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Feynman Diagrams

and Map Enumeration

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Abstract

Feynman Diagrams and Map Enumeration

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The goal of this thesis is to count how many graphs exist given a number of vertices or some other restrictions. The graphs are counted by perturbing Gaussian integrals and using the Wick lemma to interpret the perturbations in terms of graphs. Fat graphs, a specific type of graph, are central in this thesis. A method based on orthogonal polynomials to count fat graphs is presented. The thesis finishes with the formulation and some results related to the three-color problem.

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Summary in swedish

Inom Kvantfältteorin beräknas partikelinteraktioner med en mycket komplicerad oändligtdimensionell integral. 1948 föreslog den amerikanske fysikern Richard Feynman ett sätt att grafiskt approximera denna integral som än idag används flitigt. Approximationen bygger på så kallade *Feynman diagram*, som består av olika sorters linjer som sammankopplas i noder. Linjerna representerar partiklar och noderna representerar interaktioner mellan partiklarna. En naturlig fråga efter introduktionen av Feynman Diagram är "Givet ett antal partiklar, hur många diagram finns det?". Fysikaliskt kan detta översättas till "På hur många sätt kan partiklarna interagera?". För få partiklar eller få interaktioner är det enkelt att bara rita diagrammen och räkna dem, men för större system blir detta tillvägagångssätt väldigt opraktiskt. Man måste alltså kunna räkna dessa grafer på något annat sätt. I denna tes räknas graferna genom att utgå från en välkänd integral som kallas *Gaussisk*. Integranden ändras sedan väldigt lite och korrektionstermer till följd av denna ändring räknas ut. Dessa korrektionstermer kan tolkas som grafer.

Contents

1 Introduction

In Quantum Field Theory, particle interactions are often calculated using *Feynman Diagrams*. These diagrams, proposed by Richard Feynman in 1948, is a convenient way to graphically approximate integrals. The path integrals in Quantum Field Theory are often to complicated to calculate exactly, and Feynman diagrams is the preferred method of approximation. It is known how much particular diagrams contribute to the integral and it is therefore known which diagrams to consider for an approximation of a certain order. Of particular interest are the *connected* diagrams which describes *one* interaction. Disconnected diagrams describes several interactions that should be studied individually. For this reason, this thesis focuses mainly on connected diagrams.

The calculations in this thesis are rather the opposite of the graphical calculations in Quantum Field Theory. We calculate finite dimensional integrals, either exactly or to a certain order of approximation, and can from there count the corresponding graphs. It is not at all obvious how and why some integrals may be calculated graphically, so the first few sections contain definitions, examples and basic results that will justify the usage of graphs in integral calculations. In particular the related concepts of *Gaussian measures* and *the Wick lemma* are introduced and will be used throughout this thesis.

Regular graphs are counted with integrals over $\mathbb{R}$. How much a graph contributes to the integral depends on its number of *vertices*. The goal when counting regular graphs is to count how many connected graphs there are given a number of vertices. A different kind of graph, called *fat graphs*, are counted with matrix integrals over $\mathcal{H}_N$, the space of all $N \times N$ Hermitian matrices. For fat graphs, the contribution depends on the number of vertices, as well as the *genus* of the graph. Since the contribution of fat graphs depends on two things, the goal is to count how many connected fat graphs there are given any genus and any number of vertices.

After the introductional sections, it is explained how to visualize integrals as graphs using the Wick lemma. It is also shown how to count connected graphs. The concept of colored graphs is introduced and some basic properties are discussed. Fat graphs are then introduced, which are the main object of study in this thesis. In order to count fat graphs, orthogonal polynomials are introduced. Finally, the formulation of the *three-color problem* is given, and some related results are presented.

1.1 Gaussian measures

Let $A \in \mathbb{R}^{N\times N}$, be a symmetric and positive definite matrix. $d\mu(\vec{x})$ defines a *Gaussian measure* on $\mathbb{R}^N$ by [1]

$$d\mu(\vec{x}) = \exp\left(-\vec{x}^T \frac{A}{2} \vec{x}\right) d^N x \tag{1.1}$$

A is called the *covariance matrix* for the measure. A Gaussian measure where the covariance matrix is an identity matrix is called *standard*. Note that a Gaussian measure is even if and only if the covariance matrix is diagonal.

Orthogonally diagonalizing the matrix A, $A = P^T DP$, changing the integration variables to $\vec{y} = P\vec{x}$ and integrating over $\mathbb{R}^N$ gives

$$\int_{\mathbb{R}^N} d\mu(\vec{x}) = \int_{\mathbb{R}^N} \exp(-\vec{y}^T \frac{D}{2}\vec{y}) d^N y = \frac{\sqrt{2\pi}^N}{\sqrt{\det(D)}} = \frac{(2\pi)^{N/2}}{\sqrt{\det(A)}} \tag{1.2}$$

where the last equality follows from the fact that P is orthogonal, $P^T P = I$, hence

$$\det(P^T P) = 1 \implies \det(A) = \det(P^T D P) = \det(D) \tag{1.3}$$

The integral over $\mathbb{R}^N$ of the measure

$$d\tilde{\mu}(\vec{x}) = \frac{\sqrt{\det(A)}}{(2\pi)^{N/2}} d\mu(\vec{x}) \tag{1.4}$$

is equal to 1. A measure whose integral over its domain is equal to 1 is called *normalized*. From an arbitrary measure $d\Omega(v)$ on a space V, one can form a normalized measure $d\tilde{\Omega}(v)$ by

$$d\tilde{\Omega}(v) = \frac{1}{\int_V d\Omega(v)} d\Omega(v) \tag{1.5}$$

1.2 Expectation values of monomials

The expectation value of a function, $f : \mathbb{R}^N \to \mathbb{R}$, with respect to the normalized measure $d\tilde{\mu}(\vec{x})$ is defined as

$$\langle f(\vec{x}) \rangle = \int_{\mathbb{R}^N} f(\vec{x}) d\tilde{\mu}(\vec{x}) = \frac{\int_{\mathbb{R}^N} f(\vec{x}) d\mu(\vec{x})}{\int_{\mathbb{R}^N} d\mu(\vec{x})} \tag{1.6}$$

It will prove to be more practical to keep the integral in the denominator rather than normalizing the measure itself.

A monomial is a polynomial in N variables with only one term. To calculate expectation values of monomials, we start by defining the *generating function* $Z(\vec{J})$ by

$$Z(\vec{J}) = \int_{\mathbb{R}^N} e^{-\vec{x}^T \frac{A}{2}\vec{x} + \vec{x}^T \vec{J}} d^N x \implies Z(0) = \int_{\mathbb{R}^N} d\mu(\vec{x}) \tag{1.7}$$

The next step in the calculation is taking a derivative of the generating function at the origin. The result is

$$\frac{\partial^n Z}{\partial J_{i_1} \ldots \partial J_{i_n}}(0) = \int x_{i_1} \ldots x_{i_n} d\mu(\vec{x}) \iff \frac{1}{Z(0)} \frac{\partial^n Z}{\partial J_{i_1} \ldots \partial J_{i_n}}(0) = \langle x_{i_1} \ldots x_{i_n} \rangle \tag{1.8}$$

There is no requirement for the indicies to be distinct. In fact for any even measure, the expectation value of a monomial which is of odd degree in any variable is identically zero.

We note that

$$-\vec{x}^T \frac{A}{2}\vec{x} + \vec{x}^T \vec{J} = -(\vec{x} - A^{-1}\vec{J})^T \frac{A}{2}(\vec{x} - A^{-1}\vec{J}) + \frac{1}{2}\vec{J}^T A^{-1} \vec{J} \tag{1.9}$$

Looking at the definition of $Z(\vec{J})$ (1.7), and changing the variables of integration to $\vec{y} = \vec{x} - A^{-1}\vec{J}$,we can calculate $Z(\vec{J})$ explicitly. The answer is, with $B = A^{-1}$

$$Z(\vec{J}) = Z(0)e^{\frac{1}{2}\vec{J}^T B\vec{J}} \iff \frac{Z(\vec{J})}{Z(0)} = e^{\frac{1}{2}\vec{J}^T B\vec{J}}$$

The expectation values of monomials are now calculated with

$$\langle x_{i_1} \dots x_{i_n} \rangle = \frac{\partial^n (e^{\frac{1}{2}\vec{J}^T B\vec{J}})}{\partial J_{i_1} \dots \partial J_{i_n}} \Bigg|_{\vec{J}=0} \tag{1.10}$$

For the important case of quadratic monomials, the formula reads

$$\langle x_i x_j \rangle = \frac{\partial^2 (e^{\frac{1}{2}\vec{J}^T B\vec{J}})}{\partial J_i \partial J_j} \Bigg|_{\vec{J}=0} = B_{ij} \tag{1.11}$$

1.3 Graphs

A graph consists of two objects, edges and vertices. Vertices are represented by dots and edges are represented by lines. [1] The vertices are connected by the edges, and all edges must begin and end in a (not necessarily different) vertex. Each vertex has a number of edges adjacent to it. The number of edges adjacent to a vertex is called the *degree* of the vertex. The number of vertices in a graph is called the *order* of the graph. All graphs of order 2, with vertices of degree 3 are shown in figure 2.2.

1.4 The Wick lemma

The Wick lemma is essential to this thesis. It is what allows us to compute integrals using graphs. And by calculating the integrals exactly, we are able to count the graphs in terms of contribution to the integral. The Wick lemma states that for a normalized Gaussian measure, and for linear functions, $f_1, f_2, \dots, f_{2k}$

$$\langle f_1 f_2 \dots f_{2k} \rangle = \sum \langle f_{p_1} f_{q_1} \rangle \dots \langle f_{p_k} f_{q_k} \rangle \tag{1.12}$$

[1] where the sum is taken over $q_i > p_i$ and $p_{i+1} > p_i$, i.e all unique couplings. A term in the right-hand side is called a *Wick coupling.* For $2k$ functions there are $(2k-1)!!$ Wick couplings. A proof of the Wick lemma is given in Appendix A.

2 Expectation values of monomials graphically

2.1 One-dimensional integrals

The normalized standard Gaussian measure in one dimension is

$$d\mu = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \tag{2.1}$$

In one dimension, every monomial is a power of x. Since a one-dimensional Gaussian measure is even, the expectation value of an odd power of x is identically zero. We will now calculate the expectation value for even powers of x using the Wick lemma.

Let $f_1 = f_2 = \ldots = f_{2k} = x$. The Wick lemma states that

$$\langle f_1 f_2 \ldots f_{2k} \rangle = \sum_{\text{coupl.}} \langle f_{p_1} f_{q_1} \rangle \langle f_{p_2} f_{q_2} \rangle \ldots \langle f_{p_k} f_{q_k} \rangle \tag{2.2}$$

From equation (1.11) we know that

$$\langle f_{p_i} f_{q_i} \rangle = \langle x^2 \rangle = B_{11} = 1 \tag{2.3}$$

$$\tag{2.4}$$

Hence all the summands in the right-hand side are equal to 1. Since there are $(2k-1)!!$ Wick couplings of $2k$ functions

$$\langle x^{2k} \rangle = (2k-1)!! \tag{2.5}$$

A convenient way of visualizing Wick couplings is with graphs. A power x^k is represented by a vertex of degree k, each edge representing an x. A coupling $\langle f_i f_j \rangle$ is then represented by connecting two edges. Figure 2.1 shows the graphical representation of two vertices of degree 3 and figure 2.2 shows all graphs of order 2 with vertices of degree 3. Once all the graphs are found, the expectation value can be calculated with.

$$\langle x^{2k} \rangle = \sum W(\Gamma) \tag{2.6}$$

Where the sum runs over all graphs and $W(\Gamma)$ is the number of couplings that result in the graph Γ.

We will now calculate $\langle x^6 \rangle$ graphically. This can be done in numerous ways, as we are free to choose the number of vertices from 1 to 6 and may freely attach 6 edges to the vertices. This corresponds to writing x^6 as a product of 1 to 6 powers of x. In this case we represent x^6 as 2 vertices of degree 3.

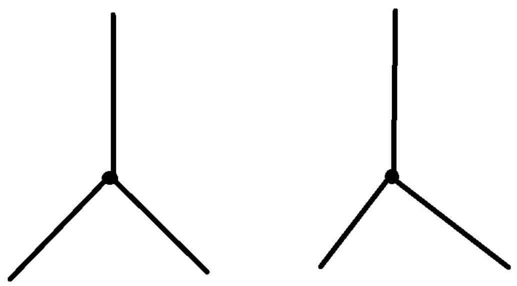

Figure 2.1: Two vertices of degree 3

These are the vertices and edges the graphs will consist of. There are two ways of combining the edges to form different graphs. If two edges from the first vertex are connected to form a *loop*, the remaining edge must be connected to an edge from the second vertex. The remaining two edges from the second vertex must then be connected to form another loop. If there is no loop in the first vertex, all three edges must be connected to an edge from the second vertex. The resulting graphs are shown in figure 2.2.

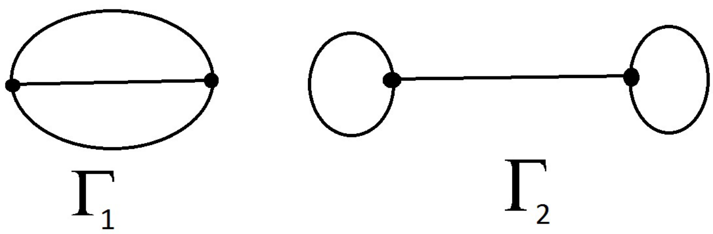

Figure 2.2: All graphs with two vertices of degree 3

Following the procedures described above, there are $3! = 6$ Wick couplings that result in the graph Γ_1, and $3^2 = 9$ Wick couplings that result in Γ_2. Equation (2.6) now reads

$$\langle x^6 \rangle = W(\Gamma_1) + W(\Gamma_2) = 15 = (6-1)!! \tag{2.7}$$

This agrees with equation (2.5). Alternatively, for a graph Γ of order ν with vertices of degree μ, $W(\Gamma)$ can be defined as the number of distinct labeled graphs one can form by labeling the edges of the vertices $\{1, 2, \ldots, \mu\}$ and label the vertices $\{1, 2, \ldots, \nu\}$. If Γ contains vertices of different degrees, say ν_1 vertices of degree μ_1, ν_2 vertices of degree μ_2 and so on, the vertices of degree μ_i are labeled by $\{1_{\mu_i}, 2_{\mu_i}, \ldots, \nu_{i\mu_i}\}$. Vertices of different degrees can not have the same label. It is this definition of $W(\Gamma)$ that will be used from here on.

For a more graphically meaningful formula we will consider expectation values of the form

$$\langle \frac{1}{\nu_1!}\left(\frac{x^{\mu_1}}{\mu_1!}\right)^{\nu_1} \frac{1}{\nu_2!}\left(\frac{x^{\mu_2}}{\mu_2!}\right)^{\nu_2} \cdot \ldots \cdot \frac{1}{\nu_n!}\left(\frac{x^{\mu_n}}{\mu_n!}\right)^{\nu_n} \rangle \tag{2.8}$$

where $\mu_i \neq \mu_j$ for $i \neq j$. The reason for this is that for ν vertices of degree μ, there are $(\mu!)^\nu \nu!$ ways to label the edges and the vertices. We have divided by the total number of labelings of the entire graph.

Not all labelings result in distinct labeled graphs. Consider the graph Γ_2 of figure 2.2. The label of the non-looped edges uniquely determines a labeled graph. Since there are 3 ways to label both of the non-looped edges, there are 9 distinct labelings of Γ_2. Figure 2.3 illustrates 2 distinct labelings of the edges of one vertex that result in the same labeled graph.

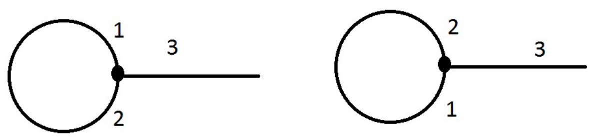

Figure 2.3: Different labelings resulting in the same graph

The quantities to calculate now are

$$\frac{W(\Gamma_i)}{\nu_1! \ldots \nu_n! (\mu_1!)^{\nu_1} \ldots (\mu_n!)^{\nu_n}} \tag{2.9}$$

which can be interpreted as the number of distinct labeled graphs one can form divided by the total number of labelings.

There are three symmetries a graph can have that makes distinct labelings result in the same labeled graph. These symmetries are loops, multiple edges between two vertices and vertex symmetry.

For a vertex with k loops, one can always switch the labels in a single loop in 2^k ways, as well as switch pairs of labels between loops in $k!$ ways. So you lose a total factor of

$$L_k = 2^k k! \tag{2.10}$$

to a loop symmetry. It is possible for a graph to have multiple loops. For a graph where one vertex has k_1 loops, another vertex has k_2 loops and so on up to a vertex with k_n loops, the total loop symmetry factor, L, equals

$$L = \prod_{i=1}^{n} L_{k_i} \tag{2.11}$$

For two vertices connected by p edges, one can switch pairs of labels in $p!$ ways. Hence the edge symmetry factor is

$$E_p = p! \tag{2.12}$$

For a graph where two vertices are connected by p_1 edges, two other vertices are connected by p_2 vertices and so on up to a pair of vertices connected by p_m edges, the total edge symmetry factor, E, is equal to

$$E = \prod_{j=1}^{m} E_{p_j} \tag{2.13}$$

The last factor is the vertex symmetry factor V. There is no nice formula to calculate V, like there are for L and E. It is defined by the number of distinct labelings of the vertices that result in the same labeled graph, disregarding the labels of the edges.

The total *symmetry factor*, S, is defined by

$$S = L \cdot E \cdot V \tag{2.14}$$

This symmetry factor is extremely important in the applications of Feynman diagrams in Quantum Field Theory [5], and it shall prove to be important in this thesis as well. As this symmetry factor corresponds to lost distinct labelings due to symmetries of the graph, we get that

$$\frac{W(\Gamma_i)}{\nu_1! \ldots \nu_n! (\mu_1!)^{\nu_1} \ldots (\mu_n!)^{\nu_n}} = \frac{1}{S(\Gamma_i)} \tag{2.15}$$

Equation (2.6) now reads

$$\frac{1}{\nu_1! \cdot \nu_2! \cdot \ldots \cdot \nu_n!} \langle \left(\frac{x^{\mu_1}}{\mu_1!} \right)^{\nu_1} \left(\frac{x^{\mu_2}}{\mu_2!} \right)^{\nu_2} \cdot \ldots \cdot \left(\frac{x^{\mu_n}}{\mu_n!} \right)^{\nu_n} \rangle = \sum_{\Gamma_i} \frac{1}{S(\Gamma_i)} \tag{2.16}$$

where the sum runs over all graphs with ν_j vertices of degree μ_j for $j = 1, 2, \ldots, n$. The left-hand side of equation (2.16) is very easy to calculate using equation (2.5). Calculating the right-hand side by drawing all graphs and finding their symmetry factors can indeed be very complicated, especially for graphs with many vertices of high degrees. We now have a simple way of summing the inverse symmetry factors of all graphs with a given set of vertices. Equation (2.16) is essentially the Wick lemma but formulated in the language of graphs rather than expectation values and couplings.

Returning to the case of 2 vertices of degree 3, the symmetry factors are

$$S(\Gamma_1) = L_0 \cdot E_3 \cdot V = 1 \cdot 3! \cdot 2 = 12 \tag{2.17}$$

$$S(\Gamma_2) = (L_1)^2 \cdot E_0 \cdot V = 2^2 \cdot 1 \cdot 2 = 8 \tag{2.18}$$

Which agrees with equation (2.15) since

$$\frac{W(\Gamma_1)}{2 \cdot 3! \cdot 3!} = \frac{6}{72} = \frac{1}{12} = \frac{1}{S(\Gamma_1)} \tag{2.19}$$

$$\frac{W(\Gamma_2)}{2 \cdot 3! \cdot 3!} = \frac{9}{72} = \frac{1}{8} = \frac{1}{S(\Gamma_2)} \tag{2.20}$$

2.2 Graph counting by integral perturbation

We now want to count the *connected* graphs that consist of k vertices of degree 4. A graph is connected if, given any 2 vertices, there is a continuous path that starts in one vertex, moves through edges and vertices, and ends in the other vertex. This is of course equivalent to the more intuitive idea of connectedness, that the graph is not expressible as a union of smaller subgraphs. A graph that is not connected is called *disconnected*. By convention the the graph without vertices is disconnected.

In the case of 2 vertices of degree 3, it just so happened that the only 2 graphs one could form were both connected. In general there is no reason to assume that all graphs are connected. In fact, given at least 3 vertices with an even number of total edges, one can always form at least 1 disconnected graph. Equation (2.16) counts all graphs, including disconnected ones, so we have to take a different approach to only count connected graphs.

We start by considering the integral

$$Z(\lambda) = \int_{\mathbb{R}} \exp\left(-\frac{x^2}{2} - \frac{\lambda}{4!}x^4\right) d\mu(x) = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \int_{\mathbb{R}} \left(\frac{x^4}{4!}\right)^k d\mu(x) \tag{2.21}$$

To make this integral useful for graph counting purposes, we divide both sides of the equation by $Z(0)$, yielding

$$\frac{Z(\lambda)}{Z(0)} = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \langle \left(\frac{x^4}{4!}\right)^k \rangle = \sum_{k=0}^{\infty} (-\lambda)^k \underbrace{\sum_{\Gamma \in g_k} \frac{1}{S(\Gamma)}}_{z_k} = \sum_{k=0}^{\infty} z_k (-\lambda)^k \tag{2.22}$$

where g_k is the set of all graphs with k vertices of degree 4.

Since the sum in z_k is over all graphs, connected and disconnected, equation (2.22) is still not the expression we are looking for. However, there is an easy way to only consider the connected graphs, and the method is somewhat surprising. In order to only consider connected graphs, we take the *logarithm* of equation (2.22).

2.3 Graphical interpretation of the logarithm

We will now show that taking the logarithm of a generating function, such as (2.22), is equivalent to only considering the connected graphs of each order. We start by splitting every graph Γ into M connected subgraphs Γ_i.

$$\Gamma = \bigcup_{i=1}^{M} (\Gamma_i) \tag{2.23}$$

The number of vertices with k loops in Γ is the sum of the number of vertices with k loops in the subgraphs. Similarly, the number of vertices connected by p edges equals the sum of the number of vertices connected by p edges in the subgraphs. Hence the loop- and edge symmetry factor of Γ is just the product of the subgraphs'. The vertex symmetry factor however is a little different. Suppose the graph Γ contains ν identical subgraphs. One can still relabel the vertices in a subgraph in the same way as if it were the entire graph, but one can also swap sets of labels from one subgraph to another. This swapping of sets of labels can be done in $\nu!$ ways and we therefore pick up an extra symmetry factor of $\nu!$. Let Γ contain ν_1 copies of Γ_1, ν_2 copies of Γ_2, and so on up to ν_P. The relation between the symmetry factor of Γ and its subgraphs' symmetry factors is

$$S(\Gamma) = \prod_{j=1}^{P} \nu_j! S(\Gamma_j)^{\nu_j} \tag{2.24}$$

Let g_k^c be the set of all connected graphs that one can form from k vertices. Define E_k as

$$E_k = \sum_{\Gamma \in g_k^c} \frac{1}{S(\Gamma)} \tag{2.25}$$

Now consider the exponent

$$\exp\left(\sum_{k=1}^{\infty} (-\lambda)^k E_k\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=1}^{\infty} (-\lambda)^k E_k\right)^n = \tag{2.26}$$

$$= \sum_{n=0}^{\infty} \sum_{\nu_1+\ldots+\nu_a=n} \frac{1}{n!} \begin{pmatrix} & n & \\ \nu_1 & \ldots & \nu_a \end{pmatrix} (-\lambda)^{l_1\nu_1+\ldots+l_a\nu_1} E_{l_1}^{\nu_1} \ldots E_{l_a}^{\nu_a} \tag{2.27}$$

where $\begin{pmatrix} & n & \\ \nu_1 & \ldots & \nu_a \end{pmatrix}$ is the multinomial coefficient which satisfies

$$\begin{pmatrix} & n & \\ \nu_1 & \ldots & \nu_a \end{pmatrix} = \frac{n!}{\nu_1!\nu_2!\ldots\nu_a!} \tag{2.28}$$

for $\nu_1+\nu_2+\ldots+\nu_a = n$. This leads to the following expression for the exponent.

$$\exp\left(\sum_{k=1}^{\infty} (-\lambda)^k E_k\right) = \sum_{n=0}^{\infty} \sum_{\nu_1+\ldots+\nu_a=n} (-\lambda)^{\vec{l}\vec{\nu}} \frac{E_{l_1}^{\nu_1} \ldots E_{l_a}^{\nu_a}}{\nu_1! \ldots \nu_a!} \tag{2.29}$$

Every graph in $E_{l_1}^{\nu_1} \dots E_{l_a}^{\nu_a}$ is of order $l_1\nu_1 + \dots + l_a\nu_a$, and the factors in the denominator is the same as the extra symmetry factors in (2.24), hence

$$\exp\left(\sum_{k=1}^{\infty}(-\lambda)^k E_k\right) = \sum(-\lambda)^{l_1\nu_1+\dots+l_a\nu_a} z_{l_1\nu_1+\dots+l_a\nu_a} = \sum_{k=0}^{\infty}(-\lambda)^k z_k = \frac{Z(\lambda)}{Z(0)} \iff \tag{2.30}$$

$$\iff \log\left(\frac{Z(\lambda)}{Z(0)}\right) = \sum_{k=1}^{\infty}(-\lambda)^k E_k \tag{2.31}$$

Thus by taking the logarithm of a generating function, one only considers the connected graphs of each order. Throughout this thesis, log refers to the natural logarithm. [2]

2.4 Counting connected graphs explicitly

We now have all the tools we need to calculate the number of connected graphs with M vertices of degree 4. The Taylor expansion of $\log(1+\lambda)$ around $\lambda = 0$ takes the form of

$$\log(1+\lambda) = -\sum_{n=1}^{\infty}\frac{(-\lambda)^n}{n} \tag{2.32}$$

Calculating the logarithm of $\frac{Z(\lambda)}{Z(0)}$ yields

$$\log\left(\frac{Z(\lambda)}{Z(0)}\right) = \log\left(1+\sum_{k=1}^{\infty}\frac{(-\lambda)^k(4k-1)!!}{k!4!^k}\right) = -\sum_{n=1}^{\infty}\frac{1}{n}\left(-\sum_{k=1}^{\infty}\frac{(-\lambda)^k(4k-1)!!}{k!4!^k}\right)^n = \tag{2.33}$$

$$= -\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sum_{k_1,\dots,k_n=1}^{\infty}(-\lambda)^{k_1+\dots+k_p}\prod_{i=1}^{n}\frac{(4k_i-1)!!}{k_i!4!^{k_i}} = \sum_{n=1}^{\infty}\frac{\lambda^n}{n}\sum_{k_1+\dots+k_P=n}\prod_{i=1}^{P}\frac{(4k_i-1)!!}{k_i!4!^{k_i}} \tag{2.34}$$

Where the sum over $k_1 + k_2 + \dots + k_P$ runs from $P = 1$ to $P = n$ and for all combinations of number $\{k_i\}_{i=1}^{P}$ such that $\sum_{i=1}^{P} k_i = n$. Comparing this to equation (2.22), we get that

$$\sum_{\Gamma\in g_n^c}\frac{1}{S(\Gamma)} = \frac{(-1)^n}{n}\sum_{k_1+k_2+\dots+k_P=n}\prod_{i=1}^{P}\frac{(4k_i-1)!!}{k_i!4!^{k_i}} \tag{2.35}$$

3 Basic colored graphs

So far we have only dealt with one-dimensional integrals. In one dimension, for any coupling, $\langle f_i f_j \rangle = \langle x^2 \rangle = 1$. So any edge can be connected with any other edge to form a graph that contributes to the integral. For multi-dimensional integrals this is not the case, as we will see.

In the application of Feynman diagrams in Quantum Field Theory, different kinds of particles are represented by different kinds of lines. Solid lines represent electrons, wavy lines represent photons etc. These are, of course, not colors but the basic idea of being able to distinguish between lines is the same.

Consider the standard Gaussian measure in $\mathbb{R}^3$

$$d\mu = \exp(-\frac{x^2 + y^2 + z^2}{2})dxdydz = \prod_{u=x,y,z} e^{-u^2/2}du \tag{3.1}$$

We get similar equations for the expectation values

$$\langle x^2 \rangle = \langle y^2 \rangle = \langle z^2 \rangle = 1 \tag{3.2}$$

$$\langle xy \rangle = \langle yz \rangle = \langle zx \rangle = 0 \tag{3.3}$$

In other words, one can only couple an edge representing an x with another edge representing an x and so on. This is graphically represented by coloring the edges of the graph. Arbitrarily choosing the colors, x-green, y-red, z-blue, the monomial $x^m y^n z^k$ correspond to a vertex with m green edges, n red edges and k blue edges. When coupling vertices of this kind, a red edge can only be coupled with another red edge and so on.

By looking at the separated measure in equation (3.1), we see that it is possible to rewrite the expectation value of a three-dimensional monomial as

$$\langle x^m y^n z^k \rangle = \langle x^m \rangle \langle y^n \rangle \langle z^k \rangle \tag{3.4}$$

So there is a clear connection between expectation values of multi-dimensional monomials and one-dimensional monomials. It is possible to count colored graphs by considering all colors individually, and hence be back to the case of non-colored graphs. There are however advantages of introducing colored graphs. One advantage is when we want to consider connected graphs. It is possible for the colored graph to be connected, even though none of the single colors connect the entire graph. Figure 3.1 shows an example of such a graph.

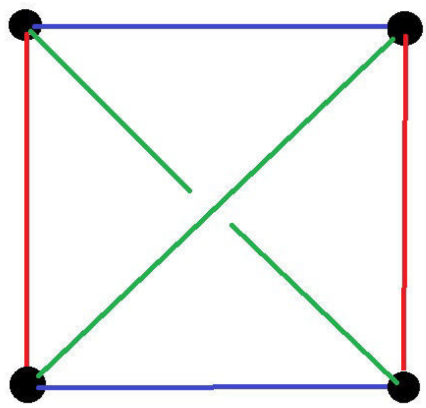

Figure 3.1: A connected colored graph

It is not at all obvious to determine how, if it's even possible, to form a particular connected colored graph given a set of disconnected one-colored graphs. It is much more practical to consider colored graphs right from the start.

Another advantage is that it allows for answers to questions like: *"In how many ways can one color the edges of a graph such that ..."*

Of course, if not all of m,n and k are even

$$\langle x^m y^n z^k \rangle = 0 \tag{3.5}$$

But if they all are even we can rename the powers

$$(m, n, k) \to (2m, 2n, 2k) \tag{3.6}$$

With this new notation, the graphically appropriate expectation value takes the form

$$\langle \frac{x^{2m}}{2m!} \frac{y^{2n}}{2n!} \frac{z^{2k}}{2k!} \rangle = \frac{\langle x^{2m} \rangle}{2m!} \frac{\langle y^{2n} \rangle}{2n!} \frac{\langle z^{2k} \rangle}{2k!} = \frac{(2m-1)!!(2n-1)!!(2k-1)!!}{2m!2n!2k!} \tag{3.7}$$

Calculating this graphically, we choose not to split the monomial into several monomials. We must therefore consider 1 vertex with $2m$ green edges, $2n$ red edges and $2k$ blue edges. The only possible graph, Γ, of such a vertex consists of m green loops, n red loops and k blue loops, hence the symmetry factor Γ is

$$S(\Gamma) = L \cdot E_0 \cdot V = L_m \cdot L_n \cdot L_k \cdot 1 \cdot 1 = 2^{m+n+k} m!n!k! = \frac{2m!2n!2k!}{(2m-1)!!(2n-1)!!(2k-1)!!} \tag{3.8}$$

Which is consistent with equations (3.7) and (2.16).

This model is easily generalized to N colors by considering integrals over the standard Gaussian measure on $\mathbb{R}^N$.

4 Matrix integrals and fat graphs

Before we discuss matrix integrals and fat graphs, the concepts of *maps* and the *Euler Characteristic* are introduced.

4.1 Surfaces, Maps and the Euler Characteristic

Throughout this thesis, all surfaces are assumed to be bounded, orientable and without boundary. The *genus* g of a surface is an integer and equal to the number of holes in the surface. There is only one surface for every non-negative genus g, up to homeomorphism. Figure 4.1 shows surfaces of genus 1,2 and 3.

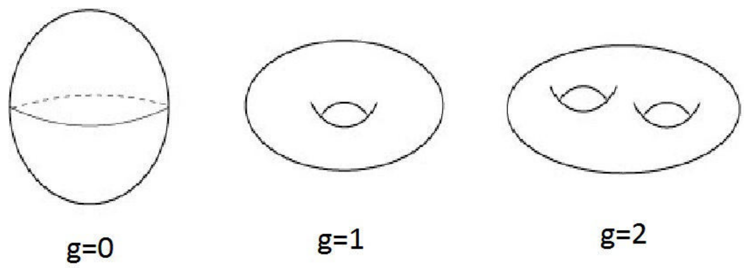

Figure 4.1: Surfaces of genus 1,2 and 3.

A *map* is a graph drawn on one or more surfaces such that [1]

- the edges do not intersect
- all the *faces* of the map are homeomorphic to an open disk

A face of a map is a region of the surface that lies inbetween the edges of the map. A map drawn on a single surface is called *connected*, whereas a *disconnected* map is drawn on several surfaces. Figure 4.2 shows a connected map with 4 vertices of degree 4, drawn on a torus.

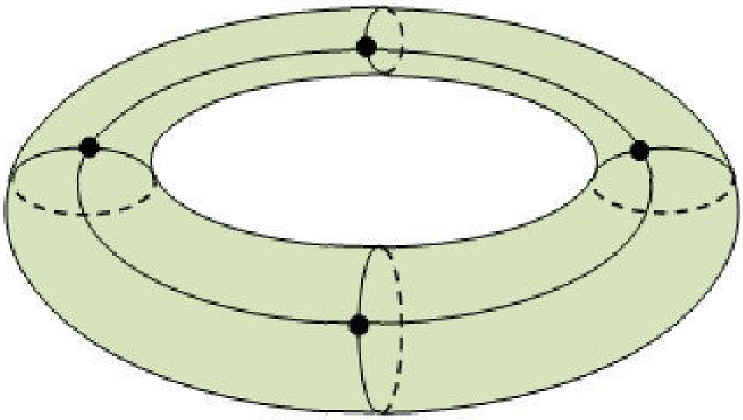

Figure 4.2: A map on a torus

Let V be the number of vertices of a map, F the number of faces and E the number of edges. The maps *Euler Characteristic*, χ, is defined as:

$$\chi = V - E + F \tag{4.1}$$

This χ depends only on the genus g of the surface that the map is drawn on. The dependence satisfies [1]

$$\chi = 2 - 2g \tag{4.2}$$

The map in figure 4.2 has 4 vertices, 8 edges and 4 faces. The Euler characteristic of the map is therefore

$$\chi = 4 - 8 + 4 = 0 \iff g = 1 \tag{4.3}$$

This agrees with equation (4.2), as the torus indeed has genus 1.

When calculating matrix integrals in terms of maps, it is the Euler characteristic that allows us to classify all maps in order of contribution to the integral.

4.2 Integrals over Hermitian matrices

A complex square matrix $M \in \mathbb{C}^{N\times N}$ is called *Hermitian* if its real part is symmetric and its imaginary part is antisymmetric. This is equivalent to requiring that $M^\dagger = M$, where $M^\dagger$ is the Hermitian conjugate of M.

Let $\mathcal{H}_N$ be the space of all $N \times N$ Hermitian matrices, and let $M = X + iY$. The antisymmetric matrix Y is uniquely determined by its strictly upper triangular part. In other words, Y has $\frac{N^2-N}{2}$ free components. X is uniquely determined by its main diagonal and its strictly upper triangular part, and hence has $\frac{N^2+N}{2}$ free components. In total M has N^2 free components, so the space $\mathcal{H}_N$ is N^2-dimensional. We equip $\mathcal{H}_N$ with the (non-standard) Gaussian measure

$$d\mu(M) = \exp\left(-\frac{1}{2}\operatorname{Tr}(M^2)\right)\prod_{m=1}^{N} dx_{mm} \prod_{m<n} dx_{mn}dy_{mn} \tag{4.4}$$

where x_{mn} and y_{mn} are the components of X and Y respectively. For $m < n$, x_{mn} and y_{mn} are in the strictly upper triangular parts of X and Y respectively.

The trace of a Hermitian matrix equals

$$\mathrm{Tr}(M^2) = \sum_{m,n=1}^{N} M_{mn}M_{nm} = \sum_{m,n=1}^{N} |M_{mn}|^2 = \tag{4.5}$$

$$= \sum_{m=1}^{N} x_{mm} + 2\sum_{m<n} x_{mn}^2 + y_{mn}^2 \tag{4.6}$$

Since the space $\mathcal{H}_N$ is of real dimension N^2, we can view the integral over $\mathcal{H}_N$ as an integral over $\mathbb{R}^{N^2}$ Rewriting the Hermitian matrix M as the $\mathbb{R}^{N^2}$-vector $(x_{11}, x_{22}, \ldots, x_{NN}, x_{12}, x_{13}, \ldots, x_{N-1,N})$, the covariance matrix A, takes the diagonal form:

$$A = \mathrm{diag}(1, 1, \ldots, 1, 2, 2, \ldots, 2) \tag{4.7}$$

with N ones and $N^2 - N$ twos. The inverse matrix B follows immediately as

$$B = \mathrm{diag}(1, 1, \ldots, 1, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \tag{4.8}$$

Thus we have from equation (1.11)

$$\langle x_{mm}^2 \rangle = \langle M_{mm}^2 \rangle = 1 \tag{4.9}$$

$$\langle x_{mn}^2 \rangle = \langle y_{mn}^2 \rangle = \frac{1}{2} \tag{4.10}$$

Thus for $m < n$

$$\langle M_{mn}M_{nm} \rangle = \langle x_{mn}^2 + y_{mn}^2 \rangle = \frac{1}{2} + \frac{1}{2} = 1 \tag{4.11}$$

$$\langle M_{mn}M_{mn} \rangle = \langle x_{mn}^2 - y_{mn}^2 + 2ix_{mn}y_{mn} \rangle = \frac{1}{2} - \frac{1}{2} + 0 = 0 \tag{4.12}$$

All other products of two elements of M are also zero as they would involve off-diagonal components of B, which are all zero. So the following formula holds

$$\langle M_{mn}M_{lk} \rangle = \delta_{nl}\delta_{mk} \tag{4.13}$$

where δ_{mn} is the Kronecker delta.

This bares some resemblance to equations (3.2) and (3.3), as a coupling $\langle M_{mn}M_{lk} \rangle$ equals either 1 or 0. The components of the matrix M are linear functions of N^2 real variables and so they satisfy the requirements of the Wick lemma, and equation (4.13) can be used to calculate every term in a Wick coupling. This means that we should be able to calculate integrals over Hermitian matrices graphically.

It is not obvious how to represent monomials in M_{mn} as vertices. We could just rewrite a monomial in M_{mn} as a homogeneous polynomial in the real variables x_{mm}, x_{mn} and y_{mn}, associate N^2 colors to the variables and follow the procedure of the previous sections. This is however not the route we wish to take. To count graphs of N^2 colors, it would be simpler to calculate integrals of monomials over the standard Gaussian measure on $\mathbb{R}^{N^2}$.

We focus on calculating expectation values of traces of powers of M, for reasons that will hopefully be clear by the end of this section. The trace of a power of M is equal to

$$\mathrm{Tr}(M^k) = \sum_{i_1, i_2, \ldots, i_k} M_{i_1 i_2} M_{i_2 i_3} \ldots M_{i_k i_1} \tag{4.14}$$

This is represented graphically as a vertex

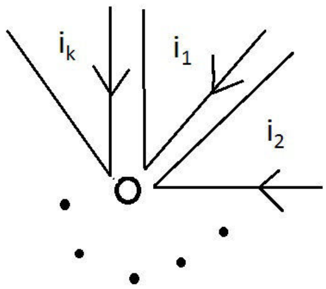

Figure 4.3: Graphical representation of $\mathrm{Tr}(M^k)$

The n:th power of $\mathrm{Tr}(M^k)$, $\mathrm{Tr}(M^k)^n$, is represented by n vertices of the kind shown in figure 4.3. Looking at equation (4.13), we see that it is important that in which order the indicies appear. That is why there are arrows on the edges. An edge with an arrow pointing away from the vertex can only be connected with a vertex with an arrow pointing towards the vertex and vice versa. The edges are coupled pairwise according to

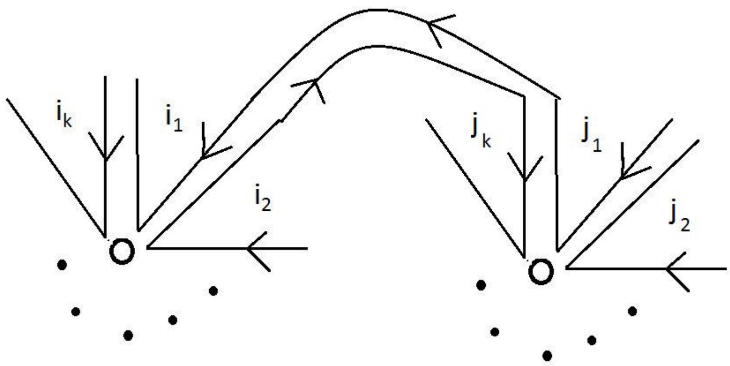

Figure 4.4: Connection of edges corresponding to a coupling

By connecting all edges in this fashion, a *fat graph*[1] is formed.

[1]A fat graph is sometimes called a *ribbon graph*.

The connection in figure 4.4 corresponds to the coupling $\langle M_{i_1 i_2} M_{j_k j_1}\rangle$. From equation (4.13), we see that this coupling is non-zero iff

$$\begin{aligned} i_2 &= j_k \\ i_1 &= j_1 \end{aligned} \tag{4.15}$$

All the contributing Wick couplings of a monomial with $2k$ matrix elements can be found with $2k$ index equalities. For a concrete example of the importance of these index equalities, we calculate 2 Wick couplings of $\langle \mathrm{Tr}(M^4)\rangle$.

$$\langle \mathrm{Tr}(M^4)\rangle = \sum_{i,j,k,l} \langle M_{ij} M_{jk} M_{kl} M_{li}\rangle = \sum_{\text{ind.}} \sum_{\text{coupl.}} \langle M_{p_1} M_{q_1}\rangle\langle M_{p_2} M_{q_2}\rangle \tag{4.16}$$

Where the indicies p_i, q_i have 2 components, the sum over ind. runs over all indicies and the sum over coupl. runs over all unique couplings. Consider first the coupling

$$\langle M_{ij} M_{jk}\rangle\langle M_{kl} M_{li}\rangle \tag{4.17}$$

For this product to be non-zero equation (4.13) gives the index equalities

$$\begin{aligned} i &= k \\ j &= j \\ i &= k \\ l &= l \end{aligned} \tag{4.18}$$

The only restriction from (4.18) is the requirement that $i = k$. When summed over the indicies

$$\sum_{\text{ind.}} \langle M_{ij} M_{jk}\rangle\langle M_{kl} M_{li}\rangle = \sum_{\text{ind.}} \delta_{ik} = \sum_{i,j,k=1}^{N} 1 = N^3 \tag{4.19}$$

N^3 is called the *contribution* of the Wick coupling.

Now consider the second Wick coupling

$$\langle M_{ij} M_{kl}\rangle\langle M_{jk} M_{li}\rangle \tag{4.20}$$

which leads to the index equalities

$$\begin{aligned} i &= l \\ j &= k \\ j &= i \\ k &= l \end{aligned} \tag{4.21}$$

This is a much stronger restriction, namely $i = j = k = l$. This Wick couplings' contribution is N. Every Wick coupling of matrices leaves a number of free indicies. The former Wick coupling leaves 3 free indicies, the latter only 1. Call this number of free indicies F, so that the contribution of a Wick coupling equals N^F

A trace of particular importance in this thesis is

$$\mathrm{Tr}(M^4)^k = \sum_{\text{ind.}} M_{i_{1_1} i_{1_2}} M_{i_{1_2} i_{1_3}} M_{i_{1_3} i_{1_4}} M_{i_{1_4} i_{1_1}} \dots M_{i_{k_4} i_{k_1}} \tag{4.22}$$

which graphically is represented as

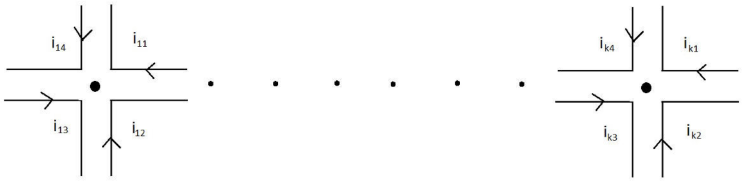

Figure 4.5: Vertices from $\mathrm{Tr}(M^4)^k$

The fat graphs from powers of $\mathrm{Tr}(M^4)$ are the simplest non-trivial fat graphs to study, and a large part of this thesis is dedicated to them.

$$\begin{aligned}\langle \mathrm{Tr}(M^4)^k\rangle &= \sum_{\text{ind.}} \langle M_{i_{1_1} i_{1_2}} M_{i_{1_2} i_{1_3}} M_{i_{1_3} i_{1_4}} M_{i_{1_4} i_{1_1}} \dots M_{i_{k_4} i_{k_1}}\rangle = \\ &= \sum_{\text{ind.}} \sum_{\text{couplings}} \langle M_{p_1} M_{q_1}\rangle \dots \langle M_{p_{2k}} M_{q_{2k}}\rangle = \sum_{\text{maps}} N^F\end{aligned} \tag{4.23}$$

Note that the same map may appear many times in the sum in equation (4.23). Graphically, a free index corresponds to a closed path in the fat graph. When drawn on a surface, these closed paths all enclose a region of the surface. We have already introduced the term *faces* for these regions, hence the contribution of a *map* equals N^F, where F is the number of faces of the map. We can now calculate the contribution of a graph in terms of its Euler characteristic. Let G be a fat graph formed by k vertices of degree 4. Before the coupling, every vertex has 4 edges, so there are $4k$ in total. After the coupling, where we always connect 2 edges to each other, the number of edges reduces to $E = \frac{4k}{2} = 2k$. From equations (4.1) and (4.2) we get that

$$\chi = 2 - 2h = V - E + F = F - k \tag{4.24}$$

where h is the genus of the surface. Let's keep the $-k$ in the left-hand side for now. Equation (4.24) lets us express the contribution of a map in terms of the genus of the surface it was drawn on.

We will now attempt to count connected maps from k vertices of degree 4. We start in the same way as for graphs, by defining a generating function in the form of an integral.

$$\frac{Z(g)}{Z(0)} = \frac{1}{Z(0)} \int_{\mathcal{H}_N} \exp\left(-\frac{1}{2}\mathrm{Tr}(M^2) - \frac{g}{N}\mathrm{Tr}(M^4)\right) dM = \sum_{k=0}^{\infty} \frac{(-g)^k}{N^k k!} \langle \mathrm{Tr}(M^4)^k\rangle \tag{4.25}$$

From here on the calculations differ from the ones already performed for graphs. There has been no discussion about the symmetry factors of these fat graphs, and that is for of a reason. The symmetry factor of graphs was a way to determine how much each of the graphs contribute to the integral. For maps the contribution is determined by the faces of the map, or the genus, h, of the surface it was drawn on. Using equations (4.23) and (4.24), we can express the generating function as

$$\frac{Z(g)}{Z(0)} = \sum_{k=0}^{\infty} \frac{(-g)^k}{N^k k!} \sum_{\text{maps}} N^F = \sum_{k=0}^{\infty} \frac{(-g)^k}{k!} \sum_{\text{maps}} N^{2-2h} \iff \tag{4.26}$$

where h is the genus of the map. The contribution of the maps is now solely determined by their genus. It is clear that the maps that contribute the most to the integrals are the ones drawn on a sphere, the surface of genus 0. Approximating a matrix integral by only considering the maps of genus 0 is called the *planar approximation.*[2] Because the contribution of a map is determined by its genus, not the number of vertices, it is convenient to rewrite equation (4.26) from a sum over vertices to a sum over genera.[3] Let G_h be the set of all maps of genus h, with vertices of degree 4, and let $\rho(G)$ be the number of times the map G arises from the couplings. The rewritten generating function takes the form

$$\frac{Z(g)}{Z(0)} = \sum_{h=0}^{\infty} \sum_{G \in G_h} N^{2-2h} \rho(G) \frac{(-g)^{k(G)}}{k(G)!} \tag{4.27}$$

Let G_h^c be the set of connected maps of genus h. By the logarithm property we get

$$\frac{1}{N^2} \log\left(\frac{Z(g)}{Z(0)}\right) = \sum_{h=0}^{\infty} N^{-2h} \underbrace{\sum_{\Gamma \in G_h^c} \rho(\Gamma) \frac{(-g)^{k(\Gamma)}}{k(\Gamma)!}}_{e_h(g)} = \sum_{h=0}^{\infty} e_h(g) N^{-2h} \tag{4.28}$$

A simple consequence of equation (4.28) is

$$\lim_{N \to \infty} \frac{1}{N^2} \log\left(\frac{Z(g)}{Z(0)}\right) = e_0(g) \tag{4.29}$$

$e_0(g)$ is the planar graph approximation. Later we will use equation (4.29) to determine $e_0(g)$ explicitly.

4.3 $\mathcal{U}_N$-invariant integrals

There is a way to simplify some integrals over $\mathcal{H}_N$ by only explicitly integrating over the matrices' eigenvalues $\{\lambda_i\}_{i=1}^N$. This reduces the integral over the N^2-dimensional space $\mathcal{H}_N$ to an integral over $\mathbb{R}^N$. There are however restrictions on what measures this approach works for.

By the spectral theorem every Hermitian matrix is unitarily diagonalizable. In other words, for a matrix $M \in \mathcal{H}_N$, there is a diagonal matrix Λ, and a unitary matrix $U \in \mathcal{U}_N$ such that

$$M = U^\dagger \Lambda U \tag{4.30}$$

[2]The name comes from the fact that a graph that can be drawn on a sphere can also be drawn on a plane.

[3]Genera is the plural of genus.

where $U^\dagger$ is the Hemritian conjugate of U. Furthermore since the eigenvalues of a Hermitian matrix are real, Λ is a real diagonal matrix. There are some degrees of freedom in choosing the matrices U and Λ. One can position the eiqenvalues on the diagonal of Λ in $N!$ ways. Any eigenvector can also be multiplied by an arbitrary factor $e^{i\phi} \in \mathcal{U}_1$.

The restriction on the measure is that it needs to be *unitary invariant*. A function $f : \mathbb{C}^{N\times N} \to \mathbb{C}$ is called unitary invariant if for any $U \in \mathcal{U}_N$

$$f(U^\dagger M U) = f(M) \tag{4.31}$$

Since the trace operator is invariant under cyclic permutations, $\mathrm{Tr}(M^k)$ is unitary invariant for all k.

The (non-Gaussian) measure

$$d\mu(M) = \exp\left(-\frac{1}{2}\,\mathrm{Tr}(M^2) - \sum_{p=2}^{M} \bar{g}_p(\mathrm{Tr}(M^{2p}))\right) dM \tag{4.32}$$

where $\bar{g}_p = \frac{g_p}{N^{p-1}}$, is also unitary invariant, as it consists of a sum of traces. So let Ω_N be the volume of the unitary group $\mathcal{U}_N$. We can now simplify the integral of (4.32) over $\mathcal{H}_N$ to an integral over $\mathbb{R}^N$.

$$\int_{\mathcal{H}_N} d\mu(M) = \frac{\Omega_N}{N!(2\pi)^N} \int_{\mathbb{R}^N} \Delta(\lambda)^2 \exp\left(-\frac{1}{2}\sum_{i=1}^{N} \lambda_i^2 - \sum_{p=2}^{P} \bar{g}_p \sum_{i=1}^{N} \lambda_i^{2p}\right) d^N\lambda \tag{4.33}$$

The factor $N!(2\pi)^N$ comes from the ambiguity of choosing a diagonalization of M. $\Delta(\lambda)$, the Jacobian for this change of variables, is the *Vandermonde determinant*, which is defined as

$$\Delta(\lambda) = \begin{vmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_N \\ \lambda_1^2 & \lambda_2^2 & & \lambda_N^2 \\ \vdots & & \ddots & \vdots \\ \lambda_1^{N-1} & \lambda_2^{N-1} & \dots & \lambda_N^{N-1} \end{vmatrix} \tag{4.34}$$

Define, with $d\mu(M)$ as in equation (4.32), the function $Z(\bar{g})$ and the associated $\bar{Z}(\bar{g})$ as

$$Z(\bar{g}) = \int_{\mathcal{H}_N} d\mu(M) = \frac{\Omega_N}{(2\pi)^N N!} \int_{R^N} \exp\left(-\frac{1}{2}\sum_{k=1}^{N} \lambda_k^2 - \sum_{p=2}^{P} \bar{g}_p \sum_{i=1}^{N} \lambda_i^{2p}\right) \Delta(\lambda)^2 d^N\lambda = \frac{\Omega_N}{(2\pi)^N N!} \bar{Z}(g) \tag{4.35}$$

The integral $Z(0)$ is also separable into a unitary part and an eigenvalue part. So the generating function $\frac{Z(\bar{g})}{Z(0)}$ takes the form

$$\frac{Z(g)}{Z(0)} = \frac{\frac{\Omega_N}{(2\pi)^N N!}\bar{Z}(\bar{g})}{\frac{\Omega_N}{(2\pi)^N N!}\bar{Z}(0)} = \frac{\bar{Z}(g)}{\bar{Z}(0)} \tag{4.36}$$

5 Orthogonal polynomials

There are several ways of calculating and approximating the generating function (4.36). One approach, which in principal could calculate (4.36) exactly, is to use *orthogonal polynomials*. This section contains a comprehensible overview of orthogonal polynomials and their application without much detail or justification. In appendix C, a more rigorous treatment of orthogonal polynomials is given.

Consider the measure

$$d\nu(\lambda) = \exp\left(-\frac{1}{2}\sum_{k=1}^{\infty}\lambda_k^2 - \sum_{p=2}^{P}\bar{g}_p\sum_{i=1}\lambda_i^{2p}\right)d^N\lambda \tag{5.1}$$

on $\mathbb{R}^N$, as it appears in (4.35). It is separable into N real components. We want to find orthogonal polynomials of one variable, so we shall find the orthogonal polynomials with respect to one of the components of the measure (5.1). This measure takes the form

$$d\mu(\lambda) = \exp(-\frac{1}{2}\lambda^2 - \sum_{p=2}^{P}\bar{g}_p\lambda^{2p})d\lambda = e^{-V(\lambda)}d\lambda \tag{5.2}$$

The expectation value of a function $f : \mathbb{R} \to \mathbb{R}$ with respect to (5.2) is defined as

$$\langle f(\lambda)\rangle = \int_{\mathbb{R}} f(\lambda)d\mu(\lambda) \tag{5.3}$$

Note that we do not normalize the measure $d\mu(\lambda)$ here. Let two polynomials $P_n(\lambda)$ and $P_m(\lambda)$, be orthogonal with respect to the measure (5.2), and let h_n be the norm of $P_n(\lambda)$. By definition, they must satisfy

$$\langle P_n(\lambda)P_m(\lambda)\rangle = h_n(\bar{g})\delta_{mn} \tag{5.4}$$

where δ_{nm} is the Kronecker delta. The measure (5.2) depends on a number of g_p:s, hence the expectation value of a function of λ also depends on the g_p:s. In particular the coefficients of the polynomial $P_n(\lambda)$ and its norm h_n, depend on the g_p:s.

The polynomials satisfy the the well-known recursion formula [2]

$$\lambda P_n(\lambda) = P_{n+1}(\lambda) - A_nP_n(\lambda) + R_nP_{n-1}(\lambda) \tag{5.5}$$

The coefficient R_n also depends on the $\bar{g}$:s in the measure. For an even measure $d\mu$, $P_{2k}(\lambda)$ is even and $P_{2k+1}(\lambda)$ is odd. By the parity of equation (5.5) it is seen that $A_k = 0$ for an even measure. There is a known relation between the norms h_n and the coefficient R_n from equation (5.5)

$$h_n = R_nh_{n-1} \tag{5.6}$$

Equation (5.6) will prove to be very important for the integral in (4.35). From partial integration of $\lambda P_n'(\lambda)P_n(\lambda)$, we get

$$nh_n = R_n\int_{\mathbb{R}} d\mu V'(\lambda)P_n(\lambda)P_{n-1}(\lambda) \tag{5.7}$$

Where by construction

$$V'(\lambda) = \lambda + \sum_{p=1}^{\infty} 2(p+1)\bar{g}_{p+1}\lambda^{2p+1} \tag{5.8}$$

All the formulas are derived in appendix C. Equations (5.2) to (5.8) contain all the information we need to calculate the generating function (4.35) in terms of h_0 and the R_n factors. Iterating equation (5.6) gives

$$h_n = R_n h_{n-1} = R_n R_{n-1} h_{n-2} = \ldots = R_n R_{n-1} \ldots R_1 h_0 \tag{5.9}$$

The Vandermonde determinant, $\Delta(\lambda)$, is easily expressed in terms of orthogonal polynomials according to,

$$\Delta(\lambda) = \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \cdots & \cdots & \lambda_2^{N-1} \\ \vdots & & \ddots & & \\ 1 & \cdots & & \cdots & \lambda_N^{N-1} \end{vmatrix} = \begin{vmatrix} P_0(\lambda_1) & P_1(\lambda_1) & P_2(\lambda_2) & \cdots & P_{N-1}(\lambda_1) \\ P_0(\lambda_2) & \ddots & & & \\ \vdots & & \ddots & & \\ P_0(\lambda_N) & & & & P_{N-1}(\lambda_N) \end{vmatrix} \tag{5.10}$$

The equality is again derived in appendix C. We can now use the explicit formula for the determinant to calculate (4.35).

$$\bar{Z}(\bar{g}) = N! h_0(\bar{g}) \ldots h_{N-1}(\bar{g}) = N! h_0(\bar{g})^N \prod_{k=1}^{N} R_k^{N-k}(\bar{g}) \tag{5.11}$$

Where equation (5.9) was used for the second equality. Equation (5.11) is the generating function (4.35) expressed only in terms h_0 and the R_n:s. The generating function for connected maps can be expressed as [2]

$$\frac{1}{N^2}\log\left(\frac{Z(\bar{g})}{Z(0)}\right) = \frac{1}{N}\log\left(\frac{h_0(\bar{g})}{h_0(0)}\right) + \frac{1}{N}\sum_{k=1}^{N}(1-k/N)\log\left(\frac{R_k(\bar{g})}{R_k(0)}\right) \tag{5.12}$$

For the planar graph approximation, we take the large-N limit of (5.12).

Now consider the integral (5.7). $V'(\lambda)$ is a polynomial in λ. Remembering the recursion formula (5.5), we can view a λ^p-term as an operator taking $P_{n-1}(\lambda)$ to $P_n(\lambda)$. This way, the integral becomes easy to visualize. Take $P_s(\lambda)$ as the s:th step in a stair, and $\lambda P_s(\lambda)$ as moving up one step, staying put and moving down one step. Each of these options is represented by 1 term in (5.5). So λ^p corresponds to all paths with p moves. A path starting at the $n-1$:th step must end at the n:th step, or else the integral (5.7) is zero. The moves contribute to the path according to

- Moving up from step k: factor 1
- Staying put on step k: factor $-A_k$ (=0 for an even measure)
- Moving down from step k: factor R_k

Thus the integral becomes a sum of all paths like: [2]

$$\int_{\mathbb{R}} P_n(\lambda)(\lambda^p P_{n-1}(\lambda))d\mu(\lambda) = h_n \sum_{\text{paths}} R_a \dots R_b(-A_c)\dots(-A_d) = h_n \alpha_n^p \tag{5.13}$$

Where the sum runs over all paths of p moves from step $n-1$ to step n.

For an even measure, there are some simplifications of this model. Firstly, as $V'(\lambda)$ is odd, all paths consist of an odd number of moves. Furthermore $A_k = 0$, so all moves must either be a move up or a move down. Inserting these constraints in equation (5.13) and inserting it into equation (5.7), we get the very useful equation [2]

$$n = R_n(g)\left(1 + \sum_{p\geq 1} 2(p+1)\bar{g}_{p+1} \sum_{paths} R_{s_1}\dots R_{s_p}\right) \tag{5.14}$$

In a path from step $n-1$ to step n in $2p+1$ moves, without standing still, there must be $p+1$ moves up and p moves down. Hence there are $\binom{2p+1}{p}$ such paths. A simple consequence of equation (5.14) is that for $g_p \equiv 0$

$$R_n(0) = n \tag{5.15}$$

6 Finding the planar graph approximation

In order to calculate the large-N limit of (5.12), we need to find an approximation of $R_k(\bar{g})$ which is continuous in $x = \frac{k}{N}$. It is shown in appendix B that

$$\frac{R_k(g/N)}{N} = r_0(k/N) + \mathcal{O}(N^{-2}) \tag{6.1}$$

where $r_0(x)$ is defined as the non-negative solution to

$$x = r_0(x)\left(1 + \sum_{p\geq 1} g_{p+1}\frac{(2p+2)!}{p!(p+1)!}r_0(x)^{p-1}\right) \tag{6.2}$$

for $x \in [0, 1]$.

We now have everything we need to calculate the planar graph approximation for an arbitrary number of vertices of even degree. We note first that, with $x = \frac{k}{N}$

$$\frac{R_k(\bar{g})}{R_k(0)} = \frac{R_k(\bar{g})/N}{k/N} = \frac{r_0(x)}{x} + \mathcal{O}(N^{-1}) \implies \log\left(\frac{R_k(\bar{g})}{R_k(0)}\right) = \log\left(\frac{r_0(x)}{x}\right) + \mathcal{O}(N^{-1}) \tag{6.3}$$

Hence, the large-N limit of (5.12) can be calculated as

$$e_0(g) = \lim_{N\to\infty}\left(\frac{1}{N}\log\frac{h_0(g)}{h_0(0)} + \frac{1}{N}\sum_{k=1}^{N}(1-k/N)\left(\frac{R_k(g)}{R_k(0)}\right) = \right. \tag{6.4}$$

$$= \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}(1-k/N)\log\left(\frac{r_0(k/N)}{k/N}\right) + \mathcal{O}(N^{-1}) = \int_0^1 (1-x)\log(r_0(x)/x)dx \tag{6.5}$$

Dealing with $r_0(x)$ in the integral is rather unpractical, since we have no explicit formula for $r_0(x)$. Equation (6.2) gives a simple expression for $x(r_0)$. Let $a^2 = r_0(1)$, and change the variable of integration to r_0. After partial integration we get

$$e_0(g) = \frac{1}{2}\log(a^2) + \frac{1}{2}\int_0^{a^2} r_0\bar{x}'(r_0)(2 - r_0\bar{x}(r_0))dr_0 \tag{6.6}$$

where $\bar{x}(r_0) = \frac{x(r_0)}{r_0}$. Since $\bar{x}(r_0)$ is a polynomial in r_0 the integral in (6.6) is easily calculated. For a polynomial of high degree however, the equation $x(a^2) = 1$ becomes difficult or impossible to solve exactly. By limiting ourselves to vertices of low degrees, it is possible to get an explicit expression for (6.6).

For the simplest non-trivial case of only considering vertices of order 4, we can calculate (6.6) explicitly. Equation (6.2) reduces to a quadratic polynomial polynomial in r_0. Dropping the subscript on g_2, we get

$$\bar{x}(r_0) = 1 + 12gr_0 \implies 1 = a^2 + 12ga^4 \iff a^2 = \frac{-1+\sqrt{1+48g}}{24g} \tag{6.7}$$

Which results in the planar graph approximation

$$e_0(g) = \frac{1}{2}\log(a^2) + \frac{1}{24}(1-a^2)(9-a^2) = -2g + 18g^2 + \ldots \tag{6.8}$$

7 Counting graphs for higher genera

This section is again limited to the case where all vertices are of degree 4. In principle $e_k(g)$ can be calculated for any $k \in \mathbb{N}$ with the method presented here. Since the calculations get very complicated for larger values of k, we only perform calculations for $k = 0, 1$.

With all vertices being of degree 4, the step-stair process becomes very simple. Every path starts from step $n-1$, and after 3 moves, ends at step n. There are only 3 such paths, with down-moves on steps $n-1, n$ and $n+1$. Hence equation (5.14) reads

$$\begin{aligned} n = R_n(g/N)(1 + 4\frac{g}{N}(R_{n-1}(g/N) + R_n(g/N) + R_{n+1}(g/N))) \iff \\ \iff \frac{n}{N} = \frac{R_n(g/N)}{N}(1 + 4g\frac{R_{n-1}(g/N) + R_n(g/N) + R_{n+1}(g/N)}{N}) \end{aligned} \tag{7.1}$$

Let $\bar{R}_k(g/N) = \frac{R_k(g/N)}{k}$, $x = \frac{n}{N}$, $\varepsilon = \frac{1}{N}$ and define the analytic function

$$r(x) = x\bar{R}_n(g/N) = \frac{R_n(g/N)}{N} \tag{7.2}$$

Equation (7.1) can now be rewritten as [2]

$$x = r(x) + 4gr(x)(r(x-\varepsilon) + r(x) + r(x+\varepsilon)) \tag{7.3}$$

Equation (7.3) shows that r is even in ε, and hence attains an expansion in the even powers of ε.

$$r(x) = \sum_{k=0}^{\infty} r_{2k}(x)\varepsilon^{2k} \tag{7.4}$$

Using equation (7.4), we can Taylor expand $r(x+\varepsilon) + r(x-\varepsilon)$ around $\varepsilon = 0$.

$$r(x+\varepsilon) + r(x-\varepsilon) = 2\sum_{n=0}^{\infty} \frac{r^{(2n)}(x)}{(2n)!}\varepsilon^{2n} = 2\sum_{n,k=0}^{\infty} \varepsilon^{2n+2k}\frac{r_{2k}^{(2n)}(x)}{(2n)!} = \tag{7.5}$$

$$= 2\sum_{N=0}^{\infty} \varepsilon^{2N} \sum_{k+p=N} \left(\frac{r_{2k}^{(2n)}(x)}{(2n)!}\right) \tag{7.6}$$

So (7.3) can be expanded in even powers of ε according to

$$x = \sum_{k=0}^{\infty} r_{2k}(x)\varepsilon^{2k} + 4g\sum_{k=0}^{\infty} r_{2k}(x)\varepsilon^{2k}\left(\sum_{k=0}^{\infty} r_{2k}(x)\varepsilon^{2k} + 2\sum_{N=0}^{\infty}\varepsilon^{2N}\sum_{k+p=N}\left(\frac{r_{2k}^{(2n)}}{(2n)!}\right)\right) \tag{7.7}$$

where the superscripts $(2n)$ is the $2n$:th derivative with respect to x.

Taking out the component of ε^{2m} we get

$$x\delta_{m0} = r_{2m} + 4g\left(\sum_{n_1+n_2=m} r_{2n_1}r_{2n_2} + 2\sum_{n_1+n_2=m} r_{2n_1}\sum_{k+p=n_2}\frac{r_{2k}^{2p}}{(2p)!}\right) \tag{7.8}$$

The first 2 components of this becomes, with ($\alpha = 1 + 48gx$):

$$x = r_0 + 4g(r_0^2 + 2r_0^2) = r_0 + 12gr_0^2 \iff r_0 = \frac{-1+\sqrt{\alpha}}{24g} \tag{7.9}$$

$$0 = r_2(1 + 24gr_0) + 4gr_0 r_0^{(2)} \iff r_2 = -\frac{4gr_0 r_0^{(2)}}{\alpha^{1/2}} \tag{7.10}$$

$$\tag{7.11}$$

The derivatives of $r_0(x)$ up to second order are

$$r_0^{(1)} = \alpha^{-1/2} \tag{7.12}$$

$$r_0^{(2)} = -\frac{1}{2}\alpha^{-3/2}\alpha' = -24g\alpha^{-3/2} \tag{7.13}$$

This gives the expression for $r_2(x)$

$$r_2(x) = -\frac{4g}{\alpha^{1/2}} r_0(x)(-24g\alpha^{-3/2}) = 4g(24g)\frac{r_0(x)}{\alpha^2}$$

The $r_0(x)$ is kept in the formula which will be practical in the logarithmic expansion of $\bar{R}_k(g/N)$. An explicit formula for $r_4(x)$ is derived in appendix D.

We now return to the exact equation (5.12). We have already introduced the quantity $\bar{R}_k(\bar{g}) = \frac{R_k(\bar{g})}{k} = \frac{R_k(\bar{g})}{R_k(0)}$. Remembering that $\varepsilon = \frac{1}{N}$, we can expand

$$\frac{R_k(\bar{g})}{R_k(0)} = \frac{r(x)}{x} = \frac{r_0(x)}{x} + \frac{1}{N^2}\frac{r_2(x)}{x} + \mathcal{O}(\frac{1}{N^4}) \tag{7.14}$$

so the expansion of $\log\left(\frac{R_k(\bar{g})}{R_k(0)}\right)$ up to N^{-4} is

$$\log\left(\frac{r_0(x)}{x} + \frac{1}{N^2}\frac{r_2(x)}{x} + \mathcal{O}(N^{-3})\right) = \log\left(\frac{r_0(x)}{x}\left[1 + \frac{1}{N^2}\frac{r_2(x)}{r_0(x)} + \mathcal{O}(N^{-4})\right]\right) = \tag{7.15}$$

$$= \log(r_0(x)/x) + \log(1 + \frac{1}{N^2}\frac{r_2(x)}{r_0(x)} + \mathcal{O}(N^{-4}) = \tag{7.16}$$

$$= \log(r_0(x)/x) + \frac{1}{N^2}\frac{r_2(x)}{r_0(x)} + \mathcal{O}(N^{-4}) \tag{7.17}$$

In order to expand equation (5.12) in powers of $\frac{1}{N}$ we need to make use of the *Euler-Maclaurin formula*. It states, for all the relevant powers in this case, that a function $f : [0,1] \to \mathbb{R}$ which is 6 times continuously differnetiable satisfies

$$\frac{1}{N}\sum_{k=1}^{N} f(k/N) = \int_0^1 f(x)dx + \frac{1}{2N}f(x)|_0^1 + \frac{1}{2!6}\frac{1}{N^2}f^{(1)}(x)|_0^1 + \mathcal{O}(N^{-4}) \tag{7.18}$$

where $f^{(k)}|_0^1 = f^{(k)}(1) - f^{(k)}(0)$. So expanding $f(k/N) = (1-k/N)\log(\bar{R}_k(g/N))$ as

$$f(x) = (1-x)\left[\log\left(\frac{r_0(x)}{x}\right) + \frac{1}{N^2}\left(\frac{r_2(x)}{r_0(x)}\right)\right] + \mathcal{O}(N^{-4}) \tag{7.19}$$

The Euler-Maclaurin formula can be used to calculate the generating function up to N^{-4}. Let $\alpha_1 = 1 + 48g$, $\omega = 24g$ and $a^2 = \frac{-1+\sqrt{1+48g}}{24g}$. Calculating this term by term, up to N^{-4} gives

$$\frac{1}{2N} f(x)|_0^1 = -\frac{1}{2N} f(0) = -\frac{\omega^2}{12N^3} \tag{7.20}$$

$$\frac{1}{12N^2} f^{(1)}(x)|_0^1 = \frac{1}{12N^2}\left(\frac{\omega}{2} - \log\left(\frac{2}{1+\sqrt{\alpha_1}}\right)\right) \tag{7.21}$$

$$\frac{1}{N^2}\int_0^1 (1-x)\frac{r_2(x)}{r_0(x)}dx = \frac{1}{24N^2}(\alpha_1 - 1 - \log(\alpha_1)) \tag{7.22}$$

$$\int_0^1 (1-x)\log\left(\frac{r_0(x)}{x}\right)dx = \frac{1}{2}\log(a^2) + \frac{3}{8} + \frac{1}{2\omega} + \frac{1}{12\omega^2} - \frac{\sqrt{\alpha}_1}{12\omega^2}(1+5\omega) \tag{7.23}$$

Combine these terms now to conclude

$$\begin{aligned} \frac{1}{N}\sum_{k=1}^{N}(1-k/N)\log(\bar{R}_k(g/N)) &= \frac{1}{2}\log(a^2) + \frac{3}{8} + \frac{1}{2\omega} + \frac{1}{12\omega^2} - \\ -\frac{\sqrt{\alpha}_1}{12\omega^2}(1+5\omega) + \frac{1}{12N^2}&\left[\frac{3}{2}\omega - \log\left(2-a^2\right)\right] - \frac{\omega^2}{12N^3} + \mathcal{O}(N^{-4}) \end{aligned} \tag{7.24}$$

The last term might appear troublesome, since our prediction is that the generating function can be expanded in even powers of $\frac{1}{N}$. There is however a term in (5.12) which we have yet to calculate. As it turns out, the terms of odd power in $\frac{1}{N}$ cancel out. The remaining term is

$$\frac{1}{N}\log\left(\frac{h_0(g/N)}{h_0(0)}\right) \tag{7.25}$$

$h_0(g/N)$ is the norm of the orthogonal polynomial of degree 0, $P_0(\lambda) \equiv 1$. The fraction $\frac{h_0(g/N)}{h_0(0)}$ is easily calculated using equation (2.5).

$$\frac{h_0(g/N)}{h_0(0)} = \sum_{k=0}^{\infty}\left(-\frac{g}{N}\right)^k \frac{(4k-1)!!}{k!} = 1 - 3\left(\frac{g}{N}\right) + \frac{105}{2}\left(\frac{g}{N}\right)^2 + \mathcal{O}(N^{-3}) \tag{7.26}$$

Taylor expanding up to N^{-4}, the formula now reads

$$\begin{aligned} \log\left(1 - 3\left(\frac{g}{N}\right) + \frac{105}{2}\left(\frac{g}{N}\right)^2\right) &= -3\left(\frac{g}{N}\right) + 48\left(\frac{g}{N}\right)^2 \iff \\ \iff \frac{1}{N}\log\left(\frac{h_0(g/N)}{h_0(0)}\right) &= -\frac{\omega}{8N^2} + \frac{\omega^2}{12N^3} + \mathcal{O}(N^{-4}) \end{aligned} \tag{7.27}$$

Adding (7.27) and (7.24) gives the expressions

$$e_0(g) = \frac{1}{2}\log\left(\frac{-1+\sqrt{\alpha_1}}{\omega}\right) + \frac{3}{8} + \frac{1}{2\omega} + \frac{1}{12\omega^2} - \frac{(1+5\omega)\sqrt{\alpha_1}}{12\omega^2} = -2g + 18g^2 - \dots \tag{7.28}$$

$$e_1(g) = -\frac{1}{12}\log\left(\frac{2\omega + 1 - \sqrt{\alpha_1}}{\omega}\right) = -g + 30g^2 - \dots \tag{7.29}$$

An explicit calculation of $e_2(g)$ is given in [2].

8 Calculating the number of basic colored fat graphs

8.1 Graphically

This section deals with fat graphs with colored edges. Since colored graphs are counted using integrals over $\mathbb{R}^N$ and fat graphs are counted using integrals over $\mathcal{H}_N$, it is straight-forward to employ the previous techniques and count colored fat graphs with integrals over k copies of $\mathcal{H}_N$, $\mathcal{H}_N^k$. The simplest of the colored fat graphs contains 2 colors and vertices of degree 2. As we will see, these graphs can be counted exactly.

Analogously with the previous sections, we represent $\mathrm{Tr}(AB)$ as the vertex

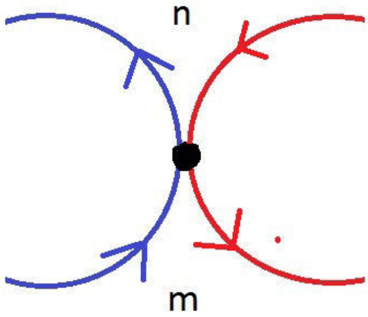

Figure 8.1: Vertex representation of $\mathrm{Tr}(AB)$

And as before, only edges of the same color may be connected and the arrows of 2 connected edges must point in the same direction. For an even number of vertices, there is a unique connected graph. A section of it is shown in figure 8.2.

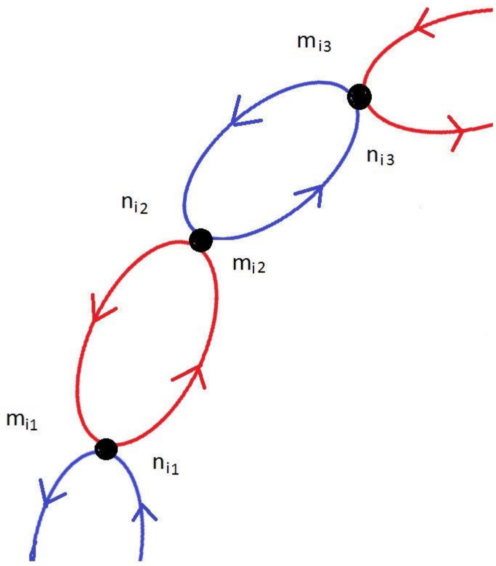

Figure 8.2: The only connected graph of order k.

There is no way to add another vertex to the graph in figure 8.2, keeping in mind the coloring rules. Hence there are no allowed graphs for an odd number of vertices. The graph 8.2 leads to the index equalities:

$$m_1 = n_2 = \ldots = n_{2k} \tag{8.1}$$

$$n_1 = m_2 = \ldots = m_{2k} \tag{8.2}$$

So its contribution is N^2. There are $(2k-1)!$ ways to couple the edges into this graph, hence we get the equality

$$\langle \mathrm{Tr}(AB)^{2k} \rangle_c = (2k-1)! N^2 \tag{8.3}$$

8.2 By integral perturbation

Following the techniques of the previous sections, we define the generating function

$$\frac{Z(g)}{Z(0)} = \frac{1}{Z(0)} \int_{\mathcal{H}_N^2} \exp\left(-\,\mathrm{Tr}(\frac{A^2}{2} + \frac{B^2}{2} + gAB) \right) d\nu(A) d\nu(B) = \tag{8.4}$$

$$= \frac{1}{Z(0)} \int_{\mathcal{H}_N^2} \exp\left(-\,\mathrm{Tr}(\frac{A^2}{2} + \frac{1}{2}(B + gA)^2) - \frac{1}{2} g^2 A^2 \right) d\nu(A) d\nu(B) = \tag{8.5}$$

$$= \frac{1}{Z(0)} \int_{\mathcal{H}_N} \exp\left(-\frac{1}{2}(1 - g^2)\,\mathrm{Tr}(A^2) \right) d\nu(A) = \frac{1}{\sqrt{(1-g^2)^N}} \tag{8.6}$$

Where equation (1.2) was used to obtain the last equality. Taking the logarithm to only consider connected graphs we get

$$\frac{Z(g)}{Z(0)} = \sqrt{1-g^2}^{-N^2} \implies \log\left(\frac{Z(g)}{Z(0)}\right) = -\frac{N^2}{2}\log(1-g^2) = \frac{N^2}{2}\sum_{k=1}^{\infty}\frac{g^{2k}}{k} = \sum_{k=1}^{\infty}\frac{(-g)^k}{k!}\langle \mathrm{Tr}(AB)^k\rangle_c \tag{8.7}$$

Finally we identify the g^k components and conclude

$$\langle \mathrm{Tr}(AB)^{2k+1}\rangle_c = 0 \tag{8.8}$$

$$\langle \mathrm{Tr}(AB)^{2k}\rangle = \frac{2k!}{2k}N^2 = (2k-1)!N^2 \tag{8.9}$$

This is exactly the same result as obtained by graphical reasoning.

9 The three-color problem

In this, the final section of the thesis, the first steps of a solution to *the three-color problem* is presented. The problem was solved by R. J. Baxter in 1969 [4]. The problem can be stated as: In how many ways is it possible to color the links of a hexagonal lattice with 3 colors, such that all links meeting at a vertex are of different colors? [3] The approach to count these graphs is very similar to before, but a different kind of approximation is used. We start by defining the generating function

$$\frac{Z(g)}{Z(0)} = \frac{1}{Z(0)} \int \exp\left(-N \operatorname{Tr}\left(\frac{A^2+B^2+C^2}{2} + g(ABC+ACB)\right)\right) d^3\nu \tag{9.1}$$

We need both ABC and ACB in the exponent in order to allow for *all* colorings. The N in the exponent allows us to use the *saddlepoint approximation.* The saddlepoint approximation works for a large N and a positive function $f(x,g)$ with saddlepoints at $x*_i$, in which case

$$\frac{\int e^{-Nf(x,g)}dx}{\int e^{-Nf(x,0)}dx} \approx \frac{\sum_i e^{-Nf(x*_i,g)}}{\sum_i e^{-Nf(x*_i,0)}} \tag{9.2}$$

where the saddlepoints $x*_i$ depend on the parameter g. The planar approximation, $e_0(g)$, takes the simple form when the function only has one saddlepoint, namely

$$e_0(g) = \lim_{N\to\infty} \frac{1}{N^2} \log\left(\frac{Z(g)}{Z(0)}\right) = -\lim_{N\to\infty} \frac{1}{N}(f(x*,g) - f(x*,0)) \tag{9.3}$$

For this reason, we are looking for the saddlepoints in the large-N limit.

Returning to the integral (9.1), let $D = AB + BA$ and complete the square which, up to cyclic permutations, equals

$$\frac{C^2}{2} + gCD = \frac{1}{2}(C+gD)^2 - \frac{1}{2}g^2D^2$$

and the remaining trace equals

$$\operatorname{Tr}(D^2) = \operatorname{Tr}((AB+BA)^2) = \operatorname{Tr}(ABAB+ABBA+BABA+BAAB) = 2\operatorname{Tr}((AB)^2+BA^2B) \tag{9.4}$$

So the integral takes the form

$$\begin{aligned}\frac{Z(g)}{Z(0)} &= \frac{1}{Z(0)} \int \exp\left(\operatorname{Tr}\left(\frac{A^2+B^2+(C+gD)^2}{2} - g^2(ABAB+BAAB)\right)\right) d^3\nu = \\ &= \frac{1}{Z(0)} \int \exp\left(\operatorname{Tr}\left(\frac{A^2+B^2}{2} - g^2(ABAB+BAAB)\right)\right) d^2\nu\end{aligned} \tag{9.5}$$

The measure in (9.5) is *not* unitary invariant in both components. In other words, the measure is not invariant under the $\mathcal{U}_N^2$ transformation

$$(A,B) \to (U_1^\dagger A U_1, U_2^\dagger B U_2) \tag{9.6}$$

for all $U_1, U_2 \in \mathcal{U}_N$. So we can not have both A and B in diagonal form simultaneously. There is however a useful transformation that allows for one of A and B to be in diagonal form. It is achieved by requiring $U_1 = U_2$ in (9.6).

$$(A, B) \to (U^\dagger A U, U^\dagger B U) \tag{9.7}$$

Clearly the measure (9.5) is invariant under this transformation, and it allows us to diagonalize one of the matrices. But before we diagonalize a matrix, we change the integration variable A to gA, so that(9.5) may be written as

$$\frac{Z(g)}{Z(0)} = \frac{1}{Z(0)} \int \exp\left(\mathrm{Tr}\left(\frac{(A/g)^2 + B^2}{2} - (ABAB + BAAB)\right)\right) d^2\nu \tag{9.8}$$

Now diagonalize A as $A_{mn} = \lambda_n \delta_{mn}$:

$$A = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N) \tag{9.9}$$

The trace of the terms including a B are now easily calculated as a quadratic form. The individual terms as quadratic forms are, with $B_{mn} = x_{mn} + iy_{mn}$

$$\mathrm{Tr}(B^2/2) = \sum_{m,n=1}^{N} \frac{B_{mn}B_{nm}}{2} = \sum_{m,n=1}^{N} \frac{1}{2}|B_{mn}|^2 = \frac{1}{2}\sum_{m=1}^{N} x_{mm}^2 + \sum_{n>m} x_{mn}^2 + y_{mn}^2 \tag{9.10}$$

$$\mathrm{Tr}(BAAB) = \sum_{m,n=1}^{N} B_{mn}\lambda_n^2 B_{nm} = \sum_{m,n=1}^{N} \lambda_n^2 |B_{mn}|^2 = \sum_{m=1}^{N} \lambda_m^2 x_{mm}^2 + \sum_{n>m} (x_{mn}^2 + y_{mn}^2)(\lambda_n^2 + \lambda_m^2) \tag{9.11}$$

$$\mathrm{Tr}(BABA) = \sum_{m,n,k,l=1}^{N} B_{mn}A_{nk}B_{kl}A_{lm} = \sum_{m=1}^{N} \lambda_m^2 x_{mm}^2 + \sum_{n>m} 2\lambda_n\lambda_m(x_{mn}^2 + y_{mn}^2) \tag{9.12}$$

So the sum of the B-traces of equation (9.8) may be rewritten as

$$Tr\left(\frac{B^2}{2} - (ABAB + BAAB)\right) = \sum_{m=1}^{N} \frac{1}{2}x_{mm}^2 - \lambda_m^2 x_{mm}^2 - \lambda_m^2 x_{mm}^2 +$$

$$+ \sum_{n>m} x_{mn}^2 + y_{mn}^2 - 2\lambda_n\lambda_m(x_{mn}^2 + y_{mn}^2) - (x_{mn}^2 + y_{mn}^2)(\lambda_n^2 + \lambda_m^2) = \tag{9.13}$$

$$= \frac{1}{2}\sum_{m=1}^{N} x_{mm}^2(1 - (\lambda_m + \lambda_m)^2) + \sum_{n>m} x_{mn}^2(1 - (\lambda_m + \lambda_n)^2) + y_{mn}^2(1 - (\lambda_n + \lambda_m)^2) \tag{9.14}$$

Rewriting the matrix B as the $\mathbb{R}^{N^2}$ vector $\vec{B} = (x_{11}, x_{22}, \ldots, x_{NN}, x_{12}, y_{12}, \ldots, y_{N-1,N})$, the sum of traces are expressible as $\vec{B}^T \Omega \vec{B}$, with the matrix $\Omega \in \mathbb{R}^{N^2 \times N^2}$

$$\Omega = \mathrm{diag}\left(\frac{1}{2}(1 - (\lambda_1 + \lambda_1)^2), \ldots, \frac{1}{2}(1 - (\lambda_N + \lambda_N)^2), 1 - (\lambda_1 + \lambda_2)^2, 1 - (\lambda_1 + \lambda_2)^2, \ldots, 1 - (\lambda_{N-1} + \lambda_N)^2\right) \tag{9.15}$$

So the determinant of Ω is

$$\prod_{m=1}^{N}\left(\frac{1}{2}-(\lambda_m+\lambda_m)^2)\right)\cdot\prod_{m\neq n}(1-(\lambda_m+\lambda_n)^2)=\frac{1}{2^N}\prod_{m}(1-(\lambda_m+\lambda_m)^2)\prod_{m\neq n}(1-(\lambda_m+\lambda_n)^2)=$$
$$=\frac{1}{2^N}\prod_{m,n}(1-(\lambda_m+\lambda_n)^2) \tag{9.16}$$

Hence integrating with respect to B gives

$$\frac{Z(g)}{Z(0)}=\frac{1}{Z(0)}\int\exp\left(-N\operatorname{Tr}\left(\frac{(A/g)^2+B^2}{2}-(ABAB+BAAB)\right)\right)d^2\nu=$$
$$=\frac{1}{Z(0)}\int\Delta(\lambda)^2\exp\left(-\frac{N}{g^2}\sum_i\lambda_i/2\right)\int\exp(-N\vec{B}^T\Omega\vec{B})dBd^N\lambda=$$
$$=\frac{1}{Z(0)}\int\Delta(\lambda)^2\frac{1}{\sqrt{\det(\Omega)}}\exp(-N/g^2\sum_i\lambda_i/2)=$$
$$=\frac{1}{Z(0)}\int\exp\left(-N\left(\sum_i(\frac{1}{g^2}\lambda_i^2/2)-\frac{1}{N}\log(\Delta(\lambda)^2)+\frac{1}{2N}\log(\det(\Omega))\right)\right)d^N\lambda \tag{9.17}$$

So we are to find saddlepoints of the function

$$f(\lambda,g)=\sum_i\frac{\lambda_i^2}{2g^2}-\frac{1}{N}\sum_{m>n}\log((\lambda_n-\lambda_m)^2)+\frac{1}{2N}\sum_{m,n}\log(1-(\lambda_m+\lambda_n)^2) \tag{9.18}$$

To find the saddlepoints, we set every partial derivative of $f(\lambda,g)$ equal to 0.

$$0=\frac{\partial V}{\partial\lambda_k}=\frac{\lambda_k}{g^2}-\frac{2}{N}\sum_{m\neq k}\frac{1}{\lambda_k-\lambda_m}+\frac{1}{N}\sum_{m=1}^{N}\frac{1}{1+\lambda_k+\lambda_m}-\frac{1}{1-\lambda_k-\lambda_m} \tag{9.19}$$

We want to find the saddlepoints in the large-N limit. Assuming that you can define an increasing differentiable function $\lambda:[0,1]\to\mathbb{R}$ such that $\lambda(k/N)=\lambda_k$, these terms converge, in the large-N limit, to the integrals

$$0=\frac{\lambda(k/N)}{g^2}-2\int_0^1\frac{1}{\lambda(k/N)-\lambda(x)}dx+\int_0^1\left(\frac{1}{1+\lambda(k/N)+\lambda(x)}-\frac{1}{1-\lambda(k/N)-\lambda(x)}\right)dx \tag{9.20}$$

Introducing the even and positive eigenvalue density $u(\lambda)=\frac{dx}{d\lambda}$, all the N saddlepoint equations can be expressed as the single equation

$$0=\frac{\lambda}{g^2}-2\int_{-2a}^{2a}\frac{u(\mu)}{\lambda-\mu}d\mu+\int_{-2a}^{2a}\left(\frac{1}{1+\mu+\lambda}-\frac{1}{1-\lambda-\mu}\right)u(\mu)d\mu \tag{9.21}$$

where the eigenvalues are assumed to lie in the interval $[-2a,2a]$, i.e $\lambda(1)=2a,\lambda(0)=-2a$. This implies that $u(\lambda)$ is normalized.

$$\int_{-2a}^{2a}u(\lambda)d\lambda=\int_{-2a}^{2a}\frac{\partial x}{\partial\lambda}(\lambda)d\lambda=x(2a)-x(-2a)=1-0=1 \tag{9.22}$$

Defining

$$W(\lambda) = \int_{-2a}^{2a} \frac{u(\mu)}{\lambda - \mu} d\mu \tag{9.23}$$

And since $u(\mu)$ is even

$$\int_{-2a}^{2a} \frac{u(\mu)}{\lambda - \mu} d\mu = \int_{-2a}^{2a} \frac{u(\mu)}{\lambda + \mu} d\mu \tag{9.24}$$

The saddle-point equation becomes, for $\lambda \in [-2a, 2a]$

$$\frac{\lambda}{g^2} = 2W(\lambda) + W(1-\lambda) - W(1+\lambda) = 2W(\lambda) + W(1-\lambda) + W(-1-\lambda) \tag{9.25}$$

From the definition of $W(\lambda)$, equation (9.23), some properties of $W(\lambda)$ can be derived.

- $W(\lambda)$ is analytic in the complex plane apart from the interval $[-2a, 2a]$
- $W(\lambda)$ goes as $1/\lambda$ as $|\lambda| \to \infty$
- $W(\lambda)$ is real for $\lambda \in \mathbb{R}$.
- for $\lambda \in [-2a, 2a]$, $\lim_{\epsilon \to 0^+} \Im(W(\lambda \pm i\epsilon)) = \mp \pi u(\lambda)$

And for $\lambda \in [-2a, 2a]$ and $\epsilon \to 0^+$, let $-\gamma_2$ be the properly oriented line between $2a + 2i\epsilon$ and $-2a + 2i\epsilon$

$$W(\lambda + i\epsilon) = \int_{-2a}^{2a} \frac{u(\mu)}{\lambda + i\epsilon - \mu} d\mu = \oint_{\gamma_1 - \gamma_2} \frac{u(\mu)}{\lambda + i\epsilon - \mu} d\mu + \int_{\gamma_2} \frac{u(\mu)}{\lambda + i\epsilon - \mu} d\mu = \tag{9.26}$$

$$= -2\pi i u(\lambda + i\epsilon) + \int_{-2a+2i\epsilon}^{2a+2i\epsilon} \frac{u(\mu)}{\lambda + i\epsilon - \mu} d\mu \tag{9.27}$$

With the change of variables $\mu' = \mu - 2i\epsilon$, the integral becomes

$$\int_{-2a}^{2a} \frac{u(\mu' + 2i\epsilon)}{\lambda - i\epsilon - \mu'} d\mu' = W(\lambda - i\epsilon) \tag{9.28}$$

So

$$W(\lambda + i\epsilon) = -2\pi i u(\lambda) + W(\lambda - i\epsilon) \iff W(\lambda + i\epsilon) - \overline{W(\lambda + i\epsilon)} = 2i\Im(W(\lambda + i\epsilon)) = \tag{9.29}$$

$$= -2\pi i u(\lambda) \iff \Im(W(\lambda + i\epsilon)) = -\pi u(\lambda) \tag{9.30}$$

where a bar denotes the complex conjugation, and $\Im$ denotes the imaginary part. From the definition of $W(\lambda)$ it is clear that

$$W(\bar{\lambda}) = \overline{W(\lambda)} \tag{9.31}$$

and hence

$$\Im(W(\lambda - i\epsilon)) = \Im(\overline{W(\lambda + i\epsilon)}) = +\pi u(\lambda) \tag{9.32}$$

So for $\epsilon > 0$ approaching 0 we get

$$\Im(W(\lambda \pm i\epsilon)) = \mp \pi u(\lambda) \tag{9.33}$$

The first property, that the function is analytic on $\mathbb{C} \setminus [-2a, 2a]$ tells us that $W(\lambda)$ is of the form

$$W(\lambda) = f(\lambda) + g(\lambda)\sqrt{\lambda^2 - 4a^2} \tag{9.34}$$

where $f(\lambda)$ and $g(\lambda)$ are analytic in the whole complex plane, and the squareroot is zero for $\lambda \in [-2a, 2a]$ This is the beginning of the solution to the three-color problem. A full solution is given in [3] and [4].

10 Summary

Using generating functions defined by integrals has shown to be a very good approach for counting graphs. The approach works for many different kinds of graphs. We have found that for graphs with vertices of degree 4

$$e_0(g) = \frac{1}{2}\log\left(\frac{-1+\sqrt{\alpha_1}}{\omega}\right) + \frac{3}{8} + \frac{1}{2\omega} + \frac{1}{12\omega^2} - \frac{(1+5\omega)\sqrt{\alpha_1}}{12\omega^2} = -2g + 18g^2 - \dots$$

$$e_1(g) = -\frac{1}{12}\log\left(\frac{2\omega+1-\sqrt{\alpha_1}}{\omega}\right) = -g + 30g^2 - \dots$$

Following the same procedure in the calculations of $e_0(g)$ and $e_1(g)$, it is possible to calculate $e_k(g)$ for any k by performing expansions of higher degrees, although the complexity of these calculations grows rapidly for larger k. For manageable calculations of $e_k(g)$ for a large k, an entirely new approach is probably needed. The solution to the three-color problem is designed to single out the planar approximation, and there is no apparent extension of the method that would allow for calculations of higher genera.

A weakness of the generating function-approach is when one wishes to count graphs with vertices of odd degree. There might be an issue with the convergence of the integral that defines the generating function. In special cases this problem can be avoided. Suppose we only allow vertices of degree 3. The intuitive definition of the generating function

$$Z(g) = \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2} - gx^3\right) dx \tag{10.1}$$

does not converge. But if we notice that there are no graphs of odd order, we can instead integrate $\exp\left(-\frac{x^2}{2} - gx^6\right)$, and view x^6 as 2 vertices of degree 3. We do not lose any graphs in this process. If we want to count graphs with vertices of different odd degrees however, this approach might be unusable.

Even though not many results were explicitly derived, this thesis contains sufficient information for many different graphical calculations that have not been performed here.

References

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[3] I. K. Kostov *Exact solution of the three-color problem on a random lattice*

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[5] M. Srednicki *Quantum Field Theory*

A Proof of the Wick lemma

It suffices to prove the lemma for monomials. Let

$$f(\vec{J}) = \frac{1}{2}\vec{J}^T B \vec{J} \tag{A.1}$$

$f(\vec{J})$ is a quadratic function of $\vec{J}$ and has the following derivatives at the origin

$$\begin{aligned} f(0) &= 0 \\ \frac{\partial f}{\partial J_i}(0) &= 0 \\ \frac{\partial^2 f}{\partial J_i \partial J_j}(0) &= B_{ij} = \langle x_i x_j \rangle \\ \frac{\partial^k f}{\partial J_{i_1} \dots \partial J_{i_k}}(0) &= 0, \quad k \geq 3 \end{aligned} \tag{A.2}$$

It is worth noting that $\frac{\partial^2 f}{\partial J_i \partial J_j}$ is constant. From (1.8) we have

$$\langle x_{i_1} \dots x_{i_{2n}} \rangle = \frac{\partial^{2n}}{\partial J_{i_1} \dots \partial J_{i_{2n}}} e^{f(\vec{J})}|_{\vec{J}=0} = \frac{\partial^{2n}}{\partial J_{i_1} \dots \partial J_{i_{2n}}} \left(\sum_{k=0}^{\infty} \frac{f(\vec{J})^k}{k!} \right) |_{\vec{J}=0} \tag{A.3}$$

From the derivatives at the origin it's clear that only second derivatives of $f(\vec{J})$ have the possibility of being nonzero. Hence the only power of $f(\vec{J})$ in the power series that contributes to the sum is when $k = n$

$$\langle x_{i_1} \dots x_{i_{2n}} \rangle = \frac{\partial^{2n}}{\partial J_{i_1} \dots \partial J_{i_{2n}}} \frac{f(\vec{J})^n}{n!} |_{\vec{J}=0} \tag{A.4}$$

With the notation

$$f_{i_1 i_2, \dots i_n}(\vec{J}) = \frac{\partial^n f}{\partial J_{i_1} \partial J_{i_2} \dots \partial J_{i_n}}(\vec{J}) \tag{A.5}$$

for n=1

$$\langle x_i x_j \rangle = \sum_{\text{coupl.}} \langle x_p x_q \rangle \tag{A.6}$$

For n=2

$$\begin{aligned} \langle x_{i_1} x_{i_2} x_{i_3} x_{i_4} \rangle &= \frac{\partial^4}{\partial J_{i_1} \partial J_{i_2} \partial J_{i_3} \partial J_{i_4}} \left(\frac{f(\vec{J})^2}{2} \right) = \\ &= \frac{\partial^3}{\partial J_{i_1} \partial J_{i_2} \partial J_{i_3}} (f_{i_4}(\vec{J}) f(\vec{J})) = \frac{\partial^2}{\partial J_{i_1} \partial J_{i_2}} ((f_{i_4 i_3} f + f_{i_4} f_{i_3}) = \\ &= f_{i_4 i_3} f_{i_2 i_1} + \frac{\partial}{J_{i_1}} (f_{i_4 i_2} f_{i_3} + f_{i_4} f_{i_3 i_2}) = f_{i_4 i_3} f_{i_2 i_1} + f_{i_4 i_2} f_{i_3 i_1} + f_{i_4 i_1} f_{i_3 i_2} = \sum_{\text{coupl.}} \langle x_{p_1} x_{q_1} \rangle \langle x_{p_2} x_{q_2} \rangle \end{aligned} \tag{A.7}$$

Suppose for induction that

$$\langle x_{i_1} x_{i_2} \dots x_{i_{2n}} \rangle = \sum_{\text{coupl.}} \langle x_{p_1} x_{q_1} \rangle \langle x_{p_2} x_{q_2} \rangle \dots \langle x_{p_n} x_{q_n} \rangle \tag{A.8}$$

holds for n and $n-1$. Then for $n+1$

$$\begin{aligned}
\langle x_{i_1} x_{i_2} \dots x_{i_{2n+2}} \rangle &= \frac{\partial^{2n+2}}{\partial J_{i_1} \partial J_{i_2} \dots J_{i_{2n+2}}} \left(\frac{f(\vec{J})^{n+1}}{(n+1)!} \right) = \\
&= \frac{1}{(n+1)!} \frac{\partial^{2n+1}}{\partial J_{i_1} \partial J_{i_2} \dots \partial J_{i_{2n+1}}} ((n+1) f_{i_{2n+2}} f^n) = \\
&= \frac{\partial^n}{\partial J_{i_1} \partial J_{i_2} \dots \partial J_{i_n}} \left(\frac{1}{n!} f_{i_{2n+2} i_{2n+1}} f^n + \frac{1}{(n-1)!} f_{i_{2n+2}} f_{i_{2n+1}} f^{n-1}\right) = \\
&= f_{i_{2n+2} i_{2n+1}} \langle x_{i_1} x_{i_2} \dots x_{i_{2n}} \rangle + \sum_{p_1, q_1} f_{i_{2n+2} p_1} f_{i_{2n+1} q_1} \langle x_{i_1} \dots \widehat{x_{p_1}} \widehat{x_{q_1}} \dots x_{i_{2n}} \rangle = \\
&= f_{i_{2n+2} i_{2n+1}} \sum_{\text{coupl.}} \langle x_{p_1} x_{q_1} \rangle \dots \langle x_{p_n} x_{q_n} \rangle + \sum_{p_1, q_1} f_{i_{2n+2} p_1} f_{i_{2n+1} q_1} \sum_{\text{coupl.}} \langle x_{p_2} x_{q_2} \rangle \dots \langle x_{p_n} x_{q_n} \rangle
\end{aligned} \tag{A.9}$$

where the hat over x_{p_1} and x_{q_1} means that they are excluded in the product. The first term is a sum over all Wick couplings where $x_{i_{2n+2}}$ is coupled with $x_{i_{2n+1}}$. The second term is a sum over all Wick couplings where $x_{i_{2n+2}}$ is *not* coupled with $x_{i_{2n+1}}$. So the sum of the two terms is the sum of all Wick couplings, and the Wick lemma is proved.

B Continuous approximation of $R_k(\bar{g})/N$

In the steps-stairs process, moving from step $k-1$ to step k in $2p+1$ steps, the highest possible step is $p+k$ and the minimum is $k+1-p$. The distance from step k is bounded. Hence for any $\frac{R_l(\bar{g})}{N}$ appearing in the path-sum

$$R_l(0) - R_k(0) = l - k \tag{B.1}$$

is bounded. Note that k runs from 1 to N, so $R_k(\bar{g})$ is not bounded in the large N-limit. From equation (5.14) we know that $\frac{R_k(\bar{g})}{N} < 1$.

So we know that

$$R_l(\bar{g}) - R_k(\bar{g}) = l - k + \mathcal{O}(1/N) \iff R_l = R_k(\bar{g}) + l - k + \mathcal{O}(1/N) \tag{B.2}$$

$$\iff \frac{R_l(\bar{g})}{N} = \frac{R_k(\bar{g})}{N} + \frac{l-k}{N} + \mathcal{O}(1/N^2) \tag{B.3}$$

Where $\bar{g}$ is a vector of all the $\bar{g}_p$:s.

$$\frac{k}{N} = \frac{R_k(\bar{g})}{N}\left(1+\sum_{p\geq 1} 2(p+1)\bar{g}_{p+1}\sum_{paths} R_{s_1}\dots R_{s_N}\right) =$$

$$= \frac{R_k(\bar{g})}{N}\left(1+\sum_{p=1} 2(p+1)g_{p+1}\sum_{paths}\left(\frac{R_k+s_1-k}{N}+\mathcal{O}(1/N^2)\right)\dots\left(\frac{R_k+s_p-k}{N}+\mathcal{O}(1/N^2)\right)\right) =$$

$$= \frac{R_k(\bar{g})}{N}\left(1+\sum_{p=1} 2(p+1)g_{p+1}\sum_{paths}\left(\frac{R_k(\bar{g})}{N}\right)^p + \frac{s_i-k}{N}\left(\frac{R_k(\bar{g})}{N}\right)^{p-1}\right) + \mathcal{O}(1/N^2) =$$

$$= \frac{R_k(\bar{g})}{N}\left(1+\sum_{p=1} 2(p+1)g_{p+1}\left(\frac{R_k(\bar{g})}{N}\right)^p\binom{2p+1}{p} + \left(\frac{R_k(\bar{g})}{N}\right)^{p-1}\sum_{paths}\frac{s_i-k}{N}\right) + \mathcal{O}(1/N^2) \tag{B.4}$$

Equation (5.14) can be expanded in the large N-limit as

$$\tag{B.5}$$

Consider the sum

$$\sum_{\text{paths}} s_i - k \tag{B.6}$$

for a path from step $k-1$ to step k in $2p+1$ steps, where the down steps are from steps s_i. Rewrite the sum as

$$\sum_{\text{paths}} s_i - k = \sum_{n=0}^{2p+1} C_n \tag{B.7}$$

where C_n is the sum of all contributions from down steps on the n:th step of the path. Let u_b and d_b be the number of up steps and downsteps taken before the n:th step respectively. Let u_a and d_a be the number of up- and down steps taken after the n:th step. They clearly satisfy the following equations

$$u_b + d_b = n-1 \tag{B.8}$$
$$u_a + d_a = 2p+1-n \tag{B.9}$$
$$d_a + d_b = p-1 \tag{B.10}$$
$$u_a + u_b = p+1 \tag{B.11}$$

The step one is on at the n:th step, s is

$$s = k-1+u_b-d_b \iff s-k = u_b-d_b-1 = d_a-u_a+1 \tag{B.12}$$

Hence the contribution from that down step is

$$u_b - d_b - 1 = 2u_b - n = 2p+2-n-2u_a \tag{B.13}$$

The number of paths with u_b up steps before the n:th step is, for $n \leq p$

$$\binom{n-1}{u_b}\binom{2p+1-n}{u_a} = \binom{n-1}{u_b}\binom{2p+1-n}{p+1-u_b} \tag{B.14}$$

So, for $n \leq p$

$$C_n = \sum_{u_b=0}^{n-1} (2u_b - n) \binom{n-1}{u_b} \binom{2p+1-n}{p+1-u_b} \tag{B.15}$$

And for $n \geq p+2$

$$C_n = \sum_{u_a=0}^{2p+1-n} (2p+2-n-2u_a) \binom{n-1}{p+1-u_a} \binom{2p+1-n}{u_a} \tag{B.16}$$

Hence, for $n \leq p \iff 2p+2-n \geq p+2$

$$c_{2p+2-n} = \sum_{u=0}^{n-1} (n-2u) \binom{2p+1-n}{p+1-u} \binom{n-1}{u} = -C_n \implies \sum_{n=0}^{2p+1} C_n = C_{p+1} \tag{B.17}$$

$$C_{p+1} = \sum_{u=1}^{p} (2u-p-1) \binom{p}{u} \binom{p}{p+1-u} = [v = p+1-u] = \tag{B.18}$$

$$= \sum_{v=1}^{p} (-2v+p+1) \binom{p}{p+1-v} \binom{p}{v} = -C_{p+1} = 0 \tag{B.19}$$

So equation (B.5) now reads

$$\frac{k}{N} = \frac{R_k(\bar{g})}{N} \left(1 + \sum_{p=1} \frac{(2p+2)!}{p!(p+1)!} g_{p+1} \left(\frac{R_k(\bar{g})}{N} \right)^p \right) + \mathcal{O}(1/N^2) \tag{B.20}$$

Clearly

$$\lim_{N \to \infty} \frac{R_k(\bar{g})}{N} = r_0(x) \left(\frac{k}{N} \right) \tag{B.21}$$

with $r_0(x)$ as defined in equation (6.2) So let

$$f = \frac{R_k(\bar{g})}{N} - r_0(k/N) \implies \lim_{N \to \infty} f = 0 \tag{B.22}$$

Plug this expression into equation (B.20)

$$\frac{k}{N} = (r_0(k/N) + f)\left(1 + \sum_{p=1}^{M} \frac{(2p+2)!}{p!(p+1)!} g_{p+1}\left(r_0(k/N)^p + pfr_0(k/N)^{p-1} + \mathcal{O}(f^2)\right) + \mathcal{O}(1/N^2) = \tag{B.23}$$

$$= \underbrace{(r_0(k/N)(1 + \sum_{p=1}^{M} \frac{(2p+2)!}{p!(p+1)!} g_{p+1} r_0(k/N)^p)}_{k/N} + f\sum_{p=1}^{M} \frac{(2p+2)!}{p!(p+1)!} g_{p+1} p r_0(k/N)^p + \tag{B.24}$$

$$+f(1 + \sum_{p=1}^{M} \frac{(2p+2)!}{p!(p+1)!} g_{p+1} r_0(k/N)^p) + \mathcal{O}(f^2) + \mathcal{O}(N^{-2}) \iff \tag{B.25}$$

$$\iff \underbrace{f\,(1 + \sum_{p=1}^{M} \frac{(2p+2)!}{p!(p+1)!} g_{p+1}(p+1) r_0(k/N)^p)}_{\text{bounded}} = \mathcal{O}(f^2) + \mathcal{O}(N^{-2}) \implies f = \mathcal{O}(f^2) + \mathcal{O}(N^{-2}) \tag{B.26}$$

Since we have allready established that $\lim_{N\to\infty} f = 0$,

$$f = \mathcal{O}(f^2) \tag{B.27}$$

makes no sense. Hence

$$f = \frac{R_k(\bar{g})}{N} - r_0(k/N) = \mathcal{O}(N^{-2}) \iff \frac{R_k(\bar{g})}{N} = r_0(k/N) + \mathcal{O}(N^{-2}) \tag{B.28}$$

C Orthogonal polynomials

There is a standard procedure for finding an orthogonal basis given an arbitrary basis. This procedure is called *Gram-Schmidt orthogonalization*. We start with the standard basis of polynomials

$$\{1, \lambda, \lambda^2, \ldots\} \tag{C.1}$$

Starting from this basis, the orthogonal polynomial of degree n in the Gram-Schmidt orthogonalization is defined as

$$P_n(\lambda) = \lambda^n - \sum_{k=0}^{n-1} \frac{\langle \lambda^n P_k(\lambda)\rangle}{\langle P_k(\lambda)^2\rangle} P_k(\lambda) \tag{C.2}$$

We introduce the notation h_n for the norm of $P_n(\lambda)$.

$$h_n := \langle P_n(\lambda)^2\rangle \tag{C.3}$$

With this notation, equation (5.4) can be extended to a relation that holds for all polynomials $P_n(\lambda)$ and $P_m(\lambda)$

$$\langle P_n(\lambda) P_m(\lambda)\rangle = h_n \delta_{nm} \tag{C.4}$$

Any orthogonal polynomial from the Gram-Schmidt process, $P_n(\lambda)$, can be expressed as

$$P_n(\lambda) = \lambda^n + Q_{n-1} \tag{C.5}$$

where Q_{n-1} is a polynomial of degree $n-1$. Hence The set of$\{P_0, P_1, \ldots, P_n\}$ forms a basis of $\mathcal{P}_n$. This means that w.r.t $d\mu$, P_{n+1} is orthogonal to $\mathcal{P}_n$. For $l = 0, 1, \ldots, n-2$, $\lambda P_l \in \mathcal{P}_{n-1}$ so $\langle P_l \lambda P_n \rangle = 0$.

$$\lambda P_n \in \mathcal{P}_{n+1} \implies \lambda P_n = \sum_{k=0}^{n+1} a_k P_k \tag{C.6}$$

and for $l = 0, 1, \ldots, n-2$

$$\langle P_l \lambda P_n \rangle = \sum_{k=0}^{n+1} a_k \langle P_l P_k \rangle = a_l h_l = 0 \iff a_l = 0 \tag{C.7}$$

The remaining terms make up the recursion formula

$$\lambda P_n(\lambda) = a_{n+1} P_{n+1}(\lambda) + a_n P_n(\lambda) + a_{n-1} P_{n-1}(\lambda) \tag{C.8}$$

Identification of the λ^{n+1} terms leads to the conclusion that $a_{n+1} = 1$. Then by just renaming the coefficients $a_n = -A_n$ and $a_{n-1} = R_n$, the recursion formula is attained.

$$\lambda P_n(\lambda) = P_{n+1}(\lambda) - A_n P_n(\lambda) + R_n P_{n-1}(\lambda) \tag{C.9}$$

For an even measure $d\mu$, $A_n \equiv 0$. For an even $d\mu$, $\langle \lambda^{2k+1} \rangle = 0$, and

$$P_0(\lambda) = 1 \tag{C.10}$$

$$P_1(\lambda) = \lambda - \frac{\langle \lambda \rangle}{h_0} 1 = \lambda \tag{C.11}$$

Suppose for induction that $P_k(-\lambda) = (-1)^k P_k(\lambda)$ for $k \in \{0, 1, \ldots, n\}$, which we know holds for n=0,1 then:

$$P_{n+1}(-\lambda) = (-\lambda)^{n+1} - \sum_{k=0}^{n} \frac{\langle (-\lambda)^{n+1} P_k(-\lambda) \rangle}{h_k} P_k(-\lambda) = (-1)^{n+1} P_{n+1}(\lambda) \tag{C.12}$$

Thus

$$(-\lambda) P_n(-\lambda) = (-1)^{n+1} (\lambda P_n(\lambda) = P_{n+1}(-\lambda)) - A_n P_n(-\lambda) + R_k P_{n-1}(-\lambda) = \tag{C.13}$$

$$= (-1)^{n+1} (P_{n+1}(\lambda) + A_{n-1}(\lambda)) - (-1)^n A_n = (-1)^{n+1} (P_{n+1}(\lambda) - A_n P_n(\lambda) + R_n(\lambda)) \iff \tag{C.14}$$

$$\iff A_k = 0 \tag{C.15}$$

Let Q_k be a polynomial of degree k. Because P_n is orthogonal to the space of all polynomials of degree $n-1$

$$h_n = \langle P_n P_n \rangle = \langle (\lambda^n + Q_{n-1}) P_n \rangle = \langle \lambda^n P_n \rangle \tag{C.16}$$

Observe that

$$\lambda P_n = \lambda^{n+1} + Q_n \implies \langle \lambda P_n P_{n+1} \rangle = \langle (\lambda^{n+1} + Q_n) P_n \rangle = h_{n+1} \tag{C.17}$$

Applying the recursion formula to λP_{n+1} yields

$$h_{n+1} = \langle P_n \lambda P_{n+1} \rangle = \langle P_n (P_{n+2} - A_{n+1} P_{n+1} + R_{n+1} P_n) \rangle = R_{n+1} h_n = h_{n+1} \tag{C.18}$$

Since one can add or subtract columns in a determinant, the Vandermonde determinant, $\Delta(\lambda)$, is expressible in terms of orthogonal polynomials as

$$\Delta(\lambda) = \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \cdots & \cdots & \lambda_2^{N-1} \\ \vdots & & \ddots & & \\ 1 & \cdots & & \cdots & \lambda_N^{N-1} \end{vmatrix} = \begin{vmatrix} P_0(\lambda_1) & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{N-1} \\ P_0(\lambda_2) & \lambda_2 & \cdots & \cdots & \lambda_2^{N-1} \\ \vdots & & \ddots & & \\ P_0(\lambda_N) & \cdots & & \cdots & \lambda_N^{N-1} \end{vmatrix} = \tag{C.19}$$

$$= \begin{vmatrix} P_0(\lambda_1) & \lambda_1 - \frac{\langle \lambda_1 P_0(\lambda_1) \rangle}{h_0} P_0(\lambda_1) & \lambda_1^2 & \cdots & \lambda_1^{N-1} \\ P_0(\lambda_2) & \lambda_2 - \frac{\langle \lambda_2 P_0(\lambda_2) \rangle}{h_0} P_0(\lambda_2) & \cdots & \cdots & \lambda_2^{N-1} \\ \vdots & & \ddots & & \\ P_0(\lambda_N) & \cdots & & \cdots & \lambda_N^{N-1} \end{vmatrix} = \tag{C.20}$$

$$= \begin{vmatrix} P_0(\lambda_1) & P_1(\lambda_1) & \lambda_1^2 & \cdots & \lambda_1^{N-1} \\ P_0(\lambda_2) & P_1(\lambda_2) & \cdots & \cdots & \lambda_2^{N-1} \\ \vdots & & \ddots & & \\ P_0(\lambda_N) & \cdots & & \cdots & \lambda_N^{N-1} \end{vmatrix} = \ldots = \begin{vmatrix} P_0(\lambda_1) & P_1(\lambda_1) & P_2(\lambda_2) & \cdots & P_{N-1}(\lambda_1) \\ P_0(\lambda_2) & \ddots & & & \\ \vdots & & \ddots & & \\ P_0(\lambda_N) & & & & P_{N-1}(\lambda_N) \end{vmatrix} \tag{C.21}$$

D Derivation of $r_2(x)$ and $r_4(x)$

With $(\alpha = 1 + 48gx)$, equation (7.8) for $m = 0, 1, 2$ take the forms

$$x = r_0 + 4g(r_0^2 + 2r_0^2) = r_0 + 12gr_0^2 \iff r_0 = \frac{-1 + \sqrt{\alpha}}{24g}$$

$$0 = r_2(1 + 24gr_0) + 4gr_0 r_0^{(2)} \iff r_2 = -\frac{4gr_0 r_0^{(2)}}{\alpha^{1/2}}$$

$$0 = r_4 \alpha^{1/2} + 4g\left(3(r_2)^2 + r_2 r_0^{(2)} + r_0 r_2^{(2)} + \frac{1}{12} r_0 r_0^{(4)}\right) \iff$$

$$\iff r_4 = -\frac{4g}{\alpha^{1/2}}\left(3(r_2)^2 + r_2 r_0^{(2)} + r_0 r_2^{(2)} + \frac{1}{12} r_0 r_0^{(4)}\right)$$

A simple consequence of this is

$$\alpha^{1/2} = 1 + 24gr_0$$

The derivatives of r_0 up to 4th order are

$$\begin{aligned} r_0' &= \alpha^{-1/2} \\ r_0^{(2)} &= -\frac{1}{2}\alpha^{-3/2}\alpha' = -24g\alpha^{-3/2} \\ r_0^{(3)} &= \frac{3}{4}\alpha^{-5/2}(\alpha')^2 \\ r_0^{(4)} &= -\frac{15}{8}\alpha^{-7/2}(\alpha')^3 = -15(24g)^3(\alpha)^{-7/2} \end{aligned}$$

Thus r_2 is

$$r_2 = -\frac{4g}{\alpha^{1/2}}r_0(-24g\alpha^{-3/2}) = 4g(24g)\frac{r_0}{\alpha^2}$$

and r_2:s derivatives up to second order are

$$\begin{aligned} r_2' = 4g(24g)\frac{r_0'\alpha^2 - 2\alpha\alpha' r_0}{\alpha^4} = 4g(24g)\frac{\alpha^{3/2} - 4\alpha(24g)r_0}{\alpha^4} = \\ = 4g(24g)(\alpha^{-5/2} - 4(24g)r_0\alpha^{-3}) \\ r_2^{(2)} = 4g(24g)(-\frac{5}{2}\alpha^{-7/2}\alpha' - 4(24g)(r_0'\alpha^{-3} - 3\alpha^{-4}\alpha' r_0)) = \\ = 4g(24g)(-5(24g)\alpha^{-7/2} - 4(24g)(\alpha^{-7/2} - 6(24g)\alpha^{-4}r_0) = \\ = \frac{(24g)^3}{2\alpha^4}(-3 + 5(24g)r_0) \end{aligned}$$

So the formula for r_4 becomes

$$\begin{aligned} r_4 = -\frac{4g}{\alpha^{1/2}}(16\cdot 24^2 g^4 r_0^2\alpha^{-4}3 + r_0\frac{1}{2}(24g)^3\alpha^{-4}(-3+5(24g)r_0) - r_0\frac{15}{12}(24g)^3\alpha^{-4}(1+24gr_0) - \\ -\frac{1}{6}(24g)^3 r_0\alpha^{-7/2}) = -\frac{1}{6}(24g)^4\alpha^{-9/2}r_0\left(2gr_0 + \frac{1}{2}(-3+5(24g)r_0) - \frac{5}{4}(1+24gr_0) - \frac{1}{6}(1+24gr_0)\right) = \\ = -\frac{1}{6}(24g)^4\alpha^{-9/2}r_0(28gr_0 - \frac{35}{12}) = \frac{7}{6}(24g)^4\alpha^{-9/2}r_0(-4gr_0 + \frac{5}{12}) = \frac{7}{72}(24g)^4\alpha^{-9/2}r_0(5 - 48gr_0) \end{aligned}$$