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Geometry of Contact Toric Manifolds in 3D

Master Thesis in Physics

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Populärvetenskaplig Sammanfattning

I denna avhandling ger vi en översikt över symplektisk geometri och kontaktgeometri för att sedan tillämpa vår kunskap på några specifika exempel. Översikten innehåller de viktigaste inslagen hos symplektisk geometri och kontaktgeometri, och de olika konstruktioner som vi kan skapa på dessa typer av manifolds (en manifold är ett topologiskt rum som lokalt ser ut som det Euklidiska rummet).

Både symplektisk geometri och kontaktgeometri härstammar från studier av klassisk mekanik. Symplektisk geometri användes först av Joseph Louis Lagrange 1808 i en uppsats om planeternas rörelser och kontaktgeometri av Sophus Lie 1872 i studiet av differentialekvationer. Senare har både symplektiska manifolds och kontakt manifolds blivit mycket viktiga inom klassisk mekanik. Rummet av koordinater och rörelsemängd har i den klassiska mekaniken visat sig vara en symplektisk manifold, och tillägget av en tidskoordinat ger oss en kontakt manifold. Kontaktgeometri har många fler tillämpningar inom områden så som geometrisk optik, klassisk mekanik, termodynamik och geometrisk kvantisering.

Anledningen till att vi studerar egenskaperna hos symplektiska manifolds och kontakt manifolds, för nurvarande, är deras tillämpning inom modern teoretisk fysik. I många fall utgör dessa manifolds bakgrundsrummet där en hel del nya teorier utvecklas, och genom att förstå deras egenskaper kan vi ytterligare öka vår kunskap i den underliggande fysiken. I denna avhandling utforskar vi några konstruktioner som är mycket populära och användbara i många tillämpningar inom teoretisk fysik.

Contents

Abstract

In this project we present some applications of symplectic and contact geometry on 3-manifolds. In the first section we introduce the notion of symplectic manifolds and various geometrical objects associated with them, such as the moment map and the Delzant polytope. We also investigate symplectic cuts which are used to create new manifolds from given symplectic manifolds. In the second section we review all the basic definitions that arise from contact geometry such as the contact structure, the contact form and the Reeb vector. Several examples are also used in order to clarify these definitions. In the final section we apply the aforementioned notions to the specific case of Lens spaces, which are 3-manifolds. We also propose a possible application on 5-manifolds which would further expand what was discussed here.

1 Introduction

The first mention of the word symplectic was made by Hermann Weyl in his book *The Classical Groups*. However, the symplectic structure was defined by Joseph Louis Lagrange in 1808 (under a different name) in a paper about the slow variations of the orbital elements of the planets in the solar system.

Symplectic geometry is an even dimensional geometry and measures the sizes of 2-dimensional objects rather than the 1-dimensional lengths and angles (which are part of Riemannian geometry). Therefore symplectic geometry is mostly associated with the field complex numbers. The concept of a symplectic structure was firstly defined in the study of classical mechanical systems (e.g. a planet orbiting the sun). In order to determine the trajectory of a such a system (for example, a unit mass moving in a straight line), we must know its position q and momentum p at any one time. This pair of real numbers (q, p) gives a point on the plane $\mathbb{R}^2$. The symplectic structure ω is defined as $\omega = \mathrm{d}p \wedge \mathrm{d}q$ and is an area form in the plane. By integrating ω we can measure the area A of a region S in the plane. This area is invariant through the evolution of the system (for a conservative dynamical system).

The space defined by the positions (q_i) and momenta (p_i) is called phase space. If N is a manifold, then the cotangent bundle $M = T^*N$ can be described by (q_i, p_i) where q_i are the coordinates of points $p \in N$ and p_i are the coordinates of 1-forms (covectors) living in T_p^*N for each point p. It can be proven that the phase space is the same thing as a symplectic manifold.

Contact geometry, on the other hand, can be thought of as the odd-dimensional counterpart of symplectic geometry. It was also motivated by the study of classical mechanics and specifically in the context of the extended phase space which includes, aside from position and momentum, the time variable. Historically, we can say that contact geometry was first mentioned in 1872 when Sophus Lie introduced the contact transformation as a geometric tool to study systems of differential equations. Contact geometry has many applications in fields such as geometrical optics, classical mechanics, thermodynamics and geometric quantization.

The counterpart of the symplectic form ω, in contact geometry, is the contact 1-form α. Without going into the details here, the 2-form $\omega = \mathrm{d}\alpha$ can be proven to be a symplectic form. This means that from any contact manifold M we can create a natural symplectic bundle of rank one smaller that the dimension of M.

2 Symplectic Geometry

In this section we introduce symplectic manifolds and their properties, the moment map, symplectic reduction, toric manifolds and the Delzant construction and finally symplectic cuts. We also give several examples, some of which will be further investigated once we have defined contact manifolds. More in depth analysis on these subjects can be found in Ana Cannas da Silva's lectures [1] and Eugene Lerman's notes [2] and paper [3].

2.1 Symplectic Manifolds

Let M be a manifold and ω a differential 2-form on M. This means that $\forall p \in M$ we have a map $\omega_p : T_pM \times T_pM \to \mathbb{R}$ which is skew-symmetric and bilinear on the tangent space to M at p, and ω_p varies smoothly with p. The 2-form ω is closed if it satisfies the differential equation $\mathrm{d}\omega = 0$, where d is the de Rham differential. ω is non-degenerate at a point p if the equation $\omega(X_p, Y_p) = 0$ for all tangent vectors $X_p \in T_pM$ implies that $Y_p = 0$, where $Y_p \in T_pM$.

Definition 2.1. A **symplectic form** ω on a manifold M is a closed 2-form which is non-degenerate at each point p of M. The pair (M, ω) is called a **symplectic manifold**.

Using linear algebra we can prove that if M is symplectic then its dimension is even $\dim T_pM = \dim M = 2n$. Moreover, the exterior power $\omega^n = \omega \wedge \cdots \wedge \omega$ is a volume form. This means that any symplectic manifold (M, ω) is canonically oriented. The form $\omega^n/n!$ is called Liouville volume of (M, ω).

Example 2.2. Let our manifold be $M = \mathbb{R}^{2n}$. A point p on the manifold can be described using the $2n$ coordinates $x_1 \dots x_n, y_1, \dots, y_n$. The standard symplectic form on $\mathbb{R}^{2n}$ is

$$\omega_0 = \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i \tag{2.1}$$

The set of vectors

$$\left\{ \left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_n}\right)_p, \left(\frac{\partial}{\partial y_1}\right)_p, \dots, \left(\frac{\partial}{\partial y_n}\right)_p \right\} \tag{2.2}$$

is a symplectic basis of the tangent space T_pM.

Example 2.3. Let $M = \mathbb{C}^n$ with coordinates $z_1, \dots, z_n$. Then the form

$$\omega_0 = \frac{\mathrm{i}}{2} \sum_{i=1}^{n} \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i \tag{2.3}$$

is symplectic. Using the identification of spaces $\mathbb{C}^n \cong \mathbb{R}^{2n}$, $z_i = x_i + y_i$, this form is equal to the previous one.

Definition 2.4. Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds of the same dimension $\dim M_1 = \dim M_2 = 2n$ and let $\varphi : M_1 \to M_2$ be a diffeomorphism. We say that φ is a **symplectomorphism** if $\varphi^*\omega_2 = \omega_1$. By the definition of the pullback this means that for the tangent vectors $X, Y \in T_pM_1$ we have

$$(\varphi^*\omega_2)_p (X, Y) = (\omega_2)_{\varphi(p)} (\mathrm{d}\varphi_p(X), \mathrm{d}\varphi_p(Y)) \tag{2.4}$$

Using this definition we would like to classify symplectic manifolds up to symplectomorphism. The Darboux theorem helps us do that locally.

Theorem 2.5 (Darboux). *Let (M,ω) be a $2n$-dimensional symplectic manifold and let p be a point in M. Then there is a coordinate chart $(\mathcal{U}, x_1, \dots x_n, y_1, \dots, y_n)$ centered at p such that*

$$\omega|_{\mathcal{U}} = \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i \tag{2.5}$$

This theorem shows us that any symplectic manifold (M^{2n}, ω) is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$, where ω_0 is the standard form that we defined before.

For the purposes of this thesis we will only be interested in Kähler manifolds which are a subset of symplectic manifolds. Therefore we give their definition here.

Definition 2.6. A Kähler manifold is a symplectic manifold (M,ω) equipped with an integrable compatible almost complex structure. The symplectic form ω is then called a Kähler form.

It follows from this definition that M is complex manifold and that for the differential forms on M we have

$$\Omega^k(M;\mathbb{C}) = \bigoplus_{\ell+m=k} \Omega^{\ell,m} \qquad \text{and} \qquad \mathrm{d} = \partial + \bar{\partial} \tag{2.6}$$

2.2 Moment Map

Definition 2.7. Let (M,ω) be a symplectic manifold. A vector field X on M is symplectic if the contraction $\iota_X\omega$ is closed. A vector field X is **Hamiltonian** if $\iota_X\omega$ is exact.

Using Cartan's formula we can show that the flow of X preserves ω since its Lie derivative is zero

$$\mathcal{L}_X\omega = \mathrm{d} \circ \iota_X\omega + \iota_X \circ \mathrm{d}\omega = 0 \tag{2.7}$$

For a Hamiltonian vector field we have $\iota_X\omega$ exact, so we can find a smooth function $H : M \to \mathbb{R}$ such that $\iota_X\omega = \mathrm{d}H$. The flow of X also preserves H since

$$\mathcal{L}_X H = \iota_X \circ \mathrm{d}H = \iota_X\iota_X\omega = 0 \tag{2.8}$$

This implies that each integral curve of X, $\{\rho_t(x) \,|\, t \in \mathbb{R}\}$ must be contained in a level set of H

$$H(x) = (\rho_t^* H)(x) = H(\rho_t(x)) \tag{2.9}$$

The function H is called the Hamiltonian function for the vector field X.

Let (M,ω) be a symplectic manifold and G a Lie group that acts on it. Also, suppose that the action of any $g \in G$ preserves the symplectic form ω. Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{g}^*$ its dual and $\langle .,. \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ the pairing between them. Every element of the Lie algebra $X \in \mathfrak{g}$ induces a vector field $X^\# \in T_pM$ on the manifold, called fundamental vector field. Since the action of G preserves the symplectic form it follows that $\iota_{X^\#}\omega$ is closed for all $X \in \mathfrak{g}$

$$\sigma^*\omega = \omega \Rightarrow \mathcal{L}_{X^\#}\omega = 0 \Rightarrow \mathrm{d} \circ \iota_{X^\#}\omega = 0 \tag{2.10}$$

Definition 2.8. A **moment map** for the G-action on (M,ω) is a map $\mu : M \to \mathfrak{g}^*$ such that

$$\mathrm{d}\langle \mu, X \rangle = \iota_{X^\#}\omega \quad , \quad \forall X \in \mathfrak{g} \tag{2.11}$$

where the function $\langle \mu, X \rangle : M \to \mathbb{R}$ is defined by $\langle \mu, X \rangle(x) = \langle \mu(x), X \rangle \equiv \mu^X(x)$ Now we can write

$$\mathrm{d}\mu^X = \iota_{X^\#}\omega \tag{2.12}$$

which means that the function μ^X is a Hamiltonian function for the vector field $X^\#$. The map μ should also be equivariant with respect to the coadjoint action on $\mathfrak{g}^*$

$$\mu(g.x) = g.\mu(x) = \mathrm{Ad}_{g^{-1}}\mu(x) \tag{2.13}$$

If such a map μ exists then the group action is called **Hamiltonian action** and the vector (M, ω, G, μ) is called Hamiltonian G-space.

Example 2.9. The simplest example of a moment map is that of a circle action $\mathbb{T}^1 = S^1$ on the complex plane $\mathbb{C}$. This can then be generalized to any manifold $\mathbb{C}^d$. The circle group $\mathbb{T}^1 = U(1)$ is defined as

$$\mathbb{T}^1 = \{\xi \in \mathbb{C} \,:\, |\xi| = 1\} \tag{2.14}$$

so an element of the group can be represented by $\mathrm{e}^{\mathrm{i}t}$, $t \in \mathbb{R}$. The Lie algebra elements of the group can be identified with the complex line $\mathfrak{t} = \{\mathrm{i}t \,:\, t \in \mathbb{R}\}$ This toric action on $\mathbb{C}$ is defined as

$$\begin{aligned} \sigma \,:\, \mathbb{T}^1 \times \mathbb{C} &\to \mathbb{C} \\ \left(\mathrm{e}^{\mathrm{i}t}, z\right) &\to \mathrm{e}^{\mathrm{i}t} z \end{aligned}$$

Each component of the moment map $\mu : \mathbb{C} \to \mathbb{R}^*$ will be given by

$$\mathrm{d}\mu^X = \iota_{X^\#}\omega \tag{2.15}$$

where ω is the standard action on $\mathbb{C}$. So we only need to compute the fundamental vector field $X^\#$ and then contract with ω. The definition of the fundamental vector field (acting on a point of the manifold) yields

$$X^\#(z) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \sigma\left(\mathrm{e}^{\mathrm{i}t}, z\right) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\mathrm{e}^{\mathrm{i}t} z\right) = \mathrm{i}z \tag{2.16}$$

Going back to the definition of the complex number $z = x + \mathrm{i}y = r\cos\theta + \mathrm{i}\, r\sin\theta$ we can easily find that $\mathrm{i}z$ can be written as

$$\mathrm{i}z = \frac{\partial z}{\partial \theta} \tag{2.17}$$

which means that

$$X^\# = \frac{\partial}{\partial \theta} \tag{2.18}$$

If we make a change to the complex coordinates z and $\bar{z}$ we get

$$X^\# = \frac{\partial z}{\partial \theta}\frac{\partial}{\partial z} + \frac{\partial \bar{z}}{\partial \theta}\frac{\partial}{\partial \bar{z}} = \mathrm{i}\left(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}\right) \tag{2.19}$$

The contraction of this vector field with ω is

$$\mathrm{d}\mu^X = \iota_{X^\#}\omega = -\frac{1}{2}\left(z\mathrm{d}\bar{z} + \bar{z}\mathrm{d}z\right) \tag{2.20}$$

So finally, the moment map is written as

$$\mu(z) = -\frac{1}{2}|z|^2 + \lambda \tag{2.21}$$

where $\lambda \in \mathbb{R}$. Another way to obtain the same result is to write the standard form in polar coordinates

$$\omega = \frac{\mathrm{i}}{2}\mathrm{d}z \wedge \mathrm{d}\bar{z} = r\mathrm{d}r \wedge \mathrm{d}\theta \tag{2.22}$$

and contract with $X^\# = \partial/\partial\theta$ to get

$$\mathrm{d}\mu^X = -r\mathrm{d}r \Rightarrow \mu = -\frac{r^2}{2} + \lambda \tag{2.23}$$

Example 2.10. Now we generalize the previous example to d dimensions. We have the toric action of $\mathbb{T}^d$ on $\mathbb{C}^d$ defined by

$$\sigma : \left(\left(\mathrm{e}^{\mathrm{i}t_1}, \ldots, \mathrm{e}^{\mathrm{i}t_d}\right), (z_1, \ldots, z_d)\right) \to \left(\mathrm{e}^{\mathrm{i}t_1} z_1, \ldots, \mathrm{e}^{\mathrm{i}t_d} z_d\right) \tag{2.24}$$

This group and its algebra are d dimensional and are defined as

$$G = \mathbb{T}^d = U(1) \times \cdots U(1) \tag{2.25}$$

and

$$\mathfrak{g} = \mathfrak{t} = \mathfrak{u}^1 \oplus \cdots \oplus \mathfrak{u}^1 \cong \mathbb{R} \oplus \cdots \oplus \mathbb{R} = \mathbb{R}^d \tag{2.26}$$

Thus we will have d linearly independent fundamental vectors $X_i^{\#}$, $i = 1, \ldots, d$ given by

$$X_i^{\#}(z) = \mathrm{i} z_i \tag{2.27}$$

where $z = (z_1, \ldots, z_d)$. Following the same procedure as in the previous example we obtain

$$X_i^{\#} = \frac{\partial}{\partial \theta_i} = \mathrm{i}\left(z_i \frac{\partial}{\partial z_i} - \bar{z}_i \frac{\partial}{\partial \bar{z}_i}\right) \tag{2.28}$$

The moment map now is $\mu : \mathbb{C}^d \to (\mathbb{R}^d)^*$ and each of its components will be

$$\mu^{X_i} = -\frac{1}{2}|z_i|^2 + \lambda_i \tag{2.29}$$

The full moment map is then

$$\mu(z) = -\frac{1}{2}\left(|z_1^2|, \ldots, |z_d|^2\right) + \lambda \tag{2.30}$$

where $\lambda = (\lambda_1, \ldots, \lambda_d)$.

Example 2.11. In this example we will have the action of an 1-dimensional group on a d-dimensional manifold. Specifically, we will have the circle group $N = \mathbb{T}^1$ acting on $\mathbb{C}^d$. We will consider the circle group as a subgroup of the d-dimensional torus $G = \mathbb{T}^d$ using the inclusion map (in this particular example we choose all the weights to be equal to 1)

$$\begin{aligned} i : N &\to G \\ \mathrm{e}^{\mathrm{i}t} &\to \left(\mathrm{e}^{\mathrm{i}t}, \ldots, \mathrm{e}^{\mathrm{i}t}\right) \end{aligned}$$

and its pullback between their dual Lie algebras

$$i^* : \mathfrak{g}^* = (\mathbb{R}^d)^* \to \mathfrak{n}^* = \mathbb{R}^* \tag{2.31}$$

Now the group N can act on $\mathbb{C}^d$ as

$$\begin{gathered} \sigma : i(N) \times \mathbb{C}^d \to \mathbb{C}^d \\ \left(i(\mathrm{e}^{\mathrm{i}t}), (z_1, \ldots, z_d)\right) \to \left(\mathrm{e}^{\mathrm{i}t} z_1, \ldots, \mathrm{e}^{\mathrm{i}t} z_d\right) \end{gathered}$$

Now we of course have just one fundamental vector field which is

$$X^{\#}(z) = \mathrm{i}\left(z_1, \ldots, z_d\right) \tag{2.32}$$

or equivalently

$$X^{\#} = \frac{\partial}{\partial \theta_1} + \cdots + \frac{\partial}{\partial \theta_d} = \mathrm{i} \sum_{i=1}^{d} \left(z_i \frac{\partial}{\partial z_i} - \bar{z}_i \frac{\partial}{\partial \bar{z}_i} \right) \tag{2.33}$$

The moment map will be the pullback by i of the moment map for the full group G, meaning that

$$\mu_N = i^* \circ \mu : \mathbb{C}^d \to \mathfrak{n}^* = \mathbb{R}^* \tag{2.34}$$

The contraction of ω by our field $X^{\#}$ is easy to calculate and we finally have

$$\mu_N(z) = -\frac{1}{2}|z|^2 + \lambda \tag{2.35}$$

where $|z|^2 = |z_1|^2 + \cdots + |z_d|^2$.

Example 2.12. This last example will be the moment map of the action of a torus $\mathbb{T}^d$ on the symplectic manifold $\mathbb{CP}^d$. We will use the $(d+1)$ homogenous coordinates and work on the coodinate neighborhood $\mathcal{U}_0$ where

$$\mathcal{U}_0 = \{[z_0 : z_1 : \cdots : z_d] \mid z_0 \neq 0\} \tag{2.36}$$

The symplectic form on $\mathbb{CP}^d$ is known to be the Fubini-Study form

$$\omega_{\text{FS}} = \frac{\mathrm{i}}{2} \partial\bar{\partial} \log \left(\frac{z_\mu \bar{z}_\mu}{z_0 \bar{z}_0} \right) \tag{2.37}$$

where the repeated index μ implies summation. This form can be easily obtained by the Kähler potential for $\mathbb{CP}^d$. The standard action of $\mathbb{T}^d$ on $\mathbb{CP}^d$ is defined as

$$\begin{gathered} \sigma : \mathbb{T}^d \times \mathbb{CP}^d \to \mathbb{CP}^d \\ \big((\mathrm{e}^{\mathrm{i}t_1}, \ldots, \mathrm{e}^{\mathrm{i}t_d}), [z_0 : z_1 : \cdots : z_d]\big) \to \big[z_0 : \mathrm{e}^{\mathrm{i}t_1} z_1 : \cdots : \mathrm{e}^{\mathrm{i}t_d} z_d\big] \end{gathered}$$

The fundamental vector fields are the same as the ones we found before

$$X_i^{\#} = \frac{\partial}{\partial \theta_i} = \mathrm{i} \left(z_i \frac{\partial}{\partial z_i} - \bar{z}_i \frac{\partial}{\partial \bar{z}_i} \right) \tag{2.38}$$

The moment map is then

$$\mu([z]) = -\frac{1}{2|z|^2} \left(|z_1|^2, |z_2|^2 \ldots, |z_d|^2\right) + \lambda \tag{2.39}$$

In the case of $d = 2$ we have

$$\mu([z]) = -\frac{1}{2} \left(\frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}, \frac{|z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2} \right) \tag{2.40}$$

with $\lambda = 0$. The image of this moment map is the triangle (polytope) on the 2D plane with vetrices at $(0,0)$, $(-1/2, 0)$ and $(0, -1/2)$.

2.3 Symplectic Reduction

The symplectic reduction is the process of creating a principal G-bundle from a Hamiltonian G-space (M, ω, G, μ), where M is symplectic, equipped with a symplectic form on the base of the bundle. This is stated in the Marsden, Weinstein, Meyer theorem for a symplectic manifold. A detailed proof of this theorem can be found in [2].

Theorem 2.13 (**Marsden - Weinstein - Meyer**). *Consider a (proper) Hamiltonian action of a Lie group G on a symplectic manifold (M, ω) with a corresponding moment map $\mu : M \to \mathfrak{g}^*$. Suppose 0 is a regular value of μ. Then $\mu^{-1}(0)$ is a submanifold of M. Moreover, the action of G on $\mu^{-1}(0)$ has zero-dimensional stabilizer groups (locally free action). Let $i : \mu^{-1}(0) \hookrightarrow M$ be the inclusion map. Then,*

- *the orbit space $M_r = \mu^{-1}(0)/G$ is a manifold*
- *$\pi : \mu^{-1}(0) \to \mu^{-1}(0)/G$ is a principal G-bundle*
- *there is a symplectic form ω_r on M_r satisfying $i^*\omega = \pi^*\omega_r$*

Definition 2.14. The pair (M_r, ω_r) is called **symplectic reduction** of (M, ω) with respect to G and μ.

Example 2.15. As we showed before, the 1-dimensional action of $N = \mathbb{T}^1$ on $\mathbb{C}^d$ yields the moment map

$$\mu(z) = -\frac{1}{2}\left(|z_1|^2 + |z_2|^2 + \cdots + |z_d|^2\right) + \lambda \tag{2.41}$$

By choosing $\lambda = 1/2$, the zero level set $Z = \mu^{-1}(0)$ is

$$Z = \left\{|z_1|^2 + |z_2|^2 + \cdots + |z_d|^2 = 1 \,|\, (z_1, \ldots, z_d) \in \mathbb{C}^d\right\} \tag{2.42}$$

which is of course the $(2d-1)$-sphere $Z = S^{2d-1}$. The orbit space $M_r = Z/N$ is the $(2d-2)$-dimensional sphere which is isomorphic to $\mathbb{CP}^{d-1}$.

$$M_r = S^{2d-1}/S^1 \cong S^{2d-2} \cong \mathbb{CP}^{d-1} \tag{2.43}$$

Therefore we have the principal bundle

$$S^1 \hookrightarrow S^{2d-1} \to \mathbb{CP}^{d-1} \tag{2.44}$$

The symplectic manifold we obtained is $(\mathbb{CP}^{d-1}, \omega_{\mathrm{FS}})$ where ω_{FS} is the Fubini-Study form we defined before. For $d = 2$ we get the Hopf bundle.

Example 2.16. Let $N = \mathbb{T}^1$ act on $\mathbb{C}^2$ by

$$\mathrm{e}^{\mathrm{i}t}(z_1, z_2) = \left(\mathrm{e}^{\mathrm{i}kt} z_1, \mathrm{e}^{\mathrm{i}t} z_2\right) \tag{2.45}$$

for $k > 2$. This means that the inclusion map $i : N \to G$, where $G = \mathbb{T}^2$ is the standard toric action on $\mathbb{C}^2$, is defined by

$$i(\mathrm{e}^{\mathrm{i}t}) = \left(\mathrm{e}^{\mathrm{i}kt}, \mathrm{e}^{\mathrm{i}t}\right) \tag{2.46}$$

The moment map $\mu_N : \mathbb{C}^2 \to \mathbb{R}$ is

$$\mu(z) = -\frac{1}{2}\left(k|z_1|^2 + |z_2|^2\right) + \lambda \tag{2.47}$$

Then, if $\lambda > 0$, zero is is a regular value of the moment map and

$$\mu^{-1}(0) = \left\{k|z_1|^2 + |z_2|^2 = 2\lambda \,|\, (z_1, z_2) \in \mathbb{C}^2\right\} \tag{2.48}$$

is a 3-dimensional ellipsoid. If we choose any point $(z_1, 0)$ we realize that the action on $\mu^{-1}(0)$ is not free. The stabilizer group at these points is

$$\mathbb{Z}_k = \left\{\mathrm{e}^{\mathrm{i}\frac{2\pi\ell}{k}} \,|\, \ell = 0, 1, \ldots, k-1\right\} \tag{2.49}$$

The resulting reduced space $\mu^{-1}(0)/S^1$ is then an orbifold called Teardrop orbifold or conehead. It has one cone singularity with cone angle $2\pi/k$, meaning that it is a point with orbifold structure group $\mathbb{Z}_k$. This example was presented here in order to illustrate that we can have extensions of the symplectic reduction procedure. In this case we obtained an orbifold, which roughly speaking is a singular manifold where each singularity is locally modeled on $\mathbb{R}^m/\Gamma$, for some finite group $\Gamma \subset \mathrm{GL}(n, \mathbb{R})$.

Example 2.17. Consider the $N = \mathbb{T}^1$ action on $\mathbb{C}^3$

$$\mathrm{e}^{\mathrm{i}t}(z_1, z_2, z_3) = \left(\mathrm{e}^{\mathrm{i}t} z_1, \mathrm{e}^{\mathrm{i}t} z_2, \mathrm{e}^{-\mathrm{i}pt} z_3\right) \tag{2.50}$$

where p is an integer greater than zero. The moment map of this action is

$$\mu(z) = -\frac{1}{2}\left(|z_1|^2 + |z_2|^2 - p|z_3|^2\right) + \lambda \tag{2.51}$$

The zero level set is

$$Z = \mu^{-1}(0) = \left\{|z_1|^2 + |z_2|^2 - p|z_3|^2 = C \,|\, (z_1, z_2, z_3) \in \mathbb{C}^3\right\} \tag{2.52}$$

where $C = 2\lambda$. The resulting reduced space $M_r = Z/N$ is

$$M_r = \left\{|z_1|^2 + |z_2|^2 - p|z_3|^2 = C \ , \ (z_1, z_2, z_3) \sim \left(\mathrm{e}^{\mathrm{i}t} z_1, \mathrm{e}^{\mathrm{i}t} z_2, \mathrm{e}^{-\mathrm{i}pt} z_3\right) \,|\, (z_1, z_2, z_3) \in \mathbb{C}^3\right\} \tag{2.53}$$

It can be proven that this manifold is diffeomorphic to the total space of the line bundle $\mathcal{O}(-p)$ over $\mathbb{CP}^1$ which is defined as

$$\mathcal{O}(-p) = \left\{(z_1, z_2, z_3) \in \mathbb{C}^3 \,|\, (z_1, z_2, z_3) \sim \left(\alpha z_1, \alpha z_2, \alpha^{-p} z_3\right) \ , \ \alpha \in \mathbb{C}\right\} \tag{2.54}$$

The real dimension of the manifold is $\dim_{\mathbb{R}} M_r = 4$ and it should be noted that the set Z is unbounded and that M_r in non-compact.

2.4 Delzant Construction

Definition 2.18. A (symplectic) **toric manifold** is a $2n$-dimensional compact connected symplectic manifold (M^{2n}, ω) equipped with an effective Hamiltonian action of a n-torus $G = \mathbb{T}^n$ and a corresponding moment map $\mu : M \to \mathbb{R}^n$.

Definition 2.19. A **Delzant polytope** Δ in $(\mathbb{R}^n)^*$ is a convex polytope satisfying

- **simplicity:** there are n edges meeting at each vertex
- **rationality:** the edges meeting at the vertex p are rational in the sense that each edge is of the form $p + tu_i$, $t \geq 0$ where $u_i \in (\mathbb{Z}^n)^*$

- **smoothness:** for each vertex, the corresponding $u_1, \dots, u_n$ can be chosen to be a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

Delzant showed that symplectic toric manifolds can be classified (as Hamiltonian spaces) by a set of polytopes. Let Δ be a Delzant polytope in $(\mathbb{R}^n)^*$ (space of the image of the moment map of a toric manifold, dual to $\mathbb{R}^n$) with d facets (a facet is a $(d-1)$-dimensional face of the polytope). Let $v_i \in \mathbb{Z}^n$, $i = 1, \dots, d$ be the primitive (v_i cannot be written as $v_i = kv_j$ where $k \in \mathbb{Z}^n$ and $|k| > 1$), outward-pointing normal vectors to the facets of Δ. The Delzant polytope Δ can then be described as an intersection of halfspaces

$$\Delta = \{x \in (\mathbb{R}^n)^* \mid \langle x, v_i \rangle \leq \lambda_i \,,\, i = 1, \dots, d \,,\, \lambda_i \in \mathbb{R}\} \tag{2.55}$$

Theorem 2.20 (Delzant theorem). *Toric manifolds are classified by Delzant polytopes. Specifically, there is a $1-1$ correspondence between a toric manifold and a polytope*

$$\begin{aligned} \{\textit{toric manifold}\} &\longrightarrow \{\textit{Delzant polytope}\} \\ (M^{2n}, \omega, \mathbb{T}^n, \mu) &\longmapsto \mu(M) \end{aligned}$$

Next we will show that the theorem is surjective (as can be found in more detail in [1] and more examples in [4]), meaning that given a Delzant polytope we can recover the corresponding toric manifold

$$\Delta^n \longrightarrow (M^{2n}, \omega, \mathbb{T}^n, \mu) \tag{2.56}$$

Let e_i with $i = 1, \dots, d$ be the standard Cartesian basis of $\mathbb{R}^d$. We define the map π that maps $\mathbb{R}^d$ to $\mathbb{R}^n$ (n is the dimension of the polytope)

$$\pi_* : \mathbb{R}^d \longrightarrow \mathbb{R}^n \tag{2.57}$$

$$e_i \longrightarrow v_i \tag{2.58}$$

Its easily proven that the map π is surjective and maps $\mathbb{Z}^d$ onto $\mathbb{Z}^n$. Thus, π_* induces a surjective map (π) between tori

$$\begin{tikzcd} \mathbb{R}^d/2\pi\mathbb{Z}^d \arrow[r, "\pi"] \arrow[d, equal] & \mathbb{R}^n/2\pi\mathbb{Z}^n \arrow[d, equal] & \\ \mathbb{T}^d \arrow[r] & \mathbb{T}^n \arrow[r] & 1 \end{tikzcd}$$

We give the following names to our groups and their algebras

$$G = \mathbb{T}^d \quad , \quad \mathfrak{g} = \mathbb{R}^d \quad , \quad H = \mathbb{T}^n \quad , \quad \mathfrak{h} = \mathbb{R}^n \tag{2.59}$$

We define a subgroup of G named N by

$$N \equiv \ker(\pi) \subset G \tag{2.60}$$

and $\mathfrak{n}$ is its Lie algebra. This enables us to create an exact sequence of tori, which in turn induces an exact sequence on their Lie algebras

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & N \arrow[r, "i"] \arrow[d] & G = \mathbb{T}^d \arrow[r, "\pi"] \arrow[d] & H = \mathbb{T}^n \arrow[r] \arrow[d] & 1 \arrow[d] \\ 0 \arrow[r] & \mathfrak{n} \arrow[r, "i_*"] & \mathfrak{g} = \mathbb{R}^d \arrow[r, "\pi_*"] & \mathfrak{h} = \mathbb{R}^n \arrow[r] & 0 \end{tikzcd}$$

The dual exact sequence of these algebras is

$$0 \longrightarrow \mathfrak{h}^* = (\mathbb{R}^n)^* \xrightarrow{\pi^*} \mathfrak{g}^* = \left(\mathbb{R}^d\right)^* \xrightarrow{i^*} \mathfrak{n}^* \longrightarrow 0$$

Now we specify our manifold M to be $\mathbb{C}^d$. This is a Kähler manifold and admits the symplectic form $\omega = \frac{i}{2}\sum \mathrm{d}z^\mu \wedge \mathrm{d}\bar{z}^\mu$. We make this into a toric manifold by equipping it with the standard Hamiltonian action of $G = \mathbb{T}^d$

$$(\mathrm{e}^{\mathrm{i}\theta_1}, \ldots, \mathrm{e}^{\mathrm{i}\theta_d}) \cdot (z_1, \ldots, z_d) = (\mathrm{e}^{\mathrm{i}\theta_1} z_1, \ldots, \mathrm{e}^{\mathrm{i}\theta_d} z_d) \tag{2.61}$$

The moment map for this action is $\mu : \mathbb{C}^d \to \left(\mathbb{R}^d\right)^*$

$$\mu(z_1, \ldots, z_d) = -\frac{1}{2}\left(|z_1|^2, \ldots, |z_d|^2\right) + \text{const.} \tag{2.62}$$

and we choose the constant to be equal to $(\lambda_1, \ldots, \lambda_d)$. If we restrict the action on $\mathbb{C}^d$ to the subgroup N we find that it is Hamiltonian with moment map

$$i^* \circ \mu : \mathbb{C}^d \to \mathfrak{n}^* \tag{2.63}$$

We will use the notation $i^* \circ \mu \equiv \mu_N$ for convenience.

$$\begin{tikzcd} \mathbb{C}^d \arrow[rr, "\mu"] \arrow[dr, "\mu_N"'] & & \mathfrak{g}^* = \left(\mathbb{R}^d\right)^* \arrow[dl, "i^*"] \\ & \mathfrak{n}^* & \end{tikzcd}$$

Let $Z = \mu_N^{-1}(0)$ be the zero-level set. It can be proven that the set Z is compact and N acts freely on Z. Then $0 \in \mathfrak{n}^*$ is a regular value of μ_N. The real dimension of Z is

$$\dim_{\mathbb{R}} Z = d + n \tag{2.64}$$

Using the Marsden-Weinstein-Meyer theorem we know that the orbit space $M_\Delta = Z/N$ is also a compact manifold and its dimension is

$$\dim_{\mathbb{R}} M_\Delta = d + n - (d - n) = 2n \tag{2.65}$$

where $d-n$ is the dimension of the group (and its algebra) N. The point-orbit map $p : Z \to M_\Delta$ is a principal N-bundle over M_Δ and the inclusion of Z is $j : Z \hookrightarrow \mathbb{C}^d$. There also exists a symplectic form ω_Δ on M_Δ obeying

$$p^*\omega_\Delta = j^*\omega \tag{2.66}$$

In diagrammatic form the principal bundle is

$$\begin{tikzcd} N \arrow[r] & Z \arrow[r, hook, "j"] \arrow[d, "p"] & \mathbb{C}^d \\ & M_\Delta & \end{tikzcd}$$

It can be proven that the symplectic manifold $(M_\Delta, \omega_\Delta)$ is a toric manifold equipped with the action of $\mathbb{T}^n$ and its moment map image is

$$\mu_\Delta(M_\Delta) = \Delta \tag{2.67}$$

Example 2.21. We give a simple example in order to illustrate the procedure of obtaining a toric manifold from a Delzant polytope. This procedure however is exactly the same for any polytope. We start from the polytope Δ defined as

$$\Delta = \left\{ (x_1, x_2) \in (\mathbb{R}^2)^* \,|\, x_1 \geq 0 \,,\; x_2 \geq 0 \,,\; x_1 + x_2 \leq 1 \right\} \tag{2.68}$$

This is a triangle with vetrices at $(0,0)$, $(1,0)$ and $(0,1)$. From the definition of Δ we easily obtain $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 1$. Also, the three normal to the faces vectors are $v_1 = (-1,0)$, $v_2 = (0,-1)$ and $v_3 = (1,1)$. The number of faces is $d = 3$ meaning that $G = \mathbb{T}^3$ and the dimension that the polytope lives in is $n = 2$, so we have $H = \mathbb{T}^2$. The pushforward of the map π between the Lie algebras of the groups is

$$\begin{aligned} \pi_* : \mathbb{R}^3 &\longrightarrow \mathbb{R}^2 \\ e_i &\longrightarrow v_i \end{aligned}$$

and it is easy to obtain it in matrix form

$$\pi_* = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \tag{2.69}$$

So the action of π_* on a general element of $\mathfrak{g}$ is

$$\pi_*(A,B,C) = (-A + C, -B + C) \tag{2.70}$$

Thus, the map π between the groups G and H is

$$\pi(a,b,c) = (a^{-1}c, b^{-1}c) \tag{2.71}$$

The group N, which is the group we are performing the reduction with, is defined by

$$N = \ker \pi = \{(a,b,c) \in G \,|\, a = b = c\} \tag{2.72}$$

This means that the inclusion map $i : N \to G$ is

$$i : a \to (a,a,a) \quad \text{and} \quad i_* : A \to (A,A,A) \tag{2.73}$$

with $a \in N$ and $A \in \mathfrak{n}$. The pullback i^* can be obtained from the definition of the inner product between the Lie algebra and its dual. We take $X = A \in \mathfrak{n}$ and $Y = i^*(C_1, C_2, C_3) \in \mathfrak{n}^*$ (where $(C_1, C_2, C_3) \in \mathfrak{g}$) and we have

$$\begin{aligned} \langle Y, X \rangle &= \langle i^*(C_1, C_2, C_3), A \rangle \\ &= \langle (C_1, C_2, C_3), i_* A \rangle \\ &= \langle (C_1, C_2, C_3), (A,A,A) \rangle \\ &= C_1 A + C_2 A + C_3 A \end{aligned}$$

so we obtain

$$i_* : (C_1, C_2, C_3) \to C_1 + C_2 + C_3 \tag{2.74}$$

Note that instead of these steps we could just calculate the fundamental vector fields for the action of the group N. The moment map of the action of G on $\mathbb{C}^3$ for this example is

$$\begin{aligned} \mu(z) &= -\frac{1}{2} \left(|z_1|^2, |z_2|^2, |z_3|^3 \right) + (\lambda_1, \lambda_2, \lambda_3) \\ &= -\frac{1}{2} \left(|z_1|^2, |z_2|^2, |z_3|^3 \right) + (0,0,1) \end{aligned}$$

Now we restrict this action to the subgroup N using the previously defined pullback of i

$$\mu_N(z) = i^* \circ \mu = -\frac{1}{2}\left(|z_1|^2 + |z_2|^2 + |z_3|^3\right) + 1 \tag{2.75}$$

The zero level set $Z = \mu_N^{-1}(0)$ is then

$$Z = \left\{|z_1|^2 + |z_2|^2 + |z_3|^2 = 2 \mid (z_1, z_2, z_3) \in \mathbb{C}^3\right\} \tag{2.76}$$

which is a 3-sphere. The resulting toric manifold is $M_\Delta = S^3/S^1 \cong \mathbb{CP}^2$ equipped with the standard toric action of $H = \mathbb{T}^2$. This result was expected since we already calculated the moment map image for $\mathbb{CP}^2$ in a previous example and found out that it was a triangle.

Example 2.22. Here we start from the $\mathbb{T}^1$ action on $\mathbb{C}^3$ from example 2.17 and proceed with drawing the corresponding Delzant polytope. The group N acting on $\mathbb{C}^3$ can be defined by

$$N = \left\{\left(\mathrm{e}^{\mathrm{i}t}, \mathrm{e}^{\mathrm{i}t}, \mathrm{e}^{-\mathrm{i}pt}\right) \in \mathbb{T}^3\right\} \cong S^1 \tag{2.77}$$

We call this action an S^1-action with weights $(1, 1, -p)$ and we will use this type of definition multiple times in the future. We recall that the moment map and the zero level set were

$$\mu(z) = -\frac{1}{2}\left(|z_1|^2 + |z_2|^2 - p|z_3|^2\right) + \lambda \tag{2.78}$$

and

$$Z = \mu^{-1}(0) = \left\{|z_1|^2 + |z_2|^2 - p|z_3|^2 = C \,|\, (z_1, z_2, z_3) \in \mathbb{C}^3\right\} \tag{2.79}$$

respectively. Since the dimension of N is 1 and the dimension of G is 3 we have $d = 3$ and $n = 2$ (using the symbols defined in this section). This means that we will obtain a polytope which is 2-dimensional and has 3 faces (and therefore 3 normal vectors to the faces). Using the exact sequence which we defined before for $d = 3$ and $n = 2$ we obtain $H = \mathbb{T}^2$. π is the map between G and H. The group N is the kernel of the map π, so we get

$$N = \ker \pi = \left\{(a, b, c) \in G \,|\, a = b, c = a^{-p}\right\} \tag{2.80}$$

From this kernel we can obtain the map π while recognizing that this map is not unique. The action on π on an element of G can be

$$\pi(a, b, c) = (ab^{-1}, b^{-p}c^{-1}) \tag{2.81}$$

We could have chosen a different form for this map, but in the end the resulting polytope would be exactly the same geometrically. The only difference would be its location on the $x - y$ plane. The push forward $\pi_* : \mathfrak{g} \cong \mathbb{R}^3 \to \mathfrak{h} \cong \mathbb{R}^2$ is then

$$\pi_*(A, B, C) = (A - B, -pB - C) \tag{2.82}$$

By its definition, π_* maps the standard basis e_i to the normal to the faces vectors u_i. Its matrix form is

$$\pi_* = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -p & -1 \end{pmatrix} \tag{2.83}$$

This means that the three normal vectors are

$$v_1 = (-1, 0) \quad , \quad v_2 = (1, -p) \quad , \quad v_3 = (0, -1) \tag{2.84}$$

The Delzant polytope corresponding to these vectors will consist of points (x, y) such that

$$-x \leq \lambda_1 \quad , \quad x - py \leq \lambda_2 \quad , \quad -y \leq \lambda_3 \tag{2.85}$$

By choosing $\lambda_1 = \lambda_3 = 0$ we obtain

$$x \geq 0 \quad , \quad y \geq 0 \quad , \quad x - py \leq \lambda \tag{2.86}$$

This polytope is non-compact and this fact was to be expected since the original manifold M_r was not compact either. We note that this means that the polytope is not considered Delzant (since Delzant's original definition applies only to compact manifolds) but it satisfies all the Delzant conditions: simplicity, rationality and smoothness.

Example 2.23. Here we will use another way to obtain the previous result. Consider the zero level set of the previous example (2.22)

$$Z = \mu^{-1}(0) = \left\{ |z_1|^2 + |z_2|^2 - p|z_3|^2 = C \,|\, (z_1, z_2, z_3) \in \mathbb{C}^3 \right\} \tag{2.87}$$

This is part of a 2-dimensional plane embedded in $\mathbb{R}^3$ (part of a plane because all its coordinates are greater than 0). For simplicity we will denote this plane as

$$x + y - pz = C \tag{2.88}$$

where x, y, z are the moduli of z_i squared. This means that $x, y, z \geq 0$. We claim that by rotating this plane to make it parallel to the $x - y$ plane we will obtain the moment map polytope. The normal vector to Z is $\mathbf{n} = (1, 1, -p)$. To rotate the plane Z to be parallel to the $x - y$ plane is equivalent to rotating $\mathbf{n}$ into $\mathbf{n}' = (0, 0, -1)$. This must be done through an $\mathrm{SL}(3, \mathbb{Z})$ transformation, meaning that the rotation matrix must have determinant equal to 1 and integer entries. Since the numbers 1, 1, and $-p$ are coprimes (more on that later) we know that it is possible to find such a transformation (which will not be unique). We choose the matrix $\mathbf{A} \in \mathrm{SL}(3, \mathbb{Z})$ as

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 0 \\ p & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \tag{2.89}$$

The position vector of a point belonging to the plane Z can be written as

$$\mathbf{v}^\intercal = \begin{pmatrix} x & y & \frac{x+y-C}{p} \end{pmatrix} \tag{2.90}$$

The inner product between the normal to the plane vector $\mathbf{n}$ and a position vector $\mathbf{v}$ is fixed and equal to C. We want this inner product to be invariant. After the rotation we obtain the new vectors $\mathbf{n}'$ and $\mathbf{v}'$ where $\mathbf{v}' = \mathbf{B}\mathbf{v}$ for some $\mathbf{B} \in \mathrm{SL}(3, \mathbb{Z})$

$$\mathbf{v}'^\intercal \cdot \mathbf{n}' = C \Rightarrow (\mathbf{B}\mathbf{v})^\intercal (\mathbf{A}\mathbf{n}) = C \Rightarrow \mathbf{v}^\intercal (\mathbf{B}^\intercal \mathbf{A})\, \mathbf{n} = C \tag{2.91}$$

from which we obtain

$$\mathbf{B} = \left(\mathbf{A}^{-1}\right)^\intercal = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & p \end{pmatrix} \tag{2.92}$$

Therefore, the points x', y', z' of the rotated plane are related to x, y, z via the equation

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & p \end{pmatrix} \begin{pmatrix} x \\ y \\ \frac{x+y-C}{p} \end{pmatrix} \tag{2.93}$$

or

$$x' = y \quad , \quad y' = \frac{x+y-C}{p} = z \quad , \quad z' = -C \tag{2.94}$$

So the rotated plane is parallel to the $x' - y'$ plane but shifted by $-C$ in the z' direction. From the previous equations we immediately obtain

$$x' \geq 0 \qquad \text{and} \qquad y' \geq 0 \tag{2.95}$$

We also have

$$y' = \frac{1}{p}(x+y-C) \geq \frac{1}{p}(y-C) \tag{2.96}$$

where we used $x \geq 0$. We also have $y = x'$ so we finally get

$$y' \geq \frac{1}{p}(x'-C) \tag{2.97}$$

The polytope defined by this equation plus the relations $x', y' \geq 0$ is the same as the one we obtained in the previous example (with $\lambda = C$). It is the polytope enclosed by the lines $x' = 0$, $y' = 0$ and $py' = x - C$.

note: We can obtain the same result by rotating the constrains of the problem. In this example the constraints are

$$x + y - pz = C \qquad \text{and} \qquad x, y, z \geq 0 \tag{2.98}$$

The first constrain can be written as

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -p \end{pmatrix} = C \Longrightarrow \begin{pmatrix} x & y & z \end{pmatrix} \mathbf{A}^{-1}\mathbf{A} \begin{pmatrix} 1 \\ 1 \\ -p \end{pmatrix} = C \tag{2.99}$$

or

$$\begin{pmatrix} x' & y' & z' \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = C \tag{2.100}$$

which gives us $z' = -C$. Doing the same for the other constrains yields the rest of the relations.

Remark 2.24. Consider a vector with n integer entries $a_1, \ldots, a_n$. We would like to find which conditions should be met for this vector, in order for us to be able to rotate it to $(0, 0, \ldots 1)$ using an $\mathrm{SL}(n, \mathbb{Z})$ transformation. We start from the 2-dimensional case. Consider a vector $\mathbf{v}$ with integer entries and a matrix $\mathbf{M} \in \mathrm{SL}(2, \mathbb{Z})$ defined by

$$\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix} \qquad \text{and} \qquad \mathbf{M} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \tag{2.101}$$

where $xw - yz = 1$ since the determinant of $\mathbf{M}$ must be equal to 1. Then we demand that

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.102}$$

From the condition

$$za + wb = 1 \tag{2.103}$$

we realize that a and b must be coprimes. If they are coprimes then $\mathbf{M}$ can take the form

$$\mathbf{M} = \begin{pmatrix} b & -a \\ z & w \end{pmatrix} \tag{2.104}$$

By using induction we can generalize this result to any dimension and find an expression for $\mathbf{M}$.

Example 2.25. Now we give the final and easiest way to calculate the normal vectors of the Delzant polytope corresponding to the action of a group N. This will be done by rotating the weights of the action in order to turn it into the action of a group N'. Then using the simpler action of N' we can obtain the map π'_*, from which we can find the original π_*. These can be easily illustrated using as an example the 1-dimensional action $N \cong \mathbb{T}^1$ with weights $(1, 1, -p)$. We want to find a matrix $\mathbf{M} \in \mathrm{SL}(3, \mathbb{Z})$ to rotate the weights $(1, 1, -p)$ into $(0, 0, 1)$. One possible matrix $\mathbf{M}$ is

$$\mathbf{M} = \begin{pmatrix} 1 & -1 & 0 \\ p & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{2.105}$$

Now we have the group N' with weights $(0, 0, 1)$

$$N' = \ker \pi' = \big\{(a, b, c) \in \mathbb{T}^3 \,|\, a = b = 1\,,\ c\big\} \tag{2.106}$$

which means that π' maps an element to $(1, 1)$ only when $a = b = 1$. Thus, the action of π' can be

$$\pi' \begin{pmatrix} a & b & c \end{pmatrix} = \begin{pmatrix} a & b \end{pmatrix} \tag{2.107}$$

which is of course not unique (we could have any combination of a and b for each entry). Then the push-forward π'_* in matrix form will be

$$\pi'_* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.108}$$

The action of π'_* on the element $(0, 0, 1)$ will of course give zero. Similarly, the action of π_* on $(1, 1, -p)$ will also give zero. So, by inserting $\mathbf{M}$ and its inverse in the inner product we can obtain π_*. We have

$$\pi'_* \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \Rightarrow (\pi'_* \cdot \mathbf{M}) \left(\mathbf{M}^{-1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = 0 \Rightarrow (\pi'_* \cdot \mathbf{M}) \begin{pmatrix} 1 \\ 1 \\ -p \end{pmatrix} = 0 \tag{2.109}$$

This means that

$$\pi_* = \pi'_* \cdot \mathbf{M} \tag{2.110}$$

From the form of π'_* we can easily see that π_* is equal to the first two rows of $\mathbf{M}$.

$$\pi_* = \begin{pmatrix} 1 & -1 & 0 \\ p & 0 & 1 \end{pmatrix} \tag{2.111}$$

The normal vectors are the columns of π_* and they produce the same (geometrically) polytope as the one we obtained before.

Similar to the Delzant polytope we can define the polyherdal cone which is more general and can be defined for a torus G. For our purposes, a cone will be a polytope where all λ_is are zero.

Definition 2.26 (Cones). Let $\mathfrak{g}^*$ be the dual Lie algebra of a torus group G. A subset $C \subset \mathfrak{g}^*$ is a rational polyhedral cone if there exists a finite set of vectors $\{v_i\}$ in the integral lattice $\mathbb{Z}_G$ of G such that

$$C = \bigcap \{n \in \mathfrak{g}^* \,|\, \langle n, v_i \rangle \leq 0\} \tag{2.112}$$

We assume that the set $\{v_i\}$ is minimal and that each vector v_i is primitive (meaning that $\forall s \in (0, 1)$ we have $s v_i \notin \mathbb{Z}_G$). A rational polyhedral cone with non-empty interior is **good** if the annihilator of a linear span of a codimension k-space, where $0 < k < \dim G$, is the Lie algebra of a subtorus H of G and the normals to the face form a basis of the integral lattice $\mathbb{Z}_H$ of H. That is, if

$$\{0\} \neq C \cap \bigcap_{j=1}^{k} \{n \in \mathfrak{g}^* \,|\, \langle n, v_{i_j} \rangle \leq 0\} \tag{2.113}$$

is a face of C for some $\{i_1, \ldots, i_k\} \in \{1, \ldots N\}$ then

$$\left\{ \sum_{j=1}^{k} a_j v_{i_j} \,|\, a_j \in \mathbb{R} \right\} \cap \mathbb{Z}_G = \left\{ \sum_{j=1}^{k} m_j v_{i_j} \,|\, m_j \in \mathbb{Z} \right\} \tag{2.114}$$

and $\{v_{i_j}\}$ is independent over $\mathbb{Z}$.

Example 2.27. Consider once more the action of the 1-dimensional group $N \cong \mathbb{T}^1$ with weights $(1, 1, -p)$ on $\mathbb{C}^3$. The cone corresponding to this action is found from the relations giving its polytope but with all λ_is equal to zero. We therefore have

$$x \geq 0 \quad , \quad y \geq 0 \quad , \quad y \geq \tfrac{1}{p} x \tag{2.115}$$

It is obvious that the relation $y \geq 0$ becomes unnecessary and the cone is enclosed between the lines $x = 0$ and $y = x/p$. This means that we only have 2 normal vectors, namely

$$v_1 = (-1, 0) \quad \text{and} \quad v_2 = (1, -p) \tag{2.116}$$

The moment map which corresponds to this cone is

$$\mu(z) = -\frac{1}{2} \left(|z_1|^2 + |z_2|^2 - p|z_3|^2 \right) \tag{2.117}$$

and the zero level set is

$$Z = \mu^{-1}(0) = \left\{ |z_1|^2 + |z_2|^2 - p|z_3|^2 = 0 \,|\, (z_1, z_2, z_3) \in \mathbb{C}^3 \right\} \tag{2.118}$$

The action of N is not free on Z at the point $(0, 0, 0)$. Therefore, Z is not a manifold since there is a singularity at 0. This is solved either by not including 0, or, as we will see later, by restricting the problem to a specific part of the moment map image.

Example 2.28. Now we consider the action of the 1-dimensional group $N \cong \mathbb{T}^1 \subset \mathbb{T}^4$ with weights $(p+q, p-q, -p, -p)$ on $\mathbb{C}^4$, where p and q are integers, greater than zero, coprimes and $p > q$. The moment map for this action is

$$\mu(z) = -\frac{1}{2} \left((p+q)|z_1|^2 + (p-q)|z_2|^2 - p|z_3|^2 - p|z_4|^2 \right) \tag{2.119}$$

where we have $\lambda_i = 0$, $\forall \lambda_i$ since we are interested in the cone and not the polytope. The zero level set is

$$Z = \mu^{-1}(0) = \left\{ (p+q)|z_1|^2 + (p-q)|z_2|^2 - p|z_3|^2 - p|z_4|^2 = 0 \,|\, (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \right\} \tag{2.120}$$

In order to find π_*, and therefore the normal to the cone vectors, we will rotate the weights of the action. Since the values of the weights are coprimes we know that it is guaranteed to find such a transformation in $\mathrm{SL}(4, \mathbb{Z})$. We will perform the transformation using the following steps

$$\begin{pmatrix} p+q \\ p-q \\ -p \\ -p \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ p-q \\ 1 \\ -p \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{2.121}$$

where we start from the fact that $p+q$ and $-p$ (entries 1 and 3) are coprimes and thus satisfying Beizout's identity

$$a\,(p+q)+b\,(-p)=1 \tag{2.122}$$

for some $a, b \in \mathbb{Z}$. A matrix that performs the full transformation is

$$\mathbf{M}=\begin{pmatrix} -p & 0 & -(p+q) & 0 \\ -(p-q)a & 1 & -(p-q)b & 0 \\ a & 0 & b & 0 \\ p\,a & 0 & p\,b & 1 \end{pmatrix} \tag{2.123}$$

As we have seen before, we can immediately obtain π_* from $\mathbf{M}$. In this case π_* will consist of the rows $1, 2$ and 4 of $\mathbf{M}$.

$$\pi_*=\begin{pmatrix} -p & 0 & -(p+q) & 0 \\ -(p-q)a & 1 & -(p-q)b & 0 \\ p\,a & 0 & p\,b & 1 \end{pmatrix} \tag{2.124}$$

The normal to the cone vectors are

$$v_1=\begin{pmatrix} -p \\ -(p-q)a \\ p\,a \end{pmatrix} \quad , \quad v_2=\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad v_3=\begin{pmatrix} -(p+q) \\ -(p-q)b \\ p\,b \end{pmatrix} \quad , \quad v_4=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{2.125}$$

Now we specify p and q in order to investigate the cone further. Lets assume $p=2$ and $q=1$. Then we have

$$a(3)+b(-2)=1 \tag{2.126}$$

from which we find $a=-1$ and $b=-2$. The map π_* takes the form

$$\pi_*=\begin{pmatrix} -2 & 0 & -3 & 0 \\ 1 & 1 & 2 & 0 \\ -2 & 0 & -4 & 1 \end{pmatrix} \tag{2.127}$$

and the 4 normal vectors are

$$v_1=\begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix} \quad , \quad v_2=\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad v_3=\begin{pmatrix} -3 \\ 2 \\ -4 \end{pmatrix} \quad , \quad v_4=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{2.128}$$

From the relations

$$\langle \mathbf{r}, v_i \rangle \leq 0 \quad , \quad \mathbf{r} \in \left(\mathbb{R}^3\right)^* \quad , \quad \mathbf{r}=(x,y,z) \tag{2.129}$$

we obtain the 4 conditions that define the inside of the cone. We have

$$\begin{aligned} -2x+y-2z \leq 0 \quad &, \quad (\mathrm{c}_1) \\ y \leq 0 \quad &, \quad (\mathrm{c}_2) \\ -3x+2y-4z \leq 0 \quad &, \quad (\mathrm{c}_3) \\ z \leq 0 \quad &, \quad (\mathrm{c}_4) \end{aligned}$$

The faces of the cone are the normal planes to the vectors v_i that are shown below

$$\begin{aligned} P_1: &\quad -2x+y-2z=0 \\ P_2: &\quad y=0 \\ P_3: &\quad -3x+2y-4z=0 \\ P_4: &\quad z=0 \end{aligned} \tag{2.130}$$

The next step is to find which plane is next to which, or said differently, find the ordering of planes P_1, P_2, P_3 and P_4. The procedure we follow is to find the intersection of planes $P_i \cap P_j$ for $i, j = (1, 2, 3, 4)$ and check if it obeys the cone conditions, meaning that it lies inside the cone.

$$
\begin{aligned}
&P_1 \cap P_2 : x = -z \longrightarrow (\mathrm{c}_3, \mathrm{c}_4) \longrightarrow ✗\\
&P_1 \cap P_3 : x = 0 \longrightarrow (\mathrm{c}_2, \mathrm{c}_4) \longrightarrow ✓\\
&P_1 \cap P_4 : 2x = y \longrightarrow (\mathrm{c}_2, \mathrm{c}_3) \longrightarrow ✓\\
&P_2 \cap P_3 : 3x = -4z \longrightarrow (\mathrm{c}_1, \mathrm{c}_4) \longrightarrow ✓\\
&P_2 \cap P_4 : y = z = 0 \longrightarrow (\mathrm{c}_1, \mathrm{c}_3) \longrightarrow ✓\\
&P_3 \cap P_4 : 3x = 2y \longrightarrow (\mathrm{c}_1, \mathrm{c}_2) \longrightarrow ✗
\end{aligned}
\tag{2.131}
$$

From these we see that the ordering of the planes is: P_1, P_3, P_2, P_4, P_1.

2.5 Symplectic Cuts

Suppose (M, ω) is an arbitrary symplectic manifold equipped with a Hamiltonian circle action having moment map $\mu : M \to \mathbb{R}^*$. Now consider the product manifold

$$M' = (M \times \mathbb{C}, \omega \oplus (-\mathrm{i})\mathrm{d}w \wedge \mathrm{d}\bar{w}) \tag{2.132}$$

where w is the coordinate on $\mathbb{C}$. If the circle action S^1 acts freely on a level set $\mu^{-1}(\varepsilon)$ then ε is a regular value of the moment map

$$\Phi(m, w) = \mu(m) - |w|^2 \tag{2.133}$$

of M' which arises from the action

$$\mathrm{e}^{\mathrm{i}\theta}(m, w) = (\mathrm{e}^{\mathrm{i}\theta} m, \mathrm{e}^{-\mathrm{i}\theta} w) \tag{2.134}$$

The manifold $M_{\mu>\varepsilon}$ (points of the original manifold for which the moment map is greater than ε) embeds as an open dense submanifold into the reduced space

$$\overline{M}_{\mu\geq\varepsilon} := \Phi^{-1}(\varepsilon)/S^1 = \left\{(m, w) \in M \times \mathbb{C} \mid \mu(m) - |w|^2 = \varepsilon\right\}/S^1 \tag{2.135}$$

and the difference $\overline{M}_{\mu\geq\varepsilon} - M_{\mu>\varepsilon}$ is symplectomorphic to the reduced space $\mu^{-1}(\varepsilon)/S^1$. Topologically, $\overline{M}_{\mu\geq\epsilon}$ is the quotient of the manifold with boundary $M_{\mu\geq\varepsilon}$ by the equivalence relation $\sim$ where $m' \sim m$ iff $\mu(m) = \mu(m') = \varepsilon$ and $m = \mathrm{e}^{\mathrm{i}\theta} m'$ for the action $\mathrm{e}^{\mathrm{i}\theta} \in S^1$ (the action is 1-dimensional but not necessarily diagonal). Using the same procedure but for

$$\mathrm{e}^{\mathrm{i}\theta}(m, w) = (\mathrm{e}^{\mathrm{i}\theta} m, \mathrm{e}^{\mathrm{i}\theta} w) \tag{2.136}$$

we can define

$$\overline{M}_{\mu\leq\varepsilon} := \left\{(m, w) \in M \times \mathbb{C} \mid \mu(m) + |w|^2 = \varepsilon\right\}/S^1 \tag{2.137}$$

The symplectic manifold $\mu^{-1}(\varepsilon)/S^1$ is embedded in both $\overline{M}_{\mu\geq\varepsilon}$ and $\overline{M}_{\mu\leq\varepsilon}$ as a codimension 2 submanifold but with opposite normal bundles. If we glue $\overline{M}_{\mu\geq\varepsilon}$ and $\overline{M}_{\mu\leq\varepsilon}$ along the reduced space $\mu^{-1}(\epsilon)/S^1$ we recover the original manifold M. The **Symplectic Cutting** is defined as the operation that produces $\overline{M}_{\mu\geq\varepsilon}$ and $\overline{M}_{\mu\leq\varepsilon}$.

Example 2.29. We start from the symplectic manifold $M = \mathbb{C}^n$ equipped with the symplectic form

$$\omega = -\mathrm{i} \sum \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j \tag{2.138}$$

We choose the diagonal Hamiltonian action on M with moment map $\mu(z) = |z|^2$ where we denote $z = (z_1, \ldots, z_n)$ and $|z|^2 = |z_1|^2 + \cdots + |z_n|^2$. Then there exists a Hamiltonian S^1 action on

$$M' = (M \times \mathbb{C}, \omega \oplus (-\mathrm{i})\mathrm{d}w \wedge \mathrm{d}\bar{w}) \tag{2.139}$$

given by

$$\mathrm{e}^{\mathrm{i}\theta}(z, w) = (\mathrm{e}^{\mathrm{i}\theta} z, \mathrm{e}^{\mathrm{i}\theta} w) \tag{2.140}$$

with a momentum map

$$\phi(z, w) = \mu(z) - |w|^2 = |z|^2 - |w|^2 \tag{2.141}$$

The ϵ-level set $\Phi^{-1}(\epsilon)$ is a disjoint union of two S^1-invariant manifolds

$$\Phi^{-1}(\epsilon) = \Big\{(z, w) \in M' \mid \mu(z) > \epsilon \quad \& \quad w = \mathrm{e}^{\mathrm{i}\theta}\sqrt{\mu(z) - \epsilon}\Big\} \bigsqcup \Big\{(z, 0) \in M' \mid \mu(z) = \varepsilon\Big\} \tag{2.142}$$

The first manifold is equivariantly diffeomorphic to the product of

$$M_{\mu > \varepsilon} = \{z \in M \mid \mu(z) > \varepsilon\} \tag{2.143}$$

and of the circle S^1 (since the contribution of w in this case is just the circle $\mathrm{e}^{\mathrm{i}\theta}$). The second manifold is diffeomorphic to the ε-level set $\mu^{-1}(\varepsilon)$. From this we deduce that the manifold $M_{\mu>\varepsilon}$ embeds into the reduced space $\overline{M}_{\mu \geq \varepsilon} = \Phi^{-1}(\varepsilon)/S^1$ as an open dense symplectic manifold. The remaining set $\overline{M}_{\mu \geq \varepsilon} - M_{\mu > \varepsilon}$ is isomorphic to the reduced space $\mu^{-1}(\varepsilon)/S^1$. The set $\overline{M}_{\mu \geq \varepsilon}$ can be thought of as a removal of the ball of radius $\sqrt{\varepsilon}$ (from the original manifold) centered at the origin and then a collapse of the fibers of the Hopf fibration in the boundary of the remaining set

$$\{z \in \mathbb{C}^n \mid |z|^2 \geq \varepsilon\} \tag{2.144}$$

This is exactly the ε-blow up of the origin of $\mathbb{C}^n$.

note: Following the same procedure we can find that for $M = \mathbb{C}^n$ the manifold $\overline{M}_{\mu \leq \varepsilon}$ is isomorphic to the projective space $\mathbb{CP}^n$ (similarly to a previous example). We can think of this $\mathbb{CP}^n$ as $\mathbb{C}^n$ blown up at infinity by an $(\infty - \varepsilon)$ amount.

Remark 2.30 (**Cuts and Polytopes**). Suppose the torus group G acts on a compact symplectic manifold (M, ω) with a moment map $\mu : M \to \mathfrak{g}^*$. Then $\mu(M)$ is a rational convex polytope in $\mathfrak{g}^*$. Now let $\mathfrak{n} \in \mathfrak{g}$ generate a circle subgroup $N = S^1_{\mathfrak{n}}$ of G. Then the action of N on (M, ω) is Hamiltonian with moment map $\mu_{\mathfrak{n}} = i^* \circ \mu : M \to \mathfrak{n}^*$ where $i : N \to G$ is the inclusion map. The actions of N and G commute. If we cut M at $\varepsilon \in \mathbb{R}$ using $\mu_{\mathfrak{n}}$ we get two Hamiltonian G-orbifolds $\overline{M}_{\mu_{\mathfrak{n} \geq \varepsilon}}$ and $\overline{M}_{\mu_{\mathfrak{n} \leq \varepsilon}}$. Their moment polytopes are

$$\Delta_1 = \mu(M) \cap \{x \in G^* \mid \langle \mathfrak{n}, x\rangle \geq \varepsilon\} \tag{2.145}$$

and

$$\Delta_2 = \mu(M) \cap \{x \in G^* \mid \langle \mathfrak{n}, x\rangle \leq \varepsilon\} \tag{2.146}$$

Example 2.31. Let $M = \mathbb{C}^2$ and take the S^1 action with weights $(1, -1)$ on M. Then we have an S^1 action on $M' = M \times \mathbb{C}$ defined as

$$\mathrm{e}^{\mathrm{i}\theta}(z_1, z_2, w) = (\mathrm{e}^{\mathrm{i}\theta} z_1, \mathrm{e}^{-\mathrm{i}\theta} z_2, \mathrm{e}^{-\mathrm{i}\theta} w) \tag{2.147}$$

The moment map for M is

$$\mu(z_1, z_2) = |z_1|^2 - |z_2|^2 \tag{2.148}$$

and for M' we have

$$\Phi(z_1, z_2, w) = |z_2|^2 - |z_2|^2 - |w|^2 \tag{2.149}$$

The reduction at the ε-level of Φ is

$$\overline{M}_{\mu\geq\varepsilon} = \Phi^{-1}(\varepsilon)/S^1 = \left\{(z_1, z_2, w) \in \mathbb{C}^2 \times \mathbb{C} \mid |z_1|^2 - |z_2|^2 - |w|^2 = \varepsilon\right\}/S^1 \tag{2.150}$$

For simplicity we set $\varepsilon = 0$ and exclude the point $(0, 0, 0)$ from consideration since the action is not free there (this will not matter because we will restrict the problem on a 3-sphere). Now $\mu^{-1}(0)$ is the set (z_1, z_2) that satisfies $|z_1|^2 = |z_2|^2$. This is the line $y = x$ on the $x - y$ plane (with $|z_1|^2 = x$ and $|z_2|^2 = y$). The set $\Phi^{-1}(0)$ is $|z_1|^2 = |z_2|^2 + |w|^2$. The set's projection on the $x - y$ plane is the area under the $y = x$ line $(x > y)$. Similarly, we define the action on $M \times \mathbb{C}$

$$\mathrm{e}^{\mathrm{i}\theta}(z_1, z_2, w) = (\mathrm{e}^{\mathrm{i}\theta} z_1, \mathrm{e}^{-\mathrm{i}\theta} z_2, \mathrm{e}^{\mathrm{i}\theta} w) \tag{2.151}$$

with moment map Ψ

$$\Psi(z_1, z_2, w) = |z_1|^2 - |z_2|^2 + |w|^2 \tag{2.152}$$

Then we have

$$\overline{M}_{\mu\leq 0} = \Psi^{-1}(0)/S^1 = \left\{(z_1, z_2, w) \in \mathbb{C}^2 \times \mathbb{C} \mid |z_1|^2 - |z_2|^2 + |w|^2 = 0\right\}/S^1 \tag{2.153}$$

The projection of the set $\Psi^{-1}(0)$ on the $x - y$ plane is the area $y > x$. Now we restrict to the 3-sphere defined by

$$S^3 = \left\{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = k\right\} \tag{2.154}$$

If for the moment we do not consider the reduction by S^1, we see that the line $\mu^{-1}(0)$ splits this sphere into two parts S^3_+ and S^3_- with common boundary at

$$|z_1|^2 = |z_2|^2 = k/2 \tag{2.155}$$

These two parts are written as

$$S^3_+ = \left\{(z_1, z_2) \in S^3 \mid |z_1|^2 \leq k/2 \leq |z_2|^2\right\} \tag{2.156}$$

and

$$S^3_- = \left\{(z_1, z_2) \in S^3 \mid |z_2|^2 \leq k/2 \leq |z_1|^2\right\} \tag{2.157}$$

Both S^3_+ and S^3_- are homeomorphic to solid tori. Their common boundary is $|z_1| = |z_2| = \sqrt{k/2}$ which is the product of two circles $S^1 \times S^1$. Thus, their common boundary is a torus. On S^3_+ we choose z_1 as the disk coordinate $z_1 := d \in D^2$ and we map z_2 to $z_2/|z_2|$ which represents a circle coordinate $z_2 := s \in S^1$. We have

$$(z_1, z_2)_{S^3_+} \mapsto (z_1, z_2/|z_2|) = (d, s) = (r\mathrm{e}^{\mathrm{i}\alpha}, \mathrm{e}^{\mathrm{i}\beta}) \tag{2.158}$$

and similarly on S^3_-

$$(z_1, z_2)_{S^3_-} \mapsto (z_1/|z_1|, z_2) = (s, d) = (\mathrm{e}^{\mathrm{i}\alpha'}, r'\mathrm{e}^{\mathrm{i}\beta'}) \tag{2.159}$$

We see that on their common boundary we need to glue the s-cycle of one solid torus to the d-cycle of the other to recover the original S^3. Now, by imposing the cut we will need to take the reduction by S^1 at the boundary of each solid torus S^3_+ and S^3_-. We start from S^3_+ and we approach its boundary $|z_1| = |z_2| = \sqrt{k/2}$. At the boundary we will have $(z_1, z_2)_{S^3}/S^1$ where S^1 is the action with weights $(1, -1)$. This means that we have an equivalence relation $\sim$ (at the boundary) given by

$$(z_1, z_2) \sim (z_1', z_2') \quad \text{if} \quad (z_1', z_2') = (\mathrm{e}^{\mathrm{i}\theta} z_1, \mathrm{e}^{-\mathrm{i}\theta} z_2) \tag{2.160}$$

at $\mu(z_1, z_2) = 0$. At the boundary we have two angles α and β representing z_1 and z_2 which are meridians and longitudes (disk coordinate and circle coordinate). The equivalence means that

$$(\alpha, \beta) \sim (\theta + \alpha, -\theta + \beta) \tag{2.161}$$

This reduces the two original dimensions to one. When θ goes from 0 to 2π, $-\theta$ goes from 0 to -2π. This is a $(p, q) = (1, -1)$ torus knot. We can therefore picture the equivalence at the boundary as a collapse (after a $2\pi q/p$ twist) of every longitude to a point, and thus all the longitudes collapsing on a meridian. Now let us picture the solid torus cut across a chosen half-plane (a meridian). This gives us a solid cylinder with identified ends. The equivalence relation then identifies further all of the boundary of the cylinder to a circle. This is homeomorphic to a solid ball D^3 with upper and lower hemispheres identified via orthogonal projection. For more information on this subject (and nice pictures) see [5]. This is proven to be the Lens space $L(1, q)$ which is by definition the 3-sphere S^3. So, the collapsing of the boundary of S^3_+ due to the symplectic cut gives us S^3. The same is true for S^3_-. After this analysis we conclude that the symplectic cut of a 3-sphere produces two S^3 (whereas the simple cut would give us 2 solid tori).

3 Contact Geometry

In this section we define contact manifolds, explain their relation to symplectic manifolds, introduce the Reeb vector field and clarify these definitions by giving several examples. We also define group actions on contact manifolds, and the cones that arise form these actions, by making use of the symplectic manifolds that are related to said contact manifolds. This section is mainly based on Geiges [6], Etnyre's lecture notes [7] and Lerman's papers [8] and [9].

3.1 Contact Manifolds

Definition 3.1 (**Hyperplane field**). Let M be a $(2n+1)-$dimensional manifold. A hyperplane field ξ on M is a smooth subbundle of the tangent bundle TM ($\xi \subset TM$) such that $\xi_p = T_pM \cap \xi$ is a $2n$-dimensional subspace of T_pM, $\forall p \in M$. ξ is a subbundle of codimension 1.

Example 3.2. Let $M = \Sigma \times S^1$ be a 3-manifold and Σ a surface. Then $\forall p = (x, \theta) \in \Sigma \times S^1$ let $\xi_p = T_x\Sigma \subset T_pM$. We see that ξ is a plane field on M.

Definition 3.3 (**Contact 1-form**). Let M be a $(2n+1)$-dimensional manifold and α a 1-form on M. At each point $p \in M$ we have a linear map

$$\alpha_p : T_pM \to \mathbb{R} \tag{3.1}$$

Then $\ker \alpha$ is either a hyperplane or all of T_pM. If we assume that α never maps all of T_pM to zero, then $\xi = \ker \alpha$ is a hyperplane field (in the previous example, ξ is defined through $\alpha = \mathrm{d}\theta$). It can be proven that locally we can always represent a plane field as the kernel of a 1-form. A 1-form that satisfies the relation

$$\alpha \wedge (\mathrm{d}\alpha)^n \neq 0 \tag{3.2}$$

which means that it vanished nowhere, is called a **contact 1-form**. The form $\Omega = \alpha \wedge (\mathrm{d}\alpha)^n$ is a top, volume form ($(2n+1)$ form). The fact that $\Omega \neq 0$ means that M is orientable.

Definition 3.4 (**Contact Manifold**). A hyperplane field ξ is called a **contact structure** if it can be defined as

$$\xi = \ker \alpha \tag{3.3}$$

where α is a contact form. The pair (M, ξ) is called a contact manifold.

Remark 3.5. The condition $\alpha \wedge (\mathrm{d}\alpha)^n$ is independent of the specific choice of α and it is only a property of $\xi = \ker \alpha$. Any other form α' defining the same hyperplane field ξ must be of the form $\lambda\alpha$ for some smooth function $\lambda : M \to \mathbb{R} \setminus \{0\}$. Then the condition for $\lambda\alpha$ becomes

$$(\lambda\alpha) \wedge (\mathrm{d}(\lambda\alpha))^n = (\lambda\alpha) \wedge (\lambda \wedge \mathrm{d}\alpha + \mathrm{d}\lambda \wedge \alpha)^n = \lambda^{n+1}\alpha \wedge (\mathrm{d}\alpha)^n \neq 0 \tag{3.4}$$

Remark 3.6. Let $\xi = \ker \alpha$ be a hyperplane field (α is not a contact form). The Frobenius integrability theorem states that ξ is integrable iff its sections are close under the Lie bracket.

$$X, Y \in \xi \Longrightarrow [X, Y] \in \xi \tag{3.5}$$

Since $\xi = \ker \alpha$, if X, Y are sections of ξ then $\alpha(X) = \alpha(Y) = 0$. Thus, if ξ is integrable then $\alpha([X, Y]) = 0$. Then from the identity

$$\mathrm{d}\alpha(X, Y) = \mathcal{L}_X (\alpha(Y)) - \mathcal{L}_Y (\alpha(X)) - \alpha([X, Y]) \tag{3.6}$$

we obtain $\mathrm{d}\alpha|_\xi = 0$ or equivalently $\alpha \wedge \mathrm{d}\alpha = 0$. The contact condition is exactly the opposite of this.

Remark 3.7. The contact condition can be also formulated as $(\mathrm{d}\alpha)^n|_\xi \neq 0$. Then $\forall p \in M$ the $2n$-dimensional subspace $\xi_p \subset T_pM$ is a vector space on which $\mathrm{d}\alpha$ defines a skew-symmetric form of maximal rank. This means that $(\xi_p, \mathrm{d}\alpha|_{\xi_p})$ is a symplectic vector space. Therefore, there exists a complex bundle structure $J : \xi \to \xi$ compatible with $\mathrm{d}\alpha$.

Remark 3.8. Given a $(2n+1)$-dimensional manifold M and a point $p \in M$, a **contact element** of M with contact point p is a $2n$-dimensional linear subspace of the tangent space to M at p (it is another name for the hyperplane ξ_p). The contact element can then be given by the kernel of a 1-form α and, as we saw before, it can also be given by the kernel of $\lambda\alpha$ with $\lambda \neq 0$. From this we realize that the space of all contact elements of M can be identified with a quotient of the cotangent bundle T^*M

$$PT^*M = T^*M/\sim \tag{3.7}$$

where for $\alpha_i \in T_p^*M$ we have

$$\alpha_1 \sim \alpha_2 \Longleftrightarrow \exists\lambda \neq 0 : \alpha_1 = \lambda\alpha_2 \tag{3.8}$$

A contact structure on M is a smooth distribution of contact elements (denoted by ξ) which is generic at each point. The genericity condition is that ξ is non-integrable. This condition of course translates to $\alpha \wedge (\mathrm{d}\alpha)^n \neq 0$.

Remark 3.9. The restriction of $\omega = \mathrm{d}\alpha$ to a hyperplane ξ is a non-degenerate 2-form. This construction provides any contact manifold (M, ξ) with a natural symplectic bundle of rank one smaller than the dimension of M.

Example 3.10. We provide the manifold $M = \mathbb{R}^{2n+1}$ with the Cartesian coordinates

$$(x_1, y_1, \ldots, x_n, y_n, z)$$

Then the 1-form

$$\alpha_1 \equiv \mathrm{d}z + \sum_{i=1}^{n} x_i \mathrm{d}y_i \tag{3.9}$$

is a contact form. The contact structure $\xi_1 = \ker \alpha_1$ is called the standard contact structure on $\mathbb{R}^{2n+1}$. Using Darboux's theorem (as we did for a symplectic manifold and the standard symplectic form) it can be proven than for a $(2n+1)$-dimensional manifold M the contact form on a neighborhood $\mathcal{U} \subset M$ can be written as the standard contact form. Let p be a point of M. Then we can define the coordinates $(x_1, y_1, \ldots, x_n, y_n, z)$ on a neighborhood $\mathcal{U} \subset M$ of p such that $p = (0, \ldots, 0)$ and

$$\alpha|_{\mathcal{U}} = \mathrm{d}z + \sum_{i=1}^{n} x_i \mathrm{d}y_i \tag{3.10}$$

There also exists a more symmetric form for the standard contact structure on $\mathbb{R}^{2n+1}$. It is given by the kernel of the contact form

$$\alpha_2 = \mathrm{d}z + \sum_{i=1}^{n} (x_i \mathrm{d}y_i - y_i \mathrm{d}x_i) \tag{3.11}$$

Definition 3.11 (**Contactomorphism**)**.** Two contact manifolds (M, ξ) and (M', ξ') are contactomorphic if there exists a diffeomorphism $\phi : M \to M'$ with $T\phi(\xi) = \xi'$, where $T\phi : TM \to TM'$ denotes the differential of ϕ. If $\xi = \ker \alpha$ and $\xi' = \ker \alpha'$, this is equivalent to saying that α and $\phi^* \alpha'$ determine the same hyperplane field, and thus it is equivalent to the existence of a function $f : M \to \mathbb{R} \setminus \{0\}$ such that $\phi^* \alpha' = f\,\alpha$. If we can find a function f such that $f = 1$ everywhere, the contactomorphism is called strict.

Remark 3.12. In the previous example (3.10) there exists a strict contactomorphism

$$\phi : \left(\mathbb{R}^{2n+1}, \alpha_1\right) \to \left(\mathbb{R}^{2n+1}, \alpha_2\right) \tag{3.12}$$

given by

$$\phi(\mathbf{x}, \mathbf{y}, z) = \left(\frac{\mathbf{x} + \mathbf{y}}{2}, \frac{\mathbf{y} - \mathbf{x}}{2}, z + \frac{\mathbf{x} \cdot \mathbf{y}}{2}\right) \tag{3.13}$$

This means that the contact structure $\xi_2 = \ker \alpha_2$ is equivalent to ξ_1.

Example 3.13. By using polar coordinates on $\mathbb{R}^{2n+1}$ we have

$$(x_i, y_i, z) \mapsto (r_i, \varphi_i, z) \quad \text{for} \quad i = 1, \ldots, n \tag{3.14}$$

Then the contact form α_2 is written as

$$\alpha_2 = \mathrm{d}z + \sum_{i=1}^{n} r_i^2 \mathrm{d}\varphi_i \tag{3.15}$$

Example 3.14. Now we use $\mathbb{R}^3$ in order to be able to visualize our results. We take the contact form on $\mathbb{R}^3$

$$\alpha_1 = \mathrm{d}z + x\mathrm{d}y \tag{3.16}$$

Then we have $\mathrm{d}\alpha_1 = \mathrm{d}x \wedge \mathrm{d}y$, so the contact condition is

$$\alpha_1 \wedge \mathrm{d}\alpha_1 = \mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y \neq 0 \tag{3.17}$$

which verifies our definition of α_1. The contuct structure $\xi_1 = \ker \alpha_1$ is spanned by

$$\left\{\frac{\partial}{\partial x}, x\frac{\partial}{\partial z} - \frac{\partial}{\partial y}\right\} \tag{3.18}$$

Thus, a vector $X \in \xi_1$ can be written as

$$X = k\frac{\partial}{\partial x} + \lambda\left(x\frac{\partial}{\partial z} - \frac{\partial}{\partial y}\right) \tag{3.19}$$

and its easy to check that $X \in \ker \alpha_1$. We see that ξ_1 lies on the $x-y$ plane when $x=0$. If we move to the point $(1,0,0)$ then ξ_1 is spanned by

$$\left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial z} - \frac{\partial}{\partial y} \right\} \tag{3.20}$$

So ξ_1 is tangent to the x-axis but it is tilted clockwise by 45°. So if we start at $(0,0,0)$ we obtain a horizontal plane (on $x-y$) and as we move along the x-axis the plane twists in a left handed manner (clockwise). When we reach ∞ the twist will be 90°.

Example 3.15. We take $\mathbb{R}^3$ again, but now we use cylindrical coordinates (r, θ, z). The contact form is then written as

$$\alpha_2 = \mathrm{d}z + r^2 \mathrm{d}\theta \tag{3.21}$$

Since $\alpha_2 \wedge \mathrm{d}\alpha_2 = 2r\mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}z \neq 0$, $\xi_2 = \ker \alpha_2$ is a contact structure. At the point (r, θ, z) the contact plane ξ_2 is spanned by

$$\left\{ \frac{\partial}{\partial r}, r^2 \frac{\partial}{\partial z} - \frac{\partial}{\partial \theta} \right\} \tag{3.22}$$

So when $r=0$ (on the z-axis) ξ_2 lies on the $x-y$ plane. As r grows and we move out on any line perpendicular to the z-axis the planes ξ_2 will twist clockwise. So in this example we get the same result as in the previous one, but now everything is symmetric about the z-axis.

Example 3.16. Let $M = S^3$ embedded into $\mathbb{R}^4$. Then the contact form on S^3 can be written as

$$\alpha_0 = (x_1 \mathrm{d}y_1 + x_2 \mathrm{d}y_2 - y_1 \mathrm{d}x_1 - y_2 \mathrm{d}x_2)\,|_{S^3} \tag{3.23}$$

using the Cartesian coordinates of $\mathbb{R}^4$. The points (x_1, x_2, y_1, y_2) are restricted on

$$x_1^2 + x_2^2 + y_1^2 + y_2^2 = 1 \tag{3.24}$$

which is the S^3 equation. Now we prove that α_0 is actually a contact form and thus can define a contact structure ξ_0. We have

$$\alpha_0 = x_i \mathrm{d}y_i - y_i \mathrm{d}x_i \quad \text{and} \quad \mathrm{d}\alpha_0 = 2\mathrm{d}x_j \wedge \mathrm{d}y_j \tag{3.25}$$

where $i, j = 1, 2$ and a repeated index implies summation. Then the contact condition is

$$\begin{aligned} \alpha_0 \wedge \mathrm{d}\alpha_0 &= (x_i \mathrm{d}y_i - y_i \mathrm{d}x_i) \wedge (2\mathrm{d}x_j \wedge \mathrm{d}y_j) \\ &= 2x_2 \cdot \mathrm{d}y_2 \wedge \mathrm{d}x_1 \wedge \mathrm{d}y_1 + 2x_1 \cdot \mathrm{d}y_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}y_2 - \\ &\quad - 2y_2 \cdot \mathrm{d}x_2 \wedge \mathrm{d}x_1 \wedge \mathrm{d}y_1 - 2y_1 \cdot \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}y_2 \end{aligned}$$

which means that $\alpha_0 \wedge \mathrm{d}\alpha_0$ is zero only when $(x_1, x_2, y_1, y_2) = (0,0,0,0)$. But this cannot happen since we are restricted on the sphere. Thus α_0 is a contact form on S^3. Now let

$$f(x_1, x_2, y_1, y_2) = x_1^2 + x_2^2 + y_1^2 + y_2^2 \tag{3.26}$$

Then S^3 can be described as

$$S^3 = \left\{ (x_1, x_2, y_1, y_2) \in \mathbb{R}^4 \mid f(x_1, x_2, y_1, y_2) = 1 \right\} \tag{3.27}$$

or equivalently

$$S^3 = \ker(f - 1) = f^{-1}(1) \tag{3.28}$$

The tangent space T_pS^3 at a point $p = (x_1, x_2, y_1, y_2)$ is given by

$$T_pS^3 = \ker \mathrm{d}f_p = \ker\left(2x_1\mathrm{d}x_1 + 2x_2\mathrm{d}x_2 + 2y_1\mathrm{d}y_1 + 2y_2\mathrm{d}y_2\right) \tag{3.29}$$

Now we will make use of the complex structure J on $\mathbb{C}^2$ in order to prove a very useful way of writing the contact structure ξ. We know that $\mathbb{R}^4 \cong \mathbb{C}^2$ where $z_j = x_j + \mathrm{i}y_j$ for $j = 1, 2$. The complex structure acts on x_j and y_j as

$$Jx_j = y_j \qquad , \qquad Jy_j = -x_j \tag{3.30}$$

The complex structure induces a complex structure on each tangent space by

$$J\frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j} \qquad , \qquad J\frac{\partial}{\partial y_j} = -\frac{\partial}{\partial x_j} \tag{3.31}$$

We will prove that the plane field at a point p

$$\xi_p = T_pS^3 \cap \xi \tag{3.32}$$

is the set of complex tangencies to S^3

$$\xi_p = T_pS^3 \cap J\left(T_pS^3\right) \tag{3.33}$$

and therefore the contact structure is $\xi = J\left(T_pS^3\right)$. Using our previous definition we have

$$J(T_pS^3) = \ker\left(\mathrm{d}f_p \circ J\right) \tag{3.34}$$

Since J takes vectors and produces vectors, $\mathrm{d}f_p \circ J$ is still an 1-form. When we act with it on the basis vectors we get

$$\mathrm{d}f_p \circ J\left(\frac{\partial}{\partial x_j}\right) = 2y_j \qquad \text{and} \qquad \mathrm{d}f_p \circ J\left(\frac{\partial}{\partial y_j}\right) = -2x_j \tag{3.35}$$

The inner product of this 1-form with a general vector X defined by

$$X = a\frac{\partial}{\partial x_1} + b\frac{\partial}{\partial x_2} + c\frac{\partial}{\partial y_1} + d\frac{\partial}{\partial y_2} \tag{3.36}$$

is

$$\langle\left(\mathrm{d}f_p \circ J\right), X\rangle = 2y_1a + 2y_2b - 2x_1c - 2x_2d \tag{3.37}$$

and therefore $\mathrm{d}f_p \circ J$ is written as

$$\mathrm{d}f_p \circ J = -2x_1\mathrm{d}y_1 - 2x_2\mathrm{d}y_2 + 2y_1\mathrm{d}x_1 + 2y_2\mathrm{d}x_2 \tag{3.38}$$

which is $\mathrm{d}f_p \circ J = -2\alpha_0$. Since α_0 and $-2\alpha_0$ obviously have the same kernel, we finally have

$$\xi_0 = J\left(T_pS^3\right) = \ker\left(\mathrm{d}f_p \circ J\right)|_{S^3} \tag{3.39}$$

Example 3.17. The previous example can be easily generalized to the sphere S^{2n+1} embedded in $\mathbb{R}^{2n+2}$. We use the Cartesian coordinates (x_i, y_i) for $i = 1, \ldots, (n+1)$. The contact form is then

$$\alpha_0 = \sum_{i=1}^{n+1}\left(x_i\mathrm{d}y_i - y_i\mathrm{d}x_i\right)|_{S^{2n+1}} \tag{3.40}$$

Again we regard S^{2n+1} as the unit sphere in $\mathbb{C}^{n+1}$ with complex structure J. Then, at each point $p \in S^{2n+1}$ the contact structure is given by

$$\xi_p = TS^{2n+1} \cap J\left(TS^{2n+1}\right) \tag{3.41}$$

If we denote with r the radial coordinate on $\mathbb{R}^{2n+2}$ we have

$$r^2 = \sum_{i=1}^{n+1}\left(x_i^2 + y_i^2\right) \qquad \text{and} \qquad r\mathrm{d}r = \sum_{i=1}^{n+1}\left(x_i \mathrm{d}x_i + y_i \mathrm{d}y_i\right) \tag{3.42}$$

and therefore

$$T_pS^3 = \ker\left(r\mathrm{d}r\right) \tag{3.43}$$

Similarly to the previous example we obtain

$$\alpha_0 = -r\mathrm{d}r \circ J \tag{3.44}$$

Example 3.18. Another important example is the space of contact elements. Let N be a smooth n-dimensional manifold. As defined before, a contact element is a hyperplane in a tangent space to N. The space of contact elements of N is the collection of pairs (p, ξ_p) consisting of a point $p \in N$ and a contact element $\xi_p \subset T_pN$. This space can be naturally identified with the projectivised cotangent bundle $\mathbb{P}T^*N$ by associating with a hyperplane $\xi_p \subset T_pN$ the linear map $\alpha_{\xi_p} : T_pN \to \mathbb{R}$ (well defined up to a multiplication by a non-zero scalar) with $\ker \alpha_{\xi_p} = \xi_p$. The space $\mathbb{P}T^*N$ is a manifold of dimension $(2n-1)$ and it carries a natural contact structure. Let π be the bundle projection map $\pi : \mathbb{P}T^*N \to N$. For an element $\alpha \equiv \alpha_{\xi_p} \in \mathbb{P}T^*N$, let ξ be the hyperplane in $T_\alpha\left(\mathbb{P}T^*N\right)$ such that $T\pi(\xi)$ is the hyperplane ξ_p in $T_{\pi(\alpha)}N = T_pN$ defined by α. Then ξ defines a contact structure on $\mathbb{P}T^*N$. Now lets explore this proposition in more detail. Let $(q_1, \ldots, q_n)$ be the local coordinates on N and $(p_1, \ldots, p_n)$ the corresponding dual coordinates in the fibres of the cotangent bundle T^*N. Then the coordinates of a covector (1-form) are given by

$$(q_1, \ldots, q_n, p_1, \ldots, p_n) = \left(\sum_{j=1}^{n} p_j \mathrm{d}q_j\right)_{(q_1,\ldots,q_n)} \tag{3.45}$$

Therefore, a point $(q_1, \ldots, q_n, [p_1 : \cdots : p_n])$ of the projectivised cotangent bundle $\mathbb{P}T^*N$ defines the hyperplane

$$\alpha \equiv \sum_{j=1}^{n} p_j \mathrm{d}q_j = 0 \tag{3.46}$$

in T_pN, where $p = (q_1, \ldots, q_n)$. By construction, the natural contact structure ξ on $\mathbb{P}T^*N$ is defined by

$$\xi = \ker\left(\sum_{j=1}^{n} p_j \mathrm{d}q_j\right) \tag{3.47}$$

Note that the kernel is well defined in terms of the coordinates on $\mathbb{P}T^*N$ but the 1-form α is not. To verify the contact condition we restrict to affine subspaces of the fibre. For example, over the open set $\{p_1 \neq 0\}$, ξ is defined in terms of the affine coordinates $p'_j = p_j/p_1$, for $j = 2, \ldots, n$ by the equation

$$\mathrm{d}q_1 + p'_2\mathrm{d}q_2 + \cdots + p'_n\mathrm{d}q_n = 0 \tag{3.48}$$

which is exactly the description of the kernel of the contact form α_1 on $\mathbb{R}^{2n-1}$. This is easy to see by setting

$$q_1 = z \quad , \quad x_i = p'_i \quad , \quad y_i = q_i \quad \text{for} \quad i = 2, \ldots, n \tag{3.49}$$

3.2 Reeb and Liouville vector fields

Definition 3.19 (Reeb vector field). Let M be a $(2n+1)$-dimensional manifold and α a contact form on it. The Reeb vector field R_α is then defined by the equations

$$\mathrm{d}\alpha\,(R_\alpha, \cdot) \equiv 0 \tag{3.50}$$

$$\alpha(R_\alpha) \equiv 0 \tag{3.51}$$

The 2-form $\mathrm{d}\alpha|_{T_pM}$ is a skew symmetric form of maximal rank $2n$. This implies that it has a 1-dimensional kernel for each $p \in M$. Then the first equation defines a unique line field $\langle R_\alpha \rangle$ on M. From the contact condition $\alpha \wedge (\mathrm{d}\alpha)^n \neq 0$ we deduce that α is non trivial on the line field R_α, so we use the second equation (normalization condition) to make R_α into a global vector field. Every contact vector field X transverse to ξ can be written as a Reeb vector field for some 1-form α.

Example 3.20. We will calculate the Reeb vector R_α on the 3-sphere S^3 (embedded in $\mathbb{C}^2$). We take

$$f(z_1, z_2) = |z_1|^2 + |z_2|^2 \tag{3.52}$$

and define the sphere as $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid f = 1\}$. The tangent space to S^3 at the point p is given by $\ker(\mathrm{d}f_p)$ as we already saw before

$$T_pS^3 = \ker\left(\bar{z}_1\mathrm{d}z_1 + z_1\mathrm{d}\bar{z}_1 + \bar{z}_2\mathrm{d}z_2 + z_2\mathrm{d}\bar{z}_2\right) \tag{3.53}$$

The contact form α is then written as $\alpha = \mathrm{d}f_p \circ J$, where J is the complex structure on $\mathbb{C}^2$ defined by

$$J\frac{\partial}{\partial z} = \mathrm{i}\,\frac{\partial}{\partial z} \qquad \text{and} \qquad J\frac{\partial}{\partial \bar{z}} = -\mathrm{i}\,\frac{\partial}{\partial \bar{z}} \tag{3.54}$$

To find α we need to act with $\mathrm{d}f_p \circ J$ on the basis vectors in order to compute its components. We find

$$\alpha = \tfrac{\mathrm{i}}{2}\left(z_1\mathrm{d}\bar{z}_1 - \bar{z}_1\mathrm{d}z_1 + z_2\mathrm{d}\bar{z}_2 - \bar{z}_2\mathrm{d}z_2\right) \tag{3.55}$$

where the 1/2 is there by convention. The Reeb vector satisfies these conditions

$$\mathrm{d}\alpha(R_\alpha, \cdot)|_{TS^3} = 0 \qquad \text{and} \qquad \alpha(R_\alpha) = 1 \tag{3.56}$$

First we write R_α in a general form

$$R_\alpha = a\frac{\partial}{\partial z_1} + b\frac{\partial}{\partial \bar{z}_1} + c\frac{\partial}{\partial z_2} + d\frac{\partial}{\partial \bar{z}_2} \tag{3.57}$$

The 2-form $\mathrm{d}\alpha$ is

$$\begin{aligned}\mathrm{d}\alpha &= \mathrm{i}\,\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \mathrm{i}\,\mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_2 \\ &= \mathrm{i}\,\mathrm{d}z_1 \otimes \mathrm{d}\bar{z}_1 - \mathrm{i}\,\mathrm{d}\bar{z}_1 \otimes \mathrm{d}z_1 + \mathrm{i}\,\mathrm{d}z_2 \otimes \mathrm{d}\bar{z}_2 - \mathrm{i}\,\mathrm{d}\bar{z}_2 \otimes \mathrm{d}z_2\end{aligned}$$

Then for $\mathrm{d}\alpha(R_\alpha, \cdot)|_{TS^3}$ we have

$$\mathrm{d}\alpha(R_\alpha, \cdot)|_{TS^3} = \mathrm{i}\,a\,\mathrm{d}\bar{z}_1 - \mathrm{i}\,b\,\mathrm{d}z_1 + \mathrm{i}\,c\,\mathrm{d}\bar{z}_2 - \mathrm{i}\,d\,\mathrm{d}z_2 \equiv 0 \tag{3.58}$$

Since this 1-form is restricted on the sphere $f = |z_1|^2 + |z_2|^2 = 1$, it should be proportional to the 1-form $\mathrm{d}f_p$ which is also zero on the sphere (S^3 is a 3-dimensional submanifold of the

4-dimensional manifold $\mathbb{C}^2$ and therefore we only have a 1-dimensional subspace normal to S^3 at each point $p \in S^3$). We have

$$\mathrm{d}\alpha(R_a, \cdot)|_{TS^3} \sim \bar{z}_1 \mathrm{d}z_1 + z_1 \mathrm{d}\bar{z}_1 + \bar{z}_2 \mathrm{d}z_2 + z_2 \mathrm{d}\bar{z}_2 \tag{3.59}$$

so we obtain

$$a \sim -\mathrm{i}\, z_1 \quad , \quad b \sim \mathrm{i}\, \bar{z}_1 \quad , \quad c \sim -\mathrm{i}\, z_2 \quad , \quad d \sim \mathrm{i}\, \bar{z}_2 \tag{3.60}$$

where the proportionality constant will of course be the same for all the components. We name this constant ε. So far we have

$$R_\alpha = \mathrm{i}\, \varepsilon \left(-z_1 \frac{\partial}{\partial z_1} + \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} - z_2 \frac{\partial}{\partial z_2} + \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \right) \tag{3.61}$$

Now we make use of the normalization condition $\alpha(R_\alpha) = 1$ and obtain

$$\alpha(R_\alpha) = -\frac{\varepsilon}{2} \left(2|z_1|^2 + 2|z_2|^2 \right) = 1 \tag{3.62}$$

and since we are on S^3 we get $\varepsilon = -1$. Therefore, the Reeb vector is

$$R_\alpha = \mathrm{i} \left(z_1 \frac{\partial}{\partial z_1} - \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + z_2 \frac{\partial}{\partial z_2} - \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \right) \tag{3.63}$$

If we write $z_j = r_j \mathrm{e}^{\mathrm{i}\theta_j}$ the Reeb vector is written as

$$R_\alpha = \frac{\partial}{\partial \theta_1} + \frac{\partial}{\partial \theta_2} \tag{3.64}$$

This is the fundamental vector coming from the diagonal action of S^1 on $\mathbb{C}^2$. A constant level set of the moment map is of course S^3.

Definition 3.21 (**Liouville Vector Field**). Let (N, ω) be a symplectic manifold of dimension $(2n+2)$. We know that ω is a closed ($\mathrm{d}\omega = 0$) and non-degenerate ($\omega^{2n+1} \neq 0$) 2-form on N. A vector field X is called Liouville vector field if

$$\mathcal{L}_X \omega = \omega \tag{3.65}$$

where $\mathcal{L}_X$ is the Lie derivative along the vector X. Using Cartan's formula we get

$$\mathcal{L}_X \omega = (\mathrm{d} \circ \iota_X + \iota_X \circ \mathrm{d})\, \omega = \mathrm{d}\left(\iota_X \omega\right) \tag{3.66}$$

Using the Liouville vector condition we obtain

$$\mathrm{d}\left(\iota_X \omega\right) = \omega \tag{3.67}$$

The 1-form $\alpha = \iota_X \omega$ defines a contact form on any hypersurface $M \subset N$ transverse to X. This can be proven as follows

$$\alpha \wedge (\mathrm{d}\alpha)^n = \iota_X\, \omega \wedge (\mathrm{d}(\iota_X \omega))^n = \iota_X\, \omega \wedge \omega^n = \tfrac{1}{n+1}\, \iota_X \omega^{n+1} \tag{3.68}$$

which is a volume form on M.

Example 3.22. Let $N = \mathbb{R}^{2n+2}$ equipped with $\omega = \sum_{i=1}^{n+1} (\mathrm{d}x_i \wedge \mathrm{d}y_i)$ be our symplectic manifold. Then

$$X = \sum_{j=1}^{n+1} \left(x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j} \right) = \frac{r}{2} \frac{\partial}{\partial r} \tag{3.69}$$

is a Liouville vector field. Using this vector field we recover the standard contact structure on S^{2n+1} by

$$\iota_X \omega = \alpha \tag{3.70}$$

3.3 Contact Hamiltonians

A vector field X on the contact manifold $(M, \xi = \ker \alpha)$ is called an **infinitesimal automorphism** of the contact structure if the local flow of X preserves ξ. We denote the flow of X by $\psi_t : M \to M$ with

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_t = X \circ \psi_t \tag{3.71}$$

The condition for X to be an infinitesimal automorphism can be written as $T\psi_t(\xi) = \xi$, which is equivalent to $\mathcal{L}_X\alpha = \lambda\alpha$ for some function $\lambda : M \to \mathbb{R}$. The local flow of X preserves α iff $\mathcal{L}_X\alpha = 0$.

Theorem 3.23. *With a fixed choice of contact form α there is a one to one correspondence between infinitesimal automorphisms X of $\xi = \ker\alpha$ and smooth functions $H : M \to \mathbb{R}^+$. This correspondence is given by*

$$\begin{aligned} &X \mapsto H_X = \alpha(X) \\ &H \mapsto X_H \quad , \quad \begin{cases} \alpha(X_H) = H \\ \iota_{X_H}\mathrm{d}\alpha = \mathrm{d}H(R_\alpha) - \mathrm{d}H \end{cases} \end{aligned}$$

with X_H defined uniquely by these equations.

The fact that X_H is uniquely defined by the above follows from the fact that $\mathrm{d}\alpha$ is non-degenerate on ξ and that $R_\alpha \in \ker(\mathrm{d}H(R_\alpha)\alpha - \mathrm{d}H)$ (which follows from $\alpha(R_\alpha) = 1$). Now lets see why the above theorem is true. Let X be an infinitesimal automorphism of ξ. We define $H_X = \alpha(X)$ and have

$$\mathcal{L}_X\alpha = \mathrm{d}\circ\iota_X\alpha + \iota_X\circ\mathrm{d}\alpha = \mathrm{d}H_X + \iota_X\mathrm{d}\alpha = \lambda\alpha \tag{3.72}$$

where we used $\iota_X\alpha = \alpha(X)$ which is true for 1-forms. We apply this equation to the Reeb vector R_a and obtain

$$\mathrm{d}H_X R_\alpha + \iota_X\mathrm{d}\alpha\, R_\alpha = \lambda\,\alpha(R_\alpha) \Longrightarrow \mathrm{d}H_X(R_\alpha) = \lambda \tag{3.73}$$

So the vector X satisfies the equations

$$\alpha(X) = H_X \qquad \text{and} \qquad \iota_X\mathrm{d}\alpha = \mathrm{d}H_X(R_\alpha)\alpha - \mathrm{d}H_X \tag{3.74}$$

From this we have that $X_{H_X} = X$. Now, going in the other direction, we start from a function $H : M \to \mathbb{R}$ and X_H as defined in the theorem. We get

$$\mathcal{L}_{X_H}\alpha = \iota_{X_H}\mathrm{d}\alpha + \mathrm{d}\left(\alpha(X_H)\right) = \mathrm{d}H(R_\alpha)\alpha \tag{3.75}$$

using the definition. So we get that X_H is an infinitesimal automorphism of ξ (because $\mathrm{d}H(R_\alpha)$ is a function).

3.4 Group Actions on Contact Manifolds

This section is based on Lerman's paper "Contact Toric Manifolds" [9]. In this paper Lerman gives the analog of the moment map and the Delzant polytope for contact manifolds. We start by giving some definitions of previously defined notions, but in a different way.

Definition 3.24. The 1-form α is contact if $\alpha_p \neq 0\ \forall p \in M$ and $\mathrm{d}\alpha|_\xi$ is non-degenerate, where $\xi = \ker\alpha$ is a codimension-1 distribution. Thus the vector bundle $\xi \to M$ has even dimensional fibers ξ_p and M is odd dimensional.

Definition 3.25. A codimension-1 distribution ζ on a manifold M is co-orientable if its annihilator $\zeta^0 \in T^*M$ is an oriented line bundle, i.e. it has a nowhere vanishing global section. It is co-oriented if one component ζ^0_+ of $\zeta^0 \setminus \{0\}$ (ζ minus the zero section) is chosen.

Definition 3.26. A co-oriented contact structure ξ on a manifold M is a co-oriented codimension-1 distribution such that $\xi^0 \setminus \{0\}$ is a symplectic submanifold of the cotangent bundle T^*M. We denote the chosen component of $\xi^0 \setminus \{0\}$ by ξ^0_+ and refer to it as symplectization of (M, ξ). **note**: We know that ξ^0_+ is a line bundle on M and that M is odd-dimensional. Therefore the line bundle is even dimensional.

Remark 3.27. The distribution $\xi \subset TM$ is a co-oriented contact structure iff there is a contact form α with $\ker \alpha = \xi$. Given ξ we choose α to be a section of the line bundle $\xi^0 \setminus \{0\} \to M$. If f is any function from M to $\mathbb{R}$, then $\mathrm{e}^f \alpha$ defines the same contact structure ξ. Thus, a co-oriented contact structure can be thought of as a conformal class of contact forms.

Lemma 3.28. *Let G be a compact group that acts properly on the manifold M preserving a co-oriented codimension-1 distribution ζ and its co-orientation. Then, the lifted action of G on T^*M preserves a component ζ^0_+ of $\zeta^0 \setminus \{0\}$. This means that there is a G-invariant 1-form β on M such that $\zeta = \ker \beta$ and $\beta(M) \subset \zeta^0_+$.*

Definition 3.29. Let the group G act on a manifold M and preserve a 1-form β. The corresponding β-moment map $\Psi_\beta : M \to \mathfrak{g}^*$ determined by β is defined as

$$\langle \Psi_\beta(x), X \rangle = \beta_x(X_M(x)) \tag{3.76}$$

$\forall x \in M$ and $\forall X \in \mathfrak{g}$. X_M is the fundamental vector field corresponding to X and is given by $X_M(x) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \left(\mathrm{e}^{tX}\right) x$. If $\mathrm{d}\beta$ is a symplectic form, then Ψ_β is a symplectic moment map.

According to the previous definition, if α is a contact form, Ψ_α could potentially be a contact moment map. But we know that $\mathrm{e}^f \alpha$ is also a contact form and therefore $\Psi_{\mathrm{e}^f \alpha} = \mathrm{e}^f \Psi_\alpha$. So we need to find a moment map that does not have this problem. Let G act on M and preserve a co-oriented contact structure ξ. The lift of the action of G to the cotangent bundle T^*M preserves a component ξ^0_+ of $\xi^0 \setminus \{0\}$. Let Φ be the moment map from the cotangent bundle to the dual algebra of G, meaning $\Phi : T^*M \to \mathfrak{g}^*$. Then Φ is given by

$$\langle \Phi(q,p), X \rangle = \langle p, X_M(q) \rangle \tag{3.77}$$

$\forall q \in M$, $\forall p \in T^*_q M$ and $\forall X \in \mathfrak{g}$. The restriction $\Psi = \Phi|_{\xi^0_+}$ of Φ on $\xi^0_+ \subset T^*M$ depends only on the action of the group and on the contact structure. We have

$$\Psi : \xi^0_+ \to \mathfrak{g}^* \tag{3.78}$$

If α is any invariant contact form with $\ker \alpha = \xi$ and $\alpha(M) \in \xi^0_+$ then

$$\langle \alpha^* \Psi(q,p), X \rangle = \langle \alpha^* \Phi(q,p), X \rangle = \langle \alpha_q, X_M(q) \rangle = \langle \Psi_\alpha(q), X \rangle \tag{3.79}$$

Therefore, $\Psi \circ \alpha = \Psi_\alpha$, meaning that $\Psi = \Phi|_{\xi^0_+}$ is a universal moment map.

Definition 3.30. Let (M, ξ) be a co-oriented contact manifold equipped with the action of a Lie group G which preserves the contact structure and its co-orientation. On M there exists an invariant 1-form α with $\ker \alpha = \xi$ and $\alpha(M) \subset \xi^0_+$. Then, the α-moment map Ψ_α for the action of G on (M, α) and the moment map Ψ for the action of G on the symplectization ξ^0_+ are related by

$$\Psi \circ \alpha = \Psi_\alpha \tag{3.80}$$

We will refer to $\Psi : \xi^0_+ \to \mathfrak{g}^*$ as the moment map for the action of a Lie group G on a co-oriented contact manifold (M, ξ), or equivalently, as the **contact moment map**.

Definition 3.31 (**Contact quotients**)**.** Let M be a manifold and G a Lie group which acts on it and preserves a 1-form β. We denote the corresponding moment map by $\Psi_\beta : M \to \mathfrak{g}^*$. If the zero level set $\Psi_\beta^{-1}(0)$ is a manifold and the action of G is free and proper on $\Psi^{-1}(0)$ then β descends to a 1-form β_0 on $M_0 := \Psi^{-1}(0)/G$. If β is a contact form, then β_0 is also contact. The manifold M_0 and the contact structure on M_0 defined by β depend only on the contact structure defined by β and not on β itself.

Definition 3.32. Let the Lie group G be a torus $\mathbb{T}^n$. The action of G on a contact manifold (M, ξ) is completely integrable if it is effective, preserves the contact structure ξ and if $2 \dim G = \dim M + 1$. A **contact toric** G **manifold** is a co-oriented contact manifold (M, ξ) equipped with a completely integrable action of a torus G. Then, the action of G on a component ξ^0_+ of $\xi^0 \setminus \{0\}$ is a completely integrable Hamiltonian action, and therefore ξ^0_+ is a symplectic toric manifold.

Definition 3.33 (**Moment Cone**)**.** Let (M, ξ) be a co-oriented contact manifold equipped with the action of a Lie group G preserving the contact structure ξ and its co-orientation. Also let $\Psi : \xi^0_+ \to \mathfrak{g}^*$ denote the corresponding moment map. The **moment map cone** is defined to be the set

$$C(\Psi) := \Psi(\xi^0_+) \cup \{0\} \tag{3.81}$$

If α is a G-invariant contact form with $\xi = \ker \alpha$ and $\alpha(M) \subset \xi^0_+$ then

$$C(\Psi) = \{t\, f \mid f \in \Psi_\alpha(M)\,,\, t \in [0, \infty)\} \tag{3.82}$$

where $\Psi_\alpha : M \to \mathfrak{g}^*$ is the α-moment map.

4 Applications

4.1 Lens space $L(p, 1)$ moment map image

Let M be $\mathbb{C}^3$ and let the S^1 group N with weights $(1, 1, -p)$ act on it. As we have already proven before, the moment map for this action on M is

$$\mu(z) = -\frac{1}{2}\left(|z_1|^2 + |z_2|^2 - p|z_3|^2\right) + \lambda \tag{4.1}$$

The zero level set is

$$Z = \mu^{-1}(0) = \left\{|z_1|^2 + |z_2|^2 - p|z_3|^2 = C \,|\, (z_1, z_2, z_3) \in \mathbb{C}^3\right\} \tag{4.2}$$

where $C = 2\lambda$. The resulting reduced space $M_r = Z/N$ is

$$M_r = \left\{|z_1|^2 + |z_2|^2 - p|z_3|^2 = C \;,\; (z_1, z_2, z_3) \sim \left(\mathrm{e}^{\mathrm{i}\theta} z_1, \mathrm{e}^{\mathrm{i}\theta} z_2, \mathrm{e}^{-\mathrm{i}p\theta} z_3\right) \mid (z_1, z_2, z_3) \in \mathbb{C}^3\right\} \tag{4.3}$$

The polytope for the non-compact manifold M_r lives in 2 dimensions and has 3 faces. The normals to the faces are

$$v_1 = (-1, 0) \quad , \quad v_2 = (1, -p) \quad , \quad v_3 = (0, -1) \tag{4.4}$$

The Delzant polytope corresponding to these vectors is the space bounded by the lines

$$x' = 0 \quad , \quad y' = 0 \quad , \quad py' = x' - C \tag{4.5}$$

where x' and y' belong in $\mathfrak{h}^*$. According to our previous definitions, the dimension of Z is $\dim Z = d + n = 5$ and the dimension of M_r is $\dim M_r = 2n = 4$. If we now restrict the manifold by fixing $|z_3|$ we will obtain a 3-manifold. Another way to see this is the following. We start from Z

$$r_1^2 + r_2^2 - p\,r_3^2 = C \tag{4.6}$$

where we used the notation $z_i = \mathrm{e}^{\mathrm{i}\varphi_i}$. Now we fix r_3 and we get

$$r_1^2 + r_2^2 = C + p\,r_3^2 = C' \Longrightarrow r_2^2 = C' - r_1^2 \tag{4.7}$$

Therefore we have $r_1, \varphi_1, \varphi_2, \varphi_3$ independent and the dimension of the manifold is 4 so far. Now we need to apply the reduction by the group $N = (1, 1, -p)$ which will bring the dimension down to 3. The reduction induces the equivalence relation

$$\left(r_1\mathrm{e}^{\mathrm{i}\varphi_1}, r_2\mathrm{e}^{\mathrm{i}\varphi_2}, r_3\mathrm{e}^{\mathrm{i}\varphi_3}\right) \sim \left(r_1\mathrm{e}^{\mathrm{i}(\varphi_1+\theta)}, r_2\mathrm{e}^{\mathrm{i}(\varphi_2+\theta)}, r_3\mathrm{e}^{\mathrm{i}(\varphi_3-p\theta)}\right) \tag{4.8}$$

This equivalence relation tells us that all the above the points are to be identified. So we can choose one representative in this class of points. To do this we fix φ_3 and for simplicity we set $\varphi_3 = 0$. But this is not the same as the equivalence relation yet. When $\theta = 2\pi k/p$ for $k = 0, \ldots, (p-1)$, we obtain the same phase for z_3 which means that we over-counted a point p times. Thus we need to further identify points by taking the reduction with respect to the discrete group $\mathbb{Z}_p$ where

$$\mathbb{Z}_p = \left\{\mathrm{e}^{\mathrm{i}2\pi k/p}\,,\ k = 0, \ldots, p-1\right\} \tag{4.9}$$

We then have

$$(z_1, z_2) \sim \left(\mathrm{e}^{2\pi\mathrm{i}k/p} z_1, \mathrm{e}^{2\pi\mathrm{i}k/p} z_2\right) \tag{4.10}$$

To summarize, the initial equivalence relation is equivalent to choosing a representative in the class of points that are to be identified, and if we over-counted, take the reduction by a discrete group. This means that instead of taking a reduction with respect to a continuous variable, we just identify the problematic points. In our case we start from our initial equivalence relation, fix φ_3 and take the reduction only when $\theta = 2\pi k/p$

$$(z_1, z_2, z_3) \sim \left(\mathrm{e}^{2\pi\mathrm{i}k/p} z_1, \mathrm{e}^{2\pi\mathrm{i}k/p} z_2, \mathrm{e}^{-2\pi\mathrm{i}k} r_3\mathrm{e}^{\mathrm{i}\varphi_3}\right) \tag{4.11}$$

In our case r_3 is also fixed, so the third entry is not important for our considerations. So far we have

$$|z_1|^2 + |z_2|^2 = C' \quad , \quad (z_1, z_2) \sim \left(\mathrm{e}^{2\pi\mathrm{i}k/p} z_1, \mathrm{e}^{2\pi\mathrm{i}k/p} z_2\right) \tag{4.12}$$

This is by definition the Lens space $L(p, 1)$. Now we would like to find where this Lens space lies in the moment map picture for M_r. Renaming again $|z_1|^2 = x$, $|z_2|^2 = y$ and $|z_3|^2 = z = \text{const.}$, the constrain that defines $L(p, 1)$ is written as

$$x + y - pz = C \Longrightarrow x + y = C' \Longrightarrow \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = C' \tag{4.13}$$

By inserting in this constraint $\mathbf{A}$ and $\mathbf{A}^{-1}$, where $\mathbf{A}$ was defined previously as

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 0 \\ p & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \tag{4.14}$$

we obtain

$$\begin{pmatrix} x & y & z \end{pmatrix} \mathbf{A}^{-1}\mathbf{A} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = C' \Longrightarrow \begin{pmatrix} x' & y' & -C \end{pmatrix} \begin{pmatrix} 0 \\ p \\ -1 \end{pmatrix} = C' \tag{4.15}$$

or

$$py' + C = C' \Longrightarrow y' = \frac{C' - C}{p} \tag{4.16}$$

Therefore, for any $y' > 0$ and $y' = \text{const.}$, we obtain the Lens space $L(p, 1)$. So $L(p, 1)$ is a horizontal line on the moment map image. This is shown in the figure below.

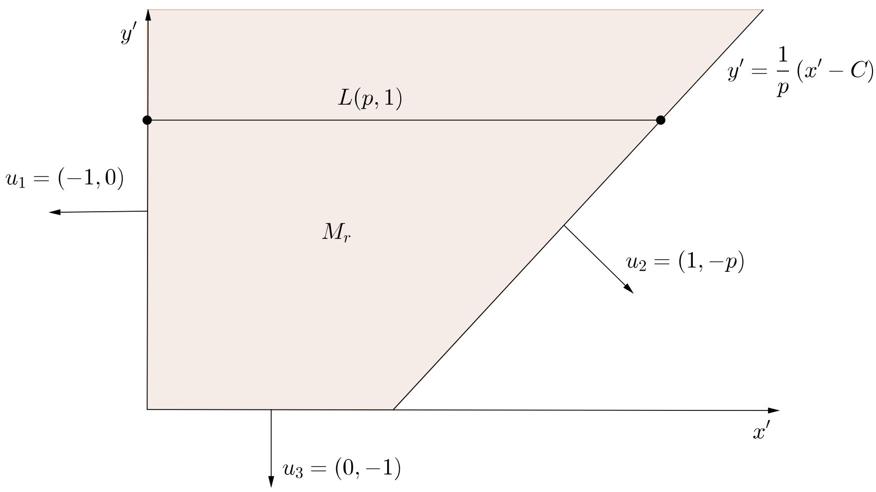

Figure 1: $L(p, 1)$ *moment map image*

It can be easily proven that if instead of $|z_3| = \text{const.}$ we take the restriction

$$|z_1|^2 + |z_2|^2 + |z_3|^2 = 1 \tag{4.17}$$

we obtain the same result; the Lens space defined by $y' = \text{const.}$. We know that geometrically, any line that intersects both $x' = 0$ and $py' = x - C$ will be the Lens space $L(p, 1)$ (if we rotate the moment map image the geometry stays the same. So we can rotate it to make the line of $L(p, 1)$ horizontal with respect to the moment map image). So we want to find when the restriction

$$a|z_1|^2 + b|z_2|^2 + c|z_3|^2 = 1 \tag{4.18}$$

is equivalent to $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$. Using (x, y, z) again and rotating the constrain with $\mathbf{A}$ we find

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 1 \Longrightarrow \begin{pmatrix} x' & y' & -C \end{pmatrix} \begin{pmatrix} b - a \\ pa + c \\ -a \end{pmatrix} = 1 \tag{4.19}$$

Thus, the line defined by

$$(b - a)x' + (pa + c)y' = 1 - a\,C \tag{4.20}$$

must intersect both $x' = 0$ and $py' = x' - C$. We also note that we must have $\gcd(a, b, c) = 1$ in order for (a, b, c) to be able to get rotated to $(1, 1, 1)$ by an $\mathrm{SL}(3, \mathbb{Z})$ transformation.

4.2 Lens space $L(p,q)$ moment map image

Let $M = \mathbb{C}^3$ again, but now $N \cong S^1$ has weights $(1, q, -p)$ where p and q are coprime integers. Its action on M is defined by

$$(z_1, z_2, z_3) \longmapsto \left(e^{i\theta} z_1, e^{iq\theta} z_2, e^{-ip\theta} z_3\right) \tag{4.21}$$

The moment map for the action is

$$\mu(z) = -\frac{1}{2}\left(|z_1|^2 + q|z_2|^2 - p|z_3|^2\right) + \lambda \tag{4.22}$$

and the zero level set is

$$Z = \mu^{-1}(0) = \left\{|z_1|^2 + q|z_2|^2 - p|z_3|^2 = C \,|\, (z_1, z_2, z_3) \in \mathbb{C}^3\right\} \tag{4.23}$$

where $C = 2\lambda$. The resulting reduced space $M_r = Z/N$ is

$$M_r = \left\{|z_1|^2 + |z_2|^2 - p|z_3|^2 = C \ , \ (z_1, z_2, z_3) \sim \left(e^{i\theta} z_1, e^{iq\theta} z_2, e^{-ip\theta} z_3\right) \mid (z_1, z_2, z_3) \in \mathbb{C}^3\right\} \tag{4.24}$$

Here we note that the action of N is not free on Z at any point $(0, z_2, 0)$. Thus the resulting space M_r is an orbifold. Here we will again restrict on $|z_3| = \text{const.}$ and therefore we will not consider the points of the form $(0, z_2, 0)$. The moment map image for M_r is now bounded by the lines

$$x' = 0 \quad , \quad y' = 0 \quad , \quad py' = qx' - C \tag{4.25}$$

as we can easily find by rotating the constrains using

$$\mathbf{A} = \begin{pmatrix} -q & 1 & 0 \\ p & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \tag{4.26}$$

The normal outward pointing vectors are now

$$v_1 = (-1, 0) \quad , \quad v_2 = (q, -p) \quad , \quad v_3 = (0, -1) \tag{4.27}$$

Following the same procedure as before, the resulting space for $|z_3| = \text{const.}$ will be

$$L(p,q) = \left\{|z_1|^2 + |z_2|^2 = C' \mid (z_1, z_2) \sim \left(e^{2\pi i k/p} z_1, e^{2\pi i kq/p} z_2\right)\right\} \tag{4.28}$$

where $k = 0, \ldots, (p-1)$. Now we perform a symplectic cut on M_r which intersects the base of the moment map image and the Lens space. This is shown in the next figure.

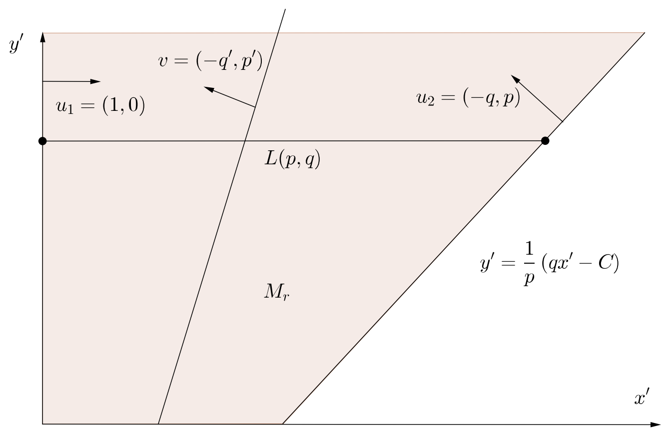

Figure 2: *$L(p,q)$ moment map image*

Lets assume that the normal vector to the cut is $v = (-q', p')$ where p' and q' are coprimes. Then the left part of the image is the Lens space $L(p', q')$ (it works similarly to the cut of the 3-sphere that we investigated before). For the right part of the cut we need to be more careful though. The normal vectors to the cut line and to the right boundary line are v and v_2 and their inner product is $v_2 \cdot v = k$ for some k. If we perform an $\mathrm{SL}(2, \mathbb{Z})$ transformation to rotate v to $(1, 0)$ we must perform the inverse (and transposed) transformation to v_2 in order to keep their inner product fixed. This transformation will also be an $\mathrm{SL}(2, \mathbb{Z})$ transformation and therefore the resulting vector will still have coprime entries. So the right part of the cut will also be some Lens space.

4.3 Outlook

Symplectic cuts on contact manifolds become even more interesting when we move to 5D contact manifolds. Let $M = \mathbb{C}^4$ and $N \cong S^1$ a circle group acting on M. Also let M_r be the manifold that comes from the reduction by N and $C(M_r)$ the moment cone that corresponds to it. The moment cone is 3-dimensional and its facets (planes forming the cone) are 2-dimensional. We then choose a 2D plane that intersects all the facets of the cone and restrict the problem there. This plane represents a 5D contact manifold. For example let $M \cong \mathbb{C}^4$ and N be the circle group with weights $(p + q, p - q, -p, -p)$ where p and q are coprimes and $p > q$. The moment cone, as we saw before, is bounded by four 2-dimensional planes. If we restrict to the plane

$$|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 = 1 \tag{4.29}$$

we obtain a contact manifold whose moment map image intersects the cone. We can now perform symplectic cuts on this manifold as we did for the Lens space. The difference is that now we perform the cuts on a 2D plane instead of a line, and therefore we have much more freedom of choice. Let us consider the cuts that are shown in the next figure.

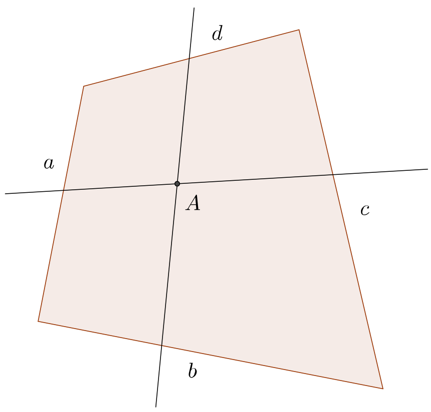

Figure 3: *Cuts on 5D contact manifold*

The line segments a, b, c and d belong to the 4 facets of the cone. These are Lens spaces which is something that we can deduce by looking at the previous application (each facet of the cone is a 2D cone and a restriction to a line would potentially give us a Lens space). The two symplectic cuts that we can see in the figure intersect at the point A and produce 4 contact manifolds. It is of great interest to investigate what exactly happens at the point A. This is something that could not happen in the case of a 3D contact manifold.

A Lens Spaces

Lens spaces are a class of 3-manifolds, but they can also be defined for higher dimensions.

Definition A.1. A Lens space $L(p,q)$ is the 3-manifold obtained by gluing the boundaries of two solid tori together such that the meridian of the first goes to a (p,q)-curve on the second. A (p,q)-curve is a curve that wraps around the longitude p times and around the meridian q times.

Lens spaces can also be defined as the quotients of the 3-sphere by the discrete group $\mathbb{Z}_p$.

Definition A.2. The 3-dimensional Lens spaces $L(p,q)$ are quotients of S^3 by $\mathbb{Z}_p$ actions. Let p and q be coprime integers and consider S^3 as the unit sphere in $\mathbb{C}^2$

$$S^3 = \left\{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\right\} \tag{A.1}$$

Then the action of $\mathbb{Z}_p$ on S^3 is

$$(z_1, z_2) \longmapsto \left(\mathrm{e}^{2\pi\mathrm{i}/p} z_1, \mathrm{e}^{2\pi\mathrm{i}q/p} z_2\right) \tag{A.2}$$

is free, as p and q are coprime. The resulting quotient space is called Lens space $L(p,q)$

$$L(p,q) = S^3/\mathbb{Z}_p \tag{A.3}$$

This definition can be easily generalized to any higher odd dimension.

Definition A.3. Let $p, q_1, \ldots, q_n$ be integers such that all q_i are coprime to p and consider S^{2n-1} as the unit sphere in $\mathbb{C}^n$. The Lens space $L(p, q_1, \ldots, q_n)$ is the quotient of S^{2n-1} by the free $\mathbb{Z}_p$-action generated by

$$(z_1, \ldots, z_n) \longmapsto \left(\mathrm{e}^{2\pi\mathrm{i}q_1/p} z_1, \ldots, \mathrm{e}^{2\pi\mathrm{i}q_n/p} z_n\right) \tag{A.4}$$

Therefore, in 3-dimensions we have $L(p, 1, q)$.

Now we explore the glued tori model for $L(p,q)$ a little further. Let $p, q \in \mathbb{Z}$ be coprime and pick $m, n \in \mathbb{Z}$ such that $mq - np = 1$. This means that the matrix

$$\mathbf{A} = \begin{pmatrix} m & p \\ n & q \end{pmatrix} \tag{A.5}$$

has $\det \mathbf{A} = 1$. Let $S^1 \times D^2$ be a solid torus thought of as a subset of $\mathbb{C}^2$

$$S^1 \times D^2 = \left\{(z, w) \in \mathbb{C}^2 \mid |z| = 1\,,\ |w| \leq 1\right\} \tag{A.6}$$

The boundary of the solid torus is a regular torus and is

$$T^2 = \partial\left(S^1 \times D^2\right) = S^1 \times \partial D^2 = S^1 \times S^1 \tag{A.7}$$

T^2 can also be described as a subset of $\mathbb{C}^2$. We have

$$T^2 = \left\{(z, w) \in \mathbb{C}^2 \mid |z| = |w| = 1\right\} \tag{A.8}$$

Now let U and V be two solid tori and let $\varphi_A : \partial U \to \partial V$ be the map

$$\varphi_A(t, s) = (t^m s^p, t^n s^q) \tag{A.9}$$

associated to the matrix $\mathbf{A}$. The Lens space $L(p,q)$ is the identification space of the disjoint union $U \sqcup V$ with respect to the partition P given by

$$P = \Big\{\{x\}, \{y\}, \{z, \varphi_A(z)\} \mid x \in \mathring{U}, y \in \mathring{V}, z \in \partial U\Big\} \tag{A.10}$$

which means that $L(p,q)$ is obtained by gluing two solid tori U and V along their boundary $\partial U \cong T^2 \cong \partial V$ using φ_A as the gluing map.

Example A.4. We take $p = 0$ and $q = 1$ and use the solid tori model. $L(0,1)$ is the result of gluing together two copies of a solid torus T together via some homeomorphism $h : \partial T \to \partial T$ which makes a meridian into a $(0,1)$ torus knot. This is of course just a meridian, so h can be chosen as the identity $i : \partial T \to \partial T$. Two meridians glued together along their border give an S^2. Doing this for all meridians (considering all gluing along the longitude S^1) we finally obtain

$$L(0,1) \cong S^1 \times S^2 \tag{A.11}$$

Example A.5. Now we choose $p = 2$ and $q = 1$. Using the quotient definition $L(2,1)$ is written as

$$L(2,1) = S^3/\mathbb{Z}_2 \tag{A.12}$$

where the action of $\mathbb{Z}_2$ is defined as

$$(z_1, z_2) \longmapsto \left(\mathrm{e}^{\pi \mathrm{i}} z_1, \mathrm{e}_2^{\pi \mathrm{i}}\right) \tag{A.13}$$

Therefore, two points z and z' are equivalent

$$(z_1', z_2') \sim (z_1, z_2) \tag{A.14}$$

if they are related by

$$(z_1', z_2') = \left(\mathrm{e}^{m\pi \mathrm{i}} z_1, \mathrm{e}^{m\pi \mathrm{i}} z_2\right) \quad \text{for} \quad m = 0, 1 \tag{A.15}$$

So the points z and $-z$ on the sphere are identified together

$$(z_1, z_2) \sim (-z_1, -z_2) \tag{A.16}$$

This means that $L(2,1)$ is S^3 with antipodal points identified. By definition, this is the 3-dimensional real projective space

$$L(2,1) \cong \mathbb{RP}^3 \tag{A.17}$$

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