On the metastability of the Standard Model

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Abstract

With the discovery of a particle consistent with the Standard Model (SM) Higgs at the Large Hadron Collider (LHC) at CERN in 2012, the final ingredient of the SM has been found. The SM provides us with a powerful description of the physics of fundamental particles, holding up at all energy scales we can probe with accelerator based experiments. However, astrophysics and cosmology show us that the SM is not the final answer, but e.g. fails to describe dark matter and massive neutrinos. Like any non-trivial quantum field theory, the SM must be subject to a so-called renormalization procedure in order to extrapolate the model between different energy scales. In this context, new problems of more theoretical nature arise, e.g. the famous hierarchy problem of the Higgs mass. Renormalization also leads to what is known as the metastability problem of the SM: assuming the particle found at the LHC is the SM Higgs boson, the potential develops a second minimum deeper than the electroweak one in which we live, at energy scales below the Planck scale. Absolute stability all the way up to the Planck scale is excluded at a confidence level of about 98 %. For the central experimental SM values the instability occurs at scales larger than $\sim 10^{10}$ GeV.

One can take two viewpoints regarding this instability: assuming validity of the SM all the way up to the Planck scale, the problem does not necessarily lead to an inconsistency of our existence. If we assume our universe to have ended up in the electroweak minimum after the Big Bang, the probability that it would have transitioned to its true minimum during the lifetime of the universe is spectacularly small. If we on the other hand demand absolute stability, new physics must modify the SM at or below the instability scale of $\sim 10^{10}$ GeV, and we can explore which hints the instability might provide us with on this new physics.

In this work, the metastability problem of the SM and possible implications are revisited. We give an introduction to the technique of renormalization and apply this to the SM. We then discuss the stability of the SM potential and the hints this might provide us with on new physics at large scales.

Sammanfattning

Standardmodellen inom partikelfysik är vår bästa beskrivning av elementarpartiklarnas fysik. År 2012 hittades en ny skalär boson vid Large Hadron Collider (LHC) på CERN, som är kompatibel med att vara Higgs bosonen, den sista saknade delen av Standardmodellen. Men även om Standardmodellen ger oss en väldigt precis beskrivning av all fysik vi ser i partikelacceleratorer, vet vi från astropartikelfysik och kosmologi att den inte kan vara hela lösningen. T.ex. beskriver Standardmodellen ej mörk materia eller neutrinernas massa. Som alla kvantfältteorier måste man renormera Standardmodellen för att få en beskrivning som fungerar på olika energiskalor. När man renormerar Standardmodellen hittar man nya problem som är mer teoretiska, t.ex. det välkända hierarkiproblemet av Higgsmassan. Renormering leder också till vad som kallas för metastabilitetsproblemet, dvs att Higgspotentialen utvecklar ett minimum som är djupare än det elektrosvaga minimum vi lever i, på högre energiskalor. Om vi antar att partikeln som hittades på CERN är Standardmodellens Higgs boson, är absolut stabilitet exkluderad med 98 % konfidens. För centrala experimentiella mätningar av Standardmodells parametrar uppkommer instabiliteten på skalor över $\sim 10^{10}$ GeV.

Det finns två olika sätt att tolka stabilitetsproblemet: Om man antar att Standardmodellen är den rätta teorien ända upp till Planckskalan, kan vi faktiskt fortfarande existera. Om vi antar att universum hamnat i det elektrosvaga minimumet efter Big Bang är sannolikheten att det har gått över till sitt riktiga minimum under universums livstid praktiskt taget noll. Dvs att vi kan leva i ett metastabilt universum. Om vi å andra sidan kräver att potentialen måste vara absolut stabil, måste någon ny fysik modifiera Standardmodellen på eller under instabilitetsskalan $\sim 10^{10}$ GeV. I så fall kan vi fundera på vilka antydningar stabilitetsproblemet kan ge oss om den nya fysiken.

Den här uppsatsen beskriver Standardmodells metastabilitetsproblem. Vi ger en introduktion till renormering och använder tekniken till Standardmodellen. Sen diskuteras stabiliteten inom Standardmodellens potential och vilka antydningar problemet kan ge oss angående ny fysik.

Contents

1 Introduction

The Standard Model of Particle Physics (SM) provides us with an astonishingly powerful description of the physics of fundamental particles at the energy scales reachable in high energy experiments. With the discovery of a particle consistent with the SM Higgs at the LHC in 2012, the last ingredient of the SM was found. Powerful as it is, we are also well aware of the incompleteness of the SM, which e.g. fails to explain dark matter, the baryon asymmetry, or neutrino masses. Answers to these questions may lie within the reach of the second LHC run started in 2015, but could well be hiding at much higher energy scales.

When extrapolating the known physics to greater energy scales, renormalization plays a crucial role. The SM cannot be solved exactly but only as a perturbation about the non-interacting theory, like any other realistic Quantum Field Theory (QFT) involving interactions. When computing beyond the leading order, one encounters infinities due to so-called loops, the production and subsequent annihilation of virtual particles. For renormalizable QFTs like the SM, a systematic and well-defined treatment of these infinities is possible: the infinities are regulated by introducing an auxiliary parameter and subsequently removing the dependency on this auxiliary parameter from all physical observables of the theory. In the course of this renormalization procedure, the parameters of the theory become dependent on the energy scales involved in a physical process. When analyzing the renormalized SM potential, one encounters the so-called (meta)stability problem: at energies much higher than experimentally testable, the potential might develop a second minimum deeper than the electroweak one.

The metastability problem has been known since the early days of the SM and been used to give powerful constraints on physics on scales larger than the ones experimentally accessible. E.g., long before the discovery of the top quark, the stability condition has been used to constrain its mass (cf. [1–7] and references therein). After the top quark had been discovered at the Tevatron in 1994, the stability bound was used to constrain the mass of the Higgs boson (cf. [8–14] and references therein).

With the discovery of a SM Higgs like particle in 2012 at the LHC all parameters of the SM are known and it appears that the model sits at a peculiar spot very close to the stability bound. While the experimental data prefers a metastable potential with the second minimum occurring at scales above $\sim 10^{10}$ GeV, the stable phase lies only a few standard deviations off the central values. This has recently led to considerations of the stability bound with improved accuracy [15, 16].

This work revisits the computation of the stability bounds on the SM potential's parameters. We compute the dominating one-loop contributions to the renormalized potential explicitly and compare the results with the renormalization group equations available in the literature. We then proceed to calculate the stability bound at two-loop order and compare with the available next-to-next-to-leading-log precision results. Finally, we discuss the interpretation of the instability problem and possible hints, the problem provides about the physics at larger scales.

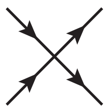
2 Renormalization

Realistic QFTs describing interacting particles cannot be solved exactly but only as a perturbation series about the non-interacting theory. When calculating beyond the leading order one encounters infinities due to loops, which in the case of renormalizable theories can be dealt with in a systematic way, the so-called renormalization procedure. Introductions to renormalization may be found in any standard QFT textbook, e.g. [17–19]. We demonstrate renormalization for a simple example, the one-loop correction to ϕ^4-theory.

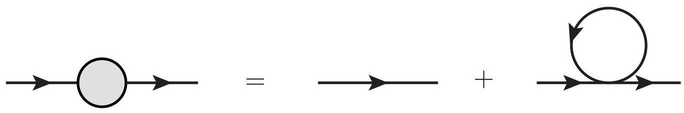
Consider a complex scalar field ϕ with quartic self-interaction and mass m. The Lagrangian is given by

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi - \frac{\lambda}{4} \left(\phi^\dagger \phi\right)^2 , \tag{2.1}$$

where throughout this work we use the metric tensor $g^{\mu\nu} = g_{\mu\nu} = \mathrm{diag}\,(+1, -1, -1, -1)$, and natural units $\hbar = c = 1$ unless noted explicitly. The Feynman-rules for this theory are[1]

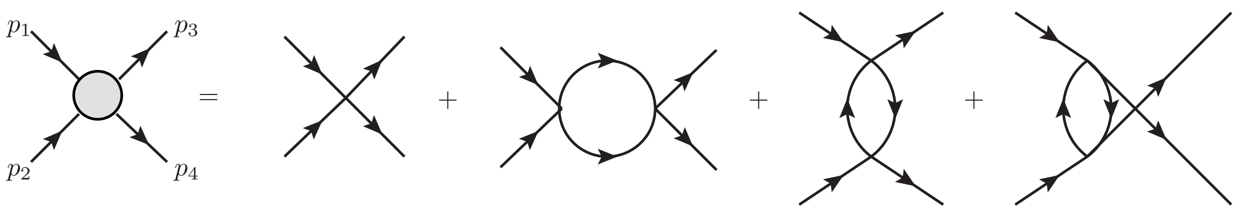
 $= \frac{i}{p^2 - m^2 + i\epsilon}$,  $= -i\lambda.$

Calculating the propagator to first non-trivial order we find



$$= \frac{i}{p^2 - m^2 + i\epsilon} + \frac{i}{p^2 - m^2 + i\epsilon} \left(-i\lambda \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \right) \frac{i}{p^2 - m^2 + i\epsilon}, \tag{2.2}$$

where p is the external momentum and k the loop-momentum. The integral over the loop-momentum is obviously divergent and we postpone its calculation for a moment. The 4-point interaction to first non-trivial order with stripped off propagators for external legs is given by



$$= -i\lambda + (-i\lambda)^2 \, i^2 \left[\frac{1}{2} V(s) + V(t) + V(u) \right] \tag{2.3}$$

where the $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, and $u = (p_1 - p_4)^2$ are the Mandelstam variables and the integral over the momentum is given by

$$V(p^2) = \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(p - k)^2 - m^2 + i\epsilon} \cdot \frac{1}{k^2 - m^2 + i\epsilon}, \tag{2.4}$$

[1] All Feynman diagrams in this work have been drawn with the Latex package AxoDraw [51]

where we encounter a similar integral as above. In fact, one typically encounters such integrals when calculating loop diagrams and it is thus worthwhile to find a general strategy to solve them.

2.1 Regularization

When calculating diagrams involving loops, one often encounters momentum-space integrals over rational functions. It is convenient to solve these by bringing them into the form

$$I(F,n) = \int \frac{d^4\,\ell}{(2\pi)^4}\,\frac{F(\ell)}{(\ell^2-\Delta)^n}, \tag{2.5}$$

where ℓ is the (possibly shifted) loop-momentum, $F(\ell)$ is a polynomial in ℓ and Δ a function of mass-dimension $[\Delta] = 2$ of external momenta and the masses. To complete the square in the denominator and find the right ℓ one often introduces an integral over Feynman parameters:

$$\frac{1}{A_1A_2\cdots A_n} = \int_0^1 dx_1\cdots dx_n\,\delta(\sum_{i=1}^{n} x_i - 1)\frac{(n-1)!}{[x_1A_1 + x_2A_2 + \cdots + x_nA_n]^n}. \tag{2.6}$$

For only two denominator factors this reduces to

$$\frac{1}{AB} = \int_0^1 dx\,\frac{1}{[xA + (1-x)B]^2}. \tag{2.7}$$

In our case we find that the integral from the propagator is already in the right form. The integral from the 4-point vertex can be rewritten with the help of Feynman parameters:

$$\begin{aligned} V(p^2) &= \int \frac{d^4k}{(2\pi)^4}\,\frac{1}{(p-k)^2+i\epsilon}\cdot\frac{1}{k^2+i\epsilon} \\ &= \int_0^1 dx \int \frac{d^4k}{(2\pi)^4}\,\frac{1}{\left[xp^2 - 2xp\cdot k + xk^2 - xm^2 + k^2 - m^2 - xk^2 + xm^2 + i\epsilon\right]^2} \\ &= \int_0^1 dx \int \frac{d^4k}{(2\pi)^4}\,\frac{1}{[(k-xp)^2 + x\,(1-x)\,p^2 - m^2 + i\epsilon]^2} \\ &= \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4}\,\frac{1}{\left[\ell^2 + x\,(1-x)\,p^2 - m^2 + i\epsilon\right]^2}, \end{aligned} \tag{2.8}$$

where we shifted the integration variable $\ell \equiv k - xp$.

Having brought integrals over loop momenta into the form of (2.5), we find that any term proportional to odd powers of ℓ in the numerator will integrate to zero by symmetry since the denominator is a function of ℓ^2. Hence, the remaining task is to calculate integrals of the form

$$I(m,n) = \int \frac{d^4\ell}{(2\pi)^4}\,\frac{\left(\ell^2\right)^m}{\left(\ell^2-\Delta\right)^n}. \tag{2.9}$$

The integral can be Wick-rotated to a Euclidean metric by identifying

$$\ell^0 \equiv i\ell_E^0 \qquad \Rightarrow \qquad \ell^2 = -\ell_E^2, \quad d^d\ell = i\,d^d\ell_E, \tag{2.10}$$

where we consider d-dimensional space-time for generality. We can rewrite the integral in (2.9)

$$I^{(d)}(m,n) = i\,(-1)^{n+m} \int \frac{d^d\ell_E}{(2\pi)^4}\,\frac{\left(\ell_E^2\right)^m}{\left(\ell_E^2+\Delta\right)^n}. \tag{2.11}$$

Since the integrand is a function of ℓ_E^2 only, it is convenient to switch to spherical coordinates

$$\begin{aligned} I^{(d)}(m,n) &= \frac{i\,(-1)^{n+m}}{(2\pi)^d} \int d\Omega_d \int_0^\infty d\ell_E\,\ell_E^{d-1}\,\frac{\left(\ell_E^2\right)^m}{\left(\ell_E^2+\Delta\right)^n} \\ &= \frac{i\,(-1)^{n+m}}{2\,(2\pi)^d} \int d\Omega_d \int_0^\infty d\left(\ell_E^2\right)\,\frac{\left(\ell_E^2\right)^{\frac{d}{2}-1+m}}{\left(\ell_E^2+\Delta\right)^n}, \end{aligned} \tag{2.12}$$

where $\int d\Omega_d$ is the surface of the d-dimensional unit sphere. In $d = 4$ dimensions $\int d\Omega_d = 2\pi^2$, hence,

$$I^{(4)}(m,n) = \frac{i(-1)^{n+m}}{16\pi^2}\int_0^\infty d\left(\ell_E^2\right)\frac{\left(\ell_E^2\right)^{1+m}}{\left(\ell_E^2+\Delta\right)^n}. \tag{2.13}$$

For $n > 2+m$ the integral $I^{(4)}(m,n)$ converges and can immediately be calculated. For $n \neq 2+m$, $I^{(4)}(n,m)$ is divergent and must be regulated. A number of different regularization schemes has been invented. We demonstrate three schemes in the following: cut-off, Pauli-Villars, and dimensional regularization.

2.1.1 Cut-off regularization

The perhaps simplest regularization scheme is the so-called cut-off regularization, where one "cuts off" the integration at some scale Λ by replacing

$$\int_0^\infty d\left(\ell_E^2\right) \to \int_0^{\Lambda^2} d\left(\ell_E^2\right). \tag{2.14}$$

We explicitly calculate some integrals in cut-off regularization, e.g. for the cases $I(0,1)$ and $I(0,2)$ encountered in the one-loop amplitudes of our ϕ^4-theory. $I(0,n)$ is convergent for $n \geq 3$. For $n = 1$ we find:

$$\begin{aligned} I_\Lambda(0,1) &= \frac{-i}{16\pi^2}\int_0^{\Lambda^2} d\left(\ell_E^2\right)\frac{\left(\ell_E^2\right)}{\left(\ell_E^2+\Delta\right)} \\ &= \frac{-i}{16\pi^2}\left[\ell_E^2 - \Delta\log\left(\ell_E^2+\Delta\right)\right]_0^{\Lambda^2} \\ &= \frac{-i}{16\pi^2}\left(\Lambda^2 - \Delta\log\left(\frac{\Lambda^2}{\Delta}+1\right)\right). \end{aligned} \tag{2.15}$$

For $\Lambda \to \infty$ this diverges quadratically $I_\Lambda(0,1) \to -i\Lambda^2/16\pi^2$. In the case $n = 2$ we find

$$\begin{aligned} I_\Lambda(0,2) &= \frac{i}{16\pi^2}\int_0^{\Lambda^2} d\left(\ell_E^2\right)\frac{\left(\ell_E^2\right)}{\left(\ell_E^2+\Delta\right)^2} \\ &= \frac{i}{16\pi^2}\left[\frac{\Delta}{\ell_E^2+\Delta} + \log\left(\ell_E^2+\Delta\right)\right]_0^{\Lambda^2} \\ &= \frac{i}{16\pi^2}\left(\frac{\Delta}{\Lambda^2+\Delta} - 1 + \log\left(\frac{\Lambda^2}{\Delta}+1\right)\right). \end{aligned} \tag{2.16}$$

Now we find only a logarithmic divergence $I(0,2) \to i\log\left(\Lambda^2/\Delta\right)/16\pi^2$. Similarly, the integrals $I(m,n)$ can be calculated for all m, n. The degree of divergence will be given by $D = 4+2m-2n$. However, while this regularization is simple to calculate it has drawbacks. E.g., it is not invariant under shifting the integrating variable $\ell \to \ell + k$ as is done when bringing the momentum space integrals into the form of (2.5). Hence, it is difficult to relate the Λ's if one has more than one integral in an amplitude, as for the case of the 4-point function (2.3).

2.1.2 Pauli-Villars regularization

In Pauli-Villars regularization one introduces a heavy field with mass M and the same quantum numbers as the field in the loop, but opposite statistics. Since e.g. scalar fields with fermionic statistics are physically meaningless one needs to remove these fields by taking the limit $M \to \infty$ after integrating. In the case of the one-loop propagator in our ϕ^4-model, Pauli-Villars regularization is implemented by replacing the propagator in the divergent integral:

$$\begin{aligned}\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{\ell^2+i\epsilon} \quad &\to \quad \int \frac{d^4\ell}{(2\pi)^4}\left[\frac{1}{\ell^2+i\epsilon}-\frac{1}{\ell^2-M^2+i\epsilon}\right] \\ &= \quad -\frac{i}{16\pi^2}\int_0^\infty d(\ell_E^2)\left[\frac{\ell_E^2}{\ell_E^2}-\frac{\ell_E^2}{\ell_E^2+M^2}\right] \\ &= \quad -\frac{i}{16\pi^2}\left[\ell_E^2-\ell_E^2+M^2\log(\ell_E^2+M^2)\right]_0^\infty \\ &= \quad -\frac{i}{16\pi^2}M^2\log\left(\frac{\ell^2\to\infty}{M^2}+1\right). \end{aligned} \tag{2.17}$$

In the limit $M\to\infty$ this is again quadratically divergent.

2.1.3 Dimensional regularization

We stated above that the integral $I\,(m,n)$ is divergent if $4+2m-2n\geq 0$. If we however generalize to d space-time dimensions, $I\,(m,n)$ is divergent for $\frac{d}{2}+m-n\geq 0$. The basic idea of dimensional regularization is to carry out integrals that are divergent in $d=4$ in some dimension $d<4$ where they are convergent, and then continue the result analytically to $d=4$. Thus, we need to consider the integral

$$I^{(d)}\,(m,n)=\frac{i\,(-1)^{n+m}}{2\,(2\pi)^d}\int d\Omega_d\int_0^\infty d\left(\ell_E^2\right)\frac{\left(\ell_E^2\right)^{\frac{d}{2}-1+m}}{\left(\ell_E^2+\Delta\right)^n} \tag{2.18}$$

The analytic continuation of the surface of a d-dimensional unit sphere is given by

$$\int d\Omega_d=\frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{2.19}$$

An simple argument for this is given in [17]: Starting from a Gaussian-integral one can write:

$$\begin{aligned}\left(\sqrt{\pi}\right)^d \quad &= \quad \left(\int dx\,e^{-x^2}\right)^d=\int d^dx\,e^{\left(-\sum_{i=1}^d x_i^2\right)}=\int d\Omega_d\int_0^\infty dx\,x^{d-1}e^{-x^2} \\ &= \quad \frac{1}{2}\int d\Omega_d\int_0^\infty d\left(x^2\right)\,\left(x^2\right)^{\frac{d}{2}-1}e^{-\left(x^2\right)}. \end{aligned} \tag{2.20}$$

This integral is the definition of the Γ-function. Hence,

$$\left(\sqrt{\pi}\right)^d=\frac{1}{2}\Gamma\left(\frac{d}{2}\right)\int d\Omega_d \Leftrightarrow \int d\Omega_d=\frac{2\pi^{d/2}}{\Gamma(d/2)} \tag{2.21}$$

Now we can rewrite our integral

$$I^{(d)}\,(m,n)=\frac{i\,(-1)^{n+m}}{(4\pi)^{d/2}\,\Gamma\left(\frac{d}{2}\right)}\int_0^\infty d\left(\ell_E^2\right)\frac{\left(\ell_E^2\right)^{\frac{d}{2}-1+m}}{\left(\ell_E^2+\Delta\right)^n}. \tag{2.22}$$

To calculate this integral we make a coordinate transformation

$$\ell_E^2\equiv\Delta\left(\frac{1}{x}-1\right), \qquad \Rightarrow d\left(\ell_E^2\right)=-\frac{\Delta}{x^2}dx. \tag{2.23}$$

This allows us to rewrite the integral

$$\begin{aligned} I^{(d)}\,(m,n) \quad &= \quad -\frac{i\,(-1)^{n+m}}{(4\pi)^{d/2}\,\Gamma\left(\frac{d}{2}\right)}\int_1^0 dx\,\frac{\Delta}{x^2}\frac{\left[\Delta\left(\frac{1}{x}-1\right)\right]^{\frac{d}{2}-1+m}}{\left[\Delta/x\right]^n} \\ &= \quad \frac{i\,(-1)^{n+m}\,\Delta^{\frac{d}{2}+m-n}}{(4\pi)^{d/2}\,\Gamma\left(\frac{d}{2}\right)}\int_0^1 dx\,x^{n-2}\left(\frac{1}{x}-1\right)^{\frac{d}{2}-1+m} \\ &= \quad \frac{i\,(-1)^{n+m}\,\Delta^{\frac{d}{2}+m-n}}{(4\pi)^{d/2}\,\Gamma\left(\frac{d}{2}\right)}\int_0^1 dx\,x^{n-m-\frac{d}{2}-1}\,(1-x)^{\frac{d}{2}+m-1}. \end{aligned} \tag{2.24}$$

The remaining integral is the Beta-integral

$$\int_0^1 dx\, x^{p-1}\,(1-x)^{q-1} = B\,(p,q) = \frac{\Gamma\,(p)\,\Gamma\,(q)}{\Gamma\,(p+q)}. \tag{2.25}$$

Hence, we arrive at a general expression for our integral

$$I^{(d)}\,(m,n) = \frac{i\,(-1)^{n+m}}{(4\pi)^{d/2}} \cdot \frac{\Gamma\left(n-m-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}+m\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma\,(n)} \left(\frac{1}{\Delta}\right)^{n-m-\frac{d}{2}} \tag{2.26}$$

We again consider explicitly the case $m=0$. Then, we find

$$I^{(d)}\,(0,n) = \frac{i\,(-1)^{n}}{(4\pi)^{d/2}} \cdot \frac{\Gamma\left(n-\frac{d}{2}\right)}{\Gamma\,(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}} \tag{2.27}$$

For the divergent cases $n<2$ we find

$$\begin{aligned} I^{(d)}\,(0,1) &= \frac{-i}{(4\pi)^{d/2}} \cdot \Gamma\left(1-\frac{d}{2}\right)\left(\frac{1}{\Delta}\right)^{1-\frac{d}{2}} = \frac{-i\Delta}{16\pi^2}\left(\frac{4\pi}{\Delta}\right)^{\varepsilon/2}\Gamma\left(-1+\frac{\varepsilon}{2}\right), & (2.28)\\ I^{(d)}\,(0,2) &= \frac{i}{(4\pi)^{d/2}} \cdot \Gamma\left(2-\frac{d}{2}\right)\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} = \frac{i}{16\pi^2}\left(\frac{4\pi}{\Delta}\right)^{\varepsilon/2}\Gamma\left(\frac{\varepsilon}{2}\right), & (2.29)\end{aligned}$$

where we introduced $d=4-\varepsilon$. Since $\Gamma\,(x)$ has poles at $-x\in\mathbb{N}$ we find $I^{(d)}\,(0,1)$ to be divergent at $d=2,4,6,8,\ldots$ and $I^{(d)}\,(0,2)$ to be divergent at $d=4,6,8,\ldots$. To check our procedure we also calculate $I^{d}\,(0,3)$, which should be regular at $d=4$:

$$I^{(d)}\,(0,3) = \frac{-i}{(4\pi)^{d/2}} \cdot \frac{\Gamma\left(3-\frac{d}{2}\right)}{\Gamma\,(3)} \left(\frac{1}{\Delta}\right)^{3-\frac{d}{2}} \xrightarrow{d\to 4} \frac{-i}{32\pi^2\Delta}. \tag{2.30}$$

To make sense of the poles in the Γ-functions we use the expansion of $\Gamma\,(x-n)$ near the poles $n=0,1,\ldots$:

$$\begin{aligned} \Gamma\,(x) &= \frac{1}{x}-\gamma+\mathcal{O}\,(x)\,, & (2.31)\\ \Gamma\,(x-1) &= -\frac{1}{x}+\gamma-1+\mathcal{O}\,(x)\,, & (2.32)\\ \Gamma\,(x-2) &= \frac{1}{2x}+\frac{3-2\gamma}{4}+\mathcal{O}\,(x)\,, & (2.33)\\ \Gamma\,(x-3) &= -\frac{1}{6x}+\frac{6\gamma-11}{36}+\mathcal{O}\,(x)\,, & (2.34)\end{aligned}$$

where $\gamma\simeq 0.57722$ is the Euler-Mascheroni constant. We also expand the reoccurring

$$\left(\frac{4\pi}{\Delta}\right)^{\varepsilon/2} = 1+\varepsilon\cdot\frac{\log\,(4\pi)-\log\Delta}{2}+\frac{\varepsilon^2}{2}\cdot\left(\frac{\log\,(4\pi)-\log\Delta}{2}\right)^2+\mathcal{O}\left(\varepsilon^3\right). \tag{2.35}$$

Thus, we can write our divergent amplitudes as

$$\begin{aligned} I^{(4-\varepsilon)}\,(0,1) &= \frac{-i\Delta}{16\pi^2}\left(1+\frac{\varepsilon}{2}\left(\log\,(4\pi)-\log\Delta\right)+\mathcal{O}\,(\varepsilon)\right)\left(-\frac{2}{\varepsilon}+\gamma-1+\mathcal{O}\,(\varepsilon)\right)\\ &= \frac{i\Delta}{16\pi^2}\left(\frac{2}{\varepsilon}-\gamma+1+\log\,(4\pi)-\log\Delta+\mathcal{O}\,(\varepsilon)\right), & (2.36)\\ I^{(4-\varepsilon)}\,(0,2) &= \frac{i}{16\pi^2}\left(1+\frac{\varepsilon}{2}\left(\log\,(4\pi)-\log\Delta\right)+\mathcal{O}\,(\varepsilon)\right)\left(\frac{2}{\varepsilon}-\gamma+\mathcal{O}\,(\varepsilon)\right)\\ &= \frac{i}{16\pi^2}\left(\frac{2}{\varepsilon}-\gamma+\log\,(4\pi)-\log\,(\Delta)+\mathcal{O}\,(\varepsilon)\right). & (2.37)\end{aligned}$$

For our calculations of loop-diagrams it is useful to have a table of integrals $I(m,n)$ in dimensional regularization for the lowest values of m. We find

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2-\Delta)^n} = \frac{i(-1)^n}{(4\pi)^{d/2}} \cdot \frac{\Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}}, \tag{2.38}$$

$$\begin{aligned}\int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^2}{(\ell^2-\Delta)^n} &= \frac{i(-1)^{n+1}}{(4\pi)^{d/2}} \cdot \frac{\Gamma\left(n-1-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}+1\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-1-\frac{d}{2}} \\ &= \frac{i(-1)^{n+1}}{(4\pi)^{d/2}} \cdot \frac{d}{2} \cdot \frac{\Gamma\left(n-1-\frac{d}{2}\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-1-\frac{d}{2}},\end{aligned} \tag{2.39}$$

$$\begin{aligned}\int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^4}{(\ell^2-\Delta)^n} &= \frac{i(-1)^{n+2}}{(4\pi)^{d/2}} \cdot \frac{\Gamma\left(n-2-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}+2\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-2-\frac{d}{2}} \\ &= \frac{i(-1)^{n}}{(4\pi)^{d/2}} \cdot \frac{d(d+2)}{4} \cdot \frac{\Gamma\left(n-2-\frac{d}{2}\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-2-\frac{d}{2}}.\end{aligned} \tag{2.40}$$

Besides being implemented comparatively easily in all orders of loops, one of the main advantages of dimensional regularization is that it conserves all symmetries of the theory explicitly, in particular gauge invariance.

2.2 Renormalization

In the section above we have found general schemes for the treatment of the divergent integrals over internal momenta occurring in the computation of diagrams involving loops. In the following we will compute such diagrams in dimensional regularization.

For our ϕ^4-theory at one-loop level, we can now write down the one-loop amplitudes for the propagator:

$$\begin{aligned}\Delta^{(1)}(p) &= \frac{i}{p^2-m^2} + \frac{i}{p^2-m^2}\left(\lambda M^{4-d}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2-m^2}\right)\frac{i}{p^2-m^2} \\ &= \frac{i}{p^2-m^2} + \frac{i}{p^2-m^2}\left(\lambda M^{4-d}\frac{-i}{(4\pi)^{d/2}}\Gamma(1-\frac{d}{2})\left(\frac{1}{m^2}\right)^{1-\frac{d}{2}}\right)\frac{i}{p^2-m^2} \\ &= \frac{i}{p^2-m^2} + \frac{i}{p^2-m^2}\left(\frac{i\lambda m^2}{16\pi^2}\left(\frac{2}{\varepsilon}-\gamma+\log\frac{4\pi M^2}{m^2}-1+\mathcal{O}(\varepsilon)\right)\right)\frac{i}{p^2-m^2},\end{aligned} \tag{2.41}$$

where we introduced an arbitrary mass scale M to keep the coupling constant λ dimensionless when changing the dimensionality of the integral. For the integral appearing in the 4-point vertex we find

$$\begin{aligned}V(p^2) &= \int_0^1 dx\, M^{4-d}\int \frac{d^d\ell}{(2\pi)^d}\frac{1}{\left[\ell^2+x(1-x)p^2-m^2+i\epsilon\right]^2} \\ &= \int_0^1 dx\,\frac{iM^{4-d}}{(4\pi)^{d/2}}\Gamma(2-\frac{d}{2})\left(\frac{1}{m^2-x(1-x)p^2}\right)^{2-\frac{d}{2}} \\ &= \frac{i}{16\pi^2}\int_0^1 dx\left(\frac{2}{\varepsilon}-\gamma+\log\left(\frac{4\pi M^2}{m^2-x(1-x)p^2}\right)+\mathcal{O}(\varepsilon)\right) \\ &= \frac{i}{16\pi^2}\left(\frac{2}{\varepsilon}-\gamma+\int_0^1 dx\,\log\left(\frac{4\pi M^2}{m^2-x(1-x)p^2}\right)+\mathcal{O}(\varepsilon)\right).\end{aligned} \tag{2.42}$$

With this, we can write down the 4-point amplitude at the one-loop level:

$$-i\mathbf{V}^{(1)} = -i\lambda + \frac{5i\lambda^2}{32\pi^2}\left(\frac{2}{\varepsilon} - \gamma + \frac{1}{5}\int_0^1 dx\, \left[\log\left(\frac{4\pi M^2}{m^2 - x(1-x)s}\right) + 2\log\left(\frac{4\pi M^2}{m^2 - x(1-x)t}\right) + \right.\right.$$
$$\left.\left. +2\log\left(\frac{4\pi M^2}{m^2 - x(1-x)u}\right)\right]\right). \tag{2.43}$$

Through the regularization procedure we have succeeded to treat the infinities in a systematic fashion, however, they are still present in the amplitudes. In order to arrive at physical observables we have to renormalize our theory. In the course of this, one has to trade the "bare" parameters of the theory (λ and m in our case) for renormalized, physical quantities. We start by rescaling the fields, which we are always free to do. In general, every field in a theory is rescaled independently.

In our ϕ^4-case there is only one field, which we rescale by:

$$\phi = Z^{1/2}\phi_r, \tag{2.44}$$

where we denote the "renormalized" field ϕ_r and Z is the field renormalization. The Lagrangian (2.1) in terms of the rescaled field is

$$\mathcal{L} = Z\partial_\mu\phi_r^\dagger\partial^\mu\phi_r - m_0^2 Z\phi_r^\dagger\phi_r - \frac{\lambda_0}{4}Z^2\left(\phi_r^\dagger\phi_r\right)^2, \tag{2.45}$$

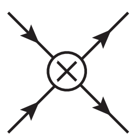
where we now denote the bare coupling λ_0 and the bare mass m_0. They are eliminated by introducing the physically measure mass m and coupling λ and defining so-called counterterms

$$\delta_Z \equiv Z - 1, \qquad \delta_m \equiv m_0 Z^2 - m^2, \qquad \delta_\lambda \equiv \lambda_0 Z^2 - \lambda. \tag{2.46}$$

The Lagrangian then becomes

$$\mathcal{L} = \partial_\mu\phi_r^\dagger\partial^\mu\phi_r - m^2\phi_r^\dagger\phi_r - \frac{\lambda}{4}\left(\phi_r^\dagger\phi_r\right)^2 + \delta_Z\partial_\mu\phi_r^\dagger\partial^\mu\phi_r - \delta_m\phi_r^\dagger\phi_r - \frac{\delta_\lambda}{4}\left(\phi_r^\dagger\phi_r\right)^2. \tag{2.47}$$

The first three terms look like our familiar ϕ^4 theory (2.1), but now in terms of the renormalized mass and coupling. The corresponding Feynman rules are the same as given below (2.1), but exchanging the bare parameters for the renormalized ones. The counterterms give rise to additional Feynman rules:

 $= i\left(p^2\delta_Z - m^2\delta_m\right),$  $= -i\delta_\lambda$

Including these in our one-loop amplitudes, the propagator (2.41) becomes

$$\Delta^{(1)}(p^2) = \frac{i}{p^2 - m^2} + \frac{i}{p^2 - m^2}\left(\frac{i\lambda m^2}{16\pi^2}\left(\frac{2}{\varepsilon} - \gamma + \log\frac{4\pi M^2}{m^2} - 1\right) + i\left(p^2\delta_Z - m^2\delta_m\right)\right)\frac{i}{p^2 - m^2}, \tag{2.48}$$

and the 4-point amplitude (2.43) becomes

$$-i\mathbf{V}^{(1)} = -i\lambda + \frac{5i\lambda^2}{32\pi^2}\left(\frac{2}{\varepsilon} - \gamma + \frac{1}{5}\int_0^1 dx\, \left[\log\left(\frac{4\pi M^2}{m^2 - x(1-x)s}\right) + 2\log\left(\frac{4\pi M^2}{m^2 - x(1-x)t}\right) + \right.\right.$$
$$\left.\left. +2\log\left(\frac{4\pi M^2}{m^2 - x(1-x)u}\right)\right]\right) - i\delta_\lambda. \tag{2.49}$$

In order to make more sense of the counterterms, one has to introduce so-called renormalization conditions. There are various renormalization schemes, which each have their advantages and drawbacks. In the following, we demonstrate two schemes, "On-shell" and "$\overline{\text{MS}}$" renormalization:

2.2.1 On-shell scheme

On-shell renormalization is perhaps the physically most intuitive renormalization scheme. One defines the loop-corrected amplitudes to be equal to physical amplitudes for on-shell particles, i.e.

$$\Delta^{(1)}(p^2 = m^2) = \frac{i}{p^2 - m^2}, \qquad -i\mathbf{V}^{(1)}(s = 4m^2, t = u = 0) = -i\lambda, \tag{2.50}$$

where the first equation specifies the location and residue of the pole of the propagator. In our case we can then immediately read off the counterterms from (2.48), (2.49)

$$\begin{aligned} \delta_Z &= 0, & (2.51)\\ \delta_m &= \frac{\lambda}{16\pi^2}\left(\frac{2}{\varepsilon} - \gamma + \log\frac{4\pi M^2}{m^2} - 1\right), & (2.52)\\ \delta_\lambda &= \frac{5\lambda^2}{32\pi^2}\left(\frac{2}{\varepsilon} - \gamma + \log\left(\frac{4\pi M^2}{m^2}\right) + \frac{2}{5}\right), & (2.53) \end{aligned}$$

using $\int_0^1 dx \, \log\left(1 - 4x\,(1-x)\right) = -2$. The advantage of this renormalization scheme is that we are left with the physical propagator for an on-shell particle[2], and λ is the physical coupling at the chosen renormalization point $s = 4m^2, t = u = 0$. The amplitudes become independent of the arbitrary energy scale M. Drawbacks of the on-shell scheme are e.g. that it is quite cumbersome to implement for more complicated theories and higher-order corrections, and more important, it is not well defined in the massless limit $m^2 \to 0$. Furthermore, in confined theories like quantum chromodynamics, the notion of on-shell particles loses its physical intuition.

2.2.2 $\overline{\text{MS}}$ scheme

A more general and easier implemented method is the class of minimal subtraction (MS) schemes. In pure minimal subtraction, one defines the counterterms to absorb only the terms proportional to $(1/\varepsilon)$ appearing in divergent quantities. The arbitrary mass scale M then remains in the amplitudes and must be dealt with by introducing an M-dependency in the renormalized couplings and masses, as we will see below.

Often more convenient is the modified minimal subtraction ($\overline{\text{MS}}$) scheme. As we saw above, the $(1/\varepsilon)$-terms are accompanied by γ and $\log(4\pi)$ terms. In $\overline{\text{MS}}$ renormalization one chooses the counterterms to also absorb the $(-\gamma + \log(4\pi))$ terms, which is equivalent to redefining the mass scale $\tilde{M}^2 = 4\pi M^2/e^\gamma$. Subtracting $\overline{\text{MS}}$-counterterms, our one-loop expressions become

$$\Delta^{(1)}(p^2) = \frac{i}{p^2 - m^2} + \frac{i}{p^2 - m^2} \cdot \frac{i\lambda m^2}{16\pi^2}\left(\log\frac{\tilde{M}^2}{m^2} - 1\right)\frac{i}{p^2 - m^2}, \tag{2.54}$$

for the propagator (2.48), and for the 4-point function (2.49)

$$\begin{aligned} -i\mathbf{V}^{(1)} = -i\lambda + \frac{i\lambda^2}{32\pi^2}\int_0^1 dx \, \Bigg[&\log\left(\frac{\tilde{M}^2}{m^2 - x\,(1-x)\,s}\right) + 2\log\left(\frac{\tilde{M}^2}{m^2 - x\,(1-x)\,t}\right) + \\ &+2\log\left(\frac{\tilde{M}^2}{m^2 - x\,(1-x)\,u}\right)\Bigg]. \end{aligned} \tag{2.55}$$

2.3 Running couplings: β-functions and anomalous dimensions

When we regularized the one-loop divergences in dimensional regularization we introduced an arbitrary mass scale M. Any physical quantity in the theory must not depend on this arbitrary parameter. To achieve this, the renormalized couplings, masses, and field renormalizations must

[2] In the particular case of ϕ^4-theory we find that the counterterm cancels the 1-loop contribution entirely and we are in fact left with $\Delta^{(1)}(p^2) = i/(p^2 - m^2)$ for any p^2.

depend on M such as to cancel the dependency in physical quantities. This can e.g. be implemented on the level of the theories' Green's functions $\Gamma^{(n_i)}$, where the n_i are the number of external fields of type i. Then, the Greens' functions have to satisfy a Callan-Symanzik equation

$$\left[M\frac{\partial}{\partial M}+\beta_{g_i}\frac{\partial}{\partial g_i}+n_i\gamma_i+m_i\gamma_{m_i}\frac{\partial}{\partial m_i}\right]\Gamma^{(n_i)}=0, \tag{2.56}$$

where $\gamma_i=\frac{M}{2}\cdot\frac{\partial Z_i}{\partial M}$ is called the anomalous dimension of the field i, describing the scaling of the fields renormalization with M, and the β-functions $\beta_{g_i}=M\frac{\partial g_i}{\partial M}$ describe the scaling of the renormalized couplings g_i. γ_{m_i} is the anomalous dimension of the mass parameters m_i. In the line of this argument and considering the general form of loop-corrections, we can understand the arbitrary mass scale M^2 as the scale of invariants built from external momenta involved in the process. Hence, the β-functions can be interpreted as the scaling of the physical couplings with the scale of involved momenta.

For our example, complex ϕ^4-theory, there is only one kind of field ϕ, one coupling λ, and one mass parameter m. From our results for the one-loop propagator and the amputated 4-point function in $\overline{\mathrm{MS}}$ renormalization (2.54), (2.55), we can immediately write down the corresponding two- and four-point Green's functions. For simplicity, we consider the massless limit. Then, the Green's functions are

$$\Gamma^{(2)} = \frac{i}{p^2} \tag{2.57}$$

$$\begin{aligned}\Gamma^{(4)} = &\left(-i\lambda+\frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left[\log\left(\frac{M^2}{-x\,(1-x)\,s}\right)+2\log\left(\frac{M^2}{-x\,(1-x)\,t}\right)+\right.\right.\\ &\left.\left.+2\log\left(\frac{M^2}{-x\,(1-x)\,u}\right)\right]\right)\prod_{i=1}^{4}\frac{i}{p_i^2}\end{aligned} \tag{2.58}$$

The corresponding Callan-Symanzik equation is

$$\left[M\frac{\partial}{\partial M}+\beta_\lambda\frac{\partial}{\partial\lambda}+n_\phi\gamma_\phi\right]\Gamma^{(n_\phi)}=0, \tag{2.59}$$

From the two-point Green's functions we can immediately conclude that the anomalous dimension is zero at the one-loop level: $\gamma_\phi=0+\mathcal{O}(\lambda^2)$. From $\Gamma^{(4)}$ we obtain

$$0=M\frac{2}{M}\cdot\frac{5i\lambda^2}{32\pi^2}+\beta_\lambda\left(-i+\mathcal{O}(\lambda)\right)+\mathcal{O}(\lambda^3) \tag{2.60}$$

where we we omitted terms $\mathcal{O}(\lambda)$ from $\partial\Gamma^{(4)}/\partial\lambda$ multiplying β_λ, and $\mathcal{O}(\lambda^3)$ terms coming from the $4\gamma_\phi\Gamma^{(4)}$ terms. From (2.60) we can read off the one-loop contribution to the β-function:

$$\beta_\lambda=\frac{5\lambda^2}{16\pi^2}+\mathcal{O}(\lambda^3). \tag{2.61}$$

After this sketch of renormalization we are ready to compute the renormalized parameters of the SM-potential. The general procedure we arrived at for doing so consists of computing the divergent contributions to convenient Green's functions in dimensional regularization, and after subtracting off $\overline{\mathrm{MS}}$ counterterms, obtaining the anomalous dimensions and β-functions of the theory by applying the Callan-Symanzik equation to these Green's functions.

3 The SM Higgs potential at the one-loop level

3.1 Model

In order to compute the effective Higgs potential we need to find the running of the quartic Higgs self-coupling λ. At the one-loop level, the dominant contributions are given by the running of λ through Higgs- and fermion-loops. Considering the size of the Yukawa couplings, the dominant fermion-contribution will arise through the top quark's Yukawa coupling. At the electroweak scale, the strong coupling g_s is also much larger than the electroweak couplings g_Y, g_2. Hence, we consider a simplified version of the SM, turning off the electroweak interaction $g_Y = g_2 = 0$, and setting all Yukawa couplings except for the top quark to zero. Since we are eventually interested in the behavior at energies close to the Planck scale we calculate in the electroweak-unbroken phase.

The Lagrangian for this simplified theory is

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\Phi + \mathcal{L}_{y_t} + \mathcal{L}_{\mathrm{QCD}}, \tag{3.1}$$

where the free Lagrangian $\mathcal{L}_0$ is given by

$$\mathcal{L}_0 = (\partial_\mu \Phi)^2 + \overline{q} i \partial\!\!\!/ q. \tag{3.2}$$

$\Phi = (\phi^+, \phi_0)^T$ is the Higgs SU(2)-doublet and q are the quark fields. We use Feynman slash notation $a\!\!\!/ \equiv \gamma^\mu a_\mu$, where γ^μ are Dirac matrices. The interactions from the Higgs sector are given by

$$\mathcal{L}_\Phi = -\mu^2 \Phi^\dagger \Phi - \lambda \left(\Phi^\dagger \Phi\right)^2, \tag{3.3}$$

$$\mathcal{L}_{y_t} = -y_t \left(\overline{t}_R \left(\Phi^c\right)^\dagger T_L + \overline{T}_L \Phi^c t_R \right), \tag{3.4}$$

where the left-handed t- and b-quarks are organized in an SU(2) doublet $T_L = (t_L, b_L)^T$ and the charge conjugated Higgs-doublet is given by $\Phi^c \equiv i\sigma_2 \Phi^* = (\phi_0^*, -\phi^-)$. The right handed top t_R is an SU(2) singlet. Left- and right-handed parts of the fields are obtained by the projectors

$$P_L = \frac{1}{2}(1 - \gamma_5), \qquad P_R = \frac{1}{2}(1 + \gamma_5), \tag{3.5}$$

where $\gamma_5 = \frac{i}{4!}\epsilon^{\mu\nu\rho\delta}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\delta$. The QCD sector is given by

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4} G^A_{\mu\nu} G^{A,\mu\nu} + g_s \overline{q} A\!\!\!/^A T^A q + \mathcal{L}_{\mathrm{gf}} + \mathcal{L}_{\mathrm{gh}}, \tag{3.6}$$

where the A^A_μ are the gluon fields, and the gluon field-strength tensor $G^A_{\mu\nu} T^A = \frac{i}{g_s}\left[D_\mu, D_\nu\right]$ is given by the covariant derivative $D_\mu = \partial_\mu - i g_s A^A_\mu T^A$, hence

$$G^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + g_s f^{ABC} A^B_\mu A^C_\nu. \tag{3.7}$$

The f^{ABC} are the structure constants of SU(3),

$$\left[T^A, T^B\right] = i f^{ABC} T^C, \tag{3.8}$$

where the T^A are the SU(3) generators for the fundamental representation and can be written in terms of the Gell-Mann matrices λ^A, using $T^A = \lambda^A/2$. The ghost Lagrangian $\mathcal{L}_{\text{gh}}$ and gauge-fixing Lagrangian $\mathcal{L}_{\text{gf}}$ in R_ξ-gauge are given by

$$\mathcal{L}_{\text{gf}} = -\frac{1}{2\xi}\left(\partial^\mu A^A_\mu\right)^2, \qquad \mathcal{L}_{\text{gh}} = \partial_\mu \bar{c}^A \partial^\mu c^A + g_s f^{ABC} \partial_\mu \bar{c}^A A^{B,\mu} c^C. \tag{3.9}$$

Renormalizing the fields

$$\Phi_0 = Z_\Phi^{1/2}\Phi_r, \quad Q_{L,0} = \left(Z_L^q\right)^{1/2} Q_{L,0}, \quad q_{R,0} = \left(Z_R^q\right)^{1/2} q_{R,r}, \quad A_0 = Z_g^{1/2} A_r, \quad c_0 = Z_c^{1/2} c_r, \tag{3.10}$$

the Lagrangian can be written as $\mathcal{L} = \mathcal{L}_r + \mathcal{L}_{\text{ct}}$, where $\mathcal{L}_r$ is our original Lagrangian (3.1) but in terms of renormalized couplings and fields, and the counterterm Lagrangian is given by

$$\begin{aligned}\mathcal{L}_{\text{ct}} = {}& \delta^{(2\Phi)}\partial_\mu \Phi^\dagger \partial^\mu \Phi - \bar{q} i \partial\!\!\!/ \left(\delta_R^{(2q)} P_R + \delta_L^{(2q)} P_L\right) q - \delta_\mu^{(2\Phi)} \Phi^\dagger \Phi - \delta_\lambda^{(4\Phi)} \left(\Phi^\dagger\Phi\right)^2 + \\ & - \delta_{y_t}^{(tb\Phi)} \left(\bar{t}_R \left(\Phi^c\right)^\dagger T_L + \overline{T}_L \Phi^c t_R\right) - \delta^{(2g)} \frac{1}{4}\left(\partial_\mu A_\nu^A - \partial_\nu A_\mu^A\right)^2 + \\ & - \delta_{g_s}^{(3g)} \frac{1}{2} f^{ABC} \left(\partial_\mu A_\nu^A - \partial_\nu A_\mu^A\right) A^{B,\mu} A^{C,\nu} - \delta_{g_s}^{(4g)} \frac{1}{4} \left(f^{ABC} A_\mu^B A_\nu^C\right)^2 + \\ & + \delta^{(2c)} \partial_\mu \bar{c}^A \partial^\mu c^A + \delta_{g_s}^{(2cg)} f^{ABC} \partial_\mu \bar{c}^A A^{B,\mu} c^C + \bar{q} A\!\!\!/^A T^A \left(\delta_{g_s,R}^{(2qg)} P_R + \delta_{g_s,L}^{(2qg)} P_L\right) q,\end{aligned} \tag{3.11}$$

where the counterterms can be written in terms of the couplings and field renormalizations

$$\begin{aligned} & \delta^{(2\Phi)} = Z_\Phi - 1, \quad \delta_R^{(2q)} = Z_R^q - 1, \quad \delta_L^{(2q)} = Z_L - 1, \\ & \delta_\mu^{(2\Phi)} = \mu_0^2 Z_\Phi - \mu^2, \quad \delta_\lambda^{(4\Phi)} = \lambda_0 Z_\Phi^2 - \lambda, \quad \delta_{y_t}^{(tb\Phi)} = y_{t,0}\sqrt{Z_\Phi Z_R^t Z_L^t} - y_t, \\ & \delta^{(2g)} = Z_g - 1, \quad \delta_{g_s}^{(3g)} = Z_g^{3/2} g_{s,0} - g_s, \quad \delta_{g_s}^{(4g)} = Z_g^2 g_{s,0}^2 - g_s^2, \quad \delta^{(2c)} = Z_c - 1, \\ & \delta^{(2cg)} = Z_c Z_g^{1/2} g_{s,0} - g_s, \quad \delta_{g_s,R}^{(2qg)} = Z_R^q Z_g^{1/2} g_{s,0} - g_s, \quad \delta_{g_s,L}^{(2qg)} = Z_L^q Z_g^{1/2} g_{s,0} - g_s. \end{aligned} \tag{3.12}$$

We note, that in general left- and right-handed parts of the quark-fields are renormalized differently. However, in this model this is only true for t- and b-quarks. For the lighter quarks u, d, c, and d we find $Z_R^q = Z_L^q$, since we set their Yukawa couplings to zero and turned electroweak interactions off. Hence, their only interaction is via QCD which does not distinguish between right- and left-handed quarks.

We want to write down the Feynman rules for our model. For Yukawa- and Φ^4-vertices it is convenient to write the states in their SU(2)-representations. We denote SU(2)-indices with small letters $a, b, \ldots$, SU(3) indices in the fundamental representation with small bold letters $\mathbf{i},\mathbf{j}, \ldots$, SU(3) indices in the adjoint representation with capital letters $A, B, \ldots$, and Lorentz indices with greek letters $\mu, \nu, \ldots$. The corresponding states are shown in Fig. 3.1.

We can rewrite the interaction terms in terms of these states, using

$$\Phi^\dagger\Phi = \Phi^{\dagger a}\Phi_a, \tag{3.13}$$

$$\left(\Phi^\dagger\Phi\right)^2 = \left(\delta_a{}^c \delta_b{}^d + \delta_a{}^d \delta_b{}^c\right) \Phi^{\dagger a} \Phi^{\dagger b} \Phi_c \Phi_d, \tag{3.14}$$

$$\left(\Phi^c\right)^\dagger Q_L = \begin{pmatrix}\phi_0 & -\phi^+\end{pmatrix} \cdot Q_L = -i \left(\sigma_2\right)_a{}^b \Phi^a \left(Q_L\right)_b, \tag{3.15}$$

$$\overline{Q}_L \Phi^c = \overline{Q}_L \begin{pmatrix}\phi_0^* \\ -\phi^-\end{pmatrix} = i\left(\sigma_2\right)_a{}^b \left(\overline{Q}_L\right)^a \Phi_b^\dagger = -i \left(\sigma_2\right)^a{}_b \Phi_a^\dagger \left(\overline{Q}_L\right)^b. \tag{3.16}$$

The corresponding Feynman rules are displayed in Figures 3.2 and 3.3.

3.2 One-loop amplitudes

Having collected all the Feynman rules of our model, we are now prepared to calculate amplitudes. We want to derive renormalization group equations at the one-loop level. To this end we need to calculate the one-loop divergent amplitudes. We compute all diagrams in Feynman-'t Hooft gauge $\xi = 1$.

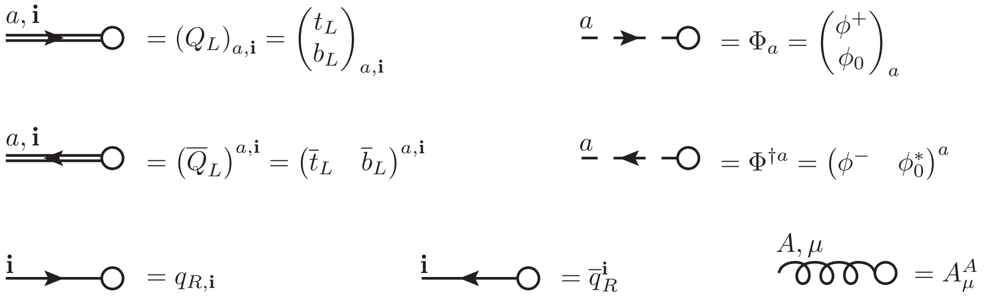


Figure 3.1: States for our Feynman rules in terms of the SU(2) multiplets.

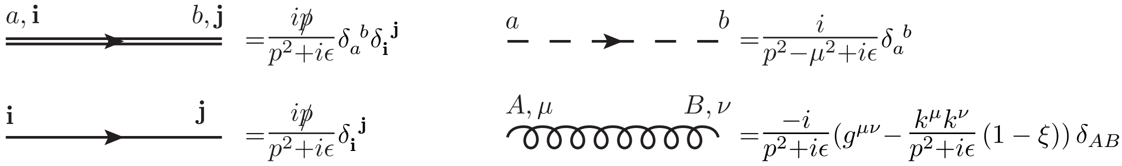
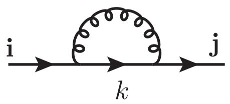


Figure 3.2: Feynman rules for propagators with momentum p going through.

3.2.1 Quark-propagators

At the one-loop level there are two different diagrams contributing to the propagator: a loop involving a gluon and a loop involving a Higgs. Since all Yukawa couplings except for the top are set to zero, the light quarks receive contributions from the gluon-loop diagram only. The left-handed top and bottom and the right-handed top also receive contributions from the Higgs-loop diagram.

We begin by calculating the gluon-loop contribution, given by



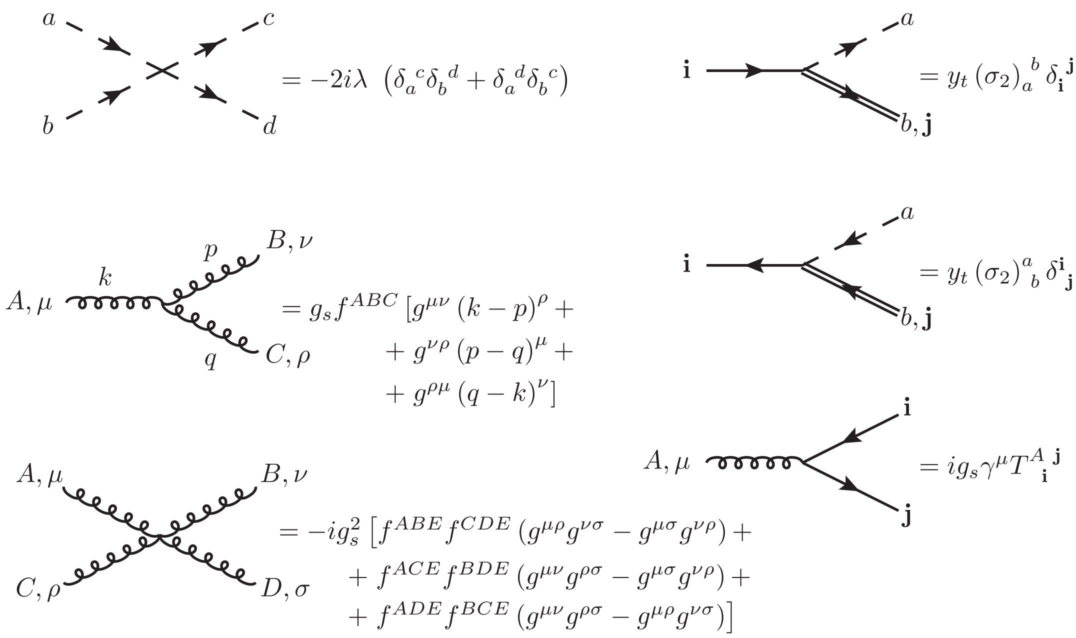
$$= \frac{i\not{p}}{p^2 + i\epsilon} \left[-i\Sigma^{(1)}_{\text{g-loop}}(p) \right]_\mathbf{i}{}^\mathbf{j} \frac{i\not{p}}{p^2 + i\epsilon},$$

where p is the external momentum. The fermion-line now represents a right- or left-handed quark, where we add a factor $\delta_a{}^b$ in the case of a left-handed doublet. The contribution of the loop is given by

$$\begin{aligned} \left[-i\Sigma^{(1)}_{\text{g-loop}}(p) \right]_\mathbf{i}{}^\mathbf{j} &= (ig_s)^2 \int \frac{d^4k}{(2\pi)^4} \gamma_\mu T^A{}_\mathbf{i}{}^\mathbf{k} \frac{i\not{k}}{k^2 + i\epsilon} \gamma_\nu T^B{}_\mathbf{k}{}^\mathbf{j} \frac{-ig^{\mu\nu}\delta_{AB}}{(p-k)^2 + i\epsilon} \\ &= g_s^2 \left[T^A T^B \delta_{AB} \right]_\mathbf{i}{}^\mathbf{j} (d-2) \int \frac{d^4k}{(2\pi)^4} \frac{\not{k}}{k^2 + i\epsilon} \frac{1}{(p-k)^2 - i\epsilon} \\ &= 2g_s^2 C_F \delta_\mathbf{i}{}^\mathbf{j} \int \frac{d^4k}{(2\pi)^4} \frac{\not{k}}{k^2 + i\epsilon} \frac{1}{(p-k)^2 - i\epsilon}, \end{aligned} \tag{3.17}$$

where $C_F \equiv T^A T^A$ is the quadratic Casimir operator of SU(3) in the fundamental representation, and we use $\gamma^\mu \gamma^\nu \gamma_\mu = -(d-2)\gamma^\nu$. We calculate the integral in dimensional regularization, compensating for dropping the dimension of the integral by introducing an arbitrary mass scale M

$$\int \frac{d^4k}{(2\pi)^4} \to M^{4-d} \int \frac{d^dk}{(2\pi)^d}, \tag{3.18}$$



$$= -2i\lambda\left(\delta_a{}^c\delta_b{}^d + \delta_a{}^d\delta_b{}^c\right)$$

$$= y_t\left(\sigma_2\right)_a{}^b\,\delta_{\mathbf{i}}{}^{\mathbf{j}}$$

$$\begin{aligned} = g_s f^{ABC}\,[g^{\mu\nu}\left(k-p\right)^\rho &+ \\ + g^{\nu\rho}\left(p-q\right)^\mu &+ \\ + g^{\rho\mu}\left(q-k\right)^\nu] & \end{aligned}$$

$$= y_t\left(\sigma_2\right)^a{}_b\,\delta^{\mathbf{i}}{}_{\mathbf{j}}$$

$$= ig_s\gamma^\mu T^A{}_{\mathbf{i}}{}^{\mathbf{j}}$$

$$\begin{aligned} = -ig_s^2\,\big[& f^{ABE}f^{CDE}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right) + \\ & + f^{ACE}f^{BDE}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}\right) + \\ & + f^{ADE}f^{BCE}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}\right)\big] \end{aligned}$$

Figure 3.3: Feynman rules for the vertices, omitting the ghost-gluon vertex. The corresponding counterterm-vertices are obtained by replacing the coupling strengths with the corresponding counterterm. Note, that for the 3-gluon vertex all momenta are incoming. The quarks in the $gq\bar{q}$-vertex can be understood as right- or left-handed quarks. For left-handed quarks in SU(2)-doublet representation, add a $\delta_a{}^b$ to the vertex, where a, b are the SU(2)-indices of the quark states.

and using a Feynman parameter. The integral then becomes

$$\begin{aligned}
\int \frac{d^4k}{(2\pi)^4}\frac{\not{k}}{k^2+i\epsilon}\frac{1}{(p-k)^2-i\epsilon} &\to M^{4-d}\int_0^1 dx \int \frac{d^dk}{(2\pi)^d}\frac{\not{k}}{\left[xp^2-2xp\cdot k+xk^2+k^2-xk^2+i\epsilon\right]^2} \\
&= M^{4-d}\int_0^1 dx \int \frac{d^dk}{(2\pi)^4}\frac{\left(\not{k}-x\not{p}\right)+x\not{p}}{\left[(k-xp)^2+x\left(1-x\right)p^2+i\epsilon\right]^2} \\
&= M^{4-d}\not{p}\int_0^1 dx\, x\int \frac{d^d\ell}{(2\pi)^d}\frac{1}{\left[\ell^2+x\left(1-x\right)p^2+i\epsilon\right]^2} \\
&= M^{4-d}\not{p}\int_0^1 dx\, x\frac{i}{(4\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)\left(\frac{1}{-x\left(1-x\right)p^2}\right)^{2-\frac{d}{2}} \\
&= \frac{i}{16\pi^2}\not{p}\int_0^1 dx\, x\left(\frac{4\pi M^2}{-x\left(1-x\right)p^2}\right)^{\varepsilon/2}\Gamma\left(\frac{\varepsilon}{2}\right) \\
&= \frac{i}{16\pi^2}\not{p}\int_0^1 dx\, x\left(\frac{2}{\varepsilon}-\gamma+\log\left(\frac{4\pi M^2}{-x\left(1-x\right)p^2}\right)+\mathcal{O}\left(\varepsilon\right)\right) \\
&= \frac{i}{32\pi^2}\not{p}\left(\frac{2}{\varepsilon}-\gamma+\log\left(\frac{4\pi M^2}{p^2}\right)-\int_0^1 dx\,2x\log\left(-x\left(1-x\right)\right)+\mathcal{O}\left(\varepsilon\right)\right),
\end{aligned} \tag{3.19}$$

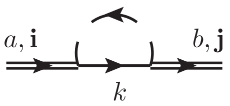
where we shifted the integral $k \to k - xp \equiv \ell$ and dropped the term proportional to ℓ in the third line since odd powers of ℓ integrate to zero from symmetry. The remaining momentum-space integral in the third line can be found in our table of integrals in dimensional regularization. The integral over the Feynman parameter in the last line can be done analytically, yielding

$$\int_0^1 dx\, 2x \log\left(-x\left(1-x\right)\right) = i\pi - 2. \tag{3.20}$$

Thus, the contribution from the gluon-loop diagram is

$$\left[-i\Sigma^{(1)}_{\text{g-loop}}(p)\right]_{\mathbf{ij}} = i\delta_{\mathbf{ij}}\not p \frac{g_s^2 C_F}{16\pi^2}\left(\frac{2}{\varepsilon} - \gamma + \log\left(\frac{4\pi M^2}{p^2}\right) + 2 - i\pi\right). \tag{3.21}$$

The T_L and t_R propagators also receive 1-loop corrections from diagrams involving a Higgs. The diagram for the T_L propagator is given by



$$= \frac{i\not p}{p^2+i\epsilon}\left[-i\Sigma^{(1)}_{L,\text{loop}}\left(p\right)\right]_{a\ \ ,\mathbf{i}}^{b,\ \mathbf{j}} \frac{i\not p}{p^2+i\epsilon},$$

where

$$\left[-i\Sigma^{(1)}_{L,\text{loop}}\left(p\right)\right]_{a\ \ ,\mathbf{i}}^{b,\ \mathbf{j}} = \delta_{\mathbf{i}}{}^{\mathbf{j}} y_t^2 \left(\sigma_2\right)^c{}_a \left(\sigma_2\right)_c{}^b \int \frac{d^4k}{\left(2\pi\right)^4} \frac{i\not k}{k^2+i\epsilon} \cdot \frac{i}{\left(p-k\right)^2 - \mu^2 + i\epsilon}. \tag{3.22}$$

We first note that

$$\left(\sigma_2\right)^c{}_a \left(\sigma_2\right)_c{}^b = -\left(\sigma_2\right)_a{}^c \left(\sigma_2\right)_c{}^b = -\left[\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix} \cdot \begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\right]_a^{\ b} = -\delta_a{}^b. \tag{3.23}$$

Thus we can rewrite

$$\left[-i\Sigma^{(1)}_{L,\text{loop}}\left(p\right)\right]_{a\ \ ,\mathbf{i}}^{b,\ \mathbf{j}} = \delta_{\mathbf{i}}{}^{\mathbf{j}} \delta_a{}^b y_t^2 \int \frac{d^4k}{\left(2\pi\right)^4} \frac{\not k}{k^2+i\epsilon} \cdot \frac{1}{\left(p-k\right)^2 - \mu^2 + i\epsilon}. \tag{3.24}$$

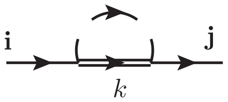
The integral is the same as in the case of the gluon-loop diagram but for an additional term μ^2 in the bosonic propagator. The computation works exactly as above and we find

$$\begin{aligned}\left[-i\Sigma^{(1)}_{L,\text{loop}}\left(p\right)\right]_{a\ \ ,\mathbf{i}}^{b,\ \mathbf{j}} &= \delta_{\mathbf{i}}{}^{\mathbf{j}} \delta_a{}^b y_t^2 M^{4-d} \not p \int_0^1 dx\, x \int \frac{d^4\ell}{\left(2\pi\right)^4} \frac{1}{\left[\ell^2 + x\left(1-x\right)p^2 - x\mu^2 + i\epsilon\right]^2} \\ &= i\delta_{\mathbf{i}}{}^{\mathbf{j}} \delta_a{}^b \not p \frac{y_t^2}{32\pi^2}\left(\frac{2}{\varepsilon} - \gamma + \int_0^1 dx\, 2x \log\left(\frac{4\pi M^2}{x\mu^2 - x\left(1-x\right)p^2}\right)\right)\end{aligned} \tag{3.25}$$

The remaining integral can again be done analytically, yielding

$$\begin{aligned}\int_0^1 dx\, 2x \log\left(\frac{4\pi M^2}{x\mu^2 - x(1-x)p^2}\right) &= \log\left(\frac{4\pi M^2}{\mu^2}\right) + 2 - \frac{\mu^2}{p^2} - \left(1 - \frac{\mu^2}{p^2}\right)\log\left(1 - \frac{p^2}{\mu^2}\right) \\ &\xrightarrow{p^2 \gg \mu^2} \log\left(\frac{4\pi M^2}{\mu^2}\right) + 2 - \log\left(-\frac{p^2}{\mu^2}\right) \\ &= \log\left(\frac{4\pi M^2}{p^2}\right) + 2 - i\pi\end{aligned} \tag{3.26}$$

The contribution to the t_R propagator from the loop involving a Higgs is identical up to the SU(2)-indices:



$$= \frac{i\not p}{p^2+i\epsilon}\left[-i\Sigma^{(1)}_{R,\text{loop}}\left(p\right)\right]_{\mathbf{i}}^{\mathbf{j}} \frac{i\not p}{p^2+i\epsilon},$$

where

$$\begin{aligned}\left[-i\Sigma^{(1)}_{R,\text{loop}}(p)\right]_{\mathbf{i}}^{\mathbf{j}} &= \delta_{\mathbf{i}}{}^{\mathbf{j}} y_t^2 \left(\sigma_2\right)^c{}_d \left(\sigma_2\right)_c{}^d \int \frac{d^4k}{\left(2\pi\right)^4} \frac{i\not k}{k^2 - i\epsilon} \cdot \frac{i}{\left(p-k\right)^2 - \mu^2 + i\epsilon} && (3.27) \\ &= i\delta_{\mathbf{i}}{}^{\mathbf{j}} \not p \frac{y_t^2}{16\pi^2}\left(\frac{2}{\varepsilon} - \gamma + \int_0^1 dx\, 2x \log\left(\frac{4\pi M^2}{x\mu^2 - x\left(1-x\right)p^2}\right)\right). && (3.28)\end{aligned}$$

We note that $\Sigma_R^{(1)}(p) = 2\Sigma_L^{(1)}(p)$. Considering the contributing diagrams we find that for the $(T_L)_a$-propagator the only contribution comes from a loop containing a t_R and a Φ_b, where the SU(2) indices $a \neq b$. However, for the t_R-propagator there is a contribution from a loop containing $(T_L)_1 = t_L$ and a $\Phi_2 = \phi_0$, and from a loop containing a $(T_L)_2 = b_L$ and a $\Phi_1 = \phi^+$. Since the contribution from each loop is as big as the total contribution in the Q_L-propagator case, the total contribution is twice as large.

It is also worth noting that we find only logarithmic divergences for the fermionic propagators, although simple power counting of the diagrams suggests linear divergences. Mathematically we found that the linear term drops out due to the symmetry of the diagram. Note that this does not rely on the fermion being massless: if the fermion had an effective mass m, e.g. from calculating in the electroweak broken phase, the only modification of the integral would have been $\Delta \to x\mu^2 - x(1-x)m^2 - x(1-x)p^2$ and we would again find no linear divergence for the mass counterterm, but only a logarithmic one. Physically, this reflects the fact that the mass of the fermion is protected by the chiral symmetry, i.e. we find an enhanced symmetry for $m = 0$.

The 1-loop divergences are absorbed in the corresponding counterterms

$$\left[-i\Sigma_{q,\text{ct}}^{(1)}(p)\right]_{\mathbf{i}}^{\ \mathbf{j}} = i\not{p}\delta_{\mathbf{i}}^{\ \mathbf{j}}\delta_{R/L}^{(2q)}, \tag{3.29}$$

to which we must add a factor $\delta_a^{\ b}$ for an Q_L propagator. We recall, that in the modified minimal subtraction scheme ($\overline{\text{MS}}$) the counterterms absorb the terms proportional to $\left(\frac{2}{\varepsilon} - \gamma + \log(4\pi)\right)$. For the $q = u, d, c, d, b_R$ quarks we thus find

$$\left[-i\Sigma_q^{(1)}(p)\right]_{\mathbf{i}}^{\ \mathbf{j}} = i\delta_{\mathbf{i}}^{\ \mathbf{j}}\not{p}\frac{g_s^2 C_F}{16\pi^2}\left(\log\left(\frac{M^2}{p^2}\right) + 2 - i\pi\right) \tag{3.30}$$

after subtracting the $\overline{\text{MS}}$-counterterm

$$\delta^{(2q)} = -\frac{g_s^2 C_F}{16\pi^2}\left(\frac{2}{\varepsilon} - \gamma + \log(4\pi)\right). \tag{3.31}$$

For the T_L propagator we find

$$\begin{aligned}\left[-i\Sigma_{T_L}^{(1)}(p)\right]_{a\ ,\mathbf{i}}^{b,\ \mathbf{j}} &= i\delta_{\mathbf{i}}^{\ \mathbf{j}}\delta_a^{\ b}\not{p}\left[\frac{g_s^2 C_F}{16\pi^2}\left(\log\left(\frac{M^2}{p^2}\right) + 2 - i\pi\right) + \right.\\ &\left. + \frac{y_t^2}{32\pi^2}\left(\log\left(\frac{M^2}{\mu^2}\right) + 2 - \frac{\mu^2}{p^2} - \left(1 - \frac{\mu^2}{p^2}\right)\log\left(1 - \frac{p^2}{\mu^2}\right)\right)\right]\\ &\xrightarrow{p^2 \gg \mu^2} i\delta_{\mathbf{i}}^{\ \mathbf{j}}\delta_a^{\ b}\not{p}\left(\frac{g_s^2 C_F}{16\pi^2} + \frac{y_t^2}{32\pi^2}\right)\left(\log\left(\frac{M^2}{p^2}\right) + 2 - i\pi\right),\end{aligned} \tag{3.32}$$

and the counterterm

$$\delta_L^{(2t)} = -\left(\frac{g_s^2 C_F}{16\pi^2} + \frac{y_t^2}{16\pi^2}\right)\left(\frac{2}{\varepsilon} - \gamma + \log(4\pi)\right). \tag{3.33}$$

For the t_R-propagator we find

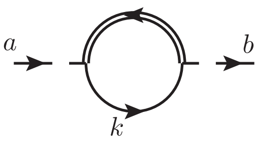
$$\begin{aligned}\left[-i\Sigma_{t_R}^{(1)}(p)\right]_{\mathbf{i}}^{\ \mathbf{j}} &= i\delta_{\mathbf{i}}^{\ \mathbf{j}}\not{p}\left[\frac{g_s^2 C_F}{16\pi^2}\left(\log\left(\frac{M^2}{p^2}\right) + 2 - i\pi\right) + \right.\\ &\left. + \frac{y_t^2}{16\pi^2}\left(\log\left(\frac{M^2}{\mu^2}\right) + 2 - \frac{\mu^2}{p^2} - \left(1 - \frac{\mu^2}{p^2}\right)\log\left(1 - \frac{p^2}{\mu^2}\right)\right)\right]\\ &\xrightarrow{p^2 \gg \mu^2} i\delta_{\mathbf{i}}^{\ \mathbf{j}}\not{p}\left(\frac{g_s^2 C_F}{16\pi^2} + \frac{y_t^2}{16\pi^2}\right)\left(\log\left(\frac{M^2}{p^2}\right) + 2 - i\pi\right),\end{aligned} \tag{3.34}$$

with the counterterm

$$\delta_L^{(2T)} = -\left(\frac{g_s^2 C_F}{16\pi^2} + \frac{y_t^2}{32\pi^2}\right)\left(\frac{2}{\varepsilon} - \gamma + \log(4\pi)\right). \tag{3.35}$$

3.2.2 Φ-propagator

The Φ-propagator receives corrections from two divergent diagrams at the one-loop level, one with a fermion- and one with an Φ-loop. We begin by calculating the contribution of the fermionic loop:



$$=\frac{i}{p^2-\mu^2+i\epsilon}\left[-i\Pi^{(1)}_{f-\text{loop}}\left(p^2\right)\right]_a^{\ b}\frac{i}{p^2-\mu^2+i\epsilon},$$

where

$$\left[-i\Pi^{(1)}_{f-\text{loop}}\left(p^2\right)\right]_a^{\ b} = -y_t^2\delta_{\mathbf{i}}{}^{\mathbf{j}}\delta_{\mathbf{j}}{}^{\mathbf{i}}\left(\sigma_2\right)_a^{\ c}\left(\sigma_2\right)^b_{\ c}\int\frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{i\not{k}}{k^2+i\epsilon}\cdot\frac{i\left(\not{k}-\not{p}\right)}{(k-p)^2+i\epsilon}\right], \tag{3.36}$$

with an extra factor (-1) for the fermionic loop. To work out the trace over the γ-matrices we have to explicitly insert the projectors since we have a left- and a right-handed quark in the loop:

$$\begin{aligned}\left[-i\Pi^{(1)}_{f-\text{loop}}\left(p^2\right)\right]_a^{\ b} &= \delta_a^{\ b}3i^2y_t^2\int\frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\left[\left(\frac{1+\gamma_5}{2}\right)\frac{\gamma^\mu k_\mu}{k^2+i\epsilon}\left(\frac{1-\gamma_5}{2}\right)\frac{\gamma^\nu(k-p)_\nu}{(k-p)^2+i\epsilon}\right]\\ &= -\delta_a^{\ b}3y_t^2\int\frac{d^4k}{(2\pi)^4}\frac{\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\right]+\mathrm{Tr}\left[-\gamma_5\gamma^\mu\gamma_5\gamma^\nu\right]}{4}\frac{k_\mu}{k^2+i\epsilon}\frac{(k-p)_\nu}{(k-p)^2+i\epsilon}\\ &= -\delta_a^{\ b}3y_t^2\int\frac{d^4k}{(2\pi)^4}\frac{2\,\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\right]}{4}\frac{k_\mu}{k^2+i\epsilon}\frac{(k-p)_\nu}{(k-p)^2+i\epsilon}\\ &= -\delta_a^{\ b}6y_t^2\int\frac{d^4k}{(2\pi)^4}g^{\mu\nu}\frac{k_\mu}{k^2+i\epsilon}\frac{(k-p)_\nu}{(k-p)^2+i\epsilon},\end{aligned} \tag{3.37}$$

where we used $\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\right]=4g^{\mu\nu}$, $\mathrm{Tr}\left[\gamma_5\gamma^\mu\gamma^\nu\right]=0$, $\{\gamma^\mu,\gamma_5\}=0$, and $(\gamma_5)^2=\mathbf{1}$. We rewrite:

$$\begin{aligned}\left[-i\Pi^{(1)}_{f-\text{loop}}(p^2)\right]_a^{\ b} &= -\delta_a^{\ b}6y_t^2\int\frac{d^4k}{2\pi^4}\frac{k}{k^2+i\epsilon}\cdot\frac{k-p}{(k-p)^2+i\epsilon}\\ &= -\delta_a^{\ b}6y_t^2\int_0^1dx\int\frac{d^4k}{(2\pi)^4}\frac{k^2-k\cdot p}{\left[xp^2-2xp\cdot k+xk^2+k^2-xk^2+i\epsilon\right]^2}\\ &= -\delta_a^{\ b}6y_t^2\int_0^1dx\int\frac{d^4\ell}{(2\pi)^4}\frac{\ell^2+(2x-1)\,\ell\cdot p-x(1-x)\,p^2}{[\ell^2+x(1-x)\,p^2+i\epsilon]^2},\end{aligned} \tag{3.38}$$

where we introduced $\ell\equiv k-xp$. We can again drop the term proportional to ℓ in the numerator since it will vanish upon integration, and calculate the remaining integral in dimensional regularization,

$$\left[-i\Pi^{(1)}_{f-\text{loop}}(p^2)\right]_a^{\ b} = -\delta_a^{\ b}6y_t^2\int_0^1dx\,M^{4-d}\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^2-x(1-x)\,p^2}{\left[\ell^2+x(1-x)\,p^2\right]^2}. \tag{3.39}$$

We define $\Delta\equiv -x(1-x)\,p^2$ and calculate the integral in two parts. We begin with the Δ-term:

$$\begin{aligned}M^{4-d}\int\frac{d^d\ell}{(2\pi)^d}\frac{\Delta}{\left[\ell^2-\Delta\right]^2} &= \Delta M^{4-d}\frac{i}{(4\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}=\frac{i\Delta}{16\pi^2}\left(\frac{4\pi M^2}{\Delta}\right)^{\varepsilon/2}\Gamma\left(\frac{\varepsilon}{2}\right)\\ &= \frac{i\Delta}{16\pi^2}\left(\frac{2}{\varepsilon}-\gamma+\log(4\pi)+\log\left(\frac{M^2}{\Delta}\right)+\ldots\right),\end{aligned} \tag{3.40}$$

where as power counting suggested we find poles at $d=4,6,8,\ldots$. We now turn to the ℓ^2 term. Power counting suggests this to be divergent for $d\geq 2$. Carrying out the integral

$$M^{4-d}\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^2}{\left[\ell^2-\Delta\right]^2}=-\frac{iM^{4-d}}{(4\pi)^{d/2}}\frac{d}{2}\Gamma\left(1-\frac{d}{2}\right)\left(\frac{1}{\Delta}\right)^{1-\frac{d}{2}}, \tag{3.41}$$

we indeed find poles at $d = 2, 4, 6, \ldots$. The quadratic divergence corresponding to the pole at $d = 2$ is independent of the external momenta and can hence be cancelled completely by the mass renormalization. We can analytically continue to the pole at $d = 4$ and find:

$$\begin{aligned} M^{4-d} \int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^2}{[\ell^2 - \Delta]^2} &= -\frac{i\Delta}{16\pi^2} \left(\frac{4\pi M^2}{\Delta}\right)^{\varepsilon/2} \left(2 - \frac{\varepsilon}{2}\right) \Gamma\left(\frac{\varepsilon}{2} - 1\right) \\ &= \frac{i\Delta}{8\pi^2} \left(\frac{2}{\varepsilon} - \gamma + \frac{1}{2} + \log\left(\frac{4\pi M^2}{\Delta}\right) + \ldots\right). \end{aligned} \tag{3.42}$$

Putting the pieces together we find for the contribution of the fermion-loop:

$$\begin{aligned} \left[-i\Pi^{(1)}_{f-\text{loop}}(p^2)\right]_a^{\ b} &= i\delta_a^{\ b} \frac{3y_t^2}{8\pi^2} p^2 \int_0^1 dx\, x\,(1-x) \left(\frac{6}{\varepsilon} - 3\gamma + 1 + 3\log\left(\frac{4\pi M^2}{-x\,(1-x)\,p^2}\right)\right) \\ &= i\delta_a^{\ b} \frac{3y_t^2}{16\pi^2} p^2 \left(\frac{2}{\varepsilon} - \gamma + \frac{1}{3} + \log\left(\frac{4\pi M^2}{p^2}\right) - \int_0^1 dx\, 6x\,(1-x) \log\left(-x\,(1-x)\right)\right). \end{aligned} \tag{3.43}$$

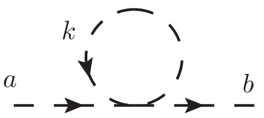
The remaining integral over the Feynman parameter can again be done analytically, yielding

$$\int_0^1 dx\, 6x\,(1-x) \log\left(-x\,(1-x)\right) = -\frac{5}{3} + i\pi, \tag{3.44}$$

which we use to rewrite the fermion-loop contribution as

$$\left[-i\Pi^{(1)}_{f-\text{loop}}(p^2)\right]_a^{\ b} = i\delta_a^{\ b} \frac{3y_t^2}{16\pi^2} p^2 \left(\frac{2}{\varepsilon} - \gamma + \log\left(\frac{4\pi M^2}{p^2}\right) + 2 - i\pi\right). \tag{3.45}$$

The contribution from the Φ-loop is



$$= \frac{i}{p^2 - \mu^2 + i\epsilon} \left[-i\Pi^{(1)}_{\Phi-\text{loop}}(p^2)\right]_a^{\ b} \frac{i}{p^2 - \mu^2 + i\epsilon},$$

where

$$\left[-i\Pi^{(1)}_{\Phi-\text{loop}}\left(p^2\right)\right]_a^{\ b} = -2i\lambda \left(\delta_a^{\ b}\delta_c^{\ c} + \delta_a^{\ c}\delta_c^{\ b}\right) \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - \mu^2 + i\epsilon}. \tag{3.46}$$

Calculating the integral as before in dimensional regularization, we find

$$\begin{aligned} \left[-i\Pi^{(1)}_{\Phi-\text{loop}}(p^2)\right]_a^{\ b} &= \delta_a^{\ b} 6\lambda M^{4-d} \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 - \mu^2 + i\epsilon} \\ &= \delta_a^{\ b} 6\lambda M^{4-d} \frac{-i}{(4\pi)^{d/2}} \Gamma\left(1 - \frac{d}{2}\right) \left(\frac{1}{\mu^2}\right)^{1-\frac{d}{2}}. \end{aligned} \tag{3.47}$$

This again has poles at $d = 2, 4, 6, \ldots$. The quadratic divergence from the pole at $d = 2$ can once more be cancelled by the mass counterterm and continuing analytically to $d = 4$ we find

$$\begin{aligned} \left[-i\Pi^{(1)}_{\Phi-\text{loop}}(p^2)\right]_a^{\ b} &= -i\delta_a^{\ b} 6\lambda \cdot \frac{\mu^2}{16\pi^2} \left(\frac{4\pi M^2}{\mu^2}\right)^{\varepsilon/2} \Gamma\left(1 - \frac{\varepsilon}{2}\right) \\ &= i\delta_a^{\ b} \frac{3\lambda\mu^2}{8\pi^2} \left(\frac{2}{\varepsilon} - \gamma + 1 + \log\left(4\pi\right) + \log\left(\frac{M^2}{\mu^2}\right)\right). \end{aligned} \tag{3.48}$$

Putting together the contributions from the Φ and the fermion-loop we find for the 1-loop correction to the Φ-propagator:

$$\begin{aligned} \left[-i\Pi^{(1)}(p^2)\right]_a^{\ b} = -i\delta_a^{\ b} \Bigg[&\frac{3y_t^2}{16\pi^2} p^2 \left(\frac{2}{\varepsilon} - \gamma + \log\left(\frac{4\pi M^2}{p^2}\right) + 2 - i\pi\right) + \\ &+ \frac{3\lambda\mu^2}{8\pi^2} \left(\frac{2}{\varepsilon} - \gamma + \log\left(\frac{4\pi M^2}{\mu^2}\right) + 1\right)\Bigg]. \end{aligned} \tag{3.49}$$

We need to add the counterterm $\left[-i\Pi_{\text{ct}}^{(1)}(p^2)\right]_a^{\ b} = i\delta_a^{\ b}\left(p^2\delta^{(2\Phi)} - \delta_\mu^{(2\Phi)}\right)$. The field renormalization $Z_\Phi = \delta^{(2\Phi)} + 1$ is only logarithmically divergent. The quadratic divergence of the renormalized mass μ^2 corresponding to the poles at $d = 2$ in dimensional regularization leads to the Hierarchy problem, which requires a finely tuned counter term to keep a small Higgs mass. After subtracting the counterterms

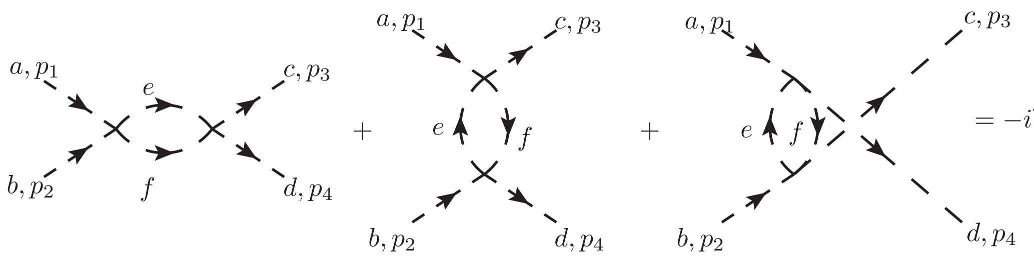
$$\delta^{(2\Phi)} = -\frac{3y_t^2}{16\pi^2}\left(\frac{2}{\varepsilon} - \gamma + \log(4\pi)\right), \qquad \delta_\mu^{(2\Phi)} = \frac{3\lambda\mu^2}{8\pi^2}\left(\frac{2}{\varepsilon} - \gamma + \log(4\pi)\right), \tag{3.50}$$

we are left with

$$\left[-i\Pi^{(1)}(p^2)\right]_a^{\ b} = i\delta_a^{\ b}\left[\frac{3y_t^2}{16\pi^2}p^2\left(\log\left(\frac{M^2}{p^2}\right) + 2 - i\pi\right) - \frac{3\lambda\mu^2}{8\pi^2}\left(\log\left(\frac{M^2}{\mu^2}\right) + 1\right)\right]. \tag{3.51}$$

3.2.3 Φ^4-vertex

There are eleven one-loop diagrams giving corrections to the Φ^4 vertex: three diagrams with a Φ-loop and eight diagrams with a fermionic loop. We begin by calculating the contribution from the Φ-loops:



$$= -i\mathbf{V}^{(1)}_{\Phi^4,\Phi-\text{loop}}(s,t,u),$$

where

$$\begin{aligned} -i\mathbf{V}^{(1)}_{\Phi^4,\Phi-\text{loop}}(s,t,u) = (-2i\lambda)^2 \Big[&\frac{1}{2}\left(\delta_a^{\ e}\delta_b^{\ f} + \delta_a^{\ f}\delta_b^{\ e}\right)\left(\delta_e^{\ c}\delta_f^{\ d} + \delta_e^{\ d}\delta_f^{\ c}\right)i^2V(s) + \\ &+ \left(\delta_a^{\ c}\delta_e^{\ f} + \delta_a^{\ f}\delta_e^{\ c}\right)\left(\delta_b^{\ d}\delta_f^{\ e} + \delta_b^{\ e}\delta_f^{\ d}\right)i^2V(t) + \\ &+ \left(\delta_a^{\ d}\delta_e^{\ f} + \delta_a^{\ f}\delta_e^{\ d}\right)\left(\delta_b^{\ c}\delta_f^{\ e} + \delta_b^{\ e}\delta_f^{\ c}\right)i^2V(u)\Big], \end{aligned} \tag{3.52}$$

with the Mandelstam-variables

$$s = (p_1 + p_2)^2, \qquad t = (p_1 - p_3)^2, \qquad u = (p_1 - p_4)^2. \tag{3.53}$$

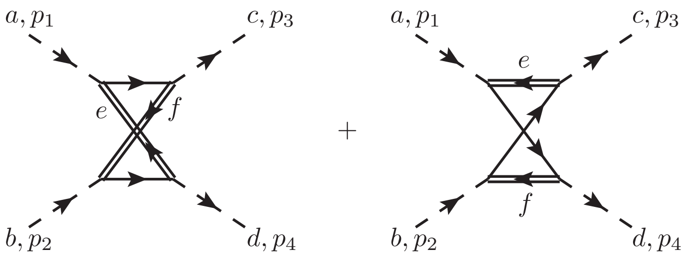
We find a symmetry factor 1/2 from the loop, which is cancelled for the t- and u-channel diagrams since we can find identical diagrams from exchanging both incoming and outgoing momenta in the diagrams. There is only one integral we need to calculate:

$$\begin{aligned} V\left(p^2\right) &\equiv \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2 - \mu^2 + i\epsilon}\cdot\frac{1}{(p-k)^2 - \mu^2 + i\epsilon} \\ &= \int_0^1 dx \int \frac{d^4k}{(2\pi)^4}\frac{1}{\left[xp^2 - 2xp\cdot k + xk^2 - x\mu^2 + k^2 - \mu^2 - xk^2 + x\mu^2 i\epsilon\right]^2} \\ &= \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4}\frac{1}{\left[\ell^2 - \left(\mu^2 - i\epsilon - x\left(1-x\right)p^2\right)\right]^2} \\ &= \frac{i}{16\pi^2}\left(\frac{2}{\varepsilon} - \gamma + \log\left(4\pi\right) + \int_0^1 dx\,\log\left(\frac{M^2}{\mu^2 - x\left(1-x\right)p^2}\right)\right). \end{aligned} \tag{3.54}$$

Working out the SU(2)-index contractions, we find for the contribution from the Φ-loop diagrams

$$\mathbf{V}^{(1)}_{\Phi^4,\Phi-\mathrm{loop}}(s,t,u) = -\frac{\lambda^2}{4\pi^2}\left[\left(\delta_a{}^c\delta_b{}^d + \delta_a{}^d\delta_b{}^c\right)\left(\frac{2}{\varepsilon} - \gamma + \int_0^1 dx\,\log\left(\frac{4\pi M^2}{\mu^2 - x(1-x)s}\right)\right) + \right.$$
$$+\left(4\delta_a{}^c\delta_b{}^d + \delta_a{}^d\delta_b{}^c\right)\left(\frac{2}{\varepsilon} - \gamma + \int_0^1 dx\,\log\left(\frac{4\pi M^2}{\mu^2 - x(1-x)t}\right)\right) +$$
$$\left. +\left(\delta_a{}^c\delta_b{}^d + 4\delta_a{}^d\delta_b{}^c\right)\left(\frac{2}{\varepsilon} - \gamma + \int_0^1 dx\,\log\left(\frac{4\pi M^2}{\mu^2 - x(1-x)u}\right)\right)\right]. \tag{3.55}$$

There are eight diagrams with a fermion-loop contributing to the Φ^4 vertex:



$$+ \ldots = -i\mathbf{V}^{(1)}_{\Phi^4,f-\mathrm{loop}}(s,t,u),$$

where the dots indicate the diagrams with exchanged external momenta, and

$$-i\mathbf{V}^{(1)}_{\Phi^4,f-\mathrm{loop}}(s,t,u) = y_t^4\delta_{\mathbf{i}}{}^{\mathbf{j}}\delta_{\mathbf{j}}{}^{\mathbf{k}}\delta_{\mathbf{k}}{}^{\mathbf{l}}\delta_{\mathbf{l}}{}^{\mathbf{i}}\times$$
$$\times\left[\left((\sigma_2)_a{}^e(\sigma_2)^d{}_e(\sigma_2)_b{}^f(\sigma_2)^c{}_f + (\sigma_2)_a{}^e(\sigma_2)^c{}_e(\sigma_2)_b{}^f(\sigma_2)^d{}_f\right)\frac{-i^4}{4}W(p_2,-p_3,p_1) + \right.$$
$$\left. +(a,p_1\leftrightarrow b,p_2) + (c,p_3\leftrightarrow d,p_4) + \begin{pmatrix} c,p_3\leftrightarrow d,p_4 \\ a,p_1\leftrightarrow b,p_2\end{pmatrix}\right], \tag{3.56}$$

taking into account an extra (-1) for the fermion-loop and a symmetry factor $1/4$. The integral over the loop-momentum is given by

$$W(p,q,r) = \int\frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{\not k}{k^2+i\epsilon}\cdot\frac{\not k+\not p}{(k+p)^2+i\epsilon}\cdot\frac{\not k+\not p+\not q}{(k+p+q)^2+i\epsilon}\cdot\frac{\not k+\not p+\not q+\not r}{(k+p+q+r)^2+i\epsilon}\right]. \tag{3.57}$$

Working out the SU(2) index-structure we can rewrite the contribution from the fermion-loop diagrams to

$$\mathbf{V}^{(1)}_{\Phi^4,f-\mathrm{loop}}(s,t,u) = -\frac{3iy_t^4}{4}\left(\delta_a{}^c\delta_b{}^d + \delta_a{}^d\delta_b{}^c\right)\left[W(p_2,-p_3,p_1) + W(p_1,-p_3,p_2) + \right.$$
$$\left. + W(p_2,-p_4,p_1) + W(p_1,-p_4,p_2)\right]. \tag{3.58}$$

We are left to evaluate the integral $W(p,q,r)$. To work out the trace over the γ-matrices, we again insert the projectors for the right- and left-handed quarks explicitly:

$$\mathrm{Tr}\left[P_L\gamma^\mu P_R\gamma^\nu P_L\gamma^\rho P_R\gamma^\sigma\right] = \mathrm{Tr}\left[P_L\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right] = \frac{1}{2}\,\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right] = 2\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right). \tag{3.59}$$

We find

$$W(p,q,r) = 2\int\frac{d^4k}{(2\pi)^4}\left[\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)\frac{k_\mu(k+p)_\nu(k+p+q)_\rho(k+p+q+r)_\sigma}{k^2(k+p)^2(k+p+q)^2(k+p+q+r)^2}\right], \tag{3.60}$$

where we suppressed the $i\epsilon$-terms in the denominator. This is a rational function in k, which in principle can be integrated. One can do the integral, but we are only interested in the divergent part of the integral. Power counting shows that there is a logarithmic divergence from the k^4 term in the numerator and all terms including external momenta are finite. Completing the square in the denominator and shifting the integration variable accordingly the integral will take the from

$$W(p,q,r) = 2\int \frac{d^4\ell}{(2\pi)^4} \frac{\left(\ell^2\right)^2 + \ell^2 F_2(p,q,r) + F_4(p,q,r)}{\left[\ell^2 - \Delta\left(p,q,r\right)\right]^4}, \tag{3.61}$$

where we dropped terms with odd powers of ℓ in the numerator, since they will integrate to zero. F_2 and F_4 are polynomials of degree 2 and 4 in the external momenta, respectively, and Δ is a polynomial of order 2 in the external momenta. Integrating this, we find

$$W(p,q,r) = \frac{i}{8\pi^2}\left(\frac{2}{\varepsilon} - \gamma - \frac{5}{6} + \log\left(4\pi\right) + \log\left(\frac{M^2}{\Delta(p,q,r)}\right)\right) + \text{finite terms}. \tag{3.62}$$

The integral depends only on bilinear combinations of the external momenta, as it must by Lorentz invariance. Hence, it is possible to rewrite the dependency on the external momenta in terms of the Mandelstam variables s,t,u. Exchanging momenta then corresponds to exchanging Mandelstam variables and we find for the fermion-loop contribution

$$\begin{aligned}\mathbf{V}^{(1)}_{\Phi^4,f-\text{loop}}(s,t,u) = \frac{3y_t^4}{8\pi^2}\left(\delta_a{}^c\delta_b{}^d + \delta_a{}^d\delta_b{}^c\right)\Bigg[&\frac{2}{\varepsilon} - \gamma - \frac{5}{6} + \log\left(4\pi\right) + \\ &+\frac{1}{2}\log\left(\frac{M^2}{\Delta(s,t,u)}\right) + \frac{1}{2}\log\left(\frac{M^2}{\Delta(s,u,t)}\right)\Bigg].\end{aligned} \tag{3.63}$$

Adding the contribution from the Φ-loop diagrams and subtracting the $\overline{\text{MS}}$ counterterm $-i\mathbf{V}^{(1)}_{\Phi^4,\text{ct}} = -2i\left(\delta_a{}^c\delta_b{}^d + \delta_a{}^d\delta_b{}^c\right)\delta_\lambda^{(4\Phi)}$ we find for the one-loop correction

$$\begin{aligned}\mathbf{V}^{(1)}_{\Phi^4} = &-\frac{\lambda^2}{4\pi^2}\Bigg[\left(\delta_a{}^c\delta_b{}^d + \delta_a{}^d\delta_b{}^c\right)\left(\int_0^1 dx\,\log\left(\frac{M^2}{\mu^2 - x\left(1-x\right)s}\right)\right) + \\ &+\left(4\delta_a{}^c\delta_b{}^d + \delta_a{}^d\delta_b{}^c\right)\left(\int_0^1 dx\,\log\left(\frac{M^2}{\mu^2 - x\left(1-x\right)t}\right)\right) + \\ &+\left(\delta_a{}^c\delta_b{}^d + 4\delta_a{}^d\delta_b{}^c\right)\left(\int_0^1 dx\,\log\left(\frac{M^2}{\mu^2 - x\left(1-x\right)u}\right)\right)\Bigg] + \\ &+\frac{3y_t^4}{16\pi^2}\left(\delta_a{}^c\delta_b{}^d + \delta_a{}^d\delta_b{}^c\right)\left[\log\left(\frac{M^2}{\Delta(s,t,u)}\right) + \log\left(\frac{M^2}{\Delta(s,u,t)} - \frac{5}{3}\right)\right],\end{aligned} \tag{3.64}$$

where $\Delta(s,t,u)$ is some first order polynomial in the Mandelstam variables. The exact form is not needed for the renormalization group equations, as we will see. The counterterm is given by

$$\delta^{(4\Phi)} = \left(3\lambda^2 - \frac{3y_t^4}{4\pi^2}\right)\cdot\left(\frac{2}{\varepsilon} - \gamma + \log(4\pi)\right). \tag{3.65}$$

If we go to the symmetric point $s=t=u$ this can be written as

$$\mathbf{V}^{(1)}_{\Phi^4} = \left(\delta_a{}^c\delta_b{}^d + \delta_a{}^d\delta_b{}^c\right)\left[\frac{3y_t^4}{8\pi^2}\left(\log\left(\frac{M^2}{\Delta(s)}\right) - \frac{5}{6}\right) - \frac{3\lambda^2}{2\pi^2}\int_0^1 dx\,\log\left(\frac{M^2}{\mu^2 - x\left(1-x\right)s}\right)\right]. \tag{3.66}$$

For $s \gg \mu^2$ we can again carry out the integral over the Feynman parameter and find

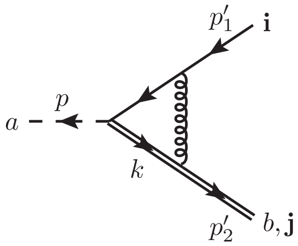
$$\mathbf{V}^{(1)}_{\Phi^4} \xrightarrow{s\gg\mu^2} \left(\delta_a{}^c\delta_b{}^d + \delta_a{}^d\delta_b{}^c\right)\left[\frac{3y_t^4}{8\pi^2}\left(\log\left(\frac{M^2}{\Delta(s)}\right) - \frac{5}{6}\right) - \frac{3\lambda^2}{2\pi^2}\left(\log\left(\frac{M^2}{s}\right) + 2 - i\pi\right)\right]. \tag{3.67}$$

3.2.4 Yukawa-vertex

The usual one-loop diagram contributing to the Yukawa vertex involving a Higgs in the loop does not exist in our model, as can be seen from the Feynman diagrams shown in Figure 3.4. The scalar propagators in the diagram cannot be closed. The reason for this is that the only possible loop involving a scalar in this diagram would be the exchange of the real or imaginary part of the neutral component of Φ only. The first correction to the Yukawa-vertex from diagrams with Higgs-loops appears at the two-loop level. This diagram is logarithmically divergent, as can be seen by power-counting, and is proportional to y_t^5. The only remaining one-loop correction comes

Figure 3.4: Feynman diagrams for the Yukawa vertex. There is no one-loop contribution, since the scalar propagators in the left diagram cannot be connected. The right diagram is the two-loop contribution to the Yukawa-vertex if the scalars do not interact and a three-loop diagram otherwise.

from a similar diagram involving a gluon. The contribution is given by the graph



$$= -i\mathbf{V}^{(1)}_{tb\Phi,g-\text{loop}}(p,p_1',p_2'),$$

where

$$\begin{aligned}
-i\mathbf{V}^{(1)}_{tb\Phi,g-\text{loop}}(p,p_1',p_2') &= y_t\,(\sigma_2)_a{}^c\,\delta_{\mathbf{k}}{}^{\mathbf{l}}\,(ig_s)^2\,\delta_c{}^b \int \frac{d^4k}{(2\pi)^4}\gamma_\mu T^A{}_{\mathbf{i}}{}^{\mathbf{k}} P_L \frac{i\not k}{k^2+i\epsilon} P_R \frac{i\left(\not k-\not p\right)}{(k-p)^2+i\epsilon}\gamma_\nu T^B{}_{\mathbf{l}}{}^{\mathbf{j}} \frac{-ig^{\mu\nu}\delta_{AB}}{(k-p_2')^2+i\epsilon} \\
&= -i\,(\sigma_2)_a{}^b\,\delta_{\mathbf{i}}{}^{\mathbf{j}} y_t g_s^3 C_F\left(\gamma^\mu P_L\gamma^\rho\gamma^\sigma\gamma_\mu\right)\int\frac{d^4k}{(2\pi)^4}\,\frac{k_\rho}{k^2+i\epsilon}\,\frac{(k-p)_\sigma}{(k-p)^2+i\epsilon}\,\frac{1}{(k-p_2')^2+i\epsilon} \\
&= -i\,(\sigma_2)_a{}^b\,\delta_{\mathbf{i}}{}^{\mathbf{j}} y_t g_s^2 4C_F\int\frac{d^4k}{(2\pi)^4}\,P_R\frac{k\cdot(k-p)}{(k^2-i\epsilon)\cdot((k-p)^2+i\epsilon)\cdot((k-p_2')^2)+i\epsilon)},
\end{aligned} \tag{3.68}$$

and where we contracted the γ-matrices using $\gamma^\mu\gamma^\rho\gamma^\sigma\gamma_\mu = 4g^{\rho\sigma}$. In order to evaluate the integral one can introduce three Feynman parameters

$$\frac{1}{ABC} = \int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{2!}{[xA+yB+zC]^3}. \tag{3.69}$$

Then, the denominator in the integral can be rewritten as

$$\begin{aligned} D^{-1} &= \left[\left(k^2 + i\epsilon\right) \cdot \left(\left(p-k\right)^2 + i\epsilon\right) \cdot \left(\left(k-p_2'\right)^2 + i\epsilon\right)\right]^{-1} \\ &= \int_0^1 dx\, dy\, dz\, \frac{\delta(x+y+z-1)}{[k^2 - 2k\cdot(yp + zp_2') + yp^2 + zp_2'^2 + i\epsilon]^3} \\ &= \int_0^1 dy\, dz\, \frac{1}{[\ell^2 + y\,(1-y)\,p^2 + z\,(1-z)\,p_2'^2 - 2yzp\cdot p_2' + i\epsilon]^3}, \end{aligned} \tag{3.70}$$

where $\ell \equiv k - yp - zp_2'$. For brevity we also define $\Delta \equiv -y\,(1-y)\,p_2'^2 - z\,(1-z)\,p^2 + 2yzp_2'\cdot p$. Rewriting the numerator of the integral in terms of ℓ, we find

$$N = k\cdot(k-p) = \ell^2 + \tilde{N}_1 + \tilde{N}_0, \tag{3.71}$$

where $\tilde{N}_1$ contains all terms proportional to ℓ, which will integrate to zero by symmetry, and $\tilde{N}_0 = -y\,(y-1)\,p^2 - z^2 p_2'^2 + z\,(2y-1)\,p\cdot p_2'$ are the terms independent of ℓ. Shifting the integration variable $k \to \ell$ we can thus rewrite the amplitude as

$$-i\mathbf{V}^{(1)}_{tb\Phi,g-\text{loop}}(p,p_1',p_2') = -i\,(\sigma_2)_a{}^b\,\delta_{\mathbf{i}}{}^{\mathbf{j}} y_t g_s^2 4C_F \int_0^1 dy\,dz \int \frac{d^4\ell}{(2\pi)^4} P_R \frac{\ell^2 + \tilde{N}_0}{[\ell^2 - \Delta]^3}. \tag{3.72}$$

Power counting suggests a divergent contribution from the ℓ^2-term and a finite contribution from $\tilde{N}_0$. Calculating the integral in dimensional regularization, we find

$$\begin{aligned} \int \frac{d^4\ell}{(2\pi)^4}\frac{\ell^2}{[\ell^2-\Delta]^3} &\to -\frac{4M^{4-d}}{d}\int \frac{d^d\ell}{(2\pi)^d}\frac{\ell^2}{[\ell^2-\Delta]^3} = \frac{4M^{4-d}}{d}\frac{i}{(4\pi)^{d/2}}\frac{d}{2}\cdot\frac{\Gamma(2-\frac{d}{2})}{\Gamma(3)}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} \\ &= \frac{i}{16\pi^2}\left(\frac{4\pi M^2}{\Delta}\right)^{\varepsilon/2}\Gamma(\frac{\varepsilon}{2}) = -\frac{i}{16\pi^2}\left(\frac{2}{\varepsilon} - \gamma + \log\left(\frac{4\pi M^2}{\Delta}\right)\right), \end{aligned} \tag{3.73}$$

$$\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{[\ell^2-\Delta]^3} = M^{4-d}\frac{-i}{(4\pi)^{d/2}}\cdot\frac{\Gamma(3-\frac{d}{2})}{\Gamma(3)}\left(\frac{1}{\Delta}\right)^{3-\frac{d}{2}} = -\frac{i}{32\pi^2}\cdot\frac{1}{\Delta}. \tag{3.74}$$

We are left to evaluate the integral over the Feynman parameters. However, we will learn nothing new from this since we know that the result has the form

$$-i\mathbf{V}^{(1)}_{tb\Phi,g-\text{loop}}(p,p_1',p_2') = (\sigma_2)_a{}^b\,\delta_{\mathbf{i}}{}^{\mathbf{j}}\frac{y_t g_s^2 C_F}{4\pi^2}\left(\frac{2}{\varepsilon} - \gamma + \log\left(\frac{4\pi M^2}{\Delta'(p,p_2')}\right) + \ldots\right), \tag{3.75}$$

where $\Delta'(p,p_2')$ is a second order function in the external momenta and "..." indicates finite terms. The appropriate counterterm has the form $-i\mathbf{V}^1_{tb\Phi,\text{ct}} = (\sigma_2)_a{}^b\,\delta_{\mathbf{i}}{}^{\mathbf{j}}\delta^{(tb\Phi)}_{y_t}$, hence, after subtracting the $\overline{\text{MS}}$ counterterm

$$\delta^{(tb\Phi)}_{y_t} = -\frac{y_t g_s^2 C_F}{4\pi^2}\left(\frac{2}{\varepsilon} - \gamma + \log(4\pi)\right), \tag{3.76}$$

we find

$$-i\mathbf{V}^{(1)}_{tb\Phi}(p,p_1',p_2') = (\sigma_2)_a{}^b\,\delta_{\mathbf{i}}{}^{\mathbf{j}}\frac{y_t g_s^2 C_F}{4\pi^2}\left(\log\left(\frac{M^2}{\Delta'(p,p_2')}\right) + \ldots\right). \tag{3.77}$$

3.3 β-functions and anomalous dimensions

In the previous sections we have calculated divergent amplitudes at the one-loop level in our model in dimensional regularization and the $\overline{\text{MS}}$ scheme. Recalling the renormalization procedure described in chapter 2, it remains to obtain the running parameters of the theory from requiring physical quantities to be independent of the arbitrary mass scale M we introduced when regularizing the divergent diagrams. The Callan-Symanzik equation for our model reads

$$\left[M\frac{\partial}{\partial M} + \beta_\lambda\frac{\partial}{\partial\lambda} + \beta_{y_t}\frac{\partial}{\partial y_t} + \sum_i n_i\gamma_i + \mu\gamma_\mu\frac{\partial}{\partial\mu}\right]\Gamma^{(n_i)} = 0, \tag{3.78}$$

where the $\Gamma^{(n_i)}$ are the Green's functions with n_i external fields of type $i = q, t_r, T_L, \Phi, \ldots$, $\gamma_i = \frac{M}{2}\frac{\partial Z_i}{\partial M}$ is the anomalous dimensions of the field i describing the scaling of the field renormalization with the renormalization scale, and the β-functions $\beta_{g_i} = M\frac{\partial g_i}{\partial M}$ describe the scaling of the renormalized couplings. γ_μ is the anomalous dimension of the mass-parameter μ. In the line of this argument and considering the form of the one-loop corrections, we can understand the arbitrary M^2 as the scale of invariants built from the external momenta involved in the process. Hence, the β-functions can be interpreted as the scaling of the physical couplings with the scale of external momenta.

We can take some shortcuts and obtain the anomalous dimensions and β-functions at the one-loop level directly from the M-dependent parts of the one-loop amplitudes. We consider e.g. the propagator for a $q = u, b, c, s, b_R$ quark,

$$\Delta_q(p) = \frac{i\not{p}}{p^2 - i\epsilon}\delta_{\mathbf{i}}^{\ \mathbf{j}}\left[1 - \frac{g_s^2 C_F}{16\pi^2}\log\left(\frac{M^2}{p^2}\right) + \ldots\right] + \ldots, \tag{3.79}$$

where the first "..." stands for finite contributions from the one-loop diagram, and the last "..." for higher order corrections. From the form of the Callan-Symanzik equation (3.78) we can directly obtain the anomalous dimension from the M-dependent part of the propagator by

$$\frac{i\not{p}}{p^2 + i\epsilon}2\gamma_q = -M\frac{\partial}{\partial M}\Delta_q, \tag{3.80}$$

which yields

$$\gamma_q^{(1)} = -\frac{M}{2}\frac{\partial}{\partial M}\left[-\frac{g_s^2 C_F}{16\pi^2}\log\left(\frac{M^2}{p^2}\right)\right] = \frac{g_s^2 C_F}{16\pi^2}. \tag{3.81}$$

Similarly, from our results for the one-loop propagators for t_R, T_L, and Φ

$$\begin{aligned}
\Delta_{t_R}(p) &= \frac{i\not{p}}{p^2}\delta_{\mathbf{i}}^{\ \mathbf{j}}\left[1 - \frac{g_s^2 C_F}{16\pi^2}\log\left(\frac{M^2}{p^2}\right) - \frac{y_t^2}{16\pi^2}\log\left(\frac{M^2}{p^2}\right) + \ldots\right] + \ldots, &(3.82)\\
\Delta_{T_L}(p) &= \frac{i\not{p}}{p^2}\delta_{\mathbf{i}}^{\ \mathbf{j}}\delta_a^{\ b}\left[1 - \frac{g_s^2 C_F}{16\pi^2}\log\left(\frac{M^2}{p^2}\right) - \frac{y_t^2}{32\pi^2}\log\left(\frac{M^2}{p^2}\right) + \ldots\right] + \ldots, &(3.83)\\
\Delta_\Phi(p^2) &= \frac{i}{p^2 - \mu^2}\delta_a^{\ b}\left[1 - \frac{p^2}{p^2 - \mu^2}\frac{3y_t^2}{16\pi^2}\log\left(\frac{M^2}{p^2}\right) + \frac{\mu^2}{p^2 - \mu^2}\frac{3\lambda}{8\pi^2}\log\left(\frac{M^2}{\mu^2}\right) + \ldots\right] + \ldots &(3.84)
\end{aligned}$$

we find the anomalous dimensions

$$\begin{aligned}
\gamma_{t_R}^{(1)} &= -\frac{M}{2}\frac{\partial}{\partial M}\left[-\left(\frac{g_s^2 C_F}{16\pi^2} + \frac{y_t^2}{16\pi^2}\right)\log\left(\frac{M^2}{p^2}\right)\right] = \frac{g_s^2 C_F}{16\pi^2} + \frac{y_t^2}{16\pi^2} = \gamma_q^{(1)} + \frac{y_t^2}{16\pi^2}, &(3.85)\\
\gamma_{T_L}^{(1)} &= -\frac{M}{2}\frac{\partial}{\partial M}\left[-\left(\frac{g_s^2 C_F}{16\pi^2} + \frac{y_t^2}{32\pi^2}\right)\log\left(\frac{M^2}{p^2}\right)\right] = \frac{g_s^2 C_F}{16\pi^2} + \frac{y_t^2}{32\pi^2} = \gamma_q^{(1)} + \frac{y_t^2}{32\pi^2}, &(3.86)\\
\gamma_\Phi^{(1)} &= -\frac{M}{2}\frac{\partial}{\partial M}\left[-\frac{3y_t^2}{16\pi^2}\log\left(\frac{M^2}{p^2}\right)\right] = \frac{3y_t^2}{16\pi^2}. &(3.87)
\end{aligned}$$

Similarly, from the one-loop vertices

$$\begin{aligned}
-i\mathbf{V}_{\Phi^4} &= -2i\left(\delta_a^{\ c}\delta_b^{\ d} + \delta_a^{\ d}\delta_b^{\ c}\right)\left[\lambda + \frac{3y_t^4}{16\pi^2}\log\left(\frac{M^2}{\Delta(s)}\right) - \frac{3\lambda^2}{4\pi^2}\log\left(\frac{M^2}{s}\right) + \ldots\right], &(3.88)\\
-i\mathbf{V}_{tb\Phi} &= (\sigma_2)_a^{\ b}\,\delta_{\mathbf{i}}^{\ \mathbf{j}}\left[y_t + \frac{y_t g_s^2 C_F}{4\pi^2}\log\left(\frac{M^2}{\Delta'(p, p_2')}\right) + \ldots\right], &(3.89)
\end{aligned}$$

where the "..." now indicates both finite contributions from the one-loop diagrams and higher order contributions, we get

$$\begin{aligned}
\beta_\lambda &= -M\frac{\partial}{\partial M}\left[\left(\frac{3y_t^4}{16\pi^2} - \frac{3\lambda^2}{4\pi^2}\right)\log\left(\frac{M^2}{\tilde{p}^2}\right) + 4\lambda\gamma_\Phi\right] = \frac{3\lambda^2}{2\pi^2} + \frac{3\lambda y_t^2}{4\pi^2} - \frac{3y_t^4}{8\pi^2}, &(3.90)\\
\beta_{y_t} &= -M\frac{\partial}{\partial M}\left[\frac{y_t g_s^2 C_F}{4\pi^2}\log\left(\frac{M^2}{\tilde{p}^2}\right)\right] + y_t\left(\gamma_{t_R} + \gamma_{T_L} + \gamma_\Phi\right) = \frac{9y_t^3}{32\pi^2} - \frac{3y_t g_s^2 C_F}{8\pi^2}. &(3.91)
\end{aligned}$$

We can already note a peculiar feature of the β-function of the quartic coupling λ: it is the only SM-coupling with an β-function that is not proportional to its coupling. This opens the possibility for λ to change sign at some scale M^2. From (3.90) we see, that a large top Yukawa coupling can drive λ negative at large scales if λ is sufficiently small at small scales.

4 Extrapolation of the SM Higgs potential

With the discovery of a particle compatible with the SM Higgs at the LHC in 2012 the final ingredient of the SM was found. We can now measure all the parameters of the SM, i.e. the physical masses and coupling strengths, at energies up to $\sim 1\,\text{TeV}$. Merging this experimental input with the theoretical predictions for the running of the parameters we can thus explore the behavior of the SM at scales much larger than the ones experimentally accessible. It has been known for a long time that the SM Higgs potential could develop a second minimum deeper than the electroweak one at energy scales below the Planck scale, or become non-perturbative at large scales, depending on the SM parameters [1–16]. If we take the Higgs-like particle discovered at the LHC to be the SM Higgs, the SM occupies a peculiar spot in the parameter space. It is metastable, but sits only a few standard deviations off the region where it would remain stable up to the Planck scale.

When investigating the stability of the SM Higgs potential, the proper object to consider is the effective potential improved by renormalization group equations. Excellent introductions to the effective potential and its computation can be found in [1,6] as well as in standard QFT textbooks, e.g., [17,18]. However, as long as the instability occurs at scales well above the electroweak scale it suffices to require $\lambda \geq 0$ as a stability criterion to good approximation [3,25]. For the perturbativity requirement we follow [12,14] and consider two bounds: if we demand $\lambda \leq \pi$ at the cut-off scale, the two-loop correction is less than 25 % of the one-loop contribution to the β-function of the quartic coupling, and the perturbative expansion is still reliable. For $\lambda = 2\pi$ at the cut-off, the two-loop correction to the one-loop β-function is 50 % and the model is on the verge of non-perturbativity.

In the previous chapter we calculated the dominant contributions to the running of the quartic Higgs-coupling λ. We found

$$M\frac{\partial\lambda}{\partial M} = \beta_\lambda = \frac{3\lambda^2}{2\pi^2} + \frac{3\lambda y_t^2}{4\pi^2} - \frac{3y_t^4}{8\pi^2}, \tag{4.1}$$

in our simplified model without electroweak interactions and all Yukawa couplings except for the top quark set to zero. In this simplified model we already find the characteristic behavior of the Higgs potential at large energies. If the quartic coupling is too large, i.e. the Higgs too heavy, λ blows up at large energies and the model becomes non-perturbative. Eventually, the quartic coupling will hit a Landau pole below the cutoff. If y_t is too large, i.e. the top too heavy, the y_t^4 term in β_λ will drive λ to negative values and the potential will become unstable. For even larger top masses, the top Yukawa coupling will become non-perturbative, which leads in turn to λ becoming non-perturbative as well. In order to compute the running couplings in our model, we also need the β-function for the strong gauge coupling g_s. It is well known for a long time, and given by $\beta_{g_s} = -7g_s^3/16\pi^2$ at the one-loop level [26].

Besides the β-functions, one needs the values of the SM parameters at some convenient scale as input when extrapolating to higher scales. Obtaining the renormalized parameters in the $\overline{\text{MS}}$-scheme from the physical observables, the so-called matching, is a non-trivial task [20–22]. Ready-to-use expressions for the $\overline{\text{MS}}$ parameters at the two-loop level for the matching scale $M = M_t$,

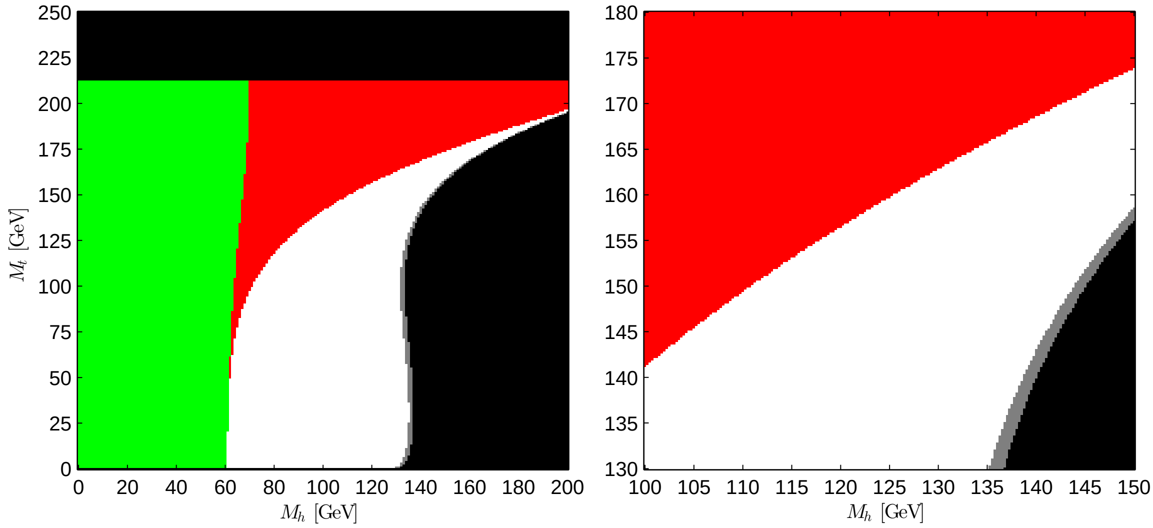


Figure 4.1: Phase space for the simplified model of chapter 3 for floating top and Higgs mass. Green regions correspond to portions of the phase space for which the quartic Higgs coupling λ turns negative for renormalization scales below the electroweak scale $M \leq v = 246\,\text{GeV}$. For such small instability scales, our simplified criterion for instability is no longer valid and one must consider the full effective potential. In the red portion of the parameter space, λ becomes negative below the Planck-scale $v \leq M \leq M_{\text{Pl}} = 1.22 \times 10^{19}\,\text{GeV}$. For the white regions, the quartic coupling stays positive all the way up to the Planck scale, and remains perturbative. In the gray portion of the parameter space, $\pi \leq \lambda(M = M_{\text{Pl}}) \leq 2\pi$ and the model is on the verge of non-perturbativity. In the black region, $\lambda(M = M_{\text{Pl}}) \geq 2\pi$ and the model becomes non-perturbative. The right panel shows parameter space around the central experimental values $M_h = 125.7\,\text{GeV}$, $M_t = 173.21\,\text{GeV}$ in more detail.

the top quark mass, are given in [16]:

$$
\begin{aligned}
\lambda(M = M_t) &= 0.12711 + 0.00206\,(M_h - 125.66) - 0.00004\,(M_t - 173.10) \pm 0.00030_{\text{th}}, && (4.2)\\
\mu(M = M_t) &= 132.03 + 0.94\,(M_h - 125.66) + 0.17\,(M_t - 173.10) \pm 0.15_{\text{th}}, && (4.3)\\
y_t(M = M_t) &= 0.93558 + 0.00550\,(M_t - 173.10) - 0.00042 \cdot \frac{\alpha_3(M_Z) - 0.1184}{0.0007} + \\
&\quad -0.00042 \cdot \frac{M_W - 80.384}{0.014} \pm 0.00050_{\text{th}}, && (4.4)\\
g_Y(M = M_t) &= 0.35761 + 0.00011\,(M_t - 173.10) - 0.00021 \cdot \frac{M_W - 80.384}{0.014}, && (4.5)\\
g_2(M = M_t) &= 0.64822 + 0.00004\,(M_t - 173.10) + 0.00011 \cdot \frac{M_W - 80.384}{0.014}, && (4.6)\\
g_s(M = M_t) &= 1.1666 + 0.00314 \cdot \frac{\alpha_3(M_Z) - 0.1184}{0.0007} - 0.00046\,(M_t - 173.10)\,, && (4.7)
\end{aligned}
$$

where all masses are measured in GeV. The experimental values are [23]

$$
\begin{aligned}
&\alpha_s(M_Z^2) = 0.1185 \pm 0.0006, \quad \frac{M_W}{\text{GeV}} = 80.385 \pm 0.015, \quad \frac{M_Z}{\text{GeV}} = 91.1876 \pm 0.0021,\\
&\frac{M_h}{\text{GeV}} = 125.7 \pm 0.4, \quad \frac{M_t}{\text{GeV}} = 173.21 \pm 0.51 \pm 0.71.
\end{aligned} \tag{4.8}
$$

The resulting phase space when extrapolating our simplified model to the Planck scale with floating masses for the Higgs and the top quark is shown in Figure 4.1.

For a more thorough analysis of the SM's stability one must include the electroweak interactions. It is well known, that the β-functions for the non-Abelian gauge couplings in the SM are negative, while the β-function for the U(1) gauge coupling is positive. Hence, at some scale the weak hypercharge U(1) coupling in the SM g_Y surpasses g_s. It is also well known that in the SM the $\mathrm{SU}(2)_L$ coupling g_2 becomes larger than g_s for scales $M \gtrsim 10^{17}$ GeV. Hence, at large scales the electroweak contributions will dominate the QCD contributions to the running of the top Yukawa and the quartic Higgs coupling. The two-loop renormalization group equations for the full SM have been known for a long time [26–28]. The three-loop β-functions for the full SM became available in the last few years [29–31].

Including electroweak interactions, but again setting all Yukawa couplings except for the top to zero, the one-loop β-functions for the gauge-couplings are given by [26]

$$\beta_{g_1}^{(1)} = \frac{g_1}{16\pi^2}\left(\frac{41}{10}g_1^2\right), \quad \beta_{g_2}^{(1)} = \frac{g_2}{16\pi^2}\left(-\frac{19}{6}g_2^2\right), \quad \beta_{g_s}^{(1)} = \frac{g_s}{16\pi^2}\left(-7g_s^2\right), \tag{4.9}$$

with the $\mathrm{U}(1)_Y$ hypercharge coupling g_Y in conventional SU(5) normalization $g_1^2 = \frac{5}{3}g_Y^2$. For the top Yukawa and the quartic Higgs coupling the β-functions are [26–28]

$$\beta_{y_t}^{(1)} = \frac{y_t}{16\pi^2}\left(\frac{9}{2}y_t^2 - \frac{17}{20}g_1^2 - \frac{9}{4}g_2^2 - 8g_s^3\right), \tag{4.10}$$

$$\beta_{\lambda}^{(1)} = \frac{1}{16\pi^2}\left(24\lambda^2 + 12y_t^2\lambda - 6y_t^4 - \left(\frac{9}{5}g_1^2 + 9g_2^2\right)\lambda + \frac{27}{200}g_1^4 + \frac{9}{20}g_1^2g_2^2 + \frac{9}{8}g_2^4\right). \tag{4.11}$$

The two-loop β-functions are given by [26–28]

$$\beta_{g_1}^{(2)} = \frac{g_1}{(16\pi^2)^2}\left[\frac{199}{50}g_1^4 + \frac{27}{10}g_1^2g_2^2 + \frac{44}{5}g_1^2g_s^2 - \frac{17}{10}y_t^2g_1^2\right], \tag{4.12}$$

$$\beta_{g_2}^{(2)} = \frac{g_2}{(16\pi^2)^2}\left[\frac{9}{10}g_1^2g_2^2 + \frac{35}{6}g_2^4 + 12g_2^2g_s^2 - \frac{3}{2}y_t^2g_s^2\right], \tag{4.13}$$

$$\beta_{g_s}^{(2)} = \frac{g_s}{(16\pi^2)^2}\left[\frac{11}{10}g_1^2g_s^2 + \frac{9}{2}g_2^2g_s^2 - 26g_s^4 - 2y_t^2g_s^2\right], \tag{4.14}$$

$$\begin{aligned}\beta_{y_t}^{(2)} = \frac{y_t}{(16\pi^2)^2}\Big[&-12y_t^4 + 6\lambda^2 - 12\lambda y_t^2 + \frac{393}{80}g_1^2y_t^2 + \frac{225}{16}g_2^2y_t^2 + 36g_s^3y_t^2 + \\ &+\frac{1187}{600}g_1^4 - \frac{9}{20}g_1^2g_2^2 + \frac{19}{15}g_1^2g_3^2 - \frac{23}{4}g_2^4 + 9g_2^2g_s^2 - 108g_s^4\Big],\end{aligned} \tag{4.15}$$

$$\begin{aligned}\beta_{\lambda}^{(2)} = \frac{1}{(16\pi^2)^2}\Big[&-312\lambda^3 + 108\left(\frac{g_1^2}{5} + g_2^2\right)\lambda^2 + \left(\frac{1887}{200}g_1^4 + \frac{117}{20}g_1^2g_2^2 - \frac{73}{8}g_2^4\right)\lambda + \\ &+\frac{305}{16}g_2^6 - \frac{289}{80}g_1^2g_2^4 - \frac{1667}{400}g_1^4g_2^2 - \frac{3411}{2000}g_1^6 + \\ &-32g_s^2y_t^4 - \frac{8}{5}g_1^2y_t^4 - \frac{9}{4}g_2^4y_t^2 + g_1^2\left(\frac{63}{10}g_2^2 - \frac{171}{100}g_1^2\right)y_t^2 \\ &+\left(\frac{17}{2}g_1^2 + \frac{45}{2}g_2^2 + 80g_s^2\right)y_t^2\lambda - 144y_t^2\lambda^2 - 3y_t^4\lambda + 30y_t^6\Big].\end{aligned} \tag{4.16}$$

The three-loop β-functions can be found in [29–31].

Figures 4.2 and 4.3 show a comparison between the running couplings found in our simplified model at the one-loop level and when including electroweak interactions at the one-loop or two-loop level. We find that the electroweak interactions make the SM potential more stable. Considering the β-functions (4.10)-(4.16), we see that there are two reasons for this: the electroweak interactions give a negative contribution to the running of y_t, hence, reducing its effect of driving λ negative, and a positive contribution to the running of λ itself. Figure 4.4 shows the phase space for floating top quark and Higgs mass. The stable phase of the model occupies a larger portion of the parameter space when including electroweak contributions, for the reasons described above. Due to the same reasons, the model also remains perturbative for larger M_h: since y_t evolves to smaller values at

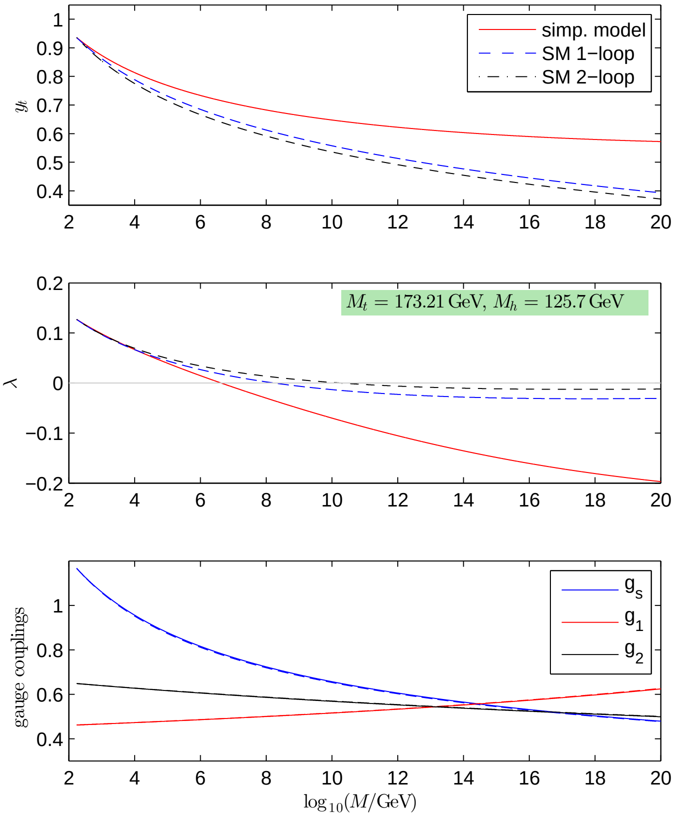


Figure 4.2: Comparison of the running couplings in the simplified model of chapter 3 at the one-loop level and when including electroweak contributions at the one-loop and the two-loop level for the central experimental values $M_t = 173.21\,\mathrm{GeV}$, $M_h = 125.7\,\mathrm{GeV}$. The top panel shows the running of the Yukawa coupling for the top quark and the middle panel the running of the quartic Higgs coupling. In both panels, the solid red line corresponds to the simplified model, the dashed blue line to the SM at the one-loop, and the dash-dotted black line to the SM at the two-loop level. The bottom panel shows the running of the gauge couplings. Blue corresponds to the strong coupling, black to the $\mathrm{SU}(2)_L$ coupling g_2 and red to the $\mathrm{U}(1)_Y$ coupling in conventional normalization $g_1^2 = \frac{5}{3} g_Y^2$. In the case of the gauge couplings, the difference between the one-loop and the two-loop level evolution is not visible at this scale.

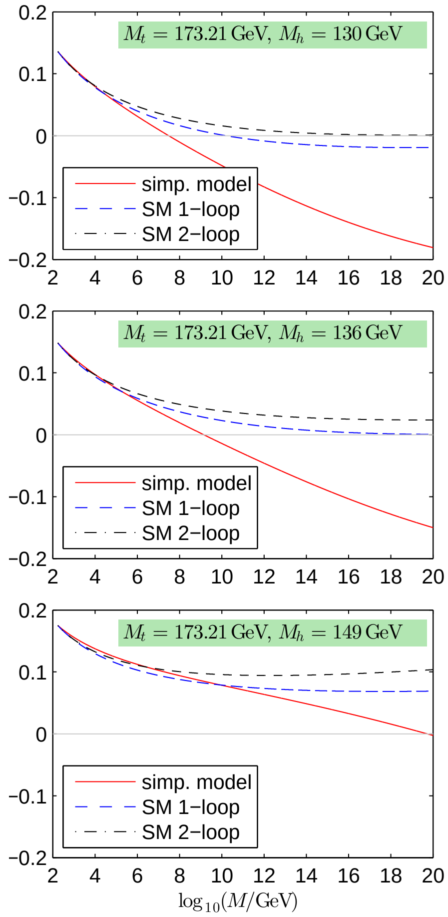


Figure 4.3: Running of the quartic Higgs coupling λ in the simplified model (red solid line), the SM at the one-loop (dashed blue line) and the SM at the two-loop level (dash-dotted black line) for different values of Higgs pole mass and the top mass fixed to the central value $M_t = 173.21\,\mathrm{GeV}$. The top panel is for $M_h = 130\,\mathrm{GeV}$, where λ stays positive up to M_{pl} for the SM two-loop case. The middle panel is for $M_h = 136\,\mathrm{GeV}$, where λ stays positive up to M_{pl} for the SM one-loop case, and the bottom panel is for $M_h = 149\,\mathrm{GeV}$, where λ stays positive for our simplified model.

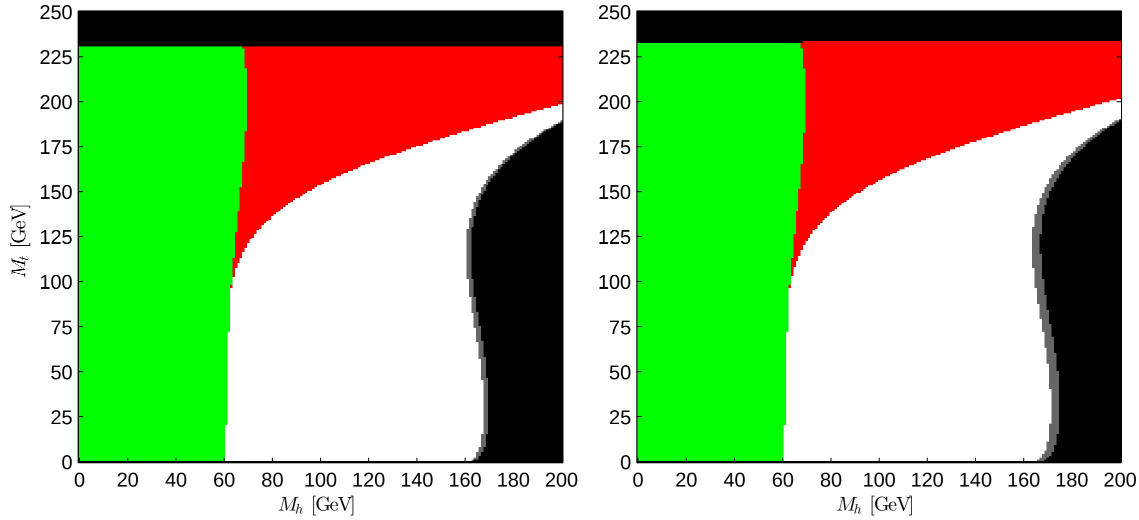


Figure 4.4: Comparison of the phase space for the SM for floating top and Higgs mass at the one-loop level (left panel) and the two-loop level (right panel). Green regions correspond to portions of the phase space for which the quartic Higgs coupling λ turns negative for renormalization scales below the electroweak scale $M \leq v = 246\,\mathrm{GeV}$. For such small instability scales, our simplified criterion for instability is no longer valid and one must consider the full effective potential. In the red portion of the parameter space, λ becomes negative below the Planck-scale $v \leq M \leq M_{\mathrm{Pl}} = 1.22 \times 10^{19}\,\mathrm{GeV}$. For the white regions, the quartic coupling stays positive all the way up to the Planck scale, and remains perturbative. In the gray portion of the parameter space, $\pi \leq \lambda(M = M_{\mathrm{Pl}}) \leq 2\pi$ and the model is on the verge of non-perturbativity. In the black region, $\lambda(M = M_{\mathrm{Pl}}) \geq 2\pi$ and the model becomes non-perturbative.

large scales than in our simplified model, the term proportional to $y_t\lambda$ in the one-loop beta-function for λ contributes less. Additionally, the $g_1^2\lambda$ and $g_2^2\lambda$ terms give negative contribution and thus keep λ from growing too large. As for our simplified model, we find that for $M_h \lesssim 60\,\mathrm{GeV}$ λ turns negative below the electroweak scale $v = 246\,\mathrm{GeV}$. Then, our simplified approach of requiring $\lambda \geq 0$ is no longer valid and one must consider the full effective potential. For $M_t \gtrsim 230\,\mathrm{GeV}$, we find an instability due to the top Yukawa coupling becoming non-perturbative, which leads to λ blowing up.

A comparison of the stable regions of the phase space between our simplified model of chapter 3 at the one-loop level and when including electroweak contribution at the one-loop and the two-loop level is shown in Figure 4.5. In the vicinity of the central experimental value for the top mass $M_t = 173.21 \pm 0.51_{\mathrm{stat.}} \pm 0.71_{\mathrm{syst.}}$ [23], one can give a linear approximation of the bound for the Higgs mass, for which the quartic coupling $\lambda(M < M_{\mathrm{Pl}}) \geq 0$. Then, the SM would be stable all the way up to the Planck scale. In our simplified model this bound is given by

$$M_h > 149\,\mathrm{GeV} + 1.89\,(M_t - 173.25\,\mathrm{GeV})\,. \tag{4.17}$$

When including electroweak contributions we find at the one-loop level

$$M_h > 136\,\mathrm{GeV} + 2.18\,(M_t - 173.25\,\mathrm{GeV})\,, \tag{4.18}$$

and at the two-loop level

$$M_h > 130\,\mathrm{GeV} + 2.12\,(M_t - 173.25\,\mathrm{GeV}) \pm 1\,\mathrm{GeV}. \tag{4.19}$$

The error on M_h for the two-loop bound is dominated by shifts to the top quark pole mass from QCD corrections [14]. For the one-loop results we give no error, but from the results one can readily conclude that for the one-loop results the uncertainty due to higher order corrections is $\sim 15\,\mathrm{GeV}$ when including electroweak contributions and $\sim 20\,\mathrm{GeV}$ for our simplified model.

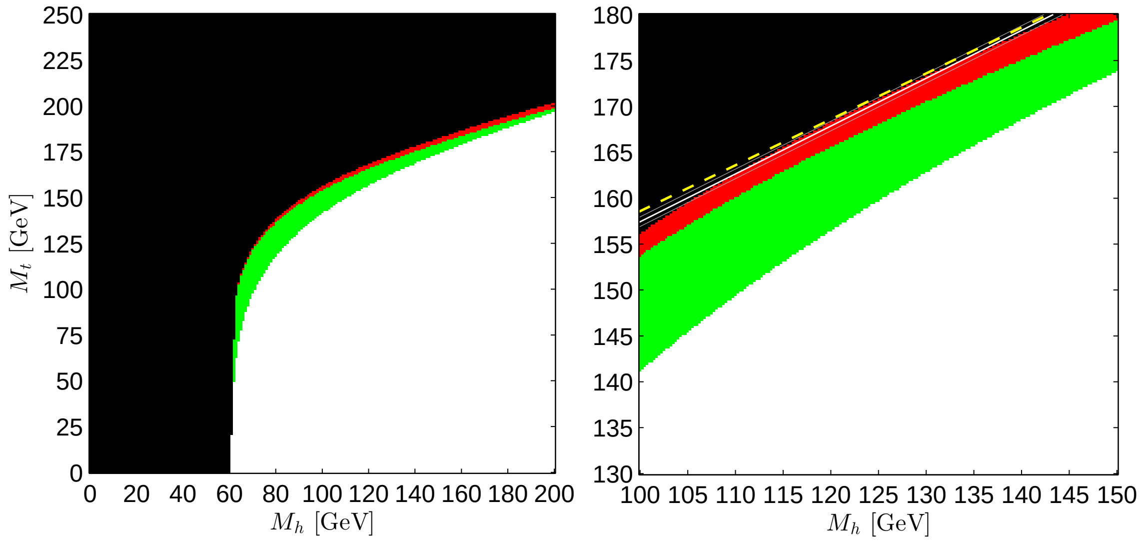


Figure 4.5: Comparison between the stable portions of phase space between our simplified model of chapter 3 and when including electroweak contributions in the SM at the one-loop and two-loop level. For the white portion of the phase space, $\lambda \geq 0$ all the way to the Planck-scale for our simplified model. The green and white regions corresponds to the region of the phase space where $\lambda \geq 0$ for the one-loop approximation. At the two-loop level, λ remains positive all the way up to the Planck scale in the red, green and white portion of the parameter space. The right panel shows the region around the central experimental values $M_h = 125.7\,\text{GeV}$, $M_t = 173.21\,\text{GeV}$ in more detail. The white solid line is the stability bound at the two-loop level reported in [9] with the thin gray lines displaying the reported error. The dashed yellow line is the bound at the three-loop level from [16], where the thickness of the line corresponds to the reported error. Both bounds are valid in the vicinity of the measured top quark mass $M_t \approx 173\,\text{GeV}$, and the SM is stable in the region of phase space below the corresponding line.

Within the errors, our bounds agree with the ones found in the literature. At the two-loop level, [9] finds

$$\begin{aligned} &M_h > 127.9\,\text{GeV} + 1.92\,(M_t - 174\,\text{GeV}) - 4.25\,\text{GeV}\left(\frac{\alpha_s - 0.124}{0.006}\right) \pm 1\,\text{GeV} \\ \Rightarrow\ &M_h > 130.4 + 1.92\,(M_t - 173.25) \pm 1\,\text{GeV}, \end{aligned} \tag{4.20}$$

for the value $\alpha_s = 0.1185$ we use and demanding stability up to a cut-off scale $M = 10^{19}\,\text{GeV}$ slightly smaller than our criterion $M_{\text{Pl}} = 1.2 \times 10^{19}\,\text{GeV}$. At the three-loop level, [16] finds

$$M_h > 129.1\,\text{GeV} + 2.0\,(M_t - 173.10\,\text{GeV}) - 0.5\,\text{GeV}\left(\frac{\alpha_s(M_Z) - 0.1184}{0.0007}\right) \pm 0.3\,\text{GeV}. \tag{4.21}$$

Both bounds are displayed in Figure 4.5 together with our regions of stability.

Comparing this with the central experimental values for the top and Higgs mass [23][1]

$$\frac{M_h}{\text{GeV}} = 125.7 \pm 0.4, \qquad \frac{M_t}{\text{GeV}} = 173.21 \pm 0.51 \pm 0.71, \tag{4.22}$$

one finds the SM to lie just outside the stability region, but being compatible with stability all the way up to the Planck-scale when deviating $\sim 2.5\,\sigma$ from the central values.

[1] Recently, an updated combined measurement for the Higgs mass $M_h = 125.09 \pm 0.24\,\text{GeV}$ was published by the CMS and ATLAS collaborations [24]. The impact on our results is negligible.

5 Interpretation of the instability. Hints to new physics?

In the previous chapter, we saw that the experimental values suggest that the SM occupies a peculiar spot in the parameter space: it lies just outside the stable region and develops a second minimum deeper than the electroweak one at scales $M \gtrsim 10^{10}\,\mathrm{GeV}$. However, experimental and theoretical uncertainties are too large as yet as to make a final statement on this instability. Absolute instability, i.e. the quartic Higgs coupling $\lambda \geq 0$ all the way up to the Planck scale, is only excluded at about 98 % confidence limit.

Even if we assume that the situation remains unchanged as experimental and theoretical uncertainties become smaller, this does not necessarily lead to an inconsistency of the SM at scales below M_{Pl}. As discussed e.g. in [10, 14–16], one can trade requiring an absolutely stable potential for demanding metastability: if we assume the universe to have fallen in the electroweak vacuum during its evolution, we can compute the quantum tunneling rate to its true minimum. Comparing the lifetime of the electroweak vacuum with the age of the universe, one can accept a metastable situation where the lifetime of the electroweak vacuum is greater than the age of the universe. The quantum tunneling probability is given by [14]

$$p = \max_{\Phi<\Lambda} \left[V_U \Phi^4 e^{-8\pi^2/3|\lambda(\Phi)|} \right], \tag{5.1}$$

where $V_U = \tau_U^4$ is the space-time volume of the past light cone of the observable universe, and $\tau_U = 13.7 \pm 0.2\,\mathrm{Gyrs}$ the lifetime of the universe. Φ is the field value of the Higgs field and the quartic coupling λ must be evaluated at this field value, as discussed above. Λ is the cut-off scale of our model. For the central experimental values of the SM and taking the cut-off scale to be $\Lambda = M_{\mathrm{Pl}}$, the Planck scale, we find $p \sim e^{-1000}$, hence, the SM has a spectacularly small probability to tunnel to the true vacuum and is sitting well in the metastable phase. Considering the entire phase space, [14] finds a metastability bound considering only quantum tunneling

$$M_h > 108.9\,\mathrm{GeV} + 3.1\,(M_t - 173.1\,\mathrm{GeV}) - 3.5\,\mathrm{GeV} \left(\frac{\alpha_s(M_Z) - 0.1193}{0.0028} \right) \pm 3\,\mathrm{GeV} \tag{5.2}$$

in the vicinity of $M_t = 173.1\,\mathrm{GeV}$.

The metastability bound given above neglects thermal or inflationary fluctuations in the early universe. If one wants to take these into account, one has to assume a scale up to which standard cosmology is valid. Assuming the electroweak vacuum to be stable against thermal fluctuations up to temperatures as large as the Planck scale, [14] finds a metastability bound

$$M_h > 122.0\,\mathrm{GeV} + 2.3\,(M_t - 173.1\,\mathrm{GeV}) - 2.3\,\mathrm{GeV} \left(\frac{\alpha_s(M_Z) - 0.1193}{0.0028} \right) \pm 3\,\mathrm{GeV}. \tag{5.3}$$

In the light of the metastability argument, the SM can survive all the way up to the Planck scale without modifications. However, other problems besides the stability problem, e.g. neutrino masses, the strong-CP problem, dark matter, the hierarchy problem of the renormalized Higgs mass, or a quantized theory of gravity lead us to believe that new physics will appear below the Planck scale. There have been many speculations whether the instability problem can provide hints on new physics. Trivially, new physics must set in at or below the scale of $M \sim 10^{10}\,\mathrm{GeV}$, if it is to keep the quartic Higgs coupling from turning negative. In the following, we will conduct a short literature review of the connection of the SM potential and some of the aforementioned problems.

5.1 Neutrino masses

Neutrinos in the SM are massless left-handed fermions. From neutrino oscillation experiments we know however, that neutrinos must have small, non-degenerate masses. Neutrino masses can most economically be introduced at low energies through the dimension five Weinberg operator [32]. At tree-level, this operator can be realized through the three so-called seesaw mechanisms [33]. In type-I seesaw, one introduces at least two heavy $\mathrm{SU}(2)_L$ singlet fermions, the so-called right-handed neutrinos. In type-II, one replaces the right-handed neutrinos with scalar $\mathrm{SU}(2)_L$ triplets, and in type-III with fermionic $\mathrm{SU}(2)_L$ triplets.

Introducing new particles to the SM will alter the renormalization group equations for scales larger than the new particles' masses. Thus, it will modify the high energy behavior of the running couplings and possibly influence the stability of the SM vacuum.

For type-I seesaw the situation looks unpromising: the heavy fermions will act similar to the top quark's effect on the quartic Higgs coupling, hence, making the model less stable [34, 35]. The case is different for type-III seesaw, although the new particles are fermionic. It has been shown in [35, 36], that type-III seesaw models can lower the absolute stability bound to $M_h = 125\,\mathrm{GeV}$ for seesaw scales as small as $\sim 1\,\mathrm{TeV}$. The situation appears even more promising for type-II seesaw, for which it has been shown in [37–39] that type-II seesaw with a seesaw scale as low as a TeV can push the stability bound well below 125 GeV.

5.2 The strong-CP problem and axions

QCD allows for a CP-breaking term $\mathcal{L}_\theta = \frac{\alpha_s}{8\pi}\theta G^a_{\mu\nu}\tilde{G}^{a,\mu\nu}$, where $G^{\mu\nu}$ ($\tilde{G}^{\mu\nu}$) is the (dual) gluon field strength tensor, and θ a free dimensionless parameter of the SM. The most sensitive test of θ is the electric dipole moment of the neutron, which restricts $\left|\overline{\theta}\right| \lesssim 10^{-10}$ [46]. $\overline{\theta} = \theta + \arg\det M$ is the effective parameter and M the quark mass matrix. That θ, which parametrizes the CP-violation of QCD, is at most of order 10^{-10} while theoretically expected to be of order 1, is the so-called strong-CP problem.

In 1977, Peccei and Quinn proposed a solution of the strong-CP problem by introducing a new global $\mathrm{U}(1)_{\mathrm{PQ}}$ symmetry broken at some scale f_a [40]. Soon after, it was realized that the Goldstone boson from breaking the $\mathrm{U}(1)_{\mathrm{PQ}}$ acquires an effective mass through its couplings to quarks, and one finds a pseudoscalar pseudo-Goldstone boson, the so-called axion [41]. The original axion used an f_a of the order of the electroweak scale v and was soon ruled out experimentally. Soon after, so-called invisible axion models where developed with much larger breaking scales $f_a \gg v$, notably the KSVZ [42] and DFSZ [43] axions. The KSVZ axion employs a heavy fermion, which is a scalar under $\mathrm{SU}(2)_L$, and an $\mathrm{SU}(3)_c$ triplet, and a heavy complex scalar charged only under the new $\mathrm{U}(1)_{\mathrm{PQ}}$ symmetry. The DFSZ axion uses a two Higgs doublet model and a heavy scalar $\mathrm{SU}(2)_L$ singlet, and ordinary quarks and leptons being charged under the new $\mathrm{U}(1)_{\mathrm{PQ}}$. In both cases, the scalars develop a non-vanishing vacuum expectation value below the breaking scale f_a. Axion models are arguably the best motivated solution to the strong-CP problem. Furthermore, they provide a viable dark matter candidate (see [44–47] for recent reviews on axions and their role in solving various problems in the SM).

Recently, an extended version of type-I seesaw including an KSVZ axion has been considered in [48]. The model was shown to provide a stable electroweak vacuum all the way up to the Planck scale for reasonable choices of the seesaw and axion parameters. In [49], it was shown that a DFSZ axion also leads to a stable Higgs potential all the way up to the Planck scale. In both cases, the threshold effects of the heavy scalars keep the quartic coupling λ from turning negative below M_{Pl}.

5.3 Quantum gravitational contributions

The discussion of metastability suffers from the problem, that the true vacuum of the SM potential occurs at $M = 10^{30}\,\mathrm{GeV} \gg M_{\mathrm{Pl}}$ when considering the renormalization group improved effective potential (cf. the left panel of Figure 5.1). Above the Planck scale, gravitational effects certainly cannot be neglected anymore. Recently, the Higgs metastability problem has been considered in

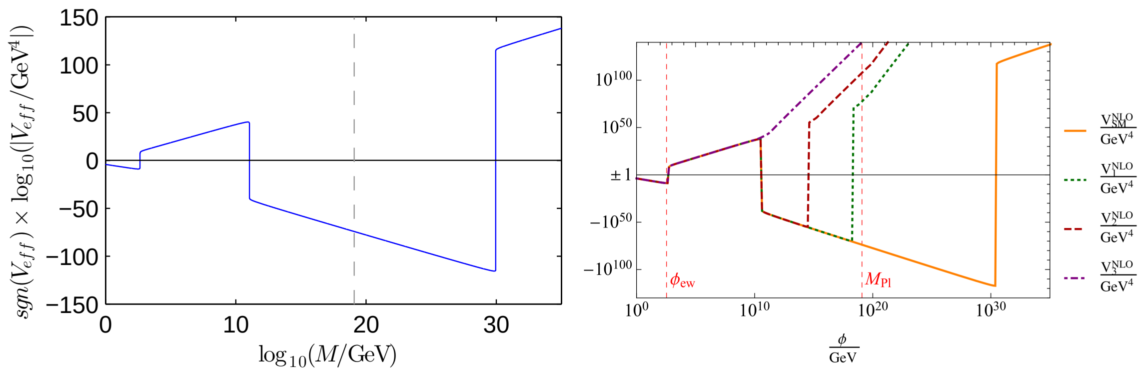


Figure 5.1: Left Panel: double logarithmic plot of the one-loop effective potential improved with two-loop renormalization group equation for the central experimental values $M_h = 125.7\,\mathrm{GeV}$, $M_t = 173.21\,\mathrm{GeV}$. The one-loop effective potential can be found in [16]. The vertical gray dashed line indicates the Planck scale.
Right panel: double logarithmic plot of the one-loop effective potential improved by two-loop renormalization group equation without gravitational corrections (solid orange line) and with gravitational corrections for different values of couplings (dashed lines). Figure taken from [50], see this paper for details.

the light of the SM coupled non-minimally to Einstein gravity [50]. The authors argue, that this induces higher dimensional counterterms $\propto \Phi^6, \Phi^8$ into the low-energy effective theory with a priori undetermined couplings. Depending on the strength of these couplings, the true minimum of the Higgs potential might be pushed below the Planck scale, or the potential might even become stable (cf. the right panel of Figure 5.1 and [50] for details).

6 Summary and conclusion

The SM provides us with a powerful description of the physics of elementary particles at the scales we can measure in the laboratory. In order to obtain a model which can be applied on different scales, we must subject the SM to a renormalization procedure. Thus, we obtain a model which accurately describes physics up to the highest scales accessible with lab experiments of $\sim 1\,\text{TeV}$. The same techniques which allow us to extend the model from the MeV to the TeV scale can be used to extend the SM to arbitrarily high energy scales and thus identify theoretical problems of the model long before such scales may become experimentally accessible. We demonstrated the fundamental concepts of renormalization, and explicitly calculated the dominant contributions to the Higgs potential at high energies. We saw, that the top Yukawa coupling drives the quartic Higgs coupling to small values and eventually negative. Hence, the SM potential develops a second minimum deeper than the electroweak one at scales larger than $M \sim 10^8\,\text{GeV}$. In a more thorough analysis including electroweak contributions and using two-loop renormalization group equations, we found the SM to develop instabilities at scales larger than $M \sim 10^{10}\,\text{GeV}$. Varying the top quark and Higgs masses, we found that the SM sits right on the boundary to the region where the potential becomes stable all the way up to the Planck scale. This scenario of absolute stability is disfavored by about $98\,\%$ confidence limit.

There are two basic possibilities to interpret the instability of the SM potential if it holds up after reducing theoretical and experimental errors. The second minimum of the potential does not necessarily point to inconsistencies of the SM below the Planck scale. If we assume the universe to have ended up in the electroweak vacuum after the Big Bang, the probability that it would have transitioned to the true vacuum during the lifetime of the universe is spectacularly small. Hence, we can accept living in a metastable vacuum. If we however demand absolute stability, new physics must appear at or below the instability scale of $M \sim 10^{10}\,\text{GeV}$. Well motivated extensions for new physics below the instability scale are e.g. the seesaw models explaining neutrino masses, and axion models solving the strong-CP problem. The instability of the SM potential gets worse if one implements realistic type-I seesaw models. On the other hand, one arrives at a potential stable all the way up to the Planck scale for type-II or type-III seesaw models. The situation looks even more promising when including axion models, which render the SM even with type-I seesaw mechanism stable all the way up to the Planck scale.

It is fascinating, that we have the theoretical tools and experimental precision to allow for a meaningful extrapolation of the SM from the TeV scale all the way up to the Planck scale. We do not know if and when new physics will appear in this "desert", but can only take the hints nature gives us at the experimentally accessible energy scales. The SM seems to lie just outside the stable region but well in the metastable phase where the lifetime of the electroweak vacuum is much larger than the age of the Universe. If we however demand absolute stability, new physics must appear at scales below $\sim 10^{10}\,\text{GeV}$.

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