University of Uppsala

Master Thesis

Geometry of BV Quantization and Mathai-Quillen Formalism

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Abstract. Mathai-Quillen (MQ) formalism is a prescription to construct a Gaussian shaped Thom form for a vector bundle. The aim of this master thesis is to formulate a new Thom form representative using geometrical aspects of Batalin-Vilkovisky (BV) quantization. In the first part of the work we review the BV and MQ formalisms both in finite dimensional setting. Finally, to achieve our purpose, we will exploit the odd Fourier transform considering the MQ representative as a function over the appropriate graded manifold.

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CHAPTER 1

Introduction

The Batalin-Vilkovisky (BV) formalism is widely regarded as the most powerful and general approach to the quantization of gauge theories. The physical novelty introduced by BV formalism is to make possible the quantization of gauge theories that are difficult to quantize with the Fadeev-Popov method. In particular, it offers a prescription to perform path integrals for these theories. In quantum field theory the path integral is understood as some sort of integral over infinite dimensional functional space. Up to now there is no suitable definition of the path integral and in practice all heuristic understanding of the path integral is done by mimicking the manipulations of finite dimensional integrals. Thus, a proper understanding of the formal algebraic manipulations with finite dimensional integrals is crucial for a better insight to the path integrals. Such formalism firstly appeared in the papers of Batalin and Vilkovisky [6, 7] while a clear geometric interpretation was given by Schwarz in [11, 14]. This thesis will largely follow the spirit of [15] where the authors described some geometrical properties of BV formalism related to integration theory on supermanifolds. On the odd tangent bundle there is a canonical way to integrate a function of top degree while to integrate over the odd cotangent bundle we always have to pick a density. Although the odd cotangent bundle does not have a nice integration property, it is however interesting because of his algebraic property due to the BV structure on it.

Characteristic classes play an essential role in the study of global properties of vector bundles. Consider a vector bundle over a certain base manifold, we would like to relate differential forms on the total space to differential forms on the basis, to do that we would like to integrate over the fiber and this is what the Thom class allows us. Basically the Thom class can be thought as a gaussian shaped differential form of top degree which has indices in the vertical direction along the fiber. Mathai and Quillen [17] obtained an explicit construction of the Thom class using Berezin integration, a technique widely used in physics literature. The physical significance of this construction was first pointed out, in an important paper of Atiyah and Jeffrey [22]. They discovered that the path integrals of a topological field theory of the Witten type [25] are integral representations of Thom classes of vector bundles in infinite dimensional spaces. In his classic work [18] Witten showed that a topological gauge theory can be constructed by twisting $\mathcal{N} = 2$ supersymmetric Yang-Mills theory. Correlation functions of the twisted theory are non other than Donaldson invariants of four-manifolds and certain quantities in the supersymmetric gauge theory considered are determined solely by the topology, eliminating the necessity of complicated integrals. In this way topological field theories are convenient testing grounds for subtle non perturbative phenomena appearing in quantum field theory.

Understanding the dynamical properties of non-abelian gauge fields is a very difficult problem, probably one of the most important and challenging problem in theoretical physics. Infact the Standard Model of fundamental interactions is based on non-abelian quantum gauge field theories. A coupling constant in such theories usually decreases at high energies and blows up at low energies. Hence, it is easy and valid to apply perturbation theory at high energies. However, as the energy decreases the perturbation theory works worse and completely fails to give any meaningful results at the energy scale called Λ. Therefore, to understand the Λ scale physics, such as confinement, hadron mass spectrum and the dynamics of low-energy interactions, we need non-perturbative methods. The main such methods are based on supersymmetry and duality. Like any symmetry, supersymmetry imposes some constraints on the dynamics of a physical system. Then, the dynamics is restricted by the amount of supersymmetry imposed, but we still have a very non-trivial theory and thus interesting for theoretical study. Duality means an existence of two different descriptions of the same physical system. If the strong coupling limit at one side of the duality corresponds to the weak coupling limit at the other side, such duality is especially useful to study the theory. Indeed, in that case difficult computations in strongly coupled theory can be done perturbatively using the dual weakly coupled theory.

The aim of this master thesis is to establish a relationship between geometrical aspects of BV quantization and the Mathai-Quillen formalism for vector bundle. We will formulate a new representative of the Thom class, called BV representative. To reach this goal we will use the odd Fourier transform as explained in [15]. However, we will generalize this construction to the case of differential forms over a vector bundle. Lastly, we will show that our BV representative is authentically a Thom class and that our procedure is consistent.

The outline of the thesis is as follows. In section 2 we discuss some basic notions of super linear algebra. We pay particular attention to the Berezinian integration in all its details, by giving detailed proofs of all our statements. In section 3 we treat the differential geometry of supermanifolds. As main examples we discuss the odd tangent and odd cotangent bundles. We also review the integration theory from a geometric point of view, following the approach of [1]. In section 4 we briefly illustrate the $\mathbb{Z}$-graded refinement of supergeometry, known as graded geometry. In section 5 we define the odd Fourier transform which will be an object of paramount importance trough all the thesis. Then, the BV structure on the odd cotangent bundle is introduced as well as a version of the Stokes theorem. Finally, we underline the algebraic aspects of integration within BV formalism. In section 6 we explain the Mathai-Quillen (MQ) formalism. Firstly, we describe topological quantum field theories, then we introduce the notions of Thom class and equivariant cohomology and eventually we give an explicit proof of the Poincaré-Hopf theorem using the MQ representative. In section 7 it is contained the original part of this work. Here, we discuss our procedure to create a BV representative of the Thom class and obtain the desired relationship between BV quantization and Mathai-Quillen formalism. Section 8 is the conlusive section of this thesis where we summarize our results and discuss open issues.

CHAPTER 2

Super Linear Algebra

Our starting point will be the construction of linear algebra in the super context. This is an important task since we need these concepts to understand super geometric objects. Super linear algebra deals with the category of super vector spaces over a field k. In physics k is $\mathbb{R}$ or $\mathbb{C}$. Much of the material described here can be found in books such as [2, 4, 5, 8, 12].

2.1. Super Vector Spaces

A super vector space V is a vector space defined over a field $\mathbb{K}$ with a $\mathbb{Z}_2$ grading. Usually in physics $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. V has the following decomposition

$$V = V_0 \oplus V_1 \tag{2.1}$$

the elements of V_0 are called even and those of V_1 odd. If d_i is the dimension of V_i we say that V has dimension $d_0|d_1$. Consider two super vector spaces V , W, the morphisms from V to W are linear maps $V \to W$ that preserve gradings. They form a linear space denoted by $\mathrm{Hom}(V, W)$. For any super vector space the elements in $V_0 \cup V_1$ are called homogeneous, and if they are nonzero, their parity is defined to be 0 or 1 according as they are even or odd. The parity function is denoted by p. In any formula defining a linear or multilinear object in which the parity function appears, it is assumed that the elements involved are homogeneous.
If we take $V = \mathbb{K}^{p+q}$ with its standard basis e_i with $1 \leq i \leq p+q$, and we define e_i to be even if $i \leq p$ or odd if $i > p$, then V becomes a super vector space with

$$V_0 = \sum_{i=1}^{p} \mathbb{K}e_i \qquad\qquad V_1 = \sum_{i=p+1}^{q} \mathbb{K}e_i \tag{2.2}$$

then V will be denoted $\mathbb{K}^{p|q}$.
The tensor product of super vector spaces V and W is the tensor product of the underlying vector spaces, with the $\mathbb{Z}_2$ grading

$$(V \otimes W)_k = \underset{i+j=k}{\oplus} V_i \otimes W_j \tag{2.3}$$

where i, j, k are in $\mathbb{Z}_2$. Thus

$$\begin{aligned}(V \otimes W)_0 &= (V_0 \otimes W_0) \oplus (V_1 \otimes W_1)\\ (V \otimes W)_1 &= (V_0 \otimes W_1) \oplus (V_1 \otimes W_0)\end{aligned} \tag{2.4}$$

For super vector spaces V, W , the so called internal Hom , denoted by $\mathbf{Hom}(V, W)$, is the vector space of all linear maps from V to W. In particular we have the following definitions

$$\mathbf{Hom}(V, W)_0 = \{T : V \to W | T \text{ preserves parity}\} \ \ (= \mathrm{Hom}(V, W));$$

$$\mathbf{Hom}(V, W)_1 = \{T : V \to W | T \text{ reverses parity}\}$$

For example if we take $V = W = \mathbb{K}^{1|1}$ and we fix the standard basis, we have that

$$\begin{aligned}\mathrm{Hom}(V, W) &= \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} |a, d \in \mathbb{K} \right\};\\ \mathbf{Hom}(V, W) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} |a, b, c, d \in \mathbb{K} \right\}\end{aligned} \tag{2.5}$$

If V is a super vector space, we write $\mathbf{End}(V)$ for $\mathbf{Hom}(V, V)$.

Example 2.1.1. *Consider purely odd superspace $\Pi\mathbb{R}^q = \mathbb{R}^{0|q}$ over the real number of dimension q. Let us pick up the basis θ^i, $i = 1, 2, ..., q$ and define the multiplication between the basis elements satisfying $\theta^i\theta^j = -\theta^j\theta^i$. The functions $\mathcal{C}^\infty(\mathbb{R}^{0|q})$ on $\mathbb{R}^{0|q}$ are given by the following expression*

$$f(\theta^1, \theta^2, ..., \theta^q) = \sum_{l=0}^{q} \frac{1}{l!}\ f_{i_1 i_2 ... i_l}\theta^{i_1}\theta^{i_2}...\theta^{i_l} \tag{2.6}$$

and they correspond to the elements of exterior algebra $\wedge^\bullet(\mathbb{R}^q)^*$. The exterior algebra

$$\wedge^\bullet (\mathbb{R}^q)^* = (\wedge^{\text{even}}(\mathbb{R}^q)^*) \bigoplus \left(\wedge^{\text{odd}}(\mathbb{R}^q)^*\right) \tag{2.7}$$

is a supervector space with the supercommutative multiplications given by wedge product. The wedge product of the exterior algebra corresponds to the function multiplication in $C^\infty(\mathbb{R}^{0|q})$.

2.1.1. Rule of Signs. The $\otimes$ in the category of vector spaces is associative and commutative in a natural sense. Thus, for ordinary vector spaces U, V, W we have the natural associativity isomorphism

$$(U \otimes V) \otimes W \simeq U \otimes (V \otimes W), \quad (u \otimes v) \otimes w \longmapsto u \otimes (v \otimes w) \tag{2.8}$$

and the commutativity isomorphism

$$c_{V,W} : V \otimes W \simeq W \otimes V, \qquad v \otimes w \longmapsto w \otimes v \tag{2.9}$$

For the category of super vector spaces the associativity isomorphism remains the same, but the commutativity isomorphism is subject to the following change

$$c_{V,W} : V \otimes W \simeq W \otimes V, \quad v \otimes w \longmapsto (-1)^{p(v)p(w)} w \otimes v \tag{2.10}$$

This definition is the source of the rule of signs, which says that whenever two terms are interchanged in a formula a minus sign will appear if both terms are odd.

2.2. Superalgebras

In the ordinary setting, an algebra is a vector space A with a multiplication which is bilinear. We may therefore think of it as a vector space A together with a linear map $A \otimes A \to A$, which is the multiplication. Let A be an algebra , $\mathbb{K}$ a field by which elements of A can be multiplied. In this case A is called an algebra over $\mathbb{K}$.
Consider a set $\Sigma \subset A$, we will denote by $A(\Sigma)$ a collection of all possible polynomials of elements of Σ. If $f \in A(\Sigma)$ we have

$$f = f_0 + \sum_{k \geq 1} \sum_{i_1,\ldots,i_k} f_{i_1,\ldots,i_k} a_{i_1} \ldots a_{i_k}, \;\; a_i \in \Sigma, \; f_{i_1,\ldots,i_k} \in \mathbb{K} \tag{2.11}$$

Of course $A(\Sigma)$ is a subalgebra of A, called a subalgebra generated by the set Σ. If $A(\Sigma) = A$, the set Σ is called a system of generators of algebra A or a generating set.

Definition 2.2.1. *A superalgebra A is a super vector space A, given with a morphism , called the product: $A \otimes A \to A$. By definition of morphisms, the parity of the product of homogeneous elements of A is the sum of parities of the factors.*

The superalgebra A is associative if $(xy)z = x(yz)\ \forall\, x, y, z \in A$. A unit is an even element 1 such that $1x = x1 = x \;\; \forall x \in A$. By now we will refer to superalgebra as an associative superalgebra with the unit.

Example 2.2.2. *If V is a super vector space,* $\mathbf{End}(V)$ *is a superalgebra. If $V = \mathbb{K}^{p|q}$ we write $M(p|q)$ for* $\mathbf{End}(V)$*. Using the standard basis we have the usual matrix representations for elements of $M(p|q)$ in the form*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.12}$$

where the letters A, B, C, D denotes matrices respectively of orders $p \times p, p \times q, q \times p, q \times q$ and where the even elements the odd ones are, respectively, of the form.

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \qquad \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \tag{2.13}$$

A superalgebra is said to be commutative if

$$xy = (-1)^{p(x)p(y)}yx \ , \ \forall x, y \in A \, ; \tag{2.14}$$

commutative superalgebra are often called supercommutative.

2.2.1. Supertrace. Let $V = V_0 \oplus V_1$ a finite dimensional super vector space, and let $X \in \mathbf{End}(V)$. Then we have

$$X = \begin{pmatrix} X_{00} & X_{01} \\ X_{10} & X_{11} \end{pmatrix} \tag{2.15}$$

where X_{ij} is the linear map from V_j to V_i such that $X_{ij}v$ is the projection onto V_i of Xv for $v \in V_j$. Now the supertrace of X is defined as

$$\mathrm{str}(X) = \mathrm{tr}(X_{00}) - \mathrm{tr}(X_{11}) \tag{2.16}$$

Let Y, Z be rectangular matrices with odd elements, we have the following result

$$\mathrm{tr}(YZ) = -\mathrm{tr}(ZY) \tag{2.17}$$

to prove this statement we denote by y_{ik}, z_{ik} the elements of matrices Y and Z respectively then we have

$$\mathrm{tr}(YZ) = \sum y_{ik} z_{ki} = -\sum z_{ki} y_{ik} = -\mathrm{tr}(ZY) \tag{2.18}$$

notice that anologous identity is known for matrices with even elements but without the minus sign. Now we can claim that

$$\mathrm{str}(XY) = (-1)^{p(X)p(Y)}\mathrm{str}(YX), \ \ X, Y \in \mathbf{End}(V) \tag{2.19}$$

2.2.2. Berezinian. Consider a super vector space V, we can write a linear transformation of V in block form as

$$W = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.20}$$

where here A and D are respectively $p \times p$ and $q \times q$ even blocks, while B and C are odd. An explicit formula for the Berezinian is

$$\mathrm{Ber}(W) = \det(A - BD^{-1}C)\det(D)^{-1} \tag{2.21}$$

notice that the Berezinian is defined only for matrices W such that D is invertible. As well as the ordinary determinant also the Berezinian enjoys the multiplicative property, so if we consider two linear transformations W_1 and W_2, like the ones that we introduced above , such that $W = W_1 W_2$ we will have

$$\mathrm{Ber}(W) = \mathrm{Ber}(W_1)\mathrm{Ber}(W_2) \tag{2.22}$$

To prove this statement firstly we define the matrices W_1 and W_2 to get

$$\begin{aligned} W_1 &= \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \; W_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \\ \Longrightarrow W &= \begin{pmatrix} A_1A_2 + B_1C_2 & A_1B_2 + B_1D_2 \\ C_1A_2 + D_1C_2 & C_1B_2 + D_1D_2 \end{pmatrix} \end{aligned} \tag{2.23}$$

Using matrix decomposition we can write for W_1

$$\begin{aligned} W_1 &= \begin{pmatrix} 1 & B_1D_1^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_1 - B_1D_1^{-1}C_1 & 0 \\ 0 & D_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ D_1^{-1}C_1 & 1 \end{pmatrix} \\ &= X_1^+ X_1^0 X_1^- \end{aligned} \tag{2.24}$$

obviously this is true also for W_2. So now we want to compute the following Berezinian

$$\mathrm{Ber}(W) = \mathrm{Ber}(X_1^+ X_1^0 X_1^- X_2^+ X_2^0 X_2^-) \tag{2.25}$$

As a first step we consider two block matrices X and Y such that

$$X = \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix}, \qquad Y = \begin{pmatrix} B & C \\ D & E \end{pmatrix} \tag{2.26}$$

we can see that X resembles the form of X_1^+. Computing the Berezinians we get

$$\mathrm{Ber}(X)\mathrm{Ber}(Y) = \det(B - CE^{-1}D)\det(E)^{-1} \tag{2.27}$$

while

$$\mathrm{Ber}(XY) = \det(B + AD - (C + AE)E^{-1}D)\det(E)^{-1} \tag{2.28}$$

after this first check we can safely write that

$$\mathrm{Ber}(W) = \mathrm{Ber}(X_1^+)\mathrm{Ber}(X_1^0 X_1^- X_2^+ X_2^0 X_2^-) \tag{2.29}$$

As a second step we consider once again two block matrices X and Y now defined as

$$X = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \qquad Y = \begin{pmatrix} C & D \\ E & F \end{pmatrix} \tag{2.30}$$

clearly now X resembles the form of X_1^0. Computing the Berezinians we get

$$\begin{aligned} \mathrm{Ber}(X)\mathrm{Ber}(Y) &= \det(A)\det(B)^{-1}\det(C - DF^{-1}E)\det(F)^{-1} \\ &= \det(AC - ADF^{-1}E)\det(BF)^{-1} \end{aligned} \tag{2.31}$$

while

$$\mathrm{Ber}(XY) = \det(AC - ADF^{-1}B^{-1}BE)\det(BF)^{-1} \tag{2.32}$$

so after this second step we conlude that

$$\mathrm{Ber}(W) = \mathrm{Ber}(X_1^+)\mathrm{Ber}(X_1^0)\mathrm{Ber}(X_1^- X_2^+ X_2^0 X_2^-) \tag{2.33}$$

Now repeating two times more the procedure done in the first two steps we get the following result

$$\mathrm{Ber}(W) = \mathrm{Ber}(X_1^+)\mathrm{Ber}(X_1^0)\mathrm{Ber}(X_1^- X_2^+)\mathrm{Ber}(X_2^0)\mathrm{Ber}(X_2^-) \tag{2.34}$$

Now we want to show the multiplicativity of $\mathrm{Ber}(X_1^- X_2^+)$ but we can't proceed as in the previous steps. In fact if we consider once again two matrices X and Y such that

$$X = \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}, \qquad Y = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \tag{2.35}$$

we have that

$$\begin{aligned} &\mathrm{Ber}(X)\mathrm{Ber}(Y) = 1 \\ &\mathrm{Ber}(XY) = \det(1 - B(1+CB)^{-1}C)\det(1+CB)^{-1} \end{aligned} \tag{2.36}$$

To guarantee the multiplicative property also in this case we have to prove that

$$\det(1 - B(1+CB)^{-1}C)\det(1+CB)^{-1} = 1 \tag{2.37}$$

We may assume that B is an elementary matrix, which it means that all but one entry of B are 0, and that one is an odd element b. By this property we see that $(CB)^2 = 0$, consequently

$$(1+CB)^{-1} = 1 - CB \tag{2.38}$$

and hence

$$1 - B(1+CB)^{-1}C = 1 - B(1-CB)C = 1 - BC \tag{2.39}$$

Now we can use the general formula

$$\det(1 - BC) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{j=1}^{\infty} \frac{(-1)^{2j+1}}{j} \operatorname{tr}((BC)^j) \right)^k = 1 - \operatorname{tr}(BC) \tag{2.40}$$

Using the same formula we get the following result

$$\det(1 + CB)^{-1} = (1 + \operatorname{tr}(CB))^{-1} = (1 - \operatorname{tr}(BC))^{-1} \tag{2.41}$$

where in the last passage we used (2.18). Eventually we can easily verify that

$$\begin{aligned} \det(1-&B(1 + CB)^{-1}C) \det(1 + CB)^{-1} \\ &= (1 - \operatorname{tr}(BC))(1 - \operatorname{tr}(BC))^{-1} = 1 \end{aligned} \tag{2.42}$$

At this point we may write

$$\begin{aligned} \operatorname{Ber}(W) &= \operatorname{Ber}(X_1^+)\operatorname{Ber}(X_1^0)\operatorname{Ber}(X_1^-)\operatorname{Ber}(X_2^+)\operatorname{Ber}(X_2^0)\operatorname{Ber}(X_2^-) \\ &= \operatorname{Ber}(X_1^+ X_1^0 X_1^-)\operatorname{Ber}(X_2^+ X_2^0 X_2^-) \\ &= \operatorname{Ber}(W_1)\operatorname{Ber}(W_2) \end{aligned} \tag{2.43}$$

If we use another matrix decomposition for the matrix W defined in (2.20) we get an equivalent definition of the Berezinian which is

$$\operatorname{Ber}(W) = \det(A) \det(D - CA^{-1}B)^{-1} \tag{2.44}$$

2.3. Berezin Integration

Consider the super vector space $\mathbb{R}^{p|q}$, it admits a set of generators $\Sigma = (t^1 \dots t^p | \theta^1 \dots \theta^q)$ with the properties

$$t^i t^j = t^j t^i \qquad 1 \leq i, j \leq p \tag{2.45}$$

$$\theta^i \theta^j = -\theta^j \theta^i \qquad 1 \leq i, j \leq q \tag{2.46}$$

in particular $(\theta^i)^2 = 0$. We will referer to the $(t^1 \dots t^p)$ as the even(bosonic) coordinates and to the $(\theta^1 \dots \theta^q)$ as the odd(fermionic) coordinates. On $\mathbb{R}^{p|q}$, a general function g can be expanded as a polynomial in the θ's:

$$g(t^1 \dots t^p | \theta^1 \dots \theta^q) = g_0(t^1 \dots t^p) + \dots + \theta^q \theta^{q-1} \dots \theta^1 g_q(t^1 \dots t^p). \tag{2.47}$$

The basic rules of Berezin integration are the following

$$\int d\theta = 0 \qquad \int d\theta\, \theta = 1 \tag{2.48}$$

by these rules, the integral of g is defined as

$$\int_{\mathbb{R}^{p|q}} [dt^1 \dots dt^p | d\theta^1 \dots d\theta^q] g(t^1 \dots t^p | \theta^1 \dots \theta^q) = \int_{\mathbb{R}^p} dt^1 \dots dt^p g_q(t^1 \dots t^p) \tag{2.49}$$

Since we require that the formula (2.49) remains true under a change of coordinates, we need to obtain the transformation rule for the integration form $[dt^1 \dots dt^p | d\theta^1 \dots d\theta^q]$. In fact, although we know how the things work in the ordinary(even) setting, we have to understand the behavior of the odd variables in this process.

2.4. Change of Coordinates

Consider the simplest transformation for an odd variables

$$\theta \longrightarrow \widetilde{\theta} = \lambda\theta, \quad \lambda \text{ constant.} \tag{2.50}$$

then the equations (2.48) imply

$$\int d\theta\, \theta = \int d\widetilde{\theta}\, \widetilde{\theta} = 1 \iff d\widetilde{\theta} = \lambda^{-1} d\theta \tag{2.51}$$

as we can see $d\theta$ is multiplied by λ^{-1}, rather than by λ as one would expect.

Now we consider the case of $\mathbb{R}^q$, where q denotes the number of odd variables, and perform the transformation

$$\theta^i \longrightarrow \widetilde{\theta}^i = f^i(\theta^1 \dots \theta^q) \tag{2.52}$$

where f is a general function. Now we can expand f^i in the following manner

$$f^i(\theta^1 \dots \theta^q) = \theta^k f^i_k + \theta^k \theta^l \theta^m f^i_{klm} + \dots \tag{2.53}$$

since the $\widetilde{\theta}^i$ variables has to respect the anticommuting relation (2.46), the function f^imust have only odd numbers of the θ^i variables in each factor. Now we compute the product

$$\begin{aligned}
\widetilde{\theta}^q \dots \widetilde{\theta}^1 &= (\theta^{k_q} f^q_{k_q} + \dots)(\theta^{k_{q-1}} f^{q-1}_{k_{q-1}} + \dots) \dots\dots (\theta^{k_1} f^1_{k_1} + \dots) \\
&= \theta^{k_q} \theta^{k_{q-1}} \dots \theta^{k_1} f^q_{k_q} f^{q-1}_{k_{q-1}} \dots f^1_{k_1} \\
&= \theta^q \theta^{q-1} \dots \theta^1 \varepsilon^{k_q \dots k_1} f^q_{k_q} f^{q-1}_{k_{q-1}} \dots f^1_{k_1} \\
&= \theta^q \theta^{q-1} \dots \theta^1 \det(F)
\end{aligned} \tag{2.54}$$

where in the last passage we used the usual formula for the determinant of the F matrix. We are ready to perform the Berezin integral in the new variables $\widetilde{\theta}$

$$\int d\widetilde{\theta}^1 \dots d\widetilde{\theta}^q\, \widetilde{\theta}^q \dots \widetilde{\theta}^1 = \int d\widetilde{\theta}^1 \dots d\widetilde{\theta}^q\, \theta^q \theta^{q-1} \dots \theta^1 \det(F) \tag{2.55}$$

preserving the validity of (2.48) implies

$$d\widetilde{\theta}^1 \dots d\widetilde{\theta}^q = \det(F)^{-1} d\theta^1 \dots d\theta^q \tag{2.56}$$

Using this result it is possible to define the transformation rule for Berezin integral under this transformation

$$\begin{aligned} t &\longrightarrow \widetilde{t} = \widetilde{t}(t^1 \dots t^p) \\ \theta &\longrightarrow \widetilde{\theta} = \widetilde{\theta}(\theta^1 \dots \theta^q) \end{aligned} \tag{2.57}$$

which is

$$\begin{aligned} &\int_{\mathbb{R}^{p|q}} [dt^1 \dots dt^p | d\theta^1 \dots d\theta^q] g(t^1 \dots t^p | \theta^1 \dots \theta^q) \\ &= \int_{\mathbb{R}^{p|q}} [d\widetilde{t}^1 \dots d\widetilde{t}^p | d\widetilde{\theta}^1 \dots d\widetilde{\theta}^q] \det\left(\frac{\partial t}{\partial \widetilde{t}}\right) \det\left(\frac{\partial \theta}{\partial \widetilde{\theta}}\right)^{-1} g(\widetilde{t}^1 \dots \widetilde{t}^p | \widetilde{\theta}^1 \dots \widetilde{\theta}^q) \end{aligned} \tag{2.58}$$

From this formula is clear that the odd variables transforms with the inverse of the Jacobian matrix determinant; the inverse of what happen in the ordinary case. At this point a question naturally arises: provided that we are respecting the original parity of the variables, what does it happen if the new variables undergo a mixed transformation ? To answer at this question we have to study a general change of coordinates of the form

$$\begin{aligned} t &\longrightarrow \widetilde{t} = \widetilde{t}(t^1 \dots t^p | \theta^1 \dots \theta^q) \\ \theta &\longrightarrow \widetilde{\theta} = \widetilde{\theta}(t^1 \dots t^p | \theta^1 \dots \theta^q) \end{aligned} \tag{2.59}$$

The Jacobian of this transformation will be a block matrix

$$\begin{aligned} W = \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ CA^{-1} & D \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D^{-1} \end{pmatrix} \begin{pmatrix} 1 & A^{-1}B \\ 0 & D - CA^{-1}B \end{pmatrix} \\ &= W^+ W^0 W^- \end{aligned} \tag{2.60}$$

where $A = \dfrac{\partial t}{\partial \widetilde{t}}$ and $D = \dfrac{\partial \theta}{\partial \widetilde{\theta}}$ are the even blocks while $B = \dfrac{\partial t}{\partial \widetilde{\theta}}$ and $C = \dfrac{\partial \theta}{\partial \widetilde{t}}$ are the odd ones. The matrix decomposition suggests that we can think at the general change of coordinates as the product of three distinct transformations represented by the three block matrices in the right hand side of (2.60). At the moment we only know how to deal with a transformation of the type (2.57) which have a Jacobian matrix like W^0. To proceed further we have to analyze the remaining transformations

$$\begin{aligned} t &\longrightarrow \widetilde{t} = \widetilde{t}(t^1 \dots t^p) \\ \theta &\longrightarrow \widetilde{\theta} = \widetilde{\theta}(t^1 \dots t^p | \theta^1 \dots \theta^q) \end{aligned} \tag{2.61}$$

$$\begin{aligned} t &\longrightarrow \widetilde{t} = \widetilde{t}(t^1 \dots t^p | \theta^1 \dots \theta^q) \\ \theta &\longrightarrow \widetilde{\theta} = \widetilde{\theta}(\theta^1 \dots \theta^q) \end{aligned} \tag{2.62}$$

Consider the case (2.61) then we rewrite the transformation as

$$\begin{aligned} t &\longrightarrow \widetilde{t} = h(t^1 \dots t^p) \\ \theta &\longrightarrow \widetilde{\theta} = g(t^1 \dots t^p | \theta^1 \dots \theta^q) \end{aligned} \tag{2.63}$$

where f and g are general functions. By formula (2.49) we know how to perform the Berezin integration for a function $F(t^1 \dots | \dots \theta^q)$. Using the new variables will give

$$\begin{aligned} &\int [d\widetilde{t}^1 \dots d\widetilde{t}^p | d\widetilde{\theta}^1 \dots d\widetilde{\theta}^q] F(\widetilde{t}^1 \dots | \dots \widetilde{\theta}^q) \\ &= \int [d\widetilde{t}^1 \dots d\widetilde{t}^p | d\widetilde{\theta}^1 \dots d\widetilde{\theta}^q] \Big[\widetilde{F}_0(\widetilde{t}^1 \dots \widetilde{t}^p) + \dots + \widetilde{\theta}^q \dots \widetilde{\theta}^1 \widetilde{F}_q(\widetilde{t}^1 \dots \widetilde{t}^p) \Big] \end{aligned} \tag{2.64}$$

where we used (2.47). As we did in (2.53) we expand g as

$$g^i(t^1 \dots | \dots \theta^q) = \theta^{k_q} g^i_{k_q}(t^1 \dots t^p) + \theta^{k_q}\theta^{l_q}\theta^{m_q} g^i_{k_q l_q m_q}(t^1 \dots t^p) + \dots \tag{2.65}$$

and similary to (2.54) we get

$$\widetilde{\theta}^q \dots \widetilde{\theta}^1 = \theta^q \dots \theta^1 \det[G(t^1 \dots t^p)] \tag{2.66}$$

Inserting this result inside (2.64) we obtain

$$\begin{aligned} &\int [d\widetilde{t}^1 \dots | \dots d\widetilde{\theta}^q] F(\widetilde{t}^1 \dots | \dots \widetilde{\theta}^q) = \\ &\qquad = \int [d\widetilde{t}^1 \dots | \dots d\widetilde{\theta}^q] \Big[\widetilde{F}_0(\widetilde{t}^1 \dots \widetilde{t}^p) + \\ &\qquad\qquad \dots + \theta^q \dots \theta^1 \det[G(t^1 \dots t^p)] \widetilde{F}_q(\widetilde{t}^1 \dots \widetilde{t}^p) \Big] \end{aligned} \tag{2.67}$$

as seen before if we want to achieve the same conclusion of (2.49) we demand that

$$[d\widetilde{t}^1 \dots | \dots d\widetilde{\theta}^q] = \det[H(t^1 \dots t^p)] \det[G(t^1 \dots t^p)]^{-1} [dt^1 \dots | \dots d\theta^q] \tag{2.68}$$

The matrix W^+ in equation (2.60) is the Jacobian of a change of coordinates which is a special case of the one that we studied in (2.63). In (2.68) we found that for this type of transformations, the integration form behaves exaclty as in (2.58). A similiar argument can be used for transformations like (2.62) and consequently for W^-. Finally, we discovered the complete picture for the mixed change of coordinates which is

$$\begin{aligned} [d\tilde{t}^1 \dots | \dots d\tilde{\theta}^q] &= \det(A)\det(D - CA^{-1}B)^{-1}[dt^1 \dots | \dots d\theta^q] \\ &= \mathrm{Ber}(W)[dt^1 \dots | \dots d\theta^q] \end{aligned} \tag{2.69}$$

where in the last passage we used the definition given in (2.44). Equation (2.69) gives rise to the rule for the change of variables in $\mathbb{R}^{p|q}$. In fact, if we express the integral of a function $g(t^1 \dots | \dots t^p)$ defined on a coordinate system $T = t^1 \dots | \dots \theta^q$ in a new coordinate system $\widetilde{T} = \tilde{t}^1 \dots | \dots \tilde{\theta}^q$ the relation is

$$\begin{aligned} &\int [dt^1 \dots | \dots d\theta^q] g(t^1 \dots | \dots \theta^q) \\ &= \int \mathrm{Ber}\left(\frac{\partial T}{\partial \widetilde{T}}\right) [d\tilde{t}^1 \dots | \dots d\tilde{\theta}^q] g(\tilde{t}^1 \dots | \dots \tilde{\theta}^q). \end{aligned} \tag{2.70}$$

2.5. Gaussian Integration

Prior to define how to perform Gaussian integration with odd variables we will recall some results using even variables. For example consider a $p \times p$ symmetric and real matrix A, then it is well known that we can always find a matrix $R \in SO(p)$ such that $R^\top A R = \mathrm{diag}(\lambda_1 \dots \lambda_p)$, where λ_i are the real eigenvalue of the matrix A. As a consequence we get

$$\begin{aligned} Z(A) &= \int dt^1 \dots dt^p \exp\Big\{ -\frac{1}{2} t^\top A t \Big\} \\ &= \int dy^1 \dots dy^p \exp\Bigg\{ -\frac{1}{2}(Ry)^\top A (Ry) \Bigg\} \\ &= \int dy^1 \dots dy^p \exp\Bigg\{ -\frac{1}{2}\sum_{i=1}^p \lambda_i (y^i)^2 \Bigg\} \\ &= \prod_{i=1}^p \int_{-\infty}^{+\infty} dy^i \exp\Bigg\{ -\frac{1}{2}\lambda_i (y^i)^2 \Bigg\} \\ &= \prod_{i=1}^p \left(\frac{2\pi}{\lambda_i}\right)^{\frac{1}{2}} = (2\pi)^{\frac{p}{2}} (\det A)^{-\frac{1}{2}} \end{aligned} \tag{2.71}$$

Moreover if we consider the case of $2p$ integration variables $\{x^i\}$ and $\{y^i\}$,$i = 1 \dots p$, and we assume that the integrand is invariant under a simoultaneous identical rotation in all (x^i, y^i) planes then we can introduce formal complex variables z^i and $\overline{z}^i$ defined as

$$z^i = \frac{x^i + iy^i}{\sqrt{2}} \qquad \overline{z}^i = \frac{x^i - iy^i}{\sqrt{2}} \tag{2.72}$$

The Gaussian integral now is

$$\int \left(\prod_{i=1}^{p} \frac{dz^i d\overline{z}^i}{2\pi i} \right) \exp\{-\overline{z}^i A_{ij} z^j\} = (\det A)^{-1} \tag{2.73}$$

in which A is an Hermitian matrix with non-vanishing determinant. Now we turn our attention to the case of odd variables where we have to compute the following integral

$$Z(A) = \int d\theta^1 \ldots d\theta^{2q} \exp\left(\frac{1}{2} \sum_{i,j=1}^{2q} \theta^i A_{ij} \theta^j \right) \tag{2.74}$$

in which A is an antisymmetric matrix. Expanding the exponential in a power series, we observe that only the term of order q which contains all products of degree $2q$ in θ gives a non-zero contribution

$$Z(A) = \frac{1}{2^q q!} \int d\theta^1 \ldots d\theta^{2q} \left(\sum_{i,j} \theta^i A_{ij} \theta^j \right)^q \tag{2.75}$$

In the expansion of the product only the terms containing a permutation of $\theta^1 \ldots \theta^{2q}$ do not vanish. Ordering all terms to factorize the product $\theta^{2q} \ldots \theta^1$ we find

$$Z(A) = \frac{1}{2^q q!} \varepsilon^{i_1 \ldots i_{2q}} A_{i_1 i_2} \ldots A_{i_{2q-1} i_{2q}} \tag{2.76}$$

The quantity in the right hand side of (2.76) is called Pfaffian of the antisymmetric matrix

$$Z(A) = \mathrm{Pf}(A) \tag{2.77}$$

As we did before, we consider two independent set of odd variables denoted by θ^i and $\overline{\theta}^i$, then we get

$$Z(A) = \int d\theta^1 d\overline{\theta}^1 \ldots d\theta^q d\overline{\theta}^q \exp\left(\sum_{i,j=1}^{q} \overline{\theta}^i A_{ij} \theta^j \right) \tag{2.78}$$

The integrand can be rewritten as

$$\exp\left(\sum_{i,j=1}^{q} \overline{\theta}^i A_{ij} \theta^j \right) = \prod_{i=1}^{q} \exp\left(\overline{\theta}^i \sum_{j=1}^{q} A_{ij} \theta^j \right) = \prod_{i=1}^{q} \left(1 + \overline{\theta}^i \sum_{j=1}^{q} A_{ij} \theta^j \right) \tag{2.79}$$

Expanding the product, we see that

$$Z(A) = \varepsilon^{j_1 \dots j_q} A_{qj_q} A_{q-1 j_{q-1}} \dots A_{1j_1} = \det(A) \tag{2.80}$$

which is,once again, the inverse of what happen in the ordinary case. As a final example in which the superdeterminant makes its appearance we shall evaluate the Gaussian integral

$$Z(M) = \int \left[dt^1 \dots | \dots d\theta^q\right] \exp\left\{ -\frac{1}{2} \begin{pmatrix} t^1 & \dots & | \dots \theta^q \end{pmatrix} M \begin{pmatrix} t^1 \\ \vdots \\ - \\ \vdots \\ \theta^q \end{pmatrix} \right\} \tag{2.81}$$

here M is a block matrix of dimension (p, q) like

$$M = \begin{pmatrix} A & C \\ C^\top & B \end{pmatrix} \tag{2.82}$$

where $A = A^\top$ and $B = -B^\top$ are the even blocks and C the odd one. The first step is to carry out the change of coordinates

$$\begin{aligned} t^i &\longrightarrow \widetilde{t}^{\,i} = t^i + A^{-1\,ij} C_{jk} \theta^k \\ \theta^i &\longrightarrow \widetilde{\theta}^{\,i} = \theta^i \end{aligned} \tag{2.83}$$

this is a transformation of (2.61) type with a unit Berezinian. Now the integral takes the form

$$Z(M) = \int \left[d\widetilde{t}^{\,1} \dots | \dots d\widetilde{\theta}^{\,q}\right] \exp\left\{ -\frac{1}{2} \begin{pmatrix} \widetilde{t}^{\,1} & \dots & | \dots \widetilde{\theta}^{\,q} \end{pmatrix} \widetilde{M} \begin{pmatrix} \widetilde{t}^{\,1} \\ \vdots \\ - \\ \vdots \\ \widetilde{\theta}^{\,q} \end{pmatrix} \right\} \tag{2.84}$$

where $\widetilde{M}$ has the diagonal block form

$$\widetilde{M} = \begin{pmatrix} A & 0 \\ 0 & B + C^\top A^{-1} C \end{pmatrix} \tag{2.85}$$

We assume that the matrices A and $B - C^\top A^{-1} C$ are nonsingular with A and B respectively symmetric and antisymmetric matrices. Then there exist real orthogonal matrices O_1 and O_2, of determinant $+1$, which transform A and B into the

$$\begin{aligned}
O_1^\top A O_1 &= \mathrm{diag}(\lambda_1 \ldots \lambda_p) \\
O_2^\top B O_2 &= \mathrm{diag}\left(\begin{pmatrix} 0 & i\mu_1 \\ -i\mu_1 & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & i\mu_q \\ -i\mu_q & 0 \end{pmatrix} \right)
\end{aligned} \tag{2.86}$$

where the $(\lambda_i\,, \mu_i)$ are respectively real eigenvalue of A and B. Next we carry out a second transformation

$$\begin{aligned}
\widetilde{t}^{\,i} &\longrightarrow \widehat{t}^{\,i} = O_{1j}^{i} \widetilde{t}^{\,j} \\
\widetilde{\theta}^{\,i} &\longrightarrow \widehat{\theta}^{\,i} = O_2^{ij} [1_q + B^{-1} C^\top A^{-1} C]_{jk}^{\frac{1}{2}} \widetilde{\theta}^{\,k}
\end{aligned} \tag{2.87}$$

whose representative matrix denoted by J has the following Berezinian

$$\mathrm{Ber}(J) = \det([1_q + B^{-1} C^\top A^{-1} C])^{-\frac{1}{2}} \tag{2.88}$$

Now plugging these transformation into (2.84) and using the integration rules founded in (2.71) and (2.77) we get

$$Z(M) = (2\pi)^{\frac{p}{2}} \det(A)^{-\frac{1}{2}} \mathrm{Pf}(B) \mathrm{Ber}(J)^{-1} \tag{2.89}$$

Since for an antisymmetric matrix B we have $\mathrm{Pf}(B)^2 = \det(B)$ we found that

$$\begin{aligned}
Z(M) &= (2\pi)^{\frac{p}{2}} \det(A)^{-\frac{1}{2}} \mathrm{Pf}(B) \mathrm{Ber}(J)^{-1} \\
&= (2\pi)^{\frac{p}{2}} \det(A)^{-\frac{1}{2}} \det(B)^{\frac{1}{2}} \det([1_q + B^{-1} C^\top A^{-1} C])^{\frac{1}{2}} \\
&= (2\pi)^{\frac{p}{2}} \mathrm{Ber}(M)^{-\frac{1}{2}}
\end{aligned} \tag{2.90}$$

CHAPTER 3

Supermanifolds

Roughly speaking, a supermanifold M of dimension $p|q$ (that is, bosonic dimension p and fermionic dimension q) can be described locally by p bosonic coordinates $t^1 \dots t^p$ and q fermionic coordinates $\theta^1 \dots \theta^q$. We cover M by open sets U_α each of which can be described by coordinates $t^1_\alpha \dots | \dots \theta^q_\alpha$. On the intersection $U_\alpha \cap U_\beta$, the t^i_α are even functions of $t^1_\beta \dots | \dots \theta^q_\beta$ and the θ^s_α are odd functions of the same variables. We call these functions gluing functions and denote them as $f_{\alpha\beta}$ and $\psi_{\alpha\beta}$:

$$\begin{aligned} t^i_\alpha &= f^i_{\alpha\beta}(t^1_\beta \dots | \dots \theta^q_\beta) \\ \theta^s_\alpha &= \psi^s_{\alpha\beta}(t^1_\beta \dots | \dots \theta^q_\beta). \end{aligned} \tag{3.1}$$

On the intersection $U_\alpha \cap U_\beta$, we require that the gluing map defined by $f^1_{\alpha\beta} \dots | \dots \psi^q_{\alpha\beta}$ is inverse to the one defined by $f^1_{\beta\alpha} \dots | \dots \psi^q_{\beta\alpha}$, and we require a compatibility of the gluing maps on triple intersections $U_\alpha \cap U_\beta \cap U_\gamma$. Thus formally the theory of supermanifolds mimics the standard theory of smooth manifolds. However, some of the geometric intuition fails due to the presence of the odd coordinates and a rigorous definition of supermanifold require the use of sheaf theory. Of course, there is a huge literature on supermanifolds and it is impossible to give complete references, nevertheless we suggest [1–5].

3.1. Presheaves and Sheaves

Let M be a topological space.

Definition 3.1.1. *We define a presheaf of rings on M a rule $\mathcal{R}$ which assigns a ring $\mathcal{R}(U)$ to each open subset U of M and a ring morphism (called restriction map) $\varphi_{U,V} : \mathcal{R}(U) \to \mathcal{R}(V)$ to each pair $V \subset U$ such that*

- $\mathcal{R}(\emptyset) = \{0\}$
- $\varphi_{U,U}$ *is the identity map*
- *if $W \subset V \subset U$ are open sets, then $\varphi_{U,W} = \varphi_{V,W} \circ \varphi_{U,V}$*

The elements $s \in \mathcal{R}(U)$ are called sections of the presheaf $\mathcal{R}$ on U. If $s \in \mathcal{R}(U)$ is a section of $\mathcal{R}$ on U and $V \subset U$, we shall write $s_{|V}$ instead of $\varphi_{U,V}(s)$.

Definition 3.1.2. *A sheaf on a topological space M is a presheaf $\mathcal{F}$ on M which fulfills the following axioms for any open subset U of M and any cover $\{U_i\}$ of U*

- *If two sections $s \in \mathcal{F}(U)$,$\check{s} \in \mathcal{F}(U)$ coincide when restricted to any U_i, $s_{|U_i} = \check{s}_{|U_i}$, they are equal, $s = \check{s}$*
- *Given sections $s_i \in \mathcal{F}(U_i)$ which coincide on the intersections, $s_{i|U_i \cap U_j} = s_{j|U_i \cap U_j}$ for every i, j there exist a section $s \in \mathcal{F}(U)$ whose restriction to each U_i equals s_i, $s_{|U_i} = s_i$*

Naively speaking sheaves are presheaves defined by local conditions. As a first example of sheaf let's consider $\mathcal{C}_M(U)$ the ring of real-valued functions on an open set U of M, then $\mathcal{C}_M$ is the sheaf of continuous functions on M. In the same way we can define $\mathcal{C}^\infty_M$ and Ω^p_M which are respectively the sheaf on differentiable functions and the sheaf of differential p-forms on a differentiable manifold M. At this point it is interesting to underline the difference between sheaves and presheaves and to do that we will use the familiar context of de-Rham theory. Let M be a differentiable manifold, and let $d : \Omega^\bullet_M \to \Omega^\bullet_M$ be the de-Rham differential. We can define the presheaves $\mathcal{Z}^p_M$ of closed differential p-forms, and $\mathcal{B}^p_M$ of exact p-forms. $\mathcal{Z}^p_M$ is a sheaf, since the condition of being closed is local: a differential form is closed if and only if it is closed in a neighbourhood of each point of M. Conversely $\mathcal{B}^p_M$ it's not a sheaf in fact if we consider $M = \mathbb{R}^2$, the presheaf $\mathcal{B}^1_M$ of exact 1-forms does not satisfy the second sheaf axiom. This situation arise when we consider the form

$$\omega = \frac{xdy - ydx}{x^2 + y^2}$$

which is defined on the open subset $U = \mathbb{R}^2 - \{(0,0)\}$. Since ω is closed on U, there is an open cover $\{U_i\}$ of U where ω is an exact form, $\omega_{|U_i} \in \mathcal{B}^1_M(U_i)$ (Poincaré Lemma). But ω it's not an exact form on U since its integral along the unit circle is different from zero. In the interesting reference [34] there is a more complete description of sheaf theory, as well of other concepts of algebraic geometry, aimed to physicists. Right now we are ready to define precisely what a supermanifold is by means of the sheaf theory.

Definition 3.1.3. *A real smooth supermanifold $\mathcal{M}$ of dimension $p|q$ is a pair $(M, \mathcal{O}_\mathcal{M})$, where M is a real smooth manifold of dimension p,*

$\mathcal{O}_{\mathcal{M}}$ is a sheaf of commutative superalgebra such that locally

$$\mathcal{O}_{\mathcal{M}}(U) \simeq \mathcal{C}^{\infty}_{M}(U) \otimes \wedge^{\bullet}(\mathbb{R}^{q})^{*} \tag{3.2}$$

where $U \subset M$ is an open subset and $\wedge^{\bullet}(\mathbb{R}^{q})^{*}$ as defined in Example 2.1.1.

Although this statement is precise, it is not as much useful when we are dealing with computations, let's follow the approach of [15] and illustrate this formal definition with a couple of concrete examples.

Example 3.1.4. *Assume that M is smooth manifold then we can associate to it the supermanifold ΠTM called odd tangent bundle, which is defined by the gluing rule*

$$\widetilde{t}^{\,\mu} = \widetilde{t}^{\,\mu}(t)\ ,\ \widetilde{\theta}^{\,\mu} = \frac{\partial \widetilde{t}^{\,\mu}}{\partial t^{\,\nu}} \theta^{\,\nu}\ , \tag{3.3}$$

where t's are local coordinates on M and θ's are glued as dt^{μ}. Here we consider the fiber directions of the tangent bundle to be fermionic rather than bosonic. The symbol Π stands for reversal of statistics in the fiber directions; in the literature, this is often called reversal of parity. The functions on ΠTM have the following expansion

$$f(t,\theta) = \sum_{p=0}^{\dim M} \frac{1}{p!} f_{\mu_1 \mu_2 \ldots \mu_p}(t) \theta^{\mu_1} \theta^{\mu_2} \ldots \theta^{\mu_p} \tag{3.4}$$

and thus they are naturally identified with the differential forms, $\mathcal{C}^{\infty}(\Pi TM) = \Omega^{\bullet}(M)$.

Example 3.1.5. *Again let M be a smooth manifold and now we associate to it another supermanifold $\Pi T^{*}M$ called odd cotangent bundle, which has the following local description*

$$\widetilde{t}^{\,\mu} = \widetilde{t}^{\,\mu}(t)\ ,\ \widetilde{\theta}_{\mu} = \frac{\partial t^{\,\nu}}{\partial \widetilde{t}^{\,\mu}} \theta_{\nu}\ , \tag{3.5}$$

where t's are local coordinates on M and θ's transform as ∂_{μ}. The functions on $\Pi T^{*}M$ have the expansion

$$f(t,\theta) = \sum_{p=0}^{\dim M} \frac{1}{p!} f^{\mu_1 \mu_2 \ldots \mu_p}(t) \theta_{\mu_1} \theta_{\mu_2} \ldots \theta_{\mu_p} \tag{3.6}$$

and thus they are naturally identified with multivector fields, $\mathcal{C}^{\infty}(\Pi T^{*}M) = \Gamma(\wedge^{\bullet} TM)$.

The use of local coordinates is extremely useful and sufficient for most purposes and we will follow this approach troughout this notes.

3.2. Integration Theory

A proper integration theory on supermanifold requires the explanation of what sort of object can be integrated. To achieve this result it will be useful to reinterpret some result from sections (2.3 ,2.4) following a geometrical approach [1]. Now let M be a compact supermanifold of dimension $p|q$, as described in section 3.1. We introduce on M a line bundle called Berezinian line bundle $\mathrm{Ber}(M)$. $\mathrm{Ber}(M)$ is defined by saying that every local coordinate system $T = t^1 \ldots | \ldots \theta^q$ on M determines a local trivialization of $\mathrm{Ber}(M)$ that we denote $\left[\mathrm{d}t^1 \ldots | \ldots \mathrm{d}\theta^q\right]$. Moreover, if $\widetilde{T} = \widetilde{t}^1 \ldots | \ldots \widetilde{\theta}^q$ is another coordinate system, then the two trivializations of $\mathrm{Ber}(M)$ are related by

$$\left[\mathrm{d}t^1 \ldots | \ldots \mathrm{d}\theta^q\right] = \mathrm{Ber}\left(\frac{\partial T}{\partial \widetilde{T}}\right)\left[\,\mathrm{d}\widetilde{t}^1 \ldots | \ldots \mathrm{d}\widetilde{\theta}^q\right]. \tag{3.7}$$

see the analogy with formula (2.70). What can be naturally integrated over M is a section of $\mathrm{Ber}(M)$. To show this, first let s be a section of $\mathrm{Ber}(M)$ whose support is contained in a small open set $U \subset M$ on which we are given local coordinates $t^1 \ldots | \ldots \theta^q$, establishing an isomorphism of U with an open set in $\mathbb{R}^{p|q}$. This being so, we can view s as a section of the Berezinian of $\mathbb{R}^{p|q}$. This Berezinian is trivialized by the section $[\mathrm{d}t^1 \ldots | \ldots \mathrm{d}\theta^q]$ and s must be the product of this times some function g:

$$s = [\mathrm{d}t^1 \ldots | \ldots \mathrm{d}\theta^q]\, g(t^1 \ldots | \ldots \theta^q). \tag{3.8}$$

So we define the integral of s to equal the integral of the right hand side of equation (3.8):

$$\int_M s = \int_{\mathbb{R}^{p|q}} \left[\mathrm{d}t^1 \ldots | \ldots \mathrm{d}\theta^q\right] g(t^1 \ldots | \ldots \theta^q). \tag{3.9}$$

The integral on the right is the naive Berezin integral (2.49). For this definition to make sense, we need to check that the result does not depend on the coordinate system $t^1 \ldots | \ldots \theta^q$ on $\mathbb{R}^{p|q}$ that was used in the computation. This follows from the rule (3.7) for how the symbol $\left[\mathrm{d}t^1 \ldots | \ldots \mathrm{d}\theta^q\right]$ transforms under a change of coordinates. The Berezinian in this formula is analogous to the usual Jacobian in the transformation law of an ordinary integral under a change of coordinates as we have seen in section (2.4). Up to now, we have defined the integral of a section of $\mathrm{Ber}(M)$ whose support is in a sufficiently small region in M. To reduce the general case to this, we pick a cover of M by small open sets U_α, each of which is isomorphic to an open set in $\mathbb{R}^{p|q}$, and we use the existence of a partition of unity. On a smooth manifold, one can find smooth functions h_α on M such that each h_α is supported in the interior of U_α and $\sum_\alpha h_\alpha = 1$. Then we write $s = \sum_\alpha s_\alpha$ where $s_\alpha = sh_\alpha$. Each s_α is supported in U_α, so its integral can be defined as in (3.9). Then we define $\int_M s = \sum_\alpha \int_M s_\alpha$. To show that this doesn't depend on the choice of the open cover or the partition of unity we can use the same kind of arguments used to define the integral of a differential form on an ordinary manifold. The way to integrate over a supermanifold M is found by noting this basic difference: on M, there is not in general a natural way to have a section of the Berezinian, on ΠTM the natural choice is always possible because of the of the behaviour of the variables in pairs. Let's study the integration on odd tangent and odd cotangent bundles.

Example 3.2.1. *On ΠTM the even part of the measure transforms in the standard way*

$$[d\widetilde{t}^1 \dots d\widetilde{t}^n] = \det\left(\frac{\partial \widetilde{t}}{\partial t}\right)[dt^1 \dots dt^n] \tag{3.10}$$

while the odd part transforms according to the following property

$$[d\widetilde{\theta}^1 \dots d\widetilde{\theta}^n] = \det\left(\frac{\partial \widetilde{t}}{\partial t}\right)^{-1}[d\theta^1 \dots d\theta^n] \tag{3.11}$$

where this transormation rules are obtained from Example 3.1.4. As we can see the transformation of even and odd parts cancel each other and thus we have

$$\int [d\widetilde{t}^1 \dots | \dots d\widetilde{\theta}^q] = \int [dt^1 \dots | \dots d\theta^q] \tag{3.12}$$

which corresponds to the canonical integration on ΠTM. Any function of top degree on ΠTM can be integrated canonically.

Example 3.2.2. *On ΠT^*M the even part transforms as before*

$$[d\widetilde{t}^1 \dots d\widetilde{t}^n] = \det\left(\frac{\partial \widetilde{t}}{\partial t}\right)[dt^1 \dots dt^n] \tag{3.13}$$

while the odd part transforms in the same way as the even one

$$[d\widetilde{\theta}^1 \dots d\widetilde{\theta}^n] = \det\left(\frac{\partial \widetilde{t}}{\partial t}\right)[d\theta^1 \dots d\theta^n] \tag{3.14}$$

where this transormation rule are obtained from Example 3.1.5. We assume that M is orientiable and choose a volume form

$$\mathrm{vol} = \rho(t)\ dt^1 \wedge \dots \wedge dt^n \tag{3.15}$$

ρ transforms as a densitity

$$\widetilde{\rho} = \det \left(\frac{\partial \widetilde{t}}{\partial t} \right)^{-1} \rho \tag{3.16}$$

Now we can define the following invariant measure

$$\int [d\widetilde{t}^1 \ldots | \ldots d\widetilde{\theta}^q] \widetilde{\rho}^2 = \int [dt^1 \ldots | \ldots d\theta^q] \rho^2 \tag{3.17}$$

Thus to integrate the multivector fields we need to pick a volume form on M.

Remark: A naive generalization of differential forms to the case of supermanifold with even coordinates t^μ and odd coordinates θ^μ leads to functions $F(t, \theta | dt, d\theta)$ that are homogeneous polynomials in $(dt, d\theta)$ (note that dt is odd while $d\theta$ is even) and such forms cannot be integrated over supermanifolds. In the pure even case, the degree of the form can only be less or equal than the dimension of the manifold and the forms of the top degree transform as measures under smooth coordinate transformations. Then, it is possible to integrate the forms of the top degree over the oriented manifolds and forms of lower degree over the oriented subspaces. On the other hand, forms on a supermanifold may have arbitrary large degree due to the presence of commuting $d\theta^\mu$ and none of them transforms as a Berezinian measure. The correct generalization of the differential form that can be integrated over supermanifold is an object ω on M called *integral form* defined as arbitrary generalized function $\omega(x, dx)$ on ΠTM, where we abbreviated the whole set of coordinates $t^1 \ldots | \ldots \theta^q$ on M as x. Basically we require that in its dependence on $d\theta^1 \ldots d\theta^q$, ω is a distribution supported at the origin. We define the integral of ω over M as Berezin integral over ΠTM

$$\int_M \omega = \int_{\Pi TM} \mathcal{D}(x, dx)\, \omega(x, dx) \tag{3.18}$$

where $\mathcal{D}(x, dx)$ is an abbreviation for $[dt^1 \ldots d(d\theta^q) | d\theta^1 \ldots d(dt^p)]$. The integral on the right hand side of equation (3.18) does not depend on the choice of coordinates in fact as we have seen before the two Berezinians, one from the change of x and the other from the change of dx cancel each other due to the twist of parity on ΠTM. On the left hand side of equation (3.18) it's crucial that ω is an integral form rather than a differential form. Because $\omega(x)$ has compact support as a function of even variables $d\theta^1 \ldots d\theta^q$ the integral over those variables makes sense. A similar approach to integrating a differential form on M would not make sense, since if $\omega(x)$ is a differential form, it has polynomial dependence on $d\theta^1 \dots d\theta^q$ and the integral over those variables does not converge.

CHAPTER 4

Graded Geometry

Graded geometry is a generalization of supergeometry. Here we are introducing a $\mathbb{Z}$-grading instead of a $\mathbb{Z}_2$-grading and many definitions from supergeometry have a related analog in the graded case. References [32, 38] are standard introductions to the subject.

4.1. Graded Linear Algebra

A graded vector space V is a collection of vector spaces V_i with the decomposition

$$V = \bigoplus_{i \in \mathbb{Z}} V_i \tag{4.1}$$

If $v \in V_i$ we say that v is a homogeneous element of V with degree $|v| = i$. Any element of V can be decomposed in terms of homogeneous elements of a given degree. A morphism $f : V \to W$ of graded vector spaces is a collection of linear maps

$$(f_i : V_i \to W_i)_{i \in \mathbb{Z}} \tag{4.2}$$

The morphisms between graded vector spaces are also referred to as graded linear maps i.e. linear maps which preserves the grading. The dual V^* of a graded vector space V is the graded vector space $(V^*_{-i})_{i\in\mathbb{Z}}$. Moreover,$V$ shifted by k is the graded vector space $V[k]$ given by $(V_{i+k})_{i\in\mathbb{Z}}$. By definition, a graded linear map of degree k between V and W is a graded linear map between V and $W[k]$. If the graded vector space V is equipped with an associative product which respects the grading then we call V a graded algebra. If for a graded algebra V and any homogeneous elements v, $\check{v} \in V$ we have the relation

$$v\check{v} = (-1)^{|v||\check{v}|}\check{v}v \tag{4.3}$$

then we call V a graded commutative algebra. A significant example of graded algebra is given by the graded symmetric space $S(V)$.

Definition 4.1.1. *Let V be a graded vector space over $\mathbb{R}$ or $\mathbb{C}$. We define the graded symmetric algebra $S(V)$ as the linear space spanned* *by polynomial functions on* V

$$\sum_l f_{a_1 a_2 \dots a_l} \; v^{a_1} v^{a_2} \dots v^{a_l} \tag{4.4}$$

where

$$v^a v^b = (-1)^{|v^a||v^b|} v^b v^a \tag{4.5}$$

with v^a *and* v^b *being homogeneous elements of degree* $|v^a|$ *and* $|v^b|$ *respectively. The functions on* V *are naturally graded and multiplication of function is graded commutative. Therefore the graded symmetric algebra* $S(V)$ *is a graded commutative algebra.*

4.2. Graded Manifolds

To introduce the notion of graded manifold we will follow closely what we have done for the supermanifolds.

Definition 4.2.1. *A smooth graded manifold* $\mathcal{M}$ *is a pair* $(M, \mathcal{O}_{\mathcal{M}})$, *where* M *is a smooth manifold and* $\mathcal{O}_{\mathcal{M}}$ *is a sheaf of graded commutative algebra such that locally*

$$\mathcal{O}_{\mathcal{M}}(U) \simeq \mathcal{C}^{\infty}_{M}(U) \otimes S(V) \tag{4.6}$$

where $U \subset M$ *is an open subset and* V *is a graded vector space.*

The best way to clarify this definition is by giving explicit examples.

Example 4.2.2. *Let us introduce the graded version of the odd tangent bundle. We denote the graded tangent bundle as* $T[1]M$ *and we have the same coordinates* t^μ *and* θ^μ *as in Example 3.1.4, with the same transformation rules. The coordinate* t *is of degree* 0 *and* θ *is of degree* 1 *and the gluing rules respect the degree. The space of functions* $\mathcal{C}^{\infty}(T[1]M) = \Omega^{\bullet}(M)$ *is a graded commutative algebra with the same* $\mathbb{Z}$*-grading as the differential forms.*

Example 4.2.3. *Moreover, we can introduce the graded version* $T^*[-1]M$ *of the odd cotangent bundle following Example 3.1.5. We allocate the degree* 0 *for* t *and degree* -1 *for* θ. *The gluing preserves the degrees. The functions* $\mathcal{C}^{\infty}(T^*[-1]M) = \Gamma(\wedge^{\bullet} TM)$ *is graded commutative algebra with degree given by minus of degree of multivector field.*

A big part of differential geometry can be readily generalized to the graded case. Integration theory for graded manifolds is the same to what we already introduced in (3.2) since we look at the underlying supermanifold structure. A graded vector fields on a graded manifold can be identified with graded derivations of the algebra of smooth functions.

Definition 4.2.4. *A graded vector field on $\mathcal{M}$ is a graded linear map*

$$X : \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})[k] \tag{4.7}$$

which satisfies the graded Leibniz rule

$$X(fg) = X(f)g + (-1)^{k|f|} f X(g) \tag{4.8}$$

for all homogeneous smooth functions f, g. The integer k is called degree of X.

A graded vector field of degree 1 which commutes with itself is called a cohomological vector field. If we denote this cohomological vector field with D we say that D endows the graded commutative algebra of functions $\mathcal{C}^\infty(\mathcal{M})$ with the structure of differential complex. Such graded commutative algebra with D is called a graded differential algebra or simply a dg-algebra. A graded manifold endowed with a cohomological vector field is called dg-manifold.

Example 4.2.5. *Consider the shifted tangent bundle $T[1]M$, whose algebra of smooth functions is equal to the algebra of differential forms $\Omega(M)$. The de Rham differential on $\Omega(M)$ corresponds to a cohomological vector field D on $T[1]M$. The cohomological vector field D is written in local coordinates as*

$$D = \theta^\mu \frac{\partial}{\partial t^\mu} \tag{4.9}$$

In this setting $\mathcal{C}^\infty(T[1]M)$ is an example of dg-algebra.

CHAPTER 5

Odd Fourier transform and BV-formalism

In this section we will derive the BV formalism via the odd Fourier transformation which provides a map from $\mathcal{C}^\infty(T[1]M)$ to $\mathcal{C}^\infty(T^*[-1]M)$. As explained in [15] the odd cotangent bundle $\mathcal{C}^\infty(T^*[-1]M)$ has an interesting algebraic structure on the space of functions and employing the odd Fourier transform we will obtain the Stokes theorem for the integration on $T^*[-1]M$. The power of the BV formalism is based on the algebraic interpretation of the integration theory for odd cotangent bundle.

5.1. Odd Fourier Transform

Let's consider a n-dimensional orientable manifold M, we can choose a volume form

$$\mathrm{vol} = \rho(t)\ dt^1 \wedge \cdots \wedge dt^n = \frac{1}{n!}\ \Omega_{\mu_1 \ldots \mu_n}(t)\ dt^{\mu_1} \wedge \cdots \wedge dt^{\mu_n} \tag{5.1}$$

which is a top degree nowhere vanishing form, where

$$\rho(t) = \frac{1}{n!}\ \varepsilon^{\mu_1 \cdots \mu_n} \Omega_{\mu_1 \ldots \mu_n}(t) \tag{5.2}$$

Since we have the volume form, we can define the integration only along the odd direction on $T[1]M$ in the following manner

$$\int [d\widetilde{\theta}^1 \ldots d\widetilde{\theta}^n] \widetilde{\rho}^{-1} = \int [d\theta^1 \ldots d\theta^n] \rho^{-1} \tag{5.3}$$

The odd Fourier transform is defined for $f(t, \theta) \in \mathcal{C}^\infty(T[1]M)$ as

$$F[f](t, \psi) = \int [d\theta^1 \ldots d\theta^n] \rho^{-1} e^{\psi_\mu \theta^\mu} f(t, \theta) \tag{5.4}$$

To make sense globally of the transformation (5.4) we assume that the degree of ψ is -1. Additionally we require that ψ_μ transforms as ∂_μ (dual to θ^μ). Thus $F[f](t, \psi) \in \mathcal{C}^\infty(T^*[-1]M)$ and the odd Fourier transform maps functions on $T[1]M$ to functions on $T^*[-1]M$. The explicit computation of the integral in the right hand side of equation

(5.4) leads to

$$F[f](t,\psi) = \frac{(-1)^{(n-p)(n-p+1)/2}}{p!(n-p)!} f_{\mu_1\dots\mu_p}\Omega^{\mu_1\dots\mu_p\mu_{p+1}\dots\mu_n}\partial_{\mu_{p+1}}\wedge\dots\wedge\partial_{\mu_n} \tag{5.5}$$

where $\Omega^{\mu_1\dots\mu_n}$ is defined as components of a nowhere vanishing top multivector field dual to the volume form (5.1)

$$\mathrm{vol}^{-1} = \rho^{-1}(t)\ \partial_1\wedge\dots\wedge\partial_n = \frac{1}{n!}\ \Omega^{\mu_1\dots\mu_n}(t)\ \partial_{\mu_1}\wedge\dots\wedge\partial_{\mu_n} \tag{5.6}$$

Equation (5.5) needs a comment, indeed the factor $(-1)^{(n-p)(n-p+1)/2}$ appearing here is due to conventions for θ-terms ordering in the Berezin integral; as we can see the odd Fourier transform maps differential forms to multivectors. We can also define the inverse Fourier transform F^{-1} which maps the functions on $T^*[-1]M$ to functions on $T[1]M$

$$F^{-1}[\widetilde{f}](t,\theta) = (-1)^{n(n+1)/2}\int[d\psi_1\dots d\psi_n]\rho^{-1}e^{-\psi_\mu\theta^\mu}\widetilde{f}(t,\psi) \tag{5.7}$$

where $\widetilde{f}(t,\psi)\in\mathcal{C}^\infty(T^*[-1]M)$. Equation (5.7) can be also seen as a contraction of a multivector field with a volume form. To streamline our notation we will denote all symbols without tilde as functions on $T[1]M$ and all symbols with tilde as functions on $T^*[-1]M$. Under the odd Fourier transform F the differential D defined in (4.9) transforms to bilinear operation Δ on $\mathcal{C}^\infty(T^*[-1]M)$ as

$$F[Df] = (-1)^n\Delta F[f] \tag{5.8}$$

and from this we get

$$\Delta = \frac{\partial^2}{\partial x^\mu\partial\psi_\mu} + \partial_\mu(\log\rho)\frac{\partial}{\partial\psi_\mu} \tag{5.9}$$

By construction $\Delta^2 = 0$ and degree of Δ is 1. To obtain formula (5.9) we need to plug the expression for D, found in (4.9), into (5.4) and to bring out the two derivatives from the Fourier transform. The algebra of smooth functions on $T^*[-1]M$ is a graded commutative algebra with respect to the ordinary multiplication of functions, but Δ it's not a derivation of this multiplication since

$$\Delta(\widetilde{f}\widetilde{g}) \neq \Delta(\widetilde{f})\widetilde{g} + (-1)^{|\widetilde{f}|}\widetilde{f}\Delta(\widetilde{g}) \tag{5.10}$$

We define the bilinear operation which measures the failure of Δ to be a derivation as

$$\{\widetilde{f},\widetilde{g}\} = (-1)^{|\widetilde{f}|}\Delta(\widetilde{f}\widetilde{g}) - (-1)^{|\widetilde{f}|}\Delta(\widetilde{f})\widetilde{g} - \widetilde{f}\Delta(\widetilde{g}) \tag{5.11}$$

A direct calculation gives

$$\{\widetilde{f}, \widetilde{g}\} = \frac{\partial \widetilde{f}}{\partial x^\mu}\frac{\partial \widetilde{g}}{\partial \psi_\mu} + (-1)^{|\widetilde{f}|}\frac{\partial \widetilde{f}}{\partial \psi_\mu}\frac{\partial \widetilde{g}}{\partial x^\mu} \tag{5.12}$$

which is very reminiscent of the standard Poisson bracket for the cotangent bundle, but now with the odd momenta.

Definition 5.1.1. *A graded commutative algebra* V *with the odd bracket* $\{\ ,\ \}$ *satisfying the following axioms*

$$\begin{aligned} \{v, w\} &= -(-1)^{(|v|+1)(|w|+1)}\{w, v\} \\ \{v, \{w, z\}\} &= \{\{v, w\}, z\} + (-1)^{(|v|+1)(|w|+1)}\{w, \{v, z\}\} \\ \{v, wz\} &= \{v, w\}z + (-1)^{(|v|+1)|w|}w\{v, z\} \end{aligned} \tag{5.13}$$

is called a Gerstenhaber algebra [9].

It is assumed that the degree of bracket $\{\ ,\ \}$ is 1.

Definition 5.1.2. *A Gerstenhaber algebra* $(V, \cdot, \{\ ,\ \})$ *together with an odd, anticommuting,* $\mathbb{R}$*-linear map which generates the bracket* $\{\ ,\ \}$ *according to*

$$\{v, w\} = (-1)^{|v|}\Delta(vw) - (-1)^{|v|}(\Delta v)w - v(\Delta w) \tag{5.14}$$

is called a BV algebra [10]. Δ *is called the odd Laplace operator (odd Laplacian).*

It is assumed that degree of Δ is 1. Here we are not showing that the bracket (5.14) respect all axioms (5.13), however to reach this result is also necessary to understand that the BV bracket enjoys a generalized Leibniz rule

$$\Delta\{v, w\} = \{\Delta v, w\} - (-1)^{|v|}\{v, \Delta w\} \tag{5.15}$$

Summarizing, upon a choice of a volume form on M the space of functions $\mathcal{C}^\infty(T^*[-1]M)$ is a BV algebra with Δ defined in (5.9). The graded manifold $T^*[-1]M$ is called a BV manifold. A BV manifold can be defined as a graded manifold $\mathcal{M}$ such that the space of function $\mathcal{C}^\infty(\mathcal{M})$ is endowed with a BV algebra structure. As a final comment we will give an alternative definition of BV algebra.

Definition 5.1.3. *A graded commutative algebra* V *with an odd, anticommuting,* $\mathbb{R}$*-linear map satisfying*

$$\begin{aligned} \Delta(vwz) = {} & \Delta(vw)z + (-1)^{|v|}v\Delta(wz) + (-1)^{(|v|+1)|w|}w\Delta(vz) \\ & - \Delta(v)wz - (-1)^{|v|}v\Delta(w)z - (-1)^{|v|+|w|}vw\Delta(z) \end{aligned} \tag{5.16}$$

is called a BV algebra.

An operator Δ with these properties gives rise to the bracket (5.14) which satisfies all axioms in (5.13). This fact can be seen easily in the following way: using the definition (5.14), to show that the second equation in (5.13) holds, we discover the relation (5.16). For a better understanding of the origin of equation (5.16) let's consider the functions $f(t)$, $g(t)$ and $h(t)$ of one variable and the second derivative which satisfies the following property

$$\frac{d^2(fgh)}{dt^2} + \frac{d^2 f}{dt^2} gh + f\frac{d^2 g}{dt^2} h + fg\frac{d^2 h}{dt^2} = \frac{d^2(fg)}{dt^2} h + \frac{d^2(fh)}{dt^2} g + f\frac{d^2(gh)}{dt^2} \quad (5.17)$$

This result can be regarded as a definition of second derivative. Basically the property (5.16) is just the graded generalization of the second order differential operator. In the case of $C^\infty(T^*[-1]M)$, the Δ as in (5.9) is of second order.

5.2. Integration Theory

Previously we discussed different algebraic aspects of graded manifolds $T[1]M$ and $T^*[-1]M$ which can be related by the odd Fourier transformation upon the choice of a volume form on M. $T^*[-1]M$ has a quite interesting algebraic structure since $C^\infty(T^*[-1]M)$ is a BV algebra. At the same time $T[1]M$ has a very natural integration theory. The goal of this section is to mix the algebraic aspects of $T^*[-1]M$ with the integration theory on $T[1]M$ using the odd Fourier transform defined in (5.1). The starting point is a reformulation of the Stokes theorem in the language of the graded manifolds. For this purpose it is useful to review a few facts about standard submanifolds. A submanifold C of M can be described in algebraic language as follows. Consider the ideal $\mathcal{I}_C \subset \mathcal{C}^\infty(M)$ of functions vanishing on C. The functions on submanifold C can be described as quotient $\mathcal{C}^\infty(C) = \mathcal{C}^\infty(M)/\mathcal{I}_C$. Locally we can choose coordinates t^μ adapted to C such that the submanifold C is defined by the conditions $t^{p+1} = 0, t^{p+2} = 0, \dots, t^n = 0$ ($\dim C = p$ and $\dim M = n$) while the rest $t^1, t^2, \dots, t^p$ may serve as coordinates for C. In this local description $\mathcal{I}_C$ is generated by $t^{p+1}, t^{p+2}, \dots, t^n$. The submanifolds can be defined purely algebraically as ideals of $\mathcal{C}^\infty(M)$ with certain regularity condition. This construction leads to a generalization for the graded settings. Let's collect some particular examples which are relevant to fulfill our task.

Example 5.2.1. *$T[1]C$ is a graded submanifold of $T[1]M$ if C is submanifold of M. In local coordinates $T[1]C$ is described by*

$$t^{p+1} = 0,\ t^{p+2} = 0\ ,\ \dots\ ,\ t^n = 0\ ,\ \theta^{p+1} = 0\ ,\ \theta^{p+2} = 0\ ,\ \dots\ ,\ \theta^n = 0 \tag{5.18}$$

thus $t^{p+1}, ..., t^n, \theta^{p+1}, ..., \theta^n$ generate the corresponding ideal $\mathcal{I}_{T[1]C}$.

Functions on the submanifold $\mathcal{C}^\infty(T[1]C)$ are given by the quotient $\mathcal{C}^\infty(T[1]M)/\mathcal{I}_{T[1]C}$. Moreover the above conditions define a natural embedding $i : T[1]C \to T[1]M$ of graded manifolds and thus we can define properly the pullback of functions from $T[1]M$ to $T[1]C$.

Example 5.2.2. *There is another interesting class of submanifolds, namely odd conormal bundle $N^*[-1]C$ viewed as graded submanifold of $T^*[-1]M$. In local coordinate $N^*[-1]C$ is described by the conditions*

$$t^{p+1} = 0,\ t^{p+2} = 0\ ,\ \dots\ ,\ t^n = 0\ ,\ \psi_1 = 0\ ,\ \psi_2 = 0\ ,\ \dots\ ,\ \psi_p = 0 \tag{5.19}$$

thus $t^{p+1}, \dots, t^n, \psi_1, \dots, \psi_p$ generate the ideal $\mathcal{I}_{N^*[-1]C}$.

All functions on $\mathcal{C}^\infty(N^*[-1]C)$ can be described by the quotient $\mathcal{C}^\infty(T^*[-1]M)/\mathcal{I}_{N^*[-1]C}$. The above conditions define a natural embedding $j : N^*[-1]C \to T^*[-1]M$ and thus we can define properly the pullback of functions from $T^*[-1]M$ to $N^*[-1]C$. At this point we can relate the following integrals over different manifolds by means of the Fourier transform

$$\int\limits_{T[1]C} [dt^1 \dots dt^p | d\theta^1 \dots d\theta^p]\, i^* \left(f(t,\theta)\right) =$$
$$= (-1)^{(n-p)(n-p+1)/2} \int\limits_{N^*[-1]C} [dt^1 \dots dt^p | d\psi^1 \dots d\psi^{n-p}]\, \rho\, j^* \left(F[f](t,\psi)\right) \tag{5.20}$$

Equation (5.20) needs some comments. On the left hand side we are integrating the pullback of $f \in \mathcal{C}^\infty(T[1]M)$ over $T[1]C$ using the well known integration rules defined troughout section (3.2). On the right hand side we are integrating the pullback of $F[f] \in \mathcal{C}^\infty(T^*[-1]M)$ over $N^*[-1]C$. We have to ensure that the measure $[dt^1 \dots dt^p | d\psi^1 \dots d\psi^{n-p}]\, \rho$ is invariant under a change of coordinates which preserve C.

Proof. Let's consider the adapted coordinates $t^\mu = (t^i, t^\alpha)$ such that t^i $(i, j, k = 1, 2, \dots, p)$ are the coordinates along C and t^α $(\alpha, \beta, \gamma =$

$p+1,\ldots,n$) are coordinates transverse to C. A generic change of variables has the form

$$\widetilde{t}^{\,i} = \widetilde{t}^{\,i}(t^j, t^\beta) \qquad\qquad \widetilde{t}^{\,\alpha} = \widetilde{t}^{\,\alpha}(t^j, t^\beta) \tag{5.21}$$

then all the transformations preserving C have to satisfy

$$\frac{\partial \widetilde{t}^{\,\alpha}}{\partial t^k}(t^j, 0) = 0 \qquad\qquad \frac{\partial \widetilde{t}^{\,i}}{\partial t^\beta}(t^j, 0) = 0 \tag{5.22}$$

These conditions follow from the general transformation of differentials

$$d\widetilde{t}^{\,\alpha} = \frac{\partial \widetilde{t}^{\,\alpha}}{\partial t^{\,k}}(t^j, t^\gamma) dt^k + \frac{\partial \widetilde{t}^{\,\alpha}}{\partial t^{\,\beta}}(t^j, t^\gamma) dt^\beta \tag{5.23}$$

$$d\widetilde{t}^{\,i} = \frac{\partial \widetilde{t}^{\,i}}{\partial t^{\,k}}(t^j, t^\gamma) dt^k + \frac{\partial \widetilde{t}^{\,i}}{\partial t^{\,\beta}}(t^j, t^\gamma) dt^\beta \tag{5.24}$$

in fact if we want that adapted coordinates transform to adapted coordinates we have to impose equations (5.22). On $N^*[-1]C$ we have the following transformations of odd conormal coordinate ψ_α

$$\widetilde{\psi}_\alpha = \frac{\partial t^\beta}{\partial \widetilde{t}^{\,\alpha}}(t^i, 0)\psi_\beta \tag{5.25}$$

Note that ψ_α is a coordinate on $N^*[-1]C$ not a section, and the invariant object will be $\psi_\alpha dt^\alpha$. Under the above transformations restricted to C our measure transforms canonically

$$[dt^1 \ldots dt^p | d\psi^1 \ldots d\psi^{n-p}]\ \rho(t^i, 0) = [d\widetilde{t}^{\,1} \ldots d\widetilde{t}^{\,p} | d\widetilde{\psi}^{\,1} \ldots d\widetilde{\psi}^{\,n-p}]\ \widetilde{\rho}(\widetilde{t}^{\,i}, 0) \tag{5.26}$$

where ρ transforms as (3.16). □

The pullback of functions on the left and right hand side consists in imposing conditions (5.18) and (5.19) respectively. Since all operations in (5.20) are covariant, (respecting the appropriate gluing rule), the equation is globally defined and independent from the choice of adapted coordinates. Let's recap two important corollaries of the Stokes theorem for differential forms emerging in the context of ordinary differential geometry. The first corollary is that the integral of an exact form over a closed submanifold C is zero and the second one is that the integral over closed form depends only on the homology class of C

$$\int_C d\omega = 0 \qquad\qquad \int_C \alpha = \int_{\widetilde{C}} \alpha \tag{5.27}$$

where α and ω are differential forms, $d\alpha = 0$, C and $\widetilde{C}$ are closed submanifolds in the same homology class. These two statements can be rewritten in the graded language

$$\int\limits_{T[1]C} [dt^1 \dots dt^p | d\theta^1 \dots d\theta^p] \; Dg = 0 \tag{5.28}$$

$$\int\limits_{T[1]C} [dt^1 \dots dt^p | d\theta^1 \dots d\theta^p] \; f = \int\limits_{T[1]\widetilde{C}} [dt^1 \dots dt^p | d\theta^1 \dots d\theta^p] \; f \tag{5.29}$$

where $Df = 0$ and we are working with pullbacks of $f, g \in C^\infty(T[1]M)$ to the submanifolds. Next we can combine the formula (5.20) with (5.28) and (5.29). Then we get the following properties to which we will refer as Ward identities

$$\int\limits_{N^*[-1]C} [dt^1 \dots dt^p | d\psi^1 \dots d\psi^{n-p}] \; \rho \; \Delta\widetilde{g} = 0 \tag{5.30}$$

$$\int\limits_{N^*[-1]C} [dt^1 \dots dt^p | d\psi^1 \dots d\psi^{n-p}] \, \rho \, \widetilde{f} = \int\limits_{N^*[-1]\widetilde{C}} [dt^1 \dots dt^p | d\psi^1 \dots d\psi^{n-p}] \, \rho \, \widetilde{f} \tag{5.31}$$

where $\Delta\widetilde{f} = 0$ and we are dealing with the pullbacks of $\widetilde{f}, \widetilde{g} \in C^\infty(T^*[-1]M)$ to $N^*[-1]C$. We can interpret these statements as a version of Stokes theorem for the cotangent bundle.

5.3. Algebraic Aspects of Integration

On the graded cotangent bundle $T^*[-1]M$ there is a BV algebra structure defined on $\mathcal{C}^\infty(T^*[-1]M)$ with an odd Lie bracket defined in (5.12) and an analog of Stokes theorem introduced in section (5.2). The natural idea here is to combine the algebraic structure on $T^*[-1]M$ with the integration and understand what an integral is in this setting. On a Lie algebra $\mathfrak{g}$ we can define the space of k-chains c_k as an element of $\wedge^k \mathfrak{g}$. This space is spanned by

$$c_k = T_1 \wedge T_2 \cdots \wedge T_k \tag{5.32}$$

where $T_i \in \mathfrak{g}$ and the boundary operator can be defined as

$$\partial(T_1 \wedge T_2 \wedge ... \wedge T_k) = \sum_{1 \le i < j \le k} (-1)^{i+j+1} [T_i, T_j] \wedge T_1 \wedge ... \wedge \widehat{T}_i \wedge ... \wedge \widehat{T}_j \wedge ... \wedge T_n \tag{5.33}$$

where $\widehat{T}_i$ denotes the omission of argument T_i. The usual Jacobi identity guarantee that $\partial^2 = 0$. A dual object called k-cochain c^k also exist, it is a multilinear map $c^k : \wedge^k \mathfrak{g} \to \mathbb{R}$ such that the coboundary operator δ is defined like

$$\delta c^k(T_1 \wedge T_2 \wedge \cdots \wedge T_k) = c^k\left(\partial(T_1 \wedge T_2 \wedge \cdots \wedge T_k)\right) \tag{5.34}$$

where $\delta^2 = 0$. This gives rise to what is usually called Chevalley-Eilenberg complex. If $\delta c^k = 0$ we call c^k a cocycle. If there exist a b^{k-1} such that $c^k = \delta b^{k-1}$ then we call c^k a coboundary. In this way we can define a Lie algebra cohomology $H^k(\mathfrak{g}, \mathbb{R})$ which consists of cocycles modulo coboundaries. We are interested in the generalization of Chevalley-Eilenberg complex for the graded Lie algebras. Let's introduce $W = V[1]$, the graded vector space with a Lie bracket of degree 1. The k-cochain is defined as a multilinear map $c^k(w_1, w_2, \dots w_k)$ with the property

$$c^k(w_1, \dots, w_i, w_{i+1}, \dots, w_k) = (-1)^{|w_i||w_{i+1}|} c^k(w_1, \dots, w_{i+1}, w_i, \dots, w_k) \tag{5.35}$$

The coboundary operator δ is acting as follows

$$\delta c^k(w_1, \dots, w_{k+1}) = \sum (-1)^{s_{ij}} c^k\left((-1)^{|w_i|+1}[w_i, w_j], w_1, \dots, \widehat{w}_i, ..., \widehat{w}_j, ..., w_{k+1}\right) \tag{5.36}$$

where s_{ij} is defined as

$$s_{ij} = |w_i|(|w_1| + \cdots + |w_{i-1}|) + |w_j|(|w_1| + \cdots + |w_{j-1}|) + |w_i||w_j| \tag{5.37}$$

The sign factor s_{ij} is called the Koszul sign; it appear when we move w_i, w_j at the beginning of the right hand side of equation (5.36). The cocycles, coboundaries and cohomology are defined as before. Now we introduce an important consequence of the Stokes theorem for the multivector fields (5.30) and (5.31).

THEOREM 5.3.1. *Consider a collection of functions $f_1, f_2 \dots f_k \in \mathcal{C}^\infty(T^*[-1]M)$ such that $\Delta f_i = 0$ for each i. Define the integral*

$$c^k(f_1, f_2, \dots f_k; C) = \int\limits_{N^*[-1]C} [dt^1 \dots dt^p | d\psi^1 \dots d\psi^{n-p}]\ \rho\ f_1(t, \psi) \dots f_k(t, \psi) \tag{5.38}$$

where C is a closed submanifold of M. Then $c^k(f_1, f_2, \dots f_k)$ is a cocycle i.e.

$$\delta c^k(f_1, f_2, \dots f_k) = 0 \tag{5.39}$$

Additionally $c^k(f_1, f_2, \dots f_k; C)$ differs from $c^k(f_1, f_2, \dots f_k; \widetilde{C})$ by a coboundary if C is homologous to $\widetilde{C}$, i.e.

$$c^k(f_1, f_2, \dots f_k; C) - c^k(f_1, f_2, \dots f_k; \widetilde{C}) = \delta b^{k-1} \tag{5.40}$$

where b^{k-1} is some $(k-1)$-cochain.

This theorem is based on the observation by A. Schwarz in [40] and the proof given here can be found in [15].

PROOF. Equation (5.38) defines properly a k-cochain for odd Lie algebra in fact

$$c^k(f_1,\dots,f_i,f_{i+1},\dots,f_k;C)=(-1)^{|f_i||f_{i+1}|}c^k(f_1,\dots,f_{i+1},f_i,\dots,f_k;C) \tag{5.41}$$

this follows from the graded commutativity of $\mathcal{C}^\infty(T^*[-1]M)$. Equation (5.30) implies that

$$0=\int\limits_{N^*[-1]C}[dt^1\dots dt^p|d\psi^1\dots d\psi^{n-p}]\ \rho\ \Delta(f_1(t,\psi)\dots f_k(t,\psi)) \tag{5.42}$$

Iterating the Δ operator property (5.11), we obtain the following formula

$$\begin{aligned}\Delta(f_1f_2...f_k)&=\sum_{i<j}(-1)^{s_{ij}}(-1)^{|f_i|}\{f_i,f_j\}f_1\dots\widehat{f_i}\dots\widehat{f_j}\dots f_k\\ s_{ij}&=(-1)^{(|f_1|+\dots+|f_{i-1}|)|f_i|+(|f_1|+\dots+|f_{j-1}|)|f_j|+|f_i||f_j|}\end{aligned} \tag{5.43}$$

where we used $\Delta f_i=0$. Combining (5.42) and (5.43) we discover that c^k defined in (5.38) is a cocycle

$$\begin{aligned}&\delta c^k(f_1,...,f_{k+1};C)\\ &\quad=\int\limits_{N^*[-1]C}[dt^1\dots dt^p|d\psi^1\dots d\psi^{n-p}]\ \rho\ \Delta(f_1(t,\psi)\dots f_k(t,\psi))=0\end{aligned} \tag{5.44}$$

where we have adopted the definition for the coboundary operator (5.36). Next we have to exhibit that the cocycle (5.38) changes by a coboundary when C is deformed continuously. Consider an infinitesimal transformation of C parametrized by

$$\delta_C t^\alpha=\varepsilon^\alpha(t^i)\qquad\qquad \delta_C\psi_i=-\partial_i\varepsilon^\alpha(t^i)\psi_\alpha \tag{5.45}$$

where the index convention is the same of section (5.2). In this way a function $f\in C^\infty(T^*[-1]M)$ changes as

$$\begin{aligned}\delta_C f(t,\psi)\big|_{N^*[-1]C}&=\varepsilon^\alpha\partial_\alpha f-\partial_i\varepsilon^\alpha(t^i)\psi_\alpha\partial_{\psi_j}f\big|_{N^*[-1]C}\\ &=-\{\varepsilon^\alpha(t^i)\psi_\alpha,f\}\big|_{N^*[-1]C}\end{aligned} \tag{5.46}$$

Using Δ we can rewrite equation (5.46) as

$$\delta_C f(x,\psi)\big|_{N^*[-1]C} = \Delta(\varepsilon^\alpha(x^i)\psi_\alpha f) + \varepsilon^\alpha(x^i)\psi_\alpha \Delta(f)\big|_{N^*[-1]C} \tag{5.47}$$

The first term vanishes under the integral. If we look at the infinitesimal deformation of $f_1 \cdots f_k$, we have

$$\delta_C\, c^k(f_1,\ldots,f_k;C) = \delta b^{k-1}(f_1,\ldots,f_k) \tag{5.48}$$

where

$$b^{k-1}(f_1,\cdots f_{k-1};C) = \int\limits_{N^*[-1]C} [dt^1 \ldots dt^p | d\psi^1 \ldots d\psi^{n-p}]\, \rho\, \varepsilon^\alpha(x^i)\psi_\alpha\, f_1 \cdots f_{k-1} \tag{5.49}$$

Under an infinitesimal change of C, c^k changes by a coboundary. If we look now at finite deformations of C, we can parameterize the deformation as a one-parameter family $C(t)$. Thus, for every t, we have the identity

$$\frac{d}{dt} c^k(f_1,\ldots,f_k;C(t)) = \delta b^{k-1}(f_1,\ldots,f_k;C(t)) \tag{5.50}$$

integrating both sides we get the formula for the finite change of C

$$c^k(f_1,\ldots,f_k;C(1)) - c^k(f_1,\ldots,f_k;C(0)) = \delta \int\limits_0^1 dt\; b^{k-1}_{C(t)} \tag{5.51}$$

This concludes the proof of Theorem 5.3.1. □

At this point we can perform the integral (5.38) in an explicit way. We assume that the functions f_i are of fixed degree and we will use the same notation for the corresponding multivector $f_i \in \Gamma(\wedge^\bullet TM)$. If we pull back the functions, the odd integration in (5.38) gives

$$c^k(f_1,\ldots,f_k;C) = \int_C i_{f_1} i_{f_2} \cdots_{f_k} \mathrm{vol} \tag{5.52}$$

where i_f is the usual contraction of a differential form with a multivector. Note that the volume form in (5.52) it is originated by the product of the density ρ with the total antisymmetric tensor ε coming from the ψ-term ordering in the integral. In our computation we also assumed that all vector fields are divergenceless. Only if $n-p = |f_1| + \cdots + |f_k|$ the integral gives rise to cocycle on $\Gamma(\wedge^\bullet TM)$ otherwise the integral is identically zero due to the property of Berezin integration. In our combination of algebraic and integration aspects on $T^*[-1]M$ we saw that BV integral produces cocycle with a specific dependence on C. This result can be used also as a definition for those kind of integrals.

5.4. Geometry of BV Quantization

The physical motivation for the introduction of BV formalism is to make possible the quantization of field theories that are difficult to quantize by means of the Fadeev-Popov method. In fact in the last years there has been the emergence of many gauge-theoretical models that exhibit the so called open gauge algebra. These models are characterized by the fact that the gauge transformation only close on-shell which means that if we compute the commutator of two infinitesimal gauge transformation we will find a transformation of the same type only modulo the equation of motion. Models with an open gauge algebra include supergravity theories, the Green-Schwarz superstring and the superparticle, among others. This formalism firstly appeared in the papers of Batalin and Vilkovisky [6, 7] while a clear geometric interpretation was given by Schwarz in [11, 14]. A short but nice description of BV formalism can also be found in [13]. Here we will try to resume some aspects of the Schwarz approach in a brief way. Let's review some facts about symplectic geometry that will be useful in the sequel.

Definition 5.4.1. *Let w be a 2-form on a manifold M, for each $p \in M$ the map $w_p : T_pM \times T_pM \to \mathbb{R}$ is skew-symmetric bilinear on the tangent space to M at p. The 2-form w is said symplectic if w is closed and w_p is symplectic for all $p \in M$ i.e. it is nondegenerate , in other words if we define the subspace $U = \{u \in T_pM \mid w_p(u, v) = 0, \forall\, v \in T_pM\}$ then $U = \{0\}$.*

The skew-symmetric condition restrict M to be even dimensional otherwise w would not be invertible.

Definition 5.4.2. *A symplectic manifold is a pair (M, w) where M is a manifold and w is a symplectic form. If $\dim M = 2n$ we will say that M is an $(n|n)$-dimensional manifold.*

The most important example of symplectic manifold is a cotangent bundle $M = T^*Q$. This is the traditional phase space of classical mechanics, Q being known as the configuration space in that context.

Example 5.4.3. *A cotangent bundle T^*Q has a canonical symplectic 2-form w which is globally exact*

$$w = d\theta \tag{5.53}$$

and hence closed. Any local coordinate system $\{q^k\}$ on Q can be extended to a coordinate system $\{q^k, p_k\}$ on T^*Q such that θ and w are locally given by

$$\theta = p_k dq^k \qquad\qquad w = dq^k \wedge dp_k \tag{5.54}$$

On any symplectic manifold it's always possible to choose a local coordinate system such that w takes the form (5.54). This property is known as Darboux's theorem. Using a general local coordinate system $(z^1, \dots, z^{2n})$ on M we can write the Poisson bracket as

$$\{F, G\} = \frac{\partial F}{\partial z^i} w^{ij}(z) \frac{\partial G}{\partial z^j} \tag{5.55}$$

where $w^{ij}(z)$ is an invertible matrix and its inverse $w_{ij}(z)$ determines

$$w = dz^i w_{ij} dz^j \tag{5.56}$$

which is exactly the symplectic form.

Definition 5.4.4. *A submanifold $L \subset M$ is called isotropic if*

$$t^i w_{ij}(x) \widetilde{t}^j = 0 \tag{5.57}$$

for every pair of tangent vectors $t, \widetilde{t} \in T_x L$.

Definition 5.4.5. *Assuming that* $\dim L = (k|k)$ *we define a Lagrangian manifold as an isotropic manifold of dimension $(k|n-k)$ where $0 \leq k \leq n$.*

Now we will drag this concepts in the supergeometry framework.

Definition 5.4.6. *Let's consider an $(n|n)$-dimensional supermanifold M equipped with an odd symplectic form w. We will refer to M as a P-manifold.*

If the P-manifold M is also equipped with a volume, by means of density ρ, we can define an odd second order differential operator Δ which is related to the divergence of a vector field on M. If the operator Δ satisfy $\Delta^2 = 0$ we will refer to M as an SP-manifold.

Example 5.4.7. *Following the spirit of this approach let's consider the odd Laplacian (5.9), basically the SP-manifold just described is exactly the same of what we called a BV manifold in section (5.1).*

Example 5.4.8. *If L is a Lagrangian submanifold of an SP-manifold it is singled out by the equation*

$$t^{k+1} = \dots = t^n = 0 \qquad\qquad \psi_1 = \dots = \psi_k = 0 \tag{5.58}$$

see the analogy with equation (5.19). The odd conormal bundle $N^*[-1]C$ described in section (5.2) is a Lagrangian submanifold.

Let's consider a function f defined on a compact SP-manifold M satisfying $\Delta f = 0$. The following expression

$$\int_L f d\sigma \tag{5.59}$$

where L is a Lagrangian submanifold of M and $d\sigma$ the volume element upon it, does not change by continuous variations of L; moreover, L can be replaced by any other Lagrangian submanifold $\widetilde{L}$ which is in the same homology class. In the case when $f = \Delta g$ the integral (5.59) vanishes. Sometimes this result is called Schwarz theorem and it's exactly the geometric counterpart of what we described in (5.2) and (5.3).

5.4.1. The BV Gauge Fixing. The problem of quantization in quantum field theory is to make sense of certain path integrals of the form

$$Z = \int_{\mathcal{M}} e^{-S/\hbar} \tag{5.60}$$

where $\mathcal{M}$ is the manifolds where the fields are evaluated and there exist a Lie group $\mathcal{G}$, which is the gauge symmetry group, acting on $\mathcal{M}$. S is the action of the gauge theory considered and it can be seen as a function on $S \in \mathrm{Fun}(\mathcal{M}/\mathcal{G})$. Gauge symmetry of S implies that the Hessian of action in any stationary point is degenerate and the perturbative expansion is not well-defined. The quantization in BV formalism is done by computing the partition function Z of (5.60) on a Lagrangian submanifold L instead of $\mathcal{M}$

$$Z = \int_{L} e^{-S/\hbar} \tag{5.61}$$

What we usually call gauge fixing here is just the choice of a specific Lagrangian submanifold, the gauge invariance of the theory is guaranteed by the invariance of the partition function Z under the change of gauge fixing condition as stated by the Schwarz theorem. Thus, the partition function Z has to be invariant under deformations of the Lagrangian submanifold L in the space of fields. For any function f, the integral $\int_L f$ is invariant under such deformations if $\Delta f = 0$. Taking $f = e^{-S/\hbar}$, we obtain the BV quantum master equation $\Delta e^{-S/\hbar} = 0$ that leads to

$$-\hbar\Delta S + \frac{1}{2}\{S, S\} = 0 \tag{5.62}$$

Eventually we can resume in its entirely the BV quantization procedure: the starting point is considering a classical action functional and constructing a solution to (5.62); we will call a function f, defined on an SP-manifold M, a quantum observables if it obeys to

$$-\hbar\Delta f + \frac{1}{2}\{f, S\} = 0 \tag{5.63}$$

and the expression

$$\langle f \rangle = \int_L f e^{-S/\hbar} \tag{5.64}$$

has the meaning of the expectation value of f and it depends only on the homology class of L. Essentially we obtain all the expectation values for the physical observables in terms of deformation of the action. The choice of a Lagrangian submanifold corresponds to a gauge fixing for the theory. Certainly in the quantization procedure we should consider ill-defined infinite-dimensional integrals however all the statements about integral (5.59) are proved rigorously only in the finite-dimensional case. In addition it is very difficult to define the notion of infinite-dimensional SP-manifold and to construct the operator Δ. Nevertheless we can use the framework of perturbation theory to quantize gauge theories using the BV formalism.

CHAPTER 6

The Mathai-Quillen Formalism

The Mathai-Quillen formalism, introduced in [17], provides a particular representative of the Thom class, using differential forms on the total space of a vector bundle. Characteristic classes are essential in the study of global properties of a vector bundle, for this reason the explicit construction given by Mathai and Quillen by means of Berezin integration, it's a higly important mathematical discovery.
However, in this section, we would like to stress the physical relevance of this formalism which is closely related to topological quantum field theories.

6.1. General Remarks on Topological Quantum Field Theories

Let's consider a quantum field theory defined over a manifold X equipped with a Riemannian metric $g_{\mu\nu}$. In general the partition function and correlation functions of this theory will depend on the background metric. We will say that a quantum field theory is topological if there exists a set of operators in the theory, known as topological observables, such that their correlation functions do not depend on the metric. If we denote these operators by $\mathcal{O}_i$ then

$$\frac{\delta}{\delta g_{\mu\nu}}\langle \mathcal{O}_{i_1} \dots \mathcal{O}_{i_n} \rangle = 0 \tag{6.1}$$

Topological quantum field theories (TQFT), largely introduced by E. Witten, may be grouped into two classes. The *Schwarz type* [20] theories have the action and the observables which are metric independent. This guarantees topological invariance as a classical symmetry of the theory and hence the quantum theory is expected to be topological. The most important example of a TQFT of Schwarz type is 3D Chern-Simons gauge theory. The *Witten type* [25] theories (also called cohomological TQFTs) have the action and the observables that may depend on the metric, but the theory has an underlying scalar symmetry carried by an odd nilpotent operator Q acting on the fields in such a way that the correlation functions of the theory do not depend on the background metric. In a cohomological theory physical

observables are Q-cohomology classes. The path integrals of a cohomological topological field theory are integral representations of Thom classes of vector bundles in infinite dimensional spaces. This was first pointed out in an important paper of Atiyah and Jeffrey [22] where they generalized the Mathai-Quillen formalism to the infinite dimensional case. Adopting this point of view gives some advantages. First of all, it provides a proof that finite dimensional topological invariants can be represented by functional integrals, the hallmark of topological field theories. Moreover, it offers some insight into the mechanism of localization of path integrals in supersymmetric quantum field theory; a very pervasive technique in modern theoretical physics.
In his classic works [18, 19] Witten showed that by changing the coupling to gravity of the fields in an $\mathcal{N} = 2$ supersymmetric theory in two or four dimensions, a TQFT theory of cohomological type was obtained. This redefinition of the theory is called twisting. We suggest to the reader the reference [29] which works out in detail all this procedure. For a brief review look at [28]. The interesting idea in supersymmetric quantum field theory is that fermionic degrees of freedom cancel bosonic degrees of freedom to such extent that the infinite dimensional path integral of the QFT reduces to a finite dimensional integral over certain geometrical spaces called moduli spaces. Now, from another point of view we know [25] that all the topologically twisted QFTs fit in the following paradigm

(1) Fields: Represented by ϕ^i. These might be, for example, connections on a principal bundle.
(2) Equations: We are interested in some equations on the fields $s(\phi^i)$, where s is a generic section. Usually these are partial differential equations.
(3) Symmetries: Typically the equations have a gauge symmetry.

The main statement, as before, is that the path integral localizes to

$$\mathcal{M} = \mathcal{Z}(s)/\mathcal{G} \tag{6.2}$$

where $\mathcal{M}$ is what we called moduli space, $\mathcal{Z}(s) = \{\phi : s(\phi) = 0\}$ and $\mathcal{G}$ is the group of symmetry. These kind of spaces are all of the form (6.2) and they share three properties. They are finite dimensional, generically noncompact and generically singular. The last two properties pose technical problems that we will not discuss.

Example 6.1.1. *As an example of what we discussed let's consider the Yang-Mills gauge theory with the following basic data*

(1) *A closed, oriented , Riemannian 4-manifold* $(X, g_{\mu\nu})$.

(2) *A principal bundle* $P \to X$ *for a compact Lie group* G *with Lie algebra* $\mathfrak{g}$.
(3) *An action of the form*

$$S = \int_X \mathrm{tr}(F \wedge *F) \tag{6.3}$$

Here our fields ϕ^i *will be* $A \in \mathcal{A} = \mathrm{Conn}(P)$. *In* $D = 4$, $* : \Omega^2(X) \to \Omega^2(X)$ *and* $*^2 = 1|_{\Omega^2(X)}$, *so that we may define the eigenspaces* $\Omega^{2,+}(X)$ *and* $\Omega^{2,-}(X)$ *with eigenvalues under* $*$ *of* $+1$ *and* 1, *respectively:*

$$\Omega^2(X) = \Omega^{2,-}(X) \oplus \Omega^{2,+}(X) \tag{6.4}$$

Our bundle of equations will be

$$\mathcal{E} = \mathcal{A} \times \Omega^{2,+}(X, \mathrm{ad}\, \mathfrak{g}) \tag{6.5}$$

where $\Omega^{2,+}(X, \mathrm{ad}\, \mathfrak{g})$ *is the space of self-dual two forms with values in the adjoint of the Lie algebra* $\mathfrak{g}$. *Then our section will be*

$$s(A) = F_A^+ = F + *F \tag{6.6}$$

Notice that $F_A^+ = 0$ *is one of the possible extremal configurations of (6.3). Eventually our group of symmetry will be*

$$\mathcal{G} = \mathrm{Aut}(P) \sim \mathrm{Map}(X, G) \tag{6.7}$$

In this particular example the path integral of the theory localizes to the moduli space of istantons $\mathcal{M}_{SD}$ *defined as*

$$\mathcal{M}_{SD} = \{A \in \mathcal{A} \,|\, F_A^+ = 0\}/\mathcal{G} \tag{6.8}$$

The lesson that we learn here is that certain versions of QFTs can be used to study the geometry of certain geometrical spaces using path integrals, from a mathematical point of view we are studying intersection theory in moduli spaces applying the language of physics. Let's see how mathematicians looks at path integrals in this context. Here, of course, we just considered an infinite dimensional situation, we had an infinite dimensional space $\mathcal{A}$, an infinite dimensional bundle $\mathcal{E}$ over it and a quotient over an infinite dimensional group $\mathcal{G}$. Anyway, just to understand the concepts, we will forget for the moment about the infinite dimensional case to avoid further technicalities.

6.2. Euler Class

Definition 6.2.1. *Consider a real vector bundle* $\pi : E \to X$ *over a manifold* X. *We will assume that* E *and* X *are orientable,* X *is closed and the rank (fiber dimension) of* E *satisfies* $\mathrm{rk}(E) = 2m \leq \dim(X) = n$. *The Euler class of* E *is an integral cohomology class* $e(E) \in H^{2m}(X)$.

For $m = 1$ the Euler class can be defined easily, see [16], but for higher rank bundle a similar construction, although possible in principle, becomes unwieldy. For an extensive discussion of the Euler class of a vector bundle we suggest [35]. In spite of this circumstance there are other three ways to compute the Euler class of a vector bundle. The first of these is done by counting the zeros of a certain section of the bundle, this is known as Hopf theorem. The second makes use of the theory of characteristic classes producing an explicit representative $e_\nabla(E)$ of $e(E)$ which depends on the curvature Ω_∇ of a connection ∇ on E. This explicit representative have the following form

$$e_\nabla(E) = \mathrm{Pf}\left(\frac{\Omega_\nabla}{2\pi}\right) \tag{6.9}$$

Finally, the third way is in terms of the Thom class of E which we will describe in the following section. When $\mathrm{rk}(E) = \dim(X)$, e.g if $E = TX$, then $H^{2m}(X) = H^n(X)$ and we can consider, instead of $e(E)$, its evaluation on the fundamental class $[X]$

$$\chi(E) = e(E)[X] \tag{6.10}$$

$\chi(E)$ is called Euler number and we can think to the pairing with fundamental class $[X]$ as the concept of integrating over the manifold X. Let's obtain the Euler number in terms of the first two descriptions of $e(E)$ that we had given before. Looking to the Hopf theorem from this perspective we can obtain the Euler number as the signed sum over the zeros of a generic section s of E

$$\chi(E) = \sum_{x_k\,:\,s(x_k)=0} \deg_s(x_k) \tag{6.11}$$

Moreover we can obtain it also from the integral

$$\chi(E) = \int_X e_\nabla(E) \tag{6.12}$$

If we match the formulae (6.11) and (6.12) we can realize that the Hopf theorem is basically an example of localization, by localization we mean that the integral of a differential form of top degree on the manifold X, in this case $e_\nabla(E)$, it's converted to a sum of a discrete set of points as in (6.11).

Example 6.2.2. *In Yang-Mills Example 6.1.1 we took the space of solutions of the self-dual equation $\mathcal{M}_{SD}$, which is a quotient of a subset of the space of all connections $\mathcal{A}$, and the path integral over the space of all connections localizes just to the path integral over the space of self-dual connections.*

To prove that (6.11) and (6.12) are basically the same thing we need the Mathai-Quillen representative of the Thom class which is roughly speaking a generalization of (6.9) that depends also on a generic section s of E. Thus, making the appropriate choice of a section, it's possible to show that the Mathai-Quillen representative gives a formula for $\chi(E)$ that smoothly interpolates between (6.11) and (6.12).

6.3. Thom Class

Given a vector bundle $\pi : E \to X$ we can define the cohomology class of the total space $H^\bullet(E)$ and the cohomology class of the base manifold $H^\bullet(X)$.The Thom class of a vector bundle allows us to relate this cohomology classes. Typically we are used to consider the standard de Rham cohomology $H^\bullet(E)$ but in this situation is better to adopt, for reasons that we will see soon, the cohomology with compact support $H_c^\bullet(E)$ which in a very natural way is defined for differential forms with compact support. In physics a more natural notion than compact support is rapid decrease: i.e., Gaussian decay at infinity. Cohomology for forms with rapid decrease is the same as cohomology with compact support [17]. From now on we will always refer to the rapid decrease cohomology group $H_{rd}^\bullet(E)$. For forms with rapid decrease there exist a push-forward map π_* called integration along the fibers. In local coordinates and for trivial bundle, this is the operation of integrating along the fibers the part of $\alpha \in \Omega_{rd}^\bullet(E)$ which contains a vertical $2m$-form (remember that $\mathrm{rk}(E) = 2m$) and considering the outcome as a differential form on X. In this way we have a globally well defined operation

$$\pi_* : \Omega_{rd}^\bullet(E) \to \Omega^{\bullet-2m}(X) \tag{6.13}$$

From this result, we can understand that there is a Poincaré lemma also for the rapid decrease cohomology

$$\pi_* : H_{rd}^\bullet(E) \simeq H^{\bullet-2m}(M) \tag{6.14}$$

This correspondence leads to the introduction of the so called Thom isomorphism, which is the inverse of π_*

$$\mathcal{T} : H^\bullet(X) \longrightarrow H_{rd}^{\bullet+2m}(E) \tag{6.15}$$

Definition 6.3.1. *The image of* $1 \in H^0(X)$ *under the Thom isomorphism determines a cohomology class* $\Phi(E) \in H_{rd}^{2m}(E)$, *called the Thom class of the oriented vector bundle* E.

Clearly $\pi_*\Phi(E) = 1$ and if we take a differential form $\alpha \in H^\bullet(X)$ the Thom isomorphism is explicitly realized as

$$\mathcal{T}(\alpha) = \pi^*(\alpha) \wedge \Phi(E) \tag{6.16}$$

The Thom class has two key properties that allow us to realize the importance of the discovery of Mathai and Quillen. Let $s : X \to E$ be any section of E, then $s^*\Phi(E)$ is a closed form and its cohomology class coincides with the Euler class $e(E)$

$$s^*\Phi(E) = e(E) \tag{6.17}$$

This perspective on the Euler class is really intriguing: provided that we can find an explicit differential form representative $\Phi_\nabla(E)$ of $\Phi(E)$, depending on a connection ∇ on E, we can use a section s to pull it back to X obtaining

$$e_{s,\nabla}(E) = s^*\Phi_\nabla(E) \tag{6.18}$$

which is an explicit representation of the Euler class $e(E)$. The second property is called localization principle and it is deeply related to what we said about the relationship between cohomology classes of the bundle and of the base space. Let $\alpha \in \Omega^\bullet(E)$ be a differential form on the

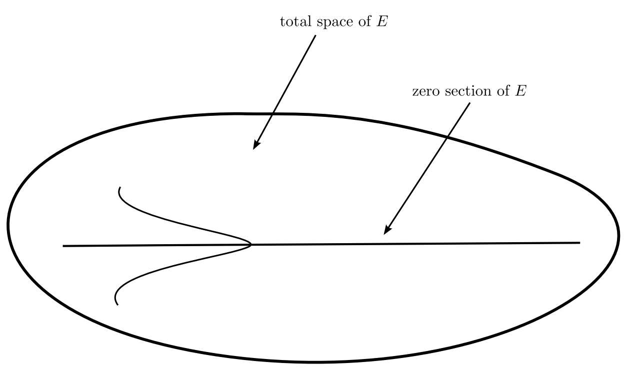

FIGURE 6.1. Mathai-Quillen construction of a Thom form

total space then

$$\int_E \alpha \wedge \Phi(E) = \int_X \alpha|_X \tag{6.19}$$

where $\alpha|_X$ is the restriction of α to the base manifold X. This result is a direct consequence of integration along the fibers which is a natural concept for differential forms with rapid decrease. Mathai and Quillen interpreted the Thom class of a vector bundle as a gaussian shaped differential form which has indices only in the vertical direction along the fiber. Basically they constructed a volume form of Gaussian-like shape along the fiber satisfying (6.19), this differential form representative is exactly what we needed to pull back, via section s, to prove (6.18).
We can also write the localization principle from the point of view of the base manifold. To do that let's pick a submanifold $i : S \to X$ and

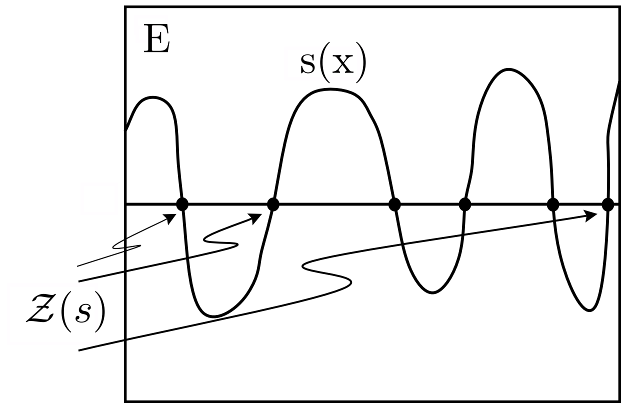

FIGURE 6.2. The zero set of a generic section

denote by $\mathcal{Z}(s)$ the zero locus of a section s of E as in fig. 6.2. If s is a generic section then

$$\int_{\mathcal{Z}(s)} i^*\mathcal{O} = \int_X s^*(\Phi(E)) \wedge \mathcal{O} \tag{6.20}$$

where $s^*(\Phi(E)) \wedge \mathcal{O} \in \Omega^\bullet(X)$. In the application to topological field theory we interpret s as $s(\phi) = D\phi$ where ϕ is a field in the space of all fields $\mathcal{C}$ and D is some differential operator. Then (6.20) is the key property which allows us to localize the integral to the subspace $D\phi = 0$. Before proceeding further, let's return for a while on our final statement from section (6.2). Let's assume that we have our representative of the Thom class satisfying (6.18) and that we choose the zero section $s_0 : X \to E$. Then, for a reason that will be clear later on, the pullback of $\Phi_\nabla(E)$ via the zero section will be

$$s_0^*\Phi_\nabla(E) = e_\nabla(E) = \mathrm{Pf}\Big(\frac{\Omega_\nabla}{2\pi}\Big) \tag{6.21}$$

Now let's assume that s is a generic section, then we can multiply it by a parameter $t \in \mathbb{R}$ and analyze the pullback $(ts)^*\Phi_\nabla$. The limit $t \to \infty$ is represented by a section which goes very large away from the zeros (fig. 6.3). When we pullback using this section the Thom form is almost zero. In fact if we consider a region where the section is large this happens because we have been taking a representative which decays really fast at infinity. When we pullback to the base manifold we get a zero contributions to the integral from the regions where the sections aren't zero. In the neighborhood of the zeros of the sections only critical points contributes to the integral. This argument clarify

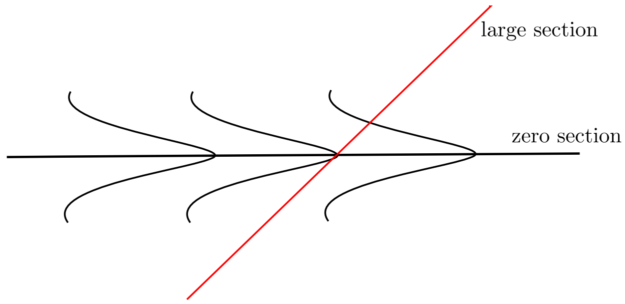

FIGURE 6.3. An arbitrary section which goes large away from the zeros

what we said before and it can be considered as an abstract proof of the Hopf theorem. A final important remark is that for any section s the pullback $s^*\Phi(E)$ is independent from the choice of s. Any two sections of E are homotopic as maps from X to E, and homotopic maps induce the same pullback map in cohomology. We can take any section of E to pullback the Thom form obtaining a differential form over the base manifold X, but when we integrate this form over the base we still get the same result. This property can be easily recovered from the Mathai-Quillen representative and it's intimately related with the property of ordinary Gaussian integration and Berezin integration.

6.4. Equivariant Cohomology

The Mathai-Quillen is better formulated in the context of equivariant cohomology, which we will briefly review here. The interested reader should consult [17, 21, 24, 39] for detailed expositions on the subject. Equivariant cohomology appears when we want to study a topological space X with the action of a group G. We will denote by $\mathfrak{g}$ the Lie algebra of G. Some of what follows is only rigorously true when G is compact, but the formal discussion can be applied to any group. In particular, in topological field theory, it is applied to infinite dimensional groups. If we consider our topological space X to be a G-manifold, then X has an action $x \to g \cdot x$, for all $x \in X$ and $g \in G$. The action of G is said to be free if, for any $x \in X$,

$$g \cdot x \; = \; x \iff g \; = \; 1 \tag{6.22}$$

that is, there are no nontrivial isotropy groups. If the action of G is free on X, then the quotient space X/G forms the base space of a principal G bundle

$$\begin{array}{ccc} X & \longleftarrow & G \\ \pi \big\downarrow & & \\ X/G & & \end{array} \tag{6.23}$$

where the quotient space is smooth. From here we can easily understand that if G acts freely, the equivariant cohomology of X is just

$$H^{\bullet}_{G}(X) = H^{\bullet}(X/G) \tag{6.24}$$

In many cases of interest to physics and mathematics, the group action is not free and for this reason we need to create an appropriate extension of the de Rham cohomology. Topologically, equivariant cohomology is usually defined as the ordinary cohomology of the space

$$X_G = EG \times_G X \tag{6.25}$$

where EG is called universal G-bundle. EG is a very special space which we can always associate to a group G, satisfying:

- G acts on EG without fixed points.
- EG is contractible.

There are also algebraic definitions of equivariant cohomology. These alternative definitions are the so-called Weil, BRST and Cartan models of equivariant cohomology which we will briefly recall here.

6.4.1. Weil Model. In the study of the differential geometry of a principal bundle P with Lie algebra $\mathfrak{g}$ we encounter the so-called Weil algebra $\mathcal{W}(\mathfrak{g})$. Let us recall its definition; for more details see, for example, [21]. The Weil algebra is a dg-algebra, see (4.2), with $\mathfrak{g}$-valued generators ω, ϕ of degrees 1 and 2 respectively. We define a differential operator $d_{\mathcal{W}}$ which acts on the generators as

$$d_{\mathcal{W}}\omega = \phi - \frac{1}{2}[\omega, \omega] \qquad d_{\mathcal{W}}\phi = -[\omega, \phi] \tag{6.26}$$

It may be seen that $d_{\mathcal{W}}$ is nilpotent, $d_{\mathcal{W}}^2 = 0$. These are the relations that are valid for a connection ω and curvature ϕ on a principal bundle P. Basically we can interpret ϕ and ω as algebraic counterparts of the curvature and the connection. Moreover, we can define a connection simply as a homomorphism $\mathcal{W}(\mathfrak{g}) \to \Omega^\bullet(P)$. We can also introduce two derivations of the Weil algebra: the interior derivative or contraction

$$\iota_a \omega^b = \delta_a^b \qquad \iota_a \phi^b = 0 \tag{6.27}$$

(where $\omega = \omega^a e_a, \phi = \phi^a e_a$, with e_a a basis for $\mathfrak{g}$) and the Lie derivative

$$\mathcal{L}_a = \iota_a d_{\mathcal{W}} + d_{\mathcal{W}} \iota_a \tag{6.28}$$

For the Weil model of equivariant cohomology we consider the algebra $\mathcal{W}(\mathfrak{g}) \otimes \Omega^\bullet(X)$. Algebraically, the replacement $X \to EG \times X$ is analogous to $\Omega(X) \to \mathcal{W}(\mathfrak{g}) \otimes \Omega(X)$. On this algebra we have the action of the operators ι_a and $\mathcal{L}_a$, now defined as $\iota_a = \iota_a \otimes 1 + 1 \otimes \iota_a$ *etc.* Here we write $\iota_a = \iota(V_a)$, with V_a the vector field on X corresponding to the Lie algebra element e_a. One then restricts to the so-called basic forms $\eta \in \mathcal{W}(\mathfrak{g}) \otimes \Omega^*(X)$ which satisfy $\iota_a \eta = \mathcal{L}_a \eta = 0$. Basic forms are horizontal and G-invariant differential forms. The equivariant cohomology groups are then defined as

$$H_G^\bullet(X) = H^\bullet((\mathcal{W}(\mathfrak{g}) \otimes \Omega^\bullet(X))_{basic}, d_T) \tag{6.29}$$

where

$$d_T = d_{\mathcal{W}} \otimes 1 + 1 \otimes d \tag{6.30}$$

is the differential.

6.4.2. Cartan Model. Following what we did previously, we start by defining the Cartan algebra $\mathcal{C}(\mathfrak{g})$ which is obtained by simply putting $\omega = 0$ in the Weil algebra and is generated by the single variable ϕ of degree two. The Cartan model is a simpler model of equivariant cohomology based on the Cartan algebra $\mathcal{C}(\mathfrak{g}) = S(\mathfrak{g}^*)$. The starting point is now the algebra $S(\mathfrak{g}^*) \otimes \Omega^\bullet(X)$, but as differential we choose

$$d_{\mathcal{C}} = 1 \otimes d - \phi^a \otimes \iota_a. \tag{6.31}$$

This operator satisfies $(d_{\mathcal{C}})^2 = -\phi^a \otimes \mathcal{L}_a$ and thus only defines a complex on the G-invariant forms. The Cartan model of equivariant cohomology is now defined as

$$H_G^\bullet(X) = H^\bullet((S(\mathfrak{g}^*) \otimes \Omega^\bullet(X))^G, d_{\mathcal{C}}). \tag{6.32}$$

It is possible to show that the definitions (6.29) and (6.32) are equivalent and agree with the topological definition. This last statement is best understood in the context of the so-called BRST model of equivariant cohomology and it was originally shown by Kalkman in [23, 24].

6.5. Universal Thom Class

We will now show how to construct a nice explicit representative for $\Phi(E)$ by first constructing a "universal" representative [17]. While E might be twisted and difficult to work with, we can replace constructions on E by equivariant constructions on a trivial bundle. Let E be an orientable real vector bundle such that $\mathrm{rk}(E) = 2m$, with standard fiber V. Since E is orientable the structure group G of the bundle can be reduced to $SO(V)$ and we will denote its Lie algebra by $\mathfrak{g}_s$. We can identify E as a bundle associated to a principal $SO(V)$ bundle $P \to X$, where P is the $SO(V)$ bundle of all orthonormal oriented frames on E

$$\pi : P \times V \to E = \frac{P \times V}{G} \tag{6.33}$$

Recall that when given a principal G-bundle $\pi : P \to X$, a differential form α on P descends to a form on X if the following two conditions are satisfied: first, given vector fields V_i , $\alpha(V_1, \ldots, V_q) = 0$ whenever one of the V_i is vertical. In this case α is said to be horizontal. Second, α is invariant under the G action. The forms that satisfy both conditions are called basic. We already encountered basic differential forms in (6.4.1). In particular, if we consider the principal bundle (6.33) we have an isomorphism

$$\Omega^\bullet(P \times_G V) \simeq \Omega^\bullet(P \times V)_{\mathrm{basic}} \tag{6.34}$$

Suppose now that P is endowed with a connection $A \in \Omega^1(P, \mathfrak{g}_s)$ and associated curvature $\Omega \in \Omega^2(P, \mathfrak{g}_s)$, and consider the Weil algebra, $\mathcal{W}(\mathfrak{g}_s)$. As $\mathfrak{g}_s = so(2m)$ the generators are antisymmetric matrices A_{ab} (of degree 1) and Ω_{ab} (of degree 2). The property that $\mathcal{W}(\mathfrak{g})$ provides a universal realization of the relations defining the curvature and connection on P gives the Chern-Weil homomorphism

$$w : \mathcal{W}(\mathfrak{g}) \to \Omega^\bullet(P) \tag{6.35}$$

defined in a natural way through the expansions

$$A = A^\alpha T_\alpha \qquad\qquad \Omega = \Omega^\alpha T_\alpha \tag{6.36}$$

where $\{T_\alpha\}_{\alpha=1,\dots,\dim(G)}$ is a basis of $\mathfrak{g}$, and $A^\alpha \in \Omega^1(P)$, $\Omega^\alpha \in \Omega^2(P)$. For $G = SO(2m)$, the map w is just the correspondence between the generators of $\mathcal{W}(\mathfrak{g}_s)$ and the entries of the antisymmetric matrices for the curvature and connection in P. The Chern-Weil homomorphism maps the universal connection and curvature in the Weil algebra to the actual connection and curvature in P. Combined with the lifting of forms from V to $P \times V$, we obtain another homomorphism

$$w \otimes \pi_2^* : \mathcal{W}(\mathfrak{g}) \otimes \Omega^\bullet(V) \to \Omega^\bullet(P \times V) \tag{6.37}$$

where $\pi_2 : P \times V \to V$ is the projection on the second factor. This is the geometric context of the Mathai-Quillen construction and the correspondence between the de Rham theory on P and the Weil algebra, $\mathcal{W}(\mathfrak{g}_s)$, suggests the following definition.

Definition 6.5.1. *A form $U \in \mathcal{W}(\mathfrak{g}_s) \otimes \Omega_{rd}(V)$ will be called a universal Thom form (in the Weil model) if it satisfies:*

(i) *U is basic*
(ii) *$QU = 0$, where $Q = d_{\mathcal{W}} + d$*
(iii) *$\int_V U = 1$*

The reason U is useful is that if we choose a connection ∇ on E compatible with the fiber metric, then we can obtain a representative $\Phi_\nabla(E)$ of the Thom class as follows. As we have noted, a connection on E, (equivalently, a connection on P) is the same thing as a choice of Weil homomorphism $w : \mathcal{W}(\mathfrak{g}_s) \to \Omega^\bullet(P)$. We then have a diagram:

$$\begin{array}{ccc}
\mathcal{W}(\mathfrak{g}_s) \otimes \Omega^\bullet(V) & \xrightarrow{w\otimes\pi_2^*} & \Omega^\bullet(P \times V) \\
\uparrow & & \uparrow \\
(\mathcal{W}(\mathfrak{g}_s) \otimes \Omega^\bullet(V))_{\text{basic}} & \xrightarrow[w\otimes\pi_2^*]{} & \Omega^\bullet(P \times V)_{\text{basic}} \\
& \searrow^{\overline{w}} & \uparrow \pi^* \\
& & \Omega^\bullet(E)
\end{array} \tag{6.38}$$

Applying the Weil homomorphism combined with the lifting of forms, $w \otimes \pi_2^*$, to $U \in \mathcal{W}(\mathfrak{g}_s) \otimes \Omega^\bullet(V)$ gives $(w \otimes \pi_2^*)(U) \in \Omega^\bullet(P \times V)$. This form is then a basic closed differential form which descends to a form in $\Omega^\bullet(E)$, that is,

$$\Phi_\nabla(E) = \overline{w}(U) \tag{6.39}$$

for some form $\Phi_\nabla(E) \in H^{2m}_{rd}(E)$. Using the defining properties of the Thom class of E described in (6.3), we see that properties (i), (ii) and (iii) suffice to prove that $\overline{w}(U)$ represents the Thom class of E. Finally, we have used here the Weil model of equivariant cohomology but it is also possible to construct a universal class in the Cartan or BRST model. For more details, see [21].

6.6. Mathai-Quillen Representative of the Thom Class

The universal Thom form U of Mathai and Quillen is an element in $\mathcal{W}(\mathfrak{g}_s) \otimes \Omega^\bullet(V)$ given by

$$U = (2\pi)^{-m}\mathrm{Pf}(\Omega)\exp\left(-\frac{1}{2}\xi^2 - \frac{1}{2}\nabla\xi^a(\Omega^{-1})_{ab}\nabla\xi^b \right) \tag{6.40}$$

In this expression the ξ^a are fiber coordinates on V, $\nabla\xi^a$ is the exterior covariant derivative of ξ^a such that $\nabla\xi^a = d\xi^a + A^a_b\xi^b$ and Ω_{ab}, A_{ab} are the antisymmetric matrices of generators in $\mathcal{W}(\mathfrak{g}_s)$. As we can see, U is $SO(V)$ invariant, then a necessary condition for (i) is that U be horizontal. This is achieved by inserting the covariant derivative, in fact $\nabla\xi$ is horizontal

$$\iota(V_a)\nabla\xi^a = 0 \tag{6.41}$$

so U is horizontal. Thus, U is basic, checking property (i). In the universal Thom form the Ω^{-1} is slightly formal, but makes perfectly sense if we expand the exponential and we combine with the Pfaffian. In this way we get a volume form on the fiber which is what we need to integrate. As stated in [17] the curvature matrix is never invertible in a geometric situation. Nevertheless, by introducing the Weil algebra and equivariant forms, we can obtain a universal algebraic situation where the curvature matrix can be assumed invertible. To prove the remaing properties it's convenient to take into account another representation of the universal Thom form based on Berezin integration. The Pfaffian of a real antisymmetric matrix K^{ab} can be written as

$$\mathrm{Pf}(K) = \int d\chi\,\exp\{\chi_a K^{ab}\chi_b/2\} \tag{6.42}$$

where χ^a is a real odd variable. See section (2.5). It is easy to write the universal Thom form as

$$U = (2\pi)^{-m}\int d\chi\,\exp\{-\xi^2/2 + \chi_a\Omega^{ab}\chi_b/2 + i\nabla\xi^a\chi_a\} \tag{6.43}$$

and the expansion of this expression leads precisely to (6.40). From this representation it is obvious that $\int_V U = 1$. The reason is that to get a top form on V we have to take the term in the exponential with the top degree of $d\xi$. This also pulls off the top form in χ. Then, since the χ and $d\xi^a$ anticommute

$$\begin{aligned}\int_V U &= \frac{1}{(2\pi)^m} \int_V \int d\chi\ e^{-\xi^2/2} \frac{i^{2m}}{(2m)!} \big(d\xi^a \chi_a\big)^{2m} \\ &= \frac{1}{(2\pi)^m} \int_V d\xi^1 \wedge \cdots \wedge d\xi^{2m}\ e^{-\xi^2/2} = 1\end{aligned} \tag{6.44}$$

It remains to show that U is closed. In order to write a manifestly closed expression for U we enlarge the equivariant cohomology complex to

$$\mathcal{W}(\mathfrak{g}_s) \otimes \Omega^\bullet(V) \otimes \Omega^\bullet(\Pi V^*) \tag{6.45}$$

and consider the following differential

$$Q_{\mathcal{W}} = d_{\mathcal{W}} \otimes 1 \otimes 1 + 1 \otimes d \otimes 1 + 1 \otimes 1 \otimes \delta \tag{6.46}$$

$d_{\mathcal{W}}$ is the Weil differential defined in (6.4.1), while δ is the de Rham differential in ΠV^*. To do that we first introduce an auxiliary bosonic field B_a, which has the meaning of a basis of differential forms for the fiber. In this way we have a pair of fields (χ_a, B_a) associated to the fiber V. The action of δ on this fields is explicitly

$$\delta \begin{pmatrix} \chi_a \\ B_a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \chi_a \\ B_a \end{pmatrix} \tag{6.47}$$

Notice that $Q_{\mathcal{W}}^2 = 0$. Expanding the action and doing the Gaussian integral on B leads to the representation

$$U = \frac{1}{(2\pi)^{2m}} \int d\chi dB e^{-Q_{\mathcal{W}}(\Psi)} \tag{6.48}$$

where the so called "gauge fermion" is given by

$$\Psi = \chi_a(i\xi^a + \frac{1}{4}A^{ab}\chi_b + \frac{1}{2}B^a) \tag{6.49}$$

Acting with δ on Ψ we get

$$\delta\Psi = B_a \frac{\partial \Psi}{\partial \chi_a} \tag{6.50}$$

then

$$Q_{\mathcal{W}}\Psi = (d_{\mathcal{W}} + d)\Psi + B_a \frac{\partial \Psi}{\partial \chi_a} \tag{6.51}$$

We can get rid of the δ-action on Ψ because it produces a term which is a total derivative with respect to χ and this term integrates to zero because of Berezin integral properties. The advantage of this representation is that

$$\int Q_{\mathcal{W}}\Big(\cdots\Big) = (d + d_{\mathcal{W}})\int\Big(\cdots\Big) \tag{6.52}$$

Since the integrand is $Q_{\mathcal{W}}$-closed, it immediately follows from (6.52) that U is Q-closed in $\mathcal{W}(\mathfrak{g}_s) \otimes \Omega^\bullet(V)$. Thus, we have finally proven that U satisfies criteria (i), (ii) and (iii) of section (6.5) and hence U is a universal Thom form.

Example 6.6.1. *Let's consider the Thom class of a trivial vector bundle $X \times V \to X$. In this simple case, the Thom class is just a normalized generator of $H^{2m}_{rd}(V)$. Using an inner product on V and an orientation, we get a volume form $d\xi^1 \wedge \ldots \wedge d\xi^{2m}$ and the Thom class is represented by $f\ d\xi^1 \wedge \ldots \wedge d\xi^{2m}$, where f is a function on V such that $\int_V f = 1$. We will restate this trivial result in a complicated way now, the purpose is to show that the choice of gauge fermion (6.49) it is not arbitrary but it has a precise geometrical meaning. Here, the cohomology complex will be*

$$\Omega^\bullet(X) \otimes \Omega^\bullet(V) \otimes \Omega^\bullet(\mathit{\Pi} V^*) \tag{6.53}$$

and the differential

$$Q = d_X \otimes 1 \otimes 1 + 1 \otimes d_V \otimes 1 + 1 \otimes 1 \otimes \delta \tag{6.54}$$

where d_X and d_V are, respectively, the ordinary de Rham differential on the base manifold X and on the fiber V while δ is defined as before. Then, the Thom class representative can be written as

$$\Phi(E) = \frac{1}{(2\pi)^{2m}}\int d\chi dB e^{-Q(\Psi)} \tag{6.55}$$

where

$$\Psi = i\chi_a\xi^a + \frac{1}{2}\chi_a B^a \tag{6.56}$$

indeed, if we compute both the Gaussian and the Berezin integral we find the expected result.

Now we would like to generalize the Example 6.6.1 to the case where the trivial bundle $X \times V \to X$ is replaced by an oriented vector bundle $E \to X$. To do this we must give E a connection ∇ and we will denote the local one form by A^{ab}. In this way to covariantize the gauge fermion (6.56) we must add a third term to it

$$\Psi = i\chi_a\xi^a + \frac{1}{2}\chi_a B^a + \frac{1}{2}\chi_a A^{ab}\chi_b \tag{6.57}$$

which is precisely the same expression for (6.49). In this way we see that the representation of (6.56) it is simply related to (6.49) by the shift: $\overline{B}_a = B_a - A_a^b \chi_b$.

6.6.1. Pullback of Thom Class. Henceforth, we will not consider anymore the universal Thom class U instead we will work with its image under the $\overline{w}$ map defined in (6.38)

$$\Phi(E) = \overline{w}(U) \in H_{rd}^{2m}(E) \tag{6.58}$$

here we dropped the ∇ symbol to gain a more convenient notation. Let be E an orientable real vector bundle of rank $2m$ with fiber V over the base manifold X, the Mathai-Quillen representative of the Thom class is defined by

$$\Phi(E) = (2\pi)^{-m} \int d\chi \exp\{-\xi^2/2 + \chi_a \Omega^{ab} \chi_b/2 + i\nabla \xi^a \chi_a\} \tag{6.59}$$

Now $\Phi(E)$ is a closed differential form in $\Omega_{rd}^{\bullet}(E)$ and Ω is really the curvature of the connection on E. Let $s : X \to E$ be a section of E. As we said previously in section (6.3), the pullback of $\Phi(E)$ trough a generic section s will descends to a differential form on X which is precisely $e_{s,\nabla}(E)$ as defined in (6.18). Using (6.59) we can represent it as a Berezin integral

$$e_{s,\nabla}(E) = (2\pi)^{-m} \int d\chi \exp\{-s^2/2 + \chi_a \Omega^{ab} \chi_b/2 + i\nabla s^a \chi_a\} \tag{6.60}$$

In our notation $e_{s,\nabla}(E)$ is obtained from (6.59) by replacing the fiber coordinate ξ by $s(x)$. As a consistency check, note that, as follows from (6.42), $e_{s=0,\nabla}(E) = e_{\nabla}(E)$, i.e., the pullback of the Mathai-Quillen form by the zero section gives back the Euler class of E. We just proved (6.21). Let us denote by x^μ, a set of local coordinates on the base manifold X. The form $e_{s,\nabla}(E)$ can be rewritten in a compact way with the help of odd real variables θ^μ

$$e_{s,\nabla}(E) = (2\pi)^{-m} \int d\chi \exp\{-s^2/2 + \chi_a \Omega_{\mu\nu}^{ab} \theta^\mu \theta^\nu \chi_b/2 + i\chi_a (\nabla_\mu s)^a \theta^\mu \} \tag{6.61}$$

where we identified $\theta^\mu \leftrightarrow dx^\mu$. If $\mathrm{rk}(E) = \dim(X)$ we can evaluate the Euler number by computing

$$\chi(E) = (2\pi)^{-m} \int_X dx d\theta d\chi \exp\{-s^2/2 + \chi_a \Omega_{\mu\nu}^{ab} \theta^\mu \theta^\nu \chi_b/2 + i\chi_a (\nabla_\mu s)^a \theta^\mu \} \tag{6.62}$$

6.6.2. Example: Poincaré-Hopf Theorem. We now work out the computation of the Euler number for the case of $E = TX$ where X is a Riemannian manifold with metric g_{ij}. We take the Levi-Civita connection on TX. Let $V = V^i\partial_i$ be a section of TX. Then, considered as an element of $\mathcal{C}^\infty(\Pi TX)$,

$$V^*(\Phi(TX)) = \frac{1}{(2\pi)^{2m}}\int d\chi\sqrt{g}\ \exp\Big[-\frac{1}{2}g_{ij}V^iV^j + i\chi_j(\nabla_k V)^j\theta^k + \frac{1}{2}\chi_i R^{ij}_{\ \ kl}\theta^k\theta^l\chi_j\Big] \tag{6.63}$$

Replacing V by tV for $t \in \mathbb{R}$ we get

$$V^*(\Phi_t(TX)) = \frac{1}{(2\pi)^{2m}}\int d\chi\sqrt{g}\ \exp\Big[-\frac{t^2}{2}g_{ij}V^iV^j + t\,i\chi_j(\nabla_k V)^j\theta^k + \frac{1}{2}\chi_i R^{ij}_{\ \ kl}\theta^k\theta^l\chi_j\Big] \tag{6.64}$$

To evaluate the Euler number we have to perform the following integral

$$\chi(TX) = \frac{1}{(2\pi)^{2m}}\int dx d\theta d\chi\sqrt{g}\exp\Big[-\frac{t^2}{2}g_{ij}V^iV^j + t\,i\chi_j(\nabla_k V)^j\theta^k + \frac{1}{2}\chi_i R^{ij}_{\ \ kl}\theta^k\theta^l\chi_j\Big] \tag{6.65}$$

Letting $t \to \infty$ we see that this integral is concentrated at the zeroes, P, of the vector field V. Let's assume that there exist a local coordinate system $\{x^i\}$ such that

$$V^i = V^i_{\ j}x^j + \mathcal{O}(x^2) \tag{6.66}$$

in the neighborhood of a zero of V, this means that we are choosing a section which vanishes linearly at $x = 0$. Using the supergeometry notation we can write

$$(\nabla V)^i = V^i_{\ j}\theta^j + \mathcal{O}(x) \tag{6.67}$$

Now we do the integral (6.65) in the neighborhood of $x = 0$. As $t \to \infty$ the Gaussian approximation gives an integral over χ and θ leading to a factor of

$$(-1)^m\sqrt{g}\det(V)$$

The bosonic Gaussian integral yields

$$\frac{1}{\sqrt{g}\det(V)}$$

Thus, the boson and fermion determinants cancel up to sign and the contribution of the fixed point is just

$$\operatorname{sign}\det(V) = \deg_V(P) \tag{6.68}$$

With this result we proved the Poincaré-Hopf theorem, which is a special case of the Hopf theorem that we introduced in (6.11). We could have also choose the limit $t \to 0$ to evaluate the integral (6.65). In that case we easily recover the result, which is

$$\chi(TX) = \mathrm{Pf}\left(\frac{R}{2\pi}\right) \tag{6.69}$$

usually this result is called Gauss-Bonnet theorem. In summary, the Mathai-Quillen representative gives a formula for the Euler character that interpolates smoothly between the Gauss-Bonnet and Poincaré-Hopf formulae for $\chi(TX)$. We can generalize this last statement also to the case of a generic vector bundle E and generic section s. Then the Mathai-Quillen representative interpolates between (6.11) and (6.12). With this example in mind we can now comprehend perfectly the final statement of section (6.2).

CHAPTER 7

BV representative of the Thom Class

The goal of this section is to formulate a new representative of the Thom class using the ideas of BV formalism. In particular, to achieve this result, we will apply the odd Fourier transform defined in (5.1). From a mathematical point of view, we are defining the odd Fourier transform for differential forms on vector bundles and obtaining consequently representatives for the Thom class of arbitrary vector bundles. This problem was raised in a paper by Kalkman [23] and this section can be seen as a solution to that.

7.1. Geometry of $T[1]E$

Let be E an orientable real vector bundle of rank $2m$ with fiber V over the base manifold X, as we already seen in (6.6.1) the Mathai-Quillen representative of the Thom class is defined by

$$\Phi(E) = (2\pi)^{-m} \int d\chi \exp\{-\xi^2/2 + \chi_a \Omega^{ab} \chi_b/2 + i\nabla\xi^a \chi_a\} \tag{7.1}$$

where ξ's are fiber coordinates, χ's are fermionic variables and $\nabla\xi^a$ is the exterior covariant derivative of ξ^a. The presence of covariant derivative may sound strange since to integrate $\Phi(E)$ along the fibers we just need to put a term of the form $d\xi^a \chi_a$ in the exponential in order to get a top form on $\Omega^\bullet(V)$. The problem of a term like this, in the formula above, is that it will affect the covariance. Both ξ and χ transforms as a section but $d\xi$ not, in fact

$$\widetilde{\xi}^a = t^a_b(x)\xi^b \Rightarrow d\widetilde{\xi}^a = \partial_\mu t^a_b(x) dx^\mu \xi^b + t^a_b(x) d\xi^b \tag{7.2}$$

Before proceeding further, we have to clarify one point. In the study of differential geometry on total space of a vector bundle $E \to X$ we would like to divide up the coordinates into two sets: basic coordinates x^μ and fiber coordinates ξ^a. Similarly, the anticommuting variables separate into $\theta^\mu = dx^\mu$ and $\widehat{\lambda}^a = d\xi^a$. At this point it is very convenient to employ the extra data of a connection ∇ on E to restore the covariance for the action of the differential d on $T[1]E$. In this context $T[1]E$ is the graded manifold that we can naturally associate to E and following Example 4.2.2 we will identify smooth functions on

$T[1]E$ with differential forms on E, $\mathcal{C}^\infty(T[1]E) = \Omega^\bullet(E)$. Using the connection ∇ we can establish the following isomorphism

$$\begin{array}{ccccccc} T[1]E & \simeq & E & \oplus & E[1] & \oplus & T[1]M \\ & & \downarrow & & \downarrow & & \downarrow \\ & & (x^\mu, \xi^a) & & \lambda^a & & \theta^\mu \end{array} \tag{7.3}$$

where λ^a is the new anticommuting fiber coordinates that transforms as a section of E. Following [27], what we just did is a reduction of the structure group of the sheaf of functions on $T[1]E$ by means of a connection. In this way, our sheaf of functions will be generated by variables $(x^\mu, \theta^\mu; \xi^a, \lambda^a)$ with $(\theta^\mu; \xi^a, \lambda^a)$ transforming linearly across patch boundaries on the base manifold X. So, we can write out explicitly the action of the differential on the variables as

$$\begin{aligned} dx^\mu &= \theta^\mu \\ d\theta^\mu &= 0 \\ \nabla\xi^a \equiv \lambda^a &= d\xi^a + A^a_{\mu\, b}\theta^\mu\xi^b \\ \nabla\lambda^a &= d\lambda^a + A^a_{\mu\, b}\theta^\mu\lambda^b = \tfrac{1}{2}\Omega^a_{b\mu\nu}\theta^\mu\theta^\nu\xi^b \end{aligned} \tag{7.4}$$

where $A^a_{\mu\, b}$ is the local expression for the connection 1-form and $\Omega^a_{b\mu\nu}$ is the local expression for the curvature 2-form. Summarizing, we discovered that in the case of the total space of a vector bundle, the fiber variable λ^a should be considered as the covariant differential $\nabla\xi^a$.

7.2. Odd Fourier Transform Revisited

Let's consider all the notations introduced in (7.1), the form $\Phi(E)$ can be rewritten as

$$\Phi(E) = (2\pi)^{-m}\int d\chi \exp\{-\xi^2/2 + \chi_a\Omega^{ab}_{\mu\nu}\theta^\mu\theta^\nu\chi_b/2 + i\lambda^a\chi_a\} \tag{7.5}$$

It is helpful to consider also the following notation

$$\Phi(E) = (2\pi)^{-m}\int d\chi \exp(\Psi) \tag{7.6}$$

where

$$\Psi = -\xi^2/2 + \chi_a\Omega^{ab}_{\mu\nu}\theta^\mu\theta^\nu\chi_b/2 + i\lambda^a\chi_a \tag{7.7}$$

From the point of view of the base manifold X, the odd Fourier transform is a way to relate $T[1]X$ and $T^*[-1]X$, see (5.1). In this subsection, we would like to look at the odd Fourier transform from the point of view of total space E and think of it as a tool to relate $T[1]E$ and $T^*[-1]E$. Again, in the study of $T^*[-1]E$ it is useful to reduce the structure group of the the sheaf of functions using the extra data of the connection previously introduced. We create the following isomorphism

$$\begin{array}{ccccc} T^*[-1]E & \simeq & E & \oplus\, E^*[-1] & \oplus\, T^*[-1]M \\ & & \downarrow & \downarrow & \downarrow \\ & & (x^\mu, \xi^a) & \sigma_a & \psi_\mu \end{array} \tag{7.8}$$

where σ_a is the dual variable with respect to λ^a introduced in (7.1). It is important to stress that (7.3) and (7.8) are examples of non canonical splitting, these splittings are possible only upon the choice of a connection ∇. Thus, the generalization of odd Fourier transform for differential forms on a vector bundle will maps functions on $T[1]E$ to functions on $T^*[-1]E$

$$F : \mathcal{C}^\infty(T[1]E) \to \mathcal{C}^\infty(T^*[-1]E) \tag{7.9}$$

and the corresponding action on the coordinates will be

$$(x^\mu, \theta^\mu; \xi^a, \lambda^a) \xrightarrow{F} (x^\mu, \psi_\mu; \xi^a, \sigma_a) \tag{7.10}$$

Being a differential forms on E, the Mathai-Quillen representative (7.5) can be identified with a smooth functions on $T[1]E$

$$\Phi(E) \in \mathcal{C}^\infty(T[1]E) \tag{7.11}$$

then it is natural to ask how it would look the odd Fourier transform of $\Phi(E)$. Precisely, we have

$$F[\Phi(E)] = \Phi_{BV}(E) = (2\pi)^{-m} \int d\theta \rho(x)^{-1} \int d\lambda \rho(\xi)^{-1} \int d\chi \exp\{\widetilde{\Psi}\} \tag{7.12}$$

where $\widetilde{\Psi}$ is expressed as

$$\widetilde{\Psi} = -\xi^2/2 + \tfrac{1}{2}\chi_a \Omega^{ab}_{\mu\nu} \theta^\mu \theta^\nu \chi_b - i\chi_a \lambda^a + \psi_\mu \theta^\mu + \sigma_a \lambda^a \tag{7.13}$$

After some manipulation, it is possible to show that

$$\Phi_{BV}(E) = (2\pi)^{-m} \int d\theta \rho_x^{-1} \rho_\xi^{-1} \exp\{-\xi^2/2 + \sigma_a \Omega^{ab}_{\mu\nu} \theta^\mu \theta^\nu \sigma_b/2 + \psi_\mu \theta^\mu\} \tag{7.14}$$

From now on, we will refer to $\Phi_{BV}(E)$ as the BV representative for the Thom class of the vector bundle E which should be considered as a smooth functions on $T^*[-1]E$. The expression (7.13) transforms nicely because of the covariantization procedure that we adopted. In conclusion, we have seen that providing a generalization of the odd Fourier transform to the case of a vector bundle we can define a new Thom class representative which is the BV representative $\Phi_{BV}(E)$ defined in (7.12).

7.3. Analysis of the BV Representative

Looking at the explicit expression for $\Phi_{BV}(E)$ it appears clearly that we have a rapid decay object, which integrates to 1 upon proper Berezin integrations. However, to be sure that the procedure that we followed to contruct this new representative is correct we have to check all the properties in the definition of the Thom class i.e. we have to check that $\Phi_{BV}(E)$ is closed which is less evident. At this stage it is worth to spend some words to prove that also the MQ representative $\Phi(E)$ is closed.

PROOF. The standard exterior derivative, acting on $\Omega(E)$, will look like

$$D = dx^\mu \frac{\partial}{\partial x^\mu} + d\xi^a \frac{\partial}{\partial \xi^a} \tag{7.15}$$

Observing what we described in (7.1), the convenient way to express D is really

$$D = \theta^\mu \frac{\partial}{\partial x^\mu} - A^a_{\mu\, b}\theta^\mu \xi^b \frac{\partial}{\partial \xi^a} + \lambda^a \frac{\partial}{\partial \xi^a} = \nabla + \lambda^a \frac{\partial}{\partial \xi^a} \tag{7.16}$$

where ∇ is the usual covariant derivative. In this way we get

$$D\Phi(E) = (2\pi)^{-m} \int d\chi \exp(\Psi)\left(\nabla\Psi + \lambda^a \frac{\partial \Psi}{\partial \xi^a}\right) \tag{7.17}$$

then

$$\begin{aligned} \lambda^a \frac{\partial \Psi}{\partial \xi^a} &= -\xi_a \lambda^a \\ \nabla\Psi &= +i\nabla_\nu A^a_{\mu\, b}\theta^\mu\theta^\nu\xi^b\chi_a = -i\xi_a \Omega^{ab}_{\mu\nu}\theta^\mu\theta^\nu\chi_b \end{aligned} \tag{7.18}$$

and

$$\nabla\Psi + \lambda^a \frac{\partial \Psi}{\partial \xi^a} = -\xi_a\lambda^a - i\xi_a\Omega^{ab}_{\mu\nu}\theta^\mu\theta^\nu\chi_b \tag{7.19}$$

where we used the Bianchi identities $\nabla\Omega = 0, \nabla A = \Omega$ and the anti-symmetry of Ω^{ab}. Using the properties of Berezin integration we can always write

$$\Phi(E) = (2\pi)^{-m} \int d\chi \exp(\Psi) + (2\pi)^{-m} \int d\chi \frac{\partial}{\partial \chi} \exp(\Psi) \tag{7.20}$$

since the last term integrates to zero. At this point we can express (7.17) as

$$\begin{aligned} D\Phi(E) &= (2\pi)^{-m} \int d\chi \exp(\Psi)\left(-\xi_a\lambda^a - i\xi_a\Omega^{ab}_{\mu\nu}\theta^\mu\theta^\nu\chi_b + i\xi_a \frac{\partial \Psi}{\partial \chi_a}\right) \\ &= 0 \end{aligned} \tag{7.21}$$

indeed

$$i\xi_a \frac{\partial \Psi}{\partial \chi_a} = \xi_a \lambda^a + i\xi_a \Omega^{ab}_{\mu\nu} \theta^\mu \theta^\nu \chi_b \tag{7.22}$$

In this way we proved that $\Phi(E)$ is a closed differential form, as it should be, since this is a defining property for the Thom class. □

To show that $\Phi_{BV}(E)$ is closed we cannot simply apply the operator D but we have to transform it to get the proper differential operator on $T^*[-1]E$. Doing that, we will obtain an odd Laplacian operator that we will denote Δ_{BV}. The Fourier transformation for (7.16) will be

$$\Delta_{BV} = \frac{\partial^2}{\partial x^\mu \partial \psi_\mu} - A^a_{\mu\, b} \xi^b \frac{\partial^2}{\partial \psi_\mu \partial \xi^a} + \frac{\partial^2}{\partial \xi^a \partial \sigma_a} \tag{7.23}$$

It is not difficult to understand how we got this formula indeed we followed the same considerations of (5.1). However, for the sake of simplicity, here we assumed that all the ρ's are constant. Thus, also $\Phi_{BV}(E)$ is closed in fact

Proof. By (5.8) we know that

$$\Delta_{BV} F[\Phi(E)] = F[D\Phi(E)] \tag{7.24}$$

To verify this statement we firstly calculate the following expression

$$\begin{aligned} \Delta_{BV} \exp(\widetilde{\Psi}) &= \exp(\widetilde{\Psi}) \left(\frac{\partial \widetilde{\Psi}}{\partial x^\mu} \frac{\partial \widetilde{\Psi}}{\partial \psi_\mu} - A^a_{\mu\, b} \xi^b \frac{\partial \widetilde{\Psi}}{\partial \xi^a} \frac{\partial \widetilde{\Psi}}{\partial \psi_\mu} + \frac{\partial \widetilde{\Psi}}{\partial \xi^a} \frac{\partial \widetilde{\Psi}}{\partial \sigma_a} \right) \\ &= \exp(\widetilde{\Psi}) \left(\theta^\mu \frac{\partial \widetilde{\Psi}}{\partial x^\mu} - A^a_{\mu\, b} \theta^\mu \xi^b \frac{\partial \widetilde{\Psi}}{\partial \xi^a} + \lambda^a \frac{\partial \widetilde{\Psi}}{\partial \xi^a} \right) \\ &= \exp(\widetilde{\Psi})(D\widetilde{\Psi}) \end{aligned} \tag{7.25}$$

then we have

$$\begin{aligned} \Delta_{BV} F[\Phi(E)] &= (2\pi)^{-m} \int d\theta \rho(x)^{-1} \int d\lambda \rho(\xi)^{-1} \int d\chi\, \Delta_{BV} \exp(\widetilde{\Psi}) \\ &= (2\pi)^{-m} \int d\theta \rho(x)^{-1} \int d\lambda \rho(\xi)^{-1} \int d\chi\, \exp(\widetilde{\Psi})(D\widetilde{\Psi}) \\ &= F[D\Phi(E)] = 0 \end{aligned} \tag{7.26}$$

where the last equality follows from (7.21). □

Finally, the BV representative $\Phi_{BV}(E)$ is effectively a representative for the Thom class of E. In the language of BV quantization, introduced in (5.4.1), the fact that $\Phi_{BV}(E)$ is closed mean that we have found a solution to the master equation. The outcome of this section is that we have found a nice relationship between two seemingly unrelated subjects such as the BV formalism and the Mathai-Quillen formalism.

CHAPTER 8

Conclusions

The focus of this thesis was the construction of a new representative, for the Thom class of a vector bundle, called BV representative. This representative has been constructed using the odd Fourier transform developed in [15, 23]. In (7.1) and (7.2) we shown how it is possible to generalize the odd Fourier transform to the case of a vector bundle. Considering the MQ Thom form as a smooth function over $T[1]E$, we obtained the corresponding BV representative defined over $T^*[-1]E$. In (7.3) we also shown that our BV representative is closed under the action of a suitable differential operator, this means that the BV representative is a solution to the BV master equation. In view of these results, chapter 7 is a proof for the following theorem

THEOREM 8.0.1. *The BV representative $\Phi_{BV}(E)$ is a multivector field representative for the Thom class of a vector bundle and it is a solution to the BV master equation.*

The present work can be seen as a first step in the process of understanding BV quantization from the point of view of Mathai-Quillen formalism. Indeed, on one hand we have the MQ construction which is a concrete prescription that can be used to calculate Euler class and its infinite dimensional generalization is very well understood, being the basis of all cohomological topological field theories. On the other hand we have the BV quantization which is, algebraically, a very powerful technique but we still have a lack of knowledge on what we are effectively computing in this formalism. Moreover, the path integral manipulations in BV formalism are understood only formally and the finite dimensional setting provides only a good heuristically comprehension of the problem. Understanding completely BV quantization is a very deep problem, far beyond the aim of this thesis, but it is also a fertile ground for future investigations and interesting discoveries.

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