolving to the final state $|-\theta\rangle$; the Grassmann variable is θ at $\tau = 0$, and $-\theta$ at $\tau = \beta$. The construction of this path integral is analogous to the bosonic case. With the time step $\epsilon = \beta/N$, hence the limit

$$e^{-\beta\hat{H}} = \lim_{N\to\infty}(1 - \beta\hat{H}/N)^N,$$

and inserting the coherent completeness relation $N-1$ times, gives the following partition function:

$$\begin{aligned}
\mathcal{Z}(\beta) &= \lim_{N\to\infty}\int d\theta^* d\theta e^{-\theta^*\theta}\langle -\theta|(1-\beta\hat{H}/N)^N|\theta\rangle \\
&= \lim_{N\to\infty}\int d\theta^* d\theta e^{-\theta^*\theta}\prod_{k=1}^{N-1}\int d\theta_k^* d\theta_k e^{-\sum_{n=1}^{N-1}\theta_n^*\theta_n}\langle -\theta|(1-\epsilon\hat{H})|\theta_{N-1}\rangle \dots \\
&\quad \times \dots \langle\theta_2|(1-\epsilon t\hat{H})|\theta_1\rangle\langle\theta_1|(1-\epsilon\hat{H})|\theta\rangle \\
&= \lim_{N\to\infty}\int\prod_{k=1}^{N} d\theta_k^* d\theta_k e^{-\sum_{n=1}^{N}\theta_n^*\theta_n}\langle\theta_N|(1-\epsilon\hat{H})|\theta_{N-1}\rangle \dots \langle\theta_1|(1-\epsilon\hat{H})|-\theta_N\rangle,
\end{aligned}$$

where we define the initial and final states as $\theta = -\theta_N = \theta_0$, $\theta^* = -\theta_N^* = \theta_0^*$. From the definition of fermionic coherent states we have $\langle\theta_k|\theta_{k-1}\rangle = 1 + \theta_k^*\theta_{k-1} = \exp(\theta_k^*\theta_{k-1})$ and $\langle\theta_k|\hat{H}|\theta_{k-1}\rangle = \langle\theta_k|(\theta_k^*\theta_{k-1} - 1/2)\omega|\theta_k\rangle$. We now evaluate each one of the matrix elements for $k = 0, \dots, N$:

$$\begin{aligned}\langle\theta_k|(1-\epsilon\hat{H})|\theta_{k-1}\rangle &= \langle\theta_k|\theta_{k-1}\rangle\left[1-\epsilon\frac{\langle\theta_k|\hat{H}|\theta_{k-1}\rangle}{\langle\theta_k|\theta_{k-1}\rangle}\right]\\ &\cong \exp(\theta_k^*\theta_{k-1})\exp\left[-\epsilon\omega\left(\theta_k^*\theta_{k-1}-\frac{1}{2}\right)\right].\end{aligned}$$

Hence, the partition function is

$$\begin{aligned}\mathscr{Z} &= e^{\beta\omega/2}\lim_{N\to\infty}\prod_{k=1}^{N}\int d\theta_k^* d\theta_k e^{-\sum_{n=1}^{N}\epsilon[\theta_n^*(\theta_n-\theta_{n-1})+\epsilon\omega\theta_n^*\theta_{n-1}]}\\ &= e^{\beta\omega/2}\lim_{N\to\infty}\prod_{k=1}^{N}\int d\theta_k^* d\theta_k e^{-\sum_{k=1}^{N}\epsilon[(1-\epsilon\omega)\theta_n^*(\theta_n-\theta_{n-1})/\epsilon+\omega\theta_n^*\theta_n]}\\ &= \int\mathscr{D}\theta^*\mathscr{D}\theta\exp\left[\beta\omega/2-\int_0^\beta d\tau\theta^*\left((1-\epsilon\omega)\frac{d}{d\tau}+\omega\right)\theta\right],\end{aligned} \tag{3.8}$$

where in the first equality we add and subtract a factor $\epsilon\omega\theta_n^*\theta_n$ in the sum of the exponential, in order to rewrite the argument of the exponential as given in the second equality. Finally, in the third equality the time step ϵ is kept in the action due to its contribution of a factor of two, when evaluating the partition function via zeta function regularization[10] that gives $\mathscr{Z}(\beta) = 2\cosh(\beta\omega/2)$ as in (3.7).

3.4 The Supersymmetric Path Integral

We derived one kind of path integral for bosons and another kind for fermions; except from commutativity and anti-commutativity of their variables, respectively, they differ by the boundary conditions imposed on their solutions.

To put the bosonic and fermionic path integrals on an equal footing, we impose therefore periodic boundary conditions on the fermionic part partition function and it is given by

$$\begin{aligned}\mathrm{Tr}(-1)^F e^{-\beta\hat{H}} &= \sum_{n=0}^{1}\langle n|(-1)^F e^{-\beta\hat{H}}|n\rangle\\ &= \int d\theta^* d\theta\langle-\theta|(-1)^F e^{-\beta\hat{H}}|\theta\rangle e^{-\theta^*\theta}\\ &= \int d\theta^* d\theta\langle\theta|e^{-\beta\hat{H}}|\theta\rangle e^{-\theta^*\theta},\end{aligned}$$

where $F = c^\dagger c$ is the fermion number operator, and $(-1)^F$ is defined as

$$(-1)^F = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}.$$

Let the operator $(-1)^F$ act on a coherent state $|\theta\rangle = |0\rangle + |1\rangle\theta$, thus the boundary condition of that state is changed accordingly

$$(-1)^F|\theta\rangle = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \theta \\ 1 \end{pmatrix} = \begin{pmatrix} -\theta \\ 1 \end{pmatrix} = |0\rangle - |1\rangle\theta = |-\theta\rangle,$$

i.e., an anti-periodic boundary condition is changed into a periodic one as $\langle -\theta|(-1)^F = \langle\theta|$ in the third equality of the trace above.

Thus, in the supersymmetric path integral we combine the bosonic and the fermionic cases into one, unified, path integral in Euclidean time:

$$\mathcal{Z}(\beta) = \int\limits_{\text{PBCs}} \mathscr{D}x\mathscr{D}\psi e^{-\int_0^\beta dtL}, \tag{3.9}$$

where $\mathscr{D}x$ and $\mathscr{D}\psi$ are the measures of the bosonic and the fermionic fields, respectively (a field is a variable with infinitely many degrees of freedom).

4 Spontaneous Breaking of Supersymmetry

In this chapter we study the trace formula $\mathrm{Tr}(-1)^F \exp(-\beta H)$, first introduced in the fermionic partition function in the previous chapter, and relate it to the *analytical index* of an operator. In this review of symmetry breaking we will be rather heuristic, hence no derivations will be found here, with the goal of merely presenting basic facts about $\mathrm{Tr}(-1)^F$ and its physical meaning in the context of supersymmetry. We follow closely [15, 16] where a more comprehensive study can be found.

We do not observe in nature, e.g., neither spin-1 electrons, nor photons with half integer spin, hence supersymmetry must be spontaneously broken. A broken symmetry implies a mechanism that gives mass to particles. As will be shown below, the index $\mathrm{Tr}(-1)^F$ takes integer values and determines whether supersymmetry is *un*broken. In other words, $\mathrm{Tr}(-1)^F$ is a mathematical tool used for identifying, and discarding, supersymmetrical models that cannot describe nature.

First we introduce some terminology that will be used frequently throughout this chapter. By *internal symmetry breaking* we mean the symmetry breaking mechanism in electroweak theory that gives mass to non-supersymmetric particles. Electroweak theory is a topic usually reviewed in introductory textbooks on quantum field theory, see for instance [11]. The concepts of internal symmetry breaking should be stated in stark contrast to *spontaneous supersymmetry breaking*, since there are certain conditions where the latter will occur. Thus, the major topics in this chapter is the formal definition of $\mathrm{Tr}(-1)^F$, and the conditions that forces us to discard a supersymmetrical model.

4.1 The Energy Spectrum

In order to associate $\mathrm{Tr}(-1)^F$ to an index and to determine whether we have unbroken supersymmetry , we need to define and study the energy spectrum of the theory.

We define a supersymmetric theory in a volume V (and take the limit $V \to \infty$ in the end) where we are mainly interested in the ground state $|0\rangle$, or zero energy state, and a few low lying states above $|0\rangle$. The definition of $\mathrm{Tr}(-1)^F$, which is called the *Witten index*, is

$$\mathrm{Tr}(-1)^F = n_B^{E=0} - n_F^{E=0}, \tag{4.1}$$

where $n_B^{E=0}$ ($n_F^{E=0}$) is the number of zero energy bosonic (fermionic) states. In supersymmetric theories the energy $E \geq |P|$, $|P|$ the magnitude of the momentum, hence $P = 0$ for the ground state. Notice that we can regulate $\mathrm{Tr}(-1)^F$ with the kernel $\exp(-\beta H)$, hence $\mathrm{Tr}(-1)\exp(-\beta H)$, and let $\beta \to 0$ which is the high temperature limit, and thus removing high energy states.

We define the Hamiltonian H in terms of the hermitian supersymmetry charges $Q_1, Q_2, \dots, Q_K$ ($K = 4$ for supersymmetry in $3+1$ dimensions):

$$\begin{aligned} &Q_1^2 = Q_2^2 = \cdots = Q_K^2 = H, \\ &Q_i Q_j + Q_j Q_i = 0, \quad \text{for } i \neq j. \end{aligned}$$

In four dimensions we define a bosonic (femionic) state $|b\rangle$ ($|f\rangle$) that satisfies the operator $\exp(2\pi \mathrm{i} J_z)|b\rangle = |b\rangle$ ($\exp(2\pi \mathrm{i} J_z)|f\rangle = -|f\rangle$). The operator $\exp(2\pi \mathrm{i} J_z)$ rotates a state counter clockwise by 2π in the x-y plane. To be more precise, $\exp(2\pi \mathrm{i} J_z)$ is $(\exp(\frac{1}{2}\pi \mathrm{i} J_z))^4$;

four successive rotations by ninety degrees in the x-y plane. In a finite volume the rotation generator J_z is not well defined[7], but $\exp(\frac{1}{2}\pi \mathrm{i} J_z)$ is well defined. Furthermore, we define the matrix

$$(-1)^F = \exp(2\pi \mathrm{i} J_z) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In $0+1$ dimensions we have to define $(-1)^F$ more abstractly (since there are no angular momentum J_z in dimensions less than two) as the commutator and the anti-commutator, respectively,

$$(-1)^F \phi = \phi(-1)^F, \quad (-1)^F \psi = -\psi(-1)^F,$$

for some Bose field ϕ and Fermi field ψ.

For any bosonic state $|b\rangle$, and for $E \neq 0$, the fermionic state is defined as $|f\rangle = (1/\sqrt{E})Q|b\rangle$, where Q is now any of the Q_i, $i = 1, \ldots, K$. The reason we are only interested in the zero energy state is due to the pair

$$Q|b\rangle = \sqrt{E}|f\rangle, \quad Q|f\rangle = \sqrt{E}|b\rangle, \tag{4.2}$$

where the second equation follows from the definition of the Hamiltonian $Q^2 = H \geq 0$, and $H|b\rangle = E|b\rangle$. The interpretation of (4.2) is that, for every non-zero energy state, there must exist Bose-Fermi pairs, hence we have the difference $n_B^{E>0} - n_F^{E>0} = 0$. An energy spectrum is shown in figure (2) where the lowest horizontal line is the ground state, hence the equation (4.1) is equal to one in this particular example.

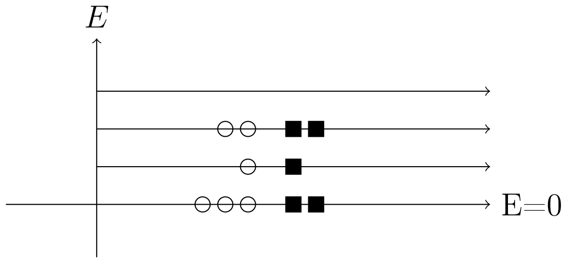

Figure 2: The bosons are indicated by circles, and the fermions by filled rectangles, in the diagram. The lowest horizontal line is the zero energy state where $\mathrm{Tr}(-1)^F = 3 - 2 = 1$, while $\mathrm{Tr}(-1)^F = 0$ for all states above the ground state. For $\mathrm{Tr}(-1)^F \neq 0$, supersymmetry is unbroken.

The parameters of the supersymmetric theory is understood as the volume V, the mass m_i of the particles, and the coupling constants g_i. Varying the parameters implies that the energy states are shifted, either up or down, in the energy spectrum. For instance, assume that we are varying some parameter so that the first state above the ground state in figure (2) is slowly moving down and, eventually, coincide with the ground state. The difference is now $n_B^{E=0} - n_F^{E=0} = 4 - 3 = 1$, hence, the same as in the original configuration. This invariance is, of course, due to the Bose-Fermi pairs in the non-zero energy states. The important property here is the following conditions:

[7]Rotate a cube lying on the x-y plane. We put a label on one of its vertical faces and apply a (discrete) rotation. If the label seems to be on the same face, we can't tell whether there has been applied a 2π rotation, or no rotation at all. A ninety degree rotation, on the other hand, will surely distinguish the initial position from the final position.

If $\text{Tr}(1-)^F = n_B^{E=0} - n_F^{E=0}$ is *not zero*, supersymmetry is *not spontaneously broken.*

We can only use $\text{Tr}(-1)^F$ to decide whether supersymmetry is unbroken. On the other hand, computing $\text{Tr}(-1)^F$ and if we get $n_B^{E=0} - n_F^{E=0} = 0$, we *cannot* draw any conclusions. We have two cases for the result $\text{Tr}(-1)^F = 0$; (i) $n_B^{E=0} = n_F^{E=0} = 0$ implies broken supersymmetry, and (ii) $n_B^{E=0} = n_F^{E=0} \neq 0$, hence unbroken supersymmetry.
In summary, the aforementioned cases are shown in figure (3) and figure (4) below.

The difference $n_B^{E=0} - n_F^{E=0}$ is equal to the index of an operator. In chapter 5, the derivation of the index, from $\text{Tr}(-1)^F \exp(-\beta H)$, is shown explicitly.

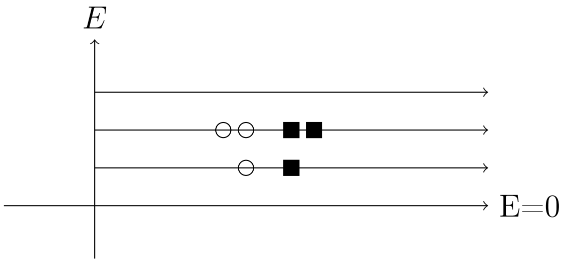

Figure 3: The difference $n_B^{E=0} - n_F^{E=0} = 0$, with $n_B^{E=0} = n_F^{E=0} = 0$, hence supersymmetry is broken. Due to $\text{Tr}(-1)^F = 0$, no conclusions can be drawn, since with $n_B^{E=0} = n_F^{E=0} \neq 0$ gives the same difference, but with a different outcome. The ground state energy for a broken supersymmetry is thus positive.

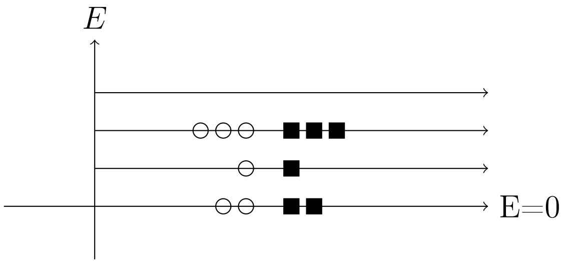

Figure 4: The difference $n_B^{E=0} - n_F^{E=0} = 0$, with $n_B^{E=0} = n_F^{E=0} \neq 0$, hence supersymmetry is unbroken. With $\text{Tr}(-1)^F = 0$ no conclusions can be drawn, since the same result of the trace is achieved with $n_B^{E=0} = n_F^{E=0} = 0$ and supersymmetry is in that case broken.

4.2 The Potential Energy

Recall that the Hamiltonian is the sum of the squares of the supersymmetry charges Q. Hence the energy E of any state is positive or zero. A state $|0\rangle$ can have zero energy if it is annihilated by the supercharge; $Q|0\rangle = 0$. If there exists an unbroken supersymmetric state, it is annihilated by Q, and it is automatically the true ground state with $E = 0$ (cf. figure (4)). On the other hand, if there does not exists a state invariant under supersymmetry (which means that $n_B^{E=0} = n_F^{E=0} = 0$), the supersymmetry is spontaneously broken, and thus the ground state energy is positive (cf. figure (3)).

In general we have a Lagrangian of the form

$$L(\phi, \dot{\phi}) = (\text{terms with derivatives}) - V(\phi),$$

where ϕ is some field. We are here only interested in the potential $V(\phi)$ and want to compare internal symmetry breaking versus supersymmetric symmetry breaking. We say that the internal symmetry is not broken if the expectation value, $\langle\phi\rangle$, of ϕ in the ground state is equal to zero; $\langle 0|\phi|0\rangle = 0$.

In figure (5) we have a potential $V(\phi) \sim \phi^2 + a$, where a is some positive constant. The ground state is the minimum of $V(\phi)$ and clearly $\langle\phi\rangle = 0$. From the discussion above that if supersymmetry is spontaneously broken, it implies that the ground state is strictly positive, hence supersymmetry is here broken while internal symmetry is unbroken.

In figure (6) the potential is $V(\phi) \sim (\phi^2 - b)^2$, with $b > 0$, hence we have two minima at $\phi = \pm\sqrt{b}$ and the expectation value is $\langle\phi\rangle \neq 0$ ($\langle\phi\rangle$ can be either at $\sqrt{b}$, or at $-\sqrt{b}$). Internal symmetry is broken[8] while supersymmetry is unbroken, since the ground state energy is exactly zero.

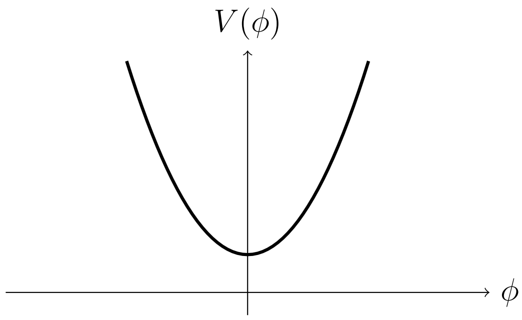

Figure 5: The potential is $V(\phi) \sim \phi^2 + a$, $a > 0$. Internal symmetry is unbroken, since the expectation value of ϕ is zero at the minimum (the ground state) of $V(\phi)$. The ground state energy is strictly positive, thus supersymmetry is spontaneously broken.

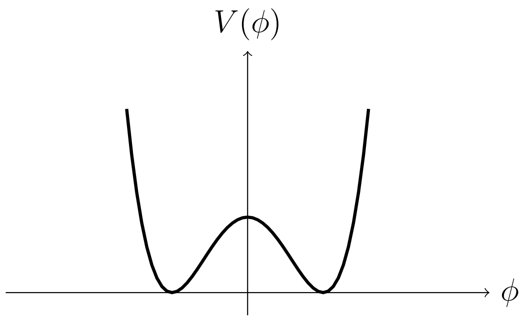

Figure 6: The potential is $V(\phi) \sim (\phi^2 - b)^2$, $b > 0$. Internal symmetry is spontaneously broken, since the expectation value $\langle\phi\rangle = \pm\sqrt{b}$. Supersymmetry is unbroken since the ground state energy is zero.

Both internal symmetry and supersymmetry can be broken if we have a potential of the form $V(\phi) \sim (\phi^2 - b)^2 + c$, $c > 0$; the potential in figure (6) shifted by an amount c in the positive V-axis.

One can also ask whether quantum corrections, i.e. fluctuations, can shift the potential in figure (5) in a negative direction, and thus restore supersymmetry. This is not possible, and in general we have that quantum corrections will not break a symmetry that is

[8]This is the simplest model where the internal symmetry is spontaneously broken, and it is called the ϕ^4-theory in quantum field theory literature.

unbroken at the tree level (zeroth order correction in perturbation), nor will they restore a broken symmetry.

In the language of particles, we know that if supersymmetry is spontaneously broken, there exists a massless fermion, a so called Goldstone fermion. If the Goldstone fermion does not already exist, fluctuations cannot shift the potential downwards and thus create massless fermions. In other words, if supersymmetry is not broken and in which all fermions have non-zero masses, the supersymmetry must be truly unbroken. This brings us back to the case $\mathrm{Tr}(-1)^F \neq 0$ where we know with certainty that the supersymmetry is (truly) unbroken.

The contrary, however, is not true. For instance, if we have a potential of the form $V(\phi) \sim \phi^2$ and given that the index $\mathrm{Tr}(-1)^F = 0$, then there is a possibility that quantum fluctuations can shift the potential so that the ground state energy becomes positive and thus break supersymmetry. This is called *dynamical breaking of supersymmetry*, and again, we refer to [15].

4.3 An Example: The Wess-Zumino Model

The following example is from [16]. The Wess-Zumino model is the simplest supersymmetric model, and the superspace potential is $W(\phi) = \frac{1}{3}g\phi^3 - (m^2/4g)\phi$. The ordinary potential is given by

$$V(\phi, \phi^*) = \left|\frac{\partial W}{\partial \phi}\right|^2 = g^2 \left|\phi^2 - \frac{m^2}{4g^2}\right|^2,$$

where ϕ is a single complex scalar field. In addition, we have a fermion field ψ in the model.

We evaluate $\mathrm{Tr}(-1)^F$ and assume, at first, that $m \neq 0$ (the trace is independent of the parameters g and m, as discussed above). Minimizing $V(\phi, \phi^*)$ we find two ground states $\langle\phi\rangle = \pm m/2g$ (the potential is similar to the potential in figure (6)). Both the scalar (boson) ϕ and the fermion ψ are massive

$$m_\phi = m_\psi = m(1 + \mathcal{O}(g^2)).$$

In each minimum of the potential, there is one spin zero state, hence bosonic, where $E = 0$. All other states with $E > 0$ are obtained by adding ϕ and ψ quanta to the bosonic ground state $|0\rangle$. Each of the two ground states contributes one to $\mathrm{Tr}(-1)^F$, hence

$$\mathrm{Tr}(-1)^F = 2.$$

Supersymmetry is not spontaneously broken in the Wess-Zumino model.

In the case $m = 0$ we have the potential $V(\phi, \phi^*) \sim \phi^4$, and there is a massless fermion (the ψ particle). We are now interested to know whether quantum fluctuations can shift the potential, so that the ground state get a positive energy, hence break the symmetry. Since $\mathrm{Tr}(-1)^F \neq 0$ this is not possible, thus the Wess-Zumino model cannot describe nature.

5 Index Theorems and Supersymmetry

In this chapter we use the supersymmetric path integral to prove index theorems, and we follow closely [14]. We stipulate three versions; the *Atiyah-Singer index theorem*, the *Euler number* (arriving at the famous Gauss-Bonnet theorem) and, finally, the *Hirzebruch signature*. Complementary calculations and topics are given in appendix A that the reader may consult before reading this chapter.

5.1 The Index of the Dirac Operator

We derive here that the Witten index introduced in the previous chaper gives a connection to the analytical index of an elliptic operator. The operators considered in this chapter are the Dirac operator $\mathbf{D}$ and the Laplacian $\Delta = \nabla^2$, hence we have here a minor repetition from chapter 2, but with a different notation and using instead the *heat equation proof* of the index theorem.

First, we define the Dirac operator $\mathbf{D}$ on some compact manifold M as

$$\mathbf{D} = \begin{pmatrix} 0 & \mathbf{D}_{\mathrm{L}} \\ \mathbf{D}_{\mathrm{R}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{D}_{\mathrm{L}} \\ -\mathbf{D}_{\mathrm{L}}^{\dagger} & 0 \end{pmatrix}.$$

where the operator $\mathbf{D}_{\mathrm{R}}$ ($\mathbf{D}_{\mathrm{L}}$) maps left(right)-hand spinors to right(left)-hand spinors; $\mathbf{D}_{\mathrm{R}} : S_{\mathrm{L}} \to S_{\mathrm{R}}$ ($\mathbf{D}_{\mathrm{L}} : S_{\mathrm{R}} \to S_{\mathrm{L}}$). The kernel of the Dirac operator is

$$\ker \mathbf{D}_{\mathrm{L,R}} = \{\psi; \mathbf{D}_{\mathrm{L,R}}\psi = 0\}.$$

The index of the Dirac operator, $\mathit{index}(\mathbf{D})$, is then defined as

$$\mathit{index}(\mathbf{D}) = \dim\ker(\mathbf{D}_{\mathrm{L}}) - \dim\ker(\mathbf{D}_{\mathrm{R}}) \in \mathbb{Z} \tag{5.1}$$

We assume here that the Dirac operator is a Fredholm operator; this implies that the number of eigenvalues are finite, hence the right hand side is equal to an integer $\mathbb{Z}$. Next, we show how the trace $\mathrm{Tr}(-1)^F e^{-\beta\hat{H}}$, for $\beta \to 0$, in the Witten index and $\mathit{index}(\mathbf{D})$ are related. To temporary free ourselves from the notation of right- and left-handedness of the operator, we introduce a generic Fredholm operator $\mathbf{A}$ and its adjoint $\mathbf{A}^\dagger$. In what follows, we briefly sketch the proofs of two theorems that can be found in [10] (theorems 12.4 and 12.5).

First, we define the eigenvalue problem $(\mathbf{A}^\dagger\mathbf{A})\phi_n = \lambda_n\phi_n$ and the associated eigenstate $\psi_n \equiv \mathbf{A}\phi_n/\sqrt{\lambda_n}$ for $\lambda_n > 0$. We now compute the eigenvalue of $\mathbf{A}\mathbf{A}^\dagger$ acting on ψ_n:

$$\begin{aligned}(\mathbf{A}\mathbf{A}^\dagger)\psi_n &= \mathbf{A}\mathbf{A}^\dagger(\mathbf{A}\phi_n/\sqrt{\lambda_n}) = \mathbf{A}(\mathbf{A}^\dagger\mathbf{A}\phi_n)/\sqrt{\lambda_n} = \mathbf{A}(\lambda_n\phi_n)/\sqrt{\lambda_n} = \lambda_n(\mathbf{A}\phi_n/\sqrt{\lambda_n}) \\ &= \lambda_n\psi_n,\end{aligned}$$

hence we get the same eigenvalue λ_n for both eigenvalue problems. Furthermore, $\{\psi_n\}$ is orthonormal

$$\langle\psi_n|\psi_m\rangle = \frac{1}{\sqrt{\lambda_n\lambda_m}}\langle\psi_n|(\mathbf{A}^\dagger\mathbf{A}|\phi_m\rangle) = \frac{1}{\sqrt{\lambda_n\lambda_m}}\lambda_m\langle\phi_n|\phi_m\rangle = \frac{\lambda_m}{\sqrt{\lambda_n\lambda_m}}\delta_{nm} = \delta_{mn}.$$

The index is derived from the difference $\mathrm{Tr}\exp(-\beta\mathbf{A}^\dagger\mathbf{A}) - \mathrm{Tr}\exp(-\beta\mathbf{A}\mathbf{A}^\dagger)$, where the first trace is over $\{\phi_n\}$ while the second is over $\{\psi_n\}$; this is the heat kernel proof of the index theorem where $h(\beta) = \exp(-\beta H)$ is a heat kernel. Let $1 \le i \le \dim\ker\mathbf{A}$ and $1 \le j \le \dim\ker\mathbf{A}^\dagger$, hence we get

$$\begin{aligned}
\mathrm{Tr}\, e^{-\beta\mathbf{A}^\dagger\mathbf{A}} - \mathrm{Tr}\, e^{-\beta\mathbf{A}\mathbf{A}^\dagger} =& \sum_{\left\{ {i, \atop \lambda_n=0} \right\}} \langle\phi_i|\phi_i\rangle + \sum_{\left\{ {n, \atop \lambda_n\neq0} \right\}} \langle\phi_n|e^{-\beta\mathbf{A}^\dagger\mathbf{A}}|\phi_n\rangle \\
&- \sum_{\left\{ {j, \atop \lambda_n=0} \right\}} \langle\psi_j|\psi_j\rangle - \sum_{\left\{ {n, \atop \lambda_n\neq0} \right\}} \langle\psi_n|e^{-\beta\mathbf{A}\mathbf{A}^\dagger}|\psi_n\rangle \\
=& \sum_i^{\dim\ker\mathbf{A}} 1 - \sum_j^{\dim\ker\mathbf{A}^\dagger} 1 + \sum_{\left\{ {n, \atop \lambda_n\neq0} \right\}} \left(\langle\phi_n|\phi_n\rangle - \langle\psi_n|\psi_n\rangle\right) \\
=&\dim\ker\mathbf{A} - \dim\ker\mathbf{A}^\dagger =: \mathit{index}(\mathbf{A}).
\end{aligned}$$

Notice that the index is independent of the parameter β. Going back to the Dirac operator $\mathbf{A}$, we define the two self-adjoint Laplace operators

$$\Delta_{\mathrm{L}} = \mathbf{D}_{\mathrm{L}}^\dagger\mathbf{D}_{\mathrm{L}}, \quad \text{and} \quad \Delta_{\mathrm{R}} = \mathbf{D}_{\mathrm{R}}^\dagger\mathbf{D}_{\mathrm{R}},$$

where we have

$$\ker\Delta_{\mathrm{L}} = \ker\mathbf{D}_{\mathrm{L}}, \quad \ker\Delta_{\mathrm{R}} = \ker\mathbf{D}_{\mathrm{R}}. \tag{5.2}$$

We show this explicitly. If $\mathbf{D}_{\mathrm{L}}\psi = 0$, then $\mathbf{D}_{\mathrm{L}}^\dagger\mathbf{D}_{\mathrm{L}}\psi = 0$ hence $\ker\mathbf{D}_{\mathrm{L}}$ equals $\ker\Delta_{\mathrm{L}}$. On the other hand, if $\Delta_{\mathrm{L}}\psi = 0$ we get

$$0 = (\psi, \Delta_{\mathrm{L}}\psi) = (\psi, \mathbf{D}_{\mathrm{L}}^\dagger\mathbf{D}_{\mathrm{L}}\psi) = (\mathbf{D}_{\mathrm{L}}\psi, \mathbf{D}_{\mathrm{L}}\psi),$$

and this shows that $\mathbf{D}_{\mathrm{L}}\psi = 0$. From the definition of the Dirac operator we have $\mathbf{D}_{\mathrm{R}} = -\mathbf{D}_{\mathrm{L}}^\dagger$, hence $\mathbf{D}_{\mathrm{R}}^\dagger = -\mathbf{D}_{\mathrm{L}}$ and furthermore, the eigenstates of Δ_{L} and Δ_{R} are paired:

$$\Delta_{\mathrm{L}}\psi = \lambda\psi \longleftrightarrow \Delta_{\mathrm{R}}\mathbf{D}_{\mathrm{L}}\psi = \lambda\mathbf{D}_{\mathrm{L}}\psi \tag{5.3}$$

We show the implication explicitly

$$\Delta_{\mathrm{R}}\mathbf{D}\psi = (\mathbf{D}_{\mathrm{R}}^\dagger\mathbf{D}_{\mathrm{R}})\mathbf{D}_{\mathrm{L}}\psi = (-\mathbf{D}_{\mathrm{L}})(-\mathbf{D}_{\mathrm{L}}^\dagger)\mathbf{D}_{\mathrm{L}}\psi = \mathbf{D}_{\mathrm{L}}(\mathbf{D}_{\mathrm{L}}^\dagger\mathbf{D}_{\mathrm{L}})\psi = \mathbf{D}_{\mathrm{L}}\Delta_{\mathrm{L}}\psi = \lambda\mathbf{D}_{\mathrm{L}}\psi.$$

It remains to show that $\mathbf{D}_{\mathrm{L}}\psi \neq 0$ in the ultimate equality:

$$0 \neq (\psi, \Delta_{\mathrm{L}}\psi) = \lambda(\psi, \psi) = (\mathbf{D}_{\mathrm{L}}\psi, \mathbf{D}_{\mathrm{L}}\psi),$$

hence, $\mathbf{D}_{\mathrm{L}}\psi \neq 0$.

The Hamiltonian H is defined in terms of the Dirac operator as

$$H = \mathbf{D}^\dagger\mathbf{D} = \begin{pmatrix} 0 & \mathbf{D}_{\mathrm{R}}^\dagger \\ \mathbf{D}_{\mathrm{L}}^\dagger & 0 \end{pmatrix}\begin{pmatrix} 0 & \mathbf{D}_{\mathrm{L}} \\ \mathbf{D}_{\mathrm{R}} & 0 \end{pmatrix} = \begin{pmatrix} \Delta_{\mathrm{R}} & 0 \\ 0 & \Delta_{\mathrm{L}} \end{pmatrix}.$$

Notice the equality between $(-1)^F$ and the gamma matrix γ_5:

$$\gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} =: (-1)^F,$$

thus we rewrite the index in a compact form as $\operatorname{Tr}\gamma_5 \exp(-\beta H)$. Hence the index theorem in the spinor notation is

$$\begin{aligned}\operatorname{Tr}\gamma_5 e^{-\beta H} &= \dim\ker\Delta_{\mathrm{L}} - \dim\ker\Delta_{\mathrm{R}} \\ &= \dim\ker\mathbf{D}_{\mathrm{L}} - \dim\ker\mathbf{D}_{\mathrm{R}} \\ &= \mathit{index}(\mathbf{D}).\end{aligned}$$

In the first equality we used, from (5.3), the identity that the eigenvalue λ is equal for both left- and right-handed spin operators $\Delta_{\mathrm{L,R}}$. The second equality follows from the auxiliary property between $\Delta_{\mathrm{L,R}}$ and $\mathbf{D}_{\mathrm{L,R}}$ as given in (5.2). Due to the equality between γ_5 and $(-1)^F$ we write the Witten index as

$$\begin{aligned}\mathit{index}(\mathbf{D}) = \operatorname{Tr}(-1)^F e^{-\beta H} &= \sum_{\left\{\substack{\text{bosonic}\\ \text{states}}\right\}} e^{-\beta\lambda_{\mathrm{B}}} - \sum_{\left\{\substack{\text{fermionic}\\ \text{states}}\right\}} e^{-\beta\lambda_{\mathrm{F}}} \\ &= n_{\mathrm{B}}(\lambda_{\mathrm{B}} = 0) - n_{\mathrm{F}}(\lambda_{\mathrm{F}} = 0),\end{aligned}$$

and reaffirm its connection to the path integral via the fermionic partition function (3.7).

We compute the index explicitly using the supersymmetric path integral introduced in section 3.4. For a Dirac operator on a d-dimensional Riemannian manifold, where the fermion fields $\psi^\mu = \psi^\mu(t)$ are coupled to an external gauge field $A_\mu(x)$, we get the supersymmetric action[14]

$$S = \int_0^\beta dt \left[\frac{1}{2} g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \frac{1}{2} g_{\mu\nu}\psi^\mu (D_t^g\psi)^\nu + \bar{\eta}^a D_t^A \eta^a - \frac{1}{2}\bar{\eta}^a F_{\mu\nu}^{ab}\psi^\mu\psi^\nu\eta^b + i\frac{\alpha}{\beta}\bar{\eta}^a\eta^a\right] \tag{5.4}$$

with the covariant derivatives

$$\begin{aligned}&(D_t^g)^\mu{}_\nu = \partial_t \delta^\mu{}_\nu + \dot{x}^\varrho \Gamma^\mu_{\varrho\nu} \quad (\text{for} \quad \mu,\nu = 1,\dots,d), \\ &D_t^A = \partial_t + \dot{x}^\alpha A_\alpha(x),\end{aligned}$$

the field strength

$$F_{\mu\nu}^{ab} = \partial_\mu A_\nu^{ab} - \partial_\nu A_\mu^{ab} + [A_\mu, A_\nu]^{ab},$$

and the Christoffel symbol

$$\Gamma^\mu_{\varrho\nu} = \frac{1}{2} g^{\mu\sigma}(\partial_\varrho g_{\nu\sigma} + \partial_\nu g_{\varrho\sigma} - \partial_\sigma g_{\varrho\nu}).$$

The interaction with the gauge field $A_\mu(x)$ give rise to isospin fields η and they are dual to the fermion fields. For a systematic construction of the interaction terms in the action

(5.4), see for instance [6]. The supersymmetric transformations of the fields in the action are given by

$$\begin{aligned}
\delta x^\mu &= \epsilon\psi^\mu, \\
\delta\psi^\mu &= -\epsilon\dot{x}^\mu, \\
\delta\eta^a &= -\epsilon\psi^\mu A_\mu^{ab}\eta^b, \\
\delta\bar{\eta}^a &= -\epsilon\bar{\eta}^b A_\mu^{ba}\psi^\mu.
\end{aligned}$$

Thus taking the variation of S, and using the Bianchi identities for the field strength $F_{\mu\nu}$ and the Riemann tensor ${R^\mu}_{\alpha\beta\varrho}$, the action fulfills *Hamilton's principle*

$$\delta S = \frac{1}{2}\int\limits_{\text{PBCs}} dt\epsilon(g_{\mu\nu}\dot{x}^\mu\psi^\nu) = \left[\frac{1}{2}\epsilon g_{\mu\nu}\dot{x}^\mu(t)\psi^\nu(t)\right]_0^{t=\beta} = 0,$$

where the last equality follows from the boundary conditions, e.g. $\psi(0) = \psi(\beta) = 0$. To be more explicit, there are also boundary terms that both depend on a total derivative in the integral, they do not contribute to the equations of motion and hence can be neglected. The computations leading to the vanishing variation of the action above are carried out exceedingly in appendix A.

Recall From elementary quantum mechanics that spherical harmonics $Y_n^m(\theta,\varphi) \sim (-1)^m \exp(\mathrm{i}m\varphi)$, where m is an integer and φ an azimuthal angle, represents angular momentum eigenfunctions. Spherical harmonics are generated by way of a generating function[2]. Any function $f(\theta,\varphi)$, where θ is a polar angle, can be expanded in terms of spherical harmonics, thus $f(\theta,\varphi)$ is evaluated over surface of a sphere in a Laplace series:

$$f(\theta,\varphi) = \sum_{m,n} a_{mn}Y_n^m(\theta,\varphi).$$

In order to get a quantum state of rank k, there is a number operator $N = \bar{\eta}\eta$ in the action S, where $\bar{\eta}$ (η) is the creation (annihilation) operator in Fock space. The index formula for all the antisymmetric tensor products of the internal space is given by the generating function

$$I(\alpha) = \sum_k I_k e^{-\mathrm{i}\alpha k}.$$

The number operator N commutes with the Hamiltonian and from Heisenberg's equations of motion we have $dN/dt = 0$, thus making it possible to implement the term $\mathrm{i}(\alpha/\beta)\bar{\eta}^a\eta^a$ into the Lagrangian in the action. The *modified heat kernel* $\tilde{h}(\beta,\alpha)$ is given by

$$\tilde{h}(\beta,\alpha) = e^{-\beta H - \mathrm{i}\alpha N},$$

and the generating function, expressed in terms of a path integral in Euclidean time, is equal to

$$I(\alpha) = \mathrm{Tr}(-1)^F e^{-\beta H} = \mathrm{Tr}\,\gamma_5(-1)^N e^{-\beta H - \mathrm{i}\alpha N} = \int\limits_{\text{PBCs}} \mathscr{D}x^\mu\mathscr{D}\psi^\mu\mathscr{D}\bar{\eta}^a\mathscr{D}\eta^a e^{-S}.$$

The operator $(-1)^N$ imposes periodic boundary conditions on the anticommutative isospin fields η, analogous to $(-1)^F$ for the fermion fields. As we shall see below, the analytical index is given by term I_k in the generating function $I(\alpha)$. Before we can compute the index theorems, we need to evaluate the path integral for various differential operators in the action; the topic of the next section.

5.2 Trace Formulas

5.2.1 Fermionic Fields with Periodic Boundary Conditions

We will evaluate the following path integral[9]

$$\mathrm{Tr}(\gamma_5 e^{-\frac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu}) = \int\limits_{\text{PBCs}} \mathscr{D}\psi^\mu \exp\left[-\frac{1}{2}\int_0^1 dt(\psi^\mu\dot{\psi}_\mu + \omega_{\mu\nu}\psi^\mu\psi^\nu)\right], \tag{5.5}$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$, $\dot{\omega}_{\mu\nu} = 0$, and the fermionic field ψ corresponds to $\gamma^\mu/\sqrt{2}$. The path integral is defined in Euclidean time, on the other hand, let $t \to -it$ here and the integral is transformed into the ordinary path integral.

The Grassmann functions $\psi^\mu(t)$ can be expanded in a Fourier series[10], e.g., a series of the form

$$\psi^\mu(t) = \sum_{n=-\infty}^{\infty} \psi_n^\mu e^{i2\pi nt} = \psi_0^\mu + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \psi_n^\mu e^{i2\pi nt}.$$

We separate the path integral into zero modes and non-zero modes in the fields. From the Gaussian integral over Grassmann variables, introduced in chapter three, we evaluate the trace as

$$\mathrm{Tr}(\gamma_5 e^{\frac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu}) = \mathscr{N}(\sqrt{2})^d \det{}^{1/2}(\partial_t + \omega^\mu{}_\nu) \int\limits_{\substack{\text{zero}\\\text{modes}}} d\psi^1 \dots d\psi^d e^{-\frac{1}{2}\omega_{\mu\nu}\psi^\mu\psi^\nu},$$

the normalization factor $\mathscr{N}$ is to be determined below, and the determinant is over the non-zero modes, all having periodic boundary conditions.

To evaluate the determinant we use a *product expansion* formula [2], see appendix B for further details on the formula. For some function $g(z)$ of a complex variable z with zeros at a_n we have the product expansion

$$g(z) = g(0)\exp\left(\frac{zg'(0)}{g(0)}\right)\prod_{n=1}^{\infty}\left(1 - \frac{z}{a_n}\right)e^{z/a_n}. \tag{5.6}$$

E.g. let $a_n = n\pi$, $n \neq 0$,

$$\sin z = z\prod_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(1 - \frac{z}{n\pi}\right)e^{z/n\pi} = z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2\pi^2}\right).$$

[9]Notice the absence of the complex number $\sqrt{-1} = \mathrm{i}$ in the integral. This definition is related to the partition function, introduced in chapter 3.4. Throughout this chapter we use the term *path integral*, which is understood implicitly as the path integral in Euclidean time.

Furthermore, we define $g(0) = 1$ and $g'(0) = b$, where b is an unknown constant. Hence, $g(z)$ is of the form

$$g(z) = \frac{\sin z}{z} e^{bz}.$$

In order to substitute a function $g(z)$ into the determinant in the solution of the path integral, we first need to solve an eigenvalue problem. From the Lagrangian in the action we get the Hamiltonian

$$H = -\frac{1}{2}\omega_{\mu\nu}\psi^\mu\psi^\nu,$$

whose eigenvalues are $\mp\frac{1}{2}\omega_{\mu\nu}$ (cf. the fermionic harmonic oscillator). Since we are working with the path integral in Euclidean time we get instead a hyperbolical function inside the functional determinant which is realized from the identities $\det(A) = \exp(\mathrm{Tr}(\ln A))$, $\det(AB) = \det(A)\det(B)$, and $\sin(\mathrm{i}x) = \mathrm{i}\sinh(x)$, where $\mathrm{Tr}\,\omega = 0$. The path integral reduces to

$$2^{-\frac{1}{2}d}\,\mathrm{Tr}(\gamma_5 e^{-\frac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu}) = \mathscr{N}\det{}^{1/2}\left(\frac{\sinh\frac{\omega^\mu{}_\nu}{2}}{\frac{\omega^\mu{}_\nu}{2}}\right)\int\limits_{\substack{\text{zero}\\\text{modes}}} d\psi^1\dots d\psi^d e^{-\frac{1}{2}\omega_{\mu\nu}\psi^\mu\psi^\nu}.$$

where the normalizing constant $\mathscr{N}$ is determined by multiplying the path integral by γ_5 and taking the limit $\omega_{\mu\nu} \to 0$. Using the identities

$$\begin{aligned}
&\lim_{\omega\to 0}\left(\frac{\sinh\frac{\omega^\mu{}_\nu}{2}}{\frac{\omega^\mu{}_\nu}{2}}\right) = \mathbb{1}_{d\times d},\\
&\gamma_5^2 = \mathbb{1} \quad (\text{a } 2^{d/2}\times 2^{d/2}\text{-dimensional unit matrix}),\\
&\gamma_5 := (-\mathrm{i})^{\frac{d}{2}}\gamma_1\gamma_2\dots\gamma_d = (-\mathrm{i})^{\frac{d}{2}}(\sqrt{2})^d(-1)^{\frac{d}{2}}\psi^d\dots\psi^1,
\end{aligned}$$

for even dimension d, we yield the left hand side of the path integral

$$2^{-\frac{1}{2}d}\,\mathrm{Tr}(\gamma_5^2) = 1,$$

and the right hand side

$$\mathscr{N}(\mathrm{i})^{\frac{d}{2}}(\sqrt{2})^d\int\limits_{\substack{\text{zero}\\\text{modes}}} d\psi^1\dots d\psi^d\psi^d\dots\psi^1.$$

Thus the normalizing constant is given by $\mathscr{N} = (-i/2)^{\frac{d}{2}}$. In the limit $\omega_{\mu\nu} \to 0$ we get the *free field contribution* from the path integral which is equal to

$$(-i)^{d/2}\int\limits_{\substack{\text{zero}\\\text{modes}}} d\psi^1\dots d\psi^d \tag{5.7}$$

The free field contribution is one of the path integrals substituted in the supersymmetrical path integral in the derivation of the Atiyah-Singer index theorem below.

For sake of clarity and in the derivation of the Euler number, we need also the general solution. Expanding the exponential in the integral up to order $d/2$, the only term that saturates the measure is of the highest order term. Hence the path integral is equal to

$$\begin{aligned}\int d\psi^1 \dots d\psi^d e^{-\frac{1}{2}\omega_{\mu\nu}\psi^\mu\psi^\nu} &= \int d\psi^1 \dots d\psi^d \frac{1}{\left(\frac{d}{2}\right)!}(-\frac{1}{2}\omega_{\mu\nu}\psi^\mu\psi^\nu)^{\frac{d}{2}} \\ &= \frac{(-1)^{\frac{d}{2}}}{2^{\frac{d}{2}}\left(\frac{d}{2}\right)!}\omega_{\mu_1\mu_2}\omega_{\mu_3\mu_4}\dots\omega_{\mu_{d-1}\mu_d}\int d\psi^1 \dots d\psi^d \psi^{\mu_1}\dots\psi^{\mu_d} \\ &= \frac{1}{2^{\frac{d}{2}}\left(\frac{d}{2}\right)!}\omega_{\mu_1\mu_2}\omega_{\mu_3\mu_4}\dots\omega_{\mu_{d-1}\mu_d}\varepsilon^{\mu_1\mu_2\dots\mu_d}.\end{aligned}$$

The following identity was substituted in the second equality

$$\psi^{\mu_1}\dots\psi^{\mu_d} = \varepsilon^{\mu_1\mu_2\dots\mu_d}\psi^1\dots\psi^d = (-1)^{\frac{d}{2}}\varepsilon^{\mu_1\mu_2\dots\mu_d}\psi^d\dots\psi^1.$$

Finally, the general solution of the path integral over the fermion fields is given by

$$\mathrm{Tr}(\gamma_5 e^{-\frac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu}) = \frac{(-\mathrm{i})^{\frac{d}{2}}}{\left(\frac{d}{2}\right)!}2^{-d/2}\varepsilon^{\mu_1\mu_2\dots\mu_d}\omega_{\mu_1\mu_2}\omega_{\mu_3\mu_4}\dots\omega_{\mu_{d-1}\mu_d}\det{}^{1/2}\left(\frac{\sinh\frac{\omega^\mu{}_\nu}{2}}{\frac{\omega^\mu{}_\nu}{2}}\right). \tag{5.8}$$

5.2.2 Fermionic Field with Anti-periodic Boundary Conditions

If we omit the gamma matrix γ_5 in the trace, in equation (5.5) above, anti-periodic boundary conditions are instead imposed on its solution. Hence, we have the path integral

$$\mathrm{Tr}(e^{-\frac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu}) = \int\limits_{\text{APBCs}} \mathscr{D}\psi^\mu \exp\left[-\frac{1}{2}\int_0^1 dt(\psi^\mu\dot{\psi}_\mu + \omega_{\mu\nu}\psi^\mu\psi^\nu)\right]. \tag{5.9}$$

This case is computed in a similar way as in the case above. The difference is that when evaluating the functional determinant, now with anti-periodic boundary conditions, the following product formula is replacing the sine-product used above:

$$\cos x = \prod_{n=1}^{\infty}\left[1 + \frac{x^2}{\pi^2(2n+1)}\right].$$

Hence we get the d-dimensional fermionic path integral

$$\mathrm{Tr}(e^{-\frac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu}) = (\sqrt{2})^d\det{}^{1/2}\left(\cosh\frac{\omega^\mu{}_\nu}{2}\right). \tag{5.10}$$

5.2.3 Isospin Fields

The isospin fields η obeys the same algebra as the spin fields ψ. Periodic boundary conditions are imposed on the solution of the path integral, due to the factor $(-1)^{N_n}$ for $n = 0$ and 1, hence the path integral is:

$$\mathrm{Tr}((-1)^{N_n}e^{-\bar{\eta}^a T^{ab}\eta^b}) = \int\limits_{\text{PBCs}} \mathscr{D}\bar{\eta}^a\mathscr{D}\eta^a \exp\left[-\int\limits_0^1 dt(\bar{\eta}^a\dot{\eta}^a + \bar{\eta}^a T^{ab}\eta^b)\right]. \tag{5.11}$$

The computations are analogous to the spin case above. We define the matrix T with elements T^{ab} and here $\mathrm{Tr}(T) \neq 0$, hence we get

$$
\begin{aligned}
\mathrm{Tr}((-1)^{N_n} e^{-\bar{\eta}^a T^{ab}\eta^b}) =& \mathscr{N}' \det(\partial_t + T) \int\limits_{\substack{\text{zero}\\\text{modes}}} d\bar{\eta}_1 d\eta_1 \dots d\bar{\eta}_n d\eta_n e^{-\bar{\eta}^a T^a{}_b \eta^b} \\
=& \mathscr{N}' \det\left(\frac{\sinh\frac{T}{2}}{\frac{T}{2}} e^{bT/2i}\right) \det(T) \\
=& \mathscr{N}' \det\left(e^{\frac{T}{2}(1+\frac{b}{i})} - e^{-\frac{T}{2}(1-\frac{b}{i})}\right), \quad n = 0, 1.
\end{aligned} \tag{5.12}
$$

To determine the normalizing constant $\mathscr{N}'$ and the constant b, we take the case where $N_1 := 1$ and let the matrix $\tilde{T} = \bar{\eta}^a T^{ab} \eta^b$ be diagonal.

Recall the partition function $\mathcal{Z}(\beta)$ of a fermionic oscillator (3.7) and its functional integral (3.8), written out as a functional determinant:

$$
\mathcal{Z}(\beta) = e^{\beta\omega/2} \det\left((1-\epsilon\omega)\frac{d}{d\tau} + \omega\right) = 2\sinh(\beta\omega/2), \quad \epsilon = \beta/N \text{ and } N \to \infty.
$$

Hence, we get the determinant of $\mathcal{Z}$:

$$
\det\left((1-\epsilon\omega)\frac{d}{d\tau} + \omega\right) = 2e^{-\beta\omega/2}\sinh(\beta\omega/2) = (1 - e^{-\beta\omega}).
$$

Solving (5.11) using similar techniques as for the fermionic partition function, gives the path integral of the isospin fields, here for the non-trivial case $N_1 = 1$,

$$
\mathrm{Tr}((-1)^{N_1} e^{-\tilde{T}}) = 2e^{-T/2}\sinh T/2 = (1 - e^{-T}), \quad n = 1.
$$

Thus, the constant $b = -\mathrm{i}$ and the normalizing constant $\mathscr{N}' = 1$ in (5.12). The path integral, for both states $n = 0$ and 1, is finally given by

$$
\mathrm{Tr}((-1)^{N_n} e^{-\bar{\eta}^a T^{ab}\eta^b}) = \det(1 - e^{-T}); \quad n = 0, 1. \tag{5.13}
$$

5.2.4 Scalar Fields

The path integral for scalar fields is derived analogous to the cases abowe. For the Riemann curvature tensor $R^\mu{}_{\nu\alpha\beta}$, we define the curvature two-form as $\mathscr{R}^\mu{}_\nu = \frac{1}{2} R^\mu{}_{\nu\alpha\beta}\psi^\alpha\psi^\beta$, and the path integral is

$$
\begin{aligned}
&\int\limits_{\text{PBCs}} \mathscr{D}x^\mu \exp\left[-\frac{1}{2}\int_0^1 dt(\dot{x}^\mu \dot{x}_\mu + x_\mu \mathscr{R}^\mu{}_\nu \dot{x}^\nu)\right] \\
&= \mathcal{N}\det{}^{-1/2}[(-\partial_t^2 + \mathscr{R}\partial_t)^\mu{}_\nu] \int\limits_{\substack{\text{zero}\\\text{modes}}} dx^1 \dots dx^d \\
&= \mathcal{N}\det{}^{-1/2}[(\partial_t + \mathscr{R})^\mu{}_\nu] \int\limits_{\substack{\text{zero}\\\text{modes}}} dx^1 \dots dx^d \\
&= (2\pi)^{-\frac{d}{2}}\det{}^{-1/2}\left(\frac{\sinh\frac{\mathscr{R}^\mu{}_\nu}{2}}{\frac{\mathscr{R}^\mu{}_\nu}{2}}\right) \int\limits_{\substack{\text{zero}\\\text{modes}}} dx^1 \dots dx^d.
\end{aligned} \tag{5.14}
$$

where the normalizing constant $\mathcal{N} = (2\pi)^{-d/2}$ is computed from the free-field integral, in the limit $\mathscr{R}^{\mu}{}_{\nu} \to 0$ (cf. $F_0(T)$ in section 3.1.4 with $m = \hbar = 1$ and $T = -\mathrm{i}$). The argument of the functional determinant, in the second equality, is computed from the hitherto argument by integration by parts with respect to the operator ∂_t.

5.3 The Atiyah-Singer Index Theorem

In the action (5.4), we can choose a gauge for the four-potential $A_\mu(x)$ and the Christoffel symbol $\Gamma^{\mu}_{\nu\varrho}$, accordingly,

$$A_\mu(x) = -\frac{1}{2}x^\nu F_{\mu\nu}, \tag{5.15a}$$

$$\psi_\mu \Gamma^{\mu}_{\nu\varrho}\psi^\nu = \frac{1}{2}R_{\alpha\varrho\mu\nu}\psi^\mu\psi^\nu x^\alpha =: \mathscr{R}_{\mu\nu}x^\mu. \tag{5.15b}$$

The gauge choice for $A_\mu(x)$ is interpreted as that we can take any point in space as origin, here for simplicity we have $A_\mu(0) = 0$. Recall from chapter 2 that for a (local) coordinate system, with origin at some point x_0 on a manifold M, we can define the *Riemann normal coordinates* which is necessary when dealing with index theorems and the Dirac operator; the metric $g_{\mu\nu}(x)$ in the normal coordinates is given by the Kronecker-delta, i.e., $g_{\mu\nu}(x_0) = \delta_{\mu\nu}$, and it is locally flat; $\partial_\sigma g_{\mu\nu}(x_0) = 0$. The Riemann tensor $R_{\alpha\varrho\mu\nu}$ in the gauge given above is a measure on how much a path on a manifold M deviates from the geodesic equation $D^g_t \dot{x}^\mu = 0$, and the classical Euler-Lagrange equations of motion derived from the Lagrangian in the action (5.4), excluding the isospin fields η, are equal to [10]

$$-g_{\alpha\mu}(D^g_t\dot{x})^\mu + \frac{1}{2}R_{\alpha\varrho\mu\nu}\psi^\mu\psi^\nu\dot{x}^\varrho = 0.$$

If the fermion fields $\psi \equiv 0$, only then are geodesics defined on the manifold M, and they are the main contribution to the action. On the other hand, for non-zero fermion fields, quantum fluctuations around a critical point (x_0, ψ_0), are included. See appendix D on how to derive the Riemann curvature tensor from the gauge choice (5.15b) and the quantum fluctuations.

The action (5.4) in components is given by

$$\begin{aligned} S = \int_0^\beta dt \Big[&\frac{1}{2}g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu + \frac{1}{2}g_{\mu\nu}(x)\psi^\mu\dot{\psi}^\nu + \frac{1}{2}g_{\mu\nu}(x)\dot{x}^\varrho\psi^\mu\Gamma^{\mu}_{\varrho\nu}(x)\psi^\nu \\ &+\bar{\eta}^a\dot{\eta}^a + \bar{\eta}^a\dot{x}^\alpha A^{aa}_\alpha(x)\eta^a - \frac{1}{2}\bar{\eta}^a F^{ab}_{\mu\nu}(x)\psi^\mu\psi^\nu\eta^b + i\frac{\alpha}{\beta}\bar{\eta}^a\eta^a\Big] \\ = \int_0^\beta dt \Big[&\frac{1}{2}\dot{x}^\mu\dot{x}_\mu + \frac{1}{2}\psi^\mu\dot{\psi}_\mu + \frac{1}{2}x_\mu\mathscr{R}^{\mu}{}_{\nu}\dot{x}^\nu \\ &+\bar{\eta}^a\dot{\eta}^a + \left\{-\frac{1}{2}\bar{\eta}^a\dot{x}^\mu x^\nu F^{aa}_{\mu\nu}(x)\eta^a\right\} - \frac{1}{2}\bar{\eta}^a F^{ab}_{\mu\nu}(x)\psi^\mu\psi^\nu\eta^b + i\frac{\alpha}{\beta}\bar{\eta}^a\eta^a\Big]. \end{aligned}$$

The term with the gauge choice for the four-potential $A^{aa}_\alpha(x)$ is put inside the curly brackets, since it will be equal to zero after re-scaling the fields. We consider the quantum fluctuations as infinitesimal variations, $\delta x^\mu(t)$ and $\delta\psi^\mu(t)$, of the scalar- and fermion-fields, respectively:

$$x^\mu(t) = x_0^\mu + \delta x^\mu(t),$$
$$\psi^\mu(t) = \psi_0^\mu + \delta\psi^\mu(t).$$

The non-zero modes of the fluctuations are periodic and the Fourier expansions of the quantum fluctuations are given by

$$\delta x^\mu(t) = \frac{1}{\sqrt{\beta}} \sum_{n=-\infty}^{\infty} \delta x_n^\mu e^{\mathrm{i}2\pi nt/\beta},$$
$$\delta\psi^\mu(t) = \frac{1}{\sqrt{\beta}} \sum_{n=-\infty}^{\infty} \delta\psi_n^\mu e^{\mathrm{i}2\pi nt/\beta}.$$

The non-zero modes in the Fourier expansions above vanish due to the periodic boundary condition in the action, thus regarding only zero modes in the expansions, we get the fields

$$x^\mu(t) = x_0^\mu + \frac{1}{\sqrt{\beta}}\delta x_0^\mu,$$
$$\psi^\mu(t) = \psi_0^\mu + \frac{1}{\sqrt{\beta}}\delta\psi_0^\mu.$$

In the limit $\beta \to 0$, the integral over x_0^μ is equivalent with that over $\delta x_0^\mu/\sqrt{\beta}$ hence the measure in the integral is $dx_0^\mu = d(\delta x_0^\mu)/\sqrt{\beta}$. By the same token we have $d\psi_0^\mu = \sqrt{\beta}d(\delta\psi_0^\mu)$.

We substitute $x^\mu(t)$ and $\psi^\mu(t)$ in the action S, keeping only terms of second order in fluctuations, and thus re-scaling the fields for $t \to \beta t$, accordingly

$$\psi^\mu \longrightarrow \frac{1}{\sqrt{\beta}}\psi^\mu \quad \Longrightarrow \quad \dot\psi^\mu \longrightarrow \frac{1}{\sqrt{\beta}}\dot\psi^\mu,$$
$$x^\mu \longrightarrow \sqrt{\beta}x^\mu \quad \Longrightarrow \quad \dot x^\mu \longrightarrow \frac{1}{\sqrt{\beta}}\dot x^\mu,$$
$$\eta \longrightarrow \eta \quad \Longrightarrow \quad \dot\eta \longrightarrow \frac{1}{\beta}\dot\eta.$$

The Dirac operator is Fredholm in Euclidean metric only, hence we use normal coordinates here and obtain the action $S \to S'$:

$$S' = \int_0^1 dt \left[\frac{1}{2}\dot x^\mu \dot x_\mu + \frac{1}{2}\psi^\mu\dot\psi_\mu + \frac{1}{2}x_\mu \mathscr{R}^\mu{}_\nu \dot x^\nu + \bar\eta^a\dot\eta^a - \frac{1}{2}\bar\eta^a\mathscr{F}^{ab}\eta^b + i\frac{\alpha}{\beta}\bar\eta^a\eta^a \right.$$
$$\left. +\beta\left\{-\frac{1}{2}\bar\eta^a\dot x^\mu x^\nu F_{\mu\nu}^{aa}(x)\eta^a\right\}\right].$$

where we define the field strength curvature two form $\mathcal{F}^{ab} \equiv F^{ab}_{\mu\nu}\psi^\mu\psi^\nu$. The index theorem is independent of the parameter $\beta \sim (\text{temperature})^{-1}$, thus the term inside the curly-brackets in S' goes to zero in the high temperature limit $\beta \to 0$.

The generating function $I(\alpha)$, expressed in terms of path integrals, is equal to

$$\begin{aligned}
I(\alpha) &= \int\limits_{\text{PBCs}} \mathscr{D}x^\mu \mathscr{D}\psi^\mu \mathscr{D}\bar{\eta}^a \mathscr{D}\eta^a e^{-S'} \\
&= \int \mathscr{D}x^\mu \exp\left[-\frac{1}{2}\int_0^1 dt(\dot{x}^\mu \dot{x}_\mu + x_\mu \mathscr{R}^\mu{}_\nu \dot{x}^\nu)\right] \times \int \mathscr{D}\psi^\mu \exp\left[-\frac{1}{2}\int_0^1 dt(\psi^\mu \dot{\psi}_\mu)\right] \\
&\times \int \mathscr{D}\bar{\eta}^a \mathscr{D}\eta^a \exp\left[-\int_0^1 dt\left(\bar{\eta}^a \dot{\eta}^a + \bar{\eta}^a\left(-\frac{1}{2}\mathcal{F} + i\alpha\right)^{ab}\eta^b\right)\right].
\end{aligned}$$

For some generic tensor $A_{\mu_1\cdots\mu_d}$ coupled to fermion fields, we have the following identity:

$$\int dx^1 \ldots dx^d \int d\psi^1 \ldots d\psi^d A_{\mu_1\cdots\mu_d}\psi^{\mu_1}\ldots\psi^{\mu_d} = (-1)^{\frac{d}{2}} \int\limits_{\text{space}} dx^{\mu_1}\ldots dx^{\mu_d} A_{\mu_1\ldots\mu_d}.$$

Thus, in the last equality the tensor is coupled to differential forms. There is a duality between the fermion fields and the differential forms; $\psi^\mu \leftrightarrow dx^\mu$. This duality will be utilized below in derivation of the Euler number. Both $\mathscr{R}^\mu{}_\nu$ and $\mathcal{F}^{ab}$ are coupled to fermion fields, hence using the path integrals (5.7), (5.13) and (5.14) from the previous section, the fermion fields are integrated out and the generating function is

$$\begin{aligned}
I(\alpha) &= \left[\left(\frac{1}{2\pi}\right)^{\frac{d}{2}} \int\limits_{\substack{\text{zero}\\\text{modes}}} dx^1 \ldots dx^d \det{}^{-1/2}\left(\frac{\sinh(\mathscr{R}^\mu{}_\nu/2)}{\mathscr{R}^\mu{}_\nu/2}\right)\right] \times \left[(-\mathrm{i})^{\frac{d}{2}} \int\limits_{\substack{\text{zero}\\\text{modes}}} d\psi^1 \ldots d\psi^d\right] \times \\
&\times \left[\det\left(1 - e^{\mathcal{F}/2 - \mathrm{i}\alpha}\right)\right] \\
&= (-1)^{\frac{d}{2}} \int\limits_{\text{space}} \det{}^{-1/2}\left(\frac{\sinh(\mathscr{R}^\mu{}_\nu/2)}{\mathscr{R}^\mu{}_\nu/2}\right)\left(\frac{-\mathrm{i}}{2\pi}\right)^{\frac{d}{2}} \operatorname{Tr}\left((-1)^N e^{-\mathrm{i}\alpha N} e^{\mathcal{F}/2}\right) \\
&= \sum_{k=0}^{\infty}(-1)^k e^{-\mathrm{i}\alpha k}\left[\int\limits_{\text{space}} \det{}^{-1/2}\left(\frac{\sinh(\mathscr{R}^\mu{}_\nu/2)}{\mathscr{R}^\mu{}_\nu/2}\right)\left(\frac{\mathrm{i}}{2\pi}\right)^{\frac{d}{2}} \operatorname{Tr}\left(e^{\mathscr{F}}\right)\right]
\end{aligned}$$

where in the second equality we have the curvature tensors, expressed in differential forms, $\mathscr{R}^\mu{}_\nu = \frac{1}{2}R^\mu{}_{\nu\alpha\beta}dx^\alpha \wedge dx^\beta$ and $\mathscr{F} = \frac{1}{2}\mathcal{F} = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$. The first two operators in the trace, associated to the path integral over the isospin fields, are extracted and put outside the integral in the third equality where the number operator N becomes the integer k in the generating function. Recall that in the path integral of the isospin case, there are two cases; $k = 0$ (trivial) and $k = 1$ (non-trivial) and in the generating function $I(\alpha)$ the *analytical index*, denoted by I_k, is equal to the expression between the brackets in the third equality above:

$$I_k = \int\limits_{\text{space}} \det{}^{-1/2}\left(\frac{\sinh(\mathscr{R}^{\mu}{}_{\nu}/2)}{\mathscr{R}^{\mu}{}_{\nu}/2}\right)\left(\frac{\mathrm{i}}{2\pi}\right)^{\frac{d}{2}} \mathrm{Tr}\left(e^{\mathscr{F}}\right); \qquad k = 1. \tag{5.16}$$

For the non-trivial case, the *Atiyah-Singer index theorem* of the twisted Dirac operator $\mathbf{D}_V = \mathbf{D} \otimes \mathscr{F}$ is equal to

$$index(\mathbf{D}_V) = \int\limits_M \hat{A}(TM) \wedge ch(\mathscr{F}) \tag{5.17}$$

where we have the $\hat{A}$-genus and the total Chern character, respectively,

$$\hat{A}(TM) = \det{}^{-1/2}\left(\frac{\sinh(\mathscr{R}^{\mu}{}_{\nu}/2)}{\mathscr{R}^{\mu}{}_{\nu}/2}\right) = \prod_{j=1}^{d/2} \frac{y_j/2}{\sinh y_j/2},$$

$$ch(\mathscr{F}) = \left(\frac{\mathrm{i}}{2\pi}\right)^{\frac{d}{2}} \mathrm{Tr}\left(e^{\mathscr{F}}\right) = \mathrm{Tr}\exp\left(\frac{\mathrm{i}}{2\pi}\mathscr{F}\right).$$

The trivial case, I_0, is just the integral of the $\hat{A}$-genus without the gauge field applied[10], hence the Chern character is equal to one.

The Atiyah-Singer index theorem, in words, is formulated as

Atiyah-Singer Index Theorem. *The topological and the analytical index are equal.*

In the following section we will find a topological invariant, the *Euler number* $\chi(M)$, equal to the analytical index given in this section.

5.4 The Euler Number

We review here, briefly, some facts about the spin complex. In special cases the spin complex is in correspondence to the de Rham complex and the Euler number $\chi(M)$ can, via the duality $\psi^{\mu} \leftrightarrow dx^{\mu}$, be defined in terms of spinors.

5.4.1 Clifford Forms and Differential Forms

When the space-time dimension d is even, there is a one-to-one correspondence between the differential form $\omega = \omega_{\mu_1\ldots\mu_d} dx^{\mu_1}\ldots dx^{\mu_d}$ and the skew-symmetrical matrix $\hat{\omega} = \hat{\omega}_{\mu_1\ldots\mu_d}\gamma^{\mu_1}\ldots\gamma^{\mu_d}$, where $\hat{\omega}$ span all the matrices on spinors [6]. Let S be the space of spinors, and S^* be its dual space, thus the space of matrices on spinors is $S \otimes S^*$. Here we take (isospin) = (spin)*. We define the space of Clifford forms C as[14]

$$C = S \otimes S^* = \left\{\hat{\omega}; \hat{\omega} = \hat{\omega}^{sc} + \hat{\omega}_{\mu}\gamma^{\mu} + \frac{1}{2!}\hat{\omega}_{\mu\nu}\gamma^{\mu}\gamma^{\nu} + \cdots + \frac{1}{d!}\hat{\omega}_{\mu_1\ldots\mu_d}\gamma^{\mu_1}\ldots\gamma^{\mu_d}\right\}$$

As a comparison, a d-form $\omega \in S \otimes \bar{S}$ is then equal to

$$\omega = \omega^{sc} + \omega_{\mu}dx^{\mu} + \frac{1}{2!}\omega_{\mu\nu}dx^{\mu}dx^{\nu} + \cdots + \frac{1}{d!}\omega_{\mu_1\ldots\mu_d}dx^{\mu_1}\ldots dx^{\mu_d}.$$

Let the modified gamma matrix $\tilde{\gamma}_5$ be defined as

$$\tilde{\gamma}_5 := (-\mathrm{i})^{\frac{d}{2}}\gamma_5 = \gamma_1 \dots \gamma_d = \frac{1}{d!}\varepsilon_{\mu_1 \dots \mu_d}\gamma^{\mu_1} \dots \gamma^{\mu_d}.$$

The transpose of $\tilde{\gamma}_5$ is

$$\tilde{\gamma}_5^\dagger = (-1)^{d(d-1)/2}\tilde{\gamma}_5.$$

The Hodge star operator $*$ gives the duality between two spaces, e.g., the duality of the Clifford form $\hat{\omega}$ is

$$*\hat{\omega} = \frac{1}{d!}\hat{\omega}^{sc}\varepsilon_{\mu_1 \dots \mu_d}\gamma^{\mu_1} \dots \gamma^{\mu_d} + \frac{1}{(d-1)!}\hat{\omega}_{\mu_1}\varepsilon_{\mu_2 \dots \mu_d}\gamma^{\mu_2} \dots \gamma^{\mu_d} + \dots + \hat{\omega}_{\mu_1 \dots \mu_d} = \tilde{\gamma}_5 \hat{\omega}^\dagger.$$

With $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$ we can find the commutation relation of two products of gamma matrices, e.g. for $(\gamma^{\mu_1} \dots \gamma^{\mu_d})$ and $(\gamma^{\mu_1} \dots \gamma^{\mu_k})$ where $k < d$:

$$\begin{aligned}
(\gamma^{\mu_1} \dots \gamma^{\mu_d})(\gamma^{\mu_1} \dots \gamma^{\mu_k}) &= (-1)^k(\gamma^{\mu_1} \dots \gamma^{\mu_{d-1}})(\gamma^{\mu_1} \dots \gamma^{\mu_k})\gamma^{\mu_d} \\
&= (-1)^k(-1)^k(\gamma^{\mu_1} \dots \gamma^{\mu_{d-2}})(\gamma^{\mu_1} \dots \gamma^{\mu_k})\gamma^{\mu_{d-1}}\gamma^{\mu_d} \\
&\vdots \\
&= (-1)^k \dots (-1)^k(\gamma^{\mu_1} \dots \gamma^{\mu_{d-(d-k)}})(\gamma^{\mu_1} \dots \gamma^{\mu_k})\gamma^{\mu_{d-(d-k)+1}} \dots \gamma^{\mu_d} \\
&= (-1)^{k(d-k)}(\gamma^{\mu_1} \dots \gamma^{\mu_k})(\gamma^{\mu_1} \dots \gamma^{\mu_k}\gamma^{\mu_{k+1}} \dots \gamma^{\mu_d}),
\end{aligned}$$

hence $(d-k)$ gamma matrices have been swapped from the left product to the far right end, and in the end the products commute.

Furthermore, a k-form $\omega = \frac{1}{k!}\omega_{\mu_1 \dots \mu_k}dx^{\mu_1} \dots dx^{\mu_k}$ commutes with $\tilde{\gamma}_5$ accordingly

$$\tilde{\gamma}_5\omega = (-1)^{k(d-k)}\omega\tilde{\gamma}_5. \tag{5.18}$$

For d even, the space of Clifford forms can be decomposed into even and odd forms, $C = C_+ + C_-$, where

$$C_+ \equiv S \otimes S_+^* = \{\omega \in S; \omega\gamma_5 = \omega\}, \quad \text{and} \quad C_- \equiv S \otimes S_-^* = \{\omega \in S; \omega\gamma_5 = -\omega\}. \tag{5.19}$$

From the definition of the gamma matrix we have $\gamma_5 = (\mathrm{i})^{d/2}\gamma_1 \dots \gamma_d = \mathrm{i}\tilde{\gamma}_5$ and from (5.18), with simplified prefactor $[(-1)^d]^k[(-1)^{-k}]^k = (-1)^k$ in the right hand side, we find $\gamma_5\omega$:

$$\begin{aligned}
\gamma_5\omega &= (-1)^k\omega\gamma_5 = (-1)^k(+\omega), &\qquad \text{on } C_+ \\
\gamma_5\omega &= (-1)^k\omega\gamma_5 = (-1)^k(-\omega), &\qquad \text{on } C_-.
\end{aligned}$$

Hence, the correspondence between differential forms ω and Clifford forms $\hat{\omega}$ is given by

$$(-1)^k\omega \leftrightarrow \gamma_5\hat{\omega}\gamma_5. \tag{5.20}$$

5.4.2 The Index as a Topological Invariant

The correspondence $\omega \leftrightarrow \hat{\omega}$ allows us to consider the Dirac operator as a difference between the exterior derivative d and the interior derivative $d^\dagger$; $\mathbf{D} = d - d^\dagger$. We define the derivatives d, and its dual $d^\dagger$, accordingly

$$\begin{aligned} d_k &: \Lambda^k \to \Lambda^{k+1}, \\ d_k^\dagger &: \Lambda^{k+1} \to \Lambda^k; \quad k = 1, \dots, d. \end{aligned}$$

Here Λ^k is the space of k-forms on the tangent space of a d-dimensional manifold M. The *Laplacian* is defined as

$$\Delta_k = d_k^\dagger d_k + d_{k-1} d_{k-1}^\dagger : \Lambda^k \to \Lambda^k.$$

From the Euler-Poincaré theorem, we define the *Euler number*

$$\chi(M) = \sum_{k=0}^{d} (-1)^k \ker \Delta_k = \sum_{k=0}^{d} (-1)^k \dim H^k_{\mathrm{dR}}(M; \mathbb{R}),$$

where $\ker \Delta_k = \{\omega; d\omega = d^\dagger\omega = 0\}$ is the space of harmonic k-forms. We define the gamma matrix $\gamma_5 = (-1)^F := (-1)^k$, hence multiplying the duality (5.20) by γ_5 on the right and using the properties of the decomposition of the space of Clifford forms (5.19), we get

$$\begin{aligned} (-1)^k \omega \leftrightarrow \gamma_5 \hat{\omega} \quad &\text{for} \quad \hat{\omega} \in S \otimes S_+^*, \\ (-1)^{k+1} \omega \leftrightarrow \gamma_5 \hat{\omega} \quad &\text{for} \quad \hat{\omega} \in S \otimes S_-^*. \end{aligned}$$

It follows that the index I_k on $C_\pm$ is equal to

$$\begin{aligned} I_+ &= index(\mathbf{D} \text{ on } C_+) = \sum_k (-1)^k \dim Harm_k^-(C_+), \quad \text{and} \\ I_- &= index(\mathbf{D} \text{ on } C_-) = \sum_k (-1)^k \dim Harm_k^+(C_-), \end{aligned}$$

where $Harm_k^+$ ($Harm_k^-$) is the space of harmonic forms in $S \otimes S_+^*$ ($S \otimes S_-^*$), i.e. $\mathbf{D}\mathbf{D}^\dagger \hat{\omega} = 0$ ($\mathbf{D}^\dagger \mathbf{D} \hat{\omega} = 0$). From Hodge's theorem we have an isomorphism between the space of harmonic forms and the de Rham cohomology groups; $Harm_k(M) \cong H^k_{\mathrm{dR}}(M; \mathbb{R})$. From the de Rham's theorem we have $\dim H^k_{\mathrm{dR}} = \dim H_k$, where H_k is the kth homology group, hence from the Euler-Poincaré theorem, the index is equal to a topological invariant which is the Euler number:

$$index(\mathbf{D}) = I_+ - I_- = \sum_k (-1)^k \dim H_k = \chi(M).$$

The far right hand side is purely topological, while the terms to the left of the third equality are analytical quantities. Now, we can compute $I_+ - I_-$ using the expression of $I_k = I_\pm$ from the previous section.

We use (5.16) and replacing $\mathrm{Tr}(e^{\mathscr{F}})$ with $\mathrm{Tr}[e^{\mathscr{F}}(1 \pm \gamma_5)/2]$ in the Chern character $ch(\mathscr{F})$. Now we denote the analytical index as $I_k = I_\pm$ and hence we get

$$I_\pm = \int \det{}^{-1/2}\left(\frac{\sinh(\mathscr{R}^\mu{}_\nu/2)}{\mathscr{R}^\mu{}_\nu/2}\right)\left(\frac{\mathrm{i}}{2\pi}\right)^{\frac{d}{2}} \mathrm{Tr}[e^{\mathscr{F}}(1 \pm \gamma_5)/2]. \tag{5.21}$$

The field strength curvature two-form is here chosen as $F_{\mu\nu}dx^\mu \wedge dx^\nu = -\frac{1}{4}R^\alpha{}_{\beta\mu\nu}\gamma_\alpha\gamma^\beta$, which is the ordinary scalar curvature represented by infinitesimal rotations on the dual space of the spinors[6]. The similarities in construction of the curvature tensors $F_{\mu\nu}$ and $R^\alpha{}_{\beta\mu\nu}$ are reviewed in appendix C. Notice that $I_+ - I_- \sim \mathrm{Tr}[\gamma_5 e^{\mathscr{F}}]$, hence using the path integral (5.8) with $\frac{1}{2}\omega^\mu{}_\nu = \frac{1}{2}\mathscr{R}^\mu{}_\nu$ gives

$$\mathrm{Tr}[e^{\mathscr{F}/2}\gamma_5] = \frac{\left(-\frac{\mathrm{i}}{2}\right)^{\frac{d}{2}}}{\left(\frac{d}{2}\right)!}\det{}^{1/2}\left(\frac{\sinh\frac{\omega^\mu{}_\nu}{2}}{\frac{\omega^\mu{}_\nu}{2}}\right)\varepsilon^{\mu_1\mu_2\ldots\mu_d}\left(\frac{1}{2}\right)^{\frac{d}{2}}\mathscr{R}_{\mu_1\mu_2}\mathscr{R}_{\mu_3\mu_4}\ldots\mathscr{R}_{\mu_{d-1}\mu_d}.$$

Substituting $\mathrm{Tr}[e^{\mathscr{F}}\gamma_5]$ in $I_+ - I_-$ yields the index of the Dirac operator

$$\begin{aligned} I_+ - I_- &= \int\left(\frac{\mathrm{i}}{2\pi}\right)^{\frac{d}{2}}\left(\frac{1}{2\mathrm{i}}\right)^{\frac{d}{2}}\left(\frac{1}{2}\right)^{\frac{d}{2}}\frac{1}{\left(\frac{d}{2}\right)!}\varepsilon^{\mu_1\mu_2\ldots\mu_d}\mathscr{R}_{\mu_1\mu_2}\mathscr{R}_{\mu_3\mu_4}\ldots\mathscr{R}_{\mu_{d-1}\mu_d} \\ &= \int\left(\frac{1}{8\pi}\right)^{\frac{d}{2}}\frac{1}{\left(\frac{d}{2}\right)!}\varepsilon^{\mu_1\mu_2\ldots\mu_d}\mathscr{R}_{\mu_1\mu_2}\mathscr{R}_{\mu_3\mu_4}\ldots\mathscr{R}_{\mu_{d-1}\mu_d} = \chi(M). \end{aligned}$$

5.4.3 Examples

Example 1. (Gauss-Bonnet theorem) For a two-sphere $S^d = S^2$ in $\mathbb{R}^3$, we have the curvature $\varepsilon^{\mu\nu}\mathscr{R}_{\mu\nu} = \varepsilon^{12}\mathscr{R}_{12} + \varepsilon^{21}\mathscr{R}_{21} = 2\varepsilon^{12}\mathscr{R}_{12} = 2RdA$, where R is the scalar curvature (equal to the radius of the sphere) and $dA = d^2x$ an infinitesimal area element on the surface of S^2. We get the *Gauss-Bonnet theorem* from $I_+ - I_-$ above:

$$\int d^2x\frac{1}{4\pi}R = \int d^2x\frac{K}{2\pi} = \chi(S^2) = 2,$$

where $K = R/2$ is the Gauss curvature and the Euler number for a sphere S^d is given[3] by the formula: $\chi(S^d) = 1 + (-1)^d$.

In the case of S^2 the Euler number gives the number of critical points; the north pole and the south pole, where a vector field cannot be defined. We can define[5] a vector field tangent to the longitudinal lines as $\partial/\partial\theta$. At the north pole the vector field is diverging (similar to a vector field of a positive charge in electrodynamics) and the index is equal to $+1$. The vector field converges at the south pole (i.e. a vector field of a negative charge) also of index $+1$, thus $\chi(S^2) = 2$.

We can also derive the Euler number from a polyhedron, homeomorphic to a manifold M. A polyhedron homeomorphic to S^2 means that the sphere can be continuously deformed by stretching out, say three, distinct points on the surface and thus shaping it into a geometrical object that looks like a pyramid; a tetrahedron. This particular polyhedron has six edges, four vertices, and four faces, thus we have triangulated S^2. Euler's theorem states that for any set $X \subset \mathbb{R}^3$ homeomorphic to a polyhedron, the Euler number is equal to

$$\chi(X) = v - e + f,$$

where v, e and f are the number of vertices, edges and faces, respectively, i.e., $\chi(S^2) = 4 - 6 + 2 = 2$. The Euler number $\chi(X)$ is independent of the polyhedron (Poincaré-Alexander theorem) thus we can subdivide the tetrahedron. A generalization of the Euler number $\chi(X)$ into n-dimensions (Hopf's theorem) is given by

$$\begin{aligned}\chi(M^n) =&(\text{no. 0-simplexes}) - (\text{no. 1-simplexes})\\ &+ (\text{no. 2-simplexes}) - \cdots + (-1)^n(\text{no. } n\text{-simplexes})\end{aligned}$$

where M^n is an n-dimensional manifold, and the first three simplexes in the alternating sum are equal to vertices, edges, and faces, respective.

The Gauss-Bonnet theorem shows that a deformation of a surface of a manifold might change the curvature K and the area form $dA = d^2x$ pointwise, but the total curvature $\int d^2x(K/2\pi)$ remains invariant. Thus the total curvature is a measure of the *genus of a surface*. From a topological point of view, given for instance a multitorus $M = \sum_g$ with g holes, the general formula for the Euler number is equal to

$$\chi(M) = 2 - 2g.$$

The curvature K involves derivatives of the metric tensor, thus $\partial_\lambda g_{\mu\nu}(x)$ are quantities to the tangent bundle TM of some manifold M. For a two-dimensional manifold M^2 the Gauss-Bonnet theorem is

$$\int d^2x \frac{K}{2\pi} = b_0 - b_1 + b_2 \tag{5.22}$$

where the *Betti numbers* $b_p(M)$ are equal to (from Hodge's theorem) the dimension of the space of harmonic p-forms:

$$b_p(M) = \dim\ker\Delta_p : \Lambda^p \to \Lambda^p$$

The Atiyah-Singer index theorem is a vast generalization of (5.22). Thus we can replace the tangent bundle by other bundles (e.g. vector bundles, principal bundles, line bundles), the Gauss curvature by, e.g., the Riemann- or the field strength curvature, and replacing the Laplacian Δ_p by other elliptic differential operators associated with the bundle.

Example 2. (The winding number) For a manifold M of dimension $d = 2$ we have the $\hat{A}$-genus $\hat{A}(TM) = 1$ and the Chern character equal to the first Chern form; $c_1(\mathscr{F}) = (\mathrm{i}/2\pi)\frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$, thus the index (5.17) is equal to

$$I_1 = \frac{\mathrm{i}}{4\pi}\int_M d^2x\epsilon^{\mu\nu}F_{\mu\nu} = \chi(M).$$

If the field strength is chosen as $\epsilon^{\mu\nu}F_{\mu\nu} \sim -\mathrm{i}R$ we can see the similarity between the Atiyah-Singer index theorem (5.17) and the Gauss-Bonnet theorem from the previous example.

We take again the manifold $M = S^2$ and get the index 2. The two-sphere can be considered as a one-dimensional complex manifold, which is called the *Riemann sphere*[5].

There are two complex planes tangent to the Riemann sphere at the poles; the u-v plane at the south pole and the x-y plane at the north pole. Define complex coordinates at the north- and south pole, respectively, as $z = x{+}\mathrm{i}y = |z|\exp(\mathrm{i}\theta)$ and $w = u{+}\mathrm{i}v = |w|\exp(\mathrm{i}\theta)$, here θ is an angle. The relation between the u-v coordinates and the x-y coordinates is given by $w = 1/z$. In the u-v plane we have a vector field (or a velocity field) defined as $dw/dt = 1$. We can stereographically project the flow associated with the vector field onto the Riemann sphere, hence a flow near the south pole $w = \lim_{z\to\infty}(1/z) = 0$. Near the north pole $z = \lim_{w\to\infty}(1/w) = 0$ we get the flow

$$\frac{dz}{dt} = \left(\frac{dz}{dw}\right)\left(\frac{dw}{dt}\right) = -\frac{1}{w^2} = -z^2.$$

If we rotate around the path $z = e^{\mathrm{i}\theta}$ about the north pole, the vector $-z^2 = -e^{\mathrm{i}2\theta}$ makes 2 circuits, thus this gives us the index 2.

In terms of physics we can realize the 4π rotation in the context of spin-$\frac{1}{2}$ system[13]. The spin operator in the z-plane is equal to

$$S_z = \frac{\hbar}{2}\left[(|+\rangle\langle+|) - (|-\rangle\langle-|)\right],$$

where the ket $|+\rangle$ ($|-\rangle$) represents spin up (down). The operator that rotates a state in the x-y plane is given by

$$\mathscr{D}_z(\theta) = \exp\left(\frac{-\mathrm{i}S_z\theta}{\hbar}\right),$$

i.e., $|\alpha\rangle_R = \mathscr{D}_z(\theta)|\alpha\rangle$ for some state $|\alpha\rangle = |+\rangle\langle+|\alpha\rangle + |-\rangle\langle-|\alpha\rangle$. A straightforward computation of $|\alpha\rangle_R$ gives

$$\exp\left(\frac{-\mathrm{i}S_z\theta}{\hbar}\right)|\alpha\rangle = e^{-\mathrm{i}\theta/2}|+\rangle\langle+|\alpha\rangle + e^{\mathrm{i}\theta/2}|-\rangle\langle-|\alpha\rangle.$$

If we substitute the angle $\theta = 2\pi$ in $|\alpha\rangle_R$ we get the rotated state

$$|\alpha\rangle_{R_z(2\pi)} \to -|\alpha\rangle,$$

thus in order to get back to the initial state, we must rotate $|\alpha\rangle$ by two revolutions.

5.5 The Hirzebruch Signature

In this section follow the same arguments as in the previous section, regarding the duality between the differentials forms and the Clifford forms. We define an operator τ acting on k-forms ω as

$$\tau\omega = (-1)^{k(k-1)/2} * \omega,$$

where $*$ is the Hodge star operator. We have now instead the correspondence $\tau\omega \leftrightarrow \gamma_5\hat{\omega}\gamma_5$, and multiplying by γ_5 on the right gives

$$\begin{aligned} \tau\omega\gamma_5 &= \tau(+\omega) = (-1)^{k(k-1)/2} * \omega \leftrightarrow \gamma_5\hat{\omega} \quad \text{for } \hat{\omega} \in S \otimes S_+^*, \\ \tau\omega\gamma_5 &= \tau(-\omega) = (-1)^{k(k+1)/2} * \omega \leftrightarrow \gamma_5\hat{\omega} \quad \text{for } \hat{\omega} \in S \otimes S_-^*. \end{aligned}$$

Notice the symmetry between $(-1)^{k(k-1)/2}$ and $(-1)^{k(k+1)/2}$, hence it follows that the Hirzebruch signature is

$$signature(M) := I_+ + I_-.$$

From (5.21) and the path integral (5.10), since $I_+ + I_- \sim \mathrm{Tr}[e^{\mathscr{F}}]$, the signature is thus equal to

$$\begin{aligned} signature(M) &= \int \det{}^{-1/2}\left(\frac{\sinh(\mathscr{R}^\mu{}_\nu/2)}{\mathscr{R}^\mu{}_\nu/2}\right)\left(\frac{\mathrm{i}}{2\pi}\right)^{\frac{d}{2}}(\sqrt{2})^d \det{}^{1/2}\left(\cosh\frac{\mathscr{R}^\mu{}_\nu}{2}\right) \\ &= \int \left(\frac{\mathrm{i}}{\pi}\right)^{\frac{d}{2}} \det{}^{1/2}\left(\frac{\mathscr{R}^\mu{}_\nu/2}{\tanh(\mathscr{R}^\mu{}_\nu/2)}\right). \end{aligned}$$

Example. For a manifold M of dimension $d = 4$, we have the Hirzebruch signature

$$signature(M) = \int_{\text{space}} \frac{1}{192\pi^2}\,\mathrm{Tr}(\mathscr{R}^\mu{}_\nu \wedge \mathscr{R}^\mu{}_\nu).$$

The result in the right hand side is realized also in terms of the $\hat{A}$-genus[3], when $d = \dim M$ is a multiple of 4:

$$\int_M \hat{A}(TM) = -\frac{1}{24}\int_M p_1(TM) = \frac{1}{24\cdot 8\pi^2}\int_M \mathrm{Tr}(\mathscr{R}^\mu{}_\nu \wedge \mathscr{R}^\mu{}_\nu).$$

It can be shown[7] that the Hirzebruch L-polynomial is related to the $\hat{A}$-genus and the Chern character. Writing the L-polynomial $L(x_i)$ in components of order $d/2$ gives

$$\begin{aligned} L(x_i) &= 2^{d/2}\prod_{i=1}^{d/2}\frac{x_i/2}{\tanh(x_i/2)} \\ &= \prod_{i=1}^{d/2} x_i \frac{e^{x_i/2}+e^{-x_i/2}}{e^{x_i/2}-e^{-x_i/2}} \\ &= \prod_{i=1}^{d/2}\frac{x_i/2}{(e^{x_i/2}-e^{-x_i/2})/2}\prod_{i=1}^{d/2}(e^{x_i/2}+e^{-x_i/2}), \end{aligned}$$

i.e., $L(x) = \hat{A}(TM)\wedge ch(E)$ for some vector bundle E. To get the result of the signature above we have, explicitly, the integral $\int_M \hat{A}_1(TM)\wedge ch_0(E)$, where $\dim E = 1$.

In conclusion, we have here demonstrated, once again, the interrelations between the Atiyah-Singer index theorem and the Hirzebruch signature.

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Svensk Sammanfattning

Supersymmetri är en symmetri som relaterar partiklar med heltalsspinn (bosoner) till partiklar med halvtalsspinn (fermioner). Som några exempel på partiklar som har hel- och halvtalsspinn kan nämnas fotoner ("ljuspartiklar"), respektive elektroner.

I denna uppsats betraktar vi supersymmetri endast i termer av *vägintegraler*. Feynman's vägintegral beskriver kvantmekaniska processer i tidsrummet där vi kan visualisera rums-axeln som den horisontella x-axeln och tidsutvecklingen med den vertikala y-axeln. Vägintegralen är lika med summan av alla "vägar" givet en start- och slutpunkt i tidsrummet. För fermioner, respektive bosoner, använder man två olika sorters vägintegraler. I den fermioniska vägintegralen använder vi anti-kommutativa variabler, e.g. $x_1x_2 = -x_2x_1$, som även kallas Grassmann variabler, samt anti-periodiska randvillkor över integralens lösning. Å andra sidan, i bosoniska vägintegraler har vi de välbekanta kommutativa variablerna samt periodiska randvillkor. I ju med att supersymmetri inte gör någon skillnad mellan fermioner och bosoner så definierar vi den *supersymmetriska vägintegralen* som en enskild vägintegral över både anti-kommutativa och kommutativa variablerna samt med periodiskt randvillkor över integralens lösning.

Man har hittills aldrig observerat till exempel elektroner som har heltalsspinn och inte heller fotoner som har halvtalsspinn. Detta innebär att de så kallade *superpartiklarna* till fermionerna, respektive bosonerna, som vi beskrev i föregående mening, endast kan observeras om supersymmetrin är spontant bruten. *Spontant symmetribrott*, även kallat dolt symmetribrott, innebär att ett kvantmekaniskt system ser ut att vara symmetriskt i högre exciterat kvanttillstånd (i samband med att vi ändrar någon variabel $x_i \to -x_i$) men som är asymmetrisk i grundtillståndet. Spontant symmetribrott ger upphov till en mekanism som ger massa till partiklarna. Detta är den berömda *Higgs-mekanismen*. På liknande sätt ger *spontant supersymmetribrott* en mekanism som ger massa till superpartiklarna.

För att bestämma huruvida supersymmetrin i grundtillståndet misslyckas att vara bruten så använder vi oss av en kvantitet som kallas för *Witten indexet* (efter den amerikanska fysikern Edward Witten). Witten indexet ges av $n_{\mathrm{B}}^{\mathrm{E}=0} - n_{\mathrm{F}}^{\mathrm{E}=0}$ där $n_{\mathrm{B}}^{\mathrm{E}=0}$ ($n_{\mathrm{F}}^{\mathrm{E}=0}$) är antalet bosoner (fermioner) i det supersymmetriska grundtillståndet. Man kan visa att Witten indexet är lika med det så kallade *analytiska indexet* av Dirac-operatorn. Det analytiska indexet är i sin tur lika med det *topologiska indexet* som ges av en integral över karakteristik klasser, där klasserna ger ett mått på ytkrökningen av ett geometriskt objekt. I termer av den supersymmetriska vägintegralen så kan vi uttrycka de fysikaliska kvantiteterna som en produkt av vägintegraler över bosoniska variabler, respektive fermioniska variabler, exklusivt i högtemperaturgränsen. Fysikaliskt innebär högtemperaturgränsen att man "gör sig av med" de högre exciterade kvanttillstånden och betraktar endast grundtillståndet. Då man utför kollisionsexperiment i partikelfysik så kan man uppnå oerhört höga temperaturer då partiklarna kolliderar med varandra inuti en accelerator, därmed är gränsvärdet för temperaturen även giltigt fysikaliskt. Resultatet av vägintegralerna visar sig vara identiska med integralerna över karakteristik klasserna då man beräknar *index teorem* i matematik. Vi kan nämna att *Atiyah-Singer index teoremet* är en stor landvinning inom matematiken och kan betraktas som en omfattande generalisering av index teorem. I Atiyah-Singer index teoremet spelar Dirac-operatorn en avgörande roll, det innebär, med andra ord, att välkända och etablerade matematiska resultat kan härledas med hjälp av supersymmetri.

Tolkningen av Witten indexet, givet någon supersymmetrisk modell, ges av följande tumregel; om $n_{\mathrm{B}}^{\mathrm{E}=0} - n_{\mathrm{F}}^{\mathrm{E}=0}$ ej är lika med noll så är supersymmetrin ej bruten och då måste man förkasta modellen ifråga. Det omvända gäller däremot inte; om indexet är lika med noll då kan vi ej dra några säkra slutsatser huruvida supersymmetrin är bruten.

Integraler över karakteristik klasser ger upphov till *topologiska tal*, en kvantitet som förblir oförändrad oavsett hur vi deformerar ett geometriskt objekt (under förutsättningen att ytan hos objektet ej slits itu). Ta till exempel ett geometriskt objekt som ser ut som en badboll; vi kan platta till den eller dra ut den och forma ytan till ett cigarrliknande objekt, men oavsett konfiguration så förblir det topologiska talet oförändrat. Om vi däremot slår ut två hål i badbollen och fäster ett handtag i form av en slang så får vi ett objekt som (topologiskt) ser ut som en badring. Topologin mellan badbollen och badringen skiljer sig avsevärt; det förstnämnda objektet har inget hål, till skillnad från det andra objektet, och därmed skiljer sig det topologiska talet sinsemellan objekten.

I termer av kvanfältteori (ett fält är en variabel med oändligt många frihetsgrader) och matematiska index teorem så talar vi om topologisk kvantfältteori, där Witten indexet ger som vi nämnde en koppling till kvantfysik. Och i rena matematiska termer ger likheten mellan det analytiska- och det topologiska index en relation mellan differentialekvationer och topologi.(Något kortfattat kan man säga att likheten mellan de båda indexen är just Atiyah-Singer index teoremet uttryckt i ord.)

Avsaknaden av observerade superpartiklar innebär att supersymmetrin ej i dagsläget kan betraktas som en fysikalisk teori; det krävs experimentella bevis innan man kan kalla supersymmetrin en teori. I skrivandets stund kan man endast betrakta supersymmetri som ett *teoretiskt ramverk* ellen en fysikalisk modell, men det är forskarnas stora förhoppning att man inom en snar framtid ska kunna avgöra om *supersymmetrisk kvantmekanik* faktiskt beskriver naturen. Möjligtvis kommer svaren om huruvida supersymmetrin stämmer överens med experimentella mätningar att uppdaga sig efter uppgraderingen av *Large Hadron Collider* (LHC) vid partikelfysiklaboratoriumet CERN som ligger utanför staden Genève i Schweiz.

A Hamilton's Principle and Supersymmetry

A.1 The Basic Lagrangian

By basic Lagrangian we mean here the Lagrangian in the action (5.4), where the gauge field $A_\mu(x)$ is switched off, hence the absence of the isospin fields η:

$$L = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - \frac{1}{2} g_{\mu\nu} \psi^\mu \left(\frac{d\psi^\nu}{dt} + \dot{x}^\lambda \Gamma^\nu_{\lambda\kappa}(x) \psi^\kappa \right).$$

The expression inside the parentheses in the Lagrangian is the covariant derivative $(D^g_t)^\mu{}_\nu$ written out in components.

To show that that there are physically realizable quantities, i.e., classical equations of motion that describes a trajectory[10] in space, or observables in the case of quantum mechanics, the action must fulfill Hamilton's principle, also called the principle of least action:

$$\delta S = \delta \int dt L = 0.$$

This is realized if the variation of the Lagrangian, δL, with respect to the variables

$$x'^\mu = x^\mu + \delta x^\mu, \quad \text{and} \quad \psi'^\mu = \psi^\mu + \delta\psi^\mu,$$

are integrated out and is identical to zero:

$$\delta S = \int_{t'}^{t''} dt L(x^\mu + \delta x^\mu, \psi^\mu + \delta^\psi; \dot{x}^\mu + \delta \dot{x}^\mu, \dot{\psi}^\mu + \delta \dot{\psi}^\psi) - \int_{t'}^{t''} dt L(x^\mu, \psi^\mu; \dot{x}^\mu, \dot{\psi}^\mu) \equiv 0.$$

We introduce the supersymmetry transformations

$$\begin{aligned} \delta x^\mu &= \epsilon \psi^\mu, \\ \delta \psi^\mu &= -\epsilon \dot{x}^\mu, \end{aligned}$$

where ϵ is a Grassmann number, and we the following metric identity:

$$\partial_\lambda g_{\mu\nu} + \partial_\nu g_{\mu\lambda} + \partial_\mu g_{\lambda\nu} = 0.$$

We can now rewrite the Lagrangian, using the metric identity, as:

$$\begin{aligned} L =& \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \frac{1}{2} g_{\mu\nu} \psi^\mu \dot{\psi}^\nu + \frac{1}{2} \dot{x}^\mu g_{\lambda\rho} \Gamma^\rho_{\mu\nu} \psi^\lambda \psi^\nu \\ =& \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \frac{1}{2} g_{\mu\nu} \psi^\mu \dot{\psi}^\nu - \frac{1}{2} \dot{x}^\mu \frac{1}{2} (\partial_\lambda g_{\mu\nu} - \partial_\nu g_{\mu\lambda} - \partial_\mu g_{\lambda\nu}) \psi^\lambda \psi^\nu \\ =& \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \frac{1}{2} g_{\mu\nu} \psi^\mu \dot{\psi}^\nu - \frac{1}{2} \dot{x}^\mu \partial_\lambda g_{\mu\nu} \psi^\lambda \psi^\nu. \end{aligned}$$

[10] In quantum mechanics we cannot define trajectories in space-time due to Heisenberg's uncertainty principle. Classical mechanics is, however, restored in the limit $\hbar \to 0$ in the path integral.

In the first equality we renamed some indices in the third term, in order to relate the Christoffel symbol $\Gamma^{\rho}{}_{\mu\nu}$ to the metric identity, used in the second equality.

A straightforward computation of the variation of the Lagrangian gives

$$\begin{aligned}
\delta L =& \frac{1}{2}\partial_\lambda g_{\mu\nu}\delta x^\lambda \dot{x}^\mu \dot{x}^\nu + \frac{1}{2}g_{\mu\nu}\delta\dot{x}^\mu\dot{x}^\nu + \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\delta\dot{x}^\nu + \frac{1}{2}\partial_\lambda g_{\mu\nu}\delta x^\lambda\psi^\mu\dot{\psi}^\nu + \frac{1}{2}g_{\mu\nu}\delta\psi^\mu\dot{\psi}^\nu \\
&+ \frac{1}{2}g_{\mu\nu}\psi^\mu\delta\dot{\psi}^\nu - \frac{1}{2}\delta\dot{x}^\mu\partial_\lambda g_{\mu\nu}\psi^\lambda\psi^\nu - \frac{1}{2}\dot{x}^\mu\partial_\lambda\partial_\beta g_{\mu\nu}\delta x^\beta\psi^\lambda\psi^\nu - \frac{1}{2}\dot{x}^\mu\partial_\lambda g_{\mu\nu}\delta\psi^\lambda\psi^\nu \\
&- \frac{1}{2}\dot{x}^\mu\partial_\lambda g_{\mu\nu}\psi^\lambda\delta\psi^\nu \\
=& \frac{1}{2}\epsilon g_{\mu\nu}\psi^\mu\ddot{x}^\nu + \frac{1}{2}\epsilon g_{\mu\nu}\dot{x}^\mu\dot{\psi}^\nu - \frac{1}{2}\epsilon\partial_\lambda\partial_\beta g_{\mu\nu}\dot{x}^\mu\psi^\beta\psi^\lambda\psi^\nu + \frac{1}{2}\epsilon\partial_\lambda g_{\mu\nu}\dot{x}^\mu\dot{x}^\lambda\psi^\nu \\
=& \frac{1}{2}\epsilon\frac{d}{dt}(g_{\mu\nu}\dot{x}^\mu\psi^\nu) - \partial_\lambda\left[\frac{1}{2}\epsilon(\partial_\beta g_{\mu\nu}\dot{x}^\mu\psi^\nu\psi^\beta\psi^\lambda - g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\psi^\lambda)\right].
\end{aligned}$$

The supersymmetric transformations were substituted in the second equality. The second and fifth term cancel each other after the substitution, so do the first and the tenth term, likewise the fourth and the seventh term. Rewriting the first term in the second equality as a total time-derivative, the extra term from the rewriting cancels the second term (after renaming the indices ν to μ and vice versa). The term with the total derivative ∂_λ can be neglected in the action, since it does not contribute to the equations of motion. Hence the variation of the action is equal to

$$\delta S = \int\limits_{\text{PBCs}} dt\frac{d}{dt}\frac{1}{2}\epsilon(g_{\mu\nu}\dot{x}^\mu\psi^\nu) \equiv 0,$$

where the invariance of δS follows from the periodicity of the variables, e.g. $\psi(0) = \psi(\beta) = 0$.

A.2 The Gauge Field Lagrangian

Here we regard the action (5.4), now including the isospin fields η. Since we derived $\delta S = 0$ above without the gauge fields $A_\mu(x)$, we include only the terms dependent on η, hence the action

$$S_{\text{gauge}} = \int\limits_0^\beta dt\left[\bar{\eta}^a\dot{\eta}^a + \bar{\eta}^a\eta^a\dot{x}^\alpha A_\alpha(x) - \frac{1}{2}\bar{\eta}^a F^{ab}_{\mu\nu}\psi^\mu\psi^\nu\eta^b + i\frac{\alpha}{\beta}\bar{\eta}^a\eta^a\right].$$

The supersymmetry transformations are given by

$$\begin{aligned}
\delta x^\mu &= \epsilon\psi^\mu, \\
\delta\psi^\mu &= -\epsilon\dot{x}^\mu, \\
\delta\eta^a &= -\epsilon\psi^\mu A^{ab}_\mu\eta^b, \\
\delta\bar{\eta}^a &= -\epsilon\bar{\eta}^b A^{ba}_\mu\psi^\mu.
\end{aligned}$$

The Bianchi identity for the field strength $F^{ab}_{\mu\nu}$ is equal to

$$D_\mu F_{\mu\nu} = \partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] = 0.$$

For simplicity, we compute the variation of the field strength first, before the substitution in the variation of the Lagrangian further below:

$$\delta F^{ab}_{\mu\nu} = (\partial_\lambda \partial_\mu A^{ab}_\nu - \partial_\lambda \partial_\nu A^{ab}_\mu + [\partial_\lambda A_\mu, A_\nu]^{ab} + [A_\mu, \partial_\lambda A_\nu]^{ab})\delta x^\lambda.$$

Taking the variation of the Lagrangian L_{gauge}, defined by the terms between the square brackets of S_{gauge} above, and substituting the supersymmetry transformations yield:

$$\begin{aligned}
\delta L_{\text{gauge}} =& (-\epsilon\bar{\eta}^b A^{ab}_\mu \psi^\mu)\dot{\eta}^a + \bar{\eta}^a \frac{d}{dt}(-\epsilon\psi^\mu A^{ab}_\mu \eta^b) + (-\epsilon\bar{\eta}^b A^{ba}_\mu \psi^\mu)\dot{x}^\alpha A^{aa}_\alpha \eta^a \\
&+ \bar{\eta}^a(\epsilon\dot{\psi}^\mu)A^{aa}_\alpha \eta^a + \bar{\eta}^a \dot{x}^\alpha \partial_\lambda A^{aa}_\alpha (\epsilon\psi^\lambda)\eta^a + \bar{\eta}^a \dot{x}^\alpha A^{aa}_\alpha(-\epsilon\psi^\mu A^{ab}_\mu \eta^b) \\
&- \frac{1}{2}(-\epsilon\bar{\eta}^b A^{ba}_\kappa \psi^\kappa)F^{ab}_{\mu\nu}\psi^\mu\psi^\nu\eta^b \\
&- \frac{1}{2}\bar{\eta}^a(\partial_\lambda\partial_\mu A^{ab}_\nu - \partial_\lambda\partial_\nu A^{ab}_\nu + [\partial_\lambda A_\mu, A_\nu]^{ab} + [A_\mu, \partial_\lambda A_\nu]^{ab})(\epsilon\psi^\lambda)\psi^\mu\psi^\nu\eta^b \\
&- \frac{1}{2}\bar{\eta}^a F^{ab}_{\mu\nu}(-\epsilon\dot{x}^\mu)\psi^\nu\eta^b - \frac{1}{2}\bar{\eta}^a F^{ab}_{\mu\nu}\psi^\mu(-\epsilon\dot{x}^\nu)\eta^b - \frac{1}{2}\bar{\eta}^a F^{ab}_{\mu\nu}\psi^\mu\psi^\nu(-\epsilon\psi^\kappa A^{ba}_\kappa \eta^a) \\
&+ i\frac{\alpha}{\beta}(-\epsilon\bar{\eta}^b A^{ba}_\mu \psi^\mu)\eta^a + i\frac{\alpha}{\beta}\bar{\eta}^a(-\epsilon\psi^\mu A^{ab}_\mu \eta^b)
\end{aligned}$$

Notice the anti-commutativity between the Grassmann number ϵ and the isospin fields, e.g. $\epsilon\bar{\eta} = -\bar{\eta}\epsilon$. We show explicitly that all, but one, terms cancel each other.

The seventh, eighth and eleventh term vanish:

$$\begin{aligned}
&\frac{1}{2}\epsilon\bar{\eta}^b\psi^\kappa\psi^\mu\psi^\nu(A^{ba}_\kappa F^{ab}_{\mu\nu})\eta^b - \frac{1}{2}\epsilon\bar{\eta}^a\psi^\kappa\psi^\mu\psi^\nu(F^{ab}_{\mu\nu}A^{ba}_\kappa)\eta^a + \frac{1}{2}\epsilon\bar{\eta}^a\psi^\kappa\psi^\mu\psi^\nu(\partial_\kappa F^{ab}_{\mu\nu})\eta^a \\
&= \frac{1}{2}\epsilon\bar{\eta}^a\psi^\kappa\psi^\mu\psi^\nu(A^{ab}_\kappa F^{ba}_{\mu\nu} - F^{ab}_{\mu\nu}A^{ba}_\kappa + \partial_\kappa F^{ab}_{\mu\nu})\eta^a = \frac{1}{2}\epsilon\bar{\eta}^a\psi^\kappa\psi^\mu\psi^\nu(\partial_\kappa F^{ab}_{\mu\nu} + [A_\kappa, F_{\mu\nu}]^{ab})\eta^a = 0.
\end{aligned}$$

where we get the first equality after renaming η^b and $\bar{\eta}^b$ to η^a and $\bar{\eta}^a$, respectively. The third equality follows from the Bianchi identity by setting $\kappa = \mu$.

Continuing with the ninth and the tenth terms:

$$-\frac{1}{2}\bar{\eta}^a\epsilon(F^{ab}_{\mu\nu}\dot{x}^\mu\psi^\nu)\eta^b - \frac{1}{2}\bar{\eta}^a\epsilon(F^{ab}_{\mu\nu}\dot{x}^\nu\psi^\mu) = \frac{1}{2}\bar{\eta}^a\epsilon(-F^{ab}_{\mu\nu}\dot{x}^\mu\psi^\nu - F^{ab}_{\mu\nu}\dot{x}^\nu\psi^\mu)\eta^b = 0.$$

The last equality follows from renaming the indices ν to μ, and vice versa, in the last term between the parentheses in the first equality. Using the anti-symmetry of the field strength; $F_{\mu\nu} = F_{\nu\mu}$, gives the second equality.

Collecting the first, second and the fourth term gives:

$$\begin{aligned}
&(-\epsilon\bar{\eta}^b A^{ba}_\mu\psi^\mu)\dot{\eta}^a + \bar{\eta}^a\frac{d}{dt}(-\epsilon\psi^\mu A^{ab}_\mu\eta^b) + \bar{\eta}^a(\epsilon\dot{\psi}^\mu)A^{aa}_\alpha\eta^a \\
&= -\epsilon\bar{\eta}^b A^{ba}_\mu\psi^\mu\dot{\eta}^a + \epsilon\bar{\eta}^a\psi^\mu A^{ab}_\mu\dot{\eta}^b + \epsilon\bar{\eta}^a\dot{\psi}^\mu A^{ab}_\mu\eta^b - \epsilon\bar{\eta}^a\dot{\psi}^\mu A^{aa}_\alpha\eta^a \\
&= -\epsilon\bar{\eta}^b A^{ba}_\mu\psi^\mu\dot{\eta}^a + \epsilon\bar{\eta}^b A^{ba}_\mu\psi^\mu\dot{\eta}^a + \epsilon\bar{\eta}^a\dot{\psi}^\mu A^{ab}_\mu\eta^b - \epsilon\bar{\eta}^a\dot{\psi}^\mu A^{ab}_\mu\eta^b = 0.
\end{aligned}$$

In the second equality we renamed, in the second term, the indices a to b, and vice versa, while $A^{aa}\eta^a = A^{ab}\eta^b$ in the last term, hence the third equality.

The third and the sixth terms are

$$-\epsilon\bar{\eta}^b A_\mu^{ba}\psi^\mu \dot{x}^\alpha A_\alpha^{aa}\eta^a + \epsilon\bar{\eta}^a A_\mu^{ab}\psi^\mu \dot{x}^\alpha A_\alpha^{bb}\eta^b = -\epsilon\bar{\eta}^b A_\mu^{ba}\psi^\mu \dot{x}^\alpha A_\alpha^{aa}\eta^a + \epsilon\bar{\eta}^b A_\mu^{ba}\psi^\mu \dot{x}^\alpha A_\alpha^{aa}\eta^a = 0,$$

after renaming a to b, and vice versa, in the second term before the first equality.

Finally, the ultimate and penultimate terms vanish, using similar arguments as above:

$$-i\frac{\alpha}{\beta}\epsilon\bar{\eta}^b A_\mu^{ba}\psi^\mu\eta^a + i\frac{\alpha}{\beta}\epsilon\bar{\eta}^a A_\mu^{ab}\psi^\mu\eta^b = -i\frac{\alpha}{\beta}\epsilon\bar{\eta}^b A_\mu^{ba}\psi^\mu\eta^a + i\frac{\alpha}{\beta}\epsilon\bar{\eta}^b A_\mu^{ba}\psi^\mu\eta^a = 0.$$

Hence we arrive at one non-vanishing term, the fifth term,

$$\partial_\lambda(\bar{\eta}^a \dot{x}^\alpha A_\alpha^{aa}\epsilon\psi^\lambda\eta^a),$$

that does not contribute to the equations of motion, thus can be neglected in the action.

B Product Expansion of an Entire Function

In this appendix we derive the formula (5.6) used in the functional determinants in the evaluations of the path integrals that are substituted in the derivation of the index theorems. This review of the theory of complex variables is brief and heuristic, further information can be found in [2].

A complex function $g(z)$, $z = x + \mathrm{i}y$, may be constructed as

$$g(z) = u(x, y) + \mathrm{i}v(x, y),$$

for real functions $u(x, y)$ and $v(x, y)$. If $g(z)$ is differentiable at $z = z_0 \in \mathbb{C}$ and in a neighborhood of z_0, we say that $g(z)$ is *analytic*[11] at $z = z_0$. If $g(z)$ is analytic everywhere in the finite complex plane we call it an *entire function*. Examples of entire functions are; $\sin z$, $\cos z$ and $\exp z$. If a function $g(z) \sim (z - z_0)^{-m}$, $m \geq 1$, we say it has a pole, or a singularity, at $z = z_0$ with multiplicity m. A function that is analytic in the finite complex plane, except at isolated poles, is called *meromorphic*. Examples of meromorphic functions are $\tan z$, $\cot z$, and ratios of polynomials.

We now introduce the *Laurent series* of a function $g(z)$:

$$\begin{aligned} g(z) &= \sum_{n=-\infty}^{\infty} b_n (z - z_0)^n \\ &= \cdots + b_{-m} \frac{1}{(z - z_0)^m} + \cdots + b_{-1} \frac{1}{(z - z_0)} \\ &\quad + b_0 + b_1 (z - z_0) + \cdots + b_m (z - z_0)^m + \ldots , \end{aligned}$$

where the constants b_n are called the residues. Without going into further details, we can regard the Laurent series as a generalized Taylor expansion in the complex plane, where we also take the singularities into account.

A generalization of the Laurent series is called *pole expansion of a meromorphic function*. Instead of just one singularity (at $z - z_0$ in the Laurent series above) we now assume that there are several poles at $z = a_n$, with $0 < |a_1| < |a_2| < \ldots$, all having multiplicity equal to one and the series

$$g(z) = g(0) + \sum_{n=1}^{\infty} b_n \{(z - a_n)^{-1} + a_n^{-1}\}$$

converges to $g(z)$ (due to Mittag-Leffler Theorem). Now it is straight forward to show the *product expansion of an entire function*.

The logarithmic derivative of $g(z)$ is given by $d/dz \ln g(z) = g'/g$ and it is meromorphic with a pole expansion. If $g(z)$ has a simple singularity at $z = a_n$, we can get rid of that critical point after multiplication by $(z - a_n)$, then $g(z) = (z - a_n) f(z)$ with analytic $f(z)$ and $f(a_n) \neq 0$. The logarithmic derivative of $g(z)$ is equal to

$$\frac{g'(z)}{g(z)} = \frac{d}{dz} (\ln(z - a_n) + \ln f(z)) = \frac{1}{(z - a_n)} + \frac{f'(z)}{f(z)},$$

[11] The names holomorphic or regular are synonyms and can be found in the literature.

and has a simple pole at $z = a_n$ with residue 1 (the constant in front of $(z - a_n)^{-1}$). The term f'/f is analytic at z_0. The pole expansion of meromorphic functions of g'/g is given by

$$\frac{g'(z)}{g(z)} = \frac{g'(0)}{g(0)} + \sum_{n=1}^{\infty} \left[\frac{1}{z - a_n} + \frac{1}{a_n} \right].$$

Integrating the pole expansion of f'/f yields

$$\begin{aligned} \int_0^z dw \frac{d}{dw} \ln g(w) &= \ln g(z) - \ln g(0) \\ &= \frac{zg'(0)}{g(0)} + \sum_{n=1}^{\infty} \left\{ \ln(z - a_n) + \frac{z}{a_n} + A \right\}, \end{aligned}$$

where A is a constant of integration we choose as $A = -\ln(-a_n)$. Hence we get

$$\ln \frac{g(z)}{g(0)} = \frac{zg'(0)}{g(0)} + \sum_{n=1}^{\infty} \left\{ \ln \left(\frac{a_n - z}{a_n} \right) + \frac{z}{a_n} \right\},$$

and exponentiating yields the product expansion (5.6):

$$g(z) = g(0) \exp \left(\frac{zg'(0)}{g(0)} \right) \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n} \right) e^{z/a_n}.$$

Two standard examples of product expansions are the following functions:

$$\begin{aligned} \sin z &= z \prod_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left(1 - \frac{z}{n\pi} \right) e^{z/n\pi} = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2 \pi^2} \right), \\ \cos z &= \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{(n - 1/2)^2 \pi^2} \right). \end{aligned}$$

C Curvature Tensors

C.1 The Riemann Curvature Tensor

The curvature tensor $R^{\mu}{}_{\kappa\lambda\nu}$ is constructed by taking a vector V, tangent to the surface of a manifold M, and parallel transport it half way around an infinitesimal parallelogram in two opposite directions[10]. The difference in direction of the vector at the final point of the parallel transportation is a measure of the curvature of the manifold. A mnemonic on the construction of the Riemann curvature tensor and it geometrical meaning is shown in figure (7) below.

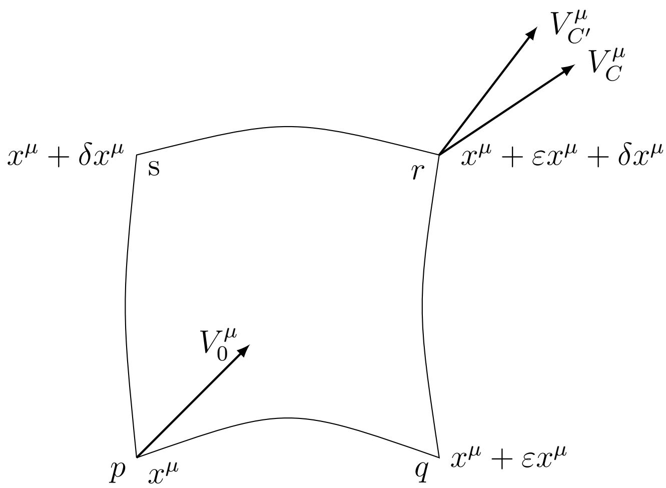

Figure 7: The Riemann curvature tensor $R^{\mu}{}_{\kappa\lambda\nu}$ is constructed by taking the parallel transport of the vector V_0 across two opposite paths; $C = pqr$ and $C' = psr$. The difference of $V^{\mu}_{C'}$ and V^{μ}_{C} at the corner r is equal to the curvature tensor.

Let $pqrs$ be a parallelogram, where p is the lower-left corner whose coordinate is x^{μ}. A vector V_0 at p parallel transported to the lower-right corner q, with coordinate $x^{\mu} + \varepsilon x^{\mu}$, ε an infinitesimal number, is given by

$$V^{\mu}_{C}(q) = V^{\mu}_{0} - V^{\kappa}_{0} \Gamma^{\mu}{}_{\nu\kappa}(p) \varepsilon x^{\nu}.$$

The infinitesimal translation from q to the upper-right corner r is equal to δx^{μ}, hence the coordinate of r is $x^{\mu} + \varepsilon x^{\mu} + \delta x^{\mu}$. The vector at the final translation point is, up to second order in ε and δ, thus equal to

$$\begin{aligned} V^{\mu}_{C}(r) &= V^{\mu}_{C}(q) - V^{\kappa}_{C}(q) \Gamma^{\mu}{}_{\nu\kappa}(q) \delta x^{\nu} \\ &= V^{\mu}_{0} - V^{\kappa}_{0} \Gamma^{\mu}{}_{\nu\kappa} \varepsilon x^{\nu} - \big[V^{\kappa}_{0} - V^{\rho}_{0} \Gamma^{\kappa}{}_{\zeta\rho}(p) \varepsilon x^{\zeta}\big]\big[\Gamma^{\mu}{}_{\nu\kappa}(p) + \partial_{\lambda} \Gamma^{\mu}{}_{\nu\kappa}(p) \varepsilon x^{\lambda}\big] \delta x^{\nu} \\ &\simeq V^{\mu}_{0} - V^{\kappa}_{0} \Gamma^{\mu}{}_{\nu\kappa} \varepsilon x^{\nu} - V^{\kappa}_{0} \Gamma^{\mu}{}_{\nu\kappa}(p) \delta x^{\nu} - V^{\kappa}_{0} \left[\partial_{\lambda} \Gamma^{\mu}{}_{\nu\kappa}(p) - \Gamma^{\rho}{}_{\lambda\kappa}(p) \Gamma^{\mu}{}_{\nu\rho}(p)\right] \varepsilon x^{\lambda} \delta x^{\nu}. \end{aligned}$$

The subscript C denotes the counter clockwise transportation through the corners pqr. Similarly, we let C' denote the clockwise translation through psr, hence the vector $V^{\mu}_{C'}(r)$ is given by

$$V^{\mu}_{C'}(r) \simeq V^{\mu}_{0} - V^{\kappa}_{0} \Gamma^{\mu}{}_{\nu\kappa} \delta x^{\nu} - V^{\kappa}_{0} \Gamma^{\mu}{}_{\nu\kappa}(p) \varepsilon x^{\nu} - V^{\kappa}_{0} \left[\partial_{\nu} \Gamma^{\mu}{}_{\lambda\kappa}(p) - \Gamma^{\rho}{}_{\nu\kappa}(p) \Gamma^{\mu}{}_{\lambda\rho}(p)\right] \varepsilon x^{\lambda} \delta x^{\nu}.$$

The difference of the two vectors $V^\mu_C(r)$ and $V^\mu_{C'}(r)$ is equal to

$$\begin{aligned}V^\mu_{C'}(r) - V^\mu_C(r) &= V^\kappa_0 \left[\partial_\lambda \Gamma^\mu{}_{\nu\kappa}(p) - \partial_\nu \Gamma^\mu{}_{\lambda\kappa}(p) - \Gamma^\rho{}_{\lambda\kappa}(p)\Gamma^\mu{}_{\nu\rho}(p) + \Gamma^\rho{}_{\nu\kappa}(p)\Gamma^\mu{}_{\lambda\rho}(p)\right]\varepsilon x^\lambda \delta x^\nu \\ &= V^\kappa_0 R^\mu{}_{\kappa\lambda\nu}\varepsilon x^\lambda \delta x^\nu.\end{aligned}$$

In summary, the Riemann curvature tensor is given by

$$R^\mu{}_{\kappa\lambda\nu}(p) = \partial_\lambda \Gamma^\mu{}_{\nu\kappa}(p) - \partial_\nu \Gamma^\mu{}_{\lambda\kappa}(p) - \Gamma^\rho{}_{\lambda\kappa}(p)\Gamma^\mu{}_{\nu\rho}(p) + \Gamma^\rho{}_{\nu\kappa}(p)\Gamma^\mu{}_{\lambda\rho}(p).$$

C.2 The Field Strength Tensor

We will be rather brief in our construction of the field strength tensor $F_{\mu\nu}$, and emphasize the similarities with the construction of the Riemann curvature tensor in the previous section. For a more comprehensive construction of the field strength tensor, see for instance [11].

The covariant derivative D_μ, in the direction n^μ, of a fermi field $\psi(x)$ is defined as

$$n^\mu D_\mu \psi = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left[\psi(x+\varepsilon n) - U(x+\varepsilon n, x)\psi(x)\right]$$

where the factor $U(x+\varepsilon n, x)$ is called the comparator. The field $\psi(x)$ transforms as $\psi(x) \to \exp(\mathrm{i}\alpha(x))\psi(x)$, i.e., a phase rotation through an angle $\alpha(x)$. On the other hand, the field $\psi(x+\varepsilon n)$ has a different transformation at the point $x+\varepsilon n$ than has $\psi(x)$ at x, hence the comparator $U(y,x)$ compensates for the phase difference under field transformations. Thus the comparator $U(y,x)$ transforms as

$$U(y,x) \to e^{\mathrm{i}\alpha(y)} U(y,x) e^{-\mathrm{i}\alpha(x)}.$$

We define $U(y,x)$ to be a pure phase; $U(y,x) := \exp[\mathrm{i}\phi(y,x)]$ for some function $\phi(y,x)$, and we set $U(y,y) = 1$. In conclusion, the objects of the forms $\psi(x)$ and $U(y,x)\psi(x)$ have the same transformation law thus the covariant derivative is well defined for a local phase transformation of the fields.

The expanded comparator $U(x+\varepsilon n, x)$ in the covariant derivative above is equal to

$$U(x+\varepsilon n, x) = 1 - \mathrm{i}e\varepsilon n^\mu A_\mu(x) + \mathcal{O}(\varepsilon^2),$$

where e is a constant[12] and $A_\mu(x)$ is the four potential. The covariant derivative is then given by

$$D_\mu\psi(x) = \partial_\mu\psi(x) + \mathrm{i}eA_\mu(x) + \mathcal{O}(\varepsilon^2) \tag{C.1}$$

To construct the field strength tensor, we expand $U(x+\varepsilon n, x)$ up to third order in the infinitesimal constant ε:

$$U(x+\varepsilon n, x) = \exp\left[-\mathrm{i}e\varepsilon n^\nu A_\mu(x+\frac{\varepsilon}{2}n) + \mathcal{O}(\varepsilon^3)\right] \tag{C.2}$$

The field strength analogue to the construction of the Riemann curvature tensor is now examined. We take comparisons around an infinitesimal parallelogram, where the initial

[12] In quantum electrodynamics the constant e is the electron charge, while in non-Abelian gauge theories the constant is a generic charge, normally denoted by the letter g.

and final point is in the lower-left corner whose coordinate is x. Let the parallelogram lie in the $(1,2)$-plane, i.e., the coordinates of the lower-right corner and the upper-right corner are equal to $x + \varepsilon\hat{1}$ and $x + \varepsilon\hat{1} + \varepsilon\hat{2}$, respectively. Here $\hat{1}$ and $\hat{2}$ are (horizontal, respectively vertical) unit vectors. The product of the four comparisons counterclockwise around the parallelogram is defined as $\mathbf{U}(x)$:

$$\mathbf{U}(x) \equiv U(x, x+\varepsilon\hat{2})U(x+\varepsilon\hat{2}, x+\varepsilon\hat{1}+\varepsilon\hat{2})U(x+\varepsilon\hat{1}+\varepsilon\hat{2}, x+\varepsilon\hat{1})U(x+\varepsilon\hat{1}, x).$$

Substituting equation (C.2) into $\mathbf{U}(x)$ yields

$$\begin{aligned}\mathbf{U}(x) = \exp\Big\{-\mathrm{i}\varepsilon e\Big[& -A_2(x+\frac{\varepsilon}{2}\hat{2}) - A_1(x+\frac{\varepsilon}{2}\hat{1}+\varepsilon\hat{2}) \\ & +A_2(x+\varepsilon\hat{1}+\frac{\varepsilon}{2}\hat{2}) + A_1(x+\frac{\varepsilon}{2}\hat{1})\Big] + \mathcal{O}(\varepsilon^3)\Big\},\end{aligned}$$

where $A_1(\bullet)$ ($A_2(\bullet)$) are the comparisons in the horizontal (vertical) directions. Expanding $U(x)$, and taking the limit $\varepsilon \to 0$, gives

$$\mathbf{U}(x) = 1 - \mathrm{i}\varepsilon^2 e\left[\partial_1 A_2(x) - \partial_1 A_2(x)\right] + \mathcal{O}(\varepsilon^3),$$

where $\partial_1 = \lim_{\varepsilon\to 0}(1/\varepsilon\hat{1})$ and $\partial_2 = \lim_{\varepsilon\to 0}(1/\varepsilon\hat{2})$. The structure between the square brackets in the final derivation of $\mathbf{U}(x)$ is of the form

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu,$$

which is equal to the electromagnetic field tensor, or the field strength tensor.

There is an alternative way of constructing the field strength tensor, namely by taking the commutator of the covariant derivative D_μ. The commutator $[D_\mu, D_\nu]$ can be interpreted as a comparison of comparisons across a small square, cf. the argument above. We derived the Abelian field strength tensor above which means that $[A_\mu, A_\nu] = 0$.

On the other hand, if we assume that $[A_\mu, A_\nu] \neq 0$ and consider a covariant derivative of the form

$$D_\mu = \partial_\mu - \mathrm{i}gA_\mu$$

where g is a constant, we get the commutator acting on a field ψ equal to

$$\begin{aligned}[D_\mu, D_\nu]\psi &= [\partial_\mu, \partial_\nu]\psi - \mathrm{i}g([\partial_\mu, A_\nu] + [A_\mu, \partial_\nu])\psi - ig[A_\mu, A_\nu]\psi \\ &= -\mathrm{i}g(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])\psi.\end{aligned}$$

Hence the non-Abelian field strength tensor is of the form

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

D Quantum Fluctuations and the Riemann Tensor

In this appendix we derive the Riemann curvature tensor, as given in the gauge choice (5.15b), from the quantum fluctuations around a critical point on a manifold. From the action (5.4) we have a term dependent on the Christoffel symbol $\Gamma^{\mu}{}_{\varrho\nu}$:

$$g_{\mu\nu}(x)\psi^{\mu}\dot{x}^{\varrho}\Gamma^{\mu}{}_{\varrho\nu}\psi^{\nu}. \tag{D.1}$$

We introduce quantum fluctuations $\delta x^{\mu} := \varepsilon x^{\mu}(t)$ and $\delta x\psi^{\mu} := \varepsilon\psi^{\mu}(t)$ around a critical point $(x_0^{\mu}, \psi_0^{\mu})$ on a manifold M, where ε is an infinitesimal number. Hence the scalar field and the fermion field are given by

$$\begin{aligned} x'^{\mu}(t) &= x_0^{\mu} + \varepsilon x^{\mu}(t), \\ \psi'^{\mu}(t) &= \psi_0^{\mu} + \varepsilon\psi^{\mu}(t). \end{aligned}$$

Substituting x'^{μ} and ψ'^{μ} into the quantities in (D.1) yields

$$\begin{aligned} g_{\mu\nu}(x_0^{\alpha} + \varepsilon x^{\alpha}(t)) &= g_{\mu\nu}(x_0^{\alpha}) + \partial_{\lambda}g_{\mu\nu}(x_0^{\alpha})\varepsilon x^{\lambda} = g_{\mu\nu}(x_0^{\alpha}) + (\Gamma^{\kappa}{}_{\lambda\mu}g_{\kappa\nu} + \Gamma^{\kappa}{}_{\lambda\nu}g_{\kappa\mu})\varepsilon x^{\lambda}; \\ \Gamma^{\mu}{}_{\varrho\nu}(x_0^{\alpha} + \varepsilon x^{\alpha}(t)) &= \Gamma^{\mu}{}_{\varrho\nu}(x_0^{\alpha}) + \partial_{\beta}\Gamma^{\mu}{}_{\varrho\nu}(x_0^{\alpha})\varepsilon x^{\beta}; \\ \dot{x}'^{\varrho} &= \varepsilon\dot{x}^{\varrho}. \end{aligned}$$

In Riemann normal coordinates we get the following simplifications:

$$\begin{aligned} g_{\mu\nu}(x_0^{\alpha}) &= \delta_{\mu\nu}; \\ \partial_{\lambda}g_{\mu\nu}(x_0^{\alpha}) &= \Gamma^{\kappa}{}_{\lambda\mu}(x_0^{\alpha}) = \Gamma^{\kappa}{}_{\lambda\nu}(x_0^{\alpha}) = \Gamma^{\mu}{}_{\varrho\nu}(x_0^{\alpha}) = 0; \\ \partial_{\beta}\Gamma^{\mu}{}_{\varrho\nu}(x_0^{\alpha}) &\neq 0. \end{aligned}$$

Substituting $g_{\mu\nu}(x_0^{\alpha} + \varepsilon x^{\alpha}(t))$, $\dot{x}'^{\varrho}$ and $\Gamma^{\mu}{}_{\varrho\nu}(x_0^{\alpha} + \varepsilon x^{\alpha}(t))$ together with x'^{μ} and ψ'^{μ} into (D.1) and keeping only terms of second order in quantum fluctuations yields

$$\begin{aligned} &[\delta_{\mu\nu}][\psi_0^{\mu} + \varepsilon\psi^{\mu}(t)][\varepsilon\dot{x}^{\varrho}][\partial_{\beta}\Gamma^{\mu}{}_{\varrho\nu}\varepsilon x^{\beta}][\psi_0^{\nu} + \varepsilon\psi^{\nu}(t)] + \mathcal{O}(\varepsilon^3) \\ =&\delta_{\mu\nu}\psi_0^{\mu}\psi_0^{\mu}\varepsilon\dot{x}^{\varrho}[\partial_{\mu}\Gamma^{\beta}{}_{\varrho\nu}\varepsilon x^{\beta}] + \mathcal{O}(\varepsilon^3) \\ =&\delta_{\mu\nu}\psi_0^{\mu}\psi_0^{\mu}\varepsilon\dot{x}^{\varrho}\left[\frac{1}{2}(\partial_{\mu}\Gamma^{\beta}{}_{\varrho\nu} - \partial_{\nu}\Gamma^{\beta}{}_{\varrho\mu})\right]\varepsilon x^{\beta} + \mathcal{O}(\varepsilon^3) \\ =&\delta_{\mu\nu}\psi_0^{\mu}\psi_0^{\mu}\varepsilon\dot{x}^{\varrho}\frac{1}{2}R^{\beta}{}_{\varrho\mu\nu}\varepsilon x^{\beta} + \mathcal{O}(\varepsilon^3) \end{aligned}$$

where in the first equality we renamed the indices μ to β, and vice versa, in the term $\partial_{\mu}\Gamma^{\beta}{}_{\varrho\nu}$. In the second equality we used the identity $\psi^{\mu}\psi^{\nu}\omega_{\mu\nu} = \psi^{\mu}\psi^{\nu}\frac{1}{2}(\omega_{\mu\nu} - \omega_{\nu\mu})$ for a generic tensor $\omega_{\mu\nu} = -\omega_{\nu\mu}$ associated to $\partial_{\mu}\Gamma^{\beta}{}_{\varrho\nu}$.

Going back to the variables $x^{\mu}(t)$ and $\psi^{\mu}(t)$ in the last equality, we arrive (after renaming, and lowering, the index β) at the gauge choice (5.15b):

$$\psi_{\mu}\Gamma^{\nu}{}_{\nu\varrho}\psi^{\nu} = \frac{1}{2}R_{\alpha\varrho\mu\nu}\psi^{\mu}\psi^{\nu}x^{\alpha}.$$

A final remark on the Riemann normal coordinates is in order here. We have assumed that the geometry is locally Euclidean, i.e., flat around some point (x_0^μ, ψ_0^μ) on the manifold M. This local property does not imply that the curvature tensor $R_{\alpha\varrho\mu\nu}$ vanishes. We can visualize the Euclidean geometry on M as an arbitrary small tangent plane on M. On the tangent plane there is no connection, thus $\Gamma^\kappa{}_{\lambda\mu} = \Gamma^\kappa{}_{\lambda\nu} = \Gamma^\mu{}_{\varrho\nu} = 0$. However, we can still move the tangent plane to an arbitrary point (x^μ, ψ^μ) on the manifold, and thereby changing the direction of the normal of the tangent plane, hence the non-vanishing $R^\beta{}_{\varrho\mu\nu} = \partial_\mu \Gamma^\beta{}_{\varrho\nu} - \partial_\nu \Gamma^\beta{}_{\varrho\mu}$. The Riemann curvature tensor is coordinate independent, thus in a general coordinate system we have the tensor

$$R^\beta{}_{\varrho\mu\nu}(x) = \partial_\mu \Gamma^\beta{}_{\varrho\nu}(x) - \partial_\nu \Gamma^\beta{}_{\varrho\mu}(x) - \Gamma^\rho{}_{\varrho\mu}(x)\Gamma^\beta{}_{\rho\nu}(x) + \Gamma^\rho{}_{\varrho\nu}(x)\Gamma^\beta{}_{\rho\mu}(x).$$

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