UPPSALA UNIVERSITET

UPTEC F 13033
Examensarbete 30 hp
5 juni 2013

(Conformal) Supersymmetric sigma models in low dimensions

Thomas Halvarsson

UPPSALA UNIVERSITET

Teknisk- naturvetenskaplig fakultet
UTH-enheten

Besöksadress:
Ångströmlaboratoriet
Lägerhyddsvägen 1
Hus 4, Plan 0

Postadress:
Box 536
751 21 Uppsala

Telefon:
018 – 471 30 03

Telefax:
018 – 471 30 00

Hemsida:
http://www.teknat.uu.se/student

Abstract

(Conformal) Supersymmetric sigma models in low dimensions

Thomas Halvarsson

The geometry of non-conformal supersymmetric non-linear sigma models in one and two dimensions are reviewed. Transformations of the Osp(1|2) subgroup of the superconformal group are derived and then used in finding geometrical constraints on the target space of an N=(1,1) sigma model reduced to an N=1 sigma model.

Handledare: Ulf Lindström
Ämnesgranskare: Maxim Zabzine
Examinator: Tomas Nyberg
ISSN: 1401-5757, UPTEC F13 033

(Conformal) Supersymmetric sigma models in low dimensions

Thomas Halvarsson

June 3, 2013

UPPSALA
UNIVERSITET

Master thesis
Supervisor: Ulf Lindström
Department of Physics and Astronomy
Division of Theoretical Physics
Uppsala University

Abstract

The geometry of non-conformal supersymmetric non-linear sigma models in one and two dimensions are reviewed. Transformations of the $Osp(1|2)$ subgroup of the superconformal group are derived and then used in finding geometrical constraints on the target space of an $N = (1,1)$ sigma model reduced to an $N = 1$ sigma model.

Contents

1 Introduction

Supersymmetry is a proposed symmetry between bosons and fermions. It asserts that every boson and every fermion have their fermionic respectively bosonic so-called superpartner which in every aspect resemble the original particles except that their respective spins differ by one half. Since also their masses should equal, superparticles would have long been discovered, but since this is not the case the symmetry needs to be broken somehow, would it still be a symmetry of nature. Unbroken supersymmetry, being the case of massless particles, is still an interesting field of research.

Non-linear sigma models were first introduced by Gellman and Lévy in 1960 to describe spinless mesons called σ-mesons. However, in today's language non-linear sigma models are understood to be a set of maps from a parameter space, or worldsheet, to a manifold, which we will call target space. Introducing D bosonic fields as maps, they can be seen as coordinates on the D-dimensional target manifold thus fixing its geometry. Different configurations of parameter space will result in different target space geometries.

This master thesis is written as a review article covering some of the more important stuff needed for doing research in this vast field. At the end I will even touch on research by analyzing the constraints needed for a dimensionally reduced sigma model to be superconformally invariant. In writing this thesis and deciding which calculations to include, I have always had in mind the nearly uninitiated student that I was when this project started. Therefore some lengthy derivations, which for some may seem trivial, are included under the motto better one too many, while other, due to their length, have been omitted nevertheless.

In sections 2, 3, 4 and 5 the geometrical background needed is treated. Section 2 comprises an introduction to real geometry in the language of manifolds. This is extended to complex geometry in section 3, which also contains a short compilation of the most important complex geometries. In section 4 generalized complex geometry is introduced, which have been shown to contain symplectic and complex geometry as special cases, thus being more general than both of them seperatly. Finally, in section 5 Minkowski space, treated as a quotiant space of the Poincaré group and the Lorentz group, is parametrized, and we take a look at how transformations work.

Section 6, 7, 8 and 9 deal with supersymmetric bosonic non-linear sigma models and how they transform under the super-Poincaré group, not including conformal transformations i.e.. Bosonic non-linerar sigma models in one and two dimensions are described in section 6. In section 7 supersymmetry is introduced and a few supersymmetric sigma models are analyzed in section 8. Section 9 deals with the geometrical constraints on target space implicated by extending and reducing the number of supersymmetries and dimensions of the sigma model.

The final sections 10, 11, 12 and 13 are dedicated to conformal theory. In section 10 non-supersymmetric conformal theory is introduced for different numbers of dimensions, and then in section 11 extended to the supersymmetric cases. In section 11 we also explicitly show how to construct superconformal transformations in one dimension. Finally, in section 12 we use this machinery on one of the reduced models in section 9, thus showing the geometrical constraints needed for the original non-reduced sigma model to be dimensionally reducable to a superconformal sigma model. In section 13 these results are discussed and some paths for further investigation are proposed.

In the appendices notations and other conventions are collected (appendix A), togehter with some lengthier derivations and calculations (appendix B), and an explicit reduction of the sigma model used in section 12 (appendix C). Finally, appendix D comprises an introduction to the main idea of supersymmetry in terms of ordinary bosonic and fermionic fields, i.e., without the use of superfields.

2 Real geometry

2.1 Manifolds

Geometry is best described in the language of manifolds. In short a manifold is a topological space which is homeomorphic[1] to $\mathbb{R}^m$ locally but not necessarily globally. This means that on every sufficiently small part U_i of our manifold we can draw a coordinate system with the help of a coordinate function φ_i, and that there exists an infinitely differentiable map ψ_{ij} between the coordinate functions of two overlapping subsets U_i and U_j of the manifold. The more formal definition reads: M is an m-dimensional differentiable manifold if

1. M is a topological space
2. There exists a family of pairs $\{(U_i, \varphi_i)\}$, called charts, on M such that the family of open sets $\{U_i\}$ covers M and for each U_i there is a homeomorphism $\varphi_i : U_i \to U_i' \in \mathbb{R}^m$
3. The map $\psi_{ij} = \varphi_i \circ \varphi_j^{-1}$ from $\varphi_j(U_i \cap U_j \neq \emptyset)$ to $\varphi_i(U_i \cap U_j)$ is infinitely differentiable

Next we introduce a differentiable map f between an m-dimensional manifold M and an n-dimensional manifold N. Taking a chart (U, φ) on M and a chart (V, ψ), f can be presented in coordinates by

$$\psi \circ f \circ \varphi^{-1} : \mathbb{R}^m \to \mathbb{R}^n. \tag{1}$$

If f is a homeomorphism and $x = \psi \circ f \circ \varphi^{-1}$ is invertible and both x and its inverse are C^∞, then f is called a diffeomorphism, and M and N are said to be diffeomorphic to each other.
If f maps from a manifold to the real numbers $\mathbb{R}$, f is called a function, and we have the coordinate presentation

$$f \circ \varphi^{-1} : \mathbb{R}^m \to \mathbb{R}. \tag{2}$$

We also define a curve on a manifold as a map from an open interval (a, b) to the manifold. We can then introduce vectors on M as tangent vectors to the curve, the set of which at point p defines the tangent space T_pM. An arbitrary vector is written $X = X^\mu \frac{\partial}{\partial x^\mu}$, where $\{e_\mu\} = \{\frac{\partial}{\partial x^\mu}\}$ are the basis vectors of T_pM. Dual vectors, or one-forms as they are also called, are defined on the cotangent space at p, denoted T_p^*M, and are written $\omega = \omega_\mu dx^\mu$, where $\{dx^\mu\}$ constitutes

[1] $f : X_1 \to X_2$ is said to be homeomorphic if it is continuous and has an inverse $f^{-1} : X_2 \to X_1$

the basis in T_p^*M. Note that T_pM and T_p^*M have the same dimension as the manifold. The inner product between a one-form and a vector is defined by

$$\langle \omega, X\rangle = \omega_\mu X^\nu \langle dx^\mu, \frac{\partial}{\partial x^\nu}\rangle = \omega_\mu X^\nu \delta_\nu^\mu = \omega_\mu X^\mu. \tag{3}$$

We generalize vectors and one-forms to objects with arbitrary number of upper and lower indices: a tensor of type (q, r) is an object that maps q elements of T_p^*M and r elements of T_pM to a real number, and is written

$$T = T^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_r} \frac{\partial}{\partial x^{\mu_1}} \cdots \frac{\partial}{\partial x^{\mu_q}} dx^{\nu_1} \dots dx^{\nu_r}. \tag{4}$$

Next we define the exterior derivative. The action of the exterior derivative d_r on an r-form

$$\omega = \frac{1}{r!}\omega_{\mu_1 \dots \mu_r} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r} \tag{5}$$

is defined by

$$d_r\omega = \frac{1}{r!}\Big(\frac{\partial}{\partial x^\nu}\omega_{\mu_1 \dots \mu_r}\Big) dx^\nu \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r}. \tag{6}$$

Usually the subscript r is dropped and the exterior derivative is thus written d. We examplify this with the (antisymmetric) two-form $\omega = \frac{1}{2}\omega_{\mu\nu} dx^\mu \wedge dx^\nu$:

$$\begin{aligned}
d\omega &= \frac{1}{2}(\omega_{\mu\nu,\rho}) dx^\rho \wedge dx^\mu \wedge dx^\mu \\
&= \frac{1}{2}\frac{1}{3!}(\omega_{\mu\nu,\rho} dx^\rho \wedge dx^\mu \wedge dx^\nu + \omega_{\mu\rho,\nu} dx^\nu \wedge dx^\mu \wedge dx^\rho + \omega_{\rho\mu,\nu} dx^\nu \wedge dx^\rho \wedge dx^\mu \\
&\quad + \omega_{\rho\nu,\mu} dx^\mu \wedge dx^\rho \wedge dx^\nu + \omega_{\nu\rho,\mu} dx^\mu \wedge dx^\nu \wedge dx^\rho + \omega_{\nu\mu,\rho} dx^\rho \wedge dx^\nu \wedge dx^\mu) \\
&= \frac{1}{2}\frac{1}{3!}(\omega_{\mu\nu,\rho} - \omega_{\mu\rho,\nu} + \omega_{\rho\mu,\nu} - \omega_{\rho\nu,\mu} + \omega_{\nu\rho,\mu} - \omega_{\nu\mu,\rho}) dx^\rho \wedge dx^\mu \wedge dx^\nu \\
&= \frac{1}{2}\frac{1}{3!}(\omega_{\mu\nu,\rho} + \omega_{\rho\mu,\nu} + \omega_{\rho\mu,\nu} + \omega_{\nu\rho,\mu} + \omega_{\nu\rho,\mu} + \omega_{\mu\nu,\rho}) dx^\rho \wedge dx^\mu \wedge dx^\nu \\
&= \frac{1}{3!}(\omega_{\mu\nu,\rho} + \omega_{\rho\mu,\nu} + \omega_{\nu\rho,\mu}) dx^\rho \wedge dx^\mu \wedge dx^\nu.
\end{aligned} \tag{7}$$

Comparing with a three-form

$$H = H_{\mu\nu\rho} dx^\mu dx^\nu dx^\rho = \frac{1}{3!} H_{\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho = \frac{1}{3!} H_{\mu\nu\rho} dx^\rho \wedge dx^\mu \wedge dx^\nu, \tag{8}$$

we see that $H = d\omega$ can be expressed as

$$H_{\mu\nu\rho} = \omega_{\mu\nu,\rho} + \omega_{\rho\mu,\nu} + \omega_{\nu\rho,\mu}. \tag{9}$$

A form ω that can be written as the exterior derivative of another form (such as H in our example) is called exact. If $d\omega = 0$, ω is called closed.

2.2 Riemannian manifolds

We define a Riemannian metric g as a type $(0,2)$ tensor field on M satisfying at each point

1. $g_p(U,V) = g_p(V,U)$
2. $g_p(U,U) \geq 0$, where equality holds only for $U = 0$

where $U, V \in T_pM$. A pseudo-Riemannian metric also satisfies the first relation but the second is now

2'. if $g_p(U,V) = 0$ for any U, then $V = 0$.

If a differentiable manifold M admits a (pseudo-)Riemannian metric g, the pair (M,g) is said to be a (pseudo-)Riemannian manifold. With the help of the metric we can define the inner product between two vectors instead of between a vector and a one-form. $g_p(U, \quad)$ is simply associated with a one-form ω_U and we get $\langle \omega_U, V\rangle = g_p(U,V)$.

2.3 The Lie derivative, the covariant derivative, torsion and curvature

The Lie derivative $\mathcal{L}_X Y$ of a vector field $Y = Y^\mu \frac{\partial}{\partial x^\mu}$ along the flow of a vector field $X = X^\mu \frac{\partial}{\partial x^\mu}$ tells us how Y changes along the flow of X, the flow being defined as a curve whose tangent in every point is parallel to the vector field. We have

$$\mathcal{L}_X Y = (X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu) e_\nu = [X,Y]. \tag{10}$$

The Lie derivative can act on an arbitrary tensor $A^{\mu_1 \cdots \mu_n}_{\nu_1 \ldots \nu_k}$ in the following way [21]

$$\begin{aligned}(\mathcal{L}_X A)^{\mu_1 \cdots \mu_n}_{\nu_1 \ldots \nu_k} =& X^\rho A^{\mu_1 \cdots \mu_n}_{\nu_1 \ldots \nu_k, \rho} + X^\rho{}_{,\nu_1} A^{\mu_1 \cdots \mu_n}_{\rho \nu_2 \ldots \nu_k} + \cdots + X^\rho{}_{,\nu_k} A^{\mu_1 \cdots \mu_n}_{\nu_1 \ldots \nu_{k-1}\rho} \\ & - X^{\mu_1}{}_{,\rho} A^{\rho \mu_2 \cdots \mu_n}_{\nu_1 \ldots \nu_k} - \cdots - X^{\mu_1}{}_{,\rho} A^{\mu_1 \cdots \mu_{n-1}\rho}_{\nu_1 \ldots \nu_k}.\end{aligned} \tag{11}$$

We exemplify this by the Lie derivative of $H^{\mu\nu}{}_\rho$:

$$(\mathcal{L}_X H)^{\mu\nu}{}_\rho = X^\kappa H^{\mu\nu}{}_{\rho,\kappa} - X^\mu{}_{,\kappa} H^{\kappa\nu}{}_\rho - X^\nu{}_{,\kappa} H^{\mu\kappa}{}_\rho + X^\kappa{}_{,\rho} H^{\mu\nu}{}_\kappa. \tag{12}$$

Noting that $\mathcal{L}_X Y$ also depends on the derivative of X, we introduce the covariant derivative ∇_X as a generalization of directional derivatives from functions to tensors. For $X = X^\mu e_\mu$ and $Y = Y^\nu e_\nu$ we have

$$\nabla_X Y = X^\mu \Big(\frac{\partial Y^\lambda}{\partial x^\mu} + Y^\nu \Gamma^\lambda{}_{\mu\nu} \Big) e_\lambda, \tag{13}$$

where the connection coefficients $\Gamma^\lambda{}_{\mu\nu}$ are defined by

$$\nabla_\mu e_\nu = e_\lambda \Gamma^\lambda{}_{\mu\nu}. \tag{14}$$

The covariant derivative describes the change of a vector Y in the direction of the vector X. The terms in the parenthesis of (13) is written

$$\nabla_\mu Y^\lambda := \frac{\partial Y^\lambda}{\partial x^\mu} + \Gamma^\lambda{}_{\mu\nu} Y^\nu. \tag{15}$$

We can generalize the covariant derivative to arbitrary tensors by

$$\begin{aligned}\nabla_\nu t^{\lambda_1\ldots\lambda_p}_{\mu_1\ldots\mu_q} =& \partial_\nu t^{\lambda_1\ldots\lambda_p}_{\mu_1\ldots\mu_q} \\ &+ \Gamma^{\lambda_1}{}_{\nu\kappa} t^{\kappa\lambda_2\ldots\lambda_p}_{\mu_1\ldots\mu_q} + \cdots + \Gamma^{\lambda_p}{}_{\nu\kappa} t^{\lambda_1\ldots\lambda_{p-1}\kappa}_{\mu_1\ldots\mu_q} \\ &- \Gamma^\kappa{}_{\nu\mu_1} t^{\lambda_1\ldots\lambda_p}_{\kappa\mu_2\ldots\mu_q} - \cdots - \Gamma^\kappa{}_{\nu\mu_q} t^{\lambda_1\ldots\lambda_p}_{\mu_1\ldots\mu_{q-1}\kappa}.\end{aligned} \tag{16}$$

We are now ready to define the torsion tensor

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]. \tag{17}$$

In components of the basis $\{e_\mu\}$ and dual basis $\{e^\mu\} = \{dx^\mu\}$ we get

$$\begin{aligned} T &= T^\lambda{}_{\mu\nu} e_\lambda e^\mu e^\nu \\ &= e^\mu \frac{\partial e^\lambda}{\partial e^\mu} e_\lambda + e^\mu e^\nu \Gamma^\lambda{}_{\mu\nu} e_\lambda - e^\mu \frac{\partial e^\lambda}{\partial e^\mu} e_\lambda - e^\nu e^\mu \Gamma^\lambda{}_{\nu\mu} e_\lambda - e^\mu \frac{\partial e^\nu}{\partial e^\mu} e_\nu + e^\mu \frac{\partial e^\nu}{\partial e^\mu} e_\nu \\ &= e^\mu e^\nu (\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}) e_\lambda, \end{aligned} \tag{18}$$

i.e.,

$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}. \tag{19}$$

We call a torsion-less connection $\Gamma^{(0)\lambda}{}_{\mu\nu}$ a Levi-Civita connection. In terms of the metric it is written

$$\Gamma^{(0)\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa} (g_{\mu\kappa,\nu} + g_{\nu\kappa,\mu} - g_{\mu\nu,\kappa}). \tag{20}$$

From (19) we see that the Levi-Civita connection is symmetric in its lower indices. We are now able to decompose the general connection into a torsionless and a torsionfull part

$$\Gamma^\lambda{}_{\mu\nu} = \Gamma^{(0)\lambda}{}_{\mu\nu} + \frac{1}{2} T^\lambda{}_{\mu\nu}. \tag{21}$$

This can be generalized to

$$\Gamma^{(\pm)\lambda}{}_{\mu\nu} = \Gamma^{(0)\lambda}{}_{\mu\nu} \pm \frac{1}{2} T^\lambda{}_{\mu\nu}, \tag{22}$$

or

$$\Gamma^{(\pm)} = \Gamma^{(0)} \pm \frac{1}{2} g^{-1} T, \tag{23}$$

with a covariant derivative $\nabla^{(\pm)}_\mu$. One source of torsion may be an antisymmetric tensor $B_{\mu\nu}$ connected to the ordinary metric $g_{\mu\nu}$ by

$$E_{\mu\nu} = g_{\mu\nu} + B_{\mu\nu}, \tag{24}$$

i.e.,

$$g_{\mu\nu} = \frac{1}{2}E_{(\mu\nu)} = \frac{1}{2}(E_{\mu\nu} + E_{\nu\mu}), \qquad B_{\mu\nu} = \frac{1}{2}E_{[\mu\nu]} = \frac{1}{2}(E_{\mu\nu} - E_{\nu\mu}), \tag{25}$$

where we implicitly have made clear our definition of symmetrization and antisymmetrization of indices. Torsion can then be interpreted as the exterior derivative of the B-field, $T = dB$, or in components

$$T_{\kappa\mu\nu} = (dB)_{\kappa\mu\nu} = B_{\kappa\mu,\nu} + B_{\mu\nu,\kappa} + B_{\nu\kappa,\mu} \tag{26}$$

The Riemann curvature tensor is defined

$$R(X,Y,Z) = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \tag{27}$$

which in our coordinates becomes

$$R^{\kappa}{}_{\lambda\mu\nu} = \partial_\mu\Gamma^{\kappa}{}_{\nu\lambda} - \partial_\nu\Gamma^{\kappa}{}_{\mu\lambda} + \Gamma^{\eta}{}_{\nu\lambda}\Gamma^{\kappa}{}_{\mu\eta} - \Gamma^{\eta}{}_{\mu\lambda}\Gamma^{\kappa}{}_{\nu\eta}. \tag{28}$$

Contracting the indices we get the Ricci tensor

$$R_{\mu\nu} := R^{\lambda}{}_{\mu\lambda\nu}, \tag{29}$$

and the scalar curvature

$$R := g^{\mu\nu}R_{\mu\nu}. \tag{30}$$

These definitions generalize in the obvious way under $\Gamma^{\kappa}{}_{\mu\nu} \to \Gamma^{(\pm)\kappa}{}_{\mu\nu}$ to

$$R^{\kappa}{}_{\lambda\mu\nu} \to R^{(\pm)\kappa}{}_{\lambda\mu\nu} \qquad R_{\mu\nu} \to R^{(\pm)}{}_{\mu\nu}. \tag{31}$$

2.4 Killing vector fields

We close this section with a short introduction to Killing vector fields. These are fields along which the metric g is constant, i.e., a vector field X is a Killing vector field if

$$\mathcal{L}_X g = 0. \tag{32}$$

Following a more detailed approach we first define an isomorphism. A diffeomorphism $f : M \to M$ on a (pseudo-)Riemannian manifold (M, g) is said to be an isomorphism if

$$\frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu}g_{\alpha\beta}(f(p)) = g_{\mu\nu}(p), \tag{33}$$

where x and y are the coordinates of p and $f(p)$ respectively. If $f : x^\mu \mapsto x^\mu + \epsilon X^\mu$ we then have

$$\frac{\partial(x^\alpha + \epsilon X^\alpha)}{\partial x^\mu}\frac{\partial(x^\beta + \epsilon X^\beta)}{\partial x^\nu}g_{\alpha\beta}(x + \epsilon X) = g_{\mu\nu}(x). \tag{34}$$

This gives us the Killing equation

$$X^\xi\partial_\xi g_{\mu\nu} + \partial_\mu X^\xi g_{\xi\nu} + \partial_\nu X^\xi g_{\mu\xi} = (\mathcal{L}_X g_{\mu\nu}) = 0. \tag{35}$$

A vector field X satisfying this equation is said to be a Killing vector field. Geometrically this means that the inner product between two vectors is constant along a Killing vector field.
We can generalize the Killing vector field by

$$\mathcal{L}_X g = c_X g, \tag{36}$$

where $c_X \in \mathbb{C}$. X is now called a homothetic Killing vector field [9].

3 Complex geometry

3.1 Complex manifolds

Complex manifolds are similarly defined as the real manifolds. To this end we introduce a complex valued function $f : \mathbb{C}^m \to \mathbb{C}$ and say that it is holomorphic if $f = f_1 + if_2$ satisfies the Cauchy-Riemann relations for each $z^\mu = x^\mu + iy^\mu$:

$$\frac{\partial f_1}{\partial x^\mu} = \frac{\partial f_2}{\partial y^\mu}, \qquad \frac{\partial f_2}{\partial x^\mu} = -\frac{\partial f_1}{\partial y^\mu} \tag{37}$$

Similarly a map $(f^1, \ldots, f^n) : \mathbb{C}^m \to \mathbb{C}^n$ is holomorphic if each function f^λ $\lambda = 1, \ldots, n$ is holomorphic. M is then said to be a complex manifold if

1. M is a topological space
2. There exists a family of pairs $\{(U_i, \varphi_i)\}$, called a chart, on M such that the family of open sets $\{U_i\}$ covers M and for each U_i there is a homeomorphism $\varphi_i : U_i \to U_i' \in \mathbb{C}^m$
3. The map $\psi_{ij} = \varphi_i \circ \varphi_j^{-1}$ from $\varphi_j(U_i \cap U_j \neq \emptyset)$ to $\varphi_i(U_i \cap U_j)$ is holomorphic

We note that the complex dimension, $dim_{\mathbb{C}} M = m$, is half the real dimension, $dim_{\mathbb{R}} M = 2m$. Therefore the tangent space $T_p M$ is spanned by $2m$ vectors

$$\Big\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}; \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^m}\Big\}, \tag{38}$$

and the cotangent space $T_p^* M$ by

$$\Big\{dx^1, \ldots, dx^m; dy^1, \ldots, dy^m\Big\}. \tag{39}$$

A linear map $J_p : T_p M \to T_p M$ can be defined by

$$J_p\Big(\frac{\partial}{\partial x^\mu}\Big) = \frac{\partial}{\partial y^\mu}, \qquad J_p\Big(\frac{\partial}{\partial y^\mu}\Big) = -\frac{\partial}{\partial x^\mu}, \tag{40}$$

which means that

$$J_p^2 = -id_{T_p M}, \tag{41}$$

where id is the identity map[2]. This defines an almost complex structure. Roughly speaking we can see it like this: an m-dimensional complex manifold M with vectors $Z = X + iY$ is a $2m$-dimensional real manifold with an almost complex structure J, telling us how to relate the m-dimensional real vector fields X and Y. We see that in the base (38) J_p takes the form

$$J_p = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} \tag{42}$$

since

$$\Big(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial y^\mu}\Big) \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} = \Big(\frac{\partial}{\partial y^\mu}, -\frac{\partial}{\partial x^\mu}\Big), \tag{43}$$

where I_m is the $m \times m$ unit matrix. We define new vectors

$$\frac{\partial}{\partial z^\mu} := \frac{1}{2}\Big(\frac{\partial}{\partial x^\mu} - i\frac{\partial}{\partial y^\mu}\Big) \tag{44}$$

$$\frac{\partial}{\partial \bar{z}^\mu} := \frac{1}{2}\Big(\frac{\partial}{\partial x^\mu} + i\frac{\partial}{\partial y^\mu}\Big), \tag{45}$$

and corresponding one-forms

$$dz^\mu := dx^\mu + i dy^\mu, \qquad d\bar{z}^\mu := dx^\mu - i dy^\mu. \tag{46}$$

These vectors and one-forms span the $2m$-dimensional complex vector space $T_pM^{\mathbb{C}}$ and its dual space $T_p^*M^{\mathbb{C}}$ respectively. Now, extending the definition of the almost complex structure to $T_pM^{\mathbb{C}}$, we find

$$J_p\frac{\partial}{\partial z^\mu} = i\frac{\partial}{\partial z^\mu}, \qquad J_p\frac{\partial}{\partial \bar{z}^\mu} = -i\frac{\partial}{\partial \bar{z}^\mu}. \tag{47}$$

This gives in these coordinates

$$J_p = \begin{pmatrix} iI_m & 0 \\ 0 & -iI_m \end{pmatrix}, \tag{48}$$

and we see that the complex manifold can be seperated into two disjoint vector spaces:

$$T_pM^{\mathbb{C}} = T_pM^+ \oplus T_pM^-, \tag{49}$$

with

$$T_pM^\pm = \{Z \in T_pM^{\mathbb{C}} | J_pZ = \pm iZ\}. \tag{50}$$

$Z = Z^\mu \frac{\partial}{\partial z^\mu} \in T_pM^+$ is called a holomorphic vector, while $Z = Z^\mu \frac{\partial}{\partial \bar{z}^\mu} \in T_pM^-$ is called an anti-holomorphic vector. $T_pM^\pm$ is called integrable if and only if

$$X, Y \in T_pM^\pm \Rightarrow [X, Y] \in T_pM^\pm, \tag{51}$$

[2]The identity map on a set M is defined such that it always returns its argument.

where [,] is the Lie bracket. Using projection operators $P^{\pm} := \frac{1}{2}(1 \mp iJ)$ this can be written

$$P^{\mp}[P^{\pm}X, P^{\pm}Y] = 0, \qquad X, Y \in T_pM. \tag{52}$$

This condition can also be expressed introducing the Nijenhuis tensor $N(X,Y)$

$$N(X,Y) := [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]. \tag{53}$$

(51) and (52) are now identical to $N(X,Y) = 0$, by a theorem proved by Newlander and Nirenberg.

3.2 Hermitian manifolds and Kähler manifolds

The Riemannian metric g of a complex manifold M is called a Hermitian metric and the pair (M,g) is said to be a Hermitian manifold if at each point $p \in M$

$$g_p(J_pX, J_pY) = g_p(X,Y) \tag{54}$$

for any $X, Y \in T_pM$ and J is the almost complex structure. Another way to define a Hermitian manifold is to demand that a complex structure is preserved by the Riemannian metric of a real manifold, i.e

$$J^t g J = g. \tag{55}$$

We define a tensor field Ω by

$$\Omega_p(X,Y) = g_p(J_pX, Y), \qquad X, Y \in T_pM, \tag{56}$$

and call it the Kähler form. Ω may also be written

$$\Omega = i g_{\mu\bar{\nu}} dz^{\mu} \wedge d\bar{z}^{\nu} = -J_{\mu\bar{\nu}} dz^{\mu} \wedge d\bar{z}^{\nu}. \tag{57}$$

A Hermitian manifold (M,g) is said to be a Kähler manifold if the corresponding Kähler form is closed ($d\Omega = 0$), and the metric g is called the Kähler metric of M. It can be shown that a Hermitian manifold is Kähler if and only if

$$\nabla_{\mu} J = 0. \tag{58}$$

The Kähler metric can locally be written

$$g_{\mu\bar{\nu}} = \frac{\partial^2 \mathcal{K}}{\partial z^{\mu} \partial \bar{z}^{\nu}}, \tag{59}$$

where $\mathcal{K}$ is a function called the Kähler potential.

3.3 Hyperkähler and pseudo-Kähler

A hyperkähler manifold is a quaternionic analogue to the Kähler manifold. Instead of one complex structure we have three complex structures I, J and K, that need to satisfy the quaternion algebra:

$$I^2 = J^2 = K^2 = -1$$

$$IJ = -JI = K, \qquad JK = -KJ = I, \qquad KI = -IK = J \tag{60}$$

This gives us a quaternion-Kähler manifold. Imposing the condition that the scalar curvature vanishes we have a hyperkähler manifold.
In a pseudo-Kähler manifold two of the structures are real, i.e, for $J^2 = -1$ we have $I^2 = K^2 = 1$.

3.4 Bihermitian geometry

Bihermitian geometry involves two complex structures

$$J_{(\pm)}^2 = -1, \tag{61}$$

with respect to which the metric should be separately Hermitian

$$J_{(\pm)}^t g J_{(\pm)} = g. \tag{62}$$

The complex structures should also be covariantly constant

$$\nabla^{(\pm)} J_{(\pm)} = 0 \tag{63}$$

with respect to a torsionful connection $\Gamma^{(\pm)}$.

3.5 Symplectic manifolds

We start this subsection by defining degeneracy of two-forms on a finite-dimensional vector space V. A two-form $f(x,y)$ on V is called degenerate if there exists a nonzero $x \in V$ such that $f(x,y) = 0$ for every $y \in V$. Else it is called non-degenerate, i.e, if $f(x,y) = 0$ for all $y \in V$ implies $x = 0$ then f is called non-degenerate.

A symplectic form ω is a closed ($d\omega = 0$) non-degenerate two-form. A smooth manifold equipped with a symplectic form is called a symplectic manifold.

4 Generalized complex geometry

Generalized complex geometry was introduced by Hitchin [11] and elaborated by Gualtieri [12]. It was found to interpolate between complex geometry and symplectic geometry and also to include bihermitian geometry. We generalize the complex structure from being an endomorphism on the tangent bundle J :

$TM \to TM$ to also include the co-tangent bundle $\mathcal{J} : TM \oplus T^*M \to TM \oplus T^*M$, still requiring

$$\mathcal{J}^2 = -1. \tag{64}$$

With $X, Y \in TM$ and $\xi, \eta \in T^*M$, an element of $TM \oplus T^*M$ can be written $X + \xi$ and the natural pairing $\mathcal{I}$ is defined by $(X + \xi, Y + \eta) = \iota_X \eta + \iota_Y \xi$, where ι_X is the interior product (also called interior or inner multiplication). This pairing needs to be Hermitian with respect to $\mathcal{J}$,

$$\mathcal{J}^t \mathcal{I} \mathcal{J} = \mathcal{I}. \tag{65}$$

Analogous to the ordinary case we can define projection operators $\Pi_\pm := \frac{1}{2}(1 \pm i\mathcal{J})$ and the integrability condition becomes

$$\Pi_\mp [\Pi_\pm (X + \xi), \Pi_\pm (Y + \eta)]_c = 0, \tag{66}$$

where we have introduced the relevant bracket called the Courant bracket:

$$[X + \xi, Y + \eta]_c := [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(\iota_X \eta - \iota_Y \xi). \tag{67}$$

It is also possible to include a closed three-form H. The H-twisted Courant bracket is then defined by

$$[X + \xi, Y + \eta]_H := [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(\iota_X \eta - \iota_Y \xi) + \iota_X \iota_Y H. \tag{68}$$

In the basis $\{\partial_\mu, dx^\mu\}$ we have

$$\mathcal{I} = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix} \tag{69}$$

$$\mathcal{J} = \begin{pmatrix} J & P \\ L & K \end{pmatrix}, \tag{70}$$

where

$$\begin{aligned} &J : TM \to TM, && P : T^*M \to TM, \\ &L : TM \to T^*M, && K : T^*M \to T^*M. \end{aligned} \tag{71}$$

We explicitly work out the constraints on (70) that follows from condition (64). We have

$$\mathcal{J}^2 = \begin{pmatrix} J & P \\ L & K \end{pmatrix} \begin{pmatrix} J & P \\ L & K \end{pmatrix} = \begin{pmatrix} J^2 + PL & JP + PK \\ LJ + KL & LP + K^2 \end{pmatrix} = \begin{pmatrix} -1_d & 0 \\ 0 & -1_d \end{pmatrix}. \tag{72}$$

From Hermicity (65) we also have

$$\mathcal{J}^t \mathcal{I} \mathcal{J} = \begin{pmatrix} J^t & L^t \\ P^t & K^t \end{pmatrix} \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix} \begin{pmatrix} J & P \\ L & K \end{pmatrix}$$

$$= \begin{pmatrix} J^tL + L^t & J^t + L^tP \\ P^tL + K^tJ & P^tK + K^tP \end{pmatrix} = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix}. \tag{73}$$

Combining the constraints from (72) and (73) we get

$$J^t = K, \qquad P^t = -P, \qquad L^t = -L. \tag{74}$$

Letting only J (and therefore also $J^t = -K$) be nonzero in (70) we get the corresponding matrix of the ordinary complex structure J in terms of generalized complex geometry

$$\mathcal{J}_J = \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix}. \tag{75}$$

For a symplectic structure ω we get

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}. \tag{76}$$

From these relations we can form a metric

$$\mathcal{G} = -\mathcal{J}_J\mathcal{J}_\omega = \begin{pmatrix} 0 & J\omega^{-1} \\ J^t\omega & 0 \end{pmatrix} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}, \tag{77}$$

where g is the ordinary Kähler metric. Now, if there exist two commuting generalized complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ (such as $\mathcal{J}_J$ and $\mathcal{J}_\omega$), and $G = -\mathcal{J}_1\mathcal{J}_2$ is a positive definite metric on $TM \oplus T^*M$, then the generalized complex geometry is called generalized Kähler.

5 Minkowski space, the Poincaré group and fields

Minkowski space, $\mathcal{M}$, can be seen as the quotient space of the Poincaré group and the Lorentz group

$$ISO(D-1,1)/SO(D-1,1). \tag{78}$$

To understand this, we introduce an equivalence relation $\sim$ between two elements g_1 and g_2 of a group G. We say that g_1 is equivalent to g_2, $g_1 \sim g_2$, if there exists an element f of the subgroup F to G such that

$$g_1 = g_2 \circ f. \tag{79}$$

G can then be seperated into equivalence classes. The set of all equivalence classes is called the left coset and is denoted G/F. Identifying G with the Poincaré group and F with the Lorentz group we let every point in the Minkowski space correspond to the (infinte) set of elements in $ISO(D-1,1)$ which are equivalent up to a Lorentz transformation. Thus we can use the translation generator P to express a point $h(x)$ in $\mathcal{M}$ by a parameter x:

$$h(x) = e^{ix^aP_a}\mathbf{1} \tag{80}$$

We let a group element $g = e^{ia^a X_a}$ with generator X and parameter a act on h from the left

$$g \circ h(x) = h(x') \circ f, \tag{81}$$

and mod out any element $f = e^{\frac{i}{2}\omega^{ab}M_{ab}}$ in the Lorentz group on the right-hand side:

$$h(x') \circ f \sim h(x') \tag{82}$$

Thus we have found a coordinate transformation

$$h(x) \to h(x') \tag{83}$$

or simply

$$x \to x'. \tag{84}$$

Fields can also be viewed as representations of the Poincaré group, thus also transforming under Poincaré transformations

$$\phi \to \phi'. \tag{85}$$

For scalar fields we choose a representation with the defining property

$$\phi'(x') = \phi(x). \tag{86}$$

Expanding under an infinitesimal transformation $x \to x' = x + a$ we have

$$\phi'(x') = \phi'(x) + a^a \partial_a \phi'(x) + \cdots = \phi(x) + \delta\phi(x) + a^a \partial_a \phi(x) + \cdots = \phi(x), \tag{87}$$

giving

$$\delta\phi(x) = -a^a \partial_a \phi(x), \tag{88}$$

where we have defined

$$\delta\phi(x) := \phi'(x) - \phi(x). \tag{89}$$

We can also write, using one of the Baker-Campbell-Haussdorff formulas (appendix A.3),

$$\begin{aligned}\phi'(x) &= e^{ia^a X_a}\phi(x)e^{-ia^a X_a} \\ &= \phi(x) + [\phi(x), -ia^a X_a] + \ldots \\ &= \phi(x) + i[a^a X_a, \phi(x)] + \ldots\end{aligned} \tag{90}$$

for any generator X_a, and thus

$$\delta\phi(x) = i[a^a X_a, \phi(x)], \tag{91}$$

or

$$i[a^a X_a, \phi(x)] = -a^a \partial_a \phi(x). \tag{92}$$

Other fields such as spinor and vector fields have representations which transform differently.

We will use the Poincaré algebra in the following form

$$
\begin{aligned}
[P_a, P_b] &= 0 \\
[M_{ab}, P_c] &= i\eta_{c[a}P_{b]} = i\eta_{ca}P_b - i\eta_{cb}P_a \\
[M_{ab}, M_{cd}] &= i\eta_{c[a}M_{b]d} - i\eta_{d[a}M_{b]c},
\end{aligned} \tag{93}
$$

where P_a is the generator of translations and M_{ab} the generator of rotations in space-time, i.e., boosts and spatial rotations. From the algebra we can deduce the coordinate changes they will generate. For an infinitesimal translation we have

$$
g = e^{ia^a P_a}, \tag{94}
$$

thus giving,

$$
\begin{aligned}
g \circ h(x) = e^{ia^a P_a} e^{ix^b P_b} &= \exp(ia^a P_a + ix^b P_b + \frac{1}{2}[ia^a P_a, ix^b P_b]) \\
&= \exp\left(i(a^a + x^a)P_a - \frac{1}{2}a^a x^b [P_a, P_b]\right) = e^{i(a^a + x^a)P_a}.
\end{aligned} \tag{95}
$$

We see that the coordinates transform as

$$
x^a \to x'^a = x^a + a^a \Rightarrow \delta x^a = a^a. \tag{96}
$$

From (92) we have

$$
\begin{aligned}
& i[a^a P_a, \phi(x)] = -a^a \partial_a \phi(x) \\
& \Rightarrow [P_a, \phi(x)] = i\partial_a \phi,
\end{aligned} \tag{97}
$$

and we define an operator

$$
\hat{P}_a := i\partial_a. \tag{98}
$$

Similarly, for a Lorentz transformation we have

$$
\begin{aligned}
g \circ h(x) &= e^{\frac{i}{2}\omega^{ab}M_{ab}} e^{ix^c P_c} \sim e^{\frac{i}{2}\omega^{ab}M_{ab}} e^{ix^c P_c} e^{-\frac{i}{2}\omega^{ab}M_{ab}} \\
&= e^{\frac{i}{2}\omega^{ab}M_{ab}}(1 + ix^c P_c + \dots)e^{-\frac{i}{2}\omega^{ab}M_{ab}} = 1 + ix^c P_c + [ix^c P_c, -\frac{i}{2}\omega^{ab}M_{ab}] + \dots \\
&= 1 + ix^c P_c - \frac{1}{2}\omega^{ab}x^c [M_{ab}, P_c] + \dots \\
&= 1 + ix^c P_c - \frac{i}{2}\omega^{ab}x^c \eta_{ca}P_b + \frac{i}{2}\omega^{ab}x^c \eta_{cb}P_a + \dots \\
&= 1 + ix^d P_d - \frac{i}{2}\omega^{ab}x^c \eta_{ca}\delta^d_b P_d + \frac{i}{2}\omega^{ab}x^c \eta_{cb}\delta^d_a P_d + \dots \\
&= e^{i(x^d - \frac{1}{2}\omega^{ab}x_a\delta^d_b + \frac{1}{2}\omega^{ab}x_b\delta^d_a)P_d},
\end{aligned} \tag{99}
$$

giving us an infinitesimal coordinate transformation

$$
x^d \to x'^d = x^d + \frac{1}{2}\omega^{ab}(-x_a\delta^d_b + x_b\delta^d_a) \Rightarrow \delta x^d = \frac{1}{2}\omega^{ab}(-x_a\delta^d_b + x_b\delta^d_a). \tag{100}
$$

We have

$$[\frac{i}{2}\omega^{ab}M_{ab},\phi(x)] = -\frac{1}{2}\omega^{ab}(-x_a\delta_b^d + x_b\delta_a^d)\partial_d\phi(x)$$
$$\Rightarrow [M_{ab},\phi(x)] = i(-x_a\partial_b + x_b\partial_a)\phi(x), \tag{101}$$

and we define an operator acting on scalar fields

$$\hat{M}_{ab} := -ix_{[a}\partial_{b]}. \tag{102}$$

6 Non-supersymmetric sigma models

A sigma model is a set of maps $X^\mu : \Sigma \to \mathcal{T}$ from a parameter space Σ with coordinates $\xi \in \Sigma, a = 1, \ldots, D$, and a target space $\mathcal{T}$ with coordinates $X^\mu \in \mathcal{T}, \mu = 0, 1, \ldots, d-1$, and an action which gives the dynamics of the system.

6.1 Bosonic model in D=2

Starting from the action of a classical string

$$S = -T\int dA, \tag{103}$$

where T is the tension of the string, inducing a metric on the world surface

$$\gamma_{ab} = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b}\eta_{\mu\nu}, \tag{104}$$

and using the fact that proper "generalized volume"

$$dV = d^p\xi\sqrt{-\det\gamma_{ab}} \tag{105}$$

is invariant under diffeomorphisms, we arrive at the Nambu-Goto action

$$S = -T\int d^2\xi\sqrt{-\det\gamma_{ab}}. \tag{106}$$

This is equivalent to the Polyakov action

$$S = -\frac{T}{2}\int d^2\xi\sqrt{-h}h^{ab}\gamma_{ab}, \tag{107}$$

where $h := \det h_{ab}$ is the independent metric of the world sheet, as can be seen from varying the action with respect to h_{ab} and setting $\delta S = 0$.
In the conformal gauge (section 10.3) the Polyakov action takes the form

$$S = \frac{T}{2}\int d^2\xi\eta_{\mu\nu}\partial_a X^\mu\partial^a X^\nu. \tag{108}$$

We generalize the target space metric $\eta_{\mu\nu}$ by $g_{\mu\nu} = g_{\mu\nu}(X)$ and introduce an antisymmetric tensor $B_{\mu\nu} = B_{\mu\nu}(X)$ in the background

$$E_{\mu\nu}(X) = g_{\mu\nu}(X) + B_{\mu\nu}(X). \tag{109}$$

In light-cone coordinates, $x^{\pm\pm} := \frac{1}{\sqrt{2}}(\xi^1 \pm \xi^2)$, we get

$$S = T \int d^2x \partial_{++} X^\mu E_{\mu\nu} \partial_= X^\nu. \tag{110}$$

In the following, we will skip the factor T for simplicity. From $\delta S = 0$ we obtain the field equations:

$$\partial_a \frac{\partial \mathcal{L}}{\partial(\partial_a X^\rho)} - \frac{\partial \mathcal{L}}{\partial X^\rho} = 0, \tag{111}$$

i.e.,

$$\begin{aligned}
\partial_{++} \frac{\partial \mathcal{L}}{\partial(\partial_{++} X^\rho)} &= \partial_{++}(E_{\rho\nu} \partial_= X^\nu) = E_{\rho\nu,\mu} \partial_{++} X^\mu \partial_= X^\nu + E_{\rho\nu} \partial_{++} \partial_= X^\nu \\
\partial_= \frac{\partial \mathcal{L}}{\partial(\partial_= X^\rho)} &= \partial_=(\partial_{++} X^\mu E_{\mu\rho}) = \partial_{++} \partial_= X^\mu E_{\mu\rho} + \partial_{++} X^\mu E_{\mu\rho,\nu} \partial_= X^\nu \\
\frac{\partial \mathcal{L}}{\partial X^\rho} &= \partial_{++} X^\mu E_{\mu\nu,\rho} \partial_= X^\nu,
\end{aligned} \tag{112}$$

which gives

$$\begin{aligned}
0 &= \partial_{++} \frac{\partial \mathcal{L}}{\partial(\partial_{++} X^\rho)} + \partial_= \frac{\partial \mathcal{L}}{\partial(\partial_= X^\rho)} - \frac{\partial \mathcal{L}}{\partial X^\rho} \\
&= (E_{\rho\mu} + E_{\mu\rho}) \partial_{++} \partial_= X^\mu + (E_{\rho\nu,\mu} + E_{\mu\rho,\nu} - E_{\mu\nu,\rho}) \partial_{++} X^\mu \partial_= X^\nu \\
&= 2 g_{\mu\rho} \partial_{++} \partial_= X^\mu + (g_{\rho\nu,\mu} + g_{\mu\rho,\nu} - g_{\mu\nu,\rho} + B_{\rho\nu,\mu} + B_{\mu\rho,\nu} - B_{\mu\nu,\rho}) \partial_{++} X^\mu \partial_= X^\nu.
\end{aligned} \tag{113}$$

Multiplying with $\frac{1}{2} g^{\kappa\rho}$ gives

$$\begin{aligned}
0 &= \partial_{++} \partial_= X^\kappa + \frac{1}{2} g^{\kappa\rho} (g_{\nu\rho,\mu} + g_{\mu\rho,\nu} - g_{\mu\nu,\rho} - B_{\nu\rho,\mu} - B_{\rho\mu,\nu} - B_{\mu\nu,\rho}) \partial_{++} X^\mu \partial_= X^\nu \\
&= \partial_{++} \partial_= X^\kappa + ({\Gamma^{(0)\kappa}}_{\mu\nu} - \frac{1}{2} g^{\kappa\rho} T_{\rho\mu\nu}) \partial_{++} X^\mu \partial_= X^\nu = \nabla^{(-)}_{++} \partial_= X^\kappa = 0,
\end{aligned} \tag{114}$$

where $T = dB$ is the torsion. This implies that the target space $\mathcal{T}$ is Riemannian with torsion.

6.2 Bosonic model in D=1

The one-dimensional bosonic sigma model we will simply state:

$$S = \frac{1}{2} \int dt g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu \tag{115}$$

7 Supersymmetry

7.1 The supersymmetry algebra

Starting from the Poincaré group (93) and an internal group

$$[B_i, B_j] = f_{ij}{}^k B_k, \tag{116}$$

it was shown in 1967 by Coleman and Mandula [18] that, under certain general assumptions and in the context of Lie algebra, the largest symmetry group containing both the Poincaré group and an internal group necessarily must be a direct product group of them, implying

$$\begin{aligned}
[P_\nu, B_I] &= 0, \\
[M_{\mu\nu}, B_I] &= 0.
\end{aligned} \tag{117}$$

By introducing a graded Lie algebra including not only commutators but also anti-commutators (and thus altering the original assumptions), it was however later realized by Haag, Lopuszanski and Sohnius [19] that the group *can* be expanded in a non-trivial way. To this end we divide the generators into an even (bosonic) and an odd (fermionic) class obeying the rules:

$$\begin{aligned}
[\text{even}, \text{even}] &= \text{even}, \\
[\text{even}, \text{odd}] &= \text{odd}, \\
\{\text{odd}, \text{odd}\} &= \text{even},
\end{aligned} \tag{118}$$

and generalize the Jacobi identity to

$$\begin{aligned}
\big[[B_1, B_2], B_3\big] + \big[[B_3, B_1], B_2\big] + \big[[B_2, B_3], B_1\big] &= 0, \\
\big[[B_1, B_2], F_3\big] + \big[[F_3, B_1], B_2\big] + \big[[B_2, F_3], B_1\big] &= 0, \\
\big\{[B_1, F_2], F_3\big\} + \big\{[B_1, F_3], F_2\big\} + \big[\{F_2, F_3\}, B_1\big] &= 0, \\
\big[\{F_1, F_2\}, F_3\big] + \big[\{F_1, F_3\}, F_2\big] + \big[\{F_2, F_3\}, F_1\big] &= 0,
\end{aligned} \tag{119}$$

where B denotes even generators and F odd. We classify the Poincaré generators and the internal group generators as even and introduce N odd generators $Q^i, i = 1, 2, \dots, N$. Using these rules we are able to derive the super-Poincaré algebra (appendices B.1, B.2).

$$\begin{aligned}
[P_a, P_b] &= 0, \\
[M_{ab}, P_c] &= i\eta_{c[a} P_{b]}, \\
[M_{ab}, M_{cd}] &= i\eta_{c[a} M_{b]d} - i\eta_{d[a} M_{b]c}, \\
[P_a, B_l] &= [M_{ab}, B_l] = 0, \\
[B_i, B_j] &= f_{ij}{}^k B_k, \\
[Q^I_\alpha, P_a] &= [\bar{Q}^I_{\dot\alpha}, P_a] = 0,
\end{aligned}$$

$$
\begin{aligned}
[Q^I_\alpha, M_{ab}] &= -\frac{i}{2}(\sigma_{ab})_\alpha{}^\beta Q^I_\beta,\\
[\bar{Q}^I_{\dot\alpha}, M_{ab}] &= \frac{i}{2}(\bar\sigma_{ab})^{\dot\beta}{}_{\dot\alpha}\bar{Q}^I_{\dot\beta},\\
\{Q^I_\alpha, Q^J_\beta\} &= X^{IJ}\epsilon_{\alpha\beta},\\
\{\bar{Q}^I_{\dot\alpha}, \bar{Q}^J_{\dot\alpha}\} &= \bar{X}^{IJ}\epsilon_{\dot\alpha\dot\beta},\\
\{Q^I_\alpha, \bar{Q}^J_{\dot\alpha}\} &= 2\delta^{IJ}P_{\alpha\dot\alpha},\\
[Q^I_\alpha, B_l] &= (S_l)^I{}_J Q^J_\alpha,\\
[\bar{Q}_{\dot\alpha}, B_l] &= (\bar{S}_l)^I{}_J \bar{Q}^J_{\dot\alpha},\\
[X^{IJ}, \mathcal{O}] &= [\bar{X}^{IJ}, \mathcal{O}] = 0,
\end{aligned}
\tag{120}
$$

where $\mathcal{O}$ is any operator and the complex constants X^{IJ} are called central charges. In this master thesis we will not discuss central charges nor internal groups. Thus the algebra simplifies greatly, the non-zero part being

$$
\begin{aligned}
[M_{ab}, P_c] &= i\eta_{c[a}P_{b]},\\
[M_{ab}, M_{cd}] &= i\eta_{c[a}M_{b]d} - i\eta_{d[a}M_{b]c},\\
[Q^I_\alpha, M_{ab}] &= -\frac{i}{2}(\sigma_{ab})_\alpha{}^\beta Q^I_\beta,\\
[\bar{Q}^I_{\dot\alpha}, M_{ab}] &= \frac{i}{2}(\bar\sigma_{ab})_{\dot\alpha}{}^{\dot\beta}\bar{Q}^I_{\dot\beta},\\
\{Q^I_\alpha, \bar{Q}^J_{\dot\alpha}\} &= 2\delta^{IJ}P_{\alpha\dot\alpha}.
\end{aligned}
\tag{121}
$$

A further simplification can be done by only considering the massless case as we will see in section 7.4.

7.2 Superspace and its operators

In the same way as the Minkowski space can be seen as the quotient space of the Poincaré group and the Lorentz group

$$
ISO(D-1,1)/SO(D-1,1), \tag{122}
$$

superspace can viewed as the quotient space of the super-Poincaré group and the Lorentz group:

$$
SISO(D-1,1)/SO(D-1,1). \tag{123}
$$

A point in this space is written

$$
h(x,\theta) = e^{i(xP+\theta Q+\bar\theta\bar{Q})} = e^{i(x^aP_a+\theta^\alpha Q_\alpha+\bar\theta_{\dot\alpha}\bar{Q}^{\dot\alpha})}, \tag{124}
$$

for two-component Weyl spinors where $\alpha = 1,2$ and $\dot\alpha = \dot1,\dot2$ (appendix A). In the same way as x^a acts as a parameter for the translation generator P_a, we now also have two anticommuting parameters, θ^α and $\bar\theta_{\dot\alpha}$, acting as paramaters for the left-handed and right-handed supersymmetric generators Q_α and $\bar{Q}^{\dot{\alpha}}$ respectively. An infintesimal transformation by Q gives

$$\begin{aligned}
&e^{i\xi^\beta Q_\beta} e^{ix^a P_a + i\theta^\alpha Q_\alpha + i\bar{\theta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}} = \\
&\qquad = e^{ix^a P_a + i(\theta^\alpha + \xi^\alpha)Q_\alpha + i\bar{\theta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}} + \frac{1}{2}[i\xi^\beta Q_\beta, i\bar{\theta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}] + \ldots} \\
&\qquad = e^{ix^a P_a + i(\theta^\alpha + \xi^\alpha)Q_\alpha + i\bar{\theta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}} - \xi^\beta\bar{\theta}^{\dot{\alpha}}(\sigma^a)_{\beta\dot{\alpha}}P_a + \ldots} \\
&\qquad = e^{i(x^a + i\xi^\alpha(\sigma^a)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}})P_a + i(\theta^\alpha + \xi^\alpha)Q_\alpha + \bar{\theta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}}. \qquad (125)
\end{aligned}$$

We see that Q_α generates a coordinate shift

$$\begin{cases} x^a \to x'^a = x^a + i\xi^\alpha(\sigma^a)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} \Rightarrow \delta x^a = i\xi^\alpha(\sigma^a)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} \\ \theta^\alpha \to \theta'^\alpha = \theta^\alpha + \xi^\alpha \Rightarrow \delta\theta^\alpha = \xi^\alpha \\ \bar{\theta}_{\dot{\alpha}} \to \bar{\theta}'_{\dot{\alpha}} = \bar{\theta}_{\dot{\alpha}} \Rightarrow \delta\bar{\theta}_{\dot{\alpha}} = 0. \end{cases} \qquad (126)$$

We get

$$\begin{aligned}
&[i\xi^\alpha Q_\alpha, \phi(x,\theta,\bar{\theta})] = -\big(\xi^\alpha\partial_\alpha + i\xi^\alpha(\sigma^a)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_a\big)\phi(x,\theta,\bar{\theta}) \\
&\Rightarrow [Q_\alpha, \phi(x,\theta,\bar{\theta})] = \big(i\partial_a - (\sigma^a)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_a\big)\phi(x,\theta,\bar{\theta}). \qquad (127)
\end{aligned}$$

Thus we can define a differential operator

$$\begin{aligned}
&\hat{Q}_\alpha := i\partial_\alpha - (\sigma^a)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_a = i\partial_\alpha - \bar{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}, \\
&\hat{Q}^\alpha = \epsilon^{\alpha\beta}\bar{Q}_\beta = -i\partial^\alpha + \bar{\theta}_{\dot{\alpha}}(\bar{\sigma}_a)^{\dot{\alpha}\alpha}\partial^a = -i\partial^a + \bar{\theta}_{\dot{\alpha}}\partial^{\alpha\dot{\alpha}}. \qquad (128)
\end{aligned}$$

An infinitesimal transformation generated by $\bar{Q}$,

$$\begin{aligned}
&e^{i\bar{\xi}_{\dot{\beta}}\bar{Q}^{\dot{\beta}}} e^{ix^a P_a + i\theta^\alpha Q_\alpha + i\bar{\theta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}} = \\
&\qquad = e^{ix^a P_a + i\theta^\alpha Q_\alpha + i(\bar{\theta}_{\dot{\alpha}} + \bar{\xi}_{\dot{\alpha}})\bar{Q}^{\dot{\alpha}} + \frac{1}{2}[i\bar{\xi}_{\dot{\beta}}\bar{Q}^{\dot{\beta}}, i\theta^\alpha Q_\alpha] + \ldots} \\
&\qquad = e^{ix^a P_a + i\theta^\alpha Q_\alpha + i(\bar{\theta}_{\dot{\alpha}} + \bar{\xi}_{\dot{\alpha}})\bar{Q}^{\dot{\alpha}} + \theta^\alpha(\sigma^a)_{\alpha\dot{\beta}}\bar{\xi}^{\dot{\beta}}P_a + \ldots} \\
&\qquad = e^{i(x^a + i\bar{\xi}^{\dot{\alpha}}\theta^\alpha(\sigma^a)_{\alpha\dot{\alpha}})P_a + i\theta^\alpha Q_\alpha + i(\bar{\theta}_{\dot{\alpha}} + \bar{\xi}_{\dot{\alpha}})\bar{Q}^{\dot{\alpha}} + \ldots}, \qquad (129)
\end{aligned}$$

gives a coordinate transformation

$$\begin{cases} x^a \to x'^a = x^a + \bar{\xi}^{\dot{\alpha}}\theta^\alpha(\sigma^a)_{\alpha\dot{\alpha}} \Rightarrow \delta x^a = i\bar{\xi}^{\dot{\alpha}}\theta^\alpha(\sigma^a)_{\alpha\dot{\alpha}} \\ \theta^\alpha \to \theta'^a = \theta^a \Rightarrow \delta\theta^\alpha = 0 \\ \bar{\theta}^{\dot{\alpha}} \to \bar{\theta}'^{\dot{\alpha}} = \bar{\theta}^{\dot{\alpha}} + \bar{\xi}^{\dot{\alpha}} \Rightarrow \delta\bar{\theta}^{\dot{\alpha}} = \bar{\xi}^{\dot{\alpha}}, \end{cases} \qquad (130)$$

and thus

$$\begin{aligned}
&[i\bar{\xi}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}, \phi(x,\theta,\bar{\theta})] = -(\bar{\xi}^{\dot{\alpha}}\partial_{\dot{\alpha}} + i\bar{\xi}^{\dot{\alpha}}\theta^\alpha(\sigma^a)_{\alpha\dot{\alpha}}\partial_a)\phi(x,\theta,\bar{\theta}) \\
&\Rightarrow -i\bar{\xi}^{\dot{\alpha}}[\bar{Q}_{\dot{\alpha}}, \phi(x,\theta,\bar{\theta})] = (-\bar{\xi}^{\dot{\alpha}}\partial_{\dot{\alpha}} - i\bar{\xi}^{\dot{\alpha}}\theta^\alpha(\sigma^a)_{\alpha\dot{\alpha}}\partial_a)\phi(x,\theta,\bar{\theta}) \\
&\Rightarrow [\bar{Q}_{\dot{\alpha}}, \phi(x,\theta,\bar{\theta})] = (-i\partial_{\dot{\alpha}} + \theta^\alpha(\sigma^a)_{\alpha\dot{\alpha}}\partial_a)\phi(x,\theta,\bar{\theta}). \qquad (131)
\end{aligned}$$

We define a differential operator

$$\hat{\bar{Q}}_{\dot{\alpha}} := -i\partial_{\dot{\alpha}} + \theta^\alpha(\sigma^a)_{\alpha\dot{\alpha}}\partial_a = -i\partial_{\dot{\alpha}} + \theta^\alpha\partial_{\alpha\dot{\alpha}},$$

$$\hat{\bar{Q}}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\hat{\bar{Q}}_{\dot{\beta}} = i\partial^{\dot{\alpha}} - (\bar{\sigma}_a)^{\dot{\alpha}\alpha}\theta_\alpha\partial^a = i\partial^{\dot{\alpha}} - \theta_\alpha\partial^{\alpha\dot{\alpha}}. \tag{132}$$

We will also need differential operators, D_α, $\bar{D}_{\dot{\alpha}}$, that anticommute with the supersymmetry operators:

$$\{D_\alpha, \hat{Q}_\beta\} = \{D_\alpha, \hat{\bar{Q}}_{\dot{\beta}}\} = \{\bar{D}_{\dot{\alpha}}, \hat{Q}_\beta\} = \{\bar{D}_{\dot{\alpha}}, \hat{\bar{Q}}_{\dot{\beta}}\} = 0. \tag{133}$$

If we take them to be

$$\begin{aligned}
D_\alpha &:= \partial_\alpha - i(\sigma^a)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_a = \partial_\alpha - i\theta^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}, \\
D^\alpha &= \epsilon^{\alpha\beta}D_\beta = -\partial^\alpha + i\bar{\theta}_{\dot{\alpha}}(\bar{\sigma}_a)^{\dot{\alpha}\alpha}\partial^a = -\partial^\alpha + i\bar{\theta}_{\dot{\alpha}}\partial^{\alpha\dot{\alpha}}, \\
\bar{D}_{\dot{\alpha}} &:= -\partial_{\dot{\alpha}} + i\theta^\alpha(\sigma^a)_{\alpha\dot{\alpha}}\partial_a = -\partial_{\dot{\alpha}} + i\theta^\alpha\partial_{\alpha\dot{\alpha}}, \\
\bar{D}^{\dot{\alpha}} &= \epsilon^{\dot{\alpha}\dot{\beta}}\bar{D}_{\dot{\beta}} = \partial^{\dot{\alpha}} - i(\bar{\sigma}_a)^{\dot{\alpha}\alpha}\theta_\alpha\partial^a = \partial^{\dot{\alpha}} - i\theta_\alpha\partial^{\alpha\dot{\alpha}},
\end{aligned} \tag{134}$$

this is indeed the case. We also have

$$\{D_\alpha, D_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0, \tag{135}$$

but note that

$$\{D_\alpha, \bar{D}_{\dot{\beta}}\} = 2i(\sigma^a)_{\alpha\dot{\beta}}\partial_a = 2i\partial_{\alpha\dot{\beta}}. \tag{136}$$

The reason for introducing these operators, which we will call the covariant derivatives, is that the supersymmetry derivatives, ∂_α and $\partial_{\dot{\alpha}}$, does not anticommute with the supersymmetry operators:

$$\partial_\beta\Big(i\xi^\alpha\hat{Q}_\alpha\phi(x,\theta,\bar{\theta})\Big) = \partial_\beta\Big(\delta\phi(x,\theta,\bar{\theta})\Big) \neq \delta\Big(\partial_\beta\phi(x,\theta,\bar{\theta})\Big). \tag{137}$$

Using instead the covariant derivatives we have

$$D_\beta\Big(\delta\phi(x,\theta,\bar{\theta})\Big) = \delta\Big(D_\beta\phi(x,\theta,\bar{\theta})\Big). \tag{138}$$

7.3 Superfields

Since θ^α and $\bar{\theta}_{\dot{\alpha}}$ anticommute with themselves, a Taylor expansion in these parameters terminates after only a few terms. The verticle line denotes the θ- and $\bar{\theta}$-free part.

$$\begin{aligned}
\phi(x,\theta,\bar{\theta}) = {} & \phi(x,\theta,\bar{\theta})| \\
& + \theta^\alpha\big(D_\alpha\phi(x,\theta,\bar{\theta})\big)| + \bar{\theta}_{\dot{\alpha}}\big(\bar{D}^{\dot{\alpha}}\phi(x,\theta,\bar{\theta})\big)| \\
& + \frac{1}{2}\theta^\alpha\theta^\beta\big(D_\alpha D_\beta\phi(x,\theta,\bar{\theta})\big)| + \theta^\alpha\bar{\theta}_{\dot{\alpha}}\big(D_\alpha\bar{D}^{\dot{\alpha}}\phi(x,\theta,\bar{\theta})\big)| + \frac{1}{2}\bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}}\big(\bar{D}^{\dot{\alpha}}\bar{D}^{\dot{\beta}}\phi(x,\theta,\bar{\theta})\big)| \\
& + \frac{1}{6}\theta^\alpha\theta^\beta\bar{\theta}_{\dot{\alpha}}\big(D_\alpha D_\beta\bar{D}^{\dot{\alpha}}\phi(x,\theta,\bar{\theta})\big)| + \frac{1}{6}\theta^\alpha\bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}}\big(D_\alpha\bar{D}^{\dot{\alpha}}\bar{D}^{\dot{\beta}}\phi(x,\theta,\bar{\theta})\big)| \\
& + \frac{1}{12}\theta^\alpha\theta^\beta\bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}}\big(D_\alpha D_\beta\bar{D}^{\dot{\alpha}}\bar{D}^{\dot{\beta}}\phi(x,\theta,\bar{\theta})\big)|.
\end{aligned} \tag{139}$$

We use that

$$\theta^\alpha\theta^\beta D_\alpha D_\beta = -\frac{1}{4}\epsilon^{\alpha\beta}\theta^\gamma\theta_\gamma\epsilon_{\alpha\beta}D^\delta D_\delta = \frac{1}{4}\epsilon^{\alpha\beta}\epsilon_{\beta\alpha}\theta^\gamma\theta_\gamma D^\delta D_\delta = \frac{1}{2}\theta^\gamma\theta_\gamma D^\delta D_\delta, \tag{140}$$

to simplify and to define new fields:

$$\begin{aligned}\phi(x,\theta,\bar\theta) &= a(x) + \theta^\alpha b_\alpha(x) + \bar\theta_{\dot\alpha}\bar c^{\dot\alpha} + \theta^\alpha\theta_\alpha d(x) + \theta^\alpha(\sigma^a)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}e_a(x)\\ &\quad + \bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}f(x) + \theta^\alpha\theta_\alpha\bar\theta_{\dot\alpha}\bar g^{\dot\alpha}(x) + \theta^\alpha\bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}h_a(x) + \theta^\alpha\theta_\alpha\bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}i(x).\end{aligned} \tag{141}$$

If we impose constraints on the superfield we can simplify further. For example, a constraint

$$\bar D_{\dot\alpha}\phi(x,\theta,\bar\theta) = 0, \tag{142}$$

gives a chiral field, while a constraint

$$D_\alpha\phi(x,\theta,\bar\theta) = 0, \tag{143}$$

gives an antichiral field. It can be shown that after a translation

$$y^a = x^a + i\theta^\alpha(\sigma^a)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}, \tag{144}$$

we can write the chiral field superfield

$$S(x,\theta) = a(y) + \theta^\alpha b_\alpha(y) + \theta^\alpha\theta_\alpha d(y), \tag{145}$$

and similarly for the antichiral superfield.

7.4 The massless case in two dimensions

A massless particle is characterized by $P^aP_a = 0$. Taking $P_a = (P, \pm P)$ this will indeed be the case in a two-dimensional Minkowski space with metric $\eta = diag(-1,+1)$. A plus corresponds to a particle moving to the right and a minus to a left-mover. Introducing light-cone coordinates

$$x^{++} := \frac{1}{\sqrt{2}}(x^0 + x^1), \qquad x^{=} := \frac{1}{\sqrt{2}}(x^0 - x^1), \tag{146}$$

with a metric

$$(\eta_{ab}) = (\eta^{ab}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{147}$$

we arrive at the algebra

$$\begin{aligned}\{Q^I_+, Q^J_+\} &= 2\delta^{IJ}P_{++}\\ \{Q^{I'}_-, Q^{J'}_-\} &= 2\delta^{I'J'}P_{=}\\ \{Q^I_+, Q^{J'}_-\} &= 0.\end{aligned} \tag{148}$$

We thus make a distinction between right- and left-moving supersymmetries. In fact it is possible to have different numbers of right-moving supersymmetries

$I = 0, 1, \ldots, q$ than left-moving $I' = 0, 1, \ldots, p$. We will label such a theory a $N = (p, q)$ theory. Analogous to (124) we write a point in this space

$$h(x, \theta_1^+, \theta_2^+, \ldots, \theta_{1'}^-, \theta_{2'}^-, \ldots) = e^{(ix^{++}P_{++} + ix^{=}P_{=} + i\theta_I^+ Q_+^I + i\theta_{I'}^- Q_-^{I'})}, \tag{149}$$

and a group element

$$e^{i\epsilon^\alpha Q_\alpha}. \tag{150}$$

Thus an infinitesimal transformation by Q_+^I becomes

$$\begin{aligned}
&e^{i\xi_J^+ Q_+^J} e^{(ix^{++}P_{++} + ix^{=}P_{=} + i\theta_I^+ Q_+^I + i\theta_{I'}^- Q_-^{I'})} \\
&= \exp\left(ix^{++}P_{++} + ix^{=}P_{=} + i(\theta_I^+ + \xi_I^+)Q_+^I + i\theta_{I'}^- Q_-^{I'} + \frac{1}{2}[i\xi_J^+ Q_+^J, i\theta_I^+ Q_+^I]\right) \\
&= \exp\left(ix^{++}P_{++} + ix^{=}P_{=} + i(\theta_I^+ + \xi_I^+)Q_+^I + i\theta_{I'}^- Q_-^{I'} + \frac{1}{2}\xi_J^+ \theta_I^+ [Q_+^J, Q_+^I]\right) \\
&= \exp\left(ix^{++}P_{++} + ix^{=}P_{=} + i(\theta_I^+ + \xi_I^+)Q_+^I + i\theta_{I'}^- Q_-^{I'} + \delta^{JI}\xi_J^+ \theta_I^+ P_{++}\right) \\
&= \exp\left(i(x^{++} - i\delta^{JI}\xi_J^+ \theta_I^+)P_{++} + ix^{=}P_{=} + i(\theta_I^+ + \xi_I^+)Q_+^I + i\theta_{I'}^- Q_-^{I'}\right).
\end{aligned} \tag{151}$$

This implies a coordinate change

$$\begin{cases} \delta x^{++} = -i\delta^{IJ}\xi_J^+ \theta_I^+ \\ \delta\theta_I^+ = \xi_I^+, \end{cases} \tag{152}$$

giving

$$\begin{aligned}
&[i\xi_I^+ Q_+^I, \phi(x, \theta)] = -(\xi_I^+ \partial_+^I - i\delta^{IJ}\xi_I^+ \theta_J^+ \partial_{++})\phi(x, \theta) \\
&\Rightarrow [Q_+^I, \phi(x, \theta)] = (i\partial_+^I + \delta^{IJ}\theta_J^+ \partial_{++})\phi(x, \theta),
\end{aligned} \tag{153}$$

where we have defined

$$\partial_+^I := \frac{\partial}{\partial\theta_I^+}, \qquad \partial_+^I \theta_J^+ = \delta_J^I, \qquad \partial_+^I \theta_{I'}^- = 0. \tag{154}$$

We get a representation

$$\hat{Q}_+^I = i\partial_+^I + \delta^{IJ}\theta_J^+ \partial_{++}. \tag{155}$$

A similar calculation gives

$$\hat{Q}_-^{I'} = i\partial_-^{I'} + \delta^{I'J'}\theta_{J'}^- \partial_{=}. \tag{156}$$

We define covariant derivatives

$$\begin{aligned}
D_+^I &: = \partial_+^I + i\delta^{IJ}\theta_J^+ \partial_{++}, \\
D_-^{I'} &: = \partial_-^{I'} + i\delta^{I'J'}\theta_{I'}^- \partial_{=},
\end{aligned} \tag{157}$$

and notice that

$$D_\pm^2 = i\partial_{\pm\pm}. \tag{158}$$

It is possible to replace δ^{IJ} in (148) by an arbitrary symmetric matrix η^{IJ}, which, if invertible, can be transformed to

$$\eta^{IJ} = \begin{pmatrix} 1_u & 0 \\ 0 & -1_t \end{pmatrix}, \tag{159}$$

for $u + t = p$. This means that we let Q_+^I square to $-P_{++}$ for some I instead of $+P_{++}$, and we call this a twisted supersymmetry. Allowing for non-invertible matrices, η^{IJ} can be further generalized to

$$\eta^{IJ} = \begin{pmatrix} 1_u & 0 & 0 \\ 0 & -1_t & 0 \\ 0 & 0 & 0_v \end{pmatrix}, \tag{160}$$

where $u + t + v = p$, i.e., we let Q_+^I square to 0 for some I. Similarly $\delta^{I'J'}$ generalizes to $\eta^{I'J'}$ for the left-moving symmetries.

7.5 One-dimensional case

We will also consider the one-dimensional case:

$$\{Q^I, Q^J\} = 2\delta^{IJ} P. \tag{161}$$

A parametrization of the superspace

$$h(t, \theta_1, \theta_2, \ldots, \theta_N) = e^{(itP + i\theta_I Q^I)}, \tag{162}$$

and an infinitesimal transformation

$$\begin{aligned} e^{(i\xi_I Q^I)} e^{(itP + i\theta_J Q^J)} &= \exp\left(itP + i(\theta_J + \xi_I \delta_J^I) Q^J + \frac{1}{2}[i\xi_I Q^I, i\theta_J Q^J]\right) \\ &= \exp\left(itP + i(\theta_J + \xi_I \delta_J^I) Q^J + \xi_I \theta_J \delta^{IJ} P\right) \\ &= \exp\left(i(t - i\xi_I \theta_J \delta^{IJ}) P + i(\theta_J + \xi_I \delta_J^I) Q^J\right) \end{aligned} \tag{163}$$

gives

$$\begin{cases} \delta t = -i\xi_I \theta_J \delta^{IJ} \\ \delta\theta_J = \xi_I \delta_J^I \end{cases} \tag{164}$$

and

$$\begin{aligned} &[i\xi_I Q^I, \phi(t,\theta)] = -(\xi_I \delta_J^I \partial_\theta^J - i\xi_I \theta_J \delta^{IJ} \partial_t) \phi(t,\theta) \\ &\Rightarrow [Q^I, \phi(t,\theta)] = (i\partial_\theta^I + \theta_J \delta^{IJ} \partial_t) \phi(t,\theta) \end{aligned} \tag{165}$$

where

$$\partial_\theta^I := \frac{\partial}{\partial \theta_I}, \qquad \partial_t := \frac{\partial}{\partial t}. \tag{166}$$

We define an operator

$$\hat{Q}^I := i\partial_\theta^I + \theta_J \delta^{IJ} \partial_t, \tag{167}$$

and a covariant derivative

$$D^I = \partial_\theta^I + i\theta_J \delta^{IJ} \partial_t, \tag{168}$$

which squares to

$$D^2 = i\partial_t. \tag{169}$$

Similar to the two-dimensional case we can twist the algebra, thus generalizing (161) to

$$\{Q^I, Q^J\} = 2\eta^{IJ} P, \tag{170}$$

where η^{IJ} is given by (159).

8 Supersymmetric sigma models

In this section we will review the calculation methods for obtaining supersymmetric sigma models. We will to some extend follow section 4 and appendices A1 and A2 in [5], but possible with even more detailed calculations.

8.1 $D = 2, N = (1, 1)$

8.1.1 The superfield and closure of the algebra

We expand the field

$$\phi^\mu(x, \theta) = X^\mu(x) + \theta^+ \psi_+^\mu(x) + \theta^- \psi_-^\mu(x) + \theta^+\theta^- F^\mu(x), \tag{171}$$

where

$$X^\mu = \phi^\mu|, \qquad \psi_\pm^\mu = D_\pm \phi^\mu|, \qquad F^\mu = -D_\pm D_\mp \phi^\mu|, \tag{172}$$

and the relavant operators and covariant derivatives are

$$\begin{aligned} \hat{Q}_\pm &= i\partial_\pm + \theta^\pm \partial_{\pm\pm}, \qquad P_{\pm\pm} = i\partial_{\pm\pm} \\ D_\pm &= \partial_\pm + i\theta^\pm \partial_{\pm\pm}. \end{aligned} \tag{173}$$

The closure of the algebra is essential The commutator of two transformations must never lead outside of the algebra or the superspace. We have

$$\delta\phi^\mu = \delta X^\mu + \theta^+ \delta\psi_+^\mu + \theta^- \delta\psi_-^\mu + \theta^+\theta^- \delta F^\mu, \tag{174}$$

but also

$$\begin{aligned} \delta\phi^\mu &= i[\epsilon Q, \phi^\mu] = i\epsilon\hat{Q}\phi^\mu = (i\epsilon^+\hat{Q}_+ + i\epsilon^-\hat{Q}_-)\phi^\mu \\ &= (-\epsilon^+\partial_+ + i\epsilon^+\theta^+\partial_{++} - \epsilon^-\partial_- + i\epsilon^-\theta^-\partial_=)\phi^\mu \\ &= (-\epsilon^+\psi_+^\mu - \epsilon^-\psi_-^\mu) + \theta^+(-i\epsilon^+\partial_{++}X^\mu - \epsilon^- F^\mu) + \theta^-(-i\epsilon^-\partial_= X^\mu + \epsilon^+ F^\mu) \\ &\quad + \theta^+\theta^-(-i\epsilon^-\partial_=\psi_+^\mu + i\epsilon^+\partial_{++}\psi_-^\mu), \end{aligned} \tag{175}$$

which means that

$$\begin{aligned}\delta X^\mu &= -\epsilon^+\psi_+^\mu - \epsilon^-\psi_-^\mu, & \delta\psi_+^\mu &= -i\epsilon^+\partial_{++}X^\mu - \epsilon^- F^\mu, \\ \delta\psi_-^\mu &= -i\epsilon^-\partial_{=}X^\mu + \epsilon^+ F^\mu, & \delta F^\mu &= -i\epsilon^-\partial_{=}\psi_+^\mu + i\epsilon^+\partial_{++}\psi_-^\mu.\end{aligned} \tag{176}$$

A second transformation gives

$$\begin{aligned}\delta_{\epsilon_2}\delta_{\epsilon_1}\phi^\mu =& (i\epsilon_2^+\epsilon_1^+\partial_{++}X^\mu + i\epsilon_2^-\epsilon_1^-\partial_{=}X^\mu + \epsilon_2^+\epsilon_1^- F^\mu - \epsilon_2^-\epsilon_1^+ F^\mu) \\ &+ \theta^+(i\epsilon_2^+\epsilon_1^+\partial_{++}\psi_+^\mu + i\epsilon_2^+\epsilon_1^-\partial_{++}\psi_-^\mu - i\epsilon_2^-\epsilon_1^+\partial_{++}\psi_-^\mu + i\epsilon_2^-\epsilon_1^-\partial_{=}\psi_+^\mu) \\ &+ \theta^-(i\epsilon_2^-\epsilon_1^+\partial_{=}\psi_+^\mu + i\epsilon_2^-\epsilon_1^-\partial_{=}\psi_-^\mu + i\epsilon_2^+\epsilon_1^+\partial_{++}\psi_-^\mu + i\epsilon_2^+\epsilon_1^-\partial_{=}\psi_+^\mu) \\ &+ \theta^+\theta^-(-\epsilon_2^-\epsilon_1^+\partial_{=}\partial_{++}X^\mu + i\epsilon_2^-\epsilon_1^-\partial_{=}F^\mu + \epsilon_2^+\epsilon_1^-\partial_{++}\partial_{=}X^\mu \\ &+ i\epsilon_2^+\epsilon_1^+\partial_{++}F^\mu),\end{aligned} \tag{177}$$

so that

$$\begin{aligned}[\delta_{\epsilon_2},\delta_{\epsilon_1}]\phi^\mu =& (2i\epsilon_2^+\epsilon_1^+\partial_{++} + 2i\epsilon_2^-\epsilon_1^-\partial_{=}X^\mu) + \theta^+(2i\epsilon_2^+\epsilon_1^+\partial_{++}\psi_+^\mu + 2i\epsilon_2^-\epsilon_1^-\partial_{=}\psi_+^\mu) \\ &+ \theta^-(2i\epsilon_2^+\epsilon_1^+\partial_{++}\psi_-^\mu + 2i\epsilon_2^-\epsilon_1^-\partial_{=}\psi_-^\mu) \\ &+ \theta^+\theta^-(2i\epsilon_2^+\epsilon_1^+\partial_{++}F^\mu + 2i\epsilon_2^-\epsilon_1^-\partial_{=}F^\mu) \\ =& (2\epsilon_2^+\epsilon_1^+P_{++} + 2\epsilon_2^-\epsilon_1^-P_{=})\phi^a,\end{aligned} \tag{178}$$

which is exactly what we would have expected since

$$\begin{aligned}[\delta_{\epsilon_2},\delta_{\epsilon_1}]\phi^\mu &= [i\epsilon_2 Q, i\epsilon_1 Q]\phi^\mu = [i\epsilon_2^+Q_+ + i\epsilon_2^-Q_-, i\epsilon_1^+Q_+ + i\epsilon_1^-Q_-]\phi^\mu \\ &= (\epsilon_2^+\epsilon_1^+\{Q_+,Q_+\} + \epsilon_2^-\epsilon_1^-\{Q_-,Q_-\})\phi^\mu = 2(\epsilon_2^+\epsilon_1^+P_{++} + \epsilon_2^-\epsilon_1^-P_{=})\phi^\mu.\end{aligned} \tag{179}$$

In fact, letting the operators work on the corresponding superfield-expansion (like (173) on (171)) they will never lead us out of the superfield, and we say that the superspace formalism is manifestly supersymmetric.

8.1.2 The sigma model

The bosonic non-supersymmetric sigma model in two dimensions reads

$$S = \int d^2x \partial_{++}X^\mu E_{\mu\nu}\partial_{=}X^\nu. \tag{180}$$

To make the sigma model supersymmetric we make an ansatz

$$S = \int d^2x d^2\theta D_+^m X^\mu E_{\mu\nu} D_-^n X^\nu, \tag{181}$$

and try to find $m, n \in \mathbb{N}$ such that the non-supersymmetric model can be recovered in some sense by reducing the supersymmetry. We try $m = 1, n = 1$

$$D_+\phi^\mu = \big(\frac{\partial}{\partial\theta^+} + i\theta^+\partial_{++}\big)\big(X^\mu + \theta^+\psi_+^\mu + \theta^-\psi_-^\mu + \theta^+\theta^- F^\mu\big)$$

$$
\begin{aligned}
&= \psi_+^\mu + \theta^- F^\mu + i\theta^+ \partial_{++} X^\mu + i\theta^+\theta^- \partial_{++}\psi_-^\mu \\
D_-\phi^\nu &= \big(\frac{\partial}{\partial\theta^-} + i\theta^-\partial_=\big)\big(X^\nu + \theta^-\psi_+^\nu + \theta^-\psi_-^\nu + \theta^+\theta^- F^\nu\big) \\
&= \psi_-^\nu - \theta^+ F^\nu + i\theta^-\partial_= X^\nu - i\theta^+\theta^-\partial_=\psi_+^\nu \qquad (182)
\end{aligned}
$$

Since

$$
D_+\phi^\mu D_-\phi^\nu = -\theta^+\theta^-\partial_{++}X^\mu\partial_= X^\nu + \ldots \qquad (183)
$$

already $m = 1, n = 1$ may return the bosonic action as one of its terms. We check this.

$$
\begin{aligned}
S &= \int d^2x d^2\theta D_+\phi^\mu E_{\mu\nu} D_-\phi^\nu \\
&= \int d^2x D_+ D_-(D_+\phi^\mu E_{\mu\nu} D_-\phi^\nu)| \\
&= \int d^2x D_+(D_-D_+\phi^\mu E_{\mu\nu} D_-\phi^\nu - D_+\phi^\mu E_{\mu\nu,\tau} D_-\phi^\tau D_-\phi^\nu \\
&\quad + D_+\phi^\mu E_{\mu\nu} D_-D_-\phi^\nu)| \\
&= \int d^2x (D_+D_-D_+\phi^\mu E_{\mu\nu} D_-\phi^\nu + D_-D_+\phi^\mu E_{\mu\nu,\tau} D_+\phi^\tau D_-\phi^\nu \\
&\quad + D_-D_+\phi^\mu E_{\mu\nu} D_+D_-\phi^\nu - D_+D_+\phi^\mu E_{\mu\nu,\tau} D_-\phi^\tau D_- X^\nu \\
&\quad + D_+\phi^\mu E_{\mu\nu,\tau\sigma} D_+\phi^\sigma D_- X^\tau D_-\phi^\nu + D_+\phi^\mu E_{\mu\nu,\tau} D_+D_-\phi^\tau D_-\phi^\nu \\
&\quad + D_+D_+\phi^\mu E_{\mu\nu} D_-D_-\phi^\nu - D_+\phi^\mu E_{\mu\nu,\tau} D_+\phi^\tau D_-D_-\phi^\nu \\
&\quad + D_+\phi^\mu E_{\mu\nu} D_+D_-D_-\phi^\nu)| \qquad (184)
\end{aligned}
$$

The seventh term gives us

$$
\int d^2x D_+D_+\phi^\mu E_{\mu\nu} D_-D_-\phi^\nu| = -\int d^2x \partial_{++}X^\mu E_{\mu\nu}\partial_= X^\nu \qquad (185)
$$

which is the bosonic action, and we conclude that we can write the two-dimensional $N = (1, 1)$ action as

$$
S = \int d^2x d^2\theta D_+\phi^\mu E_{\mu\nu} D_-\phi^\nu. \qquad (186)
$$

8.1.3 The equations of motion

The equations of motion are derived from $\delta S = 0$, leading to the Euler-Lagrange equations $D_i\Big(\frac{\partial\mathcal{L}}{\partial(D_i\phi^\mu)}\Big) - \frac{\partial\mathcal{L}}{\partial\phi^\mu} = 0$. We have

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial(D_+\phi^\mu)} &= \frac{\partial(D_+\phi^\sigma)}{\partial(D_+\phi^\mu)} E_{\sigma\tau} D_-\phi^\tau - D_+\phi^\sigma \frac{\partial E_{\sigma\tau}}{\partial(D_+\phi^\mu)} D_-\phi^\tau - D_+\phi^\sigma E_{\sigma\tau} \frac{\partial(D_-\phi^\tau)}{\partial(D_+\phi^\mu)} \\
&= \delta^\sigma_\mu E_{\sigma\tau} D_-\phi^\tau - 0 - 0 = E_{\mu\tau} D_-\phi^\tau \qquad (187)
\end{aligned}
$$

$$
\frac{\partial\mathcal{L}}{\partial(D_-\phi^\mu)} = \frac{\partial(D_+\phi^\sigma)}{\partial(D_-\phi^\mu)} E_{\sigma\tau} D_-\phi^\tau - D_+\phi^\sigma \frac{\partial E_{\sigma\tau}}{\partial(D_-\phi^\mu)} D_-\phi^\tau - D_+\phi^\sigma E_{\sigma\tau} \frac{\partial(D_-\phi^\tau)}{\partial(D_-\phi^\mu)}
$$

$$= 0 - 0 - D_+\phi^\sigma E_{\sigma\tau}\delta^\tau_\mu = -D_+\phi^\sigma E_{\sigma\mu} \tag{188}$$

$$D_+\Big(\frac{\partial\mathcal{L}}{\partial(D_+\phi^\mu)}\Big) = D_+(E_{\mu\tau}D_-\phi^\tau) = E_{\mu\tau,\kappa}D_+\phi^\kappa D_-\phi^\tau + E_{\mu\tau}D_+D_-\phi^\tau \tag{189}$$

$$D_-\Big(\frac{\partial\mathcal{L}}{\partial(D_-\phi^\mu)}\Big) = D_-(-D_+\phi^\sigma E_{\sigma\mu}) = D_+D_-\phi^\sigma E_{\sigma\mu} + D_+\phi^\sigma E_{\sigma\mu.\kappa}D_-\phi^\kappa \tag{190}$$

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\phi^\mu} &= \frac{\partial(D_+\phi^\sigma)}{\partial\phi^\mu}E_{\sigma\tau}D_-\phi^\tau + D_+\phi^\sigma\frac{\partial E_{\sigma\tau}}{\partial\phi^\mu}D_-\phi^\tau + D_+\phi^\sigma E_{\sigma\tau}\frac{\partial(D_-\phi^\tau)}{\partial\phi^\mu}\\ &= 0 + D_+\phi^\sigma E_{\sigma\tau,\mu}D_-\phi^\tau + 0 = D_+\phi^\sigma E_{\sigma\tau,\mu}D_-\phi^\tau \end{aligned}\tag{191}$$

Combining these we have

$$\begin{aligned} 0 &= D_i\Big(\frac{\partial\mathcal{L}}{\partial(D_i\phi^\mu)}\Big) - \frac{\partial\mathcal{L}}{\partial\phi^\mu}\\ &= E_{\mu\tau,\kappa}D_+\phi^\kappa D_-\phi^\tau + E_{\mu\tau}D_+D_-\phi^\tau + D_+D_-\phi^\sigma E_{\sigma\mu} + D_+\phi^\sigma E_{\sigma\mu,\kappa}D_-\phi^\kappa\\ &\quad - D_+\phi^\sigma E_{\sigma\tau,\mu}D_-\phi^\tau\\ &= (E_{\mu\tau} + E_{\tau\mu})D_+D_-\phi^\tau + (E_{\mu\tau,\sigma} + E_{\sigma\mu.\tau} - E_{\sigma\tau,\mu})D_+\phi^\sigma D_-\phi^\tau\\ &= 2g_{\mu\tau}D_+D_-\phi^\tau + (E_{\mu\tau,\sigma} + E_{\sigma\mu.\tau} - E_{\sigma\tau,\mu})D_+\phi^\sigma D_-\phi^\tau \end{aligned}\tag{192}$$

$$\begin{aligned}\Leftrightarrow 0 &= g^{\kappa\mu}g_{\mu\tau}D_+D_-\phi^\tau + \frac{1}{2}g^{\kappa\mu}(E_{\mu\tau,\sigma} + E_{\sigma\mu,\tau} - E_{\sigma\tau,\mu})D_+\phi^\sigma D_-\phi^\tau\\ &= D_+D_-\phi^\kappa\\ &\qquad + \Big[\frac{1}{2}g^{\kappa\mu}(g_{\mu\tau,\sigma} + g_{\sigma\mu,\tau} - g_{\sigma\tau,\mu}) + \frac{1}{2}g^{\kappa\mu}(B_{\mu\tau,\sigma} + B_{\sigma\mu,\tau} - B_{\sigma\tau,\mu})\Big]D_+\phi^\sigma D_-\phi^\tau\\ &= D_+D_-\phi^\tau + ({\Gamma^{(0)\kappa}}_{\sigma\tau} - \frac{1}{2}g^{\kappa\mu}T_{\mu\sigma\tau})D_+\phi^\sigma D_-\phi^\tau\\ &= D_+D_-\phi^\tau + {\Gamma^{(-)\kappa}}_{\sigma\tau}D_+\phi^\sigma D_-\phi^\tau\\ &= \nabla^{(-)}_+ D_-\phi^\tau. \end{aligned}\tag{193}$$

We have found the equations of motion:

$$\nabla^{(-)}_+ D_-\phi^\tau = 0, \tag{194}$$

which imply that the target-space geometry is Riemannian with torsion.

8.2 $D = 2, N = (1,0)$

We expand the superfield

$$\phi^\mu(x,\theta^+) = X^\mu(x) + \theta^+\psi^\mu_+(x) =: X^\mu(x) + \theta\psi^\mu(x), \tag{195}$$

where

$$X^\mu(x) = \phi^\mu|, \qquad \psi^\mu(x) = D\phi^\mu|, \tag{196}$$

and

$$\hat{Q}_+ := \hat{Q} = i\partial_\theta + \theta\partial_{++}, \qquad D_+ := D = \partial_\theta + i\theta\partial_{++}, \qquad \hat{P}_{\pm\pm} = i\partial_{\pm\pm}. \quad (197)$$

An infinitesimal transformation gives

$$\begin{aligned} \delta\phi^\mu &= [i\epsilon Q, \phi^\mu] = i\epsilon\hat{Q}\phi^\mu = (-\epsilon\partial_\theta + i\epsilon\theta\partial_{++})(X^\mu + \theta\psi^\mu) \\ &= -\epsilon\psi^\mu + i\epsilon\theta\partial_{++}X^\mu = \partial X^\mu + \theta\partial\psi^\mu, \end{aligned} \quad (198)$$

implying that

$$\begin{cases} \delta X^\mu = -\epsilon\psi^\mu \\ \delta\psi^\mu = -i\epsilon\partial_{++}X^\mu. \end{cases} \quad (199)$$

A second transformation

$$\delta_{\epsilon_2}\delta_{\epsilon_1}\phi^\mu = (-\epsilon_2\partial_\theta + i\epsilon_2\theta\partial_{++}X^\mu) = i\epsilon_2\epsilon_1\partial_{++}X^\mu + \theta(i\epsilon_2\epsilon_1\partial_{++}\psi^\mu), \quad (200)$$

shows explicitly that the algebra closes

$$\begin{aligned} [\delta_{\epsilon_2}, \delta_{\epsilon_1}] &= i(\epsilon_2\epsilon_1 - \epsilon_1\epsilon_2)\partial_{++}X^\mu + \theta(i(\epsilon_2\epsilon_1 - \epsilon_1\epsilon_2)\partial_{++}\psi^\mu) \\ &= 2i\epsilon_2\epsilon_1\partial_{++}X^\mu + \theta(2i\epsilon_2\epsilon_1\partial_{++}\psi^\mu) = 2\epsilon_2\epsilon_1\hat{P}_{++}\psi^\mu. \end{aligned} \quad (201)$$

The non-supersymmetric sigma-model reads

$$S = \int d^2x\partial_{++}X^\mu E_{\mu\nu}\partial_= X^\nu, \quad (202)$$

and we try a supersymmetric action

$$\begin{aligned} S' &= \int d^2x d\theta D\phi^\mu E_{\mu\nu}\partial_=\phi^\nu = \int d^2x D(D\phi^\mu E_{\mu\nu}\partial_=\phi^\nu)| \\ &= \int d^2x(D^2\phi^\mu E_{\mu\nu}\partial_=\phi^\nu - D\phi^\mu E_{\mu\nu,\rho}D\phi^\rho\partial_=\phi^\nu - D\phi^\mu E_{\mu\nu}\partial_= D\phi^\nu)| \\ &= \int d^2x(i\partial_{++}\phi^\mu E_{\mu\nu}\partial_=\phi^\nu - \psi^\mu E_{\mu\nu,\rho}\psi^\rho\partial_=\phi^\nu - \psi^\mu E_{\mu\nu}\partial_=\psi^\nu). \end{aligned} \quad (203)$$

We see that (202) is contained in an action

$$S = -i\int d^2x d\theta D\phi^\mu E_{\mu\nu}\partial_=\phi^\nu. \quad (204)$$

We derive the equations of motion:

$$\begin{aligned} D\frac{\partial\mathcal{L}}{\partial(D\phi^\rho)} &= D(E_{\rho\nu}\partial_=\phi^\nu) = E_{\rho\nu,\mu}D\phi^\mu\partial_=\phi^\nu + E_{\rho\nu}\partial_= D\phi^\nu \\ \partial_=\frac{\partial\mathcal{L}}{\partial(\partial_=\phi^\rho)} &= \partial_=(D\phi^\mu E_{\mu\rho}) = \partial_= D\phi^\mu E_{\mu\rho} + D\phi^\mu E_{\mu\rho,\nu}\partial_=\phi^\nu \\ \frac{\partial\mathcal{L}}{\partial\phi^\rho} &= D\phi^\mu E_{\mu\nu,\rho}\partial_=\phi^\nu, \end{aligned} \quad (205)$$

leading to

$$\begin{aligned}
0 &= D\frac{\partial\mathcal{L}}{\partial(D\phi^\rho)} + \partial_= \frac{\partial\mathcal{L}}{\partial(\partial_=\phi^\rho)} - \frac{\partial\mathcal{L}}{\partial\phi^\rho} \\
&= (E_{\mu\rho} + E_{\rho\mu})\partial_= D\phi^\mu + (E_{\rho\nu,\mu} + E_{\mu\rho,\nu} - E_{\mu\nu,\rho})D\phi^\mu\partial_=\phi^\nu \\
&= 2g_{\mu\rho}\partial_= D\phi^\mu + (g_{\rho\nu,\mu} + g_{\mu\rho,\nu} - g_{\mu\nu,\rho} + B_{\rho\nu,\mu} + B_{\mu\rho,\nu} + B_{\nu\mu,\rho})D\phi^\mu\partial_=\phi^\nu.
\end{aligned} \tag{206}$$

Multiplying with $\frac{1}{2}g^{\sigma\rho}$ gives

$$\begin{aligned}
0 &= \partial_= D\phi^\sigma + \frac{1}{2}g^{\sigma\rho}\Big(g_{\rho\nu,\mu} + g_{\mu\rho,\nu} - g_{\mu\nu,\rho} + B_{\rho\nu,\mu} + B_{\mu\rho,\nu} + B_{\nu\mu,\rho}\Big)D\phi^\mu\partial_=\phi^\nu \\
&= \partial_= D\phi^\sigma + \Big(\Gamma^{(0)\sigma}{}_{\mu\nu} - \frac{1}{2}g^{\rho\sigma}T_{\mu\nu\rho}\Big)D\phi^\mu\partial_=\phi^\nu \\
&= \partial_= D\phi^\sigma + \Big(\Gamma^{(0)\sigma}{}_{\mu\nu} + \frac{1}{2}g^{\rho\sigma}T_{\mu\nu\rho}\Big)\partial_=\phi^\mu D\phi^\nu \\
&= \partial_= D\phi^\sigma + \Gamma^{(+)\sigma}{}_{\mu\nu}\partial_=\phi^\mu D\phi^\nu \\
&= \nabla^{(+)}_= D\phi^\sigma,
\end{aligned} \tag{207}$$

where in the third line we have used that

$$\Gamma^{(0)\sigma}{}_{\mu\nu} = \Gamma^{(0)\sigma}{}_{\nu\mu}, \qquad T_{\mu\nu\rho} = -T_{\nu\mu\rho}, \tag{208}$$

and relabelled $\mu \leftrightarrow \nu$. Hence the geometry of the target space is Riemannian with torsion.

8.3 $D = 1, N = 1$

The bosonic non-supersymmetric sigma model in one dimension reads:

$$S = \int dt g_{\mu\nu}\dot{X}^\mu\dot{X}^\nu \tag{209}$$

We make an ansatz

$$S = \int dt d\theta g_{\mu\nu} D^m\phi^\mu D^n\phi^\nu, \tag{210}$$

and try to find $m, n \in \mathbb{N}$ such that the non-supersymmetric sigma model can be recovered by reducing the supersymmetry.

$$D\phi^\mu = \big(\frac{\partial}{\partial\theta} + i\theta\frac{\partial}{\partial t}\big)\big(X^\mu(t) + \theta\lambda^\mu(t)\big) = \lambda^\mu + i\theta\dot{X}^\mu \tag{211}$$

At first it seems that already $m = 1, n = 1$ may give such a term, but since $\theta\theta = 0$ we have to discard this. Next we try $n = 2$:

$$D^2\phi^\nu = i\frac{\partial}{\partial t}\phi^\nu = i\dot{X}^\nu - i\theta\dot{\lambda}^\nu \tag{212}$$

and we actually find a suitable term:

$$D\phi^\mu D^2\phi^\nu = -\theta\dot{X}^\mu\dot{X}^\nu + \dots \tag{213}$$

We perform a more formal check.

$$\begin{aligned}
S &= -\frac{1}{2}\int dtd\theta g_{\mu\nu}D\phi^\mu D^2\phi^\nu \\
&= \frac{i}{2}\int dtd\theta g_{\mu\nu}D\phi^\mu\dot{\phi}^\nu \\
&= -\frac{i}{2}\int dtD(g_{\mu\nu}D\phi^\mu\dot{\phi}^\nu)| \\
&= -\frac{i}{2}\int dt(g_{\mu\nu,\tau}D\phi^\tau D\phi^\mu\dot{\phi}^\nu + g_{\mu\nu}D^2\phi^\mu\dot{\phi}^\nu - g_{\mu\nu}D\phi^\mu D\dot{\phi}^\nu)| \\
&= \int dt(-\frac{i}{2}g_{\mu\nu,\tau}\lambda^\tau\lambda^\mu\dot{X}^\nu + \frac{1}{2}g_{\mu\nu}\dot{X}^\mu\dot{X}^\nu + \frac{i}{2}g_{\mu\nu}\lambda^\mu\dot{\lambda}^\nu)
\end{aligned} \tag{214}$$

The second term is indeed the bosonic action, and we state our result again:

$$S = \frac{i}{2}\int dtd\theta g_{\mu\nu}D\phi^\mu\dot{\phi}^\nu \tag{215}$$

9 Extending and reducing supersymmetries and dimensions

9.1 Going between $N = (1,1)$ and $N = (2,2)$ sigma models in $D = 2$

A $N = (1,1)$ sigma model in $D = 2$

$$S = \int d^2xd^2\theta D_+\phi^\mu E_{\mu\nu}(\phi)D_-\phi^\nu, \tag{216}$$

can be extended to a $N = (2,2)$ sigma model by an ansatz

$$\delta_2\phi = \epsilon^+D_+\phi^\nu J_\nu^{(+)\mu} + \epsilon^-D_-\phi^\nu J_\nu^{(-)\mu}, \tag{217}$$

that should fulfill

$$\begin{aligned}
[\delta_2^\pm(\epsilon_1^\pm), \delta_2^\pm(\epsilon_2^\pm)] &= -2i\epsilon_1^\pm\epsilon_2^\pm\partial_{\pm\pm} \\
[\delta_1, \delta_2] &= 0 \\
[\delta_2^\pm(\epsilon_1^\pm), \delta_2^\mp(\epsilon_2^\mp)] &= 0.
\end{aligned} \tag{218}$$

Then the $N = (1,1)$ action (216) is invariant if and only if

$$J^{(\pm)} \text{ are almost complex structures} \tag{219}$$

$$J^{(\pm T}GJ^{(\pm)} = G \text{ (leaves the metric invariant)} \tag{220}$$

$$J^{(\pm)\mu}_{[\lambda}J^{(\pm)\nu}_{\rho}T_{|\mu\nu|\tau]} = T_{\lambda\rho\tau} \text{ (leaves torsion invariant)} \tag{221}$$

$$N^{(\pm)\tau}_{\mu\nu} = J^{(\pm)\sigma}_{\mu}\partial_{[\sigma}J^{(\pm)\tau}_{\nu]} - (\mu \leftrightarrow \nu) = 0 \tag{222}$$

$$\nabla^{(\pm)}_{\tau}J^{(\pm)\mu}_{\nu} = 0 \tag{223}$$

This algebra closes on-shell. If, in addition, $[J^{(+)}, J^{(-)}] = 0$, the algebra also closes off-shell. If, however, $[J^{(+)}, J^{(-)}] \neq 0$, we can make the algebra close off-shell by including additional auxiliary spinorial $N = (1,1)$ fields in the Lagragian.
A $N = (2,2)$ sigma model in $D = 1$

$$S = \int d^2xd^2\theta d^2\theta K(\phi, \phi) \tag{224}$$

can also be reduced to a $N = 1$ model in $D = 2$

$$S = -2\int d^2xd^2\theta \frac{\partial^2 K}{\partial\phi^\mu\partial\bar{\phi}^\nu}D^\alpha\phi^\mu D_\alpha\bar{\phi}^\nu|. \tag{225}$$

9.2 Reduction from $N = (1,1)$ in $D = 2$ to $N = 1$ in $D = 1$

A manifest $N = (1,1)$ sigma model in $D = 2$

$$S = \int d^2xd^2\theta D_+\phi^\mu E_{\mu\nu}D_-\phi^\nu \tag{226}$$

was directly reduced to an $N = 1$ sigma model in one dimension in [5] and then compared to the most general $N = 1$ one-dimensional action. We recalculate the reduction in appendix C and obtain the following action:

$$\begin{aligned} S_R = \int dtd\theta\big[&-iG_{\mu\nu}D\hat{X}^\mu\partial_t\hat{X}^\nu - G_{\mu\nu}\hat{\psi}^\mu D\hat{\psi}^\nu - G_{\mu\nu,\rho}\hat{\psi}^\mu D\hat{X}^\nu\hat{\psi}^\rho \\ &+\frac{1}{6}T_{\mu\nu\rho}\hat{\psi}^\mu\hat{\psi}^\nu\hat{\psi}^\rho - \frac{1}{2}T_{\mu\nu\rho}D\hat{X}^\mu D\hat{X}^\nu\hat{\psi}^\rho + D(b_{\mu\nu}D\hat{X}^\mu\hat{\psi}^\nu)\big], \end{aligned} \tag{227}$$

where $\hat{X}^\mu$ are bosonic superfields and $\hat{\psi}^\mu$ are fermionic superfields. The most general $N = 1$ one-dimensional action is given in [6]

$$\begin{aligned} S_G = \int dtd\theta\big[&-\frac{i}{2}g_{ij}D\phi^i\partial_t\phi^j + \frac{1}{6}c_{ijk}D\phi^iD\phi^jD\phi^k - \frac{1}{2}h_{ab}\psi^aD\psi^b \\ &-\frac{1}{2}h_{ab}\psi^aD\phi^iA_i{}^b{}_c\psi^c + \frac{1}{6}l_{abc}\psi^a\psi^b\psi^c - if_{ia}\partial_t\phi^i\psi^a \\ &+\frac{1}{2}m_{iab}\psi^a\psi^bD\phi^i + \frac{1}{2}n_{ija}D\phi^iD\phi^j\psi^a\big], \end{aligned} \tag{228}$$

where ϕ^i are bosonic superfields, ψ^a are fermionic superfields and $A_i{}^b{}_c$ is a connection between them. Letting the bosonic and fermionic superfields point to the same targetspace, and treat their respective coordinates equally we get

$$S_G = \int dtd\theta\big[-\frac{i}{2}g_{\mu\nu}D\phi^\mu\partial_t\phi^\nu + \frac{1}{6}c_{\mu\nu\rho}D\phi^\mu D\phi^\nu D\phi^\rho - \frac{1}{2}h_{\mu\nu}\psi^\mu D\psi^\nu$$

$$
\begin{aligned}
&- \frac{1}{2} h_{\mu\kappa} A_\nu{}^\kappa{}_\rho \psi^\mu D\phi^\nu \psi^\rho + \frac{1}{6} l_{\mu\nu\rho} \psi^\mu \psi^\nu \psi^\rho - i f_{\mu\nu} \partial_t \phi^\mu \psi^\nu \\
&+ \frac{1}{2} m_{\mu\nu\rho} \psi^\nu \psi^\rho D\phi^\mu + \frac{1}{2} n_{\mu\nu\rho} D\phi^\mu D\phi^\nu \psi^\rho \big] \\
= \int dt d\theta \big[&- \frac{i}{2} g_{\mu\nu} D\phi^\mu \partial_t \phi^\nu + \frac{1}{6} c_{\mu\nu\rho} D\phi^\mu D\phi^\nu D\phi^\rho - \frac{1}{2} h_{\mu\nu} \psi^\mu D\psi^\nu \\
&- \frac{1}{2} (h_{\mu\kappa} A_\nu{}^\kappa{}_\rho + m_{\mu\nu\rho}) \psi^\mu D\phi^\nu \psi^\rho + \frac{1}{6} l_{\mu\nu\rho} \psi^\mu \psi^\nu \psi^\rho - i f_{\mu\nu} \partial_t \phi^\mu \psi^\nu \\
&+ \frac{1}{2} n_{\mu\nu\rho} D\phi^\mu D\phi^\nu \psi^\rho \big].
\end{aligned} \tag{229}
$$

Comparing the reduced action with the most general action we find that all terms except $f_{\mu\nu}\partial_t\phi^\mu\psi^\nu$ can be recovered, provided that

$$
\begin{aligned}
&g_{\mu\nu} = h_{\mu\nu} = 2G_{\mu\nu}, \qquad c_{\mu\nu\rho} = S_{\mu\nu\rho}, \qquad f_{\mu\nu} = 0, \\
&l_{\mu\nu\rho} = -n_{\mu\nu\rho} = T_{\mu\nu\rho}, \qquad h_{\mu\kappa} A_\nu{}^\kappa{}_\rho + m_{\nu\mu\rho} = 2G_{\mu\nu,\rho},
\end{aligned} \tag{230}
$$

where $S_{\mu\nu\rho}$ is totally symmetric, symmetric in two indices or zero. With $S_{\mu\nu\rho} = G_{\mu\nu,\rho}$ this is fulfilled. These results are slightly different from those obtained in [5]. We conclude that the targetspace geometry of the reduced model has additional restrictions compared to the most general case.

A reduction can also be performed via an $N = 2a$ sigma model [5]. The most general such model is

$$
S = \int dt d^2\theta \Big(D_1\phi^\mu E_{\mu\nu} D_2\phi^\nu + l_{\mu\nu} D_1\phi^\mu D_1\phi^\nu + m_{\mu\nu} D_2\phi^\mu D_2\phi^\nu \Big), \tag{231}
$$

where l and m correspond to non-Lorentz invariant terms in $D = 2$. If l and m are set to zero we are back with the dimensional reduced model (227). If, however, they are nonzero a more general model for $N = 1$ can be obtained

$$
\begin{aligned}
S = \int dt d\theta \Big[&- iG_{\mu\nu} D\phi^\mu \dot{\phi}^\nu - (G_{\mu\nu} + s_{\mu\nu}) \psi^\mu \nabla \psi^\nu + \frac{1}{3} S_{\mu\nu\tau} D\phi^\mu D\phi^\nu D^\phi D^\tau \\
&+ (H_{\mu\nu\tau} - T_{\mu\nu\tau})(D\phi^\mu D\phi^\nu \psi^\tau + \frac{1}{3} \psi^\mu \psi^\nu \psi^\tau) - 2it_{\mu\nu} \dot{\phi}^\mu \psi^\nu \Big],
\end{aligned} \tag{232}
$$

where

$$
\begin{aligned}
L_{\mu\nu\tau} &:= \frac{1}{2}(l_{\mu\nu,\tau} + l_{\nu\tau,\mu} + l_{\tau\mu,\nu}) \\
M_{\mu\nu\tau} &:= \frac{1}{2}(m_{\mu\nu,\tau} + m_{\nu\tau,\mu} + m_{\tau\mu,\nu}) \\
s_{\mu\nu} &:= l_{\mu\nu} - m_{\mu\nu} \\
t_{\mu\nu} &:= l_{\mu\nu} + m_{\mu\nu} \\
S_{\mu\nu\tau} &:= L_{\mu\nu\tau} - M_{\mu\nu\tau} \\
T_{\mu\nu\tau} &:= L_{\mu\nu\tau} + M_{\mu\nu\tau}.
\end{aligned} \tag{233}
$$

Again comparing with the most general form (228), we find that all terms can be recovered if

$$\begin{aligned}
&g_{\mu\nu} = G_{\mu\nu}, \qquad h_{\mu\nu} = G_{\mu\nu} + s_{\mu\nu}, \qquad h_{\mu\nu\tau} = G_{\mu\nu,\tau} + S_{\mu\nu\tau}, \\
&f_{\mu\nu} = t_{\mu\nu}, \qquad I_{\mu\nu\tau} = H_{\mu\nu\tau} - T_{\mu\nu\tau}, \qquad n_{\mu\nu\tau = H_{\mu}\nu\tau} - T_{\mu\nu\tau}, \\
&h_{\mu\lambda}(A_\nu)^\lambda_\tau + m_{\nu\mu\tau} = G_{\mu\nu,\tau} + s_{\mu\nu,\tau}.
\end{aligned} \tag{234}$$

But since H, S and T by construction are closed, while all couplings in the general case are arbitrary, the general case is not fully recovered. The conclusion drawn in [5] is that neither the direct reduction from $N = (1,1)$ to $N = 1$ nor a reduction via $N = 2a$ recovers the most general case. Thus restrictions not present in the general case has to be imposed on the final sigma models.

9.3 Reduction from $N = (2,0)$ in $D = 2$ to $N = 1$ in $D = 1$

The most general renormalizable Lorentz-invariant $N = (2,0)$ sigma model in $D = 2$ is given by

$$S = \int d^2x d^2\theta_+ \Big[-\frac{i}{2}(K_\mu \partial_+ \Phi^\mu - K_{\bar{\mu}} \partial_+ \Phi^{\bar{\mu}}) + f_{ab}\Psi^a\Psi^b + f_{a\bar{b}}\Psi^a\Psi^{\bar{b}} + f_{\bar{a}\bar{b}}\Psi^{\bar{a}}\Psi^{\bar{b}})\Big]. \tag{235}$$

By reducing one supersymmetry and writing in terms of real $N = (1,0)$ fields, we get the $N = (1,0)$ model in $D = 2$

$$S = -\int d^2x d\theta_+ \Big[i(G_{\mu\nu} + B_{\mu\nu}) D\phi^\mu \partial_+ \phi^\nu + G_{ab}\psi^a \nabla \psi^b \Big], \tag{236}$$

and by reducing the dimension, finally, we arrive at the $N = 1$ model in $D = 1$

$$S = \int dt d\theta \Big[i G_{\mu\nu} D\phi^\mu \dot{\phi}^\nu + \frac{1}{3} H_{\mu\nu\tau} D\phi^\mu D\phi^\nu D\phi^\tau + G_{ab}\psi^a \nabla \psi^b \Big]. \tag{237}$$

10 Conformal theory

A conformal transformation leaves the angel between any two crossing lines invariant, or in other words the metric is left invariant up to a coordinate dependent scale. This condition can be written as [8]

$$g'_{\rho\sigma}(x') = \Lambda(x) g_{\mu\nu}(x). \tag{238}$$

Note that with $\Lambda(x) = 1$ we are back with the Poincaré group, which actually can be seen as a subgroup of the conformal group. Now, any change in coordinates up to a term $\mathcal{O}(\epsilon^2)$ can be written

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x), \tag{239}$$

which corresponds to a change in the metric:

$$g_{\mu\nu} \to g_{\mu\nu} - (\partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu) \tag{240}$$

Requiring (240) to be a conformal transformation, we have

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(x) g_{\mu\nu}. \tag{241}$$

Then by taking the trace

$$f(x) = \frac{2}{D} \partial_\rho \epsilon^\rho, \tag{242}$$

where D is the dimension of the metric. We insert (241) in (242) and get

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{D} \partial_\rho \epsilon^\rho g_{\mu\nu}. \tag{243}$$

From this relation we can find (see [8] for details)

$$(D-1) \partial^\mu \partial_\mu \partial_\nu \epsilon^\nu = 0. \tag{244}$$

10.1 $D \geq 3$

In $D \geq 3$ (244) tells us that ϵ_μ is at most quadratic in x:

$$\epsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho \tag{245}$$

The first term gives an infinitesimal translation $x'^\mu = x^\mu + a^\mu$, which has D generators

$$P_\mu = -i \partial_\mu. \tag{246}$$

The second term can be divided into an even and an odd term. The odd term corresponds to an infinitesimal rotation $x'^\mu = \delta^\mu_\nu + m^\mu{}_\nu$ and have $\frac{D(D-1)}{D}$ generators

$$L_{\mu\nu} = i(x^\mu \partial_\nu - x^\nu \partial_\mu). \tag{247}$$

The even term corresponds to a new transformation, the infinitesimal scale or dilation transformation $x'^\mu = (1+\alpha) x^\mu$, which has only one generator

$$D = -i x^\mu \partial_\mu. \tag{248}$$

Finally, the last term corresponds to another new transformation, the infinitesimal Special Conformal Transformation (SCT), $x'^\mu = x^\mu + 2x^\nu b_\nu x^\mu - x^\nu x_\nu b^\mu$, where $b_\mu := c^\rho{}_{\rho\mu}$. The corresponding D generators are written

$$K_\mu = -i(2 x_\mu x^\nu \partial_\nu - x^\nu x_\nu \partial_\mu). \tag{249}$$

The finite transformations are written as follows:

$$\begin{aligned}
x'^\mu &= x^\mu + a^\mu && \text{translation} \\
x'^\mu &= M^\mu{}_\nu x^\nu && \text{rotation} \\
x'^\mu &= \alpha x^\mu && \text{dilation} \\
x'^\mu &= \frac{x^\mu - x^\nu x_\nu b^\mu}{1 - 2 b^\rho x_\rho + b^\sigma b_\sigma x^\tau x_\sigma} && \text{SCT}
\end{aligned} \tag{250}$$

We finally state the commutation relations for the generators, which define the conformal group [7]:

$$\begin{aligned}
&[D, P_\mu] = iP_\mu \\
&[D, K_\mu] = -iK_\mu \\
&[K_\mu, P_\nu] = 2i(g_{\mu\nu}D - L_{\mu\nu}) \\
&[K_\rho, L_{\mu\nu}] = i(g_{\rho\mu}K_\nu - g_{\rho\nu}K_\mu) \\
&[P_\rho, L_{\mu\nu}] = i(g_{\rho\mu}P_\nu - g_{\rho\nu}P_\mu) \\
&[L_{\mu\nu}, L_{\rho\sigma}] = i(g_{\nu\rho}L_{\mu\sigma} + g_{\mu\sigma}L_{\nu\rho} - g_{\mu\rho}L_{\nu\sigma} - g_{\nu\sigma}L_{\mu\rho})
\end{aligned} \tag{251}$$

10.2 $D = 2$

As seen from the discussion on general dimensions the condition

$$\eta_{\rho\sigma}\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu} = \Lambda(x)\eta_{\mu\nu} \tag{252}$$

and the coordinate change

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x), \tag{253}$$

lead to

$$\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = \frac{2}{D}(\partial\cdot\epsilon)\eta_{\mu\nu}. \tag{254}$$

In two dimensions and after a Wick rotation to a Euclidean metric ($\eta_{\mu\nu} = diag(+1, +1)$) this becomes

$$\partial_0\epsilon_0 = \partial_1\epsilon_1, \qquad \partial_0\epsilon_1 = -\partial_1\epsilon_0, \tag{255}$$

which we recognize as the Cauchy-Riemann relations (37). We therefore introduce complex coordinates

$$\begin{aligned}
z &= x^0 + ix^1, \qquad \epsilon = \epsilon^0 + i\epsilon^1, \qquad \partial_z = \frac{1}{2}(\partial_0 - i\partial_1) \\
\bar{z} &= x^0 - ix^1, \qquad \bar{\epsilon} = \epsilon^0 - i\epsilon^1, \qquad \partial_{\bar{z}} = \frac{1}{2}(\partial_0 + i\partial_1)
\end{aligned} \tag{256}$$

in which (255) becomes

$$\partial_z\bar{\epsilon} = -\partial_{\bar{z}}\epsilon, \qquad \partial_z\epsilon = \partial_{\bar{z}}\bar{\epsilon} \tag{257}$$

A Laurent expansion around $z = 0$ of a conformal transformation yields

$$\begin{aligned}
z' &= z + \epsilon(z) = z + \sum_{n\in\mathbb{Z}}\epsilon_n(-z^{n+1}), \qquad \epsilon_n \text{ constants}, \\
\bar{z}' &= \bar{z} + \bar{\epsilon}(\bar{z}) = \bar{z} + \sum_{n\in\mathbb{Z}}\bar{\epsilon}_n(-\bar{z}^{n+1}), \qquad \bar{\epsilon}_n \text{ constants},
\end{aligned} \tag{258}$$

and the infinite number of generators are given by

$$l_n = -z^{n+1}\partial_z, \qquad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}. \tag{259}$$

They obey the so-called Witt algebra

$$\begin{aligned} [l_m, l_n] &= (m-n)l_{m+n} \\ [\bar{l}_m, \bar{l}_n] &= (m-n)\bar{l}_{m+n} \\ [l_m, \bar{l}_n] &= 0. \end{aligned} \tag{260}$$

We interpret l_{-1}, $l_0 + \bar{l}_0$, $i(l_0 - \bar{l}_0)$, and l_{+1} as the generators of translations, dilations, rotations and special conformal transformations respectively. Note the ambiguities that arise at $z = 0$ and $z = \infty$. In fact only $\{l_{-1}, l_0, l_{+1}\}$ are globally defined on the Riemann sphere $S^2 = \mathbb{C} \cup \infty$.

Allowing for a central charge c, the Witt algebra can be extended to the Viraso algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m-1)m(m+1)\delta_{m+n,0}, \tag{261}$$

which reduces back to the Witt algebra when $m, n = -1, 0, 1$.

A field $\phi(z, \bar{z})$ is called a primary field of conformal dimension $(h, \bar{h})$ if it, under a conformal transformation $z \to f(z)$, transforms as

$$\phi(z,\bar{z}) \to \phi'(z,\bar{z}) = \Big(\frac{\partial f}{\partial z}\Big)^h \Big(\frac{\partial \bar{f}}{\partial \bar{z}}\Big)^{\bar{h}} \phi(f(z), \bar{f}(\bar{z})). \tag{262}$$

Up to second order in ϵ this means

$$\phi(z,\bar{z}) \to \phi(z,\bar{z}) + (h\partial_z\epsilon + \epsilon\partial_z + \bar{h}\partial_{\bar{z}}\bar{\epsilon} + \bar{\epsilon}\partial_{\bar{z}})\phi(z,\bar{z}), \tag{263}$$

or

$$\delta_{\epsilon,\bar{\epsilon}}\phi(z,\bar{z}) = (h\partial_z\epsilon + \epsilon\partial_z + \bar{h}\partial_{\bar{z}}\bar{\epsilon} + \bar{\epsilon}\partial_{\bar{z}})\phi(z,\bar{z}). \tag{264}$$

The Laurent expansion around $z = \bar{z} = 0$ reads

$$\phi(z,\bar{z}) = \sum_{n,\bar{m}\in\mathbb{Z}} z^{-n-h}\bar{z}^{\bar{m}-\bar{h}}\phi_{n,\bar{m}}. \tag{265}$$

Fields with only z-dependence, $\phi = \phi(z)$, are called chiral or holomorphic fields, while fields with only $\bar{z}$-dependence are called anti-chiral or anti-holomorphic.

10.3 The Polyakov action revisited

We return for a moment to the Polyakov action ((107) with (104) inserted and a string tension $T = 1$)

$$S = -\frac{1}{2}\int d^2x\sqrt{-h}h^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}. \tag{266}$$

This action is invariant under the following symmtery transformations:

1. Poincaré transformations in target space

$$\delta X^\mu = \omega^\mu{}_\nu X^\nu + a^\mu, \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}, \qquad \delta h^{ab} = 0. \tag{267}$$

2. Reparametrizations of the world sheet coordinates

$$x^a \to f^a(x), \qquad h_{ab} = \frac{\partial f^c}{\partial x^a}\frac{\partial f^d}{\partial x^b} h_{cd}. \tag{268}$$

3. Weyl transformations on the world sheet, i.e., rescaling

$$h_{ab} \to e^{g(x)} h_{ab}, \qquad \delta X^\mu = 0. \tag{269}$$

Combining reparametrizations and Weyl transformations we see that the two-dimensional Polyakov action indeed is invariant under conformal transformations (cf. (238)). This is what allowed us to choose the conformal gauge in section 6.1. The reasoning goes as follows. The world-sheet metric h_{ab} has four componenets, but since it is symmetric only three are independent. By reparametrization two of the remaining components can be choosen arbitrarly and the remaining components can be gauged away by a Weyl transformation. Thus a flat world-sheet metric, such as the Minkowski metric

$$\eta_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{270}$$

can be choosen. In section 6.1 we then chose light-cone coordinates which gave us the action (110). Here we will, however, follow the previous subsection and perform a Wick rotation to Euclidean world-sheet coordinates and then introduce complex coordinates. The bosonic non-linear sigma model then takes the form

$$S = \int dz d\bar{z} \partial_z X^\mu(z,\bar{z}) \partial_{\bar{z}} X^\nu(z,\bar{z}) E_{\mu\nu}(X). \tag{271}$$

This is conformally invariant if X^μ are primary fields with vanishing conformal dimensions $(h,\bar{h}) = (0,0)$, i.e.,

$$X'^\mu(f(z),\bar{f}(\bar{z})) = X^\mu(z,\bar{z}). \tag{272}$$

10.4 $D = 1$

Next we will look at a subgroup $Sl(2,\mathbb{R})$ of the one-dimensional conformal group Conf($\mathbb{R}$). This subgroup contains the usual translations P, dilations $\hat{D}$ and special conformal transformations K. In [3] the algebra takes the form

$$\begin{aligned} [\hat{D}, K] &= K \\ [P, \hat{D}] &= P \\ [P, K] &= \frac{1}{2}\hat{D}, \end{aligned} \tag{273}$$

and a Lagrangian

$$L = \frac{1}{2} g_{\mu\nu} \partial_t \phi^\mu \partial_t \phi^\nu + A_\mu \partial_t \phi^\mu - V(\phi), \tag{274}$$

of a non-relativistic spinning particle in a magnetic field A and with a scalar potential V, is shown to be invariant (appendix **??**) under a transformation

$$\partial_\epsilon \phi^\mu = -\epsilon a(t) \partial_t \phi + \epsilon X^\mu(t, \phi), \tag{275}$$

provided X^μ is a vector field on the target space satisfying the following conditions:

$$\begin{aligned} \nabla_{(\mu} X_{\nu)} &= \frac{1}{2} \partial_t a g_{\mu\nu} \\ \partial_t X^\nu g_{\nu\mu} + X^\nu F_{\nu\mu} &= \partial_\mu f \\ \partial_t a V + X^\mu \partial_\mu V &= -\partial_t f, \end{aligned} \tag{276}$$

where $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$, a is a generator of the $Sl(2, \mathbb{R})$ group and the function $f = f(t, \phi)$ arises because the invariance is up to a surface term. This means that a homothetic motion generated by the vector field X is needed. Another possibility is the existence of two commuting homothetic motions generated by the vector fields Y and Z on the target space.

In [4] the $Sl(2, \mathbb{R})$ algebra is parametrized by

$$\epsilon(t) = \epsilon_P + 2t\epsilon_D + t^2 \epsilon_K, \tag{277}$$

and takes the form

$$[\delta_{\epsilon_2}, \delta_{\epsilon_1}] = \delta_{(\epsilon_2 \dot{\epsilon}_1 - \epsilon_1 \dot{\epsilon}_2)}. \tag{278}$$

Defining generators

$$\delta_{\epsilon_P} = i\epsilon_P P, \qquad \delta_{2\epsilon_D t} = i\epsilon_D \hat{D}, \qquad \delta_{\epsilon_K t^2} = i\epsilon_K K, \tag{279}$$

this gives

$$[P, \hat{D}] = -2iP, \qquad [P, K] = -i\hat{D}, \qquad [\hat{D}, K] = -2iK. \tag{280}$$

With a bosonic superfield

$$\Phi^\mu = X^\mu + i\theta\lambda^\mu, \tag{281}$$

and a fermionic superfield

$$\Psi^A = i\psi^A + i\theta F^A, \tag{282}$$

the conformal transformations are asserted to be

$$\delta_\epsilon X^\mu = -\epsilon \dot{X}^\mu + \frac{1}{2} \dot{\epsilon} D^\mu(X),$$

$$\delta_\epsilon \lambda^\mu = -\epsilon\dot{\lambda}^\mu + \frac{1}{2}\dot{\epsilon}A^\mu(X,\lambda),$$
$$\delta_\epsilon \psi^A = -\epsilon\dot{\psi}^A + \frac{1}{2}\dot{\epsilon}E^A(X,\psi),$$
$$\delta_\epsilon F^A = -\epsilon\dot{F}^A + \frac{1}{2}\dot{\epsilon}B^A(X,F), \tag{283}$$

for some vector fields D, A, E and B.

11 Superconformal theory

11.1 $D = 2$, $N = (1,1)$

The $N = (1,1)$ action

$$S = \int d^2x d^2\theta D_+\phi^\mu E_{\mu\nu} D_-\phi^\mu, \tag{284}$$

is classically conformally invariant. However, going to the quantum case this is no longer the case. It can be shown [10] that (284) is conformally invariant at one-loop if there exists a function Φ such that

$$R^{(+)}_{\mu\nu} - 2\nabla_{(\mu}\nabla_{\nu)}\Phi - 2T_{\mu\nu}{}^\kappa\nabla_\kappa\Phi = 0. \tag{285}$$

11.2 $D = 1$, $N = 1$

11.2.1 Introduction

We will consider the $Osp(1|2)$ subgroup of the conformal group. This group contains the generators for translation P, dilation $\hat{D}$, special conformal transformation K, supersymmetry transformation Q, and the special superconformal transformation S, and has the following algebra:

$$\begin{array}{lllll}
[P,P]=0, & [P,\hat{D}]=2iP, & [P,K]=i\hat{D}, & [P,Q]=0, & [P,S]=-Q, \\
 & [\hat{D},\hat{D}]=0, & [\hat{D},K]=2iK, & [\hat{D},Q]=-iQ, & [\hat{D},S]=S, \\
 & & [K,K]=0, & [K,Q]=S, & [K,S]=0, \\
 & & & \{Q,Q\}=2P, & \{Q,S\}=-i\hat{D}, \\
 & & & & \{S,S\}=-2iK.
\end{array} \tag{286}$$

The generators then take the form

$$\begin{array}{lll}
P = i\partial_t, & \hat{D} = 2it\partial_t + i\theta\partial_\theta, & K = it^2\partial_t + it\theta\partial_\theta, \\
 & Q = \theta\partial_t + i\partial_\theta, & S = -it\theta\partial_t + t\partial_\theta.
\end{array} \tag{287}$$

These generators, however, cannot be promoted to operators working on superfields the way we did in equations (163)-(167). How would one, e.g., continue from

$$e^{i\xi\hat{D}}e^{i(tP+\theta Q)} = \cdots = e^{i(t+xt)P+i(\theta+\frac{1}{2}\theta\xi)Q+i\xi\hat{D}}, \tag{288}$$

to find the coordinate changes and from them the operators? One way is perhaps to consider a quotient space where we mod out not only the Lorentz group but also the dilations and the special conformal transformations. This route we will not follow here. Instead we will follow section 10.4 and let a closed homothety D^μ in the target space act on the superfields.

Extending the conformal transformations (283) it is shown in [4] that N separate $Osp(1|2)$ algebras is satisfied by the conformal transformations

$$\begin{aligned}
\delta_\epsilon X^\mu &= -\epsilon \dot{X}^\mu + \frac{1}{2}\dot{\epsilon} D^\mu(X)\\
\delta_\epsilon \lambda^\mu &= -\epsilon \dot{\lambda}^\mu + \frac{1}{2}\dot{\epsilon}(D^\mu{}_{,\nu}\lambda^\nu - \lambda^\mu)\\
\delta_\epsilon \psi^A &= -\epsilon \dot{\psi}^A - \frac{1}{2}\dot{\epsilon}(\beta+1)\psi^A\\
\delta_\epsilon F^A &= -\epsilon \dot{F}^A - \frac{1}{2}\dot{\epsilon}(\beta+2)F^A,
\end{aligned} \tag{289}$$

with

$$Q = \partial_\theta + i\theta\partial_t, \qquad D = \partial_\theta - i\theta\partial_t, \tag{290}$$

and

$$Q_i\Phi^\mu = I_i{}^\mu{}_\nu D\Phi^\nu + e_i{}^\mu{}_A \Psi^A, \qquad Q_i\Psi^A = I_i{}^A{}_B D\Psi^B - e_i{}^A{}_\mu \dot{\Phi}^\mu, \tag{291}$$

as long as

$$\mathcal{L}_D I_i{}^\mu{}_\nu = (\mathcal{L}_D - \beta)e_i{}^\mu{}_A = (\mathcal{L}_D + \beta)e_i{}^\mu{}_A = 0, \tag{292}$$

for each $i = 1, \ldots, N-1$ and β is a constant

Concentrating on the most general $N = 1$ action with only dimensionless coupling, which is cited as

$$\begin{aligned}
S = \int dt d\theta \big(&\frac{i}{2} D\Phi^\mu \dot{\Phi}^\nu + \frac{1}{6} c_{\mu\nu\rho} D\Phi^\mu D\Phi^\nu D\Phi^\rho - \frac{1}{2} h_{AB}\Psi^A D\Psi^B\\
&+ \frac{1}{6} l_{ABC}\Psi^A\Psi^B\Psi^C - i f_{\mu A}\dot{\Phi}^\mu \Psi^A + \frac{1}{2} m_{\mu AB} D\Phi^\mu \Psi^A\Psi^B + \frac{1}{2} n_{\mu\nu A} D\Phi^\mu D\Phi^\nu \Psi^A \big),
\end{aligned} \tag{293}$$

it was found that the conditions for invariance under dilations are

$$\begin{aligned}
(\mathcal{L}_D - 2)g_{\mu\nu} &= 0,\\
(\mathcal{L}_D - 2)c_{\mu\nu\rho} &= 0,\\
\mathcal{L}_D h_{AB} &= (2\beta+2)h_{AB},\\
\mathcal{L}_D l_{ABC} &= (3\beta+2)l_{ABC},\\
\mathcal{L}_D f_{\mu A} &= (\beta+2)f_{\mu A},\\
\mathcal{L}_D m_{\mu AB} &= (2\beta+2)m_{\mu AB},\\
\mathcal{L}_D n_{\mu\nu A} &= (\beta+2)n_{\mu\nu A},
\end{aligned} \tag{294}$$

and that the conditions for invariance under special conformal invariance are

$$
\begin{aligned}
D_\mu &= \partial_\mu K, \\
D^\mu c_{\mu\nu\rho} &= 0, \\
D^\mu m_{\mu AB} &= 0, \\
D^\nu(n_{\mu\nu A} - \nabla_\mu f_{\nu A}) &= 0.
\end{aligned} \tag{295}
$$

11.2.2 Construction of the $Osp(1|2)$ algebra

In this section we will review how to find the transformations of the $Osp(1|2)$ algebra. We follow [4], but with bosonic and fermionic superfields given respectively by

$$\Phi^\mu = X^\mu + \theta\lambda^\mu, \qquad \Psi^A = \psi^A + \theta F^A. \tag{296}$$

Thus, as before, the conformal transformations are parametrized by

$$\epsilon(t) = \epsilon_P + 2t\epsilon_D + t^2\epsilon_K, \tag{297}$$

giving for the $Sl(2,\mathbb{R})$ group

$$[\delta_{\epsilon_2}, \delta_{\epsilon_1}] = \delta_{(\epsilon_2\dot{\epsilon}_1 - \epsilon_1\dot{\epsilon}_2)}. \tag{298}$$

and with generators

$$\delta_{\epsilon_P} = i\epsilon_P P, \qquad \delta_{2\epsilon_D t} = i\epsilon_D \hat{D}, \qquad \delta_{\epsilon_K t^2} = i\epsilon_K K, \tag{299}$$

we get

$$[P, \hat{D}] = -2iP, \qquad [P, K] = -i\hat{D}, \qquad [\hat{D}, K] = -2iK. \tag{300}$$

We make a more general ansatz than in (283) for the conformal transformations, letting D, A, E and B depend on all the fields X, λ, ψ and F

$$
\begin{aligned}
\delta_\epsilon X^\mu &= -\epsilon\dot{X}^\mu + \frac{1}{2}\dot{\epsilon}D^\mu(X,\lambda,\psi,F), \\
\delta_\epsilon \lambda^\mu &= -\epsilon\dot{\lambda}^\mu + \frac{1}{2}\dot{\epsilon}A^\mu(X,\lambda,\psi,F), \\
\delta_\epsilon \psi^A &= -\epsilon\dot{\psi}^A + \frac{1}{2}\dot{\epsilon}E^A(X,\lambda,\psi,F), \\
\delta_\epsilon F^A &= -\epsilon\dot{F}^A + \frac{1}{2}\dot{\epsilon}B^A(X,\lambda,\psi,F).
\end{aligned} \tag{301}
$$

We also note that A and E are Grassmann-odd, which means that they anticommute with λ and ψ. These transformations are easily shown to close under (298). For example

$$
\begin{aligned}
\delta_{\epsilon_2}\delta_{\epsilon_1}X^\mu =& \delta_{\epsilon_2}(-\epsilon_1\dot{X}^\mu + \frac{1}{2}\dot{\epsilon}_1 D^\mu) \\
=& -\epsilon_1\partial_t\delta_{\epsilon_2}X^\mu
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\dot{\epsilon}_1\Big(\frac{\partial D^\mu}{\partial X^\nu}\delta_{\epsilon_2}X^\mu+\frac{\partial D^\mu}{\partial \lambda^\nu}\delta_{\epsilon_2}\lambda^\mu+\frac{\partial D^\mu}{\partial \psi^A}\delta_{\epsilon_2}\psi^A+\frac{\partial D^\mu}{\partial F^A}\delta_{\epsilon_2}F^A\Big)\\
=&\epsilon_1\dot{\epsilon}_2X^\mu+\epsilon_1\epsilon_2\ddot{X}^\mu-\frac{1}{2}\dot{\epsilon}_1\ddot{\epsilon}_2D^\mu\\
&-\frac{1}{2}\epsilon_1\dot{\epsilon}_2\Big(\frac{\partial D^\mu}{\partial X^\nu}\dot{X}^\nu+\frac{\partial D^\mu}{\partial \lambda^\nu}\dot{\lambda}^\nu+\frac{\partial D^\mu}{\partial \psi^A}\dot{\psi}^A+\frac{\partial D^\mu}{\partial F^A}\dot{F}^A\Big)\\
&+\frac{1}{2}\dot{\epsilon}_1\Big(\frac{\partial D^\mu}{\partial X^\nu}(-\epsilon_2\dot{X}^\nu+\frac{1}{2}\dot{\epsilon}_2D^\nu)+\frac{\partial D^\mu}{\partial \lambda^\nu}(-\epsilon_2\dot{\lambda}^\nu+\frac{1}{2}\dot{\epsilon}_2A^\nu)\\
&+\frac{\partial D^\mu}{\partial \psi^A}(-\epsilon_2\dot{\psi}^A+\frac{1}{2}\dot{\epsilon}_2E^A)+\frac{\partial D^\mu}{\partial F^A}(-\epsilon_2\dot{F}^A+\frac{1}{2}\dot{\epsilon}_2B^A\Big)\\
=&\epsilon_1\dot{\epsilon}_2X^\mu+\epsilon_1\epsilon_2\ddot{X}^\mu-\frac{1}{2}\dot{\epsilon}_1\ddot{\epsilon}_2D^\mu\\
&-\frac{1}{2}(\epsilon_1\dot{\epsilon}_2+\dot{\epsilon}_1\epsilon_2)\Big(\frac{\partial D^\mu}{\partial X^\nu}\dot{X}^\nu+\frac{\partial D^\mu}{\partial \lambda^\nu}\dot{\lambda}^\nu+\frac{\partial D^\mu}{\partial \psi^A}\dot{\psi}^A+\frac{\partial D^\mu}{\partial F^A}\dot{F}^A\Big)\\
&+\frac{1}{4}\dot{\epsilon}_1\dot{\epsilon}_2\Big(\frac{\partial D^\mu}{\partial X^\nu}D^\nu+\frac{\partial D^\mu}{\partial \lambda^\nu}A^\nu+\frac{\partial D^\mu}{\partial \psi^A}E^A+\frac{\partial D^\mu}{\partial F^A}B^A\Big),
\end{aligned}
\tag{302}
$$

$$
\begin{aligned}
[\delta_{\epsilon_2},\delta_{\epsilon_1}]=&(\epsilon_1\dot{\epsilon}_2-\epsilon_2\dot{\epsilon}_1)X^\mu+(\epsilon_1\epsilon_2-\epsilon_2\epsilon_1)\ddot{X}^\mu-\frac{1}{2}(\dot{\epsilon}_1\ddot{\epsilon}_2-\dot{\epsilon}_2\ddot{\epsilon}_1)D^\mu\\
&+\frac{1}{2}(\epsilon_1\dot{\epsilon}_2+\dot{\epsilon}_1\epsilon_2-(\epsilon_2\dot{\epsilon}_1+\dot{\epsilon}_2\epsilon_1))\Big(\frac{\partial D^\mu}{\partial X^\nu}\dot{X}^\nu+\frac{\partial D^\mu}{\partial \lambda^\nu}\dot{\lambda}^\nu+\frac{\partial D^\mu}{\partial \psi^A}\dot{\psi}^A+\frac{\partial D^\mu}{\partial F^A}\dot{F}^A\Big)\\
&+\frac{1}{4}(\dot{\epsilon}_1\dot{\epsilon}_2-\dot{\epsilon}_2\dot{\epsilon}_1)\Big(\frac{\partial D^\mu}{\partial X^\nu}D^\nu+\frac{\partial D^\mu}{\partial \lambda^\nu}A^\nu+\frac{\partial D^\mu}{\partial \psi^A}E^A+\frac{\partial D^\mu}{\partial F^A}B^A\Big)\\
=&-(\epsilon_2\dot{\epsilon}_1-\epsilon_1\dot{\epsilon}_2)X^\mu+\frac{1}{2}\partial_t(\epsilon_2\dot{\epsilon}_1-\epsilon_1\dot{\epsilon}_2)D^\mu\\
=&\delta_{(\epsilon_2\dot{\epsilon}_1-\epsilon_1\dot{\epsilon}_2)}X^\mu,
\end{aligned}
\tag{303}
$$

where we have used that $\epsilon_1\epsilon_2=\epsilon_2\epsilon_1$.

We want to extend our model to a supersymmetric one. First we investigate how ordinary bosonic and fermionic fields acts under a supersymmetry transformation

$$
\begin{aligned}
\delta_\zeta\Phi^\mu=i\zeta Q\Phi^\mu=i\zeta(i\partial_\theta+\theta\partial_t)(X^\mu+\theta\lambda^\mu)=-\zeta\lambda^\mu+\theta(-i\zeta\dot{X}^\mu)\\
=\delta_\zeta X^\mu+\theta\delta_\zeta\lambda^\mu.
\end{aligned}
\tag{304}
$$

Defining $\mathcal{Q}$ to act on ordinary fields we get after similar calculations on the fermionic superfields

$$
\begin{aligned}
\delta_\zeta X^\mu&=-\zeta\lambda^\mu=\zeta\mathcal{Q}X^\mu\\
\delta_\zeta\lambda^\mu&=-i\zeta\dot{X}^\mu=\zeta\mathcal{Q}\lambda^\mu\\
\delta_\zeta\psi^A&=-\zeta F^A=\zeta\mathcal{Q}\psi^A\\
\delta_\zeta F^A&=-i\zeta\dot{\psi}^A=\zeta\mathcal{Q}F^A
\end{aligned}
\tag{305}
$$

The anti-commutator of two supertransformations thus becomes

$$\delta_{\zeta_2}\delta_{\zeta_1}X^\mu = -\zeta_1\delta_{\zeta_2}\lambda^\mu = -i\zeta_2\zeta_1\dot{X}^\mu \tag{306}$$

$$\begin{aligned}[\delta_{\zeta_2},\delta_{\zeta_1}]X^\mu &= -i(\zeta_2\zeta_1 - \zeta_1\zeta_2)\dot{X}^\mu = -2i\zeta_2\zeta_1\dot{X}^\mu = 2i(\delta_{\zeta_2\zeta_1}X^\mu) = 2i(i\zeta_2\zeta_1 P)X^\mu \\ &= -2\zeta_2\zeta_1 PX^\mu \end{aligned}\tag{307}$$

$$[\delta_{\zeta_2},\delta_{\zeta_1}]X^\mu = \{\zeta_2\mathcal{Q},\zeta_1\mathcal{Q}\}X^\mu = -\zeta_2\zeta_1\{\mathcal{Q},\mathcal{Q}\}X^\mu \Rightarrow \{\mathcal{Q},\mathcal{Q}\}X^\mu = 2PX^\mu, \tag{308}$$

with similar calculations for the other fields. It is also easy to check that $[P,\mathcal{Q}] = 0$ for all fields.

Next we will incorporate the action of $\mathcal{Q}$ to the conformal transformations. For the algebra to close we will have to include the special superconformal transformations S, which we define from the commutator of K and $\mathcal{Q}$. The full algebra reads [4].

$$\begin{array}{lllll} [P,P]=0, & [P,\hat{D}] = -2iP, & [P,K] = -i\hat{D}, & [P,\mathcal{Q}] = 0, & [P,S] = -i\mathcal{Q}, \\ & [\hat{D},\hat{D}] = 0, & [\hat{D},K] = -2iK, & [\hat{D},\mathcal{Q}] = i\mathcal{Q}, & [\hat{D},S] = -iS, \\ & & [K,K] = 0, & [K,\mathcal{Q}] = iS, & [K,S] = 0, \\ & & & \{\mathcal{Q},\mathcal{Q}\} = 2P, & \{\mathcal{Q},S\} = \hat{D}, \\ & & & & \{S,S\} = 2K. \end{array}\tag{309}$$

Before we find the transformations of S, we work out the constraints that $[\hat{D},\mathcal{Q}] = i\mathcal{Q}$ puts on the vector fields D, A, E and B of (301). We have

$$[\delta_{2\epsilon_D t},\delta_\zeta]X^\mu = \epsilon_D\zeta\Big(-A^\mu + \frac{\partial D^\mu}{\partial X^\nu}\lambda^\nu - i\frac{\partial D^\mu}{\partial\lambda^\nu}\dot{X}^\nu - \frac{\partial D^\mu}{\partial\psi^A}F^A + i\frac{\partial D^\mu}{\partial F^A}\dot{\psi}\Big), \tag{310}$$

$$[\delta_{2\epsilon_D t},\delta_\zeta]X^\mu = [i\epsilon_D\hat{D},\zeta\mathcal{Q}]X^\mu = i\epsilon_D\zeta[\hat{D},\mathcal{Q}]X^\mu = i\epsilon_D\zeta(i\mathcal{Q})X^\mu = -\epsilon\delta_\zeta X^\mu = \epsilon_D\zeta\lambda^\mu \tag{311}$$

$$\Rightarrow A^\mu = -\lambda^\mu + \frac{\partial D^\mu}{\partial X^\nu}\lambda^\nu - i\frac{\partial D^\mu}{\partial\lambda^\nu}\dot{X}^\nu - \frac{\partial D^\mu}{\partial\psi^A}F^A + i\frac{\partial D^\mu}{\partial F^A}\dot{\psi} \tag{312}$$

$$\begin{aligned}[\delta_{2\epsilon_D t},\delta_\zeta]\lambda^\mu = i\epsilon_D\zeta\Big(&2\dot{X}^\mu - \frac{\partial D^\mu}{\partial X^\nu}\dot{X}^\nu - \frac{\partial D^\mu}{\partial\lambda^\nu}\dot{\lambda}^\nu - \frac{\partial D^\mu}{\partial\psi^A}\dot{\psi}^A - \frac{\partial D^\mu}{\partial F^A}\dot{F}^A \\ &+ i\frac{\partial A^\mu}{\partial X^\nu}\lambda^\nu + \frac{\partial A^\mu}{\partial\lambda^\nu}\dot{X}^\nu - i\frac{\partial A^\mu}{\partial\psi^A}F^A - \frac{\partial A^\mu}{\partial F^A}\dot{\psi}\Big),\end{aligned}\tag{313}$$

$$[\delta_{2\epsilon_D t},\delta_\zeta]\lambda^\mu = i\epsilon_D\zeta[\hat{D},\mathcal{Q}]\lambda^\mu = -\epsilon_D\zeta\mathcal{Q}\lambda^\mu = -\epsilon_D\delta_\zeta\lambda^\mu = i\epsilon_D\zeta\dot{X}^\mu \tag{314}$$

$$\begin{aligned}\Rightarrow 0 =& \dot{X}^\mu + \Big(\frac{\partial A^\mu}{\partial\lambda^\nu} - \frac{\partial D^\mu}{\partial X^\nu}\Big)\dot{X}^\nu - \frac{\partial D^\mu}{\partial\lambda^\nu}\dot{\lambda}^\nu - \Big(\frac{\partial D^\mu}{\partial\psi^A} + \frac{\partial A^\mu}{\partial F^A}\Big)\dot{\psi}^A \\ &- \frac{\partial D^\mu}{\partial F^A}\dot{F}^A + i\frac{\partial A^\mu}{\partial X^\nu}\lambda^\nu - i\frac{\partial A^\mu}{\partial\psi^A}F^A.\end{aligned}\tag{315}$$

We insert (312) in (315) and find

$$\begin{aligned}0 =&2\frac{\partial^2 D^\mu}{\partial\lambda^\nu\partial X^\rho}\lambda^\rho\dot{X}^\nu + 2i\frac{\partial^2 D^\mu}{\partial F^A\partial\lambda^\nu}\dot{X}^\nu\dot{\psi}^A - \frac{\partial D^\mu}{\partial\lambda^\nu}\dot{\lambda}^\nu + 2\frac{\partial^2 D^\mu}{\partial F^A\partial\psi^B}F^B\dot{\psi}^A\\ &-\frac{\partial D^\mu}{\partial F^A}\dot{F}^A - 2i\frac{\partial^2 D^\mu}{\partial X^\nu\partial\psi^A}F^A\lambda^\nu.\end{aligned}\tag{316}$$

If $D^\mu = D^\mu(X)$ this is satisfied and (312) becomes

$$A^\mu = A^\mu(X,\lambda) = -\lambda^\mu + \frac{\partial D^\mu}{\partial X^\nu}\lambda^\nu =: -\lambda^\mu + D^\mu{}_{,\nu}\lambda^\nu.\tag{317}$$

Same procedure on ψ^A and F^A gives

$$B^A = -F^A - \frac{\partial E^A}{\partial X^\mu}\lambda^\mu + i\frac{\partial E^A}{\partial\lambda^\mu}\dot{X}^\mu + \frac{\partial E^A}{\partial\psi^B}F^B - i\frac{\partial E^A}{\partial F^B}\dot{\psi}^B,\tag{318}$$

and

$$\begin{aligned}0 =&\dot{\psi}^A - \frac{\partial E^A}{\partial X^\mu}\dot{X}^\mu - \frac{\partial B^A}{\partial\lambda}\dot{X}^\mu - \frac{\partial E^A}{\partial\lambda^\mu}\dot{\lambda}^\mu - \frac{\partial E^A}{\partial\psi^B}\dot{\psi}^B + \frac{\partial B^A}{\partial F^B}\dot{\psi}^B\\ &-\frac{\partial E^A}{\partial F^B}\dot{F}^B - i\frac{\partial B^A}{\partial X^\mu}\lambda^\mu + i\frac{\partial B^A}{\partial\psi^B}F^B,\end{aligned}\tag{319}$$

respectively. Together we get

$$\begin{aligned}0 =& -2\frac{\partial E^A}{\partial X^\mu}\dot{X}^\mu - \frac{\partial E^A}{\partial\lambda^\mu}\dot{\lambda}^\mu - \frac{\partial E^A}{\partial F^B}\dot{F}^B + 2\frac{\partial^2 E^A}{\partial\lambda^\mu\partial X^\nu}\lambda^\nu\dot{X}^\mu\\ &+2i\frac{\partial^2 E^A}{\partial\lambda^\mu\partial F^B}\dot{\psi}^B\dot{X}^\mu + 2\frac{\partial^2 E^A}{\partial F^B\partial\psi^C}F^C\dot{\psi}^B - 2i\frac{\partial^2 E^A}{\partial X^\mu\partial\psi^B}F^B\lambda^\mu,\end{aligned}\tag{320}$$

which is satisfied by $E^A = E^A(\psi)$ and we get

$$B^A = -F^A + \frac{\partial A^A}{\partial\psi^B}F^B =: -F^A + A^A{}_{,B}F^B.\tag{321}$$

Now we are ready for the transformations of S

$$[\delta_{\epsilon_K t^2},\delta_\zeta]X^\mu = \epsilon_K\zeta\lambda^\mu = -i\epsilon_K\zeta[K,\mathcal{Q}]X^\mu = -\epsilon_K\zeta S X^\mu\tag{322}$$

$$\Rightarrow SX^\mu = -t\lambda^\mu\tag{323}$$

We introduce the antisymmetric paramater ξ and write a transformation $\delta_{\xi t} = \xi S$. Thus we get

$$\delta_{\xi t}X^\mu = -\xi t\lambda^\mu.\tag{324}$$

In the same way, acting on the λ^μ, ψ^A and F^A with $[\delta_{\epsilon_K t^2},\delta_\zeta]$ and again using $[K,\mathcal{Q}] = -iS$ gives

$$\delta_{\xi t}\lambda^\mu = -i\xi t\dot{X}^\mu + i\xi D^\mu,$$

$$\delta_{\xi t}\psi^A = -\xi t F^A,$$
$$\delta_{\xi t}F^A = -i\xi t\dot{\psi}^A + i\xi E^A. \tag{325}$$

The rest of the transformations can be checked to fulfil the algebra without further restrictions, and we conclude that the complete set of $Osp(1|2)$ transformations are given by (305), (324), (325) and

$$\begin{aligned}
\delta_\epsilon X^\mu &= -\epsilon\dot{X}^\mu + \frac{1}{2}\dot{\epsilon}D^\mu(X),\\
\delta_\epsilon\lambda^\mu &= -\epsilon\dot{\lambda}^\mu - \frac{1}{2}\dot{\epsilon}\lambda^\mu + \frac{1}{2}\dot{\epsilon}D^\mu{}_{,\nu}\lambda^\nu,\\
\delta_\epsilon\psi^A &= -\epsilon\dot{\psi}^A + \frac{1}{2}\dot{\epsilon}E^A(\psi),\\
\delta_\epsilon F^A &= -\epsilon\dot{F}^A - \frac{1}{2}\dot{\epsilon}F^A + \frac{1}{2}\dot{\epsilon}E^A{}_{,B}F^B.
\end{aligned} \tag{326}$$

12 Superconformal invariance of the reduced $D = 1$, $N = 1$ sigma model

The $D = 2$, $N = (1,1)$ action is superconformally invariant at one-loop if the function Φ described in section 11.1 exists. Thus a dimensional reduction can be performed the same way as in section C, either directly or via a $N = 2a$ sigma model, adding the corresponding constraints on the target space to the existence of Φ. However, the actions obtained from these reductions are in general not superconformally invariant (not considering the existence of Φ). It is interesting to ask whether these actions can be made superconformally invariant by the method described in section 11.2. If that is the case, what new constraints are put on the target space and how do they compare to the existence of Φ? In other words, what are the restrictions on the target space for a $N = (1,1)$ sigma model in $D = 2$ to be reducible to a $N = 1$ sigma model in $D = 1$ which is superconformally invariant?

We recall the directly reduced action (227)

$$\begin{aligned}
S_R = \int dtd\theta\big(&-iG_{\mu\nu}D\hat{X}^\mu\partial_t\hat{X}^\nu - G_{\mu\nu}\hat{\psi}^\mu D\hat{\psi}^\nu - G_{\mu\nu,\rho}\hat{\psi}^\mu D\hat{X}^\nu\hat{\psi}^\rho\\
&- \frac{1}{2}T_{\mu\nu\rho}D\hat{X}^\mu D\hat{X}^\nu\hat{\psi}^\rho + \frac{1}{6}T_{\mu\nu\rho}\hat{\psi}^\mu\hat{\psi}^\nu\hat{\psi}^\rho\big),
\end{aligned} \tag{327}$$

with bosonic superfields

$$\hat{X}^\mu(t,\theta) = X^\mu(t) + \theta\psi^\mu(t), \tag{328}$$

and fermionic superfields

$$\hat{\psi}^\mu(t,\theta) = \tilde{\psi}^\mu(t) + \theta F^\mu(t). \tag{329}$$

In components (327) expands to

$$S_R = \int dt\big(- iG_{\mu\nu,\rho}\psi^\rho\psi^\mu\dot{X}^\nu + G_{\mu\nu}\dot{X}^\mu\dot{X}^\nu + iG_{\mu\nu}\psi^\mu\dot{\psi}^\nu$$

$$
\begin{aligned}
&- G_{\mu\nu,\rho}\psi^\rho\tilde{\psi}^\mu F^\nu - G_{\mu\nu}F^\mu F^\nu + iG_{\mu\nu}\tilde{\psi}^\mu\dot{\tilde{\psi}}^\nu \\
&- G_{\mu\nu,\rho\kappa}\psi^\kappa\tilde{\psi}^\mu\psi^\nu\tilde{\psi}^\rho - G_{\mu\nu,\rho}F^\mu\psi^\nu\tilde{\psi}^\rho + iG_{\mu\nu,\rho}\tilde{\psi}^\mu\dot{X}^\nu\tilde{\psi}^\rho - G_{\mu\nu,\rho}\tilde{\psi}^\mu\psi^\nu F^\rho \\
&- \frac{1}{2}T_{\mu\nu\rho,\kappa}\psi^\kappa\psi^\mu\psi^\nu\tilde{\psi}^\rho - iT_{\mu\nu\rho}\dot{X}^\mu\psi^\nu\tilde{\psi}^\rho - \frac{1}{2}T_{\mu\nu\rho}\psi^\mu\psi^\nu F^\rho \\
&+ \frac{1}{6}T_{\mu\nu\rho,\kappa}\psi^\kappa\tilde{\psi}^\mu\tilde{\psi}^\nu\tilde{\psi}^\rho + \frac{1}{2}T_{\mu\nu\rho}\tilde{\psi}^\mu\tilde{\psi}^\nu F^\rho\big).
\end{aligned} \tag{330}
$$

We will act on one term at a time with the transformations (cf.(326))

$$
\begin{aligned}
\delta_\epsilon X^\mu &= -\epsilon\dot{X}^\mu + \frac{1}{2}\dot{\epsilon}D^\mu(X), \\
\delta_\epsilon \psi^\mu &= -\epsilon\dot{\psi}^\mu - \frac{1}{2}\dot{\epsilon}\psi^\mu + \frac{1}{2}\dot{\epsilon}D^\mu{}_{,\nu}\psi^\nu, \\
\delta_\epsilon \tilde{\psi}^\mu &= -\epsilon\dot{\tilde{\psi}}^\mu + \frac{1}{2}\dot{\epsilon}E^\mu(\tilde{\psi}) \\
\delta_\epsilon F^\mu &= -\epsilon\dot{F}^\mu - \frac{1}{2}\dot{\epsilon}F^\mu + \frac{1}{2}\dot{\epsilon}\frac{\partial E^\mu}{\partial\tilde{\psi}^\nu}F^\nu.
\end{aligned} \tag{331}
$$

We work out the first term in detail.

$$
\begin{aligned}
\delta_\epsilon\big(-iG_{\mu\nu,\rho}\psi^\rho\psi^\mu\dot{X}^\nu\big) =& -iG_{\mu\nu,\rho\sigma}\delta_\epsilon X^\sigma\psi^\rho\psi^\mu\dot{X}^\nu - iG_{\mu\nu,\rho}\delta_\epsilon\psi^\rho\psi^\mu\dot{X}^\nu \\
&- iG_{\mu\nu,\rho}\psi^\rho\delta_\epsilon\psi^\mu\dot{X}^\nu - iG_{\mu\nu,\rho}\psi^\rho\psi^\mu\partial_t\delta_\epsilon X^\nu \\
=& i\epsilon G_{\mu\nu,\rho\sigma}\dot{X}^\nu\dot{X}^\sigma\psi^\rho\psi^\mu - \frac{i}{2}\dot{\epsilon}G_{\mu\nu,\rho\sigma}D^\sigma\dot{X}^\nu\psi^\rho\psi^\mu \\
&+ i\epsilon G_{\mu\nu,\rho}\dot{X}^\nu\dot{\psi}^\rho\psi^\mu + \frac{i}{2}G_{\mu\nu,\rho}\dot{X}^\nu\psi^\rho\psi^\mu - \frac{i}{2}\dot{\epsilon}G_{\mu\nu,\rho}D^\rho{}_{,\sigma}\dot{X}^\nu\psi^\sigma\sigma^\mu \\
&+ i\epsilon G_{\mu\nu,\rho}\dot{X}^\nu\psi^\rho\dot{\psi}^\mu + \frac{i}{2}\dot{\epsilon}G_{\mu\nu,\rho}\dot{X}^\nu\psi^\rho\psi^\mu - \frac{i}{2}\dot{\epsilon}G_{\mu\nu,\rho}D^\mu{}_{,\sigma}\dot{X}^\nu\psi^\rho\psi^\sigma \\
&+ i\dot{\epsilon}G_{\mu\nu,\rho}\dot{X}^\nu\psi^\rho\psi^\mu + i\epsilon G_{\mu\nu,\rho}\ddot{X}^\nu\psi^\rho\psi^\mu - \frac{i}{2}\ddot{\epsilon}G_{\mu\nu,\rho}D^\nu\psi^\rho\psi^\mu - \frac{i}{2}\dot{\epsilon}G_{\mu\nu,\rho}D^\nu{}_{,\sigma}\dot{X}^\sigma\psi^\rho\psi^\mu \\
=& \partial_t\big(i\epsilon G_{\mu\nu,\rho}\dot{X}^\nu\psi^\rho\psi^\mu\big) \\
&- \frac{i}{2}\dot{\epsilon}\big(D^\sigma G_{\mu\nu,\rho\sigma} + D^\sigma{}_{,\mu}G_{\sigma\nu,\rho} + D^\sigma{}_{,\nu}G_{\mu\sigma,\rho} + D^\sigma G_{\mu\nu,\sigma}\big)\dot{X}^\nu\psi^\rho\psi^\mu \\
&+ i\dot{\epsilon}G_{\mu\nu,\rho}\dot{X}^\nu\psi^\rho\psi^\mu - \frac{i}{2}\ddot{\epsilon}G_{\mu\nu,\rho}D^\nu\psi^\rho\psi^\mu
\end{aligned} \tag{332}
$$

$$
\begin{aligned}
\delta_\epsilon\big(G_{\mu\nu}\dot{X}^\mu\dot{X}^\nu\big) =& \partial_t\big(-\epsilon G_{\mu\nu}\dot{X}^\mu\dot{X}^\nu\big) + \frac{1}{2}\dot{\epsilon}\big(D^\rho G_{\mu\nu,\rho} + D^\rho{}_{,\mu}G_{\rho\nu} + D^\rho{}_{,\nu}G_{\mu\rho}\big)\dot{X}^\mu\dot{X}^\nu \\
&- \dot{\epsilon}G_{\mu\nu}\dot{X}^\mu\dot{X}^\nu + \ddot{\epsilon}G_{\mu\nu}D^\mu\dot{X}^\nu
\end{aligned} \tag{333}
$$

$$
\begin{aligned}
\delta_\epsilon\big(iG_{\mu\nu}\psi^\mu\dot{\psi}^\nu\big) =& \partial_t\big(-i\epsilon G_{\mu\nu}\psi^\mu\dot{\psi}^\nu\big) + \frac{i}{2}\dot{\epsilon}\big(D^\rho G_{\mu\nu,\rho} + D^\rho{}_{,\mu}G_{\rho\nu} + D^\rho{}_{,\nu}G_{\mu\rho}\big)\psi^\mu\dot{\psi}^\nu \\
&- i\dot{\epsilon}G_{\mu\nu}\psi^\mu\dot{\psi}^\nu + \frac{i}{2}\dot{\epsilon}D^\rho{}_{,\nu\sigma}G_{\mu\rho}\dot{X}^\sigma\psi^\mu\psi^\nu + \frac{i}{2}\ddot{\epsilon}D^\rho{}_{,\nu}G_{\mu\rho}\psi^\mu\psi^\nu
\end{aligned} \tag{334}
$$

$$\delta_\epsilon(-G_{\mu\nu,\rho}\psi^\rho\tilde{\psi}^\mu F^\nu) = \partial_t(\epsilon G_{\mu\nu,\rho}\psi^\rho\tilde{\psi}^\mu F^\nu) - \frac{1}{2}\dot{\epsilon}(D^\sigma G_{\mu\nu,\rho\sigma} + D^\sigma{}_{,\rho}G_{\mu\nu,\sigma})\psi^\rho\tilde{\psi}^\mu F^\nu$$
$$- \frac{1}{2}\dot{\epsilon}(E^\sigma G_{\sigma\nu,\rho} - \frac{\partial E^\sigma}{\partial\psi^\nu}G_{\mu\sigma,\rho}\tilde{\psi}^\mu)\psi^\rho F^\nu \tag{335}$$

$$\delta_\epsilon(-G_{\mu\nu}F^\mu F^\nu) = \partial_t(\epsilon G_{\mu\nu}F^\mu F^\nu) - \frac{1}{2}\dot{\epsilon}G_{\mu\nu,\rho}D^\rho F^\mu F^\nu$$
$$- \frac{1}{2}\dot{\epsilon}G_{\mu\nu}\Big(\frac{\partial E^\mu}{\partial\psi^\rho}F^\rho F^\nu + F^\mu\frac{\partial E^\nu}{\partial\psi^\rho}F^\rho\Big) \tag{336}$$

$$\delta_\epsilon(iG_{\mu\nu}\tilde{\psi}^\mu\dot{\tilde{\psi}}^\nu) = \partial_t(-i\epsilon G_{\mu\nu}\tilde{\psi}^\mu\dot{\tilde{\psi}}^\nu) + \frac{i}{2}\dot{\epsilon}G_{\mu\nu,\rho}D^\rho\tilde{\psi}^\mu\dot{\tilde{\psi}}^\nu + \frac{i}{2}\dot{\epsilon}G_{\mu\nu}E^\mu\dot{\tilde{\psi}}^\nu$$
$$+ \frac{i}{2}\dot{\epsilon}G_{\mu\nu}\tilde{\psi}^\mu\frac{\partial E^\nu}{\partial\tilde{\psi}^\rho}\dot{\tilde{\psi}}^\rho + \frac{i}{2}\ddot{\epsilon}G_{\mu\nu}\tilde{\psi}^\mu E^\nu \tag{337}$$

$$\delta_\epsilon(-G_{\mu\nu,\rho\kappa}\psi^\kappa\tilde{\psi}^\mu\psi^\nu\tilde{\psi}^\rho) = \partial_t(-\epsilon G_{\mu\nu,\rho\kappa}\psi^\kappa\psi^\nu\tilde{\psi}^\mu\tilde{\psi}^\rho)$$
$$+ \frac{1}{2}\dot{\epsilon}(D^\sigma G_{\mu\nu,\rho\kappa\sigma} + D^\sigma{}_{,\nu}G_{\mu\sigma,\rho\kappa} + D^\sigma{}_{,\kappa}G_{\mu\nu,\rho\sigma})\psi^\kappa\psi^\nu\tilde{\psi}^\mu\tilde{\psi}^\rho$$
$$+ \frac{1}{2}\dot{\epsilon}(E^\mu G_{\mu\nu,\rho\kappa} - E^\mu G_{\rho\nu,\mu\kappa})\psi^\kappa\psi^\nu\tilde{\psi}^\rho \tag{338}$$

$$\delta_\epsilon\Big((G_{\rho\nu,\mu} - G_{\mu\nu,\rho})\tilde{\psi}^\mu\psi^\nu F^\rho\Big) = \partial_t\Big(\epsilon(G_{\rho\nu,\mu} - G_{\mu\nu,\rho})\psi^\nu\tilde{\psi}^\mu F^\rho\Big)$$
$$- \frac{1}{2}\dot{\epsilon}\Big(D^\sigma(G_{\rho\nu,\mu\sigma} - G_{\mu\nu,\rho\sigma}) + D^\sigma{}_{,\nu}(G_{\rho\sigma,\mu} - G_{\mu\sigma,\rho})\Big)\psi^\nu\tilde{\psi}^\mu F^\rho$$
$$+ \frac{1}{2}\dot{\epsilon}(G_{\rho\nu,\mu} - G_{\mu\nu,\rho})E^\mu\psi^\nu F^\rho - \frac{1}{2}\dot{\epsilon}(G_{\sigma\nu,\mu} - G_{\mu\nu,\sigma})\frac{\partial E^\sigma}{\partial\tilde{\psi}^\rho}\psi^\nu\tilde{\psi}^\mu F^\rho \tag{339}$$

$$\delta_\epsilon(iG_{\mu\nu,\rho}\tilde{\psi}^\mu\dot{X}^\nu\tilde{\psi}^\rho) = \partial_t(-i\epsilon G_{\mu\nu,\rho}\dot{X}^\nu\tilde{\psi}^\mu\tilde{\psi}^\rho) + \frac{i}{2}\dot{\epsilon}(D^\sigma G_{\mu\nu,\rho\sigma} + D^\sigma{}_{,\nu}G_{\mu\sigma,\rho})\dot{X}^\nu\tilde{\psi}^\mu\tilde{\psi}^\rho$$
$$+ \frac{i}{2}\dot{\epsilon}(G_{\mu\nu,\rho} - G_{\rho\nu,\mu})E^\mu\dot{X}^\nu\tilde{\psi}^\rho + \frac{i}{2}\ddot{\epsilon}G_{\mu\nu,\rho}D^\nu\tilde{\psi}^\mu\tilde{\psi}^\rho \tag{340}$$

$$\delta_\epsilon(-\frac{1}{2}T_{\mu\nu\rho,\kappa}\psi^\kappa\psi^\mu\psi^\nu\tilde{\psi}^\rho) = \partial_t(\frac{1}{2}\epsilon T_{\mu\nu\rho,\kappa}\psi^\kappa\psi^\mu\psi^\nu\tilde{\psi}^\rho)$$
$$- \frac{1}{4}\dot{\epsilon}(D^\sigma T_{\mu\nu\rho,\kappa\sigma} + D^\sigma{}_{,\mu}T_{\sigma\nu\rho,\kappa} + D^\sigma{}_{,\nu}T_{\mu\sigma\rho,\kappa} + D^\sigma{}_{,\kappa}T_{\mu\nu\rho,\sigma})\psi^\kappa\psi^\mu\psi^\nu\tilde{\psi}^\rho$$
$$+ \frac{1}{4}\dot{\epsilon}T_{\mu\nu\rho,\kappa}\psi^\kappa\psi^\mu\psi^\nu(\psi^\rho - E^\rho) \tag{341}$$

$$
\begin{aligned}
\delta_\epsilon\big(-iT_{\mu\nu\rho}\dot{X}^\mu\psi^\nu\tilde{\psi}^\rho\big) &= \partial_t(i\epsilon)T_{\mu\nu\rho}\dot{X}^\mu\psi^\nu\tilde{\psi}^\rho\big) \\
&- \frac{i}{2}\dot{\epsilon}(D^\sigma T_{\mu\nu\rho,\sigma} + D^\sigma{}_{,\mu}T_{\sigma\nu\rho} + D^\sigma{}_{,\nu}T_{\mu\sigma\rho})\dot{X}^\mu\psi^\nu\tilde{\psi}^\rho \\
&+ \frac{i}{2}\dot{\epsilon}T_{\mu\nu\rho}\dot{X}^\mu\psi^\nu(\tilde{\psi}^\rho - E^\rho) - \frac{i}{2}\ddot{\epsilon}_{\mu\nu\rho}D^\mu\psi^\nu\tilde{\psi}^\rho\big)
\end{aligned} \tag{342}
$$

$$
\begin{aligned}
\delta_\epsilon\big(-\frac{1}{2}T_{\mu\nu\rho}\psi^\mu\psi^\nu F^\rho\big) &= \partial_t\big(\frac{1}{2}T_{\mu\nu\rho}\psi^\mu\psi^\nu F^\rho\big) \\
&- \frac{1}{4}\dot{\epsilon}(D^\sigma T_{\mu\nu\rho,\sigma} + D^\kappa{}_{,\mu}T_{\kappa\nu\rho} + D^\kappa{}_{,\nu}T_{\mu\kappa\rho})\psi^\mu\psi^\nu F^\rho \\
&+ \frac{1}{4}\dot{\epsilon}T_{\mu\nu\rho}\psi^\mu\psi^\nu\big(F^\rho - \frac{\partial E^\rho}{\partial\tilde{\psi}^\kappa}F^\kappa\big)
\end{aligned} \tag{343}
$$

$$
\begin{aligned}
\delta_\epsilon\big(\frac{1}{6}T_{\mu\nu\rho,\kappa}\psi^\kappa\tilde{\psi}^\mu\tilde{\psi}^\nu\tilde{\psi}^\rho\big) &= \partial_t\big(-\frac{1}{6}\epsilon T_{\mu\nu\rho,\kappa}\psi^\kappa\tilde{\psi}^\mu\tilde{\psi}^\nu\tilde{\psi}^\rho\big) + \frac{1}{12}\dot{\epsilon}T_{\mu\nu\rho,\kappa}\psi^\kappa(\tilde{\psi}^\mu + 3E^\mu)\tilde{\psi}^\nu\tilde{\psi}^\rho \\
&+ \frac{1}{12}\dot{\epsilon}(D^\sigma T_{\mu\nu\rho,\kappa\sigma} + D^\sigma{}_{,\kappa}T_{\mu\nu\rho,\sigma})\psi^\kappa\tilde{\psi}^\mu\tilde{\psi}^\nu\tilde{\psi}^\rho
\end{aligned} \tag{344}
$$

$$
\begin{aligned}
\delta_\epsilon\big(\frac{1}{2}T_{\mu\nu\rho}\tilde{\psi}^\mu\tilde{\psi}^\nu F^\rho\big) &= \partial_t\big(-\frac{1}{2}\epsilon T_{\mu\nu\rho}\tilde{\psi}^\mu\tilde{\psi}^\nu F^\rho\big) \\
&+ \frac{1}{4}\dot{\epsilon}\Big(T_{\mu\nu\rho}(\tilde{\psi}^\mu + 2E^\mu) + T_{\mu\nu\sigma}\frac{\partial E^\sigma}{\partial\tilde{\psi}^\rho}\tilde{\psi}^\mu\Big)\tilde{\psi}^\nu F^\rho + \frac{1}{4}\dot{\epsilon}T_{\mu\nu\rho,\sigma}D^\sigma\tilde{\psi}^\mu\tilde{\psi}^\nu F^\rho
\end{aligned} \tag{345}
$$

Next we will eliminate the auxiliary fields F^μ by their equations of motion (where $\mathcal{L}$ is the Lagrangian):

$$
\begin{aligned}
0 =& \frac{\partial\mathcal{L}}{\partial F^\sigma} - \partial_t\Big(\frac{\partial\mathcal{L}}{\partial\dot{F}^\sigma}\Big) = -G_{\mu\nu,\rho}\psi^\rho\tilde{\psi}^\mu\delta^\nu_\sigma - G_{\mu\nu}\delta^\mu_\sigma F^\nu - G_{\mu\nu}F^\mu\delta^\nu_\sigma \\
&- G_{\mu\nu,\rho}\delta^\mu_\sigma\psi^\nu\tilde{\psi}^\rho - G_{\mu\nu,\rho}\tilde{\psi}^\mu\psi^\nu\delta^\rho_\sigma - \frac{1}{2}T_{\mu\nu\rho}\psi^\mu\psi^\nu\delta^\rho_\sigma + \frac{1}{2}T_{\mu\nu\rho}\tilde{\psi}^\mu\tilde{\psi}^\nu\delta^\rho_\sigma
\end{aligned} \tag{346}
$$

$$
\begin{aligned}
\Rightarrow F^\lambda =& \frac{1}{2}G^{\lambda\sigma}(2G_{\sigma\nu}F^\nu) \\
=& -\frac{1}{2}G^{\lambda\sigma}(G_{\mu\sigma,\rho} + G_{\sigma\rho,\mu} - G_{\mu\rho,\sigma})\psi^\rho\tilde{\psi}^\mu - \frac{1}{4}G^{\lambda\sigma}T_{\mu\nu\sigma}(\psi^\mu\psi^\nu - \tilde{\psi}^\mu\tilde{\psi}^\nu) \\
=& -\Gamma^{(0)\lambda}{}_{\mu\nu}\psi^\mu\tilde{\psi}^\nu - \frac{1}{4}T^\lambda{}_{\mu\nu}(\psi^\mu\psi^\nu - \tilde{\psi}^\mu\tilde{\psi}^\nu)
\end{aligned} \tag{347}
$$

Eliminating F and collecting terms according to their fields gives
$\underline{X}$:

$$
+\frac{1}{2}\dot{\epsilon}(D^\rho G_{\mu\nu,\rho} + D^\rho{}_{,\mu}G_{\rho\nu} + D^\rho{}_{,\nu}G_{\mu\rho} - 2G_{\mu\nu})\dot{X}^\mu\dot{X}^\nu + \ddot{\epsilon}D^\mu G_{\mu\nu}\dot{X}^\nu
$$

$$=\frac{1}{2}\dot{\epsilon}[(\mathcal{L}_D-2)G_{\mu\nu}]\dot{X}^\mu\dot{X}^\nu+\ddot{\epsilon}D^\mu G_{\mu\nu}\dot{X}^\nu \tag{348}$$

$\underline{\psi}$:

$$-\frac{i}{2}\dot{\epsilon}T_{\mu\nu\rho}\dot{X}^\mu\psi^\nu E^\rho \tag{349}$$

$\underline{\tilde{\psi}}$:

$$+\frac{i}{2}\dot{\epsilon}(G_{\mu\nu,\rho}-G_{\rho\nu,\mu})E^\mu\dot{X}^\nu\tilde{\psi}^\rho+\frac{i}{2}G_{\mu\nu}E^\mu\dot{\tilde{\psi}}^\nu+\frac{i}{2}\ddot{\epsilon}G_{\mu\nu}\tilde{\psi}^\mu E^\nu \tag{350}$$

$\underline{\psi\psi}$:

$$\begin{aligned}&-\frac{i}{2}\dot{\epsilon}[(\mathcal{L}_D-2)G_{\mu\nu,\rho}-D^\sigma{}_{,\mu\nu}G_{\rho\sigma}]\dot{X}^\nu\psi^\rho\psi^\mu+\frac{i}{2}\dot{\epsilon}(\mathcal{L}_D-2)G_{\mu\nu}\psi^\mu\dot{\psi}^\nu\\&+\frac{i}{2}\ddot{\epsilon}(D^\rho G_{\mu\rho,\nu}+D^\rho{}_{,\nu}G_{\mu\rho}]\psi^\mu\psi^\nu\end{aligned} \tag{351}$$

$\underline{\psi\tilde{\psi}}$:

$$\begin{aligned}&-\frac{i}{2}\dot{\epsilon}[D^\sigma T_{\mu\nu\rho,\sigma}+D^\sigma{}_{,\mu}T_{\sigma\nu\rho}+D^\sigma{}_{,\nu}T_{\mu\sigma\rho}-T_{\mu\nu\rho}]\dot{X}^\mu\psi^\nu\tilde{\psi}^\rho\\&-\frac{i}{2}\ddot{\epsilon}D^\mu T_{\mu\nu\rho}\psi^\nu\tilde{\psi}^\rho\end{aligned} \tag{352}$$

$\underline{\tilde{\psi}\tilde{\psi}}$:

$$\begin{aligned}&+\frac{i}{2}\dot{\epsilon}[D^\sigma G_{\mu\nu,\rho\sigma}+D^\sigma{}_{,\nu}G_{\mu\sigma,\rho}]G_{\mu\sigma,\rho}\dot{X}^\nu\tilde{\psi}^\mu\tilde{\psi}^\rho+\frac{i}{2}\dot{\epsilon}G_{\mu\nu,\rho}D^\rho\tilde{\psi}^\mu\dot{\tilde{\psi}}^\nu\\&+\frac{i}{2}\ddot{\epsilon}G_{\mu\nu,\rho}D^\nu\tilde{\psi}^\mu\tilde{\psi}^\rho+\frac{i}{2}\dot{\epsilon}G_{\mu\nu}\tilde{\psi}^\mu\frac{\partial E^\nu}{\partial\tilde{\psi}^\rho}\dot{\tilde{\psi}}^\rho\end{aligned} \tag{353}$$

$\underline{\psi\psi\psi}$:

$$\begin{aligned}&+\frac{1}{8}\dot{\epsilon}E^\sigma G_{\sigma\nu,\rho}T^\nu{}_{\rho\alpha\beta}-\frac{1}{8}\dot{\epsilon}E^\mu(G_{\rho\nu,\mu}-G_{\mu\nu,\rho})T^\rho{}_{\alpha\beta}\psi^\nu\psi^\alpha\psi^\beta\\&+\frac{1}{4}\dot{\epsilon}E^\rho T_{\mu\nu\rho,\sigma}\psi^\sigma\psi^\mu\psi^\nu\\&=\frac{1}{4}\dot{\epsilon}E^\mu(G_{\mu\kappa}\Gamma^{(0)\kappa}{}_{\rho\nu}T^\rho{}_{\alpha\beta}-T_{\mu\alpha\beta,\nu})\psi^\nu\psi^\alpha\psi^\beta\end{aligned} \tag{354}$$

$\underline{\psi\psi\tilde{\psi}}$:

$$+\dot{\epsilon}E^\mu[G_{\mu\sigma}\Gamma^{(0)\sigma}{}_{\rho\nu}\Gamma^{(0)\rho}{}_{\alpha\beta}+\frac{1}{2}(G_{\mu\alpha,\beta\nu}-G_{\beta\alpha,\mu\nu})+\frac{1}{4}T_{\mu\beta\rho}T^\rho{}_{\alpha\nu}]\psi^\nu\psi^\alpha\tilde{\psi}^\beta \tag{355}$$

$\underline{\psi\tilde{\psi}\tilde{\psi}}$:

$$-\frac{1}{4}\dot{\epsilon}[E^\mu G_{\mu\kappa}\Gamma^{(0)\kappa}{}_{\rho\nu}T^\rho{}_{\alpha\beta}+E^\mu T_{\mu\alpha\beta,\nu}-2E^\mu T_{\mu\alpha\rho}\Gamma^{(0)\rho}{}_{\nu\beta}]\psi^\nu\tilde{\psi}^\alpha\tilde{\psi}^\beta \tag{356}$$

$\underline{\tilde{\psi}\tilde{\psi}\tilde{\psi}}$

$$+\frac{1}{8}\dot{\epsilon}E^{\mu}T_{\mu\nu\rho}T^{\rho}{}_{\alpha\beta}\tilde{\psi}^{\nu}\tilde{\psi}^{\alpha}\tilde{\psi}^{\beta} \tag{357}$$

$\underline{\psi\psi\psi\psi}$:

$$\begin{aligned}
&+\frac{1}{16}\dot{\epsilon}(D^{\kappa}T_{\mu\nu\rho,\kappa}+D^{\kappa}{}_{,\mu}T_{\kappa\nu\rho}+D^{\kappa}{}_{,\nu}T_{\mu\kappa\rho}+\frac{\partial E^{\kappa}}{\partial\tilde{\psi}^{\rho}}T_{\mu\nu\kappa}-T_{\mu\nu\rho})T^{\rho}{}_{\alpha\beta}\psi^{\mu}\psi^{\nu}\psi^{\alpha}\psi^{\beta}\\
&-\frac{1}{32}\dot{\epsilon}(D^{\kappa}G_{\rho\sigma,\kappa}+\frac{\partial E^{\kappa}}{\partial\tilde{\psi}^{\rho}}G_{\kappa\sigma}+\frac{\partial E^{\kappa}}{\partial\tilde{\psi}^{\sigma}}G_{\rho\kappa})T^{\sigma}{}_{\mu\nu}T^{\rho}{}_{\alpha\beta}\psi^{\mu}\psi^{\nu}\psi^{\alpha}\psi^{\beta}
\end{aligned} \tag{358}$$

$\underline{\psi\psi\psi\tilde{\psi}}$:

$$\begin{aligned}
&-\frac{1}{4}\dot{\epsilon}\big[D^{\kappa}\partial_{\kappa}(G_{\nu\sigma}\Gamma^{(0)\sigma}{}_{\mu\alpha})+D^{\kappa}{}_{,\alpha}G_{\nu\sigma}\Gamma^{(0)\sigma}{}_{\mu\kappa}+\frac{\partial E^{\kappa}}{\partial\tilde{\psi}^{\nu}}G_{\kappa\sigma}\Gamma^{(0)\sigma}{}_{\mu\alpha}\big]T^{\nu}{}_{\beta\rho}\psi^{\alpha}\psi^{\beta}\psi^{\rho}\tilde{\psi}^{\mu}\\
&-\frac{1}{4}\dot{\epsilon}\big[D^{\kappa}G_{\nu\sigma,\kappa}\Gamma^{(0)\sigma}{}_{\mu\alpha}+\frac{\partial E^{\kappa}}{\partial\tilde{\psi}^{\sigma}}G_{\nu\kappa}\Gamma^{(0)\sigma}{}_{\mu\alpha}+\frac{\partial E^{\kappa}}{\partial\tilde{\psi}^{\nu}}G_{\kappa\sigma}\Gamma^{(0)\sigma}{}_{\mu\alpha}\big]T^{\nu}{}_{\beta\rho}\psi^{\alpha}\psi^{\beta}\psi^{\rho}\tilde{\psi}^{\mu}\\
&-\frac{1}{4}\dot{\epsilon}\big[D^{\kappa}T_{\beta\rho\mu,\alpha\kappa}+D^{\kappa}{}_{,\beta}T_{\kappa\rho\mu,\alpha}+D^{\kappa}{}_{,\rho}T_{\beta\kappa\mu,\alpha}+D^{\kappa}{}_{,\alpha}T_{\beta\rho\mu,\kappa}-T_{\beta\rho\mu,\alpha}\big]\psi^{\alpha}\psi^{\beta}\psi^{\rho}\tilde{\psi}^{\mu}\\
&+\frac{1}{4}\dot{\epsilon}\big[D^{\kappa}T_{\beta\rho\sigma,\kappa}+D^{\kappa}{}_{,\beta}T_{\kappa\rho\sigma}+D^{\kappa}{}_{,\rho}T_{\beta\kappa\sigma}+\frac{\partial E^{\kappa}}{\partial\tilde{\psi}^{\sigma}}T_{\beta\rho\kappa}\big]\Gamma^{(0)\sigma}{}_{\alpha\mu}\psi^{\alpha}\psi^{\beta}\psi^{\rho}\tilde{\psi}^{\mu}
\end{aligned} \tag{359}$$

$\underline{\psi\psi\tilde{\psi}\tilde{\psi}}$:

$$\begin{aligned}
&+\dot{\epsilon}\big[D^{\kappa}\partial_{\kappa}(G_{\rho\sigma}\Gamma^{(0)\sigma}{}_{\mu\beta})+D^{\kappa}{}_{,\beta}G_{\rho\sigma}\Gamma^{(0)\sigma}{}_{\mu\kappa}+\frac{\partial E^{\kappa}}{\partial\tilde{\psi}^{\rho}}G_{\kappa\sigma}\Gamma^{(0)\sigma}{}_{\mu\beta}\big]\Gamma^{(0)\rho}{}_{\alpha\nu}\psi^{\alpha}\psi^{\beta}\tilde{\psi}^{\mu}\tilde{\psi}^{\nu}\\
&-\frac{1}{2}\dot{\epsilon}\big[D^{\kappa}G_{\rho\sigma,\kappa}+\frac{\partial E^{\kappa}}{\partial\tilde{\psi}^{\sigma}}G_{\kappa\rho}+\frac{\partial E^{\kappa}}{\partial\tilde{\psi}^{\rho}}G_{\sigma\kappa}\big]\Gamma^{(0)\sigma}{}_{\mu\beta}\Gamma^{(0)\rho}{}_{\alpha\nu}\psi^{\alpha}\psi^{\beta}\tilde{\psi}^{\mu}\tilde{\psi}^{\nu}\\
&+\frac{1}{2}\big[D^{\kappa}G_{\mu\beta,\nu\alpha\kappa}+D^{\kappa}{}_{,\beta}G_{\mu\kappa,\nu\alpha}+D^{\kappa}{}_{,\alpha}G_{\mu\beta,\nu\kappa}\big]\psi^{\alpha}\psi^{\beta}\tilde{\psi}^{\mu}\tilde{\psi}^{\nu}\\
&-\frac{1}{16}\dot{\epsilon}\big[D^{\kappa}T_{\alpha\beta\rho,\kappa}+D^{\kappa}{}_{,\alpha}T_{\kappa\beta\rho}+D^{\kappa}{}_{,\beta}T_{\alpha\kappa\rho}+\frac{\partial E^{\kappa}}{\partial\tilde{\psi}^{\rho}}T_{\alpha\beta\kappa}-T_{\alpha\beta\rho}\big]T^{\rho}{}_{\mu\nu}\psi^{\alpha}\psi^{\beta}\tilde{\psi}^{\mu}\tilde{\psi}^{\nu}\\
&-\frac{1}{16}\dot{\epsilon}\big[D^{\kappa}T_{\mu\nu\rho,\kappa}+\frac{\partial E^{\kappa}}{\partial\tilde{\psi}^{\rho}}T_{\mu\nu\kappa}T_{\mu\nu\rho}\big]T^{\rho}{}_{\alpha\beta}\psi^{\alpha}\psi^{\beta}\tilde{\psi}^{\mu}\tilde{\psi}^{\nu}\\
&+\frac{1}{16}\big[D^{\kappa}G_{\rho\sigma,\kappa}+\frac{\partial E^{\kappa}}{\partial\tilde{\psi}^{\rho}}G_{\kappa\sigma}+\frac{\partial E^{\kappa}}{\partial\tilde{\psi}^{\sigma}}G_{\rho\kappa}\big]T^{\rho}{}_{\alpha\beta}T^{\sigma}{}_{\mu\nu}\psi^{\alpha}\psi^{\beta}\tilde{\psi}^{\mu}\tilde{\psi}^{\nu}
\end{aligned} \tag{360}$$

$\underline{\psi\tilde{\psi}\tilde{\psi}\tilde{\psi}}$:

$$\begin{aligned}
&-\frac{1}{4}\dot{\epsilon}[D^{\kappa}\partial_{\kappa}(G_{\nu\sigma}\Gamma^{(0)\sigma}{}_{\mu\rho})+D^{\kappa}{}_{,\rho}G_{\nu\sigma}\Gamma^{(0)\sigma}{}_{\mu\kappa}+\frac{\partial E^{\kappa}}{\partial\tilde{\psi}^{\nu}}G_{\kappa\sigma}\Gamma^{(0)\sigma}{}_{\mu\rho}\big]T^{\nu}{}_{\alpha\beta}\psi^{\rho}\tilde{\psi}^{\mu}\tilde{\psi}^{\alpha}\tilde{\psi}^{\beta}\\
&-\frac{1}{8}\dot{\epsilon}\big[D^{\kappa}G_{\sigma\nu,\kappa}+\frac{\partial E^{\kappa}}{\partial\tilde{\psi}^{\sigma}}G_{\kappa\nu}+\frac{\partial E^{\kappa}}{\partial\tilde{\psi}^{\nu}}G_{\sigma\kappa}\big]\Gamma^{(0)\sigma}{}_{\rho\mu}T^{\nu}{}_{\alpha\beta}\psi^{\rho}\tilde{\psi}^{\mu}\tilde{\psi}^{\alpha}\tilde{\psi}^{\beta}
\end{aligned}$$

$$+\frac{1}{8}\dot{\epsilon}\big[D^\kappa G_{\sigma\nu,\kappa}+\frac{\partial E^\kappa}{\partial\tilde{\psi}^\sigma}G_{\kappa\nu}+\frac{\partial E^\kappa}{\partial\tilde{\psi}^\nu}G_{\sigma\kappa}\big]\Gamma^{(0)\nu}{}_{\rho\mu}T^\sigma{}_{\alpha\beta}\psi^\rho\tilde{\psi}^\mu\tilde{\psi}^\alpha\tilde{\psi}^\beta$$
$$-\frac{1}{4}\dot{\epsilon}\big[D^\kappa T_{\mu\alpha\nu,\kappa}+\frac{\partial E^\kappa}{\partial\tilde{\psi}^\nu}T_{\mu\alpha\kappa}+T_{\mu\alpha\nu}\big]\Gamma^{(0)\nu}{}_{\rho\beta}\psi^\rho\tilde{\psi}^\mu\tilde{\psi}^\alpha\tilde{\psi}^\beta$$
$$+\frac{1}{12}\dot{\epsilon}\big[D^\kappa T_{\mu\alpha\beta\rho,\kappa}+D^\kappa{}_{,\rho}T_{\mu\alpha\beta,\kappa}-T_{\mu\alpha\beta,\rho}\big]\psi^\rho\tilde{\psi}^\mu\tilde{\psi}^\alpha\tilde{\psi}^\beta \tag{361}$$

$\underline{\tilde{\psi}\tilde{\psi}\tilde{\psi}\tilde{\psi}}$:

$$-\frac{1}{32}\dot{\epsilon}\big[D^\kappa\partial_\kappa(T_{\mu\nu\rho}T^\rho{}_{\alpha\beta})-2T_{\mu\nu\rho}T^\rho{}_{\alpha\beta}\big]\tilde{\psi}^\mu\tilde{\psi}^\nu\tilde{\psi}^\alpha\tilde{\psi}^\beta \tag{362}$$

For invariance under the $Osp(1|2)$ group (348)-(362) need to sum to zero according to their field compositions. For $E^\mu = E^\mu(\psi)\neq 0$ we see from (349) and (357) that $E^\mu\propto\tilde{\psi}^\mu$ Our first guess is simply $E^\mu=\beta\tilde{\psi}^\mu$ for a constant β. We get

$\underline{X}$:

$$\begin{aligned}0=&\frac{1}{2}\dot{\epsilon}(D^\rho G_{\mu\nu,\rho}+D^\rho{}_{,\mu}G_{\rho\nu}+D^\rho{}_{,\nu}G_{\mu\rho}-2G_{\mu\nu})\dot{X}^\mu\dot{X}^\nu+\ddot{\epsilon}D^\mu G_{\mu\nu}\dot{X}^\nu\\=&\frac{1}{2}\dot{\epsilon}(\mathcal{L}_D-2)G_{\mu\nu}\dot{X}^\mu\dot{X}^\nu+\ddot{\epsilon}D^\mu G_{\mu\nu}\dot{X}^\nu\end{aligned} \tag{363}$$

$\underline{\psi\psi}$:

$$\begin{aligned}0=&-\frac{i}{2}\dot{\epsilon}(\mathcal{L}_D-2)G_{\mu\nu,\rho}\dot{X}^\nu\psi^\rho\psi^\mu+\frac{i}{2}\dot{\epsilon}(\mathcal{L}_D-2)G_{\mu\nu}\psi^\mu\dot{\psi}^\nu\\&+\frac{i}{2}\ddot{\epsilon}D^\rho{}_{,\nu}G_{\mu\rho}\psi^\mu\psi^\nu+\frac{i}{2}\ddot{\epsilon}G_{\mu\nu,\rho}D^\nu\psi^\mu\psi^\rho+\frac{i}{2}\dot{\epsilon}D^\rho{}_{,\nu\sigma}G_{\mu\rho}\dot{X}^\sigma\psi^\mu\psi^\nu\end{aligned} \tag{364}$$

$\underline{\psi\tilde{\psi}}$:

$$\begin{aligned}0=&-\frac{i}{2}\dot{\epsilon}(D^\sigma T_{\mu\nu\rho,\sigma}+D^\sigma{}_{,\mu}T_{\sigma\nu\rho}+D^\sigma{}_{,\nu}T_{\mu\nu\rho}+(\beta-1)T_{\mu\nu\rho})\dot{X}^\mu\psi^\nu\tilde{\psi}^\rho\\&-\frac{i}{2}\ddot{\epsilon}D^\mu T_{\mu\nu\rho}\psi^\nu\tilde{\psi}^\rho\end{aligned} \tag{365}$$

$\underline{\tilde{\psi}\tilde{\psi}}$:

$$\begin{aligned}0=&\frac{i}{2}\dot{\epsilon}\Big(D^\sigma G_{\mu\nu,\rho\sigma}+D^\sigma{}_{,\nu}G_{\mu\sigma,\rho}+\beta(G_{\mu\nu,\rho}-G_{\rho\nu,\mu})\Big)\dot{X}^\nu\tilde{\psi}^\mu\tilde{\psi}^\rho\\&+\frac{i\beta}{2}\dot{\epsilon}(D^\rho G_{\mu\nu,\rho}+G_{\mu\nu})\tilde{\psi}^\mu\dot{\tilde{\psi}}^\nu+\frac{i}{2}\ddot{\epsilon}D^\nu G_{\mu\nu,\rho}\tilde{\psi}^\mu\tilde{\psi}^\rho\end{aligned} \tag{366}$$

$\underline{\psi\psi\psi\psi}$:

$$\begin{aligned}0=&\frac{1}{16}\dot{\epsilon}\Big(D^\kappa T_{\mu\nu\tau,\kappa}+D^\kappa{}_{,\mu}T_{\kappa\nu\tau}+D^\kappa{}_{,\nu}T_{\mu\kappa\tau}\\&-(2-\beta)T_{\mu\nu\tau}-\frac{1}{2}D^\kappa G_{\alpha\tau,\kappa}T^\alpha{}_{\mu\nu}\Big)T^\tau{}_{\rho\sigma}\psi^\mu\psi^\nu\psi^\rho\psi^\sigma\end{aligned} \tag{367}$$

$\underline{\psi\psi\psi\tilde{\psi}}$:

$$
\begin{aligned}
0 \doteq \epsilon\Big[&- \frac{1}{4}(D^\kappa T_{\mu\nu\sigma,\rho\kappa} + D^\kappa{}_{,\mu} T_{\mu\kappa\sigma,\rho} + D^\kappa{}_{,\nu} T_{\mu\kappa\sigma,\rho} + D^\kappa{}_{,\rho} T_{\mu\nu\sigma,\kappa}) \\
&+ \frac{1}{4}\Big[D^\kappa \partial_\kappa (G_{\tau\alpha}\Gamma^{(0)\alpha}{}_{\sigma\mu}) + D^\kappa{}_{,\mu}(G_{\tau\alpha}\Gamma^{(0)\alpha}{}_{\sigma\kappa}) + \beta(G_{\tau\mu,\sigma} - G_{\sigma\mu,\tau})\Big] T^\tau{}_{\nu\rho} \\
&+ \frac{1-\beta}{4} T_{\mu\nu\sigma,\rho} \\
&+ \frac{1}{4}\Big[D^\kappa T_{\mu\nu\tau,\kappa} + D^\kappa{}_{,\mu} T_{\kappa\nu\tau} + D^\kappa{}_{,\nu} T_{\mu\kappa\tau} - (1-\beta) T_{\mu\nu\tau}\Big]\Gamma^{(0)\tau}{}_{\rho\sigma} \\
&- \frac{1}{8}(D^\kappa G_{\tau\alpha,\kappa} + 2G_{\tau\alpha})(\Gamma^{(0)\tau}{}_{\mu\sigma} T^\alpha{}_{\nu\rho} - \Gamma^{(0)\tau}{}_{\nu\rho} T^\alpha{}_{\mu\sigma})\Big] \psi^\mu \psi^\nu \psi^\rho \tilde{\psi}^\sigma
\end{aligned} \tag{368}
$$

$\underline{\psi\psi\tilde{\psi}\tilde{\psi}}$:

$$
\begin{aligned}
0 \doteq \epsilon\Big[&- \frac{1}{2}(D^\kappa G_{\sigma\nu,\rho\mu\kappa} + D^\kappa{}_{,\nu} G_{\sigma\kappa,\rho\mu} + D^\kappa{}_{,\mu} G_{\sigma\nu,\rho\kappa}) \\
&- \frac{1}{2}(D^\kappa G_{\rho\tau,\mu\kappa} + D^\kappa{}_{,\mu} G_{\rho\tau,\kappa})\Gamma^{(0)\tau}{}_{\nu\sigma} - \frac{\beta}{2}(G_{\sigma\mu,\rho\mu} - G_{\rho\nu,\sigma\mu}) \\
&+ \frac{1}{2}\Big(D^\kappa (G_{\tau\nu,\rho\kappa} - G_{\rho\nu,\tau\kappa}) + D^\kappa{}_{,\nu}(G_{\tau\kappa,\rho} - G_{\rho\kappa,\tau})\Big)\Gamma^{(0)\tau}{}_{\mu\sigma} \\
&+ \beta(G_{\tau\nu,\rho} - G_{\rho\nu,\tau})\Gamma^{(0)\tau}{}_{\mu\sigma} - \frac{\beta+1}{2} T_{\rho\sigma\tau} T^\tau{}_{\mu\nu} \\
&- \frac{1}{16} D^\kappa T_{\rho\sigma\tau,\kappa} T^\tau{}_{\mu\nu} - \frac{1}{16}(D^\kappa T_{\mu\nu\tau,\kappa} + D^\kappa{}_{,\mu} T_{\kappa\nu\tau} + D^\kappa{}_{,\nu} T_{\mu\kappa\tau}) T^\tau{}_{\rho\sigma} \\
&+ \frac{1-\beta}{2} T_{\mu\nu\tau} T^\tau{}_{\rho\sigma} + \frac{1}{2}(D^\kappa G_{\tau\alpha,\kappa} + 2G_{\tau\alpha})\Gamma^{(0)\tau}{}_{\mu\rho}\Gamma^{(0)\alpha}{}_{\nu\sigma} \\
&+ \frac{1}{16}(D^\kappa G_{\tau\alpha,\kappa} + 2G_{\tau\alpha}) T^\tau{}_{\mu\nu} T^\alpha{}_{\rho\sigma}\Big] \psi^\mu \psi^\nu \tilde{\psi}^\rho \tilde{\psi}^\sigma
\end{aligned} \tag{369}
$$

$\underline{\psi\tilde{\psi}\tilde{\psi}\tilde{\psi}}$:

$$
\begin{aligned}
0 \doteq \epsilon\Big[&\frac{1}{12}(D^\kappa T_{\nu\rho\sigma,\mu\kappa} + D^\kappa{}_{\mu} T_{\nu\rho\sigma,\kappa}) \\
&- \frac{1}{4}\Big[D^\kappa \partial_\kappa (G_{\tau\alpha}\Gamma^{(0)\alpha}{}_{\nu\mu}) + D^\kappa{}_{,\mu}(G_{\tau\alpha}\Gamma^{(0)\alpha}{}_{\nu\kappa}) \\
&- \beta(G_{\tau\mu,\nu} - G_{\nu\mu,\tau}) + (D^\kappa G_{\alpha\tau,\kappa} + 2G_{\alpha\tau})\Gamma^{(0)\alpha}{}_{\mu\nu}\Big] T^\tau{}_{\rho\sigma} \\
&+ \frac{3\beta+1}{12} T_{\nu\rho\sigma,\mu} - \frac{\beta+1}{2} T_{\nu\rho\tau}\Gamma^{(0)\tau}{}_{\mu\sigma} - \frac{1}{4} D^\kappa T_{\nu\rho\tau,\kappa}\Gamma^{(0)\tau}{}_{\mu\sigma}\Big] \psi^\mu \tilde{\psi}^\nu \tilde{\psi}^\rho \tilde{\psi}^\sigma
\end{aligned} \tag{370}
$$

$\underline{\tilde{\psi}\tilde{\psi}\tilde{\psi}\tilde{\psi}}$:

$$
0 = \frac{1}{16}\epsilon\Big(D^\sigma T_{\mu\nu\rho,\sigma} + (4\beta+3) T_{\mu\nu\rho} - \frac{1}{2} D^\sigma G_{\rho\kappa,\sigma} T^\kappa{}_{\mu\nu}\Big) T^\rho{}_{\alpha\beta} \tilde{\psi}^\mu \tilde{\psi}^\nu \tilde{\psi}^\alpha \tilde{\psi}^\beta \tag{371}
$$

13 Summary and discussion

After reviewing one- and two-dimensional bosonic non-linear sigma models and the geometrical constraints they put on the target space by requiring invariance under the super-Poincaré group, we reviewed conformal and superconformal theory. Led by an article by Maloney et al. [4], we constructed the transformations of the $Osp(1|2)$ subgroup of the superconformal group in one dimension (305), (324), (325) and (326), which turned out to require the target space to include a bosonic vector field $D^\mu(X)$ and also a fermionic vector field $E^A(\tilde{\psi})$ associated with it. We then used these transformations on a directly reduced model of a two-dimensional $N = (1,1)$ sigma model (227). In this way we found the geometrical constraints on the target space needed for the already classically superconformally invariant $N = (1,1)$ model to be reducable to a one-dimensional superconformally invariant $N = 1$ model. These were found to be rather complicated as seen by setting the sum of (348)-(362) to zero. (349) and (357) forced us to the restriction $E^\mu \propto \tilde{\psi}^\mu$ and in (363)-(371) we have set $E^\mu = \beta\tilde{\psi}^\mu$ for any constant β. This is where this master thesis ends, but to continue a little bit further we note that in (352) there seems to be one term missing. If we could add a term

$$-\frac{i}{2}D^\sigma{}_{,\rho}T_{\mu\nu\rho}\dot{X}^\mu\psi^\nu\tilde{\psi}^\rho, \tag{372}$$

we would get the much nicer looking constraint

$$(\mathcal{L}_D - 1)T_{\mu\nu\rho} = 0. \tag{373}$$

By letting $E^\mu = D^\mu{}_{,\nu}\tilde{\psi}^\nu$ or even $E^\mu = D^\mu{}_{,\nu}\tilde{\psi}^\nu - \beta\tilde{\psi}^\mu$ (cf. second line in (289)) this term actually arises from (349). Since $\frac{\partial E^\mu}{\partial X^\nu} = 0$ from (320) we then also need $D^\mu{}_{,\nu\rho} = 0$. However, preliminary calculations show that not all "missing" terms can be recovered, suggesting one of the following options

1. there are no simpler expressions than (363)-(371), and the analysis of the constraints and whether they are consistent has to continue from there
2. there have been calculation errors
3. there are errors in the method used, either in the derivation of the $Osp(1|2)$ transformations or in their application to the reduced sigma model

A complete recalculation to exclude possible calculation errors is needed. An analysis of how the fermionic superfields of the reduced model relate to the fermionic superfields of the most general model (228) is also welcome.

A Notations and conventions

We will denote the flat Minkowski space metric η and use the sign convention that

$$\eta_{ab} = \eta^{ab} = diag(-1, +1, +1, +1). \tag{374}$$

The Pauli matrices (including our convention for a σ^0-matrix) are

$$\sigma^0 = \bar{\sigma}^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma^1 = -\bar{\sigma}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\sigma^2 = -\bar{\sigma}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = -\bar{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{375}$$

A space-time derivative will be written in either of the two forms:

$$\frac{\partial}{\partial x^a} = \partial_a, \qquad \frac{\partial}{\partial x_a} = \partial^a. \tag{376}$$

A.1 Spinors

In the so-called Weyl or chiral representation

$$\gamma^a = \begin{pmatrix} 0 & \sigma^a \\ \bar{\sigma}^a & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{377}$$

the four-component Dirac spinor Ψ_D breaks up into two two-component parts

$$\Psi_D = \begin{pmatrix} \xi_\alpha \\ \chi^{\dagger\dot{\alpha}} \end{pmatrix}, \tag{378}$$

with spinor indices $\alpha = 1, 2$ and $\dot{\alpha} = \dot{1}, \dot{2}$. We define the conjugate spinor field by $\bar{\Psi}_D := \Psi^\dagger \gamma^0$ which then reads

$$\bar{\Psi}_D = \Psi_D^\dagger \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \chi^\alpha & \xi^\dagger_{\dot{\alpha}} \end{pmatrix}. \tag{379}$$

We call the field ξ a 'left-handed Weyl spinor' since the left-handed projection operator $P_L = (1 - \gamma_5)/2$ projects out this part of the Dirac spinor

$$P_L \Psi_D = \begin{pmatrix} \xi_\alpha \\ 0 \end{pmatrix}, \tag{380}$$

and similarly for the right-handed projection operator $P_R = (1 + \gamma_5)/2$ with the field $\chi^\dagger$. We also define the four-component Majorana field as a Dirac field where $\xi = \chi$, i.e

$$\Psi_M = \begin{pmatrix} \xi_\alpha \\ \xi^{\dagger\dot{\alpha}} \end{pmatrix}, \qquad \bar{\Psi}_M = \begin{pmatrix} \xi^\alpha & \xi^\dagger_{\dot{\alpha}} \end{pmatrix}. \tag{381}$$

Spinor indices are antisymmetric, which means that two adjacent indices give a minus sign when they are interchanged

$$\xi^\alpha \chi^\beta = -\chi^\beta x^\alpha \qquad \xi^\alpha \chi_\beta = -\chi_\beta \xi^\alpha, \qquad \xi_\alpha \chi_\beta = -\chi_\beta \xi_\alpha. \tag{382}$$

This will also apply when one or both indices are dotted, e.g. $\xi^\alpha \chi^{\dagger\dot\beta} = -\chi^{\dagger\dot\beta}\xi^\alpha$, or, in our convention, when both indices are the same. We introduce the following notation:

$$\begin{aligned} \xi\chi &:= \xi^\alpha \chi_\alpha = -\chi_\alpha \xi^\alpha \\ \xi^\dagger \chi^\dagger &:= \xi_{\dot\alpha}\chi^{\dot\alpha} = -\xi^{\dot\alpha}\chi_{\dot\alpha} \end{aligned} \tag{383}$$

We also note that

$$\xi\xi = \xi^\alpha \xi_\alpha = -\xi_\alpha \xi^\alpha, \tag{384}$$

while, of course,

$$\xi^\alpha \xi^\alpha = 0 = \xi_\alpha \xi_\alpha. \tag{385}$$

We introduce the antisymmtric symbol ϵ with the convention

$$\begin{aligned} &\epsilon^{12} = -\epsilon^{21} = \epsilon_{21} = -\epsilon_{12} = 1, \qquad \epsilon^{11} = \epsilon^{22} = \epsilon_{11} = \epsilon_{22} = 0 \\ &\epsilon^{\dot1\dot2} = -\epsilon^{\dot2\dot1} = \epsilon_{\dot2\dot1} = -\epsilon_{\dot1\dot2} = 1, \qquad \epsilon^{\dot1\dot1} = \epsilon^{\dot2\dot2} = \epsilon_{\dot1\dot1} = \epsilon_{\dot2\dot2} = 0, \end{aligned} \tag{386}$$

or

$$(\epsilon^{\alpha\beta}) = -(\epsilon_{\alpha\beta}) = (\epsilon^{\dot\alpha\dot\beta}) = -(\epsilon_{\dot\alpha\dot\beta}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{387}$$

We have

$$\epsilon^{\alpha\beta}\epsilon_{\beta\alpha} = \epsilon^{\dot\alpha\dot\beta}\epsilon_{\dot\beta\dot\alpha} = 2. \tag{388}$$

We also write

$$\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = \epsilon_{\gamma\beta}\epsilon^{\beta\alpha} = \delta^\alpha_\gamma, \qquad \epsilon^{\dot\alpha\dot\beta}\epsilon_{\dot\beta\dot\gamma} = \epsilon_{\dot\gamma\dot\beta}\epsilon^{\dot\beta\dot\alpha} = \delta^{\dot\alpha}_{\dot\gamma}. \tag{389}$$

This antisymmetric symbol can be used to raise and lower spinor indices the following way

$$\xi_\alpha = \epsilon_{\alpha\beta}\xi^\beta, \qquad \xi^\alpha = \epsilon^{\alpha\beta}\xi_\beta, \qquad \chi^\dagger_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\chi^{\dagger\dot\beta}, \qquad \chi^{\dagger\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\chi^\dagger_{\dot\beta}. \tag{390}$$

We note that now (383) can be written

$$\xi\chi = \xi^\alpha\chi_\alpha = \xi^\alpha\epsilon_{\alpha\beta}\chi^\beta = -\chi^\beta\epsilon_{\alpha\beta}\xi^\alpha = \chi^\beta\epsilon_{\beta\alpha}\xi^\alpha = \chi^\beta\xi_\beta = \chi\xi, \tag{391}$$

and similarly for dotted indices. We can explicitly write out the components of (384)

$$\begin{aligned} \xi\xi &= \xi^\alpha\epsilon_{\alpha\beta}\xi^\beta = \xi^1\epsilon_{11}\xi^1 + \xi^1\epsilon_{12}\xi^2 + \xi^2\epsilon_{21}\xi^1 + \xi^2\epsilon_{22}\xi^2 \\ &= -\xi^1\xi^2 + \xi^2\xi^1 = 2\xi^2\xi^1. \end{aligned} \tag{392}$$

We will always (at least in theory) move the antisymmetric symbol to a position immediately to the left of the index we want to raise or lower. The reason is the following

$$\xi^\alpha\xi_\alpha = \xi^\alpha\epsilon_{\alpha\beta}\xi^\beta = \xi^\alpha\xi^\beta\epsilon_{\alpha\beta} = -\xi^\alpha\xi^\beta\epsilon_{\beta\alpha} \neq -\xi^\alpha\xi_\alpha. \tag{393}$$

For objects with two spinor indices we obviously run into trouble, but we may think of it like this

$$(\sigma^a)^{\boldsymbol{\alpha\dot{\alpha}}} = \text{"}(\sigma^a)(\epsilon^{\alpha\beta})_{\boldsymbol{\beta}}(\epsilon^{\dot{\alpha}\dot{\beta}})_{\boldsymbol{\dot{\beta}}}\text{"} = \epsilon^{\alpha\beta}(\sigma^a)_{\boldsymbol{\beta\dot{\beta}}}\epsilon^{\dot{\alpha}\dot{\beta}} = -\epsilon^{\alpha\beta}(\sigma^a)_{\boldsymbol{\beta\dot{\beta}}}\epsilon^{\dot{\beta}\dot{\alpha}}, \tag{394}$$

where we used quotation marks to indicate that this is just a way to think, and bold letters indicate the actual indices of σ^a. We see that

$$(\sigma^a)_{\alpha\dot{\alpha}} = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}(\bar{\sigma}^a)^{\dot{\alpha}\alpha}, \qquad (\bar{\sigma}^a)^{\dot{\alpha}\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}(\sigma^a)_{\beta\dot{\beta}}. \tag{395}$$

It is often convenient to introduce a spinor notation also for vectors. We do this for a vector V_a accordingly:

$$V_{\alpha\dot{\alpha}} := (\sigma^a)_{\alpha\dot{\alpha}}V_a. \tag{396}$$

We then have

$$V_a = -\frac{1}{2}(\bar{\sigma}_a)^{\dot{\alpha}\alpha}V_{\alpha\dot{\alpha}}, \tag{397}$$

since

$$(\sigma^a)_{\alpha\dot{\alpha}}(\bar{\sigma}_a)^{\dot{\beta}\beta} = -2\delta_\alpha^\beta\delta_{\dot{\alpha}}^{\dot{\beta}}. \tag{398}$$

Other useful relations are

$$\begin{aligned}
(\sigma^a)_{\alpha\dot{\alpha}}(\sigma_a)_{\beta\dot{\beta}} &= -2\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} \\
(\bar{\sigma}^a)^{\dot{\alpha}\alpha}(\bar{\sigma}_a)^{\dot{\beta}\beta} &= -2\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}} \\
\left[(\sigma^a)_{\alpha\dot{\gamma}}(\bar{\sigma}^b)^{\dot{\gamma}\beta} + (\sigma^b)_{\alpha\dot{\gamma}}(\bar{\sigma}^a)^{\dot{\gamma}\beta}\right] &= -2\eta^{ab}\delta_\alpha^\beta \\
\left[(\bar{\sigma}^a)^{\dot{\beta}\gamma}(\sigma^b)_{\gamma\dot{\alpha}} + (\bar{\sigma}^b)^{\dot{\beta}\gamma}(\sigma^a)_{\gamma\dot{\alpha}}\right] &= -2\eta^{ab}\delta_{\dot{\alpha}}^{\dot{\beta}}.
\end{aligned} \tag{399}$$

We also define

$$\begin{aligned}
(\sigma_{ab})_\alpha{}^\beta &:= -\frac{1}{4}\big((\sigma_a)_{\alpha\dot{\gamma}}(\bar{\sigma}_b)^{\dot{\gamma}\beta} - (\sigma_b)_{\alpha\dot{\gamma}}(\bar{\sigma}_a)^{\dot{\gamma}\beta}\big) \\
(\bar{\sigma}_{ab})^{\dot{\alpha}}{}_{\dot{\beta}} &= -\frac{1}{4}\big((\bar{\sigma}_a)^{\dot{\alpha}\gamma}(\sigma_b)_{\gamma\dot{\beta}} - (\bar{\sigma}_b)^{\dot{\alpha}\gamma}(\sigma_a)_{\gamma\dot{\beta}}\big).
\end{aligned} \tag{400}$$

A.2 The supersymmetric parameter θ

$$\theta\theta = \theta^\alpha\theta_\alpha = \theta^\alpha\epsilon_{\alpha\beta}\theta^\beta, \qquad \bar{\theta}\bar{\theta} = \bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} = \bar{\theta}_{\dot{\alpha}}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}_{\dot{\beta}} \tag{401}$$

leads to

$$\begin{array}{ll}
\theta_\alpha\theta_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\theta\theta, & \bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} = -\frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta}, \\
\theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta, & \bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta}.
\end{array} \tag{402}$$

We also have

$$\theta_\alpha\bar{\theta}_{\dot{\beta}} = \frac{1}{2}(\sigma^a)_{\alpha\dot{\beta}}\bar{\theta}_{\dot{\gamma}}(\bar{\sigma}_a)^{\dot{\gamma}\gamma}\theta_\gamma. \tag{403}$$

We will write the spinor derivatives in the following way

$$\frac{\partial}{\partial\theta^\alpha} := \partial_\alpha, \qquad \frac{\partial}{\partial\theta_\alpha} := \partial^\alpha, \qquad \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} := \partial^{\dot{\alpha}}, \qquad \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} := \partial_{\dot{\alpha}}. \tag{404}$$

They will work in the following way

$$\begin{array}{ll}
\partial_\alpha\theta^\beta = \delta_\alpha^\beta & \partial_{\dot\alpha}\bar\theta^{\dot\beta} = \delta_{\dot\alpha}^{\dot\beta} \\
\partial_\alpha\theta_\beta = -\epsilon_{\alpha\beta} & \partial_{\dot\alpha}\bar\theta_{\dot\beta} = -\epsilon_{\dot\alpha\dot\beta} \\
\partial_\alpha\bar\theta^{\dot\beta} = 0 & \partial_{\dot\alpha}\theta^\beta = 0 \\
\partial_\alpha\bar\theta_{\dot\beta} = 0 & \partial_{\dot\alpha}\theta_\beta = 0 \\
\partial^\alpha\theta^\beta = -\epsilon^{\alpha\beta} & \partial^{\dot\alpha}\bar\theta^{\dot\beta} = -\epsilon^{\dot\alpha\dot\beta} \\
\partial^\alpha\theta_\beta = \delta_\beta^\alpha & \partial^{\dot\alpha}\bar\theta_{\dot\beta} = \delta_{\dot\beta}^{\dot\alpha} \\
\partial^\alpha\bar\theta^{\dot\beta} = 0 & \partial^{\dot\alpha}\theta^\beta = 0 \\
\partial^\alpha\bar\theta_{\dot\beta} = 0 & \partial^{\dot\alpha}\theta_\beta = 0
\end{array} \tag{405}$$

We have e.g.

$$\frac{\partial}{\partial\theta^\alpha}(\theta_\beta) = \frac{\partial}{\partial\theta^\alpha}(\epsilon_{\beta\gamma}\theta^\gamma) = \epsilon_{\beta\gamma}\frac{\partial}{\partial\theta^\alpha}(\theta^\gamma) = \epsilon_{\beta\gamma}\delta_\alpha^\gamma = \epsilon_{\beta\alpha} = -\epsilon_{\alpha\beta}. \tag{406}$$

The indices are raised and lowered with an extra minus sign

$$\begin{array}{ll}
\epsilon^{\alpha\beta}\partial_\beta = -\partial^\alpha & \epsilon^{\dot\alpha\dot\beta}\partial_{\dot\beta} = -\partial^{\dot\alpha} \\
\epsilon_{\alpha\beta}\partial^\beta = -\partial_\alpha & \epsilon_{\dot\alpha\dot\beta}\partial^{\dot\beta} = -\partial_{\dot\alpha}.
\end{array} \tag{407}$$

Since the spinor derivative carries its own spinor index it will anticommute with every other object that carries an odd number of spinor indices. For example

$$\partial_\alpha\partial_\beta = -\partial_\beta\partial_\alpha, \qquad \partial_\alpha\partial_{\dot\beta} = -\partial_{\dot\beta}\partial_\alpha, \tag{408}$$

but

$$\partial_\alpha\partial_a = \partial_a\partial_\alpha. \tag{409}$$

The product rule becomes

$$\begin{aligned}
\partial_\alpha(\theta^\beta\theta\gamma) &= \partial_\alpha(\theta^\beta)\theta^\gamma + \partial_\alpha\theta^\beta(\theta^\gamma) \\
&= \partial_\alpha(\theta^\beta)\theta^\gamma - \theta^\beta\partial_\alpha(\theta^\gamma) \\
&= \delta_\alpha^\beta\theta^\gamma - \delta_\alpha^\gamma\theta^\beta.
\end{aligned} \tag{410}$$

We easily work out

$$\begin{array}{ll}
\frac{\partial}{\partial\theta^\alpha}(\theta\theta) = \partial_\alpha(\theta\theta) = 2\theta_\alpha, & \frac{\partial}{\partial\bar\theta^{\dot\alpha}}(\bar\theta\bar\theta) = \partial_{\dot\alpha}(\bar\theta\bar\theta) = -2\bar\theta_{\dot\alpha}, \\
\frac{\partial}{\partial\theta_\alpha}(\theta\theta) = \partial^\alpha(\theta\theta) = -2\theta^\alpha, & \frac{\partial}{\partial\bar\theta_{\dot\alpha}}(\bar\theta\bar\theta) = \partial^{\dot\alpha}(\bar\theta\bar\theta) = 2\bar\theta^{\dot\alpha}.
\end{array} \tag{411}$$

The same rule applies for a spinor ψ, e.g.,

$$\partial_\alpha(\psi\theta) = \partial_\alpha(\theta\psi) = \psi_\alpha, \qquad \partial^\alpha(\psi\theta) = \partial^\alpha(\theta\psi) = -\psi^\alpha. \tag{412}$$

Integration turns out to be equal to a derivative. We have

$$d^2\theta = -\frac{1}{4}d\theta^\alpha d\theta^\beta\epsilon_{\alpha\beta}, \qquad d^2\bar\theta = -\frac{1}{4}d\bar\theta_{\dot\alpha}\bar\theta_{\dot\beta}\epsilon^{\dot\alpha\dot\beta}, \tag{413}$$

which leads to

$$
\begin{aligned}
\int d^2\theta\ \theta\theta &= -\frac{1}{4}\int d\theta^\alpha d\theta^\beta \epsilon_{\alpha\beta}\theta\theta = -\frac{1}{4}\epsilon_{\alpha\beta}\int d\theta^\alpha d\theta^\beta \theta\theta \\
&= -\frac{1}{4}\epsilon_{\alpha\beta}\frac{\partial}{\partial\theta_\alpha}\frac{\partial}{\partial\theta_\beta}(\theta\theta) = -\frac{1}{4}\epsilon_{\alpha\beta}\frac{\partial}{\partial\theta_\alpha}(-2\theta^\beta) \\
&= \frac{1}{2}\epsilon_{\alpha\beta}\epsilon^{\beta\gamma}\delta^\alpha_\gamma = \frac{1}{2}\delta^\gamma_\alpha\delta^\alpha_\gamma \\
&= 1. \qquad (414)
\end{aligned}
$$

Similarly

$$\int d^2\bar{\theta}\bar{\theta}\bar{\theta} = 1. \qquad (415)$$

A.3 The Baker-Campbell-Haussdorff formulas

There are two very useful formulas which go under the name Baker-Campbell-Haussdorff. The first is

$$
\begin{aligned}
& e^{-B}Ae^{B} = \sum_{n=0}^{\infty}\frac{1}{n!}[A,B]_{(n)} \qquad (416) \\
& [A,B]_{(0)} = A, \qquad [A,B]_{(n+1)} = \Big[[A,B]_{(n)},B\Big]
\end{aligned}
$$

and thus

$$e^{-B}Ae^{B} = A + [A,B] + \dots \qquad (417)$$

The second formula reads

$$
\begin{aligned}
& e^{A}e^{B} = \exp\Big[\sum_{n=1}^{\infty}\frac{1}{n!}C_n(A,B)\Big] \qquad (418) \\
& C_1 = A+B, \qquad C_2 = [A,B] \\
& C_3 = \frac{1}{2}\Big[[A,B],B\Big] + \frac{1}{2}\Big[A,[A,B]\Big] \\
& C_4 = \Big[\big[A,[A,B]\big],B\Big].
\end{aligned}
$$

and thus

$$e^{A}e^{B} = A + B + \frac{1}{2}[A,B] + \frac{1}{12}\Big[[A,B],B\Big] + \frac{1}{12}\Big[A,[A,B]\Big] + \dots \qquad (419)$$

B Derivations

B.1 Derivation of the superalgebra

We want to derive the superalgebra. To this end we change to the more convenient spinor notation (appendix A.1):

$$P_{\alpha\dot{\alpha}} = (\sigma^a)_{\alpha\dot{\alpha}}P_a \in (1/2, 1/2),$$

$$\begin{aligned}
M_{\alpha\beta} &= \frac{1}{2}(\sigma^{ab})_{\alpha\beta}M_{ab} \in (1,0), \\
\bar{M}_{\dot{\alpha}\dot{\beta}} &= -\frac{1}{2}(\bar{\sigma}^{ab})_{\dot{\alpha}\dot{\beta}}M_{ab} \in (0,1), \\
Q^I_\alpha &\in (1/2, 0), \\
\bar{Q}^I_{\dot{\alpha}} &\in (0, 1/2).
\end{aligned} \tag{420}$$

We see that in accordance with (118)

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\} \in (1/2, 1/2), \tag{421}$$

which leads to

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\} = C^{IJ}P_{\alpha\dot{\alpha}}, \tag{422}$$

for some complex components C^{IJ}. Taking the adjoint we get

$$\{\bar{Q}^I_{\dot{\alpha}}, Q^J_\alpha\} = \{Q^J_\alpha, \bar{Q}^I_{\dot{\alpha}}\} = C^{JI}P_{\alpha\dot{\alpha}} \stackrel{!}{=} \bar{C}^{IJ}P_{\alpha\dot{\alpha}}, \tag{423}$$

showing that C^{IJ} is Hermitian ($C^{IJ} = \bar{C}^{JI}$) which enables us to choose a basis where C^{IJ} is diagonal. For future convenience we also rescale the generators Q^I such that

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\} = 2\delta^{IJ}P_{\alpha\dot{\alpha}}. \tag{424}$$

Next we have

$$\{Q^I_\alpha, Q^J_\beta\} \in (1/2 - 1/2, 0) \oplus (1/2 + 1/2, 0) = (0,0) \oplus (1,0). \tag{425}$$

Since $\{Q^I_\alpha, Q^J_\beta\} = \{Q^J_\beta, Q^I_\alpha\}$ we can write this with one part that is antisymmetric in both types of indices and one part that is symmetric:

$$\{Q^I_\alpha, Q^J_\beta\} = X^{IJ}\epsilon_{\alpha\beta} + Y^{IJ}M_{\alpha\beta}, \tag{426}$$

where X^{IJ} and $\epsilon_{\alpha\beta}$ are antisymmetric in their respective indices and Y^{IJ} and $M_{\alpha\beta}$ are symmetric. To determine X^{IJ} and Y^{IJ} we move on to the next commutator. We have

$$[Q^I_\alpha, P_{\beta\dot{\beta}}] \in (1/2 - 1/2, 1/2) = (0, 1/2), \tag{427}$$

since there is no generator $(1/2 + 1/2, 1/2) = (1, 1/2)$ in this representation. Thus

$$[Q^I_\alpha, P_{\beta\dot{\beta}}] = Z^I{}_J\epsilon_{\alpha\beta}\bar{Q}^J_{\dot{\beta}}, \tag{428}$$

with adjoint

$$[\bar{Q}^I_{\dot{\alpha}}, P_{\beta\dot{\beta}}] = \bar{Z}^I{}_J\epsilon_{\dot{\alpha}\dot{\beta}}Q^J_\beta, \tag{429}$$

for some complex constants $Z^I{}_J$, which we will determine with help of the generalized Jacobi identities (119).

$$0 = \big[[P_{\alpha\dot{\alpha}}, P_{\beta\dot{\beta}}], Q^I_\gamma\big] + \big[[Q^I_\gamma, P_{\alpha\dot{\alpha}}], P_{\beta\dot{\beta}}\big] + \big[[P_{\beta\dot{\beta}}, Q^I_\gamma], P_{\alpha\dot{\alpha}}\big]$$

$$
\begin{aligned}
&= 0 + Z^I{}_J \epsilon_{\gamma\alpha}[\bar{Q}^J_{\dot\alpha}, P_{\beta\dot\beta}] - Z^I{}_J \epsilon_{\gamma\beta}[\bar{Q}^J_{\dot\beta}, P_{\alpha\dot\alpha}] = \\
&= Z^I{}_J \bar{Z}^J{}_K \epsilon_{\dot\alpha\dot\beta}(\epsilon_{\gamma\alpha} Q^K_\beta + \epsilon_{\gamma\beta} Q^K_\alpha) \\
&= Z^I{}_J \bar{Z}^J{}_K \epsilon_{\dot\alpha\dot\beta}\epsilon_{\gamma\delta}(\delta^\delta_\gamma \delta^\epsilon_\beta + \delta^\delta_\beta \delta^\epsilon_\alpha) Q^K_\epsilon
\end{aligned} \tag{430}
$$

$$
\Rightarrow Z^I{}_J \bar{Z}^J{}_K = (Z\bar{Z})^I{}_K = 0 \tag{431}
$$

$$
\begin{aligned}
0 &= \{[P_{\alpha\dot\alpha}, Q^I_\beta], Q^J_\gamma\} + \{[P_{\alpha\dot\alpha}, Q^J_\gamma], Q^I_\beta\} + \big[\{Q^I_\beta, Q^J_\gamma\} P_{\alpha\dot\alpha}\big] = \\
&= -Z^I{}_J \epsilon_{\beta\alpha}\{Q^J_\gamma, \bar{Q}^K_{\dot\alpha}\} - Z^J{}_K \epsilon_{\gamma\alpha}\{Q^I_\beta, \bar{Q}^I_{\dot\alpha}\} + [X^{IJ}\epsilon_{\beta\gamma} + Y^{IJ} M_{\beta\gamma}, P_{\alpha\dot\alpha}] = \\
&= -Z^I{}_K \epsilon_{\beta\alpha} 2\delta^{JK} P_{\gamma\dot\alpha} - Z^J{}_K \epsilon_{\gamma\alpha} 2\delta^{IK} P_{\beta\dot\alpha} + Y^{IJ}\frac{1}{2}(\sigma^{ab})_{\beta\gamma}(\sigma^c)_{\alpha\dot\alpha}[M_{ab}, P_c] \\
&= -2\delta^{JK} Z^I{}_K \epsilon_{\beta\alpha} P_{\gamma\dot\alpha} - 2\delta^{IK} Z^J{}_K \epsilon_{\gamma\alpha} P_{\beta\dot\alpha} + \frac{1}{2} Y^{IJ}(\sigma^{ab})_{\beta\gamma}(\sigma^c)_{\alpha\dot\alpha}(i\eta_{ca}P_b - i\eta_{cb}P_a) \\
&= -2Z^{IJ}\epsilon_{\beta\alpha}P_{\gamma\dot\alpha} - 2Z^{JI}\epsilon_{\gamma\alpha}P_{\beta\dot\alpha} + \frac{1}{2}Y^{IJ}(\sigma^c)_{\alpha\dot\alpha}\big(i(\sigma^{ab})_{\beta\gamma}\eta_{ca}P_b - i(\sigma^{ba})_{\beta\gamma}\eta_{ca}P_b\big) \\
&= -2Z^{IJ}\epsilon_{\beta\alpha}P_{\gamma\dot\alpha} - 2Z^{JI}\epsilon_{\gamma\alpha}P_{\beta\dot\alpha} + iY^{IJ}(\sigma^c)_{\alpha\dot\alpha}(\sigma^{ab})_{\beta\gamma}\eta_{ca}P_b.
\end{aligned} \tag{432}
$$

In the last line we have used that $(\sigma^{ba})_{\alpha\beta} = -(\sigma^{ab})_{\alpha\beta}$. Multiplying with $\epsilon^{\beta\gamma}$ and noting that $(\sigma^{ab})_\alpha{}^\alpha = 0$ we get

$$
\begin{aligned}
0 &= 2Z^{IJ}\delta^\gamma_\alpha P_{\gamma\dot\alpha} - 2Z^{JI}\delta^\beta_\alpha P_{\beta\dot\alpha} + iY^{IJ}(\sigma^{ab})_\alpha{}^\alpha(\sigma^c)_{\alpha\dot\alpha}\eta_{ca}P_b \\
&= 2Z^{IJ}P_{\alpha\dot\alpha} - 2Z^{JI}P_{\alpha\dot\alpha},
\end{aligned} \tag{433}
$$

which gives

$$
Z^{IJ} = Z^{JI}. \tag{434}
$$

From (431) and (434) we conclude that $Z = 0$ and thus also $Y = 0$. We have

$$
[Q^I_\alpha, M_{\alpha\beta}] \in (1 - 1/2, 0) = (1/2, 0), \tag{435}
$$

since there is no generator $(3/2, 0)$. We write

$$
\begin{aligned}
[Q^I_\alpha, M_{ab}] &= (b_{ab})_\alpha{}^\beta Q^I_\beta \\
[\bar{Q}^I_{\dot\alpha}, M_{ab}] &= (\bar{b}_{ab})^{\dot\beta}{}_{\dot\alpha}\bar{Q}^I_{\dot\beta},
\end{aligned} \tag{436}
$$

and get

$$
\begin{aligned}
0 &= \{[M_{ab}, Q^I_{\dot\gamma}], \bar{Q}^J_{\dot\gamma}\} + \{[M_{ab}, \bar{Q}^J_{\dot\gamma}], Q^I_\gamma\} + \big[\{Q^I_\gamma, \bar{Q}^J_{\dot\gamma}\}, M_{ab}\big] \\
&= -(b_{ab})_\gamma{}^\beta\{Q^I_\beta, \bar{Q}^J_{\dot\gamma}\} - (\bar{b}_{ab})^{\dot\beta}{}_{\dot\gamma}\{\bar{Q}^J_{\dot\beta}, Q^I_\gamma\} + 2\delta^{IJ}[P_{\gamma\dot\gamma}, M_{ab}] \\
&= -(b_{ab})_\gamma{}^\beta 2\delta^{IJ}P_{\beta\dot\gamma} - (\bar{b}_{ab})^{\dot\beta}{}_{\dot\gamma} 2\delta^{IJ}P_{\gamma\dot\beta} - 2\delta^{IJ}\sigma^c{}_{\gamma\dot\gamma}[M_{ab}, P_c] \\
&= -2(b_{ab})_\gamma{}^\beta\delta^{IJ}P_{\beta\dot\gamma} - 2(\bar{b}_{ab})^{\dot\beta}{}_{\dot\gamma}\delta^{IJ}P_{\gamma\dot\beta} - 2\delta^{IJ}(\sigma^c)_{\gamma\dot\gamma}(i\eta_{ca}P_b - i\eta_{cb}P_a) \\
&= -2(b_{ab})_\gamma{}^\beta\delta^{IJ}\delta^{\dot\beta}_{\dot\gamma}P_{\beta\dot\beta} - 2(\bar{b}_{ab})^{\dot\beta}{}_{\dot\gamma}\delta^{IJ}\delta^\beta_\gamma P_{\beta\dot\beta}
\end{aligned}
$$

$$+ i\delta^{IJ}\Big((\sigma_a)_{\gamma\dot{\gamma}}(\bar{\sigma}_b)^{\dot{\beta}\beta} - (\sigma_b)_{\gamma\dot{\gamma}}(\bar{\sigma}_a)^{\dot{\beta}\beta}\Big)P_{\beta\dot{\beta}} \tag{437}$$

This is zero if

$$\begin{cases} 2(b_{ab})_\gamma{}^\beta \delta_{\dot{\gamma}}^{\dot{\beta}} = i(\sigma_a)_{\gamma\dot{\gamma}}(\bar{\sigma}_b)^{\dot{\beta}\beta} \\ 2(\bar{b}_{ab})^{\dot{\beta}}{}_{\dot{\gamma}}\delta_\gamma^\beta = -i(\bar{\sigma}_a)^{\dot{\beta}\beta}(\sigma_b)_{\gamma\dot{\gamma}}. \end{cases} \tag{438}$$

Multiplying the first line with $\delta_{\dot{\beta}}^{\dot{\gamma}}$ and the second line with δ_β^γ we get

$$\begin{cases} 4(b_{ab})_\gamma{}^\beta = i(\sigma_a)_{\gamma\dot{\gamma}}(\bar{\sigma}_b)^{\dot{\beta}\beta} \\ 4(\bar{b}_{ab})^{\dot{\beta}}{}_{\dot{\gamma}} = -i(\bar{\sigma}_a)^{\dot{\beta}\gamma}(\sigma_b)_{\gamma\dot{\gamma}}. \end{cases} \tag{439}$$

Since $M_{ab} = -M_{ba}$ we also have $b_{ab} = -b_{ba}$ and $\bar{b}_{ab} = -\bar{b}_{ba}$, and we get

$$\begin{cases} 4(b_{ab})_\gamma{}^\beta = 2(b_{ab})_\gamma{}^\beta - 2(b_{ba})_\gamma{}^\beta \overset{!}{=} \frac{i}{2}(\sigma_a)_{\gamma\dot{\gamma}}(\bar{\sigma}_b)^{\dot{\gamma}\beta} - \frac{i}{2}(\sigma_b)_{\gamma\dot{\gamma}}(\bar{\sigma}_a)^{\dot{\gamma}\beta} = -2i(\sigma_{ab})_\gamma{}^\beta \\ 4(\bar{b}_{ab})^{\dot{\beta}}{}_{\dot{\gamma}} = 2(\bar{b}_{ab})^{\dot{\beta}}{}_{\dot{\gamma}} - 2(\bar{b}_{ba})^{\dot{\beta}}{}_{\dot{\gamma}} \overset{!}{=} -\frac{i}{2}(\bar{\sigma}_a)^{\dot{\beta}\gamma}(\sigma_b)_{\gamma\dot{\gamma}} + \frac{i}{2}(\bar{\sigma}_b)^{\dot{\beta}\gamma}(\sigma_a)_{\gamma\dot{\gamma}} = 2i(\bar{\sigma}_{ab})^{\dot{\beta}}{}_{\dot{\gamma}} \end{cases} \tag{440}$$

or

$$\begin{cases} (b_{ab})_\gamma{}^\beta = -\frac{i}{2}(\sigma_{ab})_\gamma{}^\beta \\ (\bar{b}_{ab})^{\dot{\beta}}{}_{\dot{\gamma}} = \frac{i}{2}(\bar{\sigma}_{ab})^{\dot{\beta}}{}_{\dot{\gamma}}. \end{cases} \tag{441}$$

The remaining commutator is

$$[Q_\alpha^I, B_l] = (S_l)^I{}_J Q_\alpha^J, \tag{442}$$

where $(S_l)^I{}_J$ can be shown to form a representation of the internal group. The complex constants X^{IJ} can be shown to commute with every other operator and we call them central charges. We can now write down the full N-extended super-Poincaré algebra (120).

B.2 Derivation of the superalgebra: an alternative way

We want to derive

$$[Q_\alpha^i, M_{\mu\nu}] = (b_{\mu\nu})_\alpha^\beta Q_\beta^i. \tag{443}$$

To this end we use the following generalized Jacobi identity

$$\big[[M_{\mu\nu}, M_{\tau\sigma}], Q_\alpha^i\big] + \big[[Q_\alpha^i, M_{\mu\nu}], M_{\tau\sigma}\big] + \big[[M_{\tau\sigma}, Q_\alpha^i], M_{\mu\nu}\big] = 0. \tag{444}$$

First term is

$$\begin{aligned} &[(\eta_{\tau\mu}M_{\nu\sigma} - \eta_{\tau\nu}M_{\mu\sigma} - \eta_{\sigma\mu}M_{\nu\tau} + \eta_{\sigma\nu}M_{\mu\tau}), Q_\alpha^i] \\ =&\eta_{\tau\mu}[M_{\nu\sigma}, Q_\alpha^i] - \eta_{\tau_\nu}[M_{\mu\sigma}, Q_\alpha^i] - \eta_{\sigma\mu}[M_{\nu\tau}, Q_\alpha^i] + \eta_{\sigma\nu}[M_{\mu\tau}, Q_\alpha^i] \\ =&-\eta_{\tau\mu}(b_{\nu\sigma})_\alpha^\beta Q_\beta^i + \eta_{\tau\nu}(b_{\mu\sigma})_\beta^\alpha Q_\beta^i + \eta_{\sigma\mu}(b_{\nu\tau})_\alpha^\beta Q_\beta^i - \eta_{\sigma\nu}(b_{\mu\tau})_\alpha^\beta Q_\beta^i \\ =&\big(-\eta_{\tau\mu}(b_{\nu\sigma})_\alpha^\beta + \eta_{\tau\nu}(b_{\mu\sigma})_\alpha^\beta + \eta_{\sigma\mu}(b_{\nu\tau})_\alpha^\beta - \eta_{\sigma\nu}(b_{\mu\tau})_\alpha^\beta\big)Q_\beta^i, \end{aligned} \tag{445}$$

second term

$$[(b_{\mu\nu})_\alpha^\beta Q_\beta^i, M_{\tau\sigma}] = (b_{\mu\nu})_\alpha^\beta [Q_\beta^i, M_{\tau\sigma}] = (b_{\mu\nu})_\alpha^\beta (b_{\tau\sigma})_\beta^\gamma Q_\gamma^i = (b_{\mu\nu})_\alpha^\gamma (b_{\tau\sigma})_\gamma^\beta Q_\beta^i, \tag{446}$$

and the third term

$$[-(b_{\tau\sigma})_\alpha^\beta Q_\beta^i, M_{\mu\nu}] = -(b_{\tau\sigma})_\alpha^\beta [Q_\beta^i, M_{\mu\nu}] = -(b_{\tau\sigma})_\alpha^\beta (b_{\mu\nu})_\beta^\gamma Q_\gamma^i = -(b_{\tau\sigma})_\alpha^\gamma (b_{\mu\nu})_\gamma^\beta Q_\beta^i. \tag{447}$$

Equation (444) then equals

$$\begin{aligned}
&\big(-\eta_{\tau\mu}(b_{\nu\sigma})_\alpha^\beta + \eta_{\tau\nu}(b_{\mu\sigma})_\alpha^\beta + \eta_{\sigma\mu}(b_{\nu\tau})_\alpha^\beta - \eta_{\sigma\nu}(b_{\mu\tau})_\alpha^\beta \\
&+ (b_{\mu\nu})_\alpha^\gamma (b_{\tau\sigma})_\gamma^\beta \ - (b_{\tau\sigma})_\alpha^\gamma (b_{\mu\nu})_\gamma^\beta \big) Q_\beta^i \\
=&\big(-\eta_{\tau\mu}(b_{\nu\sigma})_\alpha^\beta + \eta_{\tau\nu}(b_{\mu\sigma})_\alpha^\beta + \eta_{\sigma\mu}(b_{\nu\tau})_\alpha^\beta - \eta_{\sigma\nu}(b_{\mu\tau})_\alpha^\beta \ + [b_{\mu\nu}, b_{\tau\sigma}]_\alpha^\beta \big) Q_\beta^i,
\end{aligned} \tag{448}$$

which is zero if

$$\begin{aligned}
[b_{\mu\nu}, b_{\tau\sigma}]_\alpha^\beta &= +\eta_{\tau\mu}(b_{\nu\sigma})_\alpha^\beta - \eta_{\tau\nu}(b_{\mu\sigma})_\alpha^\beta - \eta_{\sigma\mu}(b_{\nu\tau})_\alpha^\beta + \eta_{\sigma\nu}(b_{\mu\tau})_\alpha^\beta \\
&= \eta_{\tau[\mu}(b_{\nu]\sigma})_\alpha^\beta - \eta_{\sigma[\mu}(b_{\nu]\tau})_\alpha^\beta.
\end{aligned} \tag{449}$$

$(b_{\mu\nu})_\alpha^\beta$ corresponds to a representation of the Lorentz algebra and we choose the $(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$-representation. We arrive at

$$[Q_\alpha^i, M_{\mu\nu}] = \frac{1}{2}(\sigma_{\mu\nu})_\alpha^\beta Q_\beta^i.$$

C Reduction from $N = (1, 1)$ to $N = 1$

The two-dimensional $N = (1, 1)$ sigma model reads

$$\begin{aligned}
&S = \int d^2x d^2\theta D_+ \phi^\mu E_{\mu\nu} D_- \phi^\nu, \\
&\phi^\mu(x, \theta) = X^\mu(x) + \theta^+ \psi_+^\mu(x) + \theta^- \psi_-^\mu(x) + \theta^+\theta^- F^\mu(x), \\
&D_\pm = \partial_\pm + i\theta^\pm \partial_{\pm\pm}, \qquad Q_\pm = i\partial_\pm + \theta^\pm \partial_{\pm\pm}.
\end{aligned} \tag{450}$$

Let all fields be independent of the spatial coordinate of the worldsheet. Thus $\partial_{\pm\pm} = \partial_t$. Define

$$\begin{aligned}
&\theta := \frac{1}{\sqrt{2}}(\theta^+ + \theta^-), \qquad \tilde{\theta} := \frac{1}{\sqrt{2}}(\theta^+ - \theta^-), \\
&D := \frac{1}{\sqrt{2}}(D_+ + D_-) = \partial_\theta + i\theta\partial_t \\
&\tilde{D} := \frac{1}{\sqrt{2}}(D_+ - D_-) = \partial_{\tilde{\theta}} + i\tilde{\theta}\partial_t \\
&\psi^\mu := \frac{1}{\sqrt{2}}(\psi_+^\mu + \psi_-^\mu), \qquad \tilde{\psi}^\mu := \frac{1}{\sqrt{2}}(\psi_+^\mu - \psi_-^\mu).
\end{aligned} \tag{451}$$

The superfields become

$$\begin{aligned}\phi^\mu(t,\theta^+,\theta^-) &= X^\mu(t)+\theta^+\psi_+^\mu(t)+\theta^-\psi_-^\mu(t)+\theta^+\theta^-F^\mu(t)\\ &= X^\mu(t)+\theta\psi^\mu(t)+\tilde\theta\tilde\psi^\mu(t)-\theta\tilde\theta F^\mu(t).\end{aligned} \tag{452}$$

Let $\tilde\theta=0$ and define new bosonic superfields $\hat X^\mu(t,\theta)$ and new fermionic superfields $\hat\psi^\mu(t,\theta)$:

$$\begin{aligned}\phi^\mu|_{\tilde\theta=0} &= X^\mu(t)+\theta\psi^\mu(t)=:\hat X^\mu(t,\theta)\\ D\phi^\mu|_{\tilde\theta=0} &= \psi^\mu(t)+i\theta\partial_t X^\mu(t)=D\hat X^\mu(t,\theta)\\ \tilde D\phi^\mu|_{\tilde\theta=0} &= \tilde\psi^\mu(t)+\theta F^\mu(t)=:\hat\psi^\mu(t,\theta)\\ D\tilde D\phi^\mu|_{\tilde\theta=0} &= F^\mu(t)+i\theta\partial_t\tilde\psi^\mu(t)=D\hat\psi^\mu(t,\theta).\end{aligned} \tag{453}$$

We call the reduced action S_R,

$$\begin{aligned}S_R &= \int dtd\theta^+d\theta^-D_+E_{\mu\nu}D_-\phi^\nu=-\int dtd\theta d\tilde\theta\big[\frac{1}{\sqrt2}(D+\tilde D)\phi^\mu E_{\mu\nu}\frac{1}{\sqrt2}(D-\tilde D)\phi^\nu\big]\\ &= -\frac12\int dtd\theta\tilde D\big[D\phi^\mu E_{\mu\nu}D\phi^\nu-D\phi^\mu E_{\mu\nu}\tilde D\phi^\nu+\tilde D\phi^\mu E_{\mu\nu}D\phi^\nu-\tilde D\phi^\mu E_{\mu\nu}\tilde D\phi^\nu\big]|_{\tilde\theta=0}\\ &= -\frac12\int dtd\theta\big[-D\tilde DE_{\mu\nu}D\phi^\nu-D\phi^\mu E_{\mu\nu,\rho}\tilde D\phi^\rho D\phi^\nu+D\phi^\mu E_{\mu\nu}D\tilde D\phi^\nu\\ &\quad+D\tilde D\phi^\mu E_{\mu\nu}\tilde D\phi^\nu+D\phi^\mu E_{\mu\nu,\rho}\tilde D\phi^\rho\tilde D\phi^\nu+D\phi^\mu E_{\mu\nu}\tilde D^2\phi^\nu\\ &\quad+\tilde D^2\phi^\mu E_{\mu\nu}D\phi^\nu-\tilde D\phi^\mu E_{\mu\nu,\rho}\tilde D\phi^\rho D\phi^\nu+\tilde D\phi^\mu E_{\mu\nu}D\tilde D\phi^\nu\\ &\quad-\tilde D^2\phi^\mu E_{\mu\nu}\tilde D\phi^\nu+\tilde D\phi^\mu E_{\mu\nu,\rho}\tilde D\phi^\rho\tilde D\phi^\nu+\tilde D\phi^\mu E_{\mu\nu}\tilde D^2\phi^\nu\big]|_{\tilde\theta=0}\\ &= -\frac12\int dtd\theta\big[-D\hat\psi^\mu E_{\mu\nu}D\hat X^\nu-D\hat X^\mu E_{\mu\nu,\rho}\hat\psi^\rho D\hat X^\nu+D\hat X^\mu E_{\mu\nu}D\hat\psi^\nu\\ &\quad+D\hat\psi^\mu E_{\mu\nu}\hat\psi^\nu+D\hat X^\mu E_{\mu\nu,\rho}\hat\psi^\rho\hat\psi^\nu+D\hat X^\mu E_{\mu\nu}i\partial_t\hat X^\nu\\ &\quad+i\partial_t\hat X^\mu E_{\mu\nu}D\hat X^\nu-\hat\psi^\mu E_{\mu\nu,\rho}\hat\psi^\rho D\hat X^\nu+\hat\psi^\mu E_{\mu\nu}D\hat\psi^\nu\\ &\quad-i\partial_t\hat X^\mu E_{\mu\nu}\hat\psi^\nu+\hat\psi^\mu E_{\mu\nu,\rho}\hat\psi^\rho\hat\psi^\nu+\hat\psi^\mu E_{\mu\nu}i\partial_t\hat X^\nu\big]\end{aligned} \tag{454}$$

For greater clearity we label the rows A,B,C,D, and columns 1,2,3, and analyse the terms:
A1+A3:

$$-D\hat\psi^\mu E_{\mu\nu}D\hat X^\nu+D\hat X^\mu E_{\mu\nu}D\hat\psi^\nu=(E_{\mu\nu}-E_{\nu\mu})D\hat X^\mu D\hat\psi^\nu=2b_{\mu\nu}D\hat X^\mu D\hat\psi^\nu, \tag{455}$$

A2:

$$-D\hat X^\mu E_{\mu\nu,\rho}\hat\psi^\rho D\hat X^\nu=b_{\mu\nu,\rho}D\hat X^\mu D\hat X^\nu\hat\psi^\rho, \tag{456}$$

B1+C3:

$$D\hat\psi^\mu E_{\mu\nu}\hat\psi^\nu+\hat\psi^\mu E_{\mu\nu}D\hat\psi^\nu=(E_{\nu\mu}+E_{\mu\nu})\hat\psi^\mu D\hat\psi^\nu=2G_{\mu\nu}\hat\psi^\mu D\hat\psi^\nu, \tag{457}$$

B2+C2:

$$D\hat{X}^\mu E_{\mu\nu,\rho}\hat{\psi}^\rho\hat{\psi}^\nu - \hat{\psi}^\mu E_{\mu\nu,\rho}\hat{\psi}^\rho D\hat{X}^\nu = (E_{\mu\nu,\rho}+E_{\nu\mu,\rho})D\hat{X}^\mu\hat{\psi}^\rho\hat{\psi}^\nu = 2G_{\mu\nu,\rho}\hat{\psi}^\mu D\hat{X}^\nu\hat{\psi}^\rho, \tag{458}$$

B3+C1:

$$D\hat{X}^\mu E_{\mu\nu} i\partial_t\hat{X}^\nu + i\partial_t\hat{X}^\mu E_{\mu\nu} D\hat{X}^\nu = D\hat{X}^\mu(E_{\mu\nu}+E_{\nu\mu})i\partial_t\hat{X}^\nu = 2iG_{\mu\nu}D\hat{X}^\mu\partial_t\hat{X}^\nu, \tag{459}$$

D1+D3:

$$-i\partial_t\hat{X}^\mu E_{\mu\nu}\hat{\psi}^\nu + \hat{\psi}^\mu E_{\mu\nu} i\partial_t\hat{X}^\nu = -i\partial_t\hat{X}^\mu(E_{\mu\nu}-E_{\nu\mu})\hat{\psi}^\nu = -2ib_{\mu\nu}\partial_t\hat{X}^\mu\hat{\psi}^\nu, \tag{460}$$

D2:

$$\begin{aligned}\hat{\psi}^\mu E_{\mu\nu,\rho}\hat{\psi}^\rho\hat{\psi}^\nu &= -E_{\mu\nu,\rho}\hat{\psi}^\mu\hat{\psi}^\nu\hat{\psi}^\rho = -b_{\mu\nu,\rho}\hat{\psi}^\mu\hat{\psi}^\nu\hat{\psi}^\rho \\ &= -\frac{1}{3}(b_{\mu\nu,\rho}-b_{\rho\nu,\mu}-b_{\mu\rho,\nu})\hat{\psi}^\mu\hat{\psi}^\nu\hat{\psi}^\rho = -\frac{1}{3}T_{\mu\nu\rho}\hat{\psi}^\mu\hat{\psi}^\nu\hat{\psi}^\rho,\end{aligned} \tag{461}$$

where $T = db$. The action becomes

$$\begin{aligned}S_R = -\frac{1}{2}\int dtd\theta\big[&2b_{\mu\nu}D\hat{X}^\mu D\hat{\psi}^\nu + b_{\mu\nu,\rho}D\hat{X}^\mu D\hat{X}^\nu\hat{\psi}^\rho + 2G_{\mu\nu}\hat{\psi}^\mu D\hat{\psi}^\nu \\ &+ 2G_{\mu\nu,\rho}\hat{\psi}^\mu D\hat{X}^\nu\hat{\psi}^\rho + 2iG_{\mu\nu}D\hat{X}^\mu\partial_t\hat{X}^\nu - 2ib_{\mu\nu}\partial_t\hat{X}^\mu\hat{\psi}^\nu - \frac{1}{3}T_{\mu\nu\rho}\hat{\psi}^\mu\hat{\psi}^\nu\hat{\psi}^\rho\big] \\ = \int dtd\theta\big[&-b_{\mu\nu}D\hat{X}^\mu D\hat{\psi}^\nu - \frac{1}{2}b_{\mu\nu,\rho}D\hat{X}^\mu D\hat{X}^\nu\hat{\psi}^\rho - G_{\mu\nu}\hat{\psi}D\hat{\psi}^\nu \\ &- G_{\mu\nu,\rho}\hat{\psi}^\mu D\hat{X}^\nu\hat{\psi}^\rho - iG_{\mu\nu}D\hat{X}^\mu\partial_t\hat{X}^\nu + ib_{\mu\nu}\partial_t\hat{X}^\mu\hat{\psi}^\nu + \frac{1}{6}T_{\mu\nu\rho}\hat{\psi}^\mu\hat{\psi}^\nu\hat{\psi}^\rho\big].\end{aligned} \tag{462}$$

Noting that

$$\begin{aligned}D(b_{\mu\nu}D\hat{X}^\mu\hat{\psi}^\nu) &= b_{\mu\nu,\rho}D\hat{X}^\rho D\hat{X}^\mu\hat{\psi}^\nu + b_{\mu\nu}D^2\hat{X}^\mu\hat{\psi}^\nu - b_{\mu\nu}D\hat{X}^\mu D\hat{\psi}^\nu \\ &= \frac{1}{2}(b_{\nu\rho,\mu}+b_{\rho\mu,\nu})D\hat{X}^\mu D\hat{X}^\nu\hat{\psi}^\rho + ib_{\mu\nu}\partial_t\hat{X}^\mu\hat{\psi}^\nu - b_{\mu\nu}D\hat{X}^\mu D\hat{\psi}^\nu,\end{aligned} \tag{463}$$

and that

$$-\frac{1}{2}(b_{\nu\rho,\mu}+b_{\rho\mu,\nu})D\hat{X}^\mu D\hat{X}^\nu\hat{\psi}^\rho - \frac{1}{2}b_{\mu\nu,\rho}D\hat{X}^\mu D\hat{X}^\nu\hat{\psi}^\rho = \frac{1}{2}T_{\mu\nu\rho}D\hat{X}^\mu D\hat{X}^\nu\hat{\psi}^\rho, \tag{464}$$

we finally arrive at

$$\begin{aligned}S_R = \int dtd\theta\big[&-iG_{\mu\nu}D\hat{X}^\mu\partial_t\hat{X}^\nu - G_{\mu\nu}\hat{\psi}^\mu D\hat{\psi}^\nu - G_{\mu\nu,\rho}\hat{\psi}^\mu D\hat{X}^\nu\hat{\psi}^\rho \\ &+ \frac{1}{6}T_{\mu\nu\rho}\hat{\psi}^\mu\hat{\psi}^\nu\hat{\psi}^\rho - \frac{1}{2}T_{\mu\nu\rho}D\hat{X}^\mu D\hat{X}^\nu\hat{\psi}^\rho + D(b_{\mu\nu}D\hat{X}^\mu\hat{\psi}^\nu)\big].\end{aligned} \tag{465}$$

D An alternative introduction to supersymmetry

We will now introduce the basic concepts of supersymmetry by means of fields in ordinary four-dimensional space-time. We will follow the outline of Martin [20]. To this end we write down an action for a left-handed Weyl fermion ψ (appendix A.1) together with a boson described by a complex scalar field ϕ.

$$S = \int d^4x(\mathcal{L}_{\text{boson}} + \mathcal{L}_{\text{fermion}}) \tag{466}$$

For simplicity we will only consider the massless non-interaction case, thus only including the kinectic terms

$$\mathcal{L}_{\text{boson}} = -\partial^\mu \phi^* \partial_\mu \phi, \qquad \mathcal{L}_{\text{fermion}} = i\psi^\dagger_{\dot{\alpha}} (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} \partial_\mu \psi_\alpha \tag{467}$$

We want a supersymmetry transformation Q to change a boson into a fermion and vice versa, i.e., roughly

$$Q|\text{boson} >= |\text{fermion} >, \qquad Q|\text{fermion} >= |\text{boson} > . \tag{468}$$

Using the language of fields we thus write an infinitesimal change of the scalar field

$$\delta\phi = \epsilon^\alpha \psi_\alpha, \qquad \delta\phi^* = \epsilon^\dagger_{\dot{\alpha}} \psi^{\dagger\dot{\alpha}}, \tag{469}$$

where ϵ is an infintesimal and anticommuting Weyl fermion inserted because bosons and fermions obey opposite statistics. It then follows that

$$\begin{aligned}\delta\mathcal{L}_{\text{boson}} &= -\partial^\mu \delta\phi^* \partial_\mu \phi - \partial^\mu \phi^* \partial_\mu \delta\phi = -\partial^\mu (\epsilon^\dagger_{\dot{\alpha}} \psi^{\dagger\dot{\alpha}}) \partial_\mu \phi - \partial^\mu \phi^* \partial_\mu (\epsilon^\alpha \phi_\alpha) \\ &= -\epsilon^\dagger_{\dot{\alpha}} \partial^\mu \psi^{\dagger\dot{\alpha}} \partial_\mu \phi - \epsilon^\alpha \partial^\mu \psi_\alpha \partial_\mu \phi^*.\end{aligned} \tag{470}$$

We want our action S to be invariant under a supersymmetry transformation

$$\delta S = \int d^4x(\delta\mathcal{L}_{\text{boson}} + \delta\mathcal{L}_{\text{fermion}}) = 0, \tag{471}$$

i.e., if we can find the infinitesimal transformation for ψ such that $\delta\mathcal{L}_{\text{boson}} + \delta\mathcal{L}_{\text{fermion}} = 0$ up to a surface term, our problem is be solved. If we take

$$\delta\psi_\alpha = -i(\sigma^\mu)_{\alpha\dot{\alpha}} \epsilon^{\dagger\dot{\alpha}} \partial_\mu \phi, \qquad \delta\psi^\dagger_{\dot{\alpha}} = i\epsilon^\alpha (\sigma^\mu)_{\alpha\dot{\alpha}} \partial_\mu \phi^*, \tag{472}$$

this will indeed be the case. We have

$$\begin{aligned}\delta\mathcal{L}_{\text{fermion}} &= i\delta\psi^\dagger_{\dot{\alpha}} (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} \partial_\mu \psi_\alpha + i\psi^\dagger_{\dot{\alpha}} (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} \partial_\mu \delta\psi_\alpha \\ &= i\Big[i\epsilon^\beta (\sigma^\nu)_{\beta\dot{\alpha}} \partial_\nu \phi^*\Big] (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} \partial_\mu \psi_\alpha + i\psi^\dagger_{\dot{\alpha}} (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} \partial_\mu \Big[-i(\sigma^\nu)_{\alpha\dot{\beta}} \epsilon^{\dagger\dot{\beta}} \partial_\nu \phi\Big] \\ &= -\epsilon^\beta (\sigma^\nu)_{\beta\dot{\alpha}} (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} \partial_\mu \psi_\alpha \partial_\nu \phi^* + \psi^\dagger_{\dot{\alpha}} (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} (\sigma^\nu)_{\alpha\dot{\beta}} \epsilon^{\dagger\dot{\beta}} \partial_\nu \partial_\mu \phi.\end{aligned} \tag{473}$$

Using that

$$
\begin{aligned}
(\sigma^\mu)_{\alpha\dot\alpha}(\sigma^\nu)^{\dot\alpha\beta}\partial_\mu\partial_\nu\phi &= \frac{1}{2}\big[(\sigma^\mu)_{\alpha\dot\alpha}(\sigma^\nu)^{\dot\alpha\beta}\partial_\mu\partial_\nu\phi + (\sigma^\nu)_{\alpha\dot\alpha}(\sigma^\mu)^{\dot\alpha\beta}\partial_\nu\partial_\mu\phi\big] \\
&= \frac{1}{2}\big[(\sigma^\mu)_{\alpha\dot\alpha}(\sigma^\nu)^{\dot\alpha\beta} + (\sigma^\nu)_{\alpha\dot\alpha}(\sigma^\mu)^{\dot\alpha\beta}\big]\partial_\mu\partial_\nu\phi, \qquad (474)
\end{aligned}
$$

where we have made use of the commutation of partial derivatives $\partial_\mu\partial_\nu = \partial_\nu\partial_\mu$, and that

$$
(\sigma^\mu)_{\alpha\dot\alpha}(\bar\sigma^\nu)^{\dot\alpha\beta} - (\sigma^\mu)_{\alpha\dot\alpha}(\bar\sigma^\nu)^{\dot\alpha\beta} = -2\eta^{\mu\nu}\delta^\beta_\alpha, \qquad (475)
$$

the first term gives (suppressing the indices)

$$
\begin{aligned}
-\epsilon\sigma^\nu\bar\sigma^\mu\partial_\mu\psi\partial_\nu\phi^* &= -\partial_\mu(\epsilon\sigma^\nu\bar\sigma^\mu\psi\partial_\nu\phi^*) + \epsilon\sigma^\nu\bar\sigma^\mu\psi\partial_\mu\partial_\nu\phi^* \\
&= -\partial_\mu(\epsilon\sigma^\nu\bar\sigma^\mu\psi\partial_\nu\phi^*) + \frac{1}{2}\epsilon(\sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu)\psi\partial_\mu\partial_\nu\phi^* \\
&= -\partial_\mu(\epsilon\sigma^\nu\bar\sigma^\mu\psi\partial_\nu\phi^*) + \frac{1}{2}\epsilon(-2\eta^{\mu\nu})\psi\partial_\mu\partial_\nu\phi^* \\
&= -\partial_\mu(\epsilon\sigma^\nu\bar\sigma^\mu\psi\partial_\nu\phi^*) - \epsilon\psi\partial_\mu\partial^\mu\phi^* \\
&= -\partial_\mu(\epsilon\sigma^\nu\bar\sigma^\mu\psi\partial_\nu\phi^*) - \partial_\mu(\epsilon\psi\partial^\mu\phi^*) + \epsilon\partial_\mu\psi\partial^\mu\phi^*, \qquad (476)
\end{aligned}
$$

and the second term

$$
\begin{aligned}
\psi^\dagger\bar\sigma^\mu\sigma^\nu\partial_\nu\partial_\mu\phi &= \frac{1}{2}\psi^\dagger(\bar\sigma^\mu\sigma^\nu + \bar\sigma^\nu\sigma^\mu)\epsilon^\dagger\partial_\mu\partial_\nu\phi \\
&= \frac{1}{2}\psi^\dagger(-2\eta^{\mu\nu})\epsilon^\dagger\partial_\mu\partial_\nu\phi \\
&= -\psi^\dagger\epsilon^\dagger\partial_\mu\partial^\mu\phi \\
&= -\partial_\mu(\psi^\dagger\epsilon^\dagger\partial^\mu\phi) + \partial_\mu\psi^\dagger\epsilon^\dagger\partial^\mu\phi \\
&= -\partial_\mu(\epsilon^\dagger\psi^\dagger\partial^\mu\phi) + \epsilon^\dagger\partial_\mu\psi^\dagger\partial^\mu\phi, \qquad (477)
\end{aligned}
$$

Together we have

$$
\delta\mathcal{L}_{\text{fermion}} = \epsilon\partial^\mu\psi\partial_\mu\phi^* + \epsilon^\dagger\partial^\mu\psi^\dagger\partial_\mu\phi - \partial\Big(\epsilon\sigma^\nu\bar\sigma^\mu\psi\partial_\nu\phi^* + \epsilon\psi\partial^\mu\phi^* + \epsilon^\dagger\psi^\dagger\partial^\mu\phi\Big), \qquad (478)
$$

and we see that this cancels the bosonic lagrangian (470) up to a surface term.

We also need to verify that the algebra closes under these transformations, i.e., that the commutator of two supersymmetry transformations necessarily is a symmetry of the theory. For the bosonic field we have

$$
\begin{aligned}
[\delta_{\epsilon_2}, \delta_{\epsilon_1}]\phi &= \delta_{\epsilon_2}(\delta_{\epsilon_1}\phi) - \delta_{\epsilon_1}(\delta_{\epsilon_2}\phi) \\
&= -\Big(\epsilon_1(\sigma^\mu\epsilon_2^\dagger)_\alpha - \epsilon_2(\sigma^\mu\epsilon_1^\dagger)_\alpha\Big)(i\partial_\mu\phi), \qquad (479)
\end{aligned}
$$

and we get (up to a factor) the space-time translation $P_\mu = i\partial_\mu$ which of course is a symmetry of the theory. In the following we will use the notation

$$
(\sigma^\mu\epsilon)_{\dot\alpha} := (\sigma^\mu)_{\alpha\dot\alpha}\epsilon^\alpha, \qquad (\epsilon^\dagger\bar\sigma^\mu)^\alpha := \epsilon^\dagger_{\dot\alpha}(\epsilon^\mu)^{\dot\alpha\alpha}. \qquad (480)
$$

The fermionic field transforms according to

$$
\begin{aligned}
[\delta_{\epsilon_2}, \delta_{\epsilon_1}]\psi_\alpha &= \delta_{\epsilon_2}(\delta_{\epsilon_1}\psi_\alpha) - \delta_{\epsilon_1}(\delta_{\epsilon_2}\psi_\alpha) \\
&= \delta_{\epsilon_2}\big(-i(\sigma^\mu\epsilon_1^\dagger)_\alpha\partial_\mu\phi\big) - \delta_{\epsilon_1}\big(-i(\sigma^\mu\epsilon_2^\dagger)_\alpha\partial_\mu\phi\big) \\
&= -i(\sigma^\mu\epsilon_1^\dagger)_\alpha\partial_\mu(\epsilon_2\psi) + i(\sigma^\mu\epsilon_2^\dagger)_\alpha\partial_\mu(\epsilon_1\psi) \\
&= (\sigma^\mu)_{\alpha\dot{\alpha}}(\epsilon_2^{\dagger\dot{\alpha}}\epsilon_1^\beta - \epsilon_1^{\dagger\dot{\alpha}}\epsilon_2^\beta)(i\partial_\mu)\psi_\beta.
\end{aligned} \tag{481}
$$

Since the ψ we started with and the ψ at the end have different indices this is not yet a symmetry, but using the Fierz rearrangement identity

$$
\chi_\alpha\xi^\beta\eta_\beta = -\xi_\alpha\eta^\beta\chi_\beta - \eta_\alpha\chi^\beta\xi_\beta, \tag{482}
$$

we can rewrite this

$$
\begin{aligned}
(\sigma^\mu)_{\alpha\dot{\alpha}}(\epsilon_2^{\dagger\dot{\alpha}}\epsilon_1^\beta - \epsilon_1^{\dagger\dot{\alpha}}\epsilon_2^\beta)(i\partial_\mu)\psi_\beta &= (\sigma^\mu\epsilon_2^\dagger)_\alpha\epsilon_1^\beta(i\partial_\mu\psi)_\beta - (\sigma^\mu\epsilon_1^\dagger)_\alpha\epsilon_2^\beta(i\partial_\mu\psi)_\beta \\
&= -\epsilon_{1\alpha}(i\partial_\mu\psi)^\beta(\sigma^\mu\epsilon_2^\dagger)_\beta - (i\partial_\mu\psi)_\alpha(\epsilon_2^\dagger\bar{\sigma}^\mu)^\beta\epsilon_{1\beta} \\
&+ \epsilon_{2\alpha}(i\partial_\mu\psi)^\beta(\sigma^\mu\epsilon_1^\dagger)_\beta + (i\partial_\mu\psi)_\alpha(\epsilon_1^\dagger\bar{\sigma}^\mu)^\beta\epsilon_{2\beta} \\
&= \Big[\epsilon_2^\beta(\sigma^\mu)_{\beta\dot{\alpha}}\epsilon_1^{\dagger\dot{\alpha}} - \epsilon_1^\beta(\sigma^\mu)_{\beta\dot{\alpha}}\epsilon_2^{\dagger\dot{\alpha}}\Big](i\partial_\mu)\psi_\alpha \\
&+ \Big[\epsilon_{1\alpha}\epsilon_{2\dot{\alpha}}^\dagger - \epsilon_{2\alpha}\epsilon_{1\dot{\alpha}}^\dagger\Big](\bar{\sigma}^\mu)^{\dot{\alpha}\beta}(i\partial_\mu)\psi_\beta.
\end{aligned} \tag{483}
$$

The first term is again the translation operator (times a factor), but the rest of the terms only vanish by use of the equations of motion $\bar{\sigma}^\mu\partial_\mu\psi = 0$, i.e., they only vanish on-shell. By introducing an auxiliary field F, we can however make the algebra close even off-shell. Let F be a complex scalar field that transforms as

$$
\delta F = -i\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\psi, \qquad \delta F^* = i\partial_\mu\psi^\dagger\bar{\sigma}^\mu\epsilon. \tag{484}
$$

A lagrangian

$$
\mathcal{L}_{\text{auxiliary}} = F^*F, \tag{485}
$$

then transforms as

$$
\delta\mathcal{L}_{\text{auxiliary}} = i\partial_\mu\psi^\dagger\bar{\sigma}^\mu\epsilon F - i\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\psi F^*. \tag{486}
$$

Letting the fermionic field mix with the auxiliary field under a transformation

$$
\delta\psi_\alpha = -i(\sigma^\mu\epsilon^\dagger)_\alpha\partial_\mu\phi + \epsilon_\alpha F, \qquad \delta\psi_{\dot{\alpha}}^\dagger = i(\epsilon\sigma^\mu)_{\dot{\alpha}}\partial_\mu\phi^* + \epsilon_{\dot{\alpha}}^\dagger F^*, \tag{487}
$$

the fermionic langrangian becomes

$$
\begin{aligned}
\delta\mathcal{L}_{\text{fermion}} &= i(\delta\psi_{\dot{\alpha}}^\dagger)(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}\partial_\mu\psi_\alpha + i\psi_{\dot{\alpha}}^\dagger(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}\partial_\mu(\delta\psi_\alpha) \\
&= i\Big(i(\epsilon\sigma^\nu)_{\dot{\alpha}}\partial_\nu\phi^* + \epsilon_{\dot{\alpha}}^\dagger F^*\Big)(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}\partial_\mu\psi_\alpha, \\
&+ i\psi_{\dot{\alpha}}^\dagger(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}\partial_\mu\Big(-i(\sigma^\nu\epsilon^\dagger)_\alpha\partial_\nu\phi + \epsilon_\alpha F\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \epsilon\partial^\mu\psi\partial_\mu\phi^* + \epsilon^\dagger\partial^\mu\psi^\dagger\partial_\mu\phi \\
&- \partial_\mu\Big(\epsilon\sigma^\nu\bar{\sigma}^\mu\psi\partial_\nu\phi^* + \epsilon\psi\partial^\mu\phi^* + \epsilon^\dagger\psi^\dagger\partial^\mu\phi\Big) \\
&+ i\epsilon^\dagger_{\dot{\alpha}}F^*(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}\partial_\mu\psi_\alpha + i\psi^\dagger_{\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}\epsilon_\alpha\partial_\mu F, \qquad (488)
\end{aligned}
$$

where we have used (478) in the last line. The last two terms can be written as

$$
i\epsilon^\dagger F^*\bar{\sigma}^\mu\partial_\mu\psi + i\psi^\dagger\bar{\sigma}^\mu\epsilon\partial_\mu F = i\epsilon^\dagger F^*\bar{\sigma}^\mu\partial_\mu\psi + \partial_\mu(i\psi^\dagger\bar{\sigma}^\mu\epsilon F) - i\partial_\mu\psi^\dagger\bar{\sigma}^\mu\epsilon F, \quad (489)
$$

so the full fermionic lagrangian reads

$$
\begin{aligned}
\delta\mathcal{L}_{\text{fermion}} &= \epsilon\partial^\mu\psi\partial_\mu\phi^* + \epsilon^\dagger\partial^\mu\psi^\dagger\partial_\mu\phi \\
&- \partial\Big(\epsilon\sigma^\nu\bar{\sigma}^\mu\psi\partial_\nu\phi^* + \epsilon\psi\partial^\mu\phi^* + \epsilon^\dagger\psi^\dagger\partial^\mu\phi - i\psi^\dagger\bar{\sigma}^\mu\epsilon F\Big) \\
&+ i\epsilon^\dagger F^*\bar{\sigma}^\mu\partial_\mu\psi - i\partial_\mu\psi^\dagger\bar{\sigma}^\mu\epsilon F. \qquad (490)
\end{aligned}
$$

We see that this still cancels the bosnic langranigan but now also the auxiliary one up to surface terms. The commutator of two transformations of the fermionic field (483) now gets the following additional terms

$$
\delta_{\epsilon_2}(\epsilon_{1\alpha}F) - \delta_{\epsilon_1}(\epsilon_{2\alpha}F) = -i\epsilon_{1\alpha}\epsilon^\dagger_{2\dot{\beta}}(\bar{\sigma}^\mu)^{\dot{\beta}\beta}\partial_\mu\psi_\beta + i\epsilon_{2\alpha}\epsilon^\dagger_{1\dot{\beta}}(\bar{\sigma}^\mu)^{\dot{\beta}\beta}\partial_\mu\psi_\beta, \quad (491)
$$

which cancel the last two terms in (483). Finally we have

$$
\begin{aligned}
\delta_{\epsilon_2}\delta_{\epsilon_1}F &= \delta_{\epsilon_2}\Big(-(i\epsilon_1^\dagger\bar{\sigma}^\mu\partial_\mu\psi)\Big) \\
&= -i\epsilon^\dagger_{1\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}\partial_\mu\Big(-i(\sigma^\nu)_{\alpha\dot{\beta}}\epsilon_2^{\dagger\dot{\beta}}\partial_\nu\phi + i\epsilon^\dagger_{2\dot{\alpha}}F\Big) \\
&= -\epsilon_1^\dagger\bar{\sigma}^\mu\sigma^\nu\epsilon_2^\dagger\partial_\mu\partial_\nu\phi + i\epsilon_1^\dagger\bar{\sigma}^\mu\epsilon_2\partial_\mu F \\
&= \epsilon_1^\dagger\epsilon_2^\dagger\partial^\mu\partial_\mu\phi + \epsilon_1^\dagger\bar{\sigma}^\mu\epsilon_2(i\partial_\mu F), \qquad (492)
\end{aligned}
$$

so that

$$
\begin{aligned}
(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_2}\delta_{\epsilon_1})F &= (\epsilon_1^\dagger\epsilon_2^\dagger - \epsilon_2^\dagger\epsilon_1^\dagger)\partial^\mu\partial_\mu\phi + (\epsilon_1^\dagger\bar{\sigma}^\mu\epsilon_2 - \epsilon_2^\dagger\bar{\sigma}^\mu\epsilon_1)(i\partial_\mu F) \\
&= (\epsilon_1^\dagger\bar{\sigma}^\mu\epsilon_2 - \epsilon_2^\dagger\bar{\sigma}^\mu\epsilon_1)(i\partial_\mu F), \qquad (493)
\end{aligned}
$$

since $\epsilon_1^\dagger\epsilon_2^\dagger = \epsilon^\dagger_{1\dot{\alpha}}\epsilon_2^{\dagger\dot{\alpha}} = \epsilon^\dagger_{2\dot{\alpha}}\epsilon_1^{\dagger\dot{\alpha}} = \epsilon_2^\dagger\epsilon_1^\dagger$. Thus we conclude that the langragian

$$
\mathcal{L} = -\partial^\mu\phi^*\partial_\mu\phi + i\psi^\dagger\bar{\sigma}^\mu\partial\psi + F^*F \qquad (494)
$$

really has a supersymmetry off-shell.

References

[1] M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory*, Westview Press (1995)

[2] M. Nakahara, *Geometry, Topology and Physics*, Taylor & Francis Group (2003)

[3] G. Papadopoulos, *Conformal and Superconformal Mechanics*, Class.Quant.Grav.17:3715-3742 (2000); arXiv:hep-th/0002007

[4] A. Maloney, M. Spradlin and A. Strominger, *Superconformal Multi-Black Hole Moduli Spaces in Four Dimensions*, JHEP 0204:003 (2002); arXiv:hep-th/9911001

[5] M. Göteman, *The geometry of supersymmetric non-linear sigma models in $D \leq 2$ dimensions*, Uppsala University Master thesis (2008)

[6] R. A. Coles and G. Papadopoulos, *The geometry of the one-dimensional supersymmetric non-linear sigma models*, Class. Quantum Grav. 7 (1990) 427-438

[7] P. Di Francesco, P. Mathieu and D. Sénéchal, *Conformal Field Theory*, Springer-Verlag New York (1997)

[8] R. Blumenhagen and E. Plauschinn, *Introduction to Conformal Field Theory*, Heidelberg (2009)

[9] A. Schenkel, *QFT on homothetic Killing twist deformed curved spacetimes*, Gen.Rel.Grav.43:2605-2630 (2011); arXiv:1009.1090 [math-ph]

[10] C.M. Hull, *Actions For (2,1) Sigma-Models and Strings*, Nucl.Phys. B509 (1998) 252-272; arXiv:hep-th/9702067

[11] N. Hitchin, *Generalized Calabi-Yau manifolds*, Quart.J.Math.Oxford Ser.54:281-308 (2003); arXiv:math/0209099 [math.DG]

[12] M. Gualtieri, *Generalized complex geometry*, University of Oxford DPhil thesis (2003); arXiv:math/0401221 [math.DG]

[13] M. B. Green, J.H. Schwarz and E. Witten, *Superstring Theory. Vol. 1: Introduction*, Cambridge University Press (1987)

[14] J. Polchinski, *String Theory. Vol 1: An introduction to the Bosonic String*, Cambridge University Press (1998)

[15] J. Polchinski, *String Theory. Vol 2: Superstring Theory and Beyond*, Cambridge University Press (1998)

[16] K. Baker, M, Baker and J.H. Schwarz, *String Theory and M-theory - A Modern Introduction*, Cambridge University Press (2006)

[17] B. Zwiebach, *A First Course in String Theory 2nd ed.*, Cambridge University Press (2009)

[18] S. Coleman and J. Mandula, *All Possible Symmetries of the S Matrix*, Phys. Rev. 159, 12511256 (1967)

[19] R. Haag, J.T. Lopuszanski and M. Sohnius, *All possible generators of supersymmetries of the S-matrix*, Nucl. Phys. B 88, 257-274 (1975)

[20] S.P. Martin, *A Supersymmetry Primer*, (1997); arXiv:hep-ph/9709356

[21] M. Fecko *Differential Geometry and Lie Groups for Physicists*, Cambridge University Press (2006)