The Grassmannian σ-model in SU(2) Yang-Mills Theory

David Marsh

Undergraduate Thesis
Supervisor: Antti Niemi

Department of Theoretical Physics
Uppsala University

UPPSALA
UNIVERSITET

Contents

Chapter 1

Introduction

"The Universe embraces with sensual presence. No thought, no word is needed to behold sky, sun, moon, stars. It is not in reflection but in experience that one inhales the earthy smells of damp woods, becomes dizzy under the starry night sky, basks in the sun's warmth, rolls in ocean breakers, tastes salt. The sensual universe is open, available, immediate.

But beneath its sensuality, the Universe has an interior landscape that lies hidden. Though veiled, the inner nature of the cosmos draws us. It invites us to find the underlying source of the motions of the sun, moon, and stars, search out for the primal form of all matter, and discover the unity beneath the diversity of the earthy substances that excite the senses. The inner nature of the cosmos is a landscape of excuisite beauty which appropriately underlies the exterior splendor of our experience. This interior landscape of matter and motion which lies both in and beyond the senses – the physical face of the universe – is mathematical."

David Oliver, "The Shaggy Steed Of Physics" [1]

1.1 Introduction

The scoreboard of the quantum theory of strong nuclear interactions, Quantum Chromodynamics (QCD), is certainly impressive. Not only is its predictions in the perturbative regime of the theory confirmed by numerous carefully settled experiments, including evidence of the running coupling constant, violations of the Bjorken scaling and predictions of the jet structure at the accelerators [2]. It is also supported by substantial numerical evidence from computer simulations, yielding predictions of e.g. the masses of the low-lying mesons to an accuracy of approximately 10%. The triumph is further accentuated by the 2004 recognition from the Nobel committee, rewarding the founders Gross, Wilczek and Politzer the physics prize of that year.

Yet the theory carries an abundance of puzzles and mysteries for its contemporary explorers. Many physicists believe that fundamentally new physics will be needed in order to fully understand it. The importance of this problem is underscored by the Clay Mathematics Institute prescribing it as one of the Millennium Problems, with a designated million dollar prize for anyone managing to solve it properly [3].

At the fundamental, high-energy level of the theory, it contains quarks, particles nobody is ever to see, and the QCD version of the photon: the gluons. Also, at high energies, or equivalently short distances, these particles appear as wore they free, a phenomenon called asymptotic freedom. But yet the freedom is deceptive. Once a particle is viewed at low energies or long distances, the freedom turns into slavery. Why have you never seen a free quark? Where are those gluons the theory predicts? They are trapped forever in the interior of matter. Never will they live freely at large!

The essence of quark and colour confinement has also changed quantum field theory. Computing this or that order in perturbative Feynman diagrams, as worked so nicely in quantum electrodynamics (QED) and in the perturbative regime of QCD, just doesn't cut it anymore. New phenomenon occur, unforeseeable from perturbation theory, and new methods have to be devised to attack these problems. One now well understood example is how the quark and gluon condensate in the vacuum of the theory spontaneously breaks the chiral symmetry of the Lagrangian, resulting in pions and other pseduscalar mesons emerging. Another, not so well understood, example is the nature of confinement. Quantitative insight in the non-perturbative regime of QCD is primarily gained through numerical simulations of the theory, where space-time is replaced by a lattice.

The following text will be devoted to a different, recently discovered phenomenon, that might be another step along the correct path. *Spin-charge separation* means that the gluon fields, usually described by a colour charged spin-1 field under some particular circumstances separate into a charged spinless particle and a chargeless spinning particle. The particle carrying the spin is called the spinon and the charged particle is called the holon. Under normal circumstances, these two particles are tightly bound by a strong, confining force to the gluon known from QCD. In the presence of a gluon condensate, like the vacuum of QCD, this force might become weak so that the spinon and the holon can separate.

Except for the dazzling thought that what we regard as fundamental particles might be appropriately regarded as composite particles, the suggestion might also lead to a deeper understanding of the theory of quark confinement, a point we will come to in section 4.1.2. Interestingly, a similar approach has recently been applied to the theory of high-temperature superconductors, possibly indicating that the solutions to both these notoriously difficult problems might come in one blow.

This *Examensarbete* will deal with some consequences of looking at the $SU(2)$ Yang-Mills theory through the glasses of spin-charge separation, following and continuing the work commenced by Faddeev and Niemi [4]. The underlying, mathematical side to the physics of spin-charge separation is intimately connected with the so called Grassmann manifold of two dimensional planes through the origin of a four dimensional vector space. We will find that the SU(2) field strength has a natural interpretation in terms of coordinates of this manifold and that the dynamics of the pure Yang-Mills theory has an important contribution from fields taking their values in this manifold. We will also see that even the confining force between the spinons and holons stem from the rotational invariance of vectors spanning the plane in the Grassmannian. If true, the significance of the discovery of spin-charge separation in Yang-Mills theory can hardly be overestimated. Suitingly, the physical significance of the problem is accompanied by a well hidden but emerging mathematical elegance.

This *Examensarbete* is disposed with two introductory chapters on Yang-Mills theory, non-linear sigma models and the Grassmannian. Thereafter the SU(2) Yang-Mills Lagrangian is decomposed and spin-charge separation discussed, followed by the verification of the claimed appearance of the Grassmannian in this context. Finally we will discuss the coupling between the Grassmannian and a different manifold appearing in the decomposition (a sphere), and reformulate the coupling between these manifolds to a simplified and geometrically transparent form.

Chapter 2

Some Quantum Field Theory

This chapter will treat some of the features of field theory appearing in this *Examensarbete*, namely gauge theories and non-linear sigma models. The choice is idiosyncratic and done of the basis of recalling certain facts in these two areas that will come to use later. Extensive treatments can be found in e.g. [5] - [10] .

2.1 Gauge Theories

Many are the areas of physics that have benefited, or are completely dependent upon, gauge theory. Gauge theory is a formal development of the thought that physics should be independent of our descriptions of it and that the symmetries of our universe should be local. The $U(1)$ invariant theory of the electromagnetic interaction is the most famous example of such a theory. The electroweak theory is built $SU(2) \times U(1)$ and the theory of strong interactions, quantum chromodynamics, is built on $SU(3)$.

Despite the success of these theories, they are in some ways unsettling. The problem being the fundamental redundancy that is the very heart and soul of the theories. Physics is not described in terms of the quantities that really matters, but something that carries much more information. This feature leads to some problems and difficulties, one to which we will turn is the quantization in the path integral formalism of general gauge theories. A method devised by Faddeev and Popov to come to terms with this problem will be recalled, explaining the existence of a so called ”gauge-fixing term” in the Lagrangian of $SU(2)$ Yang-Mills theory, that will be treated later.

Further we will turn to a brief review of non-linear sigma models, in preparation of the treatment of the Grassmannian sigma model in chapter 4.

2.1.1 Basic Yang-Mills Theory

Suppose a Lagrangian is invariant under global transformations of the fields of G, which here is assumed to be special unitary. It is morally reasonable to demand of a theory that these transformations need not to be done globally, but at each space-time point separately (but continuously), i.e. locally. With this requirement a gauge field and a covariant derivative are necessary.

The generators of G are hermitian and satisfy some commutations relations

$$[T_a, T_b] = i f_{abc} T_c \tag{2.1}$$

where f_{abc} are the structure constants. For example, for SU(2) the structure constants are given by ϵ_{abc}. Near the unit element of G can be expressed as

$$U = e^{-i\theta^a \; T_a} \tag{2.2}$$

Suppose that we have some field, perhaps describing an electron, that transforms under $U(x) \in G$ in the defining representation, i.e.

$$\psi \rightarrow U \; \psi \tag{2.3}$$

$$\bar{\psi} \rightarrow \bar{\psi} \; U^\dagger \tag{2.4}$$

So, derivative terms like $\partial\psi$ will not transform covariantly any more. How could it, since by the definition of the derivative we are comparing the values of the field at two different space-time points, but we are allowed to make gauge transformations locally, so the difference can not be well defined any more. The way out is, in analogy with differential geometry, to introduce a covariant derivative with a connection that takes care of the transformation of the coordinate system from one point to another.

Denote $\psi(x + dx) = \psi + d\psi$, and we think of $\psi + \delta\psi$ to be "equal" to $\psi(x)$, when measured at $x + dx$, through the notion of parallel transport. The term $\psi + \delta\psi$ compensates for the how the coordinate system might change from one point to the other, while $\psi + d\psi$ is affected by both how the coordinate systems change and how the field itself changes over the distance [9]. The natural choice of a covariant derivative is therefore

$$D\psi = \psi + d\psi - (\psi + \delta\psi) = (d - \delta)\psi \tag{2.5}$$

The $\delta\psi$ term should be linear in ψ as well as the displacement. There are numerous variations on how to define it exactly, and we will not be faithful to a single one during in this text. Here we choose

$$\delta\psi = igT^a A^a_\mu dx^\mu \psi \tag{2.6}$$

Here g is the coupling constant and A^a_μ are the components of the gauge field. Denote $T^a A_\mu = \mathcal{A}_\mu$.

To make $D_\mu\psi$ transform like covariantly, an inhomogeneous transformation law for $\mathcal{A}_\mu$ has to be implemented.

$$\psi \to U\ \psi \tag{2.7}$$

$$\bar{\psi} \to \bar{\psi}\ U^\dagger \tag{2.8}$$

$$\mathcal{A}_\mu = U\mathcal{A}_\mu U^\dagger + \frac{i}{g}U\partial_\mu U^\dagger \tag{2.9}$$

The so called minimally coupled Lagrangian involving gauge fields and electrons obeying the Dirac equation, but not the dynamics of the gauge fields, looks like

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu(\partial_\mu - ig\mathcal{A}_\mu) + m\right)\psi \tag{2.10}$$

The Yang-Mills Field Tensor (the equivalent of curvature in differential geometry) is

$$\mathcal{F}_{\mu\nu} = \partial\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu - ig[\mathcal{A}_\mu, \mathcal{A}_\nu] \tag{2.11}$$

In components

$$F_{\mu\nu c} = \partial A_{\nu c} - \partial_\nu A_{\mu c} + gf_{abc}\ A^a_\mu A^b_\nu \tag{2.12}$$

The interpretation of this quantity as a curvature can be found through the commutator of the covariant derivative

$$[D_\mu, D_\nu] = -ig\mathcal{F}_{\mu\nu} \tag{2.13}$$

The physical meaning of this quantity is that if $\mathcal{F}_{\mu\nu} \neq 0$, then parallel transport of a spinor around a closed loop will have a physical effect.

In the abelian case like U(1), gauge transformation does not affect the field strength. The $\vec{E}$ and $\vec{B}$ fields are unaffected. For a non-abelian theory this is no longer true. Since $[D_\mu, D_\nu]\psi \to U[D_\mu, D_\nu]\psi$ the field strength transforms as

$$\mathcal{F}_{\mu\nu} \to U\mathcal{F}_{\mu\nu}U^\dagger \tag{2.14}$$

The gauge invariant Yang-Mills Lagrangian for the dynamics of the gauge bosons is given by

$$\mathcal{L}_{YM} = -\frac{1}{2}tr(\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}) = \frac{1}{4}F^{\mu\nu a}F_{\mu\nu a} \tag{2.15}$$

A theory only involving the gauge bosons is called *pure*. Note that it contains quadratic, qubic and quartic terms, signalling the self-interaction of the gauge bosons. The gauge bosons are themselfs charged and interact with each other, something photons would never think of.

2.1.2 Faddeev, Popov and Ghosts

The path integral formulation of quantum field theory is treated extensively in several text-books [5] - [10]. We will not spend to much time rewriting what already is written there. This section will be devoted to review one feature of the Lagrangian appearing in the later chapters, namely the existence of a gauge-fixing term in the Lagrangian.

The path integral we want to consider is given by

$$\int \mathcal{D}A \; e^{\frac{i}{4}\int d^4x (F^a_{\mu\nu})^2} \tag{2.16}$$

The partition function can be expressed

$$\mathcal{Z}[J] = \int \mathcal{D}A \; e^{i\int d^4x \frac{1}{2}A_\mu K^{\mu\nu}A_\nu - V(A) + J_\mu A^\mu} \tag{2.17}$$

A change of variables in the path integral

$$\tilde{A}_\mu = A_\mu + K^{-1}_{\mu\nu}J^\nu \tag{2.18}$$

Think of space-time discretized and consider the interesting quantity

$$\tilde{A}_{\mu x}K^{-1}_{xy}{}^{\mu\nu}\tilde{A}_{\nu\ y} = AKA + 2JA + JK^{-1}J \tag{2.19}$$

On the right hand side, both vector and space-time indices are implicit. So, in particular

$$\frac{1}{2}AKA + JK = \tilde{A}K^{-1}\tilde{A} - \frac{1}{2}JK^{-1}J \tag{2.20}$$

Note that the change of variables is linear with unit Jacobian. Finally, the fields $A_\mu(x)$ can be replaced by functional derivatives $-i\frac{\delta}{\delta J^\mu(x)}$ in the potential.

We arriwe at the "central identity" of quantum field theory, which evidently tells us to invert the operator K

$$\int \mathcal{D}A\ e^{\frac{i}{2}\int d^4x(A_\mu K^{\mu\nu}A_\nu - V(A_\mu) + J\cdot A)} = \mathcal{Z}[0]\ e^{\int -V(\delta/\delta J)}\ e^{\frac{-i}{2}\int\int J\ K^{-1}J} \tag{2.21}$$

This turns out to be easier said than done, since K is not invertible. To see this, back to classical electromagnetism where $A_\mu \sim A_\mu + \partial_\mu\Lambda(x)$, where the equivalence is with respect to giving rise to the same electric and magnetic fields. In particular $0 \sim \partial_\mu\Lambda$. Since 0 gives zero when acted on by K , so must also all function of the above form. Explicitly this can be checked by simply plugging it into

$$\mathcal{L} = F_{\mu\nu}F^{\mu\nu} = 2A_\mu(\partial^\mu\partial^\nu - \partial^2 g^{\mu\nu})A_\nu = A_\mu K^{\mu\nu}A_\nu \tag{2.22}$$

In other words, the operator can't be inverted since

$$Ker(K) \neq \{0\} \tag{2.23}$$

Fixing the gauge is therefore a different wording for choosing a representative in each equivalence class to make K invertible. We will do this through a procedure devised by Faddeev and Popov.

Consider the action $S(A)$ that is invariant under transformations g in the gauge group G, $S(A_g) = S(A)$. The measure of the path integral is also invariant $\mathcal{D}A = \mathcal{D}A_g$. It is the gauge transformed field, of course, that is denoted A_g. We want to break the path integral

$$\int \mathcal{D}A\ e^{iS(A)} \tag{2.24}$$

into two parts, one in which all the g dependence is, and the other one, in which the physics is. How this will come out we can already foresee. Since we are integrating over all field values A, we are at each space-time point integrating over the gauge group and over the "representatives". So the volume of the gauge group, which is finite for a compact continuous group, is going to fall out at each point, which gives an over all, infinite constant. Since any overall factor cancels out in the amplitudes and correlation functions, this does not matter for the physics.

Define the Faddeev-Popov determinant, Δ, such that

$$1 = \Delta(A) \int \mathcal{D}g \delta(f(A_g)) \tag{2.25}$$

The function $f(A)$ will serve as gauge condition. This determinant is gauge invariant. To see this, gauge transform it:

$$1/\Delta(A_{g'}) = \int \mathcal{D}g\ \delta(f(A_{gg'})) = \int \mathcal{D}g''\ \delta(f(A_{g''})) = 1/\Delta(A) \tag{2.26}$$

The 1 is then plugged into the integral

$$\int \mathcal{D}A\ e^{iS(A)} = \int \mathcal{D}A\ e^{iS(A)}\Delta(A) \int \mathcal{D}g\delta(f(A_g)) = \tag{2.27}$$

$$= \int \mathcal{D}g \int \mathcal{D}A\ e^{iS(A)}\Delta(A)\delta(f(A_g)) = \tag{2.28}$$

$$= (\int \mathcal{D}g) \int \mathcal{D}A\ e^{iS(A)}\Delta(A)\delta(f(A)) \tag{2.29}$$

In (2.29), a change of variables have been made.

Picking up the battle against the Yang-Mills integral, we recall the gauge transformation (this time with the coupling constant absorbed in A):

$$A_g = gAg^{-1} - i(\partial g)g^{-1} \tag{2.30}$$

infinitesimally looks like

$$A^a_\mu \rightarrow A^a_\mu - f^{abc}\theta^b A^c_\mu + \partial_\mu \theta^a \tag{2.31}$$

and pick the function f such that

$$f(A) = \partial A - \sigma \tag{2.32}$$

Keeping the effects of $\delta(f(A))$ in mind, we informally write

$$\int \mathcal{D}g\ \delta(f(A_g)) = \int \mathcal{D}g\ \delta(\partial^\mu(A^a_\mu - f^{abc}\theta^b A^c_\mu + \partial_\mu\theta^a - \sigma)) \rightarrow \tag{2.33}$$

$$\rightarrow \int \mathcal{D}g\ \delta(\partial^\mu(-f^{abc}\theta^b A^c_\mu + \partial_\mu\theta^a)) \tag{2.34}$$

Defining K^{ab} as

$$K^{ab} = \partial^\mu (f^{abc} A_\mu^c - \partial_\mu \delta^{ab}) \delta^4(x-y) \tag{2.35}$$

and recalling the identity

$$\int dx \; \delta(Kx) = \frac{1}{detK} \tag{2.36}$$

We find $\Delta(A) = detK$. While ordinary Gaussian integrals render the reciprocal of the determinant of the operator, the determinant itself is proportional to a Grassmannian functional integral [5]

$$\Delta(A) = \int \mathcal{D}c\mathcal{D}c^\dagger \; e^{iS_{ghost}(c,c^\dagger)} \tag{2.37}$$

with

$$S_{ghost} = \int \int d^4x d^4y \; c^\dagger(x) K^{ab}(x,y) c_b(y) = \tag{2.38}$$

$$= \int d^4x \; \partial c_a^\dagger D c_a(x) = \int d^4x \mathcal{L}_{ghost} \tag{2.39}$$

where D is the covariant derivative and the fields $c, c^\dagger$ are ghost fields, not corresponding to physical particles. Being scalar fermions they disobey the spin-statistics theorem.

A final point is to replace $\delta(\partial A - \sigma)$ by a Gaussian integral. Since the interesting part of the partition function is independent of σ (i.e. representative), and running through all representatives can be done how we like it, integration over sigma can be performed as

$$Z = \int \mathcal{D}\sigma \int \mathcal{D}c \int \mathcal{D}c^\dagger \; e^{i/2\xi \int d^4x \sigma(x)^2} \int \mathcal{D}A \; e^{i(S(A)+S_{ghost}(c,c^\dagger))} \delta(\partial A - \sigma) = \tag{2.40}$$

$$= \int \mathcal{D}c \int \mathcal{D}c^\dagger \int \mathcal{D}A \;\; e^{i\int d^4x \left((F_{\mu\nu}^a)^2 - \frac{1}{2\xi}(\partial A)^2 + \mathcal{L}_{ghost}\right)} \tag{2.41}$$

The parameter ξ is a gauge parameter. In what follows, we will not be considering this path integral any more, but only the Lagrangian itself. The lesson learned from this section is that in order to quantize a gauge theory, gauge fixing terms and ghosts should be added to the classical Lagrangian.

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{gauge\ fix} + \mathcal{L}_{ghost} \tag{2.42}$$

2.1.3 Geometric Digression

Although it will not come to much use later on, it is nice to warm up on differential forms in the formalism of gauge theory. The covariant derivative is now chosen to be (note the slight change in notation)

$$D = \partial + A \tag{2.43}$$

where the anti-hermitian gauge field $A_\mu dx^\mu$ is taken to be a matrix one-form

$$A^2 = A_\mu A_\nu dx^\mu \wedge dx^\nu = \frac{1}{2}[A_\mu, A_\nu]\ dx^\mu \wedge dx^\nu \tag{2.44}$$

The field strength must be a linear combination of the two possible two-forms constructed out of A, namely dA and $A^2 = A \wedge A$. In this notation the gauge fields transform as

$$A \to UAU^\dagger + UdU^\dagger \tag{2.45}$$

where the U are zero-forms. Looking on how the candidate two–forms transform, we find

$$dA \to UdAU^\dagger + dUAU^\dagger - UAdU^\dagger + dUdU^\dagger \tag{2.46}$$

$$A^2 \to UA^2U^\dagger + UAdU^\dagger - dUAU^\dagger \tag{2.47}$$

The combination $A^2 + dA$ happens to transform very simply however

$$F = dA + A^2 \to UFU^\dagger \tag{2.48}$$

Even better, since $d^2 = 0$, define

$$D = d + A \tag{2.49}$$

then

$$D^2 = (d + A)^2 = d^2 + dA + Ad + A^2 = \tag{2.50}$$

$$= d(A) - Ad + Ad + A^2 = d(A) + A^2 = F \tag{2.51}$$

Neat.

2.2 The non–linear sigma models

The non–linear sigma model is one of those exceedingly successful concepts that manage to find their way into many different areas of physics. The Heisenberg anti-ferromagnet in solid state physics, string world-sheets embedded in the Calabi-Yau manifolds of string theory and various aspects of quantum field theory all include non-linear sigma models. In particular, this *Examensarbete* is about a particular type of non–linear sigma model.

2.2.1 The Linear Sigma Model

The linear sigma model is an $O(N)$ invariant theory for N real scalar fields with a ϕ^4 interaction. Starting from the ordinary ϕ^4 theory and promoting the fields to N-vectors of scalar fields one has the choice of picking sign of the m^2 term. A positive choice gives an ordinary ϕ^4 theory, while the negative gives a more interesting theory, exhibiting the phenomenon of spontaneous symmetry breakdown. In Minkowski space-time the Lagrangian density is then given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i + \frac{1}{2}m^2(\phi^i)^2 - \tag{2.52}$$

$$-\frac{\lambda}{8}((\phi^i)^2)^2 = \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i - \frac{\lambda}{8}((\phi^i)^2 - a^2)^2 \tag{2.53}$$

with $i = 1, \ldots, N$ and letting the second equality prescribe the relation between a and m. Also, in the second equality an uninteresting constant has been dropped.

The potential is the well known *"mexican hat"* which is the canonical example of a theory with spontaneous symmetry breaking. The ground state for this theory is given by the states lying on the $N-1$ sphere with radius a. The O(N) symmetry of the lagrangian is spontaneously broken by the groundstate to O(N-1). To see this note that the potential is

$$V(\phi^2) = \partial_j\phi^i\partial_j\phi^i + \frac{\lambda}{8}\left((\phi^i)^2 - a^2\right)^2 \tag{2.54}$$

which obviously is minimized for a spatially homogeneous field with $(\phi^i)^2 = a^2$. Write

$$\begin{gathered} \phi^i(x) = \pi^i(x) \quad i = 1, \ldots, N-1 \\ \phi^N(x) = a + \sigma(x) \end{gathered} \tag{2.55}$$

to shift the field and expand around the ground state in the N:th direction. Plugging this back into the Lagrangian yields:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\vec{\pi}\partial^\mu\vec{\pi} + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{\lambda}{2}a^2\sigma^2 - -\frac{\lambda}{2}a\sigma^3 - \tag{2.56}$$

$$-\frac{\lambda}{2}a\sigma\vec{\pi}^2 - \frac{\lambda}{4}\vec{\pi}^2\sigma^2 - \frac{\lambda}{8}\vec{\pi}^4 - \frac{\lambda}{8}\sigma^4 \tag{2.57}$$

This gives rise to $N-1$ massless "pions" and one massive sigma particle corresponding to degrees of freedom staying "in the gutter" and "climbing the walls" respectively. This is an example of Goldstone theorem, stating that every broken continuous symmetry leaves behind a massless particle. Since $O(N)$ is $\frac{N(N-1)}{2}$ dimensional and $O(N-1)$ is $\frac{(N-1)(N-2)}{2}$ dimensional, there are $N-1$ broken symmetries in the example above and that explains the $N-1$ massless pions.

The linear sigma model is renormalizeable in $d \leq 4$. Superficially this is because the fields have dimension $\frac{d-2}{2}$ (from the derivative term) which gives a dimensionless coupling constant λ for $d = 4$.

2.2.2 The Historical Example

So, the linear sigma model that aims to describe pions at low energies contains both the pions and a scalar sigma particle. But this sigma particle is massive and would show up experimentally only as very broad resonances high above the typical energies considered in the low-energy treatment of the pions [10]. Would it not then be nice if the theory could be reformulated such that the sigmas don't show up in the theory? It turns out that that is possible.

The non-linear sigma model is obtained by formally taking the $\lambda \to \infty$ limit of the linear sigma model and in that way forcing all the fields (with finite action), not just the classical ground state, to lie on the $N-1$ sphere with radius a [11]. The Lagrangian density becomes:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i \tag{2.58}$$

with $\sum_i \phi^i\phi^i = a^2$. By rescaling $\phi \mapsto a\phi$, the radius a is interpreted as the inverse coupling constant. This should be expected, since letting the radius grow large would be the same as making the theory more like the free theory, which could be expressed as a decrease in the coupling constant. Let $\vec{n}$ denote an N-component unit vector and write the Lagrangian as

$$\mathcal{L} = \frac{1}{2g^2}|\partial_\mu \vec{n}|^2 \tag{2.59}$$

Solve the constraint for the N:th component of the unit vector and express the N-1 degrees of freedom in terms of π-fields. The lagrangian has a clear geometrical interpretation:

$$\mathcal{L} = \frac{1}{2g^2}\left(|\partial_\mu \vec{\pi}|^2 + \frac{(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2}{1-\vec{\pi}^2}\right) = \frac{1}{2}g_{ab}(\pi)\partial_\mu\pi^a\partial^\mu\pi^b \tag{2.60}$$

with $g_{ab} = \frac{1}{g^2}(\delta^{ab} + \frac{\pi^a\pi^b}{1-\vec{\pi}^2})$ being the round metric tensor of the sphere, induced from $\mathbb{R}^N$. ISo, the scalar fields takes on values of the points on the $N-1$ sphere and the model is called the O(N) non-linear sigma model, or sometimes simply the non-linear sigma model.

This way, the model can be interpreted as describing the phenomenological behaviour of the Goldstone Bosons, without reference to any extra, massive sigma field. In other words: it is an effective theory at low energies of some unknown theory that is better behaved at high energies.

The non-linear sigma models are renormalizeable in less than or equal to 2 dimensions, in more than 2 dimension they can arise as effective field theories interpreted as describing the phenomenological behaviour of the Goldstone Bosons, without reference to any extra, massive sigma field.

New physics then comes in when the 2-point correlation function is of the same order as the curvature of the manifold, which is called UV-completion. Comparing this case with the linear sigma model, it's clear that the massive sigma particles have a large effect on the physics on small scales.

2.2.3 Geometric Treatment

The O(N) non-linear sigma model in d dimensions is a field theory from the d-dimensional base manifold (e.g. Minkowski or Euclidean space) to the sphere S^{N-1}. But why just consider the sphere? There is a manifold of manifolds out there, clearly the concept can be generalized.

Let us define a (non-linear) σ model as a map from a d dimensional base manifold (M, g_0) (here taken to be Euclidean), to a *target space*, which is another Riemannian manifold, T, with the metric tensor g.

$$\phi : M \to T \tag{2.61}$$

$$x^a \mapsto \phi^i \tag{2.62}$$

Here $a = 0, \dots, d$ and $i = 1, \dots, D$.

As in the example treated above with $(T, g) = (S^{N-1}, g_{ab} = \frac{1}{g^2}(\delta^{ab} + \frac{\pi^a \pi^b}{1-\vec{\pi}^2}))$, the action will look like

$$S = \int d^d x \; g_{ij}(\phi) \; \partial_a \phi^i \partial_a \phi^j \tag{2.63}$$

The classical equations of motion extremizes the action.

$$0 = \delta S = \int d^d x \; \left(g_{ij\;,k}(\phi) \partial_a \phi^i \partial_a \phi^j \; \delta\phi^k + 2 g_{ij} \partial_a \phi^i \partial_a \delta\phi^j \; \right) = \tag{2.64}$$

$$= \int d^d x \; \delta\phi^k \left(g_{ij\;,k} \partial_a \phi^i \partial_a \phi^j - 2\partial_a (g_{ik} \partial_a \phi^i) \right) \tag{2.65}$$

We find

$$\partial_a (g_{ij} \partial_a \phi^i) - \frac{1}{2} g_{ik,j} \; \partial_a \phi^i \partial_a \phi^k = 0 \tag{2.66}$$

$$\Longrightarrow \; g_{ij} \; \partial^2 \phi^i + \partial_a \phi^i \partial_a \phi^k (g_{ij,k} - \frac{1}{2} g_{ik,j}) = 0 \tag{2.67}$$

$$\Longrightarrow \; g_{ij} \partial^2 \phi^i + \partial_a \phi^i \partial \phi^k \frac{1}{2} (g_{ij,k} + g_{kj,j} - g_{ik,j}) = 0 \tag{2.68}$$

Hitting (2.68) with the inverse metric we obtain an equation, as good looking as the geodesic equation, but with a different interpretation:

$$0 = \partial^2 \phi^i + \frac{1}{2} g^{ij} (g_{kj,l} + g_{lj,k} - g_{kl,i}) \partial_a \phi^k \phi^l \tag{2.69}$$

$$0 = \partial^2 \phi^i + \Gamma^i_{kl} \partial_a \phi^k \partial_a \phi^l \tag{2.70}$$

In general a potential term $V(\vec{\phi})$. can also be subtracted from the lagrangian. We will not consider this case, since it will not appear in the subsequent application to the SU(2) Yang-Mills theory. It will suffice to say that in adding a potential term, the interpretation as Goldstone bosons is lost and the strucutre of the manifold as a homogeneous space (see section 3.2) goes down with it.

Another important but not applicable point in this context is that when doing the functional integration over the paths, the measure transforms as $D\pi\sqrt{g}$ where g denotes the determinant of the metric, as usual when performing integrals over curved spaces.

Chapter 3

Basic properties of the Grassmannian

A Grassmannian, or equivalently a Grassmann manifold is the set of all k-dimensional subspaces of an n-dimensional vector space, V. It is often denoted $G_k(V)$ or $G_{k,n}$. The definition makes sense over any field. The real Grassmannian $G_{k,n}$ can be shown to be a compact manifold of dimension $k(n-k)$, see below. There are many ways to look at a Grassmannian and the range of applicability span from robotics via computer graphics to algebraic geometry and topology. In this text it will figure as the target manifold of a sigma model. In this chapter, a few useful features of this manifold will be considered.

Plücker (or sometimes, Grassmann) coordinates were first introduced in enumerative geometry as coordinates for lines in projective 3-space, but since there's a one-to-one correspondance between those lines and 2-dimensional subspaces of a 4-dimensional vector space, they also provide coordinates for the Grassmannian $G_{2,4}$. Plücker coordinates actually works as coordinates for any Grassmannian.

Many texts treat the first three sections of this chapter, see for instance [14], [15], [17], [18].

3.1 Standard Local Coordinates

Recall a few facts about the projective spaces. The real projective space of dimension N, denoted $\mathbb{R}P^N$, is defined as the set of unoriented rays through the origin of a vector space of dimension $N + 1$, which will be taken to be $\mathbb{R}^{N+1}$. By the above definition it is the simplest example of a (real) Grassmannian

$$\mathbb{R}P^N = G_{1,N+1} \tag{3.1}$$

There are infinitely many ways of supplying this manifold with an atlas. The *homogeneous coordinates*, defined as any of the $N + 1$ coordinates of the points the line passes through except the origin, is however *not* one of them. Clearly, if $\updownarrow \in \mathbb{R}P^N$ and $x = (x_1, \ldots, x_{N+1})$ are homogeneous coordinates of $\updownarrow$, then so is λx, $\lambda \in \mathbb{R}$. One may write

$$x \sim \lambda x \tag{3.2}$$

where the equivalence relation is that it determines the same point of the projective space. Clearly, this setup does not meet the bijective requirement on coordinates in local charts.

To supply the chart with local coordinates, some choice of representative of the equivalence of homogeneous coordinates. One way this can be done is through *inhomogeneous coordinates*. These are defined by the process of wall–building. Put up a N dimensional (hyper) wall at $x_i = 1$. All lines not parallel to the wall, will hit it at some point. In particular, suppose $\updownarrow$ is not parallel to the wall and will collide with it at a certain point $(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_{N+1})$. Throw away the i:th coordinate, which is one for all elements hitting the wall, and call the remaining N vector the inhomogeneous coordinate of $\updownarrow$ in the i:th chart. Suppose that $\updownarrow$ is parallel to the plane. Then no coordinate can be given to the line in this chart. On the other hand, since it goes through the origin $x_i = 0$. This can not hold for all $i = 1, \ldots, N + 1$, since there is no ray with that homogeneous coordinate.

Less talk and more math, given any homogeneous coordinate of $\updownarrow$, if $x_i \neq 0$, the local inhomogeneous coordinates in the i:th chart are given by

$$(x_1/x_i, x_2/x_i, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots, x_{N+1}/x_i) \tag{3.3}$$

Different ways of finding an atlas can be obtained, for example, since the sphere S^N is the manifold of oriented lines through the origin of $\mathbb{R}^{N+1}$, the real projective space is obtained from the sphere by identification of antipodal points. Examples will be found throughout this text.

Returning to Grassmannian, one way to represent an element of $G_{k,n}$ is by taking k vectors spanning the element and make them rows in a matrix. This provides a useful approach for supplying the manifold with an atlas. We do this explicitly here for $G_{2,4}$. Let V denote the vector space and $\{e_1, \ldots, e_4\}$ be a basis of it.

Given $W \in G_{2,4}$, let w_1, w_2 be a basis of W. Represent W by a 2×4 matrix, A, whose row vectors are w_1, w_2 with respect to the basis of the vector space. Another matrix, B, represent the same element if $A = gB$ for some $g \in GL(2, \mathbb{R})$. This divides denotes the set of all non-singular 2×4 matrices, $M_{2,4}$, into equivalence classes. It follows that $G_{2,4}$ can be written as the *coset space*

$$M_{2,4}/GL(2, \mathbb{R}) \tag{3.4}$$

Let I be 2 numbers of the set $\{1, 2, 3, 4\}$ and define U_I to be the set of all $W \in G_{2,4}$ such that there exists a matrix representative A for W whose Ith 2×2 minor is non-singular. This property is independent of representative for W. Any $W \in U_I$ can be represented uniquely by a matrix W^I whose Ith 2×2 minor is the identity matrix. For example, an element of U_I for $I = \{1, 3\}$ is of the form

$$\begin{pmatrix} 1 & * & 0 & * \\ 0 & * & 1 & * \end{pmatrix} \tag{3.5}$$

where the stars are some arbitrary real numbers. On the other hand, any matrix that looks like this determines a unique plane in R^4, so there's a bijection $\phi_I : U_I \to R^4$ for each I. The stars are the coordinates of U_I. This shows that $G_{2,4}$ is 4 dimensional.

Since not all minors of a matrix representative for an arbitrary $W \in G_{2,4}$ can be zero, $W \in U_I$ for some I. Also $\phi_I(U_I \cap U_{I'})$ is an open set.

3.2 Plücker coordinates

There's a better way to assign coordinates to the Grassmannian that continues where the above reasoning stops. The following argument will show how to look upon $G_{2,4}$ as a quadric in $\mathbb{R}P^5$.

Take $W \in G_{2,4}$. As mentioned above, it is the row space of a 2×4 matrix. The sextuple of 2×2 minors of this matrix determines W. The sextuple is called the Plücker coordinates for W. Explicitly, if x_{ij} denotes a matrix element of a matrix representative A of W, then

$$p_{ij} := x_{1i}x_{2j} - x_{1j}x_{2i} \tag{3.6}$$

These Plücker coordinates are sometimes called *global coordinates* for $G_{2,4}$, while the procedure mentioned in the first approach leads to a different set coordinates, called the *standard local coordinates* for the Grassmannian. Note that forming a 4×4 matrix with the entries p_{ij}, gives an anti-symmetric matrix with $\frac{4 \cdot 3}{2}$ independent entries.

Changing the basis of W has the effect of multiplying A on the left with a non-singular 2×2 matrix, $g \in GL(2, \mathbb{R})$, which in turn affects p_{ij} by multiplication of a nonzero constant.

$$g : p_{ij} \mapsto detg \; p_{ij} \tag{3.7}$$

For $G_{2,4}$ there are $\left(\begin{array}{c} 4 \\ 2 \end{array} \right) = 6$ minors and by what's just said, a map

$$\pi : G_{2,4} \longrightarrow \mathbb{R}P^{6-1} \tag{3.8}$$

by sending W to its Plücker coordinates, arranged in some specific order.

$\mathbb{R}P^5$ is much bigger than $G_{2,4}$ though, so this map will not be surjective. In fact it's easy to see what demands one have to place on the Plücker coordinates in $\mathbb{R}P^5$ in order for them to correspond to a point in $G_{2,4}$.

Take A as above and assume for a moment that the left-most 2×2 minor is non-singular. Multiply A by the inverse of this minor to get the matrix:

$$A' = \left(\begin{array}{cccc} 1 & 0 & a & b \\ 0 & 1 & c & d \end{array} \right) = \left(\begin{array}{cccc} 1 & 0 & (32) & (42) \\ 0 & 1 & (13) & (14) \end{array} \right) \tag{3.9}$$

with the assignments $(xy) = \frac{P_{xy}}{P_{12}}$. The minor formed out of the 3:rd and 4:th columns is $(32)(14) - (42)(13) = (34) \Rightarrow$

$$p_{12}p_{34} = p_{13}p_{24} + p_{14}p_{32} \tag{3.10}$$

This is the (in some parts of mathematics, famous) Plücker relation. It is easy to see that it holds even when the first minor is singular.

A point $P_I \in P^5$ belongs to the image of π if and only if the coordinates of P_I satisfy the Plücker relation. This means that $G_{2,4}$ is what is called a projective subvariety of $\mathbb{R}P^5$.

It's also possible to define the Plücker embedding of $G_{2,4}$ in $\mathbb{R}P^5$ without explicit reference to the coordinates. This is preferred by many mathematicians, and we will also find it useful in applications later. This would be done by letting π define a map from G to the projective space over $\wedge^2 V$ with $W \in G_{2,4} \to P[\wedge^k W]$. The exterior vectors $v_I = v_{i_1} \wedge v_{i_2}$ form a basis of $\wedge^2 V$. An arbitrary vector $\tau \in \wedge^2 V$ can be written as $\tau = \sum_I P_I v_I$. It is called decomposable if it lies in the image of π, i.e. if it can be written as $w_1 \wedge w_2$ for some vectors $w \in V$. So, that an element of $\wedge^2 V$ is decomposable says that P_I satisfies the Plücker relation, and vice versa.

3.3 The Grassmannian as a homogeneous space

Here a standard treatment of the Grassmannian as a homogeneous space will follow [15]. From the concept of homogeneous spaces, it is intuitively clear that the $G_{2,4}$ can be represented as $O(n)/[O(k) \times O(n-k)]$, since any plane can be transformed to any other by means of some $O(4)$ transformation, and the transformations in and orthogonal to the plane, $O(2) \times O(2)$ is the stabilizer. In this section this will be proven in a few more words. One benefit will be the development of some new techniques for dealing with the Grassmannian, which will prove useful in chapter 4.

Let G be a Lie group and M a manifold. The *action* σ of G on M is a map $\sigma : M \to M$ satisfies the two conditions

$$\sigma(e, p) = p \quad \forall p \in M \tag{3.11}$$

$$\sigma(g_1, g_2(p)) = \sigma(g_1 g_2, p) \tag{3.12}$$

The action σ is *transitive* if for any pair of points $p_1, p_2 \in M$ $\exists g \in G$ such that $\sigma(g, p_1) = p_2$

The *orbit* of p under σ is the subset of M:

$$Gp := \{\sigma(g, p) | g \in G\} \tag{3.13}$$

The *isotropy subgroup* (little group, stabilizer) of $p \in M$ is a subgroup of G:

$$H(p) := \{g \in G \; | \sigma(g, p) = p\} \tag{3.14}$$

H(p) is a Lie subgroup for all points in M. If G is a Lie group and H any subgroup then G/H is a coset space that admitts differentiable structure and G/H becomes a manifold called a homogeneous space with $\dim G/H = \dim G - \dim H$.

If G is a Lie group which acts transitively on M then $G/H(p)$, for any point p gives a homogeneous space and if "certain technical requirements" are fullfilled, then there is a continuous map between M and $G/H(p)$, which has a continuous inverse (i.e. a homeomorphism). We write

$$G/H(p) \cong M \tag{3.15}$$

Consider now the Grassmannian. Take $A \in G_{k,n}$ and let P_A be a $n \times n$ matrix that projects $x \in R^n$ on A. In this section we will for clarity use the "bra–ket" formalism of Dirac.

Let $\{|e_1>, \dots, |e_n>\}$ be an orthonormal basis of R^n and $\{|f_1>, \dots, |f_k>\}$ an orthonormal basis of A. The second pair of basis vectors can be expressed in terms of the first pair as $|f_a>= f_{ai}|e_i>$ for $a = 1, \dots, k$, summing over $i = 1 \dots, n$.

The projection operator P_A acting on $x \in R^n$ can be written:

$$P_a|x>= |f_a><f_a|x>= f_{ai}f_{aj}|e_i><e_j|x>= x_j f_{ai}f_{aj}|e_i> \tag{3.16}$$

So the components of P_A can be written as the matrix with components

$$(P_A)_{ij} = f_{ki}f_{kj} \tag{3.17}$$

with respect to the $\{e_i\}$-basis. It is easy to see that $P_a^2 = P_a$, $P_A^T = P_A$ and $trP_A = k$.

Any matrix that satisfies these requirements determines a unique element of $G_{k,n}$.

Now, take $g \in O(n)$ and let $P_B = gP_Ag^{-1}$. Then P_B satisfies the same 3 conditions as P_A and determines an element $B \in G_{k,n}$. Denote this action of $O(n)$ on $G_{k,n}$ as $\sigma(g, A) = B$. Since any k-dimensional ON-basis can be reached from $\{|f_i>\}$ under the action of some element in $O(n)$, it's clear that $O(n)$ acts transitively on $G_{k,n}$.

Take C to be the element in $G_{k,n}$ spanned by the k first vectors in the standard basis $\{|e_i>\}$. An element M of the isotropy group at A can be represented as a matrix as $M = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$, where $g_1 \in O(k)$ and since $M \in O(n)$, $g_2 \in O(n-k)$.

The isotropy group is isomorphic to $O(k) \times O(n-k)$. So by the above general theory:

$$G_{k,n} \cong O(n)/[O(k) \times O(n-k)] \tag{3.18}$$

The dimension of $G_{k,n}$ is then $\dim G_{k,n} = \dim O(n) - \dim[O(k) \times O(n-k)] = 1/2n(n-1) - (1/2k(k-1) + 1/2(n-k)(n-k-1)) = k(n-k)$. It also follows that $G_{k,n}$ is compact.

3.4 The complexified four-vector

In a recent article by Faddeev and Niemi [4], another set of coordinates were introduced for the $G_{2,4}$. Although we will postpone the treatment of the setting in which those coordinates arise until the next chapter, for future convenience we here inspect some of their features.

Take a point in the Grassmannian $G_{2,4}$. It can be represented by a plane in a $d = 4$ vector space. Let this plane be determined by two orthonormal unit vector that spans it: e^1_μ e^2_μ.

Define a *complex* unit vector:

$$e_\mu = \frac{1}{\sqrt{2}}(e^1_\mu + ie^2_\mu) \tag{3.19}$$

and let $\bar{e}_\mu$ denote its complex conjugate. Then

$$e_\mu e_\mu = \bar{e}_\mu \bar{e}_\mu = 0 \tag{3.20}$$

$$\bar{e}_\mu e_\mu = 1 \tag{3.21}$$

The inverted expressions are:

$$e^1_\mu = \frac{1}{\sqrt{2}}(e_\mu + \bar{e}_\mu) \tag{3.22}$$

$$e^2_\mu = \frac{-i}{\sqrt{2}}(e_\mu - \bar{e}_\mu) \tag{3.23}$$

From the definition of the Plücker coordinates, we have:

$$P_{\mu\nu} = e^1_\mu e^2_\nu - e^1_\nu e^2_\mu = i(e_\mu \bar{e}_\nu - \bar{e}_\mu e_\nu) = ie_{[\mu}\bar{e}_{\nu]} \tag{3.24}$$

Now, like with the electromagnetic field tensor, define:

$$P_{0i} = E_i \tag{3.25}$$

$$P_{ij} = \epsilon_{ijk} B_k \Leftrightarrow B_i = \frac{1}{2}\epsilon_{ijk} P_{jk} \tag{3.26}$$

A few properties are now easily extractable from the orthonormality conditions:

$$(i)\ E^2 = 2e_0\bar{e}_0 \quad (e_0\bar{e}_0 \leq 1/2) \tag{3.27}$$

$$(ii)\ B^2 = 1 - 2e_0\bar{e}_0 \tag{3.28}$$

$$(iii)\ E_i B_i = 0 \tag{3.29}$$

$$(iv)\ E^2 + B^2 = 1 \tag{3.30}$$

$$(v)\ \vec{S} := \vec{E} \times \vec{B} \rightarrow S_i = e_0\bar{e}_i + \bar{e}_0 e_i \tag{3.31}$$

$$(vi)\ S^2 = E^2 B^2 = 2e_0\bar{e}_0(1 - 2e_0\bar{e}_0) \tag{3.32}$$

$$(vii)\ S_i + iE_i = 2\bar{e}_0 e_i \tag{3.33}$$

In particular we note that

$$|\vec{E}\ | = \sqrt{2}|e_0| \tag{3.34}$$

Introduce an angle η as the phase of the zeroth component of the vector.

$$e_0 = \frac{1}{\sqrt{2}} e^{i\eta} |\vec{E}\ | = e^{i\eta}|e_0| \tag{3.35}$$

Using this, we write

$$e_i = \frac{S_i + iE_i}{\sqrt{2}|\vec{E}\ |} \tag{3.36}$$

The entire four-vector can then be expressed in terms of the electric and magnetic coordinates as

$$e_a = \frac{e^{i\eta}}{\sqrt{2}} \begin{pmatrix} |\vec{E}\ | \\ \frac{S_i + iE_i}{|\vec{E}\ |} \end{pmatrix} = e^{i\eta}\hat{e}_a \tag{3.37}$$

At this point, one might worry about how well defined these coordinates are. From the above expression, they work fine as long $\vec{E} \neq 0$, but what about when this does not hold? Clearly, since e_a was not originally defined in terms of the electric and magnetic fields but in terms of the orthonormal vectors e^1, e^2, it is still well defined. The above representation of it in terms of the electric and magnetic fields is not however. Instead, given ($\vec{E} = 0$, $\vec{B}$), e_a can still be defined by "re-inventing" the orthonormal pair e^1, e^2 as spanning the orthogonal plane of $\vec{B}$. Then

$$e_a = \frac{i}{2\sqrt{2}} \begin{pmatrix} 0 \\ \vec{B} \times (\vec{e}^1 - i\vec{e}^2) \end{pmatrix} \tag{3.38}$$

A nice aspect of $G_{2,4}$ is that there exists another set of coordinates in which (3.37) still holds, with a slightly different meaning though. This is a consequence of a duality we will later find useful. The reasoning goes like this: take a plane and consider two of the vectors that span it, construct the Plücker coordinates and find out whether $\vec{E}$ is non-zero. If that is the case, leave it there. If that is not true, take a pair of vectors orthogonal to the plane and construct their matrix M and the corresponding *dual* Plücker coordinates. This time the dual $\vec{E}$ is definitely different from zero. This defines the dual complex four-vector

$$\star e_a \tag{3.39}$$

which (by symmetry) can equally well be used in the subsequent dynamics instead of e_a.

To expand a bit further on this duality, note that the operation "going from vectors spanning the plane to vectors orthogonal to the plane" is given by the dual Hodge star operator, acting on $\wedge^2(\mathbb{R}^4)$. We will return to this point in chapter 4, but for now we take it as a fact [14]. From special relativity we recall that the dual field tensor, which is the Hodge dual of the electromagnetic field tensor, has several wonderful properties. One of them is that it changes $\vec{E}$ fields for $\vec{B}$ fields. Now we are done, since not both $\vec{E}$ and $\vec{B}$ can be zero (since that would correspond to the spanning vectors being linearly dependent and we would be considering a line in $\mathbb{R}^4$ instead of a plane).

Note also that the two Lorentz (SO(4)) invariants in Euclidean space time one can construct from the field tensor and its dual

$$\frac{1}{2} F_{\mu\nu} F_{\mu\nu} = \vec{E}^{\,2} + \vec{B}^{\,2} \tag{3.40}$$

$$F_{\mu\nu}(\star F_{\mu\nu}) = \vec{E} \cdot \vec{B} \tag{3.41}$$

correspond to the properties (iii) and (iv) above for the Plücker tensor [19]. In [4], angles are assigned to the vectors. Define $\hat{k} := \frac{\vec{E}}{|\vec{E}|}$ $\hat{l} := \frac{\vec{B}}{|\vec{B}|}$ $\hat{m} := \hat{k} \times \hat{l}$ and the angle γ:

$$\vec{E} = cos\gamma\hat{k} \quad \vec{B} = sin\gamma\hat{l} \tag{3.42}$$

Express these three unit vectors in spherical coordinates with $\hat{k} = \hat{r}$.

$$\hat{k} = \begin{pmatrix} \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{pmatrix} \tag{3.43}$$

Then $\hat{l}$ and $\hat{m}$ becomes some linear combination of $\hat{\phi}$ and $\hat{\theta}$. This gives the oppurtunity to define a new angle ξ corresponding to the fourth dimension of the Grassmannian. Let:

$$\hat{l} = \cos\xi\hat{\theta} + \sin\xi\hat{\phi} = \begin{pmatrix} \cos\xi\cos\phi\cos\theta - \sin\xi\sin\phi \\ \cos\xi\sin\phi\cos\theta + \sin\xi\cos\phi \;\; c \\ -\cos\xi\sin\theta \end{pmatrix} \tag{3.44}$$

$$\hat{m} = -\sin\xi\hat{\theta} + \cos\xi\hat{\phi} = \begin{pmatrix} -\sin\xi\cos\phi\cos\theta - \cos\xi\sin\phi \\ -\sin\xi\sin\phi\cos\theta + \cos\xi\cos\phi \;\; c \\ \sin\xi\sin\theta \end{pmatrix} \tag{3.45}$$

Summing it up:

$$\begin{aligned}
\vec{E} &= \begin{pmatrix} \cos\gamma\cos\phi\sin\theta \\ \cos\gamma\sin\phi\sin\theta \\ \cos\gamma\cos\theta \end{pmatrix} \\
\vec{B} &= \begin{pmatrix} \sin\gamma\cos\xi\cos\phi\cos\theta - \sin\gamma\sin\xi\sin\phi \\ \sin\gamma\cos\xi\sin\phi\cos\theta + \sin\gamma\sin\xi\cos\phi \;\; c \\ -\sin\gamma\cos\xi\sin\theta \end{pmatrix} \\
\vec{S} &= \begin{pmatrix} -\cos\gamma\sin\gamma\sin\xi\cos\phi\cos\theta - \cos\gamma\sin\gamma\cos\xi\sin\phi \\ -\cos\gamma\sin\gamma\sin\xi\sin\phi\cos\theta + \cos\gamma\sin\gamma\cos\xi\cos\phi \;\; c \\ \cos\gamma\sin\gamma\sin\xi\sin\theta \end{pmatrix}
\end{aligned}$$

with $\gamma \in [0, 2\pi]$, $\phi \in [0, 2\pi]$, $\xi \in [0, 2\pi]$ and $\theta \in [0, \pi]$.

Up until this point, what is represented by the angles or the E, B and S-fields are actually not, as we would recall from section 3.1, planes in a $d = 4$ vector space. What is represented is 2-frames. But not all different 2-frames correspond to different planes, as was discussed in section 3.3. To see how a harmless rotation in the plane of the spanning vectors works, consider

$$e^2_\mu \longmapsto \sin\eta e^1_\mu + \cos\eta e^2_\mu \tag{3.46}$$

$$e^1_\mu \longmapsto \cos\eta e^1_\mu - \sin\eta e^2_\mu \tag{3.47}$$

By applying the Euler identities, the transformed can be expressed as

$$\tilde{e}^1_\mu = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(e^1_\mu + ie^2_\mu)e^{i\eta} + \frac{1}{\sqrt{2}}e^{-i\eta}(e^1_\mu + ie^2_\mu)\right) = \frac{1}{\sqrt{2}}\left(e^{i\eta}e_\mu + e^{-i\eta}\bar{e}_\mu\right) \tag{3.48}$$

So, the rotation of the vectors correspond to multiplication by a phase of the complexified vectors. The same works for e^2_μ, so in conclusion, e_μ and $\bar{e}_\mu$ are determined up to:

$$e_\mu \longmapsto e^{i\eta}e_\mu \tag{3.49}$$

$$\bar{e}_\mu \longmapsto e^{-i\eta}\bar{e}_\mu \tag{3.50}$$

Note also that this invariance doesn't affect the Plücker coordinates (since the determinant of the expression is one, or that the Plücker coordinates are products of bared and unbared quantities), so it doesn't affect the E and B-fields and hence, the invariance doesn't propagate to the angles. A similar argument can be made about the normal vectors and the dual expressions above. We will later see that this invariance is the source of the confining internal $U(1)$ force.

Chapter 4

The Grassmannian σ Model in SU(2) Yang-Mills Theory

In the focal point of this *Examensarbete* is the Grassmannian sigma model appearing under very particular circumstances, namely in the formalism of spin-charge separated variables in SU(2) Yang-Mills theory. In the following chapters, we will at length discuss these matters, starting with the decomposition performed in [4], in which the appearance of this model was first supposed, yet not explicitly proven. We will discuss way two essentially different ways to prove this fact from geometry. The first will lead us to a set of embedding conditions, discussed in section 4.4, the other will force us to involve quaternions. Finally we will end up with the essentially new result, that the geometrically constructed Grassmannian sigma model really appears in the Yang-Mills Lagrangian.

4.1 The SU(2) Yang-Mills Theory in spin-charge separated variables

4.1.1 Variables

We will be considering SU(2) Yang-Mills theory over Euclidean space $(\mathbb{R}^4, \delta)$, and have space-time indices like $a, b, \ldots$.

The SU(2) Yang-Mills gauge field A can be written in terms of the linear combinations

$$A = A^i_a \frac{\sigma^i}{2} dx^a = A_a \frac{\sigma^3}{2} dx^a + X^+_a \frac{\sigma^-}{2} dx^a + X^-_a \frac{\sigma^+}{2} dx^a \tag{4.1}$$

with $a, b = 0, \ldots, 3$. the σ are the Pauli matrices

$$\sigma^{\ 1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^{\ 2} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^{\ 3} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.2}$$

In components the $\sigma^{\ \pm}$ becomes

$$\sigma^{\ +} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \sigma^{\ -} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{4.3}$$

$$A_a = A^3_a \tag{4.4}$$

$$X^{\pm}_a = A^1_a \pm i A^2_a \tag{4.5}$$

$$\sigma^{\pm} = \frac{1}{2}(\sigma^1 \pm i\sigma^2) \tag{4.6}$$

With the choice of covariant derivative $D^+ = \partial + iA$, the finite gauge transformation looks like

$$A' = UAU^{-1} - (\partial U)U^{-1} \tag{4.7}$$

Under an infinitesimal gauge transformation, $U = e^{i\theta_a \sigma^a/2}$, the gauge fields transform like

$$\delta_\theta A^i_a = (A^i_a)' - (A^i_a) = \partial_a \theta^i + \epsilon^{ijk} A^j_a \theta^k \tag{4.8}$$

Consider gauge transformations along the $\sigma^{\ 3}$ direction only. These transformations form an U(1) subgroup of SU(2), which here is denoted $U_C(1)$. For an infinitesimal $h \in U_C(1)$ $\ h = e^{i\omega^3 \sigma^3/2}$, according to (4.8), the gauge fields now transform as

$$\delta_h A_a = \partial_a \omega \tag{4.9}$$

$$\partial_h X^{\pm} = \mp i \omega X^{\pm}_a \tag{4.10}$$

$$\tag{4.11}$$

Under finite gauge transformations:

$$A_a \rightarrow A_a + \partial_a \omega \tag{4.12}$$

$$X^{\pm} \rightarrow e^{\mp i\omega} X^{\pm} \tag{4.13}$$

So A^a transforms as the $U_C(1)$ gauge field and the off-diagonal components as charged fields.

In these coordinates the field strength is

$$F^3_{ab} = \partial_a A^3_b - \partial_b A^3_a + \epsilon^{3jk} A^j_a A^k_b = \tag{4.14}$$

$$= \partial_a A_b - \partial_b A_a + \frac{i}{2}(X^+_a X^-_b - X^+_b X^-_a) = \tag{4.15}$$

$$= F_{ab} + P_{ab} \tag{4.16}$$

where F is the ordinary $U(1)$ field strength and P_{ab} is

$$P_{ab} = \frac{i}{2}(X^+_a X^-_b - X^+_b X^-_a) = A^1_a A^2_b - A^1_b A^2_a \tag{4.17}$$

The corresponding covariant derivatives are

$$D^{\pm}_a = \partial_a \pm i A_a \tag{4.18}$$

$$F^{\pm}_{ab} = D^{\pm}_a X^{\pm}_b - D^{\pm}_b X^{\pm}_a \tag{4.19}$$

For the discovery of the spin-charge separation, a change of variables is essential

$$X^+_a = A^1_a + i A^2_a = \psi_1 e_a + \psi_2 \bar{e}_a \tag{4.20}$$

$$X^- = \psi^*_2 e_a + \psi^*_1 \bar{e}_a \tag{4.21}$$

Here $\psi_{1,2}$ are complex valued scalars while e_a are four-component vectors that satisfy

$$e_a \; e_a = 0 \tag{4.22}$$

$$e_a \; \bar{e}_a = 1 \tag{4.23}$$

We re-express the anti-symmetric tensor P_{ab} in these coordinates to obtain

$$P_{ab} = \frac{i}{2}(|\psi_1|^2 - |\psi_2|^2)(e_a \bar{e}_b - e_b \bar{e}_a) = \tag{4.24}$$

$$= \rho^2 t_3 \frac{i}{2}(e_a \bar{e}_b - e_b \bar{e}_a) = \rho^2 t_3 H_{ab} \tag{4.25}$$

Where t_3 is the third component of the vector

$$\vec{t} = \frac{1}{\rho^2}(\psi^*, \psi^*_2)\vec{\sigma} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \cos\phi \sin\theta \\ \sin\phi \sin\theta \\ \cos\theta \end{pmatrix} \tag{4.26}$$

For future reference we will introduce a parametrization of the complex fields

$$\psi_1 = \rho e^{i\xi} \cos\frac{\theta}{2} e^{-i\phi/2} \tag{4.27}$$

$$\psi_2 = \rho e^{i\xi} \sin\frac{\theta}{2} e^{i\phi/2} \tag{4.28}$$

As previously noted, under the transformation of $g \in GL(2,\mathbb{R})$ acting on the 2×4 matrix with A^1, A^2 as row-vectors, P_{ab} transforms as

$$P_{ab} \to \det g \; P_{ab} \tag{4.29}$$

Apparently, P_{ab} is invariant under SL(2 $\mathbb{R}$) transformations, but the decomposition (4.20) is only invariant under $U_I(1) \subset SL(2,\mathbb{R})$ acting as

$$A_a \to A_a \tag{4.30}$$

$$\psi_1 \to e^{i\lambda}\psi_1 \tag{4.31}$$

$$\psi_2 \to e^{-i\lambda}\psi_2 \tag{4.32}$$

$$\phi \to \phi - 2\lambda \tag{4.33}$$

$$t_3 \to t_3 \tag{4.34}$$

$$e \to e^{-i\lambda} e \tag{4.35}$$

$$t_\pm := t_1 \pm i t_2 \to e^{\mp i\lambda} t_\pm \tag{4.36}$$

The covariant derivative for this transformation is chosen to be

$$C_a = i\bar{e}_b \partial_a e_b \tag{4.37}$$

which is invariant under $U_C(1)$ transformations and transforms as a gauge field under $U_I(1)$. In summary, the $U_C(1)$ transformation is

$$A_a \to A_a + \partial_a \omega \tag{4.38}$$

$$\psi_1 \to e^{-i\omega}\psi_1 \tag{4.39}$$

$$\psi_2 \to e^{-i\omega}\psi_2 \tag{4.40}$$

$$e_a \to e_a \tag{4.41}$$

Note that for $U_C(1)$ the gauge field degrees of freedom are described by the multiplet

$$A^i \sim (A_a, \psi_1, \psi_2, e_a) \tag{4.42}$$

The first field is the gauge field, the second two are charged but spin-less complex fields that transform as had they the same charge under $U_C(1)$

. The last component of the multiplet transform as a vector under SO(4) transformations, which are the Lorentz transformations in Euclidean space. It is charge-less however. Considering the $U_I(1)$ transformations we find the multiplet structure

$$A^i \sim (C_a, \psi_1, \psi_2, A_a) \tag{4.43}$$

with C_a being the gauge field, the scalars fields carrying being oppositely charged and the last component is unaffected by this transformation.

4.1.2 The Physics of Spin-Charge Separation

Recall that the charge concept comes from how a field transform under gauge transformations. For the $U_C(1)$ case above, the SU(2) gauge field is replaced by an ordinary U(1) gauge field ($\sim$ chromo-photon) and charged off-diagonal fields. The off-diagonal fields are further decomposed into the complex scalars ψ and four-vectors e_a. Since the ψ:s still transform under $U_C(1)$, they are charged but being scalars they are spin-less. The four-vectors e_a still carry the index of the $X_a^{\pm}$ though, and transform under Lorentz transformations (which in $(\mathbb{R}^4, \delta)$ are given by SO(4), instead of SO(3, 1) as in Minkowski space). The transformation is not simply a vector transformation, but a transformation in a projective representation of the Lorentz group. This is a fact discussed in detail in [4]. So these are the variables of the spin-charge separation.

It is known that a compact U(1) symmetry, like $U_I(1)$, exhibits confinement in a strong coupling domain, which is separated from a weakly coupled, deconfined domain by a first order phase transition [20]. From the transformation laws (4.30)–(4.36) and the decomposition (4.20), it is clear that the spin carrying fields and charge carrying fields are oppositely charged with respect to $U_I(1)$. One would therefore expect that these under normal circumstances are tightly bound into a gluon field, but that maybe, they could separate in a low energy, finite density environment.

Strikingly, a similar approach has been suggested to another notoriously difficult problem in theoretical physics, namely the theory of high-energy superconductors [28]. Since nearly fifty years, the conventional superconductivity has been successfully described by the BCS–mechanism where the electrons condensate to Cooper pairs. In the case of high temperature superconductivity, a general theory is known called the $t - J$ model, which most solid state physicists hold for true. Yet, it offers no clue to how the BCS mechanism could be implemented in this theory to explain high temperature superconductivity.

The radical suggestion is that in the highly correlated environment of the high-T_c superconductor, the electrons decompose into spinons, uncharged spin-carrying particles and holons or chargons, carrying charge but not spin. Under normal circumstances the spinon and holon are tightly bound to a pointlike electron, in agreement with high energy, low density experiments. The confining force is a compact, internal U(1) force, just like $U_I(1)$ in the above gluon decomposition. Under high correlation however, a decoupling could occur.

This decomposition of the electron is fully analogous to that of the gluons. Similar holons and spinons fall out, they too need a finite density and they too are bound together by a similar U(1) interaction, which consistently with particle physics makes these fields seem pointlike in the asymptotic short distance limit.

Discussing Yang–Mills theory, the decomposition has the further advantage of perhaps having the prospect of new insights in the nature of confinement. The empirical result that the energy of two quarks grows linearly with the separation

$$E \sim \sigma L \tag{4.44}$$

suggests the qualitative picture of strings holding them together. The world-sheet of the string would be an area, giving rise to the conjectured area law of Yang-Mills theory, saying that a quantity called the expectation value of the Wilson loop should be proportional to σLT. Confinement is explained by the string breaking when the energy of the separation becomes large enough to create a new particle pair.

Models explaining this behaviour qualitatively have been devised. A particularly popular one was devised by 't Hooft and Mandelstam, and is called the dual superconductor model since it is based on the BCS-mechanism discussed above. In this model, chromo-magnetic monopoles condensate in the vaccuum to "Cooper pairs". The Meissner effect, that usually expels magnetic flux from an ordinary superconductor, would then expel chromo-electric flux from the vacuum. Chromo-electric-particles in the condensate would find that the flux emitted would be squeezed into thin tubes connecting the particles, in which there could be no superconductivity. The collapsed flux tubes would essentially be strings, and thus deserves the alternate name of QCD strings.

One fundamental difficulty however, is to produce the medium of magnetic Cooper pairs that would support these strings. Despite thirty years efforts of the theoretical community, no such condensation has been found. It can be argued, that this medium must fall out from the gauge fields of the Yang-Mills theory, but yet there has been no BCS like mechanism to do the job [21].

The decomposition at hand might hold the key to that mystery as well. Perhaps it is not a BCS like mechanism and a Cooper pair condensate that is needed, but a spin-charge separation mechanism and a holon condensate? The above decomposition offer a such a medium, through the demand that

$$< \rho^2 > \neq 0 \tag{4.45}$$

a holon condensate appears. Furthermore, we will soon show that this decomposition, when applied to the pure Yang-Mills Lagrangian, gives rise to an O(3) non-linear sigma model coupled to a Grassmannian sigma model. Both these models are known to support stable, knotted solitons, just like strings.

4.2 Appearance of the ostensible model

We continue, following [4], with an attack on the SU(2) Yang-Mills Lagrangian with the off-diagonal gauge fields fixed.

$$\mathcal{L}_{YM} = \frac{1}{4}(F^i_{ab})^2 + \frac{\xi}{2}|D^+_a X^+_a|^2 + \mathcal{L}_{ghost} \tag{4.46}$$

with ξ being the gauge fixing parameter. The covariant derivative is

$$D^+_a = \partial_a + iA_a \tag{4.47}$$

We will only consider the classical Lagrangian with gauge fixing term, hence the ghost term is dropped.

$$\mathcal{L}_{YM} = \mathcal{L}_0 + \mathcal{L}_{gauge\ fix} = \frac{1}{4}(F^i_{ab})^2 + \frac{\xi}{2}|D^+_a X^+_a|^2 \tag{4.48}$$

Consider first $\mathcal{L}_0$.

$$(F^i_{ab})^2 = (\mathcal{F}_{ab} + P_{ab})^2 + (F^1_{ab})^2 + (F^2_{ab})^2 = \tag{4.49}$$

$$= (\mathcal{F}_{ab} + P_{ab})^2 + F^+_{ab}F^-_{ab} \tag{4.50}$$

We use the coordinates of the previous section

$$
\begin{aligned}
F^+_{ab}F^-_{ab} = (D^+_a X^+_b - D^+_b X^+_a)(D^-_a X^-_b - D^-_b X^-_a) = && (4.51)\\
= 2|D^+_a X^+_b|^2 - 2D^+_b X^+_a D^-_a X^-_b = && (4.52)\\
= 2|D^+_a X^+_b|^2 + 2X^+_a D^-_b D^-_a X^-_b - \partial_b(X^+_a D^-_a X^-_b) = && (4.53)\\
= 2|D^+_a X^+_b|^2 + 2X^+_a D^-_a D^-_b X^-_b + 2iX^+_a \mathcal{F}_{ab} X^-_b + && (4.54)\\
+2\partial_a(X^+_a D^-_b X^-_b) - \partial_b(X^+_a D^-_a X^-_b) = && (4.55)\\
= 2|D^+_a X^+_b|^2 + 2|D^+_a X^+_a|^2 + 2\mathcal{F}_{ab}P_{ab} - && (4.56)\\
-\partial_b(X^+_a D^-_a X^-_b - X^+_b D^-_a X^-_a) && (4.57)
\end{aligned}
$$

The last term is to be evaluated on the boundary and we drop it. Consider

$$
\begin{aligned}
(\mathcal{F}_{ab} + P_{ab})^2 + 2P_{ab}\mathcal{F}_a b = (\mathcal{F}_{ab} + 2P_{ab}\ )^2 - 3(P_{ab}\ )^2 = && (4.58)\\
= (\mathcal{F}_{ab} + 2P_{ab}\ )^2 - \frac{3}{2}(|\psi_1|^2 - |\psi_2|^2) && (4.59)
\end{aligned}
$$

All in all

$$
\mathcal{L}_0 = \frac{1}{4}(\mathcal{F}_{ab} + 2P_{ab})^2 + \frac{1}{2}|D^+_a X^+_b|^2 - \frac{1}{2}|D^+_a X^+_a|^2 - \frac{3}{8}\rho^4 t_3^2 \tag{4.60}
$$

The full Lagrangian then becomes

$$
\mathcal{L} = \frac{1}{4}(\mathcal{F}_{ab} + 2P_{ab})^2 + \frac{1}{2}|D^+_a X^+_b|^2 + \frac{\xi - 1}{2}|D^+_a X^+_a|^2 - \frac{3}{8}\rho^4 t_3^2 \tag{4.61}
$$

In [4], two choices of the gauge fixing parameter are discussed. We will continue with $\xi = 1$.

Consider the second term of the Lagrangian

$$
|D^+_a X^+_b|^2 = |D_a\psi_1|^2 + |D_a\psi_2|^2 + \rho^2|D_a e_b|^2 + \frac{1}{2}\rho^2\left(t_+(D^*\bar{e}_b)^2 + t_-(D_a e_b)^2\right) \tag{4.62}
$$

With the covariant derivatives defined as

$$
\begin{aligned}
D_a\psi_1 = (\partial_a + iA_a - iC_a)\psi_1 && (4.63)\\
D_a\psi_2 = (\partial_a + iA_a + iC_a)\psi_2 && (4.64)\\
D_a e_b = (\partial_a + iC_a)e_b && (4.65)
\end{aligned}
$$

The terms of present interest are

$$
\rho^2|D_a e|^2 = \rho^2|(\partial_a - iC_a)e|^2 = \frac{\rho^2}{2}\left((\partial\vec{E}\ )^2 + (\partial\vec{B}\ )^2\right) \tag{4.66}
$$

In the electric and magnetic coordinates introduced in the previous chapter. We will find the coupling between the both models from

$$\frac{\rho^2}{2}\left(t_+(D_a^*\bar{e}_b)^2 + t_-(D_a e_b)^2\right) = \frac{\rho^2}{2}\{n_+(\partial_a \hat{e}_b^*)^2 + n_-(\partial_a \hat{e}_b)^2\} \tag{4.67}$$

In chapter 5, this equation will be further interpreted. For the sake of completeness, we state the full pure Yang–Mills Lagrangian in spin-charge separated variables [4].

$$\mathcal{L}_{YM} = \frac{1}{4}\mathcal{F}_{ab}^2 + \frac{1}{2}\rho^2 J_a^2 + \frac{1}{8}\rho^2(D_a^{\hat{C}}\vec{n})^2 + \frac{\rho^2}{4}\{(\partial\vec{E}\ )^2 + (\partial\vec{B}\ )^2\} \tag{4.68}$$

$$+\frac{1}{4}\rho^2\{n_+(\partial_a\hat{\bar{e}}_b)^2 + n_-(\partial_a\hat{e}_b)^2\} + \frac{3}{8}(1-n_3)\rho^4 - \frac{3}{8}\rho^4 \tag{4.69}$$

where

$$\mathcal{F}_{ab} = \partial_a J_b - \partial_b J_a + \frac{1}{2}\vec{n}\cdot\partial_a\vec{n}\times\partial_b\vec{n} - \{\partial_a(n_3\hat{C}_a - \partial_b(n_3\hat{C}_a)\} - 2\rho^2 n_3 H_{ab} \tag{4.70}$$

$$J_a = \frac{i}{2\rho^2}\{\psi_1^* D_a\psi_1 - \psi_1\bar{D}_a\psi_1^* + \psi_2^* D_a\psi_2 - \psi_2\bar{D}_a\psi_2^*\} \tag{4.71}$$

4.3 The Grassmannian Sigma Model

In [4], the terms (4.66) was interpreted as the Grassmannian sigma model corresponding to the manifold $G_{2,4}$. We will in this second half of the chapter briefly review the standard ways of treating the Grassmannian sigma model, and thereafter discuss the verification of this claim. We will both discuss explicit embedding equations and more sophisticated methods, ending up with the verification of this claim.

4.3.1 Gauge Formalism

Let M denote the $k \times n$ matrix whose rows are some spanning vectors of $p \in G_{k,n}$. As stated many times before, the vectors can be chosen to be orthonormal so that

$$M^T M = 1_n \tag{4.72}$$

$$M = \begin{pmatrix} e^1 \\ e^2 \end{pmatrix} \quad e^1, e^2 \in \mathbb{R}^n \tag{4.73}$$

The Lagrangian density of the Grassmannian sigma model in 2 dimensions can be defined as [13], [12]:

$$L = Tr\left((D_\mu M)^T D_\mu M\right) = (D_\mu e^1)^2 + (D_\mu e^2)^2 \tag{4.74}$$

where the covariant derivative $D_\mu M = \partial_\mu M - MA_\mu$ contains the composite gauge field

$$A_\mu = M^T \partial_\mu M \quad A_\mu^T = -A_\mu \tag{4.75}$$

The Lagrangian density is invariant under global $O(n)$ transformations and local $O(k)$ transformations in the way:

$$M \mapsto M' = KM \quad K \in O(n) \tag{4.76}$$

and

$$M \mapsto M' = Mh(x) \quad h(x) \in O(k) \tag{4.77}$$

by the construction of the covariant derivative.

The classical equations of motion in this formulation of the model are given by

$$D_\mu D_\mu M + M(D_\mu M)^T D_\mu M = 0 \tag{4.78}$$

with the constraint $M^T M = 1$. These equations are higly non-linear and not easily solved.

4.3.2 Projector Formalism

A second text book formulation of the sigma model is constructed by the use of the representation of the Grassmannian in terms of projection matrices, as used in chapter 3 to show the transitivity of the action of $O(n)$.

Recall that the projection matrix P corresponding to the plane spanned by the orthonormal vectors $|f_i>$, is defined by

$$P = |f_i >< f_i| \tag{4.79}$$

where the subscript refers to the row vector of the matrix. By orthonormality of these vectors, the projection matrix really is a orthogonal projection matrix, that is:

$$P^2 = P = P^T \tag{4.80}$$

The Lagrangian now takes the form

$$L = Tr(\partial_\mu P \partial_\mu P) \tag{4.81}$$

together with the constraint $P^2 = P$. The equations of motion becomes

$$[\partial_\mu \partial_\mu P, P] = 0 \tag{4.82}$$

Though useful in other circumstances, these both formulation will play no role in the rest of the *Examensarbete*.

4.3.3 Induced Metric Formalism

Though not a text book model, we will in this section discuss another formalism which is perhaps the simplest of them all, at least conceptually. Though not used explicitly in the next sections deduction of the embedding equations, the same techniques that apply there are used here.

First of all, recall a few facts about induced maps [15]. Given two manifolds M, N and a smooth map between them

$$f : M \to N \tag{4.83}$$

This map affects the tangent and cotangent spaces of the manifolds by maps which are called *induced* from f. To see this, think of elements of the tangent space of a point p in M, T_pM. The elements of this vector space are the tangent vectors at the point p and they could be defined as the equivalence classes of curves through p with respect to tangency. The map f will move point the point $p + \delta$ in the neighbourhood of p to $f(p + \delta) \in N$. This will also affect the equivalence classes of curves (read: tangent vectors) passing through p and $p + \delta$. The effect will be a transformation with the Jacobian of f. This map is called the differential map denoted as

$$f_* : T_pM \to T_{f(p)}N \tag{4.84}$$

Explicitly, if $\{\frac{\partial}{\partial x^i}\}$ is a basis of T_pM and $\{\frac{\partial}{\partial y^i}\}$ one of $T_{f(p)}N$ (local coordinates for the chart being x and y, respectively). Write

$$V = V^i \frac{\partial}{\partial x^i} \in T_pM \tag{4.85}$$

$$W = W^j \frac{\partial}{\partial y^j} \in T_{f(p)=y}N \tag{4.86}$$

Then the differential map is

$$W^j = V^i \frac{\partial}{\partial x^i} y^j(x) \tag{4.87}$$

This map is naturally extended to all tensors of type $(p, 0)$. A similar map should be constructed for the cotangent space, but since no assumption has been made on invertibility of f and differentiability of the inverse, it can not go the "same way" as f and f_*. On the other hand, a map going the other way can be defined with only the Jacobian of f, this is the pullback of f denoted

$$f^* : T^*_{f(p)}N \to T^*_pM \tag{4.88}$$

Explicitly, for

$$\omega = \omega_i dy^i \in T^*_{f(p)} \tag{4.89}$$

$$f^*\omega = \xi_j dx^j \in T^*_p \tag{4.90}$$

the coordinate transformation is given by

$$\xi_j = \omega_i \frac{\partial y^i}{\partial x^j} \tag{4.91}$$

Also this map naturally extends to tensors of type $(0, q)$. In particular, the metric tensor is a symmetric tensor of type $(0, 2)$. If M is embedded in the Riemannian manifold (N, g) by f (i.e. f preserves the tangent space and the point structure of the manifold M, i.e. it is an injective immersion), then the pullback determines the induced metric tensor on M.

$$g_M = f^* g_N \tag{4.92}$$

$$g_{M\ \mu\nu} = g_{N\alpha\beta} \frac{\partial f^\alpha}{\partial x^\mu} \frac{\partial f^\beta}{\partial x^\nu} \tag{4.93}$$

Turning to the particular manifold of interest, the $G_{2,4}$, it is embedded in $\mathbb{R}P^5$ by the Plücker equation:

$$\tilde{p_1}\tilde{p_2} = \tilde{p_3}\tilde{p_4} + \tilde{p_5}\tilde{p_6} \tag{4.94}$$

with , $\tilde{p_1} = p_{12}$, $\tilde{p_2} = p_{34}$, $\tilde{p_3} = p_{13}$, $\tilde{p_4} = p_{24}$, $\tilde{p_5} = p_{14}$, $\tilde{p_6} = p_{32}$, being the Plücker coordinates corresponding to the homogeneous coordinates of $\mathbb{R}P^5$.

The r:th chart of $\mathbb{R}P^5$ is given by all points such that $\tilde{p_r} \neq 0$. Define the inhomogeneous coordinates by $q_i = \frac{\tilde{p_i}}{\tilde{p_r}}$, remove the r:th coordinate and relabel the coordinates such that $i = 1, \ldots, 5$. The Plücker relation now takes the form

$$q_1 = q_2 q_3 + q_4 q_5 \tag{4.95}$$

(possibly after some redefinition, rearrangement and use of the antisymmetry of the two Plücker indices). This equation defines the embedding of $G_{2,4}$ in $\mathbb{R}P^5$. To be more explict: each point in the Grassmannian corresponds uniquely to some coordinates $(q_1, \ldots, q_5)$ in $\mathbb{R}P^5$ for some specified chart of $\mathbb{R}P^5$ (if it is in that chart), but each point in $\mathbb{R}P^5$ does not correspond to a point in the Grasmanian. The constraint equation is the Plücker equation which will now be solved to, for each chart, get a metric on the embedded four dimensional Grassmannian.

Let $(q_2, \ldots, q_5)$ be the natural coordinates for the Grassmannian and induce the metric of P^5 on $G_{2,4}$. Define:

$$f : (q_2, q_3, q_4, q_5) \mapsto (q_2 q_3 + q_4 q_5, q_2, q_3, q_4, q_5) \tag{4.96}$$

The induced metric is then given by

$$g_{G\mu\nu} = g_{P\alpha\beta} \frac{\partial f^\alpha}{\partial x^\mu} \frac{\partial f^\beta}{\partial x^\nu} = \delta_{\alpha\beta} \frac{\partial f^\alpha}{\partial x^\mu} \frac{\partial f^\beta}{\partial x^\nu} \tag{4.97}$$

But from the definition of f we have

$$g_{\mu\nu} = \frac{\partial f^1}{\partial x^\mu} \frac{\partial f^1}{\partial x^\nu} + \delta_{\mu\nu} \tag{4.98}$$

Writing the matrix elements explicitly:

$$(g_{\mu\nu}) = \begin{pmatrix} 1+q_3^2 & q_3 q_2 & q_3 q_5 & q_3 q_4 \\ q_2 q_3 & 1+q_2^2 & q_2 q_5 & q_2 q_4 \\ q_5 q_3 & q_5 q_2 & 1+q_5^2 & q_5 q_4 \\ q_4 q_3 & q_4 q_2 & q_4 q_5 & 1+q_4^2 \end{pmatrix} \tag{4.99}$$

As seen above there is a certain "conjugacy" between the numbers 2 & 3 and 4 & 5 respectively, in the metric above. Therefore we introduce a conjugacy operator: $\tilde{}$: $\tilde{2} = 3, \tilde{3} = 2, \tilde{4} = 5, \tilde{5} = 4$ and write the metric as

$$g_{\mu\nu} = \delta_{\mu\nu} + q_{\tilde{\mu}} q_{\tilde{\nu}} \tag{4.100}$$

The inverse matrix and the Cristoffel symbols are given by rather complicated expressions though, and we do not display them here. We will later calculate these quantities for the oriented Grassmannian in which case a more interesting and simpler for of the metric tensor can be obtained.

From chapter 2, we have that the Lagrangian of the Grassmannian sigma model in these coordinates is given by

$$\mathcal{L} = g_{ab}\partial q^a \partial q^b = \sum_{ab}(\delta_{ab} + q_{\tilde{a}} q_{\tilde{b}})\partial q^a \partial q^b) = \tag{4.101}$$

$$= \partial q^a \partial q^a + \left(\sum_a q_{\tilde{a}} \partial q^a \right)^2 \tag{4.102}$$

4.4 Embedding Equations

Leaving the general discussion of Grassmanian sigma models, we now turn to the situation at hand. On the one hand, we have a contribution to the Yang–Mills Lagrangian expressed in the Plücker coordinates

$$(\partial \vec{E}\ )^2 + (\partial \vec{B}\ )^2 \tag{4.103}$$

with two subsidiary conditions

$$\begin{cases} \vec{E}\ \cdot \vec{B}\ = 0 \\ \vec{E}\ ^2 + \vec{B}\ ^2 = 1 \end{cases} \tag{4.104}$$

Call the $\mathbb{R}^6$ of Plücker coordinates p-space. From section 2, we note that the Lagrangian implies that the metric tensor of p-space is flat (at least in the neighbourhood of S^5).

$$g_{ab}\partial\phi^a\partial\phi^b = \delta_{ab}\partial\phi^a\partial\phi^b = \tag{4.105}$$

$$= \sum_{i=1}^{6}(\partial p^i)^2 = (\partial \vec{E}\ )^2 + (\partial \vec{B}\ )^2 \tag{4.106}$$

The two constraints (4.104) can be checked to correspond to the Klein quadric (Plücker equation) and S^5, respectively. This is slightly unfortunate, since the intersection of the Klein quadric with the five-sphere is a different manifold, denoted $\tilde{G}_{2,4}$. Being the brother (two-fold covering) of $G_{2,4}$, $\tilde{G}_{2,4}$ is the manifold of *oriented* two dimensional planes in $\mathbb{R}^4$. One might ask, is it the $\tilde{G}_{2,4}$ or $G_{2,4}$ that currently is at hand from the spin-charged separated Lagrangian?

To find out, recall the decomposition of P_{ab} in equation (4.25). It was written

$$P_{ab} = \lambda H_{ab} \tag{4.107}$$

where $\lambda = \rho^2 \cos\theta$. The tensor P_{ab} properly supplied coordinates for the coset space

$$\frac{M_{2,4}}{SL(2,\mathbb{R})} \tag{4.108}$$

While under a general linear transformations g of the vectors spanning the plane

$$P_{ab} \mapsto \det g P_{ab} \tag{4.109}$$

$$\Longrightarrow \begin{cases} \lambda \mapsto \det g \lambda \\ H_{ab} \mapsto H_{ab} \end{cases} \tag{4.110}$$

So

$$H_{ab} \sim \frac{M_{2,4}}{GL(2,\mathbb{R})} \tag{4.111}$$

In particular

$$[H_{ab}] \sim [-H_{ab}] \Leftrightarrow (\vec{E}\,, \vec{B}\,) \sim -(\vec{E}\,, \vec{B}\,) \tag{4.112}$$

Which gives the additional constraint of identifying antipodal points in S^5, to obtain $G_{2,4}$ as the Klein quadric in $\mathbb{R}P^5$

$$\begin{cases} \vec{E}\, \cdot \vec{B}\, = 0 \\ \vec{E}\,^2 + \vec{B}\,^2 = 1 \\ (\vec{E}\,, \vec{B}\,) \sim -(\vec{E}\,, \vec{B}\,) \end{cases} \tag{4.113}$$

In words, the unoriented Grassmannian $G_{2,4}$, is the one appearing in the Lagrangian. In later sections of this chapter the oriented Grassmannian $\tilde{G}_{2,4}$ will play an important role though, mostly through its product space factorization as $S^2 \times S^2$.

By solving the constraints in (4.113), the metric tensor from p-space $(\mathbb{R}^6, \delta)$ can be induced on $G_{2,4}$. That approach will turn out to give involved expressions for the metric tensor, and not seldom, chart dependent expressions.

For reasons that soon will become clear, the induction need only to be done half–way, which simplifies the procedure.

Denote the manifold induced in p-space by the Klein quadric (but the other conditions still "unsolved" for) M. On the one hand, the five-dimensional manifold M is embedded in p-space by the Plücker equation as a Klein quadric. Let

$$h : M \rightarrow \ p - space \tag{4.114}$$

The pullback determines the induced metric

$$h^* : T^*_{g(p)}M \rightarrow T^*_p(p - space) \tag{4.115}$$

Explicitly, since p-space is Euclidean

$$g^M_{\mu\nu} = \delta_{\alpha\beta}\frac{\partial h^\alpha}{\partial x^\mu}\frac{\partial h^\beta}{\partial x^\nu} \tag{4.116}$$

where $\mu,\ \nu = 1,\ldots 5$ and $\alpha, \beta = 1, \ldots, 6$. With some chart dependent assignment of indices such that $p_1 \neq 0$, the constraint is

$$p_6 = \frac{1}{p_1}(p_2p_5 - p_3p_4) \tag{4.117}$$

Explicitly, elimination of p_6 gives

$$g^M_{\alpha\beta} = \tag{4.118}$$

$$\begin{pmatrix} 1+\frac{(p_2p_5-p_3p_4)^2}{p_1^4} & -\frac{p_5(p_2p_5-p_3p_4)}{p_1^3} & \frac{p_4(p_2p_5-p_3p_4)}{p_1^3} & \frac{p_3(p_2p_5-p_3p_4)}{p_1^3} & -\frac{p_2(p_2p_5-p_3p_4)}{p_1^3} \\ & 1+\frac{p_5^2}{p_1^2} & -\frac{p_4p_5}{p_1^2} & -\frac{p_3p_5}{p_1^2} & +\frac{p_2p_5}{p_1^2} \\ & & 1+\frac{p_4^2}{p_1^2} & \frac{p_3p_4}{p_1^2} & -\frac{p_2p_5}{p_1^2} \\ & & & 1+\frac{p_3^2}{p_1^2} & -\frac{p_2p_3}{p_1^2} \\ & & & & 1+\frac{p_2^2}{p_1^2} \end{pmatrix} \tag{4.119}$$

On the other hand, $P_{ab} = A^1_{[a}A^2_{b]} \equiv A_aB_b - B_aA_a$ with $(A, B) \in (\mathbb{R}^4 \times \mathbb{R}^4, \delta \times \delta)$, M is embedded in $\mathbb{R}^4 \times \mathbb{R}^4$ by some smooth mapping f

$$f : M \rightarrow \mathbb{R}^4 \times \mathbb{R}^4 \tag{4.120}$$

The embedding corresponds to a particular choice of representatives for the equivalence classes in the coset space $\mathbb{R}^4 \times \mathbb{R}^4/SL(2, \mathbb{R}) \approx M_{2,4}/SL(2, \mathbb{R})$. In the same way as above, f determines an induced metric tensor on M

$$g^M_{\mu\nu} = \delta_{\alpha\beta}\frac{\partial f^\alpha}{\partial x^\mu}\frac{\partial f^\beta}{\partial x^\nu} \tag{4.121}$$

with $\mu, \nu = 1, \ldots, 5$ and $\alpha, \beta = 1, \ldots, 8$.

The claim that the SU(2) Yang-Mills Lagrangian contains a Grassmannian sigma model hence is equivalent to saying that there exists a solution f such that the both forms of metric tensor on M agree. Denote

$$f(p_1, \ldots, p_5) = \begin{pmatrix} f_a^1 \\ f_a^2 \end{pmatrix} = \begin{pmatrix} A_a' \\ B_a' \end{pmatrix} \tag{4.122}$$

where the primed quantities correspond to the choice of representative in $\mathbb{R}^4 \times \mathbb{R}^4$

With $a = 1, \ldots, 4$, for each pair α, β, we have

$$\frac{\partial f_a^1}{\partial p^\alpha}\frac{\partial f_a^1}{\partial p^\beta} + \frac{\partial f_a^2}{\partial p^\alpha}\frac{\partial f_a^2}{\partial p^\beta} = g^M_{\alpha\beta} \tag{4.123}$$

On the other hand, since A' and B' should be obtainable by acting with

$$g(A_a, B_a) = \begin{pmatrix} a(A,B) & b(A,B) \\ c(A,B) & d(A,B) \end{pmatrix} \in SL(2, \mathbb{R}) \tag{4.124}$$

on the 2×4 matrix

$$X = \begin{pmatrix} A_a \\ B_a \end{pmatrix} \tag{4.125}$$

so for any $X \in M_{2,4}$ the equations could be formulated as

$$\frac{\partial(aA_a + bB_a)}{\partial p^\alpha}\frac{\partial(aA_a + bB_a)}{\partial p^\beta} + \frac{\partial(cA_a + dB_a)}{\partial p^\alpha}\frac{\partial(cA_a + dB_a)}{\partial p^\beta} = g^M_{\alpha\beta} \tag{4.126}$$

with the constraints

$$\begin{cases} ad - bc = 1 \\ A_{[i}B_{j]} = p_{ij} = p_{k(ij)} \end{cases} \tag{4.127}$$

where the last equality follows from the chart implicit formulation, $k = 1, \ldots, 5$. This is a set of 15 partial differential equations in 5 variables p with the complication that the 3 functions: a, b, c are explicitly dependent on the 8 variables A_a, B_a and the implicit p dependence is only fixed after choosing representative.

Finding a solution to this equation would mean to prove that the two terms from the Lagrangian indeed are the Grassmannian sigma model. To show that there are no solutions to this equation would be to rule out that possibility. Explicitly:

$$\frac{\partial A'_a}{\partial p^1}\frac{\partial A'_a}{\partial p^1}+\frac{\partial B'_a}{\partial p^1}\frac{\partial B'_a}{\partial p^1}=1+\frac{(p_2p_5-p_3p_4)^2}{p_1^4} \tag{4.128}$$

$$\frac{\partial A'_a}{\partial p^1}\frac{\partial A'_a}{\partial p^2}+\frac{\partial B'_a}{\partial p^1}\frac{\partial B'_a}{\partial p^2}=-\frac{p_5(p_2p_5-p_3p_4)}{p_1^3} \tag{4.129}$$

$$\frac{\partial A'_a}{\partial p^1}\frac{\partial A'_a}{\partial p^3}+\frac{\partial B'_a}{\partial p^1}\frac{\partial B'_a}{\partial p^3}=\frac{p_4(p_2p_5-p_3p_4)}{p_1^3} \tag{4.130}$$

$$\frac{\partial A'_a}{\partial p^1}\frac{\partial A'_a}{\partial p^4}+\frac{\partial B'_a}{\partial p^1}\frac{\partial B'_a}{\partial p^4}=\frac{p_3(p_2p_5-p_3p_4)}{p_1^3} \tag{4.131}$$

$$\frac{\partial A'_a}{\partial p^1}\frac{\partial A'_a}{\partial p^5}+\frac{\partial B'_a}{\partial p^1}\frac{\partial B'_a}{\partial p^5}=-\frac{p_2(p_2p_5-p_3p_4)}{p_1^3} \tag{4.132}$$

$$\frac{\partial A'_a}{\partial p^2}\frac{\partial A'_a}{\partial p^2}+\frac{\partial B'_a}{\partial p^2}\frac{\partial B'_a}{\partial p^2}=1+\frac{p_5^2}{p_1^2} \tag{4.133}$$

$$\frac{\partial A'_a}{\partial p^2}\frac{\partial A'_a}{\partial p^3}+\frac{\partial B'_a}{\partial p^2}\frac{\partial B'_a}{\partial p^3}=-\frac{p_4p_5}{p_1^2} \tag{4.134}$$

$$\frac{\partial A'_a}{\partial p^2}\frac{\partial A'_a}{\partial p^4}+\frac{\partial B'_a}{\partial p^2}\frac{\partial B'_a}{\partial p^4}=-\frac{p_3p_5}{p_1^2} \tag{4.135}$$

$$\frac{\partial A'_a}{\partial p^2}\frac{\partial A'_a}{\partial p^5}+\frac{\partial B'_a}{\partial p^2}\frac{\partial B'_a}{\partial p^5}=\frac{p_2p_5}{p_1^2} \tag{4.136}$$

$$\frac{\partial A'_a}{\partial p^3}\frac{\partial A'_a}{\partial p^3}+\frac{\partial B'_a}{\partial p^3}\frac{\partial B'_a}{\partial p^3}=1+\frac{p_4^2}{p_1^2} \tag{4.137}$$

$$\frac{\partial A'_a}{\partial p^3}\frac{\partial A'_a}{\partial p^4}+\frac{\partial B'_a}{\partial p^3}\frac{\partial B'_a}{\partial p^4}=\frac{p_3p_4}{p_1^2} \tag{4.138}$$

$$\frac{\partial A'_a}{\partial p^3}\frac{\partial A'_a}{\partial p^5}+\frac{\partial B'_a}{\partial p^3}\frac{\partial B'_a}{\partial p^5}=-\frac{p_2p_4}{p_1^2} \tag{4.139}$$

$$\frac{\partial A'_a}{\partial p^4}\frac{\partial A'_a}{\partial p^4}+\frac{\partial B'_a}{\partial p^4}\frac{\partial B'_a}{\partial p^4}=1+\frac{p_3^2}{p_1^2} \tag{4.140}$$

$$\frac{\partial A'_a}{\partial p^4}\frac{\partial A'_a}{\partial p^5}+\frac{\partial B'_a}{\partial p^4}\frac{\partial B'_a}{\partial p^5}=-\frac{p_2p_3}{p_1^2} \tag{4.141}$$

$$\frac{\partial A'_a}{\partial p^5}\frac{\partial A'_a}{\partial p^5}+\frac{\partial B'_a}{\partial p^5}\frac{\partial B'_a}{\partial p^5}=1+\frac{p_2^2}{p_1^2} \tag{4.142}$$

with the constraints (4.127).

The symmetries of the equations and the constraints can be used to simplify the set further. Several *Ansatze* can be made, with guidance of the geometry.

Despite one or two attempts (in particular by generalising the inhomogeneous coordinates of real projective space to M), a solution has not yet been found. The final remark we will make on this is that the assertion can be proved with strikingly much more ease and beauty, which is the course we will follow.

4.5 Hodge Star Decomposition

The way to circumvent the equations (4.128 - 4.142) involves decomposing the oriented Grassmannian $\tilde{G}_{2,4}$ into its constituent spheres. This will be done in two ways, the simplest one employs the dual operator often called the Hodge star.

Before our real argument, we would like to give a hand waving argument as a motivation. Recall, that $G_{2,4}$ can be written as the homogenous space

$$\frac{SO(4)}{SO(2)\times SO(2)} \tag{4.143}$$

under the ordinary action of the orthogonal group.

$$S^3 \simeq \frac{SO(4)}{SO(3)},\ S^2 \simeq \frac{SO(3)}{SO(2)} \tag{4.144}$$

and locally, by the Hopf fibration

$$S^3 \simeq S^2 \times S^1 \tag{4.145}$$

Naively plugging this into (11), we find that $G_{2,4}$ should look something like

$$S^2 \times S^2 \tag{4.146}$$

We will now proceed to make these relations exact for the two-fold covering of $G_{2,4}$, the oriented Grassmanian $\tilde{G}_{2,4}$, by a method known from e.g. [27].

Take $p \in \tilde{G}_{2,4}$. $\{e_i\}_{i=1}^4$ is a basis of the four dimensional real vector space, V, over which the Grassmannian is constructed. All points in the Grassmannian can be represented as decomposable elements in $\Lambda^2(V)$ by simply wedging two linearly independent vectors spanning the plane as above. This statement is equivalent to the Plücker coordinates satify the Plücker equation (or that the "electric" and "magnetic" components are orthogonal). Representing p by $u\wedge v \in \Lambda^2(V)$, we choose six of the exterior products between the basis vectors of V to constitute the basis of $\Lambda^2(V)$. To make it definite, for $e_{ij} = e_i \wedge e_j$ take $(e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34})$ as a basis.

$$p \sim c_{ij} e_{ij} = u_i v_j e_{ij} \tag{4.147}$$

This way the coordinates are the Plücker coordinates.

The dual operator sends a plane in V to the plane orthogonal to it. In differential geometry, the Hodge dual star operator is introduced as a map

$$\star : \Lambda^n(V) \to \Lambda^{d-n}(V) \tag{4.148}$$

where $\dim V = d$, $n \leq d$. In this case, its action on the plane is just the above [14].

Needless to say, no plane is dual to itself. If we make a change of coordinates, to the eigenbasis of the dual operator, then the product manifold structure emerges.

$$\star e_{12} = \frac{1}{2}\epsilon_{ij12} e_{ij} = e_{34} \tag{4.149}$$

$$\star e_{34} = e_{12} \tag{4.150}$$

$$\star e_{13} = -e_{24} \tag{4.151}$$

$$\star e_{24} = -e_{13} \tag{4.152}$$

$$\star e_{14} = e_{23} \tag{4.153}$$

$$\star e_{23} = e_{14} \tag{4.154}$$

$$\tag{4.155}$$

So construct the eigenbasis

$$f_1^+ = \frac{1}{\sqrt{2}}(e_{12} + e_{34}) \tag{4.156}$$

$$f_1^- = \frac{1}{\sqrt{2}}(e_{12} - e_{34}) \tag{4.157}$$

$$\tag{4.158}$$

$$f_2^+ = \frac{1}{\sqrt{2}}(e_{13} - e_{24}) \tag{4.159}$$

$$f_2^+ = \frac{1}{\sqrt{2}}(e_{13} + e_{24}) \tag{4.160}$$

$$\tag{4.161}$$

$$f_3^+ = \frac{1}{\sqrt{2}}(e_{14} + e_{23}) \tag{4.162}$$

$$f_3^- = \frac{1}{\sqrt{2}}(e_{14} - e_{23}) \tag{4.163}$$

Expressing the representative of the generic point p in this basis yields

$$
\begin{aligned}
p \sim & u_i v_j e_{ij} = p_{12}e_{12} + p_{13}e_{13} + p_{14}e_{14} + p_{23}e_{23} + p_{24}e_{24} + p_{34}e_{34} = && (4.164)\\
= & \frac{1}{\sqrt{2}}((E_1 + B_1)f_1^+ + (E_2 + B_2)f_2^+ + (E_3 + B_3)f_3^+ + && (4.165)\\
& + (E_1 - B_1)f_1^- + (E_2 - B_2)f_2^- + (E_3 - B_3)f_3^-) = && (4.166)\\
= & \frac{1}{\sqrt{2}}(\vec{\xi}^+ + \vec{\xi}^-) \in S_+^2 \times S_-^2 && (4.167)
\end{aligned}
$$

Where the standard identification of the electric and magnetic fields have been made (3.25), (3.26). They also satisfy the normalization and orthogonality conditions

$$
\begin{cases} \vec{E}^{\,2} + \vec{B}^{\,2} = 1 \\ \vec{E} \cdot \vec{B} = 0 \end{cases} \Leftrightarrow \begin{cases} \vec{\xi}^{+\,2} = \vec{\xi}^{-\,2} = 1 \\ \vec{\xi}^+ \cdot \vec{\xi}^- = 0 \end{cases} \tag{4.168}
$$

that fixes the the Grassmannian to the intersection of the Klein quadric with S^5. We have that any point in the oriented Grassmannian can be written as

$$
\begin{aligned}
\vec{\xi}^+ &= \vec{E} + \vec{B} && (4.169)\\
\vec{\xi}^- &= \vec{E} - \vec{B} && (4.170)\\
p \in \tilde{G}_{2,4} \quad \sim \quad (\vec{\xi}^+(p), \vec{\xi}^-(p)) &\in S_+^2 \times S_-^2 && (4.171)
\end{aligned}
$$

4.6 Some Complex Differential Geometry

In obtaining the rationale for standardly embed these spheres in a Euclidean space, we will use some notions from complex differential geometry, which here will be reviewed [15], [16].

In complex analysis functions from $\mathbb{C}$ to $\mathbb{C}$ are studied. A holomorphic (analytic) function is a function that satisfies Cauchy-Riemanns equations. Geometrically this means that the value of the complex derivative at a point is the same no matter what direction it is approached from. Alternatively, given a complex function f, analyticity (Cauchy–Riemanns equations) follows from

$$
\frac{\partial f}{\partial \bar{z}} = 0 \tag{4.172}
$$

after expanding f and z into real and imaginary components.

Quickly leaping forwards, a function $f:\ \mathbb{C}^m \rightarrow \mathbb{C}^n$ is called holomorphic if all the n complex components of f satisfies Cauchy-Riemanns equations with respect to all the m complex coordinates. A complex manifold is a manifold where the transition functions between the coordinates (which are maps to $\mathbb{C}^m$ now) of overlapping charts are holomorphic, instead of just smooth. Going about showing that a manifold is complex can be done by introducing some particular set of charts (with complex coordinates) on the manifold and verifying that they give holomorphic transition functions. Alternatively it can be done by taking a $2m$ dimensional real manifold and "push it" to satisfy certain constraints, giving a complex manifold. Since the second way includes a good way to introduce holomorphic one-forms and the notion of complex structure which will be of interest in this section and the next, that is what we will do.

Take the manifold M such that $dim_{\mathbb{R}}M = 2m$, and consider the tangent space of $p \in M$, T_pM. Let it be spanned by the orthonormal basis

$$\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^m}\} \tag{4.173}$$

and let the cotangent space at p, T_p^*M be spanned by

$$\{dx^1, \ldots, dx^m, dy^1, \ldots, dy^m\} \tag{4.174}$$

An almost complex structure, J, is a real tensor field of type (1, 1), i.e.

$$J:\ TM\ \rightarrow\ TM \tag{4.175}$$

such that at the point $p \in M$, with tangent space basis as above

$$J_p(\frac{\partial}{\partial x^\mu}) = \frac{\partial}{\partial y^\mu} \tag{4.176}$$

$$J_p(\frac{\partial}{\partial y^\mu}) = -\frac{\partial}{\partial x^\mu} \tag{4.177}$$

Applying it twice to the tangent space explains the name

$$J_p^2 = -1 \tag{4.178}$$

So, by analogy with $\mathbb{R}^2$, $J_p \sim i$.

If the manifold is a complex one, so that complex coordinates can be defined and come with holomorphic transition functions on overlaps of charts, then Cauchy-Riemanns equations can be used to show that the feature [4.176], [4.177] persists after a change of coordinates. In other words, to define the tensor field globally requires a complex manifold. The (almost) complex structure is then the tensor field that have the components

$$J_p = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{4.179}$$

where the 1:s are the $m \times m$ unit matrices, at any point $p \in M$.

The almost complex structure looks very nice in a particular basis of the *complexified* tangent space, denoted $(T_pM)^{\mathbb{C}}$. A vector $X \in (T_pM)^{\mathbb{C}}$ if it consists of vectors $u, v \in T_pM$ such that $X = u + iv$. If T_pM is a real vector space of dimension $2m$, then that is the complex dimension of $(T_pM)^{\mathbb{C}}$ as well. Linear operators on T_pM are taken to be linearly extended to $(T_pM)^{\mathbb{C}}$, i.e. if f is a linear operator on T_pM ,then it acts on $X = u + iv \in (T_pM)^{\mathbb{C}}$ like

$$f(u + iv) = f(u) + if(v) \tag{4.180}$$

Now, a basis of this space worth considering is given by

$$\frac{\partial}{\partial z^\mu} := \frac{1}{2}\{\frac{\partial}{\partial x^\mu} - i\frac{\partial}{\partial y^\mu}\} \tag{4.181}$$

$$\frac{\partial}{\partial \bar{z}^\mu} := \frac{1}{2}\{\frac{\partial}{\partial x^\mu} + i\frac{\partial}{\partial y^\mu}\} \tag{4.182}$$

(4.183)

For $\mu = 1, \ldots, m$ The corresponding basis of the complexified cotangent space is given by

$$dz^\mu = dx^\mu + idy^\mu \tag{4.184}$$

$$d\bar{z}^\mu = dx^\mu - idy^\mu \tag{4.185}$$

It is easy to verify that this is an orthonormal basis

$$< dz^\nu, \frac{\partial}{\partial \bar{z}^\mu} >=< d\bar{z}^\nu, \frac{\partial}{\partial z^\mu} >= 0 \tag{4.186}$$

$$< dz^\nu, \frac{\partial}{\partial z^\mu} >=< d\bar{z}^\nu, \frac{\partial}{\partial \bar{z}^\mu} >= \delta^\nu_\mu \tag{4.187}$$

To verify the promise of a good looking complex structure on $(T_pM)^{\mathbb{C}}$, just extend the operator J_p to $(T_pM)^{\mathbb{C}}$, and see what it does to these basis vectors

$$J_p\frac{\partial}{\partial z^\mu} = i\frac{\partial}{\partial z^\mu} \tag{4.188}$$

$$J_p\frac{\partial}{\partial \bar{z}^\mu} = -i\frac{\partial}{\partial \bar{z}^\mu} \tag{4.189}$$

Or, to paraphrase the above equations; this is the eigenbasis of the complex structure on the complexified tangent space. It factorizes the vector space into two orthogonal eigenspaces corresponding to the eigenvalues $\pm i$. So the almost complex structure is diagonal in this basis

$$J_p = idz^\mu \otimes \frac{\partial}{\partial z^\mu} - id\bar{z}^\mu \otimes \frac{\partial}{\partial \bar{z}^\mu} \tag{4.190}$$

In components (with respect to this basis) the $2m \times 2m$ matrix looks like

$$J_p = \begin{pmatrix} i\,1 & 0 \\ 0 & -i\,1 \end{pmatrix} \tag{4.191}$$

In conclusion, $(T_pM)^{\mathbb{C}}$ can be written as the direct sum of these two (disjoint) subspaces, each of complex dimension m

$$(T_pM)^{\mathbb{C}} = T_pM^{\ +} \oplus T_pM^{\ -} \tag{4.192}$$

$$T_pM^{\ \pm} = \{X \in (T_pM)^{\mathbb{C}} \ : J_pX = \pm iX\} \tag{4.193}$$

(where dropping the $\mathbb{C}$ is done to improve transparency). Any elements of the complexified vector space can be written (uniquely) as the sum of two vectors, of which one is in $T_pM^{\ +}$ and the other in $T_pM^{\ -}$.

$$X \in (T_pM)^{\mathbb{C}} \implies X = X_+ + X_- \ , \quad X_\pm \in T_pM^{\ \pm} \tag{4.194}$$

A vector in $T_pM^{\ +}$ is called *holomorphic*, while one in the orthogonal complement is called *anti-holomorphic*. A similar decomposition holds for coplexified vector fields over M, once J is defined over the manifold, and naturally also for the complex forms.

The complexification of the space of q-forms, $\Omega_p^q(M)^{\mathbb{C}}$, follows the above definition for vector spaces in general:

$$\xi \in \Omega_p^q(M)^{\mathbb{C}} \implies \xi = \omega + i\eta \ , \quad \omega, \eta \in \Omega_p(M) \tag{4.195}$$

Take $\omega \in \Omega_p^q(M)^{\mathbb{C}}$, ($q \leq 2m$) such that ω is a functional of r holomorphic and s anti-holomorphic vectors. Naturally $r+s=q$. The bidegree of ω is the pair of numbers (r,s). The set of (r,s)-forms over the manifold is denoted $\Omega_p^{(r,s)}(M)$. For example dz^μ has bidegree $(1,0)$ and $d\bar{z}^\mu$ is an example of a $(0,1)$-form. A general (r,s)-form in this basis takes the form

$$\omega = \frac{1}{r!s!}\omega_{\mu_1\ldots\mu_r\nu_1\ldots\nu_s}\ dz^{\mu_1}\ldots dz^{\mu_r} d\bar{z}^{\nu_1}\ldots d\bar{z}^{\nu_s} \tag{4.196}$$

where the wedges are implicit.

A element of $\Omega^q_p(M)^{\mathbb{C}}$is in general not a "pure" (r,s)-form though, but made up of a linear combination forms of different different bidegree, as long as $r+s=q$ still holds.

$$\omega = \sum_{r+s=q} \omega^{(r,s)} \tag{4.197}$$

$$\Omega^q_p(M)^{\mathbb{C}} = \oplus_{r+s=q}\ \Omega^{(r,s)}_p(M) \tag{4.198}$$

In particular, we have that

$$\Omega^1(M)^{\mathbb{C}} = \Omega^{(1,0)}(M) \oplus \Omega^{(0,1)}(M) \tag{4.199}$$

4.7 Quaternionic Decomposition

We now put the theory to use by constructing a second way of showing $\tilde{G}_{2,4} \simeq S^2 \times S^2$. Given $p \in \tilde{G}_{2,4}$, take an ordered pair of orthonormal unit vectors, $e^1,\ e^2$ such that the span of these is the plane $\mathbb{R}^4$ (with the correct orientation) corresponding to p.

Identify $e^1,\ e^2$ with the quaternions, H in the standard way, i.e.

$$e_a = e_0 + ie_1 + je_3 + ke_3 = e_0 + \vec{e} \tag{4.200}$$

with $i^2 = j^2 = k^2 = ijk = -1$. Since for $p, q \in (H, \delta)$

$$p = a + \vec{u} \tag{4.201}$$

$$q = b + \vec{v} \tag{4.202}$$

$$pq = ab - \vec{u}\cdot\vec{v} + a\vec{v} + b\vec{u} + \vec{u}\times\vec{v} \tag{4.203}$$

we find that the real part vanish from the quaternionic products

$$\left(\frac{e^2}{e^1}\right)_R = e^2\bar{e}^1 \tag{4.204}$$

$$\left(\frac{e^2}{e^1}\right)_L = \bar{e}^1 e^2 \tag{4.205}$$

Since the resulting quaternions are of norm 1, this defines two maps from $\tilde{G}_{2,4}$ to S^2. How are we to interpret this result?

Given an oriented plane $p \in \tilde{G}_{2,4}$, define a linear map $J_p:\ \mathbb{R}^4 \to \mathbb{R}^4$ consisting of positive rotations by $\frac{\pi}{2}$ in the plane p as well as in the unique oriented plane $p^\dagger$ orthogonal to p. Extend this map linearly to the entire space. Since $(J_p)^2 = -1$, J_p defines an almost complex structure on H. This almost complex structure is immediately extended to the entire space $(\mathbb{R}^4, \delta)$, and we drop the "almost". Different points in the $\tilde{G}_{2,4}$ may determine the same complex structure. Since for vectors v

$$\forall v \in p: \; J_p(v) \in p \tag{4.206}$$

and similarly for $p^\dagger$, the planes p and $p^\dagger$have complex dimension one, or in other words, are complex lines with respect to J_p (like the complex plane under multiplication with i) . Points that share complex structure with p are similarly complex lines with respect to J_p.

How many complex structures are there on $\mathbb{R}^4$? Take the "North Pole" quaternion $u = 1$ and act on it with J_p. Since $J_p$will map u to its orthogonal complement and also preserve its length,

$$J_p(u) = x \in S^2 \tag{4.207}$$

where S^2 is the unit sphere in the completely imaginary subspace of H, or equivalently, the subspace of all $\sqrt{-1}$ in H. Since the plane with ordered basis (u, x) obviously determines the same complex structure as p, we can identify J_p with $x \in S^2$. Since all complex structures can be represented by a point in S^2, this S^2 is the space of complex structures on $\mathbb{R}^4$.

Furthermore, if $x \in S^2$ is the complex structure of the plane p with the ordered basis (e^1, e^2) then the action of x on the unit vectors should be

$$x : \begin{cases} e^1 & \mapsto e^2 \\ e^2 & \mapsto -e^1 \end{cases} \tag{4.208}$$

By the ordinary rules of quaternion multiplication (4.203)

$$(e^2\bar{e}^1)e^1 = e^2 \tag{4.209}$$

$$(e^2\bar{e}^1)e^2 = e^2(-\bar{e}^2e^1) = -e^1 \tag{4.210}$$

and with the definitions of electric and megnetic field (3.25), (3.26).

$$x = e^2\bar{e}^1 = e_0^1\vec{e}^2 - e_0^2\vec{e}^1 - \vec{e}^1 \times \vec{e}^2 = \tag{4.211}$$

$$= \vec{E} \; - \vec{B} \tag{4.212}$$

In other words, the sphere of complex structure on $\mathbb{R}^4$ are naturally given coordinates through the Plücker version of the electric and magnetic fields.

Given $x \in S^2$, how many planes in H share this complex structure? Let p be the oriented plane spanned by (u, x), then $J^{-1}(x)$ are the set of points in the $\tilde{G}_{2,4}$ with the given complex structure. As previously stated this is the set of planes in H that are one dimensional complex subspaces with respect to the complex structure J_p, or the set of complex lines with respect to J_p. But this is just the one dimensional complex projective space, $\mathbb{C}P^1 \simeq S^2$. Thus, $\tilde{G}_{2,4}$ is an S^2 trivial bundle over S^2, which is just the product space

$$\tilde{G}_{2,4} \simeq S^2 \times S^2 \tag{4.213}$$

Explicitly, for $y \in S^2$ the coordinates of this sphere is

$$y = \bar{e}^1 e^2 = e_0^1 \vec{e}^2 - e_0^2 \vec{e}^1 + \vec{e}^1 \times \vec{e}^2 = \tag{4.214}$$

$$= \vec{E} + \vec{B} \tag{4.215}$$

4.7.1 Verification

Even though the two spheres above play different roles in terms of Plücker coordinates and geometrical interpretation, they are both embedded in the complex subspace of H. Since we started with $(\mathbb{R}^4, \delta)$, or similarly (H, δ), this sphere carry the standardly induced metric tensor of the sphere. We have

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 + d\alpha^2 + \sin^2\alpha d\beta^2 \tag{4.216}$$

$$\tag{4.217}$$

$$x = \begin{pmatrix} 0 \\ \vec{\xi}^- \end{pmatrix} = \begin{pmatrix} 0 \\ \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{pmatrix} \tag{4.218}$$

$$y = \begin{pmatrix} 0 \\ \vec{\xi}^+ \end{pmatrix} \begin{pmatrix} 0 \\ \cos\beta\sin\alpha \\ \sin\beta\sin\alpha \\ \cos\alpha \end{pmatrix} \tag{4.219}$$

With the conditions $(\vec{\xi}^{\pm})^2 = 1$ naturally satisfied. Its a quick thing to obtain the Christoffel symbols from this metric

$$\Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta \tag{4.220}$$

$$\Gamma^{\phi}{}_{\theta\phi} = \cot\theta \tag{4.221}$$

$$\Gamma^{\alpha}{}_{\beta\beta} = -\sin\alpha\cos\alpha \tag{4.222}$$

$$\Gamma^{\beta}{}_{\alpha\beta} = \cot\alpha \tag{4.223}$$

Giving the classical equations of motion

$$\begin{cases} \partial^2\phi + 2\cot\theta\partial\phi\partial\theta & = 0 \\ \partial^2\theta - \sin\theta\cos\theta(\partial\phi)^2 & = 0 \\ \partial^2\beta + 2\cot\alpha\partial\beta\partial\alpha & = 0 \\ \partial^2\alpha - \sin\alpha\cos\alpha(\partial\beta)^2 & = 0 \end{cases} \tag{4.224}$$

The independent non-zero components of the Riemann curvature and the scalar curvature are

$$R_{\theta\phi\theta\phi} = \sin^2\theta \tag{4.225}$$

$$R_{\alpha\beta\alpha\beta} = \sin^2\alpha \tag{4.226}$$

$$\mathcal{R} = -4 \tag{4.227}$$

For the case of unoriented planes, there are sign ambiguities of the elements of the both spheres, so we identify antipodal points to obtain

$$G_{2,4} \simeq \mathbb{R}P^2 \times \mathbb{R}P^2 \tag{4.228}$$

This ambiguity matches the one previously discussed for S^5 and $\mathbb{R}P^5$. Expressed in the "electric" and "magnetic" coordinates the metric of p–space can consistently be taken as

$$g_{\mu\nu}(\vec{E}\,,\vec{B}\,) = \delta_{\mu\nu} \tag{4.229}$$

which agrees shows that the geometrically constructed Grassmannian agrees with the form deduced by Faddeev and Niemi [4]. We conclude that the Grassmannian sigma model can be written as

$$(\partial\vec{E}\,)^2 + (\partial\vec{B}\,)^2 \tag{4.230}$$

with the constraints

$$\begin{cases} \vec{E}\,\cdot\vec{B}\, = 0 \\ \vec{E}\,^2 + \vec{B}\,^2 = 1 \\ (\vec{E}\,,\vec{B}\,) \sim -(\vec{E}\,,\vec{B}\,) \end{cases} \tag{4.231}$$

Chapter 5

The Interaction

Reconsider the Lagrangian in spin-charge separated variables (4.69). As has been discussed at length the fourth term involves the Grassmannian sigma model of the manifold $G_{2,4}$. But there is more to it, the Lagrangian also contains the terms

$$(\partial\vec{n})^2 + (\vec{n}\cdot(\partial_a\vec{n}\times\partial_b\vec{n}))^2 \tag{5.1}$$

for the unit vector $\vec{n}$. This is a form of the previously discussed O(3) non-linear sigma model, originally suggested to be related to low–energy SU(2) Yang–Mills theory by Faddeev [22]. Moreover, in [23] it was suggested that this form of the O(3) sigma model supports stable solitons in the form of closed, knotted strings, which later have been verified by numerical and analytical analysis [24], [25]. The idea is suggestive in that it favours the interpretation of QCD strings which, in absence of quarks, close on themselves to form knots. In [4] Faddeev and Niemi argue that also the Grassmannian sigma model $G_{2,4}$ supports such knots as solutions.

5.1 The Interaction Terms

Note that the O(3) non-linear sigma model and the $G_{2,4}$ sigma model are not uncoupled. They interact with each other in a rather intricate way through the terms (4.67), which we reproduce here for sake of convenience

$$\frac{\rho^2}{2}\left(t_+(D_a^*\bar{e}_b)^2 + t_-(D_ae_b)^2\right) = \frac{\rho^2}{2}\{n_+(\partial_a\hat{e}_b^*)^2 + n_-(\partial_a\hat{e}_b)^2\} \tag{5.2}$$

By plugging in the expression for e in terms of the electric and magnetic fields, Faddeev and Niemi obtain (note the different choice of normalization from [4])

$$\frac{\rho^2 n_+}{128\ \vec{S}^{\ 2}} < \partial_a(\vec{E}\ + \vec{B}\ ), \vec{E}\ - \vec{B}\ - 2i\vec{S}\ >< \partial_a(\vec{E}\ - \vec{B}\ ), \vec{E}\ + \vec{B}\ - 2i\vec{S}\ > \tag{5.3}$$

$$\frac{\rho^2 n_-}{128\ \vec{S}^{\ 2}} < \partial_a(\vec{E}\ + \vec{B}\ ), \vec{E}\ - \vec{B}\ + 2i\vec{S}\ >< \partial_a(\vec{E}\ - \vec{B}\ ), \vec{E}\ + \vec{B}\ + 2i\vec{S}\ > \tag{5.4}$$

We will follow the steps of [4] and consider the limits of this expression, then we will go on and re-write it in a more compact and suggestive form.

5.1.1 Limits

The limits to be considered are the $\vartheta \to 0,\ \pi/2$ with ϑ

$$\begin{cases} \vec{E} & = \cos\vartheta \hat{k} \\ \vec{B} & = \sin\vartheta \hat{l} \end{cases} \tag{5.5}$$

With the obvious notation $\vec{E}\ = E\hat{k},\ \vec{S}\ = S\hat{m}$, consider the factor

$$\begin{aligned} \frac{1}{S} < \partial(\vec{E}\ + \vec{B}\ ), \vec{E}\ - \vec{B}\ - 2i\vec{S}\ >= \\ = \frac{1}{EB} < (\partial E)\hat{k}\ + E\partial\hat{k}\ + (\partial B)\hat{l}\ + B(\partial\hat{l}\ ), E\hat{k}\ - B\hat{l}\ - 2i\hat{m}\ >= \\ \frac{1}{EB}\left(\partial(E^2) - 2EB(\partial\hat{k}\ )\cdot\hat{l}\ - 2iEB\hat{m}\ \cdot(E\partial(\hat{k}\ ) + B\partial(\hat{l}\ ))\right) = \\ -2\left((\partial\hat{k}\ )\cdot\hat{l}\ + i\hat{m}\ \cdot(E\partial\hat{k}\ + B\partial\hat{l}\ )\right) \end{aligned}$$

So, in the $\vartheta \to 0$ limit

$$-2\partial\hat{k}\ (\hat{l}\ + i\hat{m}\ ) \tag{5.6}$$

and in the (hardly defined limit of leaving the chart, which any way is suggestive), $\vartheta \to \pi/2$

$$2\partial\hat{l}\ \cdot(\hat{k}\ - i\hat{m}\ ) \tag{5.7}$$

Note that for scalars $X,\ Y$

$$dX = \partial_a X dx^a \quad dY = \partial_a Y dx^a \tag{5.8}$$

$$\partial_a X \partial_a Y = (dX, dY) \tag{5.9}$$

where the (inverse Euclidean) inner product in the cotangent space is denoted by parenthesis. Identify $\hat{k} \ = \hat{r}$.

$$(\hat{l} \ + i\hat{m} \ )\partial_a \hat{k} \ dx^a = d\theta + i \sin\theta d\phi \tag{5.10}$$

A similar relation holds for the second case. So, in these limits, the coupling between the models are related to some certain one-forms on the sphere, which in [4] are called *"the unique (anti-) holomorphic forms on the sphere"*. Further it was suggested that the total coupling engage the holomorphic one-forms on $\tilde{G}_{2,4}$. This is an unfortunate wording however, for reasons that will become clear in the following sections.

5.2 Holomorphic forms and Dolbeault Cohomology

If one was to apply the exterior derivative to a complex q-form with bidegree (r, s), what would happen? Since

$$d : \Omega^q_p(M)^{\mathbb{C}} \to \Omega^{q+1}_p(M)^{\mathbb{C}} \tag{5.11}$$

d will send the (r, s)-form to forms of bidegree $(r + 1, s)$ and $(r, s + 1)$. The exterior derivative is in this way decomposed into so called *Dolbeault operators*:

$$d = \partial + \bar{\partial} \tag{5.12}$$

$$\partial : \Omega^{(r,s)}_p(M) \to \Omega^{r+1,s}(M) \tag{5.13}$$

$$\bar{\partial} : \Omega^{(r,s)}_p(M) \to \Omega^{r,s+1}(M) \tag{5.14}$$

For $\omega \in \Omega^{(1,0)}$ the expression for the components look like

$$\partial\omega = \frac{\partial \omega_\nu}{\partial z^\mu} dz^\mu \wedge dz^\nu \tag{5.15}$$

$$\bar{\partial}\omega = \frac{\partial \omega_\nu}{\partial \bar{z}^\mu} d\bar{z}^\mu \wedge dz^\nu \tag{5.16}$$

Since $d^2 = 0$ (in the sense of acting on some form)

$$(\partial + \bar{\partial})^2 = \partial^2 + (\partial\bar{\partial} + \bar{\partial}\partial) + \bar{\partial}^2 = 0 \tag{5.17}$$

Since these mappings have different destinations, the first and last term must vanish by themselves, as must the expression in parenthesis.

$$\partial^2 = \bar{\partial}^2 = (\bar{\partial}\partial + \partial\bar{\partial}) = 0 \tag{5.18}$$

A form $\omega \in \Omega^{r,0}(M)$ that satisfies

$$\bar{\partial}\omega = 0 \tag{5.19}$$

is called a *holomorphic r-form*. A holomorphic 0-form on $M = \mathbb{C}$, is then what previously was defined as a holomorphic function.

The set of (r,s)- forms satisfying $\bar{\partial}\omega = 0$ is called the *(r, s)-cocycle*, and is denoted $Z^{r,s}_{\bar{\partial}}(M)$. The *(r, s)- coboundary* on the other hand is denoted $B^{r,s}_{\bar{\partial}}(M)$ and consist of the $\bar{\partial}$-exact (r, s) forms, i.e. the $\omega \in \Omega^{(r,s)}_p(M)$ that for some $\eta \in \Omega^{r,s-1}(M)$ can be written as

$$\omega = \bar{\partial}\eta \tag{5.20}$$

The (r,s)-coboundary is trivially a part of the (r, s)-cocycle, but is factored out in the *(r, s):th $\bar{\partial}$-cohomology class*

$$H^{r,s}_{\bar{\partial}}(M) = \frac{Z^{r,s}_{\bar{\partial}}(M)}{B^{r,s}(M)} \tag{5.21}$$

in which the elements are equivalence classes of $\bar{\partial}$-closed forms, only differing by an exact $\bar{\partial}$-form. Since for the simplest possible complex manifold $\mathbb{C}^n$ has empty $\bar{\partial}$-cohomology classes, this concept tells something about how topologically trivial or non-trivial a complex manifold is. Since it is the holomorphic one-forms that appear in the decomposition, the complex dimension of $H^{1,0}(M)$ is of present interest. This quantity is so important in mathematics and physics that it has its own name, the Hodge number $h^{1,0}$.

Let's back a bit and recall that for the exterior derivative itself, we say that an r-form $\omega \in \Omega^r(M)$ is closed if $d\omega = 0$, i.e.

$$\omega \in ker(d) \subset \Omega^{r+1} \tag{5.22}$$

The set of closed r-forms is called the *r-cocycle group*, $Z_r(M)$. The exact forms constitute a subset of the closed forms, called the *coboundary group*, denoted $B^r(M)$.

The *r:th de Rham cohomology group* is defined as

$$H^r(M) = Z^r(M)/B^r(M) \tag{5.23}$$

In words, the elements of the cohomology groups are equivalence classes of closed forms, where the equivalence relation is *"differing only by an exact form"*. Two forms in the same equivalence class are called *cohomologous.*

Recall that a dual to an inner-product space is the space of linear functionals on this space. An example is the 'bras' in the 'bra' and 'ket' formalism of quantum mechanics. The inner product can then by Riesz lemma be expressed either as a map from the vector space times the vector space to the field, or alternatively like a functional from the dual space acting on a vector in the vector space. The dual space is often called the co-space. For instance the linear functionals of tangent vectors are the covectors. What is discussed above is another example, the de Rham cohomology groups are dual to the so called homology groups on the manifold. This duality is seemingly deep, because while the cohomology is defined from forms and exterior derivatives, i.e. typical concepts from differential geometry, the homology groups are topological constructs, independent of the details of geometry.

Stating it simply, the homology groups can be understood as the following. Take a (sufficiently nice) manifold. A simplex is the oriented face of a polyhedron, i.e. poyhedrons are points or oriented lines, planes, volumes etc. Make a triangulation of the topological space of the manifold using simplexes mapped to the manifold. The triangulation will be a set of simplexes in $\mathbb{R}^n$ that in some sense *nicely* fits together and, up to some continuously invertible continuous transformation, *is* the original topological space of the manifold. This thing is called a *simplicial complex.*

An *r-chain* is a formal linear combination of simplexes of the same kind (the coefficients used in this linear combination differ depending on what one is interested in. Here we will mostly consider real coefficients and simply ignore the complication of torsion). For instance, if p_i, p_j are zero simplexes (points) and $(p_i p_j)$ denotes the one-simplex (oriented line) from point i to point j, a zero-chain (with for example real, complex or integer coefficients) is given by $c^0 = 2p_1 + 7p_j$. The simplicial complex K

$$K = \{p_0, p_1, p_2, p_3, (p_0p_1), (p_0p_2), (p_0p_3), (p_1p_2), (p_1p_3), (p_2p_3), \tag{5.24}$$
$$(p_0p_1p_2), (p_0p_1p_3), (p_0p_2p_3), (p_1p_2p_3)\} \tag{5.25}$$

corresponds to the surface of a tetrahedron and is a triangulation of the sphere, S^2.

Introduce a boundary operator that takes an oriented simplex to its (oriented) boundary. For instance, an oriented line from point A to point B, (AB), would have the oriented boundary $B - A$. The boundary is an homoeomorphic nilpotent operator from the set of r-chains to the set of $r - 1$ chains. Intuitively this corresponds to the concept that a boundary does not have a boundary.

The *r-cycles* are the r-chains that lies in the kernel of the boundary operator, or in other words, are boundaryless. The set of all r-chains is denoted $Z_r(K)$, is siply called *the chain group*.

A simplex is called an *r-boundary* if it is in the image of the boundary operator on $r+1$ chains. The set of all r-boundaries with addition constitute a group called the *boundary group*, denoted $B_r(K)$. By now there will be no surprise that for the n-dimensional simplex K, the *r:th (simplicial) homology group* is defined as

$$H_r(K) = Z_r(K)/B_r(K) \tag{5.26}$$

The elements are equivalence classes called *homology classes*.

In all nice cases, the r-simplexes can be mapped to r-dimensional subspaces of the manifold, in which case the homology is called singular. It can be shown though, that the simplicial and singular homology groups are isomorphic, that is, there is a bijective map that preserves group composition (in this case addition) between the groups. We will follow the physics convntion of using the same notation for the both types of homology, and will not dwell further on these matters.

It is *this* thing, the r:th singular homology group, $H_r(M)$ that is dual to the r:th de Rham cohomology group. The inner product is the bilinear map

$$\Lambda : H_r(M) \times H^r(M) \to \mathbb{R} \tag{5.27}$$

Explicitly it is given by integrating the r-form over the subset of the manifold given by the image of the homology class. Stokes theorem ensures that this is a well-defined product.

The homology groups might seem like a very abstract construct, but in fact they can tell a lot about the structure of the manifold at hand. For instance, if the connectedness of a manifold is of interest, just calculate $H_0(M)$. It gives the number of components of the manifold by taking on factor of $\mathbb{Z}, \mathbb{R}, \mathbb{C}$, or whatever field is used, for each connected component of the manifold. The homology groups are topological invariants, that is they are invariant under homeomorphisms, that is continuous maps that have continuous inverses. This is a property not shared by the r-chains or the the r-boundaries. Homology groups also provide the basis for a definition of a generalized Euler Charachteristic. We will be interested in the holomorphic one-forms, and for reasons that soon will become clear, we will pay interest to $H_1(G_{2,4})$.

The *Betti number* of a simplicial complex K is defined as

$$b_r = dimH_r(K) \tag{5.28}$$

Naturally, for a manifold M corresponding to the simplicial complex K the above duality gives

$$b_r = b^r = dimH^r(M) \tag{5.29}$$

So these numbers are to de Rham cohomology what the Hodge numbers are to the Dolbeault cohomology. One might think that the two concepts should be intervened, after all, the Dolbeault operators are made out of the exterior derivative that is used to define the de Rham cohomology. It turns out, however, that not much can be said unless the manifold at hand possesses some extra structure, namely that it is Kähler.

We will use in this chapter that the Grassmannian, $\tilde{G}_{2,4}$, admits a complex structure. This is in fact an unusual property of the family of Grassmannn manifolds, it has been shown that $\tilde{G}_{k,n}$ does not admit a weakly complex structure for $3 \leq k \leq n-3$, except for $\tilde{G}_{3,6}$ and a few unknown cases [26]. In the case of $\tilde{G}_{2,4}$, which is the case of current interest, it is a classical result that it is a Hermitian symmetric space, and then also almost complex. We will now make sure, that supplied with the round metric tensor, it is also Kähler.

Being Kähler means that the metric is hermitian and that a form called the Kähler form can be defined. This form also has to be closed. This holds for $\tilde{G}_{2,4}$. To see this, we simply calculate the Kähler form explicitly. The hermitian metric is given by

$$g_{\mu\nu} = g(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}) \tag{5.30}$$

$$g_{\bar{\mu}\bar{\nu}} = g(\frac{\partial}{\partial \bar{z}^\mu}, \frac{\partial}{\partial \bar{z}^\nu}) \tag{5.31}$$

$$g_{\mu\bar{\nu}} = g(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial \bar{z}^\nu}) \tag{5.32}$$

$$\tag{5.33}$$

For $\tilde{G}_{2,4} \simeq S^2 \times S^2$, we have

$$ds^2 = dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2 \tag{5.34}$$

where

$$dz_1 = d\theta - i \sin\theta \; d\phi \tag{5.35}$$
$$dz_2 = d\alpha - i \sin\alpha \; d\beta \tag{5.36}$$

So upon arranging the lines and columns like $1, \bar{1}, 2, \bar{2}$, the components of the metric in this basis is

$$g = \begin{pmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 \end{pmatrix} \tag{5.37}$$

The Kähler form is

$$\Omega = i g_{\mu\bar{\nu}} dz^{\mu} \wedge d\bar{z}^{\bar{\nu}} = \frac{i}{2} \left(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 \right) = \tag{5.38}$$
$$= \sin\theta \; d\phi \wedge d\theta + \sin\alpha \; d\beta \wedge d\alpha \tag{5.39}$$

Clearly

$$d\Omega = \partial_i(\sin\theta) \; d\theta^i \wedge d\phi \wedge d\theta + \partial_i(\sin\alpha) \; d\theta^i \wedge d\beta \wedge d\alpha = 0 \tag{5.40}$$

So, $\tilde{G}_{2,4}$ is Kähler.

The Betti numbers then relate to the Hodge numbers

$$b^p = \sum_{r+s=p} h^{r,s} \tag{5.41}$$

In particular, by the symmetry of Hodge numbers: $h^{r,s} = h^{s,r}$, we have

$$b^1 = h^{1,0} + h^{0,1} = 2h^{1,0} \tag{5.42}$$

So the problem of finding the dimension of the space of holomorhic one-forms on $\tilde{G}_{2,4}$ turns out to be equivalent to finding the dimension of its first de Rham cohomology class.

Investigation of the cohomology the unoriented Grassmannians is usually conducted by means of Schubert calculus. So will be done in the appendix, for a sense of completion and because it gives us the opportunity to use a method for cell decomposition that involves nice, but perhaps superfluous and high brow mathematics. For the oriented $\tilde{G}_{2,4}$, there is a way of obtaining the cohomology that is simpler than the cell decomposition.

Let M be the product manifold $M = M_1 \times M_2$. The de Rham cohomology group $H^p(M_1)$ and $H^p(M_2)$ have dimensions $b^p(M_1)$ and $b^p(M_2)$ respectively. Choose representatives such that $\{\omega_i^p\}, i = 1, \ldots, b^p(M_1)$ and $\{\eta_i^p\}, i = 1, \ldots, b^p(M_2)$ are basis vectors of these groups.

An element in $H^r(M)$ can be written as a sum of products of $H^p(M_1)$ and $H^{r-p}(M_2)$ for $0 \leq p \leq r$ since $\omega_i^p \wedge \eta_j^{r-p}$ gives a basis for the subspace of closed but not exact forms. To see this, suppose the opposite. Then we can write

$$\omega_i^p \wedge \eta_j^{r-p} = d(\alpha^{p-1} \wedge \beta^{r-p} + \gamma^p \wedge \delta^{r-p-1}) = \tag{5.43}$$

$$= d\alpha^{p-1} \wedge \beta^{r-p} + (-1)^{p-1}\alpha^{p-1} \wedge d\beta^{r-p} + \tag{5.44}$$

$$+ d\gamma^p \wedge \delta^{r-p-1} + (-1)^p\gamma^p \wedge d\delta^{r-p-1} \tag{5.45}$$

But then $\delta^{r-p-1} = \alpha^{p-1} = 0$, which gives a contradiction. This gives the *Künneth formula*

$$H^r(M) = \oplus_{p+q=r} \left(H^p(M_1) \otimes H^q(M_2)\right) \tag{5.46}$$

where the product is a tensor product, not a direct product. In terms of Betti numbers

$$b^r(M) = \sum_{p+q=r} b^p(M_1)\, b^q(M_2) \tag{5.47}$$

Recall that

$$\tilde{G}_{2,4} \simeq S^2 \times S^2 \tag{5.48}$$

We now have a ready to use formula to apply to this structure.

$$b^1(\tilde{G}_{2,4}) = 2b^0(S^2)\, b^1(S^2) \tag{5.49}$$

The Betti numbers for S^2 are text-book examples. One way of finding them is through homology. Given some triangulation of the manifold, e.g. the one given by (5.25), the homology groups can be calculated. Only considering the dimension of the free abelian part (i.e. torsionless part), the Betti numbers are obtained. For the record, the simplicial homology of S^2 with $\mathbb{Z}$ coefficients is

$$H_0(S^2) \simeq \mathbb{Z} \tag{5.50}$$
$$H_1(S^2) \simeq \{0\} \tag{5.51}$$
$$H_2(S^2) \simeq \mathbb{Z} \tag{5.52}$$
$$\tag{5.53}$$

We can now read off the Betti number

$$b^1(S^2) = 0 \tag{5.54}$$
$$\tag{5.55}$$

So, finally we obtain

$$h^{1,0}(S^2) = 0 \tag{5.56}$$
$$h^{1,0}(\tilde{G}_{2,4}\ ) = \frac{1}{2} b^1(\tilde{G}_{2,4}\ ) = b^0(S^2)\ b^1(S^2) = 0 \tag{5.57}$$
$$\tag{5.58}$$

This tells us that when we investigate quantities supposedly being the holomorphic one-forms of these manifolds, the room for non-trivial ones is non-existing. We must conclude that the wording *"unique holomorphic one-form"* on the sphere is unfortunate, because no such form exist. The forms appearing in the interaction terms *do* have a simple interpretation in terms of holomorphic forms though, just not on the sphere.

Recall that these forms look like

$$dz = d\theta - i \sin\theta \ d\phi \tag{5.59}$$
$$\tag{5.60}$$

Recall that the plane has only one holomorphic coordinate z. The corresponding coordinate on the sphere is obtained by stereographic projection.

$$x = \cot(\theta/2) \cos\phi \tag{5.61}$$
$$y = \cot(\theta/2) \sin\phi \tag{5.62}$$
$$z = x + iy = e^{i\phi} \cot(\theta/2) \tag{5.63}$$

So the induced one-form is

$$dz = e^{i\phi} \left(i \cot(\theta/2) d\phi - \frac{1}{2\sin^2\theta} d\theta \right) = (d\theta - i \sin\theta d\phi)(-e^{i\phi} \frac{1}{2\sin^2\theta/2}) \tag{5.64}$$

which is well-defined on the local chart $S^2 - \{North\ Pole,\ South\ Pole\}$. Similarly, the forms of the interaction are proportional to this expression and indeed holomorphic on the local chart with coordinates (θ, ϕ). It is well known that this chart can not be extended to the entire sphere to constitute a good coordinate system, but breaks down at the North and South Poles. This is related to the fact that this form should not be regarded as a holomorphic one-form on the sphere, since there are no such, but on $S^2 - \{North\ Pole,\ South\ Pole\}$. Alternatively, the expression *local sections of the (anti-)holomorhic cotangent bundle,* $T^{*1,0}M$, could be used. Even more explicitly, this can be seen by changing to the *non-coordinate basis.*

The round metric on the sphere is given by:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = d\theta^2 + \sin^2\theta d\phi^2 \tag{5.65}$$

The metric tensor can be decomposed by use of the *zweibeins*

$$e^1{}_1 = 1 \tag{5.66}$$

$$e^2{}_2 = \sin\theta \tag{5.67}$$

such that

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \delta_{ab}\ e^a_\mu\ e^b_\nu dx^\mu dx^\nu = \tag{5.68}$$

$$= \delta_{ab}\ e^a\ e^b = (e^1)^2 + (e^2)^2 \tag{5.69}$$

where the pair $(e^1 = d\theta,\ e^2 = \sin\theta d\phi)$ constitutes an orthonormal basis of the cotangent space T^*S^2. Taking $dz = e^1 + ie^2$ to be the holomorphic form of the chart in analogy with the holomorphic form on $\mathbb{R}^2$, the metric in the local coordinates is just the product of the holomorphic form and its complex conjugate (the anti-holomorphic form).

$$ds^2 = dz\ d\bar{z} \tag{5.70}$$

So, the expression (5.82) is just the holomorphic one-form on S^2_+ with North and South Poles excluded. The holomorphic one-forms of the both spheres minus their poles, are then given by

$$dz_1 = d\theta + i\sin\theta\ d\phi \tag{5.71}$$

$$dz_2 = d\beta + i\sin\beta\ d\alpha \tag{5.72}$$

5.3 The Form of the interaction

Take a generic point in $\tilde{G}_{2,4}$ corresponding to the spherical coordinates $(\phi, \theta, \alpha, \beta)$

$$(\vec{\xi}^{+}(\phi,\theta),\ \vec{\xi}^{-}(\alpha,\beta)) \in S^2 \times S^2 \tag{5.73}$$

From the definition of these vectors in terms of Plücker coordinates, i.e.

$$\vec{\xi}^{\pm} = \vec{E} \pm \vec{B} \tag{5.74}$$

we have

$$\vec{\xi}^{+} \cdot \vec{\xi}^{-} = \cos 2\vartheta \tag{5.75}$$

where ϑ is defined through the magnitudes of the "electric" and "magnetic" fields in equation (5.5). It is clear that the purely electric and purely magnetic cases correspond to parallel and anti-parallel unit vectors, respectively, and that these were the cases investigated in [4].

Identify:

$$\vec{\xi}^{+} = \hat{r} = \begin{pmatrix} \cos\phi \sin\theta \\ \sin\phi \sin\theta \\ \cos\theta \end{pmatrix} \tag{5.76}$$

where the components are taken with respect to the standard Cartesian basis. Expressing $\vec{\xi}^{-}$ in the spherical basis of the S^2_+ gives

$$\vec{\xi}^{-} = \cos 2\vartheta\ \hat{r} + \sin 2\vartheta(\cos\eta\ \hat{\phi} + \sin\eta\ \hat{\theta}) \tag{5.77}$$

where the degrees of freedom previously described by the pair (α, β) now is replaced by the angles (ϑ, η).

We have for $a = 1, \ldots, 4$

$$\partial_a(\hat{r})dx^a = \partial_i(\hat{r})d\theta^i = \sin\theta\ d\phi\ \hat{\phi} - d\theta\ \hat{\theta} \tag{5.78}$$

In preparation for what is to come, consider the product

$$\partial_a(\vec{\xi}^+) \cdot (\vec{\xi}^- + i\vec{\xi}^+ \times \vec{\xi}^-)dx^a = \tag{5.79}$$

$$= \partial_i(\hat{r})d\theta^i \cdot \Big(\sin 2\vartheta(\cos\eta\hat{\phi} + \sin\eta\hat{\theta}) - i(\hat{r} \times \sin 2\vartheta(\cos\eta\hat{\phi} + \sin\eta\hat{\theta}))\Big) = \tag{5.80}$$

$$= \sin 2\vartheta\,(-(\sin\eta + i\cos\eta)d\theta + \sin\theta(\cos\eta - i\sin\eta)d\phi) = \tag{5.81}$$

$$= \sin 2\vartheta\; e^{-i(\eta+\frac{\pi}{2})}\,(d\theta + i\sin\theta d\phi) \tag{5.82}$$

Recall the form of the interaction

$$\frac{\rho^2 n_+}{128\;\vec{S}^{\,2}} < \partial_a(\vec{p} + \vec{q}), \vec{p} - \vec{q} - 2i\vec{S} >< \partial_a(\vec{p} - \vec{q}), \vec{p} + \vec{q} - 2i\vec{S} > \tag{5.83}$$

$$\frac{\rho^2 n_-}{128\;\vec{S}^{\,2}} < \partial_a(\vec{p} + \vec{q}), \vec{p} - \vec{q} + 2i\vec{S} >< \partial_a(\vec{p} - \vec{q}), \vec{p} + \vec{q} + 2i\vec{S} > \tag{5.84}$$

Using (5.74) to rewrite this we obtain

$$\frac{\rho^2 n_+}{128\;(\sin\vartheta\;\cos\vartheta)^2} < \partial_a(\vec{\xi}^+), \vec{\xi}^- + i(\vec{\xi}^+ \times \vec{\xi}^-) >< \partial_a(\vec{\xi}^-), \vec{\xi}^+ + i(\vec{\xi}^+ \times \vec{\xi}^-) > \tag{5.85}$$

$$\frac{\rho^2 n_-}{128\;(\sin\vartheta\;\cos\vartheta)^2} < \partial_a(\vec{\xi}^+), \vec{\xi}^- - i(\vec{\xi}^+ \times \vec{\xi}^-) >< \partial_a(\vec{\xi}^-), \vec{\xi}^+ - i(\vec{\xi}^+ \times \vec{\xi}^-) > \tag{5.86}$$

which is variations of the form (5.79). We therefore treat them similarly and expand the terms with derivative factor on the S^2_+ in terms of the angles $(\phi, \theta, \vartheta, \eta)$ as above, while the terms with derivatives on S^2_- are expressed in $(\alpha, \beta, \vartheta, \eta')$.

So, (5.85) and (5.86) becomes

$$\frac{1}{32}\rho^2 n_+\; e^{i(\eta-\eta')}\;(dz_1, d\bar{z}_2) \tag{5.87}$$

$$\frac{1}{32}\rho^2 n_-\; e^{-i(\eta-\eta')}\;(d\bar{z}_1, dz_2) \tag{5.88}$$

The full interaction between the Grassmannian sigma model and the O(3) sigma model is given by the sum of (5.87) and (5.88).

$$\frac{1}{16}\,\rho^2\,\Re\{n_+ e^{i(\eta-\eta')}\,(dz_1, d\bar{z}_2)\} = \tag{5.89}$$

$$\frac{1}{16}\rho^2\,\sin\tilde{\theta}\Re\{e^{i(\tilde{\phi}-2\tilde{\eta}+\eta-\eta')}\,(dz_1, d\bar{z}_2)\} \tag{5.90}$$

where the angles $(\tilde{\theta}, \tilde{\phi}, \tilde{\eta})$ correspond to degrees of freedom of the O(3) sigma model, and are defined without tilde in [4].

In conclusion, we have shown that the $G_{2,4}$ Grassmannian sigma model Lagrangian can be written in the form:

$$\mathfrak{L}_o = \frac{\rho^2}{4}\left((\partial\vec{p}\,)^2 + (\partial\vec{q}\,)^2\right) \qquad \begin{cases} \vec{E}\cdot\vec{B} & = 0 \\ \vec{E}^{\,2} + \vec{B}^{\,2} & = 1 \\ (\vec{E}\,,\vec{B}\,) & \sim -(\vec{E}\,,\vec{B}\,) \end{cases} \tag{5.91}$$

We have also found that the interaction between the both non-linear sigma models are given by

$$\mathfrak{L}_{\text{int}} = 8\;k\;\rho^2\;\sin\tilde{\theta}\;\Re\{e^{i(\tilde{\phi}-2\tilde{\eta}+\eta-\eta')}\,(dz_1, d\bar{z}_2)\} \tag{5.92}$$

Chapter 6

Discussion

6.1 Discussion

We have shown that the Grassmannian sigma model $G_{2,4}$ appears in the spin-charge separated $SU(2)$ Yang-Mills theory and that the interaction can be interpreted as involving the inner product of the holomorphic one-forms on the local charts of the constituent spheres of $\tilde{G}_{2,4}$. These results are parts of a larger programme on understanding and interpreting the structure emerging from the Yang-Mills Lagrangian. One of the important features of this decomposition, but according to [4] not the only one, is the new means of explaining QCD strings, discussed in chapter 4. The feature of spin-charge separation can easily be generalized from SU(2) to any SU(N) theory, and might be a feature of an even more general class of quantum field theories.

Prospects apart, the crucial point of judging this theory will be simulations in Lattice QCD. There a clear verdict can be obtained whether this phenomenon can be manifest in a low-energy environment or not. We are still waiting for results from such a simulation.

6.2 Acknowledgements

I would like to thank my supervisor prof. Antti Niemi for presenting me this problem and for support. I also owe a thanks to Sergey Slizovskiy for discussions and translations and to Maxim Zabzine for discussions. I have also benefited from communications with Oleg Viro, Lee Rudolph and Christian

Ohm as well as Andreas Bredthauer and Jonas Persson. I would also like to thank Michael Peskin for help with references in the early stages of this work. Finally I would like to thank Marit Strömberg for support.

Appendix:
Schubert decomposition of the Grassmannian

Although not relevant for the results in the *Examensarbete* the following decomposition of the unoriented Grassmannian is nice and can be used to find the homology of this manifold [18], [17]. It will also give us the opportunity to involve some q-mathematics, which is nice.

Bruhat decomposition

Let V_i be the subspace of a four dimensional vector space V spanned by the linearly independent set $\{v_1, \ldots, v_i\}$. This determines a *complete flag*

$$0 \subset V_1 \subset V_2 \subset V_3 \subset V_4 = V$$

For $W \in G_{2,4}$ consider the following string of subspaces

$$0 \subset V_1 \cap W \subset V_2 \cap W \subset V_3 \cap W \subset V_4 \cap W = W$$

At any step the dimension of $V_j \cap W$ is at most one larger than that of $V_{j-1} \cap W$. The jumps will occur on two places along the ladder, call this sequence the *gap sequence*, $\Gamma_W = (\gamma_1, \gamma_2)$.

Suppose for example that $\Gamma_W = (2, 4)$. This gives a way to construct a basis w_1, w_2 of W. Take w_1 to be a nonzero vector in $V_2 \cap W$, $w_1 = \alpha_{11}v_1 + \alpha_{12}v_2$. Since $\alpha_{12} \neq 0$ we can divide by it to set it equal to 1. So, redifining $w_1 = \alpha_{11}v_1 + v_2$. This determines w_1 uniquely. Pick $w_2 \in V_4 \cap W$ but not in $V_2 \cap W$. Since the last coeficcient is different from zero and by subtracting a multiple of w_1, write $w_2 = \alpha_{21}v_1 + \alpha_{23}v_3 + v_4$. This completely determines w_2. So W can be represented uniquely as the row space of the so called *Bruhat matrix* of W

$$\begin{pmatrix} \alpha_{11} & 1 & 0 & 0 \\ \alpha_{21} & 0 & \alpha_{23} & 1 \end{pmatrix}$$

For any choice of the α:s, this matrix will correspond to a W having gap sequence $(2, 4)$. This gives a homoeomorphism from the open Bruhat set:

$\Omega^o_{24} := \{W : \Gamma_W = (2,4)\} \leftrightarrow R^3$. The set $\Omega_{24} := \{W : \gamma_1 \leq 2, \gamma_2 \leq 4\}$ is the closure of Ω^o_{24} in $G_{2,4}$. It is called a *Schubert variety.*

The Grassmannian variety can be written as a disjoint union of open Bruhat cells

$$G_{2,4} = \bigcup_{\bar{a}} \Omega^o_{\bar{a}}$$

for $\bar{a} : 1 \leq a_1 < a_2 \leq 4$, the sum being over all possible gap sequences.

The $G_{2,4}$

The possible gap sequnces for $G_{2,4}$ are $(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)$. The matrices, with stars for the undetermined spots would look like this:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 1 \end{pmatrix},$$
$$\begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{pmatrix}, \begin{pmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \end{pmatrix}$$

Note that each cell can be associated with a Young diagram with one box per star.

q-mathematics

There's a nice way to do the same thing with a different machinery. Quantum calculus can be divided into two modified versions of ordinary calculus, such that ordinary calculus is obtained when some limit is taken. One obtains what is called *h-calculus* if one defines the derivative as $\frac{f(x+h)-f(x)}{h}$ without taking the limit on h. Its relative, *q-calculus*,will be of present importance and arises from the definition of derivative as: $\frac{f(qx)-f(x)}{qx-x}$, which reduces to the ordinary derivative as $q \to 1$. A lot of ordinary calculus can be reproduced in this setting. The number q is sometimes chosen to be a phase, but here a different choice will eventually be made.

First consider the q-derivative of x^n:

$$(x^n)' = \frac{(qx)^n - x^n}{qx - x} = \frac{q^n - 1}{q - 1} x^{n-1} = [n] x^{n-1}$$

with the definition of the q-integer

$$[n] := \frac{q^n - 1}{q - 1} = 1 + q + q^2 + \ldots + q^{n-1}$$

This way $(x^n)^{(n)} = [1][2]\ldots[n] = [n]!$.

An interesting thing is when the connection to *finite fields*, that is fields with finitely many elements, is explored. If q is a power of a prime number, then there's a unique field with q elements called F_q. If q is a prime number, the finite filed is just the integers modulo q. The main thought is that a lot of formulas for counting structures on finite sets have q-version saying how to count structures on projective spaces over q. This will at the end give the Schubert decomposition of $G_{2,4}$.

Examples

The number of lines through the origin of an n-dimensional vector space over F_q, i.e. the number of points in the corresponding projective space, is $[n]$. This is because specifying a line means picking one nonzero vector. There are $q^n - 1$ of these, but they don't determine the lines uniquely of course, because multiplying the vector with a nonzero number from the field gives the same line. So, the total number of lines must be $\frac{q^n-1}{q-1}$.

The number of m-dimensional subspaces over an n-dimensional vector space V over F_q is the q-binomial coefficient $\frac{[n]!}{[m]![n-m]!}$. This is because specifing the m-dimensional subspace corresponds to first choose a line L, there are $[n]$ ways of doing this, then choose another one in V/L, there are $[n-1]$ ways of doing that. Going all the way to the required m dimesnsion gives $\frac{[n]!}{[n-m]!}$, but then the same subspace has been counted several times, because the order in which the lines are plucked doesnt matter. The number of m-flags of the subspace is $[m]!$, as can be seen by continuing the reasoning of example one. So finally we arrive at $\frac{[n]!}{[m]![n-m]!}$

The first example can be further sophisticated by using advanced mathematical techniques, whose proper treatment is well beyond the scope of this text, but some results will be stated and put to use to the benefit of the gullible. The number of lines through the origin in an n-dimensional vector space over the finite field F_q is $[n] = \frac{q^n-1}{q-1} = 1+q+q^2+\ldots+q^{(n-1)}$. Think of the second equality as marking the actual set of such lines, the space seems to be decomposed into a set of one line plus a set of q lines and so on. To establish this explicitly, pick a maximal flag. There is one line contained in the one dimensional subspace. That is the 1 in the formula. There are q lines through the origin not in the first subspace but in the second. That's the q. There are q^2 lines through the origin not contained in the first two subspaced but in the third. That's the q^2, and so on up to the full vector space.

So this establishes a one to one correspondance between the finite sets they count. It is further claimed to work for all fields, not just finite ones, and it is this property that makes the whole thing worth while. So, the real projective space of dimension $n-1$, i.e. the set of lines through the origin of an n-dimensional real vector space can be decomposed into the union of open balls (cells), writing $\mathbb{R}P^{(n-1)} = \mathbb{R}^0 + \mathbb{R}^1 + \mathbb{R}^2 + \cdots + \mathbb{R}^{n-1}$, a result perhaps familiar.

The *Euler charachetristic* of this space is claimed to be found by adding the contributions from the different cells, with $|\mathbb{R}^j| = (-1)^j$, so for the projective space the total Euler charachteristic will be 0 if n is even and 1 if n is odd.

The main point however, is that building the space from cells, homology can be computed from a chain complex with one generator for each cell and a "differential" saying how the cells of dimension n are glued to the dimensions of dimension $n-1$. For complex this is particularily nice, since all cells are even dimensional and the differentials vanish. The n:th integer homology group is just $\mathbb{Z}^k$ with k being the number of cells of dimension n. It also works nicely if homology over real coefficient are considered or if one simply cannot be bothered about factors of $\mathbb{Z}_k$. We will adopt the latter attitude.

Applied to the Grassmannian

In the second example above the cardinality of $G_{k,n}$ was calculated, with the vector space being over the finite field, F_q. Let's do it for $G_{2,4}$ over the real numbers. The q-binomial coefficient is

$$\frac{[4]!}{[2]![2]!} = \frac{1(1+q)(1+q+q^2)(1+q+q^2+q^3)}{1(1+q)\ 1(1+q)} = 1+q+2q^2+q^3+q^4$$

So, since a bijective proof was used to find this formula as well, we have $G_{2,4} = \mathbb{R}^0 + \mathbb{R}^1 + 2\,\mathbb{R}^2 + \mathbb{R}^3 + \mathbb{R}^4$. This is the Schubert cell decomposition, as seen above. The Euler characteristic of $G_{2,4}$ is 2, and up to $\mathbb{Z}_n$ coefficients, the homology is given by

$$\begin{aligned} H_0 = H_4 &= \mathbb{Z} \\ H_1 = H_3 &= \mathbb{Z} \\ H_2 &= \mathbb{Z}^2 \end{aligned}$$

Bibliography

[1] Oliver, D. *The Shaggy Steed of Physics.* Second edition. Springer, Meadows of Dan, 2003.

[2] Martin, B.R. and Shaw, G. *Particle Physics.* Second edition. John Wiley & Sons Ltd., West Sussex, 1997.

[3] *http://www.claymath.org/millennium/Yang-Mills_ Theory*

[4] Faddeev, L.D. and Niemi, A.J. *Spin-charge separation, conformal covariance and the SU(2) Yang-Mills Theory.* arxiv:hep-th/0608111

[5] Peskin, M. and Schroeder, D. *An Introduction to Quantum Field Theory.* Westview Press, 1995.

[6] Zee, A. *Quantum Field Theory in a nutshell.* Princeton University Press, Princeton, 2003.

[7] Weinberg, S. *The quantum theory of fields I, II.* Cambridge University Press, Cambridge, 1996.

[8] Ramond, P. *Field Theory – a modern primer.* Westview Press, 1990.

[9] Ryder, L.H. *Field Theory.* Cambridge University Press, Cambridge, 1985.

[10] Gross, F. *Relativistic Quantum Mechanics and Field Theory.* John Wiley & Sons Inc., 1993

[11] Coleman, S. *Aspects of Symmetry.* Cambridge University Press, Cambridge, 1985.

[12] Brezin, E. and Hikami, S. and Zinn-Justin, J. *Generalized non-linear σ models with gauge invariance.* Nuclear Physics B165 (1980) 528 - 544. Hikami, S. *Non-linear σ model of Grassmann manifold and non-abelian gauge field with scalar coupling.* Progress of Theoretical Physics Vol. 64 4 (1980).

[13] Zakrzewski, W.J. *Low dimensional sigma models.* Institute Of Physics Publishing Ltd., Bristol, 1989.

[14] Hodge, W.V.D. and Pedoe, D. *Methods of Algebraic Geometry I* Cambridge University Press, Cambridge, 1953.

[15] Nakahara, M. *Geometry, Topology and Physics.* Second Edition. Institute Of Physics Publishing Ltd., 2003.

[16] Eguchi and T. Hanson and A.J. Gilkey, P.B. *Gravitation, Gauge Theories and Differential Geometry.* Physics Reports 66 6 (1980) 213 - 293.

[17] Griffiths, P. and Harris, J. *Principles of Algebraic Geometry.* John Wiley & Sons Inc., 1978.

[18] Chipalkatti, J.V. *Notes on Grassmannians and Schubert Varieties.* Queens Papers in Pure and Applied Mathematics, 2000.

[19] Rindler, W. *Introduction to Special Relativity.* Second Edition. Oxford University Press, Oxford, 1991.

[20] Chernodub, M.N. and Polikarov, M.I. in *Confinement, duality and non-perturbative aspects of QCD.* P. van Baal, Ed., Plenum Press, New York 1998.

[21] Niemi, A.J. *Could spin-charge separation be the source of confinement?.* AIP Conf.Proc.806: 114 -123, 2006. arxiv:hep-ph/0510288.

[22] Faddeev, L., *Quantization of Solitons*, preprint IAS Print-75-QS70, 1975; *Einstein and several contemporary tendencies in the field theory of elementary particles.* Relativity, Quanta and cosmology, vol.1 , 1979.

[23] Faddeev, L.D. and Niemi, A.J. *Knots and Particles.* Nature 387 (1997) 58

[24] Battye, R. and Sutcliffe P., *Solitons, Links and Knots*, Proc.Roy.Soc. London A455 (1999); *To be or knot to be?*, Phys.Rev.Lett. 81 (1998) 4798; Hietarinta, J and Salo, P., *Faddeev-Hopf knots: Dynamics of Linked Un-Knots*, Phys.Lett B451 (1999) 60.

[25] Lim, F and Yang, Y., *Existence of energy minimizers as stable knotted solitons in the Faddeev model*, Comm. Math. Phys. 249 (2004) 273.

[26] Sankaran, P. *Non-existence of almost complex structures on Grassmann manifolds.* Proceedings of the American Mathematical Society, 113 1 (1991).

[27] Glushakov, A.N. *Two Questions in the exterior geometry of Plücker embeddings for Grassmanian manifolds.* Zap.Nauchn. Semin. 246 (1997) 5 and Kozlov, S.E. *Geometry of real Grassmannian Manifolds. Part I, II.* Zap. Nauchn.Semin. 246 (1997) 108.

[28] Andersson, P.W., *The resonating valence bond state in La_2CuO_4 and superconductivity*, Science 235 (1987) 1196 and Faddeev, L.D. and Takhtajan, L., *What is the spin of a spin wave?*, Phys.Lett. A85 (1981) 375 and Lee, P.A., Nagosa, N. and Wen, X.-G., *Dopin Mott insulator: Physics of High Temperature Superconductivity*, arxiv:cond-mat/0410445.