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Bundles & Gauges, a Math-Physics Duality

the case of Gravity

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Abstract

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- the case of Gravity -

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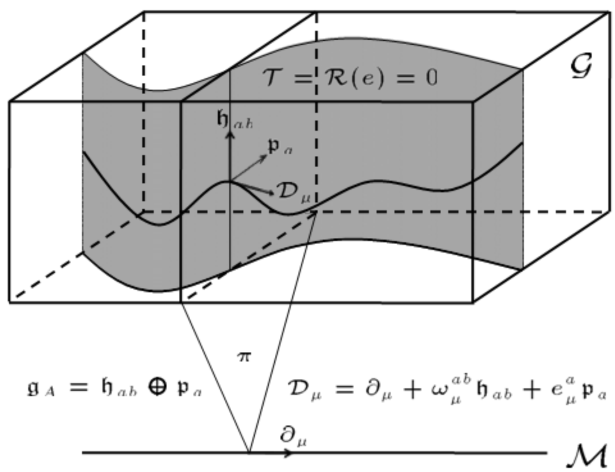

Master thesis

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Abstract

A modern and straight forward summary of the necessary tools and concepts needed to understand and work with gauge theory in a fibre bundle formalism. Due to the aim of being a quick but thorough introduction full derivations are rarely included, but references to such are given where they have been omitted. General Relativity, although being a geometric theory, in the sense that the gravitational force is described by the curvature of space-time, may not be *derived* from geometry like the other fundamental forces as in Yang-Mills theory. Thus, a possibility of unification lies in a geometrical *derivation* of gravity from gauge principles. By applying the presented formalism to the case of Gravity such a derivation is pursued along the lines of nonlinear realizations of the gauge group.

Contents

1 introduction

Before any proper introduction and historical background are given we first introduce the two main areas which will harmonize in this thesis, namely the theory of fibre bundles and the concept of gauge theories. Since fibre bundles is an entirely mathematical framework and the sole purpose of gauge theories is to describe reality, the two subjects are introduced as the separate entities they really are. The title of this thesis, indicating their correspondence, will become clear as the reader moves on.

Firstly the concept of manifolds is briefly mentioned because fibre bundles will be introduced as a natural generalization of that concept. Throughout this thesis all manifolds are assumed to be differentiable and Haussdorf and all maps are assumed to be smooth.

1.1 manifold

From a geometrical point of view we may start the mathematical representation of our physical world with the notion of a **manifold** $\mathcal{M}$. This manifold represents space, or at least the part of space we wish to include in our description. Examples of 2-dimensional manifolds are the ordinary cylinder and the 2-sphere S^2.

Although the manifold may have a very complicated topology globally, its defining property is to be *locally* homeomorphic to $\mathbb{R}^m$, m being the dimension of our manifold. The *homeomorphism* φ_i from $\mathcal{M}$ to an open subspace U_i of $\mathbb{R}^m$ in called a **chart**

$$\varphi_i \colon \mathcal{M} \to U_i \subset \mathbb{R}^m \tag{1.1}$$

and allows us to assign coordinates to the manifold by using those of U_i, which we may choose freely. If the manifold globally differs in topology from $\mathbb{R}^m$ we need to apply a set of charts $\{U_i, \varphi_i\}$, called an **open covering** of $\mathcal{M}$, such that the whole of $\mathcal{M}$ is covered and transitions between any two charts U_i and U_j is carried out by differentiable, or *smooth*, **transition functions** ψ_{ij} defined by

$$\psi_{ij} = \varphi_i \circ \varphi_j^{-1} \colon U_j \to U_i \tag{1.2}$$

The transition functions need to be defined wherever there is an overlap between two charts $U_i \cap U_j \neq \emptyset$ and need to obey the following consistency conditions

$$\begin{aligned} \psi_{ii} &= id_{U_i} \\ \psi_{ij} &= \psi_{ji}^{-1} \\ \psi_{ik} &= \psi_{ij} \circ \psi_{jk} \end{aligned} \tag{1.3}$$

where id_{U_i} is the *identity map* on U_i. The third and most restrictive of the consistency conditions, called the *cocycle condition*, apply if there is an overlap between three charts $U_i \cap U_j \cap U_k \neq \emptyset$. Considering our example-manifolds, one chart is enough to cover the cylinder while the sphere needs two charts to avoid coordinate singularities at the poles.

With a satisfying representation of space we now need a framework with additional complexity to describe forces, particles and other things we wish to incorporate in our description. Just like a plane may contain more structure than a line one might argue that the most natural way to achieve additional structure is to assign an additional manifold to every point of the manifold $\mathcal{M}$. The mathematics of such a structure is called the theory of *fibre bundles*. Fibre bundles provide a framework to naturally incorporate *gauge theories*, possibly covering all the complexity of our physical world.

1.2 fibre bundle

The manifold representing space $\mathcal{M}$ is called the **base** while the manifold defined at each point of $\mathcal{M}$ are identical copies of a manifold $\mathcal{F}$, called the **fibre**. Furthermore, all these manifolds together are made to form a **total space** which is also a manifold, denoted by $\mathcal{U}$.

To make the total space a well defined manifold several maps have to be introduced, one of which is a *surjection* $\pi\colon \mathcal{U} \to \mathcal{M}$ called the **projection**. The subset of elements $\{u\} \in \mathcal{U}$ which are projected down to a specific point $p \in \mathcal{M}$ is called the *fibre at* p, denoted $\mathcal{F}_p$

$$\pi^{-1}(p) = \mathcal{F}_p \subset \mathcal{U} \tag{1.4}$$

Note that although the fibres at different points are all isomorphic to the typical fibre $\mathcal{F}$ they consist of elements $u \in \mathcal{U}$ wheres the typical fibre is a manifold of its own with elements $f \in \mathcal{F}$.

Exactly like we needed charts $\{U_i, \varphi_i\}$ to perform calculations on $\mathcal{M}$ we need a set of diffeomorphisms to *locally* map $\mathcal{U}$ onto the direct product $U_i \times \mathcal{F}$. This is accomplished by the *inverse* of the **local trivializations**, the local trivialization is defined the other way around:

$$\phi_i\colon U_i \times \mathcal{F} \to \pi^{-1}(U_i) \subset \mathcal{U} \tag{1.5}$$

so that $\phi_{i,p}(f) \mapsto \pi^{-1}(p)$ where $\phi_{i,p}$ is the restriction of the local trivialization to $p \in U_i$.

In addition, exactly like we needed smooth coordinate transformations from U_i to U_j where they overlap $U_i \cap U_j$ we now introduce the **transition functions**

$$t_{ij}(p) \equiv \phi_{i,p}^{-1} \circ \phi_{j,p} \tag{1.6}$$

to smoothly paste the direct products $\{U_i \times \mathcal{F}\}$ together, forming a covering of the total space $\mathcal{U}$. The transition functions are elements of a *Lie group* $\mathcal{G}$ called the **structure group** which acts on $\mathcal{F}$ on the left. The given definitions form a **fibre bundle** $(\mathcal{U}, \pi, \mathcal{M}, \mathcal{F}, \mathcal{G})$, also denoted $\mathcal{U} \xrightarrow{\pi} \mathcal{M}$.

In addition to the definition of the fibre bundle and its constituents we also want to define a *local* map called a **section** s_i from $U_i \subset \mathcal{M}$ to the total space

$$s_i \colon U_i \to \mathcal{U} \quad \text{such that} \quad \pi \circ s_i = id_{U_i} \tag{1.7}$$

The set of sections on U_i is denoted $\Gamma(U_i, \mathcal{U})$, *in some cases* the section may be extended to the whole manifold and is then referred to as a **global section** $s \in \Gamma(\mathcal{M}, \mathcal{U})$, but in general there are topological obstructions to this extension. Note that by U_i we refer to both the subset of $\mathbb{R}^m$ and the corresponding subset of the manifold $\mathcal{M}$.

1.3 gauge theory

Consider the simplest action for a charged complex scalar field

$$S = \int d^4x (\partial_\mu \Psi \partial^\mu \bar{\Psi} - m^2 \Psi \bar{\Psi}) \tag{1.8}$$

This action remains invariant under multiplication of the fields by a complex constant since Ψ and $\bar{\Psi}$ occurs in pairs in the *Lagrangian*. A *Lie group* whose action leaves the Lagrangian invariant is called a **symmetry group**. In the example considered the symmetry group is $U(1)$, causing the transformations

$$\Psi \to e^{i\Lambda} \Psi \quad , \quad \bar{\Psi} \to e^{-i\Lambda} \bar{\Psi} \tag{1.9}$$

where Λ is the **transformation parameter** of the Lie group.

So far this is a **global symmetry** of the action since the group operation is performed identically everywhere. If the action remains invariant under a *space-time dependent* group operation the symmetry is promoted to a **local symmetry**. In our example this corresponds to making the transformation parameter space-time dependent, $\Lambda \to \Lambda(x)$, and this is where the story of gauge theories begins.

A **gauge theory** is a *field theory* where the action remains invariant under *local* transformations of the symmetry group, which in this case is referred to as the **gauge group**. The transformations under the action of the gauge group are called **gauge transformations** and invariance under such transformations is called **gauge invariance**.

But our action (1.8) is not invariant under gauge transformations since a local group action fails to commute with the differential operators

$$\partial_\mu e^{i\Lambda(x)} \Psi \;\neq\; e^{i\Lambda(x)} \partial_\mu \Psi \tag{1.10}$$

This is compensated for by introducing a new field A_μ called a **gauge field**, and replacing the ordinary derivatives ∂_μ by **covariant derivatives** $\mathcal{D}_\mu$ defined by

$$\mathcal{D}_\mu \Psi \equiv (\partial_\mu - iA_\mu)\Psi \tag{1.11}$$

such that the action of this new derivative transform covariantly under the gauge transformations $\mathcal{D}_\mu \to e^{i\Lambda(x)}\mathcal{D}_\mu$. That is; the change of the fields transform in the same way as the fields themselves, and we now have

$$\mathcal{D}_\mu e^{i\Lambda(x)}\Psi = e^{i\Lambda(x)}\mathcal{D}_\mu \Psi \tag{1.12}$$

In order to satisfy this condition it can be shown that A_μ need to transform as

$$A_\mu \to A_\mu + \partial_\mu \Lambda(x) \tag{1.13}$$

under gauge transformations. Rather than manually specifying the values of the gauge field A_μ one introduces an additional, gauge invariant, term in the Lagrangian

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{1.14}$$

where $F^{\mu\nu} \equiv \partial_{[\mu}A_{\nu]}$ is the **field strength** of the electromagnetic force. The *equations of motion* derived from this term gives the field equations for A_μ, specifying its values to be used in the covariant derivative $\mathcal{D}_\mu$. The new *locally* gauge invariant action becomes

$$S = \int d^4x(\mathcal{D}_\mu \Psi \mathcal{D}^\mu \bar{\Psi} - m^2 \Psi \bar{\Psi} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}) \tag{1.15}$$

Much in the same way the demand for local gauge invariance naturally leads to the introduction of the other fundamental forces in the standard model, a remarkable aspect of gauge theories!

2 fibre bundles & gauge theory

The first widely recognized gauge theory was formulated by Wolfgang Pauli in the 1940s.[1] The formulation was based on work by Weyl, Fock, Klein and others where the symmetry properties of the electromagnetic field had been related to the $U(1)$ group. Pauli later worked out a generalization of the theory to higher dimensional internal spaces in the search for a unification including the nuclear forces. But he abandoned the idea because he saw no way to give masses to the corresponding gauge bosons.

In early 1950s Chen Ning Yang and Robert Mills independently of Pauli formulated what would be recognized as Yang-Mills (YM) theory.[2] Their theory extended the gauge concept to the case of non-abelian gauge groups, in particular $SU(n)$. It took until the year of 1960 before the theory found applications in particle physics due to the concept of particles acquiring mass through spontaneous symmetry breaking. After this breakthrough YM-theory proved successful in the formulation of both electroweak unification and quantum chromodynamics.

The theory of fibre bundles is purely mathematical but we will study the theory from a physical point of view, throughoutly relating the mathematical objects to the physics they represent in the context of gauge theories. For full derivations, we refer to M. Nakahara (1990), which is the main source of the material presented in this chapter. However, our aim is to give a more friendly and straight to the point presentation, where *the point* is gauge theory applications.

Before any mathematical treatment a justification of the 'duality' between gauge theory and fibre bundles may be given in terms of a short historical remark concerning the following statement by Mayer:

> A reading of the Yang-Mills paper shows that the geometric meaning of the gauge potentials must have been clear to the authors, since they use the gauge covariant derivative and the curvature form of the connection, and indeed, the basic equations in that paper will coincide with the ones derived from a more geometric approach...[3]

But a reply by Yang falsifies this conclusion:

> What Mills and I were doing in 1954 was generalizing Maxwell's theory. We knew of no geometrical meaning of Maxwell's theory, and we were not looking in that direction.[4]

The interesting and amusing fact that Mayer were convinced that Yang and Mills were using fibre bundles although they were just generalizing gauge theory shows the striking correspondence between the two concepts!

Realizing the gauge theories strong connection with geometry Yang consulted the mathematician Jim Simons who pointed Yang in the direction of fibre bundles. But when Yang tried to familiarize himself with the subject he found that 'The language of modern mathematics is too cold and abstract for a physicist' and invited Simons to give him and his colleagues lectures on fibre bundles. Simons kindly accepted and gave a series of lectures on the subject which we will review in this chapter.

2.1 vector bundles & fields

Given an m-dimensional manifold $\mathcal{M}$ a **vector bundle** $\mathcal{V} \xrightarrow{\pi} \mathcal{M}$ is a fibre bundle whose fibre is a vector space. The prime example is the *tangent bundle* $\mathcal{TM}$ where the fibre is a copy of $\mathbb{R}^m$ defined at each point of $\mathcal{M}$ and the structure group is $GL(m, \mathbb{R})$. Specifying a point in $\mathbb{R}^m$ defines a vector, hence sections of $\mathcal{TM}$ are smooth vector fields on $\mathcal{M}$ which we denote by $\mathcal{X}(\mathcal{M}) = \Gamma(\mathcal{M}, \mathcal{TM})$.

Sections on vector bundles also pointwisely obey the usual rules for vector addition and multiplications with scalars, scalars being smooth functions $f(p)$ on $\mathcal{M}$. Corresponding to the null vector is the **null section** s_0 with the property $\phi_i^{-1}(s_0(p)) = (p, 0)$ in any local trivialization.

In the theory of fibre bundles a complex scalar field $\Psi(x)$ defined on $U_i \subset M$ is represented by a local section of a complex line bundle $s_i \in \Gamma(U_i, \mathbb{C})$. More generally, any field or wave function may be represented by a section of a vector bundle. In the following we shall describe how the gauge transformations of such a field is incorporated in the theory.

2.2 principal bundles & gauge transformations

A **principal bundle** $\mathcal{P} \xrightarrow{\pi} \mathcal{M}$ or $\mathcal{P}(\mathcal{M}, \mathcal{G})$ is a fibre bundle whose fibre is identical to its structure group $\mathcal{G}$, thus $\mathcal{P}(\mathcal{M}, \mathcal{G})$ is often referred to as a **$\mathcal{G}$-bundle** on $\mathcal{M}$. On a principal bundle it is possible to introduce the *right action* of $\mathcal{G}$ on $\mathcal{P}$ defined by

$$\mathcal{P} \times \mathcal{G} \to \mathcal{P} \quad \text{such that} \quad ug = \phi_i(p, f_i g) \qquad \forall \begin{array}{l} g \in \mathcal{G} \\ u \in \pi^{-1}(p) \end{array} \tag{2.1}$$

This is true in any local trivialization ϕ_i and corresponding $f_i \in \mathcal{F}$ since the right action commutes with the left action

$$\phi_i(p, f_i g) = \phi_i(p, t_{ij} f_j g) = \phi_j(p, f_j g) \tag{2.2}$$

Principal bundles play a crucial role in the description of gauge theories since the fibre of $\mathcal{P}(\mathcal{M}, \mathcal{G})$ is identified with the *symmetry group* of the gauge theory, and the right action on $\mathcal{P}$ is identified with the corresponding gauge transformations. We will now see how this is done.

Gauge transformations of a complex scalar field $\Psi(p)$ defined on a manifold $\mathcal{M}$ are represented by sections on a principal bundle $\mathcal{P} \xrightarrow{\pi} \mathcal{M}$ with typical fibre $U(1)$. Since a section is a map $\sigma\colon \mathcal{M} \to \mathcal{P}$ the value of a section at a point $\sigma(p)$ corresponds to an element of $U(1)$ through a local trivialization

$$\sigma(p) = \phi_i\big(p, g(p)\big) \qquad g \in U(1) \tag{2.3}$$

To be able to represent the action of the identity element $e \in \mathcal{G}$ across the whole base manifold we need to introduce what is called the **canonical local trivialization** ϕ_i^0. The canonical trivialization is given *with respect to a local section* σ_i by

$$\sigma_i(p) = \phi_i^0(p, e) \tag{2.4}$$

Unless the principal bundle is a direct product space $\mathcal{P} = \mathcal{M} \times \mathcal{F}$ such sections may only be defined locally. Once a canonical trivialization is defined all other local sections $\tilde{\sigma}_i(p)$ may be expressed naturally in terms of the original section $\sigma_i(p)$ and a right action $g_i(p) \in \mathcal{G}$ in the following way

$$\tilde{\sigma}_i(p) = \sigma_i(p)g_i(p) = \phi_i^0(p, e)g_i(p) = \phi_i^0(p, g_i) \tag{2.5}$$

The different sections $\tilde{\sigma}_i$ correspond to different gauges and the gauge transformations between them are carried out by the right action g_i of the structure group, $U(1)$ in our case. If g_i is constant the transformation is global, while a local gauge transformation is carried out by a point dependent action $g_i(p)$, which we may represent as $e^{i\Lambda(x)}$ given a chart and a coordinate system.

2.3 associated bundles & field transformations

Having seen how sections on vector bundles represent fields, and how sections on principle bundles may describe gauge transformations, we now present how a principle bundle and a vector bundle may be *associated* in a way which naturally let the gauge transformations act on the fields.

Consider a $\mathcal{G}$-bundle $\mathcal{P} \xrightarrow{\pi} \mathcal{M}$ and a manifold $\mathcal{V}$. A suitable representation ρ of $\mathcal{G}$ allows the group to act on $\mathcal{V}$, for example by some matrix representation. A right action may now be defined on elements (u, v) in the product space $\mathcal{P} \times_\rho \mathcal{V}$ by

$$(u, v) \to \big(ug, \rho(g^{-1})v\big) \quad \text{for} \quad \begin{array}{c} g \in \mathcal{G} \\ u \in \mathcal{P} \\ v \in \mathcal{V} \end{array} \tag{2.6}$$

the **associated vector bundle** $\mathcal{E}$ is then defined by identifying the points related by such a right action

$$(u, v) \sim \big(ug, \rho(g^{-1})v\big) \quad \Longrightarrow \quad (ug, v) = \big(u, \rho(g)v\big) \quad \forall g \in \mathcal{G} \tag{2.7}$$

We denote the elements of $\mathcal{E}$ by $[(u, v)]$ and introduce the projection $\pi_{\mathcal{E}}\colon \mathcal{E} \to \mathcal{M}$ defined by $\pi_{\mathcal{E}}(u, v) = \pi(u)$. Then $\mathcal{E} \xrightarrow{\pi_{\mathcal{U}}} \mathcal{M}$ is a fibre bundle with the same structure group $\mathcal{G}$ as its associated principal bundle $\mathcal{P}(\mathcal{M},\mathcal{G})$. The transition functions t_{ij} of course needs to be considered in the chosen representation $\rho(t_{ij})$.

Consider the action

$$S = \int d^4x(\partial_\mu\Psi\partial^\mu\bar{\Psi} - m^2\Psi\bar{\Psi}) \tag{2.8}$$

were the complex scalar field $\Psi(p)$ is described by a section $s(p)$ on an associated vector bundle $(\mathcal{E},\pi_{\mathcal{E}},\mathcal{M},\mathbb{C},U(1))$ i.e. the manifold part $\mathcal{V}$ is a complex line bundle $\mathbb{C}$. Take a local section $\sigma_i(p)$ on the principal bundle $\mathcal{P}(\mathcal{M},U(1))$ associated with $\mathcal{V}$ and employ the canonical trivialization ϕ_i^0 so that

$$(\phi_i^0)^{-1}\big(\sigma_i(p)\big) = (p,e) \qquad \forall p \in U_i \tag{2.9}$$

which makes $\sigma_i(p)$ correspond to e, the identity element of $U(1)$. We now define a space-time dependent group action expressed by a section as $\tilde{\sigma}_i(p) = \sigma_i(p)g_i(p)$, where $g_i \in U(1)$. To make $\tilde{\sigma}_i$ act on $\Psi(p)$ we choose the representation $\rho\colon g_i(p) \to e^{i\Lambda(p)}$ and define a base section $\boldsymbol{e}$ on the associated vector bundle

$$\boldsymbol{e} = \big[\big(\sigma_i(p)\,,\,1\big)\big] \quad \in \quad \mathcal{E} = \mathcal{P}\times_\rho\mathcal{V} \tag{2.10}$$

where 1 has been chosen as the basis vector in the complex line bundle. We express our field $\Psi(p)$ as a section in that basis

$$\Psi(p)\boldsymbol{e} = \big[\big(\sigma_i(p)\,,\,\Psi(p)\big)\big] \quad , \quad \Psi \in \mathcal{V} \tag{2.11}$$

$\tilde{\sigma}(p)$ now corresponds to a local gauge transformation according to

$$\begin{aligned}\Psi'(p)\boldsymbol{e} &\equiv \big[\big(\tilde{\sigma}(p)\,,\,\Psi(p)\big)\big] = \big[\big(\sigma_i(p)g_i(p)\,,\,\Psi(p)\big)\big] \\ &\overset{\rho}{=} \big[\big(\sigma_i(p)\,,\,e^{i\Lambda(p)}\Psi(p)\big)\big] = e^{i\Lambda(p)}\Psi(p)\boldsymbol{e}\end{aligned} \tag{2.12}$$

where $\Psi'(p)$ is the transformed field.

In this way sections on a principal bundle $\mathcal{P}$ can represent gauge transformations on a field Ψ, which in turn is represented by a section on a vector bundle associated with $\mathcal{P}$.

2.4 connection one-forms & gauge potentials

So far we have left the sections describing the field and its gauge transformations completely arbitrary. To approach our initial definition of a gauge theory, with added gauge fields to ensure the gauge invariance of the Lagrangian, we introduce a connection one-form $\boldsymbol{\omega}$ on the principal bundle, called an Ehresmann connection, which we will connect to the gauge potential A_μ presented earlier.

First we introduce the **fundamental vector field**, denoted $\boldsymbol{a}$, which is generated by an element a in the Lie algebra $\mathfrak{g}$ of the $\mathcal{G}$-bundle $\mathcal{P}(\mathcal{M},\mathcal{G})$

$$\boldsymbol{a}f(u) = \frac{d}{dt}f(ue^{ta})\Big|_{t=0} \tag{2.13}$$

where ue^{ta} defines a curve in $\mathcal{P}$ which lies entirely within $\mathcal{G}_p$ since $\pi(u) = \pi(ue^{ta}) = p$. Hence the above equation defines $\boldsymbol{a}$ to be tangent to $\mathcal{G}_p$ at every point $u \in \mathcal{P}$.

Consider the tangent space $\mathcal{T}_u\mathcal{P}$ of a point $u \in \mathcal{P}$ and a separation of $\mathcal{T}_u\mathcal{P}$ into a vertical $\mathcal{V}_u\mathcal{P}$ and a horizontal $\mathcal{H}_u\mathcal{P}$ subspace

$$\mathcal{T}_u\mathcal{P} = \mathcal{V}_u\mathcal{P} \oplus \mathcal{H}_u\mathcal{P} \tag{2.14}$$

so that every vector $\boldsymbol{x} \in \mathcal{T}_u\mathcal{P}$ may be decomposed into a vertical $\boldsymbol{x}^V$ and a horizontal $\boldsymbol{x}^H$ part

$$\boldsymbol{x} = \boldsymbol{x}^V + \boldsymbol{x}^H. \tag{2.15}$$

The vertical direction is canonically defined by $\boldsymbol{a}$, but the horizontal direction is defined only through the *choice* of a connection. The **connection one-form** is a Lie algebra valued one-form $\boldsymbol{\omega} \in \mathfrak{g} \otimes \Omega^1(\mathcal{P})$ which defines such a separation by projecting elements in $\mathcal{T}_u\mathcal{P}$ onto $\mathfrak{g} \simeq \mathcal{V}_u\mathcal{P}$. This projection is unique under the following requirements

1. $\boldsymbol{\omega}(\boldsymbol{a}) = a \qquad a \in \mathfrak{g}$
2. $R_g\boldsymbol{\omega} = a\mathbf{d}_{g^{-1}}\boldsymbol{\omega} \equiv g^{-1}\boldsymbol{\omega}g \qquad g \in \mathcal{G}$

where R_g is the right action by g and $\mathbf{d}_{g^{-1}}$ is the adjoint action defined as above.

Given an open covering $\{U_i\}$ of $\mathcal{M}$ and local sections σ_i the *pullback* by this section σ_i^* may be used to define a *local* connection $\mathcal{A}_i \in \mathfrak{g} \otimes \Omega^1(U_i)$ corresponding to $\boldsymbol{\omega}$ by

$$\mathcal{A}_i \equiv \sigma_i^*\boldsymbol{\omega} \tag{2.16}$$

This local form of the Ehresmann connection $\boldsymbol{\omega}$ is identified with the **gauge potential** up to some Lie algebra factor. Since $\mathcal{A}_i$ is a $\mathfrak{g}$-valued one-form it is possible to expand it in the dual basis basis dx^μ and in terms on the *Lie algebra generators* $\mathfrak{g}_a$

$$\mathcal{A}_i = \mathcal{A}_{i\mu}dx^\mu = A_{i\mu}^{\;\;a}\mathfrak{g}_a dx^\mu \tag{2.17}$$

Given $\mathcal{A}_i$ on the whole open covering $\{U_i\}$ of $\mathcal{M}$ the global connection $\boldsymbol{\omega}$ may be constructed, but for $\boldsymbol{\omega}$ to be *uniquely* defined throughout $\mathcal{P}$ we must have $\boldsymbol{\omega}_i = \boldsymbol{\omega}_j$ on $U_i \cap U_j$. It can be shown[5] that this constraint forces the local connections to transform as

$$\mathcal{A}_j = t_{ij}^{-1}\mathcal{A}_i t_{ij} + t_{ij}^{-1}\mathbf{d}(t_{ij}) \tag{2.18}$$

where $\mathbf{d}$ is the *de Rham differential*. On a $U(1)$-bundle where the transition functions are just complex numbers $t_{ij} = e^{i\Lambda(p)}$ the transformation of $\mathcal{A}_i$ may be written in components as

$$\mathcal{A}_{j\mu} = e^{-i\Lambda(p)}\mathcal{A}_{i\mu}e^{i\Lambda(p)} + e^{-i\Lambda(p)}\partial_\mu(e^{i\Lambda(p)}) = \mathcal{A}_{i\mu} + i\partial_\mu\Lambda(p) \tag{2.19}$$

where $\mathcal{A}_i$ can be recognized as the gauge potential for the electromagnetic field.

The local form of the connection depend on the choice of local section σ_i where the choice of section correspond to a particular gauge. If two sections σ_i and $\tilde{\sigma}_i$ are related by an element $g_i \in \mathcal{G}$ as $\tilde{\sigma}_i = \sigma_i g_i$ their corresponding connections are related as

$$\tilde{\mathcal{A}}_i = g_i^{-1} \mathcal{A}_i g_i + g_i^{-1} \mathbf{d}(g_i) \tag{2.20}$$

which in local coordinates becomes the gauge transformations of the gauge potential. Note that $\mathcal{A}_i$ is defined only locally on a chart U_i and any global information about the bundle is contained in the global connection $\boldsymbol{\omega}$ defined on $\mathcal{P}$, alternatively in $\{\mathcal{A}_i\}$, a 'covering' of local connections.

2.5 field strengths & curvatures

The exterior covariant derivative $\mathbf{d}_\omega$ of a general vector valued r-form $\boldsymbol{\phi} \in \Omega^r(\mathcal{P}) \otimes \mathcal{V}$ acting on vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{r+1} \in \mathcal{T}_u\mathcal{P}$ may be defined as

$$\mathbf{d}_\omega \boldsymbol{\phi}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{r+1}) \equiv \mathbf{d}\boldsymbol{\phi}(\boldsymbol{x}_1^H, \ldots, \boldsymbol{x}_{r+1}^H) \tag{2.21}$$

where $\boldsymbol{x}^H \in \mathcal{H}_u\mathcal{P}$ is the horizontal part of $\boldsymbol{x}$.

Now the definition of the **curvature two-form** $\Omega \in \Omega^2(\mathcal{P}) \otimes \mathfrak{g}$ is given as the exterior covariant derivative of the connection one-form $\boldsymbol{\omega}$

$$\Omega \equiv \mathbf{d}_\omega \boldsymbol{\omega} \tag{2.22}$$

from which one may derive the **structure equation**[7]

$$\Omega(\boldsymbol{x}, \boldsymbol{y}) = \mathbf{d}\boldsymbol{\omega}(\boldsymbol{x}, \boldsymbol{y}) + [\boldsymbol{\omega}(\boldsymbol{x}), \boldsymbol{\omega}(\boldsymbol{y})] \qquad \boldsymbol{x}, \boldsymbol{y} \in \mathcal{T}_u\mathcal{P} \tag{2.23}$$

where the bracket for vector valued forms α and β is an extension of the Lie bracket $[\boldsymbol{x}, \boldsymbol{y}]_\mathfrak{g}$ defined by

$$[\alpha \otimes \boldsymbol{x}, \beta \otimes \boldsymbol{y}] \equiv \alpha \wedge \beta \otimes [\boldsymbol{x}, \boldsymbol{y}]_\mathfrak{g} \tag{2.24}$$

Just like the connection one-form the curvature two-form has a local description $\mathcal{F}_i$ defined via the pull-back of a local defining section σ_i

$$\mathcal{F}_i = \sigma_i^* \Omega \tag{2.25}$$

where $\mathcal{F}_i$ is in one-to-one correspondence with the **field strength** in the context of gauge theories. Being defined in a vector bundle the structure equation for $\mathcal{F}_i$ may be written

$$\mathcal{F}_i = \mathbf{d}\mathcal{A}_i + \mathcal{A}_i \wedge \mathcal{A}_i \tag{2.26}$$

Under gauge transformations and transition between overlapping charts U_i and U_j the field strength transform in the adjoint representation

$$\mathcal{F}_i = g_i^{-1}\mathcal{F}_i g_i \tag{2.27}$$

$$\mathcal{F}_j = t_{ij}^{-1}\mathcal{F}_j t_{ij} \tag{2.28}$$

The symmetry in the relations stems from the fact that both operations essentially are a change of defining section σ_i on the principal bundle. As $\mathcal{A}_i$ and $\mathcal{F}_i$ always will denote local objects we drop the i-index from here on.

From differential geometry we know that a two-form may be expanded in a local coordinate basis as

$$\mathcal{F} = \frac{1}{2}\mathcal{F}_{\mu\nu}dx^\mu \wedge dx^\nu \tag{2.29}$$

so that the following relation holds for the components of $\mathcal{F}$

$$\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu] \tag{2.30}$$

Since both forms in the above expression are $\mathfrak{g}$-valued they can be expanded also in terms of the Lie algebra $\{\mathfrak{g}_a\}$ as $\mathcal{A}_\mu = A_\mu{}^a\mathfrak{g}_a$ and $\mathcal{F}_{\mu\nu} = F_{\mu\nu}{}^a\mathfrak{g}_a$. Using $[\mathfrak{g}_a, \mathfrak{g}_b] = f_{ab}{}^c\mathfrak{g}_c$ we write down an equation in purely scalar valued quantities

$$F_{\mu\nu}{}^a = \partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a + f_{bc}{}^a A_\mu{}^b A_\nu{}^c. \tag{2.31}$$

Another important identity involving the curvature is the **Bianchi identity** $\mathbf{d}_\omega\Omega = 0$ which in is its local form becomes

$$\mathcal{F} = \mathbf{d}\mathcal{F} + [\mathcal{A}, \mathcal{F}] = 0 \tag{2.32}$$

In the case of our $U(1)$ example we only have one generator, which commutes, thus the structure equation and the Bianchi identity simplifies to

$$\mathcal{F} = \mathbf{d}\mathcal{A} \tag{2.33}$$

$$\mathbf{d}\mathcal{F} = 0 \tag{2.34}$$

the corresponding relations for the components becomes

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.35}$$

$$\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0 \tag{2.36}$$

2.6 parallel transport & horizontal lifts

Knowing that we can represent vector fields on $\mathcal{M}$ by sections in the tangent bundle $\mathcal{TM}$ we want to extend the notion of *parallel transport* of vectors to parallel transport of *sections* on a more general bundle. The generalization important to physics is parallel transport on a *principal bundle* $\mathcal{P}(\mathcal{M}, \mathcal{G})$ and how it is inherited by the bundles associated with it.

Consider a principal bundle $\mathcal{P}(\mathcal{M},\mathcal{G})$ and a curve $\gamma\colon [0,1] \to \mathcal{M}$. A **horizontal lift** is defined to be a corresponding curve $\tilde{\gamma}\colon [0,1] \to \mathcal{P}$ such that it lies vertically above the original curve, $\pi(\tilde{\gamma}) = \gamma$, and that the tangent vector to $\tilde{\gamma}(t)$ always lies within $\mathcal{H}_{\tilde{\gamma}(t)}\mathcal{P}$. Globally the latter condition is ensured using the constraint of a vanishing connection one-form $\boldsymbol{\omega}(\boldsymbol{t}) = 0$, where $\boldsymbol{t}$ is the tangent vector to $\tilde{\gamma}(t)$.

Choosing a local section σ_i (i. e. gauge fixing) a horizontal lift may be explicitly constructed *locally* using the local connection one-form $\mathcal{A}_{i\mu} = \sigma_i^*\omega$. The construction makes use of the fact that the lift $\tilde{\gamma}(t) \in \mathcal{P}$ is related to the chosen section σ_i by a corresponding right action g_i by[6]

$$\tilde{\gamma}(t) = \sigma_i\big(\gamma(t)\big)g_i\big(\gamma(t)\big) \quad , \quad g_i\big(\gamma(t)\big) = \mathcal{T}\exp\Big(\int_{\gamma(0)}^{\gamma(t)} \mathcal{A}_{i\mu}\big(\gamma(t)\big)dx^\mu\Big) \tag{2.37}$$

where σ_i is taken to be a section such that $\sigma_i\big(\gamma(0)\big) = \tilde{\gamma}(0)$ that is $g_i\big(\gamma(0)\big) = e$. $\mathcal{T}$ is the time-ordering operator which arranges all terms $\mathcal{A}_{i\mu}(t_1)$ and $\mathcal{A}_{i\nu}(t_2)$ in order of increasing time, from right to left. This is needed since the terms in general do not commute. In the following $g_i\big(\gamma(t)\big)$ and $\sigma_i\big(\gamma(t)\big)$ will be denoted by just $g_i(t)$ and $\sigma_i(t)$, but keep in mind the dependency on the original curve γ.

Parallel transport of a point $u_0 \in \mathcal{P}$ along a curve $\gamma(t)\colon [0,1] \to \mathcal{M}$, such that $u_0 \in \pi^{-1}(\gamma(0))$, is ensured by transporting u_0 along a horizontal lift $\tilde{\gamma}(t)$ where $\tilde{\gamma}(0) = u_0$ and $\tilde{\gamma}(1) = u_1$ where u_1 becomes the parallel transported point. This (horizontal) action *commutes* with the (vertical) right action which allows us to intuitively transport points around in the principal bundle. Mathematically, the combination of right action and parallel transport defines a unique transitive action on $\mathcal{P}$, i. e. the group action takes any point $u \in \mathcal{P}$ to any other point in the bundle in exactly one way.

In order to work with fields we now want to parallel transport elements $[(u,v)]$ in a vector bundle $\mathcal{E}$ associated to $\mathcal{P}$. Employ the canonical trivialization to a section $\sigma_i \in \Gamma(U_i,\mathcal{P})$ and choose a section $s = [(\sigma_i, v)] \in \Gamma(U_i,\mathcal{E})$, choosing a section s corresponds to fixing the gauge. Now consider a curve $\gamma(t)\colon [0,1] \to \mathcal{M}$ and a corresponding horizontal lift $\tilde{\gamma}(t) = \sigma_i(t)g_i(t)$

$$\begin{aligned} s(t) &= \big[\big(\sigma_i(t)\ ,\ v(t)\big)\big] = \big[\big(\tilde{\gamma}(t)g_i^{-1}(t)\ ,\ v(t)\big)\big] \\ &= \big[\big(\tilde{\gamma}(t)\ ,\ g_i^{-1}(t)v(t)\big)\big] = \big[\big(\tilde{\gamma}(t)\ ,\ \tilde{v}(t)\big)\big] \end{aligned} \tag{2.38}$$

where the second line is obtained using the defining identification (2.7) of associated vector bundles. The section $s(p)$ is *parallel transported* along a curve $\gamma(t)\colon [0,1] \to \mathcal{M}$ if $\tilde{v}(t) = \tilde{v}(\gamma(t))$ is constant w. r. t. a horizontal lift $\tilde{\gamma}(t) \in \mathcal{P}$.

It can be shown[7] that this definition depends only on the curve $\gamma(t)$ and the connection $\boldsymbol{\omega}$, and not on the choice of horizontal lift and local trivialization. To conclude; the constraint on $\tilde{\gamma}(t)$ to 'lie horizontally' in $\mathcal{P}$ incorporates the structure of $\mathcal{P}$ into the parallel transport of a section $s(t) \in \mathcal{E}$ along a curve $\gamma(t)$ in $\mathcal{M}$. How this works becomes visible in the above equation where $g_i^{-1}(t)$ acts on $v(t)$ to compensate for the change of basis along $\gamma(t)$.

2.7 connection coefficients & representations

In the previous section we saw a remarkable result of formulating gauge theory in a fibre bundle framework; the introduction of a connection one-form $\boldsymbol{\omega}$ in the principle bundle $\mathcal{P}(\mathcal{M}, \mathcal{G})$ completely specifies parallel transport in any associated vector bundle $\mathcal{E}$, up to representations. Here the result is made more explicit in terms of matrices and coefficients.

First the choice of a local section σ_i over $U_i \subset \mathcal{M}$ specifies a local connection $\mathcal{A}$ by (2.16). Then a **matrix representation** ρ of the Lie algebra generators $\{\mathfrak{g}_a\}$ is chosen to represent the group action on the vector bundle

$$\rho(\mathfrak{g}_a) = (\mathfrak{g}_a)^{\alpha}{}_{\beta} \tag{2.39}$$

where α and β are matrix indices. The representation is chosen such that the elements of $\mathcal{G}$ may act on vectors in $\mathcal{E}$ by usual matrix multiplication, this forces an action on an n-dimensional associated vector bundle to be represented by $n \times n$ matrices. If we now expand $\mathcal{A}$ in terms of $\{(\mathfrak{g}_a)^{\alpha}{}_{\beta}\}$ we get a natural matrix representation of the local connection one-form by carrying out the summation over the Lie algebra index:

$$\rho(\mathcal{A}) = A^a (\mathfrak{g}_a)^{\alpha}{}_{\beta} = \mathcal{A}^{\alpha}{}_{\beta}. \tag{2.40}$$

Without further derivation we present the remarkable result in our matrix representation in the associated vector bundle

$$\nabla e_\beta = \mathcal{A}^{\alpha}{}_{\beta} e_\alpha. \tag{2.41}$$

$\{\boldsymbol{e}_\alpha\}$ being a basis of $\mathcal{E}$. Given local coordinates $\{x_\mu\}$, and expanding the connection in the corresponding coordinate basis $\mathcal{A} = \mathcal{A}_\mu dx^\mu$ the covariant derivative along a basis vector ∇_μ is given by the matrix representation of the components $\mathcal{A}_\mu$

$$\nabla_\mu \boldsymbol{e}_\beta = A^{\alpha}_{\mu\beta} \boldsymbol{e}_\alpha \tag{2.42}$$

where $\{A^{\alpha}_{\mu\beta}\}$ are a set of connection coefficients in $\mathcal{E}$ and thus completely specifies the covariant derivative on the associated vector bundle.

2.8 frames, frame fields & frame bundles

Even if we have been working with frame fields throughout this presentation a more rigorous treatment in terms of the presented notions will help us polish up the story.

Considering a k-dimensional vector bundle $\mathcal{V}$ over a manifold $\mathcal{M}$ equipped with an atlas $\{U_i, \phi_i\}$ so that $\phi_i^{-1}\colon \pi^{-1}(U_i) \to U_i \times \mathbb{R}^k$ we may choose k *linearly independent* local sections $\{\boldsymbol{e}_\alpha\}$ over U_i, $\alpha = 1, 2, \ldots, k$. These sections are said to define a **frame field** over U_i. We will refer to the frame field at a particular point p as the **frame** at that point $\{\boldsymbol{e}_\alpha(p)\}$.

Given this setup we can express any vector $\boldsymbol{v}_p$ at p as an element in $\mathcal{F}_p \simeq \mathbb{R}^k$, the coordinates in $\mathbb{R}^k$ become the components v_p^α of the vector $\boldsymbol{v}_p = v_p^\alpha \boldsymbol{e}_\alpha$. Using the local trivialization we may extend this notion from $p \in U_i$ to the whole open subset U_i so that a vector *field* $\boldsymbol{v}_i$ over U_i may be expressed as

$$\phi_i\big(p, v_i^\alpha(p)\big) = \boldsymbol{v}_i(p) \qquad \forall p \in U_i \tag{2.43}$$

and by definition we have

$$\phi_i(p, \{0^1, 0^2, \ldots, 1^\alpha, \ldots, 0^k\}) = \boldsymbol{e}_\alpha(p) \tag{2.44}$$

To extend our notion outside the open subset U_i we use a matrix representation ρ of the transition functions ψ_{ij} given by

$$\rho(\psi_{ij}) = X^\alpha_\beta \qquad \psi_{ij} \in GL(k, \mathbb{R}) \tag{2.45}$$

so that on a chart overlap $U_i \cap U_j \neq \emptyset$ the frame field $\{\boldsymbol{e}'_\beta\}$ over U_j is given by

$$\boldsymbol{e}'_\beta(p) = X^\alpha_\beta(p)\boldsymbol{e}_\alpha(p) \qquad \forall p \in U_i \cap U_j \tag{2.46}$$

The covariance of vectors under a change of basis

$$\boldsymbol{v} = v_i^\alpha \boldsymbol{e}_\alpha = v_j^\beta \boldsymbol{e}'_\beta \tag{2.47}$$

forces its components to transform by the inverse of X^α_β

$$v_j^\beta = X^\beta_\alpha v_i^\alpha \qquad X^\alpha_\gamma X^\gamma_\beta = \delta^\alpha_\beta \tag{2.48}$$

Now when we have extended the notion from a point p to an open subset U_i and further out into neighbouring subsets one might be tempted to claim that we have extended the notion to the whole manifold $\mathcal{M}$. There are however severe topological restrictions to this extension, see for example the 'Hairy Ball Theorem'. We won't go down that rabbit hole here, but remember that the only section which may always be globally defined is the null section.

We may however extend our notion in another direction by considering a *change of frame* at p, naturally this action is also given through a matrix representation of an element in $\mathcal{G} = GL(k, \mathbb{R})$.

$$\tilde{\boldsymbol{e}}_\beta(p) = Y^\alpha_\beta \boldsymbol{e}_\alpha(p) \qquad \begin{array}{l} p \in U_i \\ Y^\alpha_\beta \in \rho(\mathcal{G}) \end{array} \tag{2.49}$$

or by its inverse Y^β_α for the components v_p^α of vectors $\boldsymbol{v}_p \in \mathcal{F}_p$. From a given frame at p we may, by the above relation, construct *any* other frame at p due to the *transitivity* of the action of $\mathcal{G}$ on $\mathcal{F}$. By considering all frames at all points $p \in U_i$ we are thus led to the direct product space $U_i \times_\rho \mathcal{G}$ where we may describe the transition between any frame *fields* as a section $Y^\alpha_\beta(p)$ over U_i

$$\tilde{\boldsymbol{e}}_\beta(p) = Y^\alpha_\beta(p)\boldsymbol{e}_\alpha(p) \qquad \begin{array}{l} \forall p \in U_i \\ Y^\alpha_\beta \in U_i \times_\rho \mathcal{G} \end{array} \tag{2.50}$$

For this to be true even outside U_i we must form a bundle using the same transition functions as in $\mathcal{V}$ when we glue the pieces $U_i \times_\rho \mathcal{G}$ and $U_j \times_\rho \mathcal{G}$ together. By doing so the above relation still holds for frame fields stretching across open subsets. Following this line of reason we are naturally led to a *principal bundle associated with* $\mathcal{V}$ which contains all its possible frame fields, such a bundle is called a **frame bundle** $\mathcal{FM}$. Given a frame field on $\mathcal{V}$ any other frame field may be defined through a section in $\mathcal{FM}$.

3 the case of Gravity

Gravity as a gauge theory has been subject to waves of intense studies since the birth of gauge theory itself. The driving force behind this attention is the idea of a unifying theory, putting gravity on more or less equal footing with the other fundamental forces, viz. electromagnetism, the strong force and the weak force. General Relativity (GR), although being a fully satisfactory theory of gravity has little in common with the current description of the other three forces. In fact it is arguable whether gravity should be considered a force at all.

The 'other' three forces are considered mediated by fields which are naturally described by gauge fields exactly as they have been presented in this thesis, and all their interactions takes place in the background of space-time. In this picture a force is something which alters a trajectory in space-time. GR, on the contrary, describes gravity as the dynamical properties of space-time itself and its 'interaction' do not alter the trajectory in space-time but rather changes space-time itself.

However, this seemingly fundamental difference might just be a consequence of the chosen starting point of the description. It might be the case that all four forces can be ascribed to the properties of space-time, which is the idea of unification which Einstein spent most of his career pursuing. It might just as well be that gravity may be formulated as a gauge theory, which is the idea we will elaborate on in this chapter.

3.1 tensor bundles in general

The simplest tensor bundle over $\mathcal{M}$ is the tangent bundle $\mathcal{TM}$. Recall the definition of the tangent space, or the *tangent bundle at a point*, $\mathcal{T}_p\mathcal{M}$ as the union of the tangent vectors at $p \in \mathcal{M}$ to all curves in $\mathcal{M}$ passing through that point. Instead of tangent vectors we may speak of the parameter derivative $\frac{d}{dt}$ of the curve $\gamma(t)$, thus the tangent bundle is constructed solely by calculus on the base manifold. Introducing coordinates $\{x^\mu\}$ on $\mathcal{M}$ naturally induces base vectors in $\mathcal{TM}$ by describing the curves by these coordinates $\gamma(x^\mu(t))$ so that their parameter derivatives become $\frac{\partial}{\partial x^\mu}\frac{\partial x^\mu}{\partial t}$, dropping the dependence on a specific γ and extending to the whole base manifold result in the coordinate *frame field*[1] $\frac{\partial}{\partial x^\mu} = \boldsymbol{\partial}_\mu$ on $\mathcal{M}$. Here the μ index indicates, not different components, but different basis *vectors*. This intimate connection with the base manifold makes it possible to consider a tangent bundle as soon as you have defined $\mathcal{M}$. Furthermore as soon as coordinates are chosen on $\mathcal{M}$ a natural set of base vectors are induced on $\mathcal{TM}$.

Given $\mathcal{TM}$ its dual, the **cotangent bundle** $\mathcal{T}^*\mathcal{M}$, is canonically defined as

[1] Commonly referred to as just the *coordinate basis* but here and throughout we refer to all point dependent vectors as vector *fields* to make explicit their point dependency.

a map

$$\begin{aligned} \mathcal{T}^*_p\mathcal{M} \colon \mathcal{T}_p\mathcal{M} &\longrightarrow \mathbb{R} \qquad \forall p \in \mathcal{M} \\ \boldsymbol{d}^\mu \colon \boldsymbol{\partial}_\nu &\mapsto \delta^\mu_\nu \end{aligned} \tag{3.1}$$

where the elements $\boldsymbol{d}^\mu$ thus serve as a basis in the cotangent bundle and is referred to as the **dual frame field**, and arbitrary elements in $\mathcal{T}^*\mathcal{M}$ expressible in that basis are dual vectors or the **one-forms** on $\mathcal{M}$. Beware of our unorthodox notation $\boldsymbol{d}^\mu \equiv dx^\mu$ which is used to consistently distinguish between tensor and scalar valued objects.

Regarding vectors as $(1,0)$ tensors and one-forms as $(0,1)$ tensors the notion of tangent and cotangent bundles are easily generalized to arbitrary (a,b) tensor bundles $\mathcal{T}^{a,b}\mathcal{M}$ as multilinear maps from vectors and one-forms to the real numbers

$$\mathcal{T}^{a,b}_p\mathcal{M} \colon \underbrace{\mathcal{T}_p\mathcal{M} \times \mathcal{T}_p\mathcal{M} \times \ldots}_{b \text{ factors}} \times \underbrace{\mathcal{T}^*_p\mathcal{M} \times \mathcal{T}^*_p\mathcal{M} \times \ldots}_{a \text{ factors}} \longrightarrow \mathbb{R} \quad \forall p \in \mathcal{M} \tag{3.2}$$

Given a manifold $\mathcal{M}$ one can immediately construct arbitrary tensor bundles, and given coordinates they inherit frame fields so that an element $\boldsymbol{T} \in \mathcal{T}^{a,b}\mathcal{M}$ may be expanded as

$$\boldsymbol{T} = T^{\mu_1 \ldots \mu_a}_{\nu_1 \ldots \nu_b} \boldsymbol{\partial}_{\mu_1} \otimes \ldots \otimes \boldsymbol{\partial}_{\mu_a} \otimes \boldsymbol{d}^{\nu_1} \otimes \ldots \otimes \boldsymbol{d}^{\nu_b} \tag{3.3}$$

3.2 tensor bundles in general relativity

The space-time manifold $\mathcal{M}$ in General Relativity comes equipped with a covariant derivative ∇_μ with connection coefficients $\Gamma^\lambda_{\nu\mu}$ which expresses *the change* of basis vector $\boldsymbol{\partial}_\nu$ in the μ-direction as a vector $\Gamma^\lambda_{\nu\mu}\boldsymbol{\partial}_\lambda$ in the coordinate basis

$$\nabla_\mu \boldsymbol{\partial}_\nu = \Gamma^\lambda_{\nu\mu}\boldsymbol{\partial}_\lambda \tag{3.4}$$

The dual frame field $\boldsymbol{d}^\mu$ is related to the coordinate frame field by

$$\boldsymbol{\partial}_\nu \boldsymbol{d}^\mu = \delta^\mu_\nu \tag{3.5}$$

in order to find how the covariant derivative ∇_μ acts on the dual frame field we note that $\nabla_\mu(\boldsymbol{d}^\nu\boldsymbol{\partial}_\nu) = \nabla_\mu(\delta^\nu_\nu) = 0$ and use the Leibniz rule of the covariant derivative

$$\begin{aligned} \nabla_\mu\big(\boldsymbol{d}^\nu\boldsymbol{\partial}_\nu\big) &= \big(\nabla_\mu\boldsymbol{d}^\nu\big)\boldsymbol{\partial}_\nu + \boldsymbol{d}^\nu\big(\nabla_\mu\boldsymbol{\partial}_\nu\big) \\ &= \big(\nabla_\mu\boldsymbol{d}^\nu\big)\boldsymbol{\partial}_\nu + \boldsymbol{d}^\nu\Gamma^\kappa_{\nu\mu}\boldsymbol{\partial}_\kappa \\ &= \big(\nabla_\mu\boldsymbol{d}^\nu + \boldsymbol{d}^\lambda\Gamma^\nu_{\lambda\mu}\big)\boldsymbol{\partial}_\nu = 0 \quad \Longrightarrow \\ \nabla_\mu\boldsymbol{d}^\lambda &= -\Gamma^\lambda_{\mu\nu}\boldsymbol{d}^\nu \end{aligned} \tag{3.6}$$

Note that $\boldsymbol{\partial}_\mu$ and $\boldsymbol{d}^\mu$ are viewed as (dual) basis vectors and not as differential operators, the only operator acting on anything here is the covariant derivative.

Knowing how the covariant derivative act on the frame field and its dual we use the Leibniz rule, exemplified below, to perform covariant differentiation of a $(1,1)$ tensor field $\boldsymbol{T} = T^{\lambda}{}_{\kappa}\boldsymbol{\partial}_{\lambda} \otimes \boldsymbol{d}^{\kappa}$

$$\begin{aligned}\nabla_\mu \boldsymbol{T} &= \big(\nabla_\mu T^{\lambda}{}_{\kappa}\big)\boldsymbol{\partial}_{\lambda} \otimes \boldsymbol{d}^{\kappa} + T^{\lambda}{}_{\kappa}\big(\nabla_\mu \boldsymbol{\partial}_{\lambda}\big) \otimes \boldsymbol{d}^{\kappa} + T^{\lambda}{}_{\kappa}\boldsymbol{\partial}_{\lambda} \otimes \big(\nabla_\mu \boldsymbol{d}^{\kappa}\big) \\ &= \big(\partial_\mu T^{\lambda}{}_{\kappa}\big)\boldsymbol{\partial}_{\lambda} \otimes \boldsymbol{d}^{\kappa} + T^{\lambda}{}_{\kappa}\big(\Gamma^{\nu}_{\lambda\mu}\boldsymbol{\partial}_{\nu}\big) \otimes \boldsymbol{d}^{\kappa} + T^{\lambda}{}_{\kappa}\boldsymbol{\partial}_{\lambda} \otimes \big(-\Gamma^{\kappa}_{\nu\mu}\boldsymbol{d}^{\nu}\big) \\ &= \big(\partial_\mu T^{\lambda}{}_{\kappa} + \Gamma^{\lambda}_{\nu\mu}T^{\nu}{}_{\kappa} - \Gamma^{\nu}_{\kappa\mu}T^{\lambda}{}_{\nu}\big)\boldsymbol{\partial}_{\lambda} \otimes \boldsymbol{d}^{\kappa}\end{aligned} \tag{3.7}$$

where the last step contains just reordering and relabeling of dummy indices. Note how the covariant derivative reduces to ordinary partial differentiation on scalar fields since scalars fields are unaffected by a change in curvature. From (3.4) we have now derived the well known relation for the components of a $(1,1)$ tensor field

$$\nabla_\mu T^{\lambda}{}_{\kappa} = \partial_\mu T^{\lambda}{}_{\kappa} + \Gamma^{\lambda}_{\nu\mu}T^{\nu}{}_{\kappa} - \Gamma^{\nu}_{\kappa\mu}T^{\lambda}{}_{\nu} \tag{3.8}$$

where generalization to arbitrary rank tensor fields follow the same pattern by generating a minus sign for each form-index and a plus sign for each vector-index. The most common special case is the covariant derivative acting on a vector $\boldsymbol{v} = v^{\lambda}\boldsymbol{\partial}_{\lambda}$

$$\nabla_\mu v^{\lambda} = \partial_\mu v^{\lambda} + \Gamma^{\lambda}_{\nu\mu}v^{\nu} \tag{3.9}$$

Note that the connection coefficients are not tensors by themselves and thus can not give a coordinate independent description of the curvature. Objects providing such a description may however be defined through the above expression. $\nabla_\mu v^{\lambda}$ measures the change of a vector $\boldsymbol{v}$ along the μ-direction *compared to* if it had been parallel transported. The commutator between covariant derivatives $[D_\mu, D_\nu]$ acting on $\boldsymbol{v}$ thus measures the difference between parallel transporting $\boldsymbol{v}$ along the μ-direction followed by the ν-direction, versus doing it in the opposite order. Using the above definition we get

$$[D_\mu, D_\nu]v^{\lambda} = \underbrace{\big(\partial_\mu \Gamma^{\lambda}_{\nu\kappa} - \partial_\nu \Gamma^{\lambda}_{\mu\kappa} + \Gamma^{\lambda}_{\mu\tau}\Gamma^{\tau}_{\nu\kappa} - \Gamma^{\lambda}_{\nu\tau}\Gamma^{\tau}_{\mu\kappa}\big)}_{R^{\lambda}{}_{\kappa\mu\nu}}v^{\kappa} - \underbrace{\big(\Gamma^{\tau}_{\mu\nu} - \Gamma^{\tau}_{\nu\mu}\big)}_{T_{\mu\nu}{}^{\tau}}D_\tau v^{\lambda} \tag{3.10}$$

where we have singled out two objects which one can prove transforms as tensors. These are the Riemann tensor $R^{\lambda}{}_{\kappa\mu\nu}$ and the torsion tensor $T_{\mu\nu}{}^{\lambda}$. The connection in GR is assumed to be *torsion-free* so that the above expression reduces to

$$[D_\mu, D_\nu]v^{\lambda} = \big(\partial_\mu \Gamma^{\lambda}_{\nu\kappa} - \partial_\nu \Gamma^{\lambda}_{\mu\kappa} + \Gamma^{\lambda}_{\mu\tau}\Gamma^{\tau}_{\nu\kappa} - \Gamma^{\lambda}_{\nu\tau}\Gamma^{\tau}_{\mu\kappa}\big)v^{\kappa} = R^{\lambda}{}_{\kappa\mu\nu}v^{\kappa} \tag{3.11}$$

Note that the torsion-free condition forces the connection coefficients to be symmetric in their lower indices

$$T_{\mu\nu}{}^{\lambda} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} = 0 \quad \Longrightarrow \quad \Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu} \tag{3.12}$$

The metric $\boldsymbol{g}$ in GR is a $(0,2)$ type tensor field and may thus be expanded as

$$\boldsymbol{g} = g_{\mu\nu}\boldsymbol{d}^{\mu} \otimes \boldsymbol{d}^{\nu} \tag{3.13}$$

In addition to the torsion free condition the connection in GR is demanded to be compatible with the metric $\nabla_\mu \boldsymbol{g} = 0$, this is a very reasonable constraint since it ensures that the scalar product between two parallel transported vectors remains the same. The metric compatibility together with the torsion-free condition is the defining properties of the *Levi-Civita connection* ∇. The two constraints kills exactly enough degrees of freedom to allow the connection coefficients to be solved for in terms of the metric[9] to produce another well known relation

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa} (\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\kappa\mu} - \partial_\kappa g_{\mu\nu}) \tag{3.14}$$

The fact the connection and the metric are dependent dynamical variables allows the theory of General Relativity to be formulated completely in terms of the metric. When we consider alternative descriptions of Gravity this will no longer be the case.

3.3 general bundles in general relativity

Instead of describing the curvature of space-time by bundles directly inherited from the base manifold $\mathcal{M}$ we will in the following sections develop a formalism of more general fibre bundles. However, as previously stated, a fibre bundle which we want to relate to gravity bears a special relation to space-time which has huge impact on its construction. Because, in order to have any connection with the curvature on $\mathcal{M}$, which is what we want to describe, we need to introduce a connection on our fibre bundle which is compatible with the metric, or at least, the curvature of space-time. From another point of view; the connection is required to induce a sensible curvature on $\mathcal{M}$.

In the following we will present a fibre bundle formalism which is more detached from the base manifold but still keep contact with its curvature. This construction allows for a closer analogy with Yang-Mills theory, and actually paves way for including YM as a special case of the theory.

3.4 orthonormal frame & vielbein

Instead of the coordinate induced frame field $\{\boldsymbol{\partial}_\mu\}$ we may choose a frame field $\{\boldsymbol{e}_a\}$ dictated by orthonormality w. r. t. the metric $\boldsymbol{g}$

$$\boldsymbol{g}(\boldsymbol{e}_a, \boldsymbol{e}_b) = \eta_{ab} \tag{3.15}$$

where η_{ab} are the components of the flat Minkowski metric $\boldsymbol{\eta}$. Now, we know that the transformations which preserve the Minkowski metric are the *Lorentz transformations*

$$\Lambda_a{}^c \Lambda_b{}^d \eta_{cd} = \eta_{ab} \tag{3.16}$$

to preserve orthonormality we therefore restrict the change of basis to the following form

$$\boldsymbol{e}'_a = \Lambda_a{}^b \boldsymbol{e}_b \tag{3.17}$$

where ${\Lambda_a}^b$ are the inverse Lorentz transformations. We call $\{\boldsymbol{e}_\mu\}$ the **orthonormal frame field** which, instead of being related to the coordinates, is related to the curvature of $\mathcal{M}$. The relation between our two frame fields define the **vielbein field** e^a_μ

$$\boldsymbol{\partial}_\mu = e^a_\mu \boldsymbol{e}_a \qquad e^\mu_a \in \rho(GL(4,\mathbb{R})) \tag{3.18}$$

Since in GR the vielbein field takes values in the matrix representation ρ of the $GL(4,\mathbb{R})$ group it automatically has an inverse e^μ_a, satisfying

$$e^\mu_a e^a_\nu = \delta^\mu_\nu \qquad e^\mu_a e^b_\nu = \delta^b_a \tag{3.19}$$

Using e^μ_a the definition (3.15) may be expressed in components

$$e^a_\mu e^b_\nu \eta_{ab} = g_{\mu\nu} \tag{3.20}$$

from where we see that the vielbein may be used as an alternative to the metric as the dynamical variable of GR. The inverse vielbein field also relates the dual basis of the coordinate frame $\{\boldsymbol{d}^\mu\}$ and the orthonormal frame $\{\boldsymbol{\theta}^a\}$ by

$$\boldsymbol{d}^\mu = e^\mu_a \boldsymbol{\theta}^a \qquad \boldsymbol{\theta}^a \boldsymbol{e}_b = \delta^a_b \tag{3.21}$$

Considering the $(1,1)$ tensor field $\boldsymbol{T}$ previously expressed in the coordinate frame we may now transform its components into the new basis or express the tensor in mixed components using the vielbein and its inverse

$${T_\nu}^\mu = e^\mu_a {T_\nu}^a = e^b_\nu {T_b}^\mu = e^b_\nu e^\mu_a {T_b}^a \tag{3.22}$$

3.5 Lorentz bundle & spin connection

Having partially freed ourself from the coordinate dependency by the formalism introduced in the previous section we are in a better position to connect GR with our developed notion of fibre bundles.

Since the change between orthonormal frames is carried out by Lorentz transformations (3.17) one is naturally led to consider a frame bundle $\mathcal{FM}$ with the Lorentz group $\mathcal{G} = SO(1,3)$ as its structure group.

In order to consider differentiation a connection one-form $\boldsymbol{\omega}$ is introduced in the bundle to relate the group-space at nearby space-time points. The connection takes values in the Lie algebra of $\mathcal{G}$ and needs to be represented by a matrix to act on the frame field $\{\boldsymbol{e}_a\}$ in $\mathcal{V}$, the representation space of $SO(1,3)$. We take the *adjoint representation* and identify the resulting object $\omega^a_{\mu b}$ as the **spin connection**, where a and b are matrix indices in $\mathcal{V}$. The spin connection is used to define the covariant derivative $\mathcal{D}_\mu$ in the orthonormal basis, just like $\Gamma^\lambda_{\mu\nu}$ in the coordinate basis

$$\mathcal{D}_\mu {T_b}^a = \partial_\mu {T_b}^a + \omega^a_{\mu c} {T_b}^c - \omega^c_{\mu b} {T_c}^a \tag{3.23}$$

Expressing an arbitrary vector $\boldsymbol{v}$ in a coordinate basis $v^\mu \boldsymbol{\partial}_\mu$ and in an orthonormal basis $v^a \boldsymbol{e}_a$ we demand the action of the corresponding covariant derivatives ∇_μ, as in (3.9), and $\mathcal{D}_\mu$ to result in the same vector $\boldsymbol{v}'$. This leads to the following relation between the two formalisms

$$\begin{aligned}\Gamma^\nu_{\mu\lambda} &= e^\nu_a \partial_\mu e^a_\lambda + e^\nu_a e^b_\lambda \omega^a_{\mu b} \\ \omega^a_{\mu b} &= e^a_\nu e^\lambda_b \Gamma^\nu_{\mu\lambda} - e^\lambda_b \partial_\mu e^a_\lambda\end{aligned} \tag{3.24}$$

which can be manipulated into the **vielbein postulate**, which states the vanishing of the total covariant derivative of the vielbein

$$\nabla_\mu e^a_\nu = \partial_\mu e^a_\nu + \omega^a_{\mu b} e^b_\nu - \Gamma^\lambda_{\mu\nu} e^a_\lambda = 0 \tag{3.25}$$

Although we refer to R. Wald (1984) for a complete derivation of (3.24) and (3.25) we want to point out that the vielbein and its inverse may always be used to switch between the frame fields, and that covariant differentiation of mixed objects is always carried out using the corresponding connection coefficients, as in the above relation.

Since the spin connection behaves like a one-form we suppress the μ index and use differential geometry language to present the previous results in this new formalism. First we note that the de Rham differential $\mathbf{d}$ of the vielbein $\boldsymbol{e}^a$ is not a covariant object. But just as we corrected the partial derivative by the use of the spin connection we may also construct an external covariant derivative $\mathbf{d}_\omega$ such that its action on $\boldsymbol{e}^a$ yields a covariant object

$$\mathbf{d}_\omega \boldsymbol{e}^a \equiv \mathbf{d}\boldsymbol{e}^a + \boldsymbol{\omega}^a{}_b \wedge \boldsymbol{e}^b \tag{3.26}$$

This is nothing but the torsion tensor $T_{\mu\nu}{}^\lambda$ expressed in a mixed basis, using the vielbein, so that it may be viewed as a two-form which is vector valued in $\mathcal{V}$

$$e^a_\lambda T_{\mu\nu}{}^\lambda = T_{\mu\nu}{}^a = \boldsymbol{T}^a = \mathbf{d}_\omega \boldsymbol{e}^a \tag{3.27}$$

Similarly the Riemann tensor may be expressed as a two-form

$$e^a_\lambda a^\kappa_b R^\lambda{}_{\kappa\mu\nu} = R^a{}_{b\mu\nu} = \boldsymbol{R}^a{}_b \tag{3.28}$$

and together with the torsion identity we get the **Mauer-Cartan structure equations** which completely specifies the curvature of $\mathcal{M}$

$$\begin{aligned}\boldsymbol{T}^a &= \mathbf{d}\boldsymbol{e}^a + \boldsymbol{\omega}^a{}_b \wedge \boldsymbol{e}^b \\ \boldsymbol{R}^a{}_b &= \mathbf{d}\boldsymbol{\omega}^a{}_b + \boldsymbol{\omega}^a{}_c \wedge \boldsymbol{\omega}^c{}_b\end{aligned} \tag{3.29}$$

As when introducing the fibre bundle formalism we sometimes drop the matrix indices too and signal that the objects takes values in the Lie algebra by calligraphic letters, using our corrected differential $\mathbf{d}_\omega$ the structure equations may be written as

$$\begin{aligned}\mathcal{T} &= \mathbf{d}_\omega \boldsymbol{e} \\ \mathcal{R} &= \mathbf{d}\boldsymbol{\omega} + \boldsymbol{\omega} \wedge \boldsymbol{\omega}\end{aligned} \tag{3.30}$$

Note that $\mathbf{d}_\omega \boldsymbol{\omega}$ makes no sense since $\boldsymbol{\omega}$ is not a tensor and thus not covariant to begin with. Using our shorthand notation we write down the Bianchi identities as

$$\begin{aligned} \mathbf{d}_\omega \mathcal{T} &= \mathcal{R} \wedge \boldsymbol{e} \\ \mathbf{d}_\omega \mathcal{R} &= 0 \end{aligned} \tag{3.31}$$

Note that $\mathbf{d}\mathcal{R} \neq 0$. Besides being good looking this notation makes it easier to overview the overall symmetries in the theory, but for the underlying understanding and for practical computations it is usually useless.

3.6 Poincare gauge theory

When we describe gravity in terms of the vielbein field e_a^μ instead of the metric components $g_{\mu\nu}$ one should note that the vielbein has 16 independent components wheres the metric, being symmetric, is reduced to having 10 independent components. On the other hand we know that the metric is all we need to describe space-time dynamics. The excess 6 degrees of freedom constitutes our freedom to choose reference frame, which as we know should not alter the physics. We have in a way introduced unnecessary information in the description which we then describe as a gauge theory like in the previous section. Thus, in order to capture the dynamics of space-time we must also include the vielbein in the description. The basic idea of the Poincare gauge theory (PGT) is to also treat the vielbein $\boldsymbol{e}$ as a gauge field.

The most natural way of doing so is to extend the gauge group $\mathcal{G}$ to the Poincare group $ISO(1,3)$, which is the semi-direct product of the Lorentz group, with generators $\mathfrak{h}_{ij}$, and the generators of translation $\mathfrak{p}_a$. These generators satisfy the Poincare algebra

$$\begin{aligned} [\mathfrak{p}_a, \mathfrak{p}_\mathfrak{b}] &= 0 \\ [\mathfrak{h}_{ij}, \mathfrak{p}_a] &= (\eta_{ia}\delta_k^b - \eta_{ka}\delta_i^b)\mathfrak{p}_b \\ [\mathfrak{h}_{ij}, \mathfrak{h}_{kl}] &= \eta_{ij}\mathfrak{h}_{jl} + \eta_{jl}\mathfrak{h}_{ik} - \eta_{il}\mathfrak{h}_{jk} - \eta_{jk}\mathfrak{h}_{il} \end{aligned} \tag{3.32}$$

where $\boldsymbol{\eta}$ is the Minkowski metric. The total gauge field $\mathcal{A}$ may then be decomposed into a Lorentz and a translational part, which may be expanded in the generators

$$\mathcal{A} = \boldsymbol{\omega}^{ij}\mathfrak{h}_{ij} + \boldsymbol{e}^a\mathfrak{p}_a \tag{3.33}$$

Construct the covariant derivative

$$\mathcal{D}_\mu = \partial_\mu + \omega_\mu^{ij}\mathfrak{h}_{ij} + e_\mu^a\mathfrak{p}_a \tag{3.34}$$

and the corresponding field strengths

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = R_{\mu\nu}^{ij}\mathfrak{h}_{ij} + T_{\mu\nu}^a\mathfrak{p}_a \tag{3.35}$$

which naturally also breaks up into two parts. The essence in PGT lies in introducing a constraint of the form

$$T^a_{\mu\nu} = 0 \tag{3.36}$$

which cuts the degrees of freedom of the theory down to 10 to match with General Relativity. The above constraint also allows ω^{ij}_μ to be solved for in terms of e^a_μ [10]

$$\omega_{\mu ab} = \frac{1}{2} e^\nu_a e^\lambda_b (\Omega_{\mu\nu\lambda} - \Omega_{\nu\lambda\mu} + \Omega_{\lambda\mu\nu}) \qquad \Omega_{\mu\nu\lambda} = (\partial_\mu e^c_\nu - \partial_\nu e^c_\mu) e_{\lambda c} \tag{3.37}$$

and makes the spin connection dependent on the vielbein in the same way as the Levi-Civita connection depend on the metric. It is precisely this feature of PGT which makes it differ from standard gauging in Yang-Mills theory. In YM the gauge fields are always *independent* wheres in the case of PGT the connection $\boldsymbol{\omega}$ has to agree with the curvature of space-time which is dictated by the vielbein $\boldsymbol{e}$.

3.7 Einstein-Cartan action

The shift of focus from the metric $\boldsymbol{g}$ to a gauge field containing the vielbein $\boldsymbol{e}$ and spin connection $\boldsymbol{\omega}$ leads us to revisit the Einstein-Hilbert (EH) action of General Relativity. When a cosmological constant Λ is added to the picture this action looks like

$$S_{EH} = \frac{1}{2k} \int_{\mathcal{M}} (R - 2\Lambda) \sqrt{|g|} dx^4 \qquad \begin{aligned} k &= 8\pi G \\ g &= \det \boldsymbol{g} \end{aligned} \tag{3.38}$$

where R is the Ricci scalar and G the gravitational constant. Let us now do a bit of rewriting to reach a form where the relation to $\boldsymbol{e}$ and $\boldsymbol{\omega}$ becomes clear. First, define a metric weighted Levi-Civita symbol by

$$\varepsilon_{\mu\nu\lambda\kappa} = \sqrt{|g|} \epsilon_{\mu\nu\lambda\kappa} \tag{3.39}$$

where $\epsilon_{\mu\nu\lambda\kappa}$ is the usual Levi-Civita symbol. This allows us to replace the metric volume form $\sqrt{|g|} dx^4$ in the action by $\frac{1}{4!} \varepsilon_{\mu\nu\lambda\kappa} dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\kappa$. Using the Riemann tensor written as a $\mathfrak{gl}(4, \mathbb{R})$ valued two-form $\mathcal{R}^{\lambda\kappa}$ bring the action into the following form

$$S_{EH} = \frac{1}{4k} \int_{\mathcal{M}} \varepsilon_{\mu\nu\lambda\kappa} \big(dx^\mu \wedge dx^\nu \wedge \mathcal{R}^{\lambda\kappa} - \frac{\Lambda}{6} dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\kappa \big) \tag{3.40}$$

since all terms are $GL(4, \mathbb{R})$ invariant we can freely change to any basis. Going to the orthonormal basis so that $dx^\mu \to e^a$, and noting that in this frame $\varepsilon_{abcd} = \epsilon_{abcd}$ since $\sqrt{|g|} = \sqrt{|\eta|} = 1$, yields the following form of the action

$$S_{EH} = \frac{1}{4k} \int_{\mathcal{M}} \epsilon_{abcd} e^a \wedge e^b \wedge \big(\mathcal{R}^{cd} - \frac{\Lambda}{6} e^c \wedge e^d \big) \tag{3.41}$$

This is still nothing but the Einstein-Hilbert action in an orthonormal basis. But now we just replace the Levi-Civita connection by the spin connection and its corresponding curvature two-form $\mathcal{R}_\omega$, and relax the constraint of vanishing torsion, to obtain the Einstein-Cartan (EC) action

$$S_{EC} = \frac{1}{4k}\int_{\mathcal{M}} \epsilon_{abcd} e^a \wedge e^b \wedge \left(\mathcal{R}^{cd}_\omega - \frac{\Lambda}{6} e^c \wedge e^d\right) \tag{3.42}$$

Apart from utilizing $\boldsymbol{e}$ and $\boldsymbol{\omega}$ instead of the metric $\boldsymbol{g}$ the main difference between the EH and the EC formulations of gravity is of course the allowance for torsion. The torsion is however confined to matter fields, this can be seen by looking at the corresponding equations of motion for the *free field theory* obtained by varying $\boldsymbol{e}$ and $\boldsymbol{\omega}$ respectively

$$\begin{aligned} \epsilon_{abcd} e^b \wedge \left(\mathcal{R}^{bc}_\omega - \frac{\Lambda}{3} e^c \wedge e^d\right) = 0 \\ \epsilon_{abcd} \mathbf{d}_\omega \left(e^c \wedge e^d\right) = 0 \end{aligned} \tag{3.43}$$

These equations reduces to Einsteins equations if there exist an inverse of the vielbein field. That is, there exist a field e^μ_a such that

$$e^\mu_a e^b_\mu = \delta^b_a \tag{3.44}$$

This demand is equivalent to adding a metric structure on $\mathcal{M}$ because an inverse vielbein field may be used to define a *nonsingular* metric $g_{\mu\nu}$ by

$$g_{\mu\nu} = e^a_\mu e_{\nu a} \tag{3.45}$$

Under these assumptions (3.43) reduce to usual equations of general relativity since the inverse vielbein may be used to eliminate e^a_ν from the equations

$$\begin{aligned} & e^\nu_a e^\lambda_b \mathcal{R}^{ab}_{\mu\nu} \equiv \mathcal{R}^\lambda_\mu = 0 \\ & \mathcal{T}^a_{\mu\nu} = 0 \end{aligned} \tag{3.46}$$

However, in the presence of matter fields the torsion will in general not vanish since it is not a constraint of the theory like in PGT or GR. Due to the absence of such a constraint the gauge fields $\boldsymbol{e}$ and $\boldsymbol{\omega}$ *remain independent* rendering EH much more similar to YM-theory than the other gravitational theories which so far have been presented.

3.8 de Sitter gauge theory

A possible geometrical approach on de Sitter gravity on a 4-dimensional spacetime manifold $\mathcal{M}$ lies in combining the spin connection $\boldsymbol{\omega}$ and the vielbein field $\boldsymbol{e}$ into a unified $SO(2,3)$ connection $\mathcal{A}$ by

$$\mathcal{A} = \boldsymbol{\omega} + \boldsymbol{e}. \tag{3.47}$$

It is known, as will be shown in the following, that in order to correctly mimic the properties of $\boldsymbol{\omega}$ and $\boldsymbol{e}$ in gravitational theories the connection $\mathcal{A}$ should not transform under the full symmetry of $\mathcal{G} = SO(2,3)$. Instead a symmetry broken by a nonlinear realization of $\mathcal{G}$ is considered. This is accomplished by a separation of $\mathcal{G}$ into its Lorentz subgroup $\mathcal{H} = SO(1,3)$ and the quotient space $\mathcal{K} = \mathcal{G}/\mathcal{H}$. In terms of the Lie algebras this splitting is written

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p} \tag{3.48}$$

where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of $\mathcal{G}$ and $\mathcal{H}$ respectively, and $\mathfrak{p}$ contains the generators of $\mathcal{G}$ which are not included in its sub-Lie algebra $\mathfrak{h}$. Let $\mathfrak{g}_A$, $\mathfrak{h}_i$ and $\mathfrak{p}_a$ be the generators of the corresponding Lie algebras, then

$$\{\mathfrak{g}_A\} = \{\mathfrak{h}_i, \mathfrak{p}_a\}. \tag{3.49}$$

This splitting in the Lie algebra is invariant under Lorentz transformations. In the following the splitting is assumed to correspond to a *Cartan decomposition* so that the Lie algebra commutation relations may be generally written as

$$\begin{aligned} [\mathfrak{h}_i, \mathfrak{h}_j] &= f_{ij}{}^k \mathfrak{h}_k \\ [\mathfrak{h}_i, \mathfrak{p}_a] &= f_{ia}{}^b \mathfrak{p}_b \\ [\mathfrak{p}_a, \mathfrak{p}_b] &= f_{ab}{}^i \mathfrak{h}_i \end{aligned} \tag{3.50}$$

Now turn the attention to the gauge fields corresponding to the presented symmetries. Let barred symbols denote the usual, linear, Yang-Mills (YM) fields of the full, unbroken, $SO(2,3)$ symmetry and expand in terms of the Lie algebra generators

$$\bar{\mathcal{A}} = \bar{\mathcal{A}}^A \mathfrak{g}_A = \bar{H}^i \mathfrak{h}_i + \bar{K}^a \mathfrak{p}_a \tag{3.51}$$

A new gauge field $\mathcal{A}$, corresponding to a nonlinear realization of $SO(2,3)$ is introduced according to the following identification[11]

$$\bar{\mathcal{A}} = k^{-1}\big(\mathbf{d} + \underbrace{H^i \mathfrak{h}_i + K^a \mathfrak{p}_a}_{\mathcal{A}}\big)k \qquad k = e^{\xi^a \mathfrak{p}_a} \tag{3.52}$$

where $\mathbf{d}$ is the de Rham differential and the parameters $\xi \in \mathbb{R}^{1,3}$ are a set of 4 scalar *fields* which, until contact with gravity is made, are left arbitrary. The new gauge fields comprising $\mathcal{A}$ may be expressed in terms of ξ and the barred fields as

$$H^i_\mu = \bar{H}^i_\mu \tag{3.53}$$

$$K^a_\mu = \bar{K}^a_\mu + \partial_\mu \xi^a + f^a{}_{ib} \bar{H}^i_\mu \xi^b \tag{3.54}$$

The geometrical importance of ξ as a Goldstone field will not be pursued here, but note that the Lorentz gauge fields are independent of ξ, thus the subgroup $SO(1,3)$ still generates linear transformations.

In the neighbouhood of the identity of $\mathcal{G}$ a group element may be expressed as $g = kh$. Specifying k by the scalar fields ξ thus gives a unique correspondence between g and an element $h \in \mathcal{H}$. Without referring to a specific k this relation may generally be written as

$$gk = k'h \qquad k, k' \in \mathcal{K} \tag{3.55}$$

Using the above correspondence the new fields H^a and K^i has the following transformation properties under the action of $g \in \mathcal{G}$

$$\tilde{H}^i = hH^ih^{-1} + hdh^{-1} \tag{3.56}$$

$$\tilde{K}^a = hK^ah^{-1} \tag{3.57}$$

Concluding that K^a transform *covariantly* and do *not* mix with H^i the following splitting in the full covariant derivative $\widehat{\mathcal{D}}_\mu$ is sensible

$$\widehat{\mathcal{D}}_\mu = \underbrace{\partial_\mu + H^i_\mu \mathfrak{h}_i}_{\mathcal{D}_\mu} + K^a_\mu \mathfrak{p}_a = \mathcal{D}_\mu + K_\mu \tag{3.58}$$

In a nonlinear realization the generators $\mathfrak{p}_a$ no longer generates symmetries, thus the only gauge degrees of freedom which are left is the Lorentz symmetry. This can also be seen from a constructed relation

$$hk = \underbrace{hkh^{-1}}_{k'} h \tag{3.59}$$

by identifying k' in comparing with (3.55). Here the gauge transformation of an element $h \in \mathcal{H}$ is seen to be independent of k, thus also on ξ, leaving the Lorentz symmetry unbroken. From this fact $\mathcal{D}_\mu$ in (3.58) is identified as the usual covariant derivative, i.e. the covariant derivative w.r.t. the remaining symmetries.

$$\mathcal{D}_\mu = \partial_\mu + H_\mu \tag{3.60}$$

The gauge fields K_μ also transform covariantly (3.57) thus ensuring the necessary covariance of the object $\widehat{D}_\mu$ under $\mathcal{G}$. In a linear realization the total of $\widehat{D}_\mu$ would of course also transform covariantly, but not separately in $\mathcal{D}_\mu$ and K_μ as in this case.

In order to make contact with gravity note that the spin connection $\omega_{\mu\dot{b}}^{\dot{a}}$ is nothing but the adjoint representation ρ of the Lorentz gauge field H^i_μ. The dotted indices corresponds to the *orthonormal frame* on $\mathcal{M}$ and are manipulated with the Minkowski metric $\eta_{\dot{a}\dot{b}}$. The relation between H^a_μ and the vierbein $e^{\dot{a}}_\mu$ has to be imposed by hand, in contrast with the previous relation, through identifying the scalar field $\xi \in \mathbb{R}^{1,3}$ with the coordinates $\{x^\mu\}$ in $\mathcal{M}_{[13]}$

$$\xi^a = \delta^a_\mu x^\mu \tag{3.61}$$

This identification is possible because of the agreement in transformation properties of $e^{\hat{a}}_{\mu}$ and K^a_{μ}, and in the dimension of the corresponding spaces $\dim \mathcal{M} = \dim \mathcal{G}/\mathcal{H}$. By this argumentation the connection $\mathcal{A}$ is identified with the spin connection and vielbein:

$$\left.\begin{array}{l} H^i_{\mu} \overset{\rho}{=} \omega^{\dot{a}}_{\mu\dot{b}} \\ K^a_{\mu} \simeq e^{\hat{a}}_{\mu} \end{array}\right\} \Longrightarrow \mathcal{A}_{\mu} = \omega_{\mu} + e_{\mu} \tag{3.62}$$

Considering a principal bundle $\mathcal{P}(\mathcal{M}, \mathcal{G})$ the above identifications puts the base $\mathcal{M}$ in a one-to-one correspondence with the $\mathcal{G}/\mathcal{H}$ part of the structure group

$$\mathcal{M} \simeq \mathcal{G}/\mathcal{H} \tag{3.63}$$

The dots on the indices are from here on dropped and $\dot{a}\dot{b}$ will whenever possible be denoted by just $\hat{a}$ so that the new commutation relations are just the ones in (3.50) with i-indices replaced by $\hat{a}$. But keep in mind the new indices intimate connection with space-time as they correspond to the orthonormal frame field on $\mathcal{M}$. Taking values in the Lie algebra $\mathfrak{g} = \mathfrak{so}(2,3) = \mathfrak{so}(1,3) \oplus \mathfrak{p}$ the gauge fields e^a_{μ} and $\omega^a_{\mu b}$ may still be expanded in the generators so that $\widehat{\mathcal{D}}_{\mu}$ may be rewritten as

$$\widehat{\mathcal{D}}_{\mu} = \partial_{\mu} + \omega^{\hat{a}}_{\mu}\mathfrak{h}_{\hat{a}} + e^a_{\mu}\mathfrak{p}_a \qquad \omega^{\hat{a}} = \omega^{ab}_{\mu} = \eta^{bc}\omega^a_{\mu c} \tag{3.64}$$

The full curvature $\widehat{\mathcal{F}}_{\mu\nu}$ is related to $\widehat{\mathcal{D}}_{\mu}$ in the usual sense

$$[\widehat{\mathcal{D}}_{\mu}, \widehat{\mathcal{D}}_{\nu}] = \widehat{\mathcal{F}}^A_{\mu\nu}\mathfrak{g}_A = \mathcal{F}^{\hat{a}}_{\mu\nu}\mathfrak{h}_{\hat{a}} + \widetilde{\mathcal{F}}^a_{\mu\nu}\mathfrak{p}_a \tag{3.65}$$

By calculating how the nonlinear realization and the splitting (3.64) in $\widehat{\mathcal{D}}_{\mu}$ carries through in the curvature two important results are found, the first being

$$\mathcal{F}^{\hat{a}}_{\mu\nu} = \underbrace{\partial_{\nu}\omega^{\hat{a}}_{\mu} - \partial_{\mu}\omega^{\hat{a}}_{\nu} + f_{\hat{b}\hat{c}}{}^{\hat{a}}\omega^{\hat{b}}_{\nu}\omega^{\hat{c}}_{\mu}}_{\mathcal{R}^{\hat{a}}_{\mu\nu}} + f_{ab}{}^{\hat{a}}e^a_{\nu}e^b_{\mu} \tag{3.66}$$

where $\mathcal{R}^{\hat{a}}_{\mu\nu}\mathfrak{h}_{\hat{a}} = [\mathcal{D}_{\mu}, \mathcal{D}_{\nu}]$ is the curvature with respect to the Lorentz subgroup $\mathcal{H}$ and the remaining term will be shown to correspond to a cosmological constant. The second result is obtained via an investigation of $\widehat{\mathcal{F}}^a_{\mu\nu}$

$$\widetilde{\mathcal{F}}^a_{\mu\nu} = \partial_{\mu}e^a_{\nu} - \partial_{\nu}e^a_{\mu} + f_{\hat{c}b}{}^a\omega^{\hat{c}}_{\nu}e^b_{\mu} = \underbrace{\mathcal{D}_{\mu}e^a_{\nu} - \mathcal{D}_{\nu}e^a_{\mu}}_{\mathcal{T}^a_{\mu\nu}} \tag{3.67}$$

where the torsion tensor $\mathcal{T}^a_{\mu\nu}$ is identified. The choice of gauge (3.61) singles out a specific 6-dimensional hypersurface in the 10-dimensional group space of $\mathcal{P}(\mathcal{M}, \mathcal{G})$. On this hypersurface $\widetilde{\mathcal{F}}^a_{\mu\nu} = \mathcal{T}^a_{\mu\nu}$ and there is no need for this identification to be imposed from outside.

Attempting to construct an action in complete analogy with Yang-Mills theory

$$\tilde{I} = \int tr(\star\mathcal{F} \wedge \mathcal{F}) \tag{3.68}$$

presents us with the problem that the *Hodge star operator* $\star$ depends explicitly on the metric. Introducing another sort of dual $*$ which instead acts on the Lie algebra indices, according to $*\mathcal{F}^{\hat{a}} = \epsilon_{\hat{a}\hat{b}}\mathcal{F}^{\hat{a}} = \mathcal{F}_{\hat{b}}$. The metric in the orthonormal basis is just η_{ab} which allows an action of the form

$$I = \int tr(*\mathcal{F} \wedge \mathcal{F}) = \int \epsilon_{\hat{a}\hat{b}}\mathcal{F}^{\hat{a}}_{\mu\nu}\mathcal{F}^{\hat{b}}_{\lambda\kappa}\epsilon^{\mu\nu\lambda\kappa}d^4x \tag{3.69}$$

This action is not invariant w. r. t. the full gauge group $\mathcal{G}$ due to the nonlinear realization but rather w. r. t. the linearly realized subgroup $\mathcal{H}$. Substituting the relation $\mathcal{F}^{\hat{a}}_{\mu\nu} = \mathcal{R}^{\hat{a}}_{\mu\nu} + f_{ab}{}^{\hat{a}}e^a_\nu e^b_\mu$ into the above expression the following decomposition of the action is obtained

$$I = \int \epsilon_{\hat{a}\hat{b}}\big(\mathcal{R}^{\hat{a}}_{\mu\nu}\mathcal{R}^{\hat{b}}_{\lambda\kappa} + 2\mathcal{R}^{\hat{a}}_{\mu\nu}e^b_\lambda e^a_\kappa f_{ab}{}^{\hat{b}} + e^a_\mu e^b_\nu e^c_\lambda e^d_\kappa f_{ab}{}^{\hat{a}} f_{cd}{}^{\hat{b}}\big)\epsilon^{\mu\nu\lambda\kappa}d^4x \tag{3.70}$$

Note that the action, also in this expanded form, consists solely of $\mathcal{H}$-covariant objects ($\mathcal{R}$, $\boldsymbol{e}$ and scalars) due to the nonlinear realization. Normally $\mathcal{R}$ is covariant but none of the gauge fields themselves.

The last two terms are precisely the Einstein-Cartan action so that we may write

$$I = \int tr\big(*\mathcal{R} \wedge \mathcal{R}\big) + I_{EC} \tag{3.71}$$

The first term is a topological term called the *Euler characteristic*, being topological this term do not contribute to the equations of motion. Thus, de Sitter gravity constitutes a geometric derivation of the Einstein-Cartan theory from gauge principles in a way which bears striking similarities with the formulation of Yang-Mills theory. Not being invariant under the full gauge group still differentiates between the two theories.

4 epilogue & outlook

The de Sitter gauge theory of gravity in the way which has been presented in the previous section was invented and developed by Mansouri and MacDowell during the late 1970s. What one may regard as unsatisfactory in their theory is the failure of their action to be invariant under the full symmetry group $SO(2,3)$, which is the starting point for unifying the vielbein and spin connection.

The theory can be set up for a spontaneous symmetry breaking in a way presented by Stelle and West[15] a couple of years later. Their action regains the full symmetry by the introduction of a new field which in a way compensates for the deficiency in the Mansouri-MacDowell formulation. However, the sole purpose of the introduced field is to break the symmetry, it has no physical interpretation like the Higgs mechanism in the standard model. Thus one may still ask if there exist a more natural, geometrically derivable, cause behind the symmetry breaking.

Before addressing this question we introduce the notion of an extended tangent space which, apart from being mathematically interesting, opens up new doors for physical theories.

4.1 extended tangent bundle

We introduce the formalism of tangent bundles with higher dimensionality than the base space. Our reference is a paper by Chamseddine from 2010[16] but we keep the results slightly more general in order to connect it to our outlook.

Let $\mathcal{M}$ be a smooth m-dimensional manifold covered by an atlas with the coordinate basis $\boldsymbol{e}_\mu = \partial/\partial x^\mu$, a metric and a metric inverse is then defined by

$$\boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu = g_{\mu\nu} \qquad g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu \tag{4.1}$$

Following Chamseddine we now consider an N-dimensional tangent bundle $\mathcal{TM}$, where $N \geq m$, spanned by orthonormal vector fields $\boldsymbol{v}_A$ with respect to the Minkowski metric

$$\boldsymbol{v}_A \cdot \boldsymbol{v}_B = \eta_{AB} \qquad \eta^{AC}\eta_{CB} = \delta^A_B \tag{4.2}$$

so that local Lorentz transformations Λ^B_A preserve orthonormality

$$\tilde{\boldsymbol{v}}_A = \Lambda^B_A \boldsymbol{v}_B \qquad \Lambda^A_C \Lambda^B_D \eta_{AB} = \eta_{CD} = \tilde{\boldsymbol{v}}_C \cdot \tilde{\boldsymbol{v}}_D \tag{4.3}$$

Note that $\boldsymbol{\eta}$ is constant whereas $\boldsymbol{g}$ generally is not. Greek indices run from 0 to $m-1$ and capital Latin indices run from 0 to $N-1$.

We expand the coordinate basis $\boldsymbol{e}_\mu$ in terms of the orthonormal basis $\boldsymbol{v}_A$

$$\boldsymbol{e}_\mu = e^A_\mu \boldsymbol{v}_A \tag{4.4}$$

the vielbein e_μ^A then gives an expression for $g_{\mu\nu}$

$$g_{\mu\nu} = e_\mu^A e_\nu^B \eta_{AB} \tag{4.5}$$

Note that an inverse relation is not canonically defined since, in general, N may be larger than d resulting in e_μ^A not being invertible. We may, however, define a useful object e_A^μ by

$$e_A^\mu = g^{\mu\nu}\eta_{AB}e_\nu^B \tag{4.6}$$

with the properties

$$e_A^\mu e_\nu^A = \delta_\nu^\mu \quad \text{but} \quad e_A^\mu e_\mu^B \neq \delta_A^B \tag{4.7}$$

In order to find an expression for $e_A^\mu e_\mu^B$ along a canonical yet non-trivial path we have to restrict ourselves to the case where $N = d + 1$.[16] In this case $\mathcal{TM}$ may be spanned by $\{\boldsymbol{e}_\mu, \boldsymbol{n}\}$, where $\boldsymbol{n}$ is a mutually orthogonal base vector. In this new basis $\boldsymbol{v}_A$ may be expanded as

$$\boldsymbol{v}_A = v_A^\nu \boldsymbol{e}_\nu + n_A \boldsymbol{n}. \tag{4.8}$$

Using this expression for $\boldsymbol{v}_A$ in (4.4) and the orthogonality condition of $\boldsymbol{n}$ we find

$$\boldsymbol{e}_\mu = e_\mu^A v_A^\nu \boldsymbol{e}_\nu + \underbrace{e_\mu^A\ n_A}_{0} \boldsymbol{n} \quad \Longrightarrow \quad e_\mu^A v_A^\nu = \delta_\mu^\nu \tag{4.9}$$

By the relation $e_A^\mu e_\nu^A = \delta_\nu^\mu$ we identify the components v_A^ν with those of e_A^ν and we obtain

$$\boldsymbol{v}_A = e_A^\nu \boldsymbol{e}_\nu + n_A \boldsymbol{n}. \tag{4.10}$$

Writing out the metric η_{AB} in this expansion we get

$$\eta_{AB} = \boldsymbol{v}_A \cdot \boldsymbol{v}_B = e_A^\mu e_B^\nu g_{\mu\nu} + n_A n_B R \tag{4.11}$$

multiplying this equation by η^{CB}, relabel $C \to B$, and solving for the first term on the RHS we finally obtain an expression for $e_A^\mu e_\mu^B$

$$e_A^\mu e_\mu^B = \delta_A^B - n_A n^B R \equiv P_B^A \tag{4.12}$$

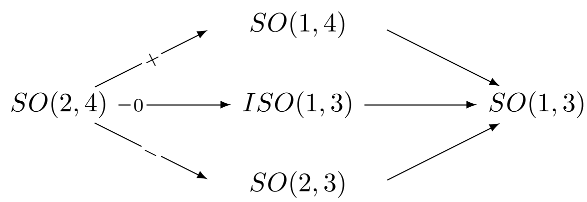
where we see that it takes the form of a projection operator P_B^A satisfying

$$P_C^A P_B^C = P_B^A \tag{4.13}$$

In the pursuit of this result we applied a certain frame $\{\boldsymbol{e}_\mu, \boldsymbol{n}\}$ which can be seen as an analog to the Darboux frame in Riemannian geometry. A big difference, however, is that no embedding of $\mathcal{M}$ is needed due to our extension of the tangent space.

4.2 symmetry reduction of $SO(2,4)$

The Lie group $\mathcal{G} = SO(2,4)$ is a common supergroup of; the de Sitter, the anti-de Sitter and the Poincare symmetry groups. It is known that the action of $SO(2,4)$ on a 6-dimensional vector space which keeps a certain vector $\boldsymbol{n}$ invariant is subject to the following symmetry reductions:



where $+$, $-$ and 0 indicates positive, negative and null norm of $\boldsymbol{n}$ respectively. All subgroups are further reducible to the Lorentz group, which represent symmetries that should apply to space-time.

4.3 generalized Cartan geometry

In the light of the possibility of an extended tangent space and the symmetry reductions of $SO(2,4)$ one may anticipate a theory involving a Cartan $SO(2,4)$ connection where the corresponding *Klein geometry*[17] is dependent on the *change* of curvature along a direction in space-time. This generalized theory would reduce to the symmetry of $SO(1,4)$, $SO(2,3)$ or $ISO(1,3)$ in the case of constant curvature.

If there is no infinitesimal change in curvature along a given direction the relevant Klein geometry would be $ISO(1,3)/SO(1,3)$ resulting in Minkowski space and Poincare symmetry. If the change is positive or negative the symmetry group would be $SO(1,4)$ or $SO(2,3)$ respectively, with a curvature of the corresponding Klein geometry matching that of space-time, infinitesimally and given a direction.

Considering the extended tangent bundle this *curvature dependent* symmetry breaking from $SO(2,4)$ to one of its natural subgroups would be represented by the norm of the extra base vector $\boldsymbol{n}$ introduced in the two previous sections. The extended tangent space, being 5-dimensional, allows for topological actions of the Chern-Simons type to be considered for the theory even though space-time would still be 4-dimensional.

Another interesting feature of this theory result from the extension of the local connection one-form into the $\boldsymbol{n}$-direction of the tangent space. Since the norm of $\boldsymbol{n}$ is related to the scalar curvature this extension become significant only at the length scales of particles, where the curvature changes rapidly, resulting in a dramatical change in the behaviour of gravity at the quantum level.

Investigating such a *two-folded* spontaneously broken gauge theory, from $SO(2,4)$ down to $SO(1,3)$, the common stabilizing subgroup of $SO(1,4)$, $SO(2,3)$ and $ISO(1,3)$, would to be an interesting generalization and natural line of research in the field of gravitational gauge theories.

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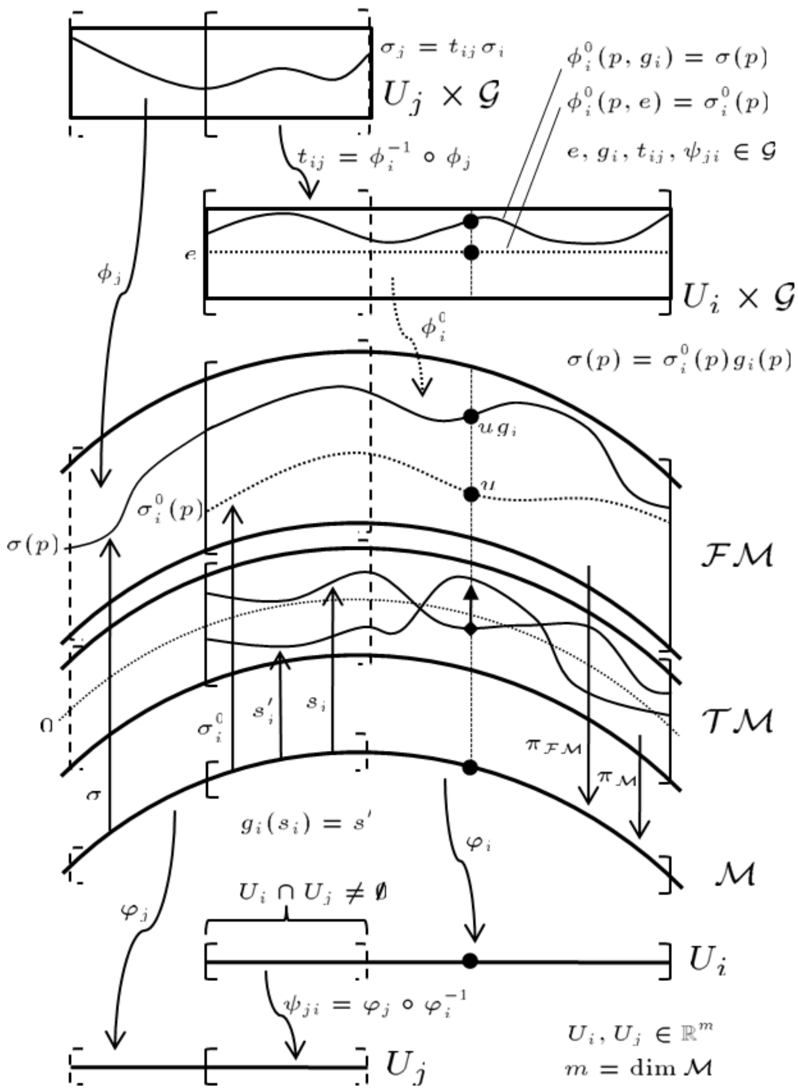

$\sigma_j = t_{ij}\sigma_i$
$U_j \times \mathcal{G}$
$\phi_i^0(p, g_i) = \sigma(p)$
$\phi_i^0(p, e) = \sigma_i^0(p)$
$e, g_i, t_{ij}, \psi_{ji} \in \mathcal{G}$
$t_{ij} = \phi_i^{-1} \circ \phi_j$
e
$U_i \times \mathcal{G}$
ϕ_j
ϕ_i^0
$\sigma(p) = \sigma_i^0(p) g_i(p)$
$u g_i$
u
$\sigma_i^0(p)$
$\sigma(p)$
$\mathcal{FM}$
$\mathcal{TM}$
0
σ_i^0
s_i'
s_i
$\pi_{\mathcal{FM}}$
$\pi_{\mathcal{M}}$
σ
$g_i(s_i) = s'$
φ_i
$\mathcal{M}$
$U_i \cap U_j \neq \emptyset$
φ_j
U_i
$\psi_{ji} = \varphi_j \circ \varphi_i^{-1}$
$U_i, U_j \in \mathbb{R}^m$
$m = \dim \mathcal{M}$
U_j