The black hole information paradox and the trace anomaly

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Abstract

We review the arguments leading to the black hole information paradox and hawking radiation. We then consider the trace anomaly of the renormalized stress tensor and its relation to hawking radiation. Finally we study how the inclusion of an effective action corresponding to the trace anomaly changes the qualitative properties of the 2D dilaton gravity CGHS-model, and in particular properties related to the information paradox. We conclude that in this particular model the standard argument for information loss cannot be made since any evaporation black hole background spacetime with a non-singular endstate is necessarily non-globally hyperbolic.

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1 Introduction

Hawkings argument that unitary evolution of quantum states is incompatible with the existence of black holes started a long debate about how to reconcile the ideas of quantum mechanics and general relativity, that touches upon the fundamental aspects of our theoretical constructions. Within the framework of einsteins general relativity the area theorem for black holes states that the area of the event horizon cannot decrease if the stress tensor for the matterfields satisfy the null energy condition. Hawking[7][9] argued that state of the quantum field corresponding to vacuum on $\mathscr{I}^-$ should be that of outgoing thermal radiation flux on $\mathscr{I}^+$, and conjectured that this radiation would cause the black hole to evaporate and disappear, thus violating the null energy condition. The hawking flux was later related to the trace anomaly of the renormalized stress tensor by Christensen and Fulling[18]. This motivated several classical dilaton-gravity models where an effective action corresponding to the trace anomaly was incorporated with the anticipation of inducing evaporation of black holes. Two such models are the RST[22][23][24] and BPP[25] models which are modifications of the CGHS-model. In these models the mechanism for evaporation is connected to the presence of boundaries of the spacetime, however its existence and detailed behavior depends on the specific choice of effective action for the trace anomaly.

In section 2,3 and 4 we introduce some basic concepts and notation relating to the description of spacetime, and in particular black holes. In section 5 we introduce quantum field theory in curved spacetime and review the derivation of hawking radiation as well as the information loss problem in evaporation spacetimes. Section 6 introduces weyl rescalings and the conformal gauge. Section 7 is about renormalization of the stress tensor and in particular the trace anomaly, its relation to hawking radiation and how we can use it to induce backreaction through an effective action approach. Finally in section 8 we review and analyse two modifications of the CGHS dilaton-gravity model, the RST and BPP models, where backreaction has been induced, and discuss how this affects the information loss problem. Section 9 is a short discussion of the main results.

2 The observer and the theory

A central object in a theory is the observer. However while the observer is an object in a theory it is also the entity using the theory. In the theory the observer or possibly the multiple observers could either be considered as a class of objects distinct from observed systems, or to be of the same class. Let us then consider two general principles that must be taken into consideration when one evaluates theoretical constructions.

Principle of ignorance Every observer interprets its measurement results by a theory. Using the theory the observer can deduce propositions about events from the measurement.The assessment of objective reality of such an event can not be made independently of the theory and is thus subjective.

Self reference problem An observer can not construct a description of a system including the observer itself with unlimited precision since such a description would have to include itself.

The logic of the self reference problem implies that any theory including the observer itself as a part of the state will be limited to descriptions of macroscopic parameters. The principle of ignorance introduces a distinction between operational events, that is events that directly relates to operational procedures, and non-operational, or *gauge*, events, which exist only as deductions from the theory.

2.1 What's in a point

Let us now consider how we may describe the observer. In a classical theory such as general relativity where one has a notion of locality and the observer is very small compared to the system, the observer and its measurement equipment is often treated as a point. If on the other hand the system is small compared to the observer and the measurement equipment the observer may be left out of the description altogether and only a structure corresponding to the measurement equipment may be present in the theoretical description. We shall call a structure relative to which one can measure properties of the system a *background structure*.

The measurement is a process in which a set of numbers are assigned to the experiment. If we describe the observer as a point it may be natural to associate these numbers to the point where the observer is located. In the second case when the measurement device is not treated as a point assigning the measurement result to a point is not necessarily possible. For this to be possible we must assume that we have a background structure that enables localisation of the measured dynamical quantities.

Now it is once again important to stress the the difference between operationally defined properties and gauge properties. While a background structure may enable an operational definition of locality, and thus also a operational definition of non-locality, theories formulated in terms of gauge events may also be local or non-local but in this case it has no physical significance.

3 Spacetime

In this section we introduce the concepts and notation relating to spacetime which are used in the theoretical constructions we shall consider in later sections. The spacetime is a concept usually used to denote the set of events. As before we can make the distinction between operationally defined events and gauge-events. In special relativity, SR, events have a operational interpretation in terms of being defined by the readings of clocks and positions of rods. As a contrast the events of general relativity are all gauge as is illustrated by the *hole argument*. In quantum theory only the preparation and measurement events are operationally defined.

In classical mechanics it is assumed that position and time can be continuously specified and that physical properties can be assigned to points. Further, the idea of distances between events leads to the introduction of metric, and this motivated the description of spacetime by a manifold M. Connected to this description of spacetime is the concept of fields, used to model continuous matter distributions. A field theory is a construction where a system is modeled as a spacetime manifold M and a collection of functions $F : M \to N$.

In special relativity every event is assumed to be in one-to-one correspondence with points in $\mathbb{R}^4$, that is the spacetime is described by a manifold M_S which is isomorphic to $\mathbb{R}^4$. The realization that there is no way to operationally establish an absolute notion of simultaneity, and the adoption of the einstein synchronization convention, which sets light speed to a constant, leads to the introduction of a metric η_{ab} that is invariant under poincare transformations. Thus the symmetry group of the spacetime is the poincare group. If we also require that spatial orientation and time orientation are physically meaningful, the symmetry group of the spacetime is the orthocronous lorentz group. Matterfields are described by tensor fields on M_S.

In general relativity the spacetime is described as a general 4-dimensional hausdorff C^∞ manifold M_G, that is a set that has the local differentiable structure of R^4 but not necessarily the same topology. Also in GR matterfields are described by tensor fields on M_G.

The assumption that spacetime can be described as a 4-dimensional manifold is only motivated by compatibility with the idea from classical mechanics that position can be specified continuously and that locally spacetime was described as $\mathbb{R}^4$. Also in quantum mechanics and conventional quantum field theories the spacetime is described as a manifold. Accepting the description of spacetime by a manifold, the choice of topology is yet another not motivated by physical considerations.

As already said above a central issue for spacetime theories is the concept of localisation of objects, that is the association of an object with an event. While the association of an object with a operational event has a physical meaning the association of an object with a gauge event has none. In the context of a quantum theory where only preparation and measurement events are operationally defined the rationale for describing the spacetime as a manifold is that it is a classical background structure.

3.1 Diffeomorphism invariance and background fields

In a spacetime M described as a manifold with metric g_{ab} and matterfields Φ where all events are gauge, the value of g_{ab} and Φ at a point of M has no physical meaning. However if one stipulates that the relations between g_{ab} and Φ are meaningful, even though their values in a specific point is not, two field configurations on M related by an active diffeomorphism represent the same physical state. Therefore the symmetry group of the spacetime in this case is the active diffeomorphism group $Diff(M)$ which acts on M transitively and freely. The action of $Diff(M)$ on g_{ab} and matterfields Φ is not generally transitive or free. In other words for any active diffeomorphism $\Lambda : M \to M$ the solutions $(M, g_{ab}, \Phi) \simeq (\Lambda M, \Lambda_* g_{ab}, \Lambda\Phi)$ and the physical states are the equivalence classes $[M, g_{ab}, \Phi] \equiv \{(M', g'_{ab}, \Phi') \mid (M', g'_{ab}, \Phi') \simeq M, g_{ab}, \Phi\}$

A diffeomorphism Λ such that $\Lambda_* g_{ab} = g_{ab}$ is called an *isomorphism* and those such that $(\Lambda^* g)_{ab} = \Omega^2 g_{ab}$ are called *conformal transformations*. The set of isometries on M forms a group $Isom(M)$ and the conformal transformations forms a group $Conf(M)$.

When the spacetime is described by a manifold and the dynamical entities are tensor fields one way to introduce operationally defined events is to introduce a background tensor field F, which is non-dynamically interacting, as a reference frame. Then events can be given an operational definition by their relation to F. The symmetry group of the spacetime is now reduced to the subgroup of $Diff(M)$ which preserves F.

In GR it is assumed that no non-dynamically interacting entities exist and as a consequence the symmetry group of the spacetime M_G is $Diff(M_G)$. In SR on the other hand the metric η_{ab} is a background tensor field, thought of as realized operationally by the rods and clocks, and the symmetry group is the subgroup of $Diff(M_S)$ that preserves η_{ab}, that is the poincare group.

We shall call spacetimes with reduced symmetries associated to background tensor fields *relational spacetimes*. Further we can consider spacetimes with both gauge events and operationally defined events constructed by having a background field F covering all of M except a subset $H \subset M$. For example one could have a metric g_{ab} which is a background field on $\overline{H}$ and a dynamical field on H. The symmetry group of such a spacetime would be the subgroup of $P(M) \subset Diff(M)$ that acts as the isometry group of g_{ab} on $\overline{H}$ and acts as subgroup of $Diff(H)$ that preserves F on $\partial\overline{H}$ on H. The physical states are thus the equivalence classes of field configurations under the action of $P(M)$.

Active diffeomorphisms preserve tensor relations in the sense that if $T_1 = T_2$ at $p \in M$ and $\Lambda : M \to M$ is a diffeomorphism then $\Lambda^* T_1 = \Lambda^* T_2$ at $\Lambda(p)$.

We may however adopt an alternative view of diffeomorphisms. Consider the diffeomorphism Λ mapping tensors at the point p to tensors at the point $\Lambda(p)$. We then introduce a coordinate system x^μ in a neighborhood U of p and a coordinate system y^μ in a neighborhood V of $\Lambda(p)$. Now we may use Λ to define a new coordinate $\tilde{x^\mu}$ system in the neighborhood $\phi^{-1}(V)$ of p by setting $\tilde{x^\mu}(q) = y^\mu(\Lambda(q))$ for $q \in \phi^{-1}(V)$. We thus associate the points related by the diffeomorphism with the same coordinates. The *passive* view of diffeomorphims is that they leave p and the tensors at p unchanged but change the coordinates. The components of the tensor $\Lambda^* T$ at $\Lambda(p)$ in the y^μ coordinate system in the active view are the same as the components of T at p in the $\tilde{x^\mu}$ coordinate system in the passive view. The passive view is however drastically different as long as coordinates do not have physical significance, since in this case all sensible theories should be invariant under coordinate transformations. If one however adopt the view that events should be associated to coordinates rather than points, the passive view appears to be identical to the active.

In any theory where objects are defined on a manifold M and the theory is assumed to be invariant under diffeomorphisms no object that is coordinate dependent, or point dependent, can be an observable. Hence observables can not be associated to a point on the M but must be associated to M itself. Thus observables in a diffeomorphism invariant theory are in this sense global. However if we consider theories where the measurement devises are included observables can always be defined in terms of quantities relative to the measurement device.

3.2 Causality conditions and orientability

So far we have not considered any restrictions on the properties of the metrics we consider. Causality however introduces a kind of restriction on the properties of the metric and certain assumptions related to causality implies that the *causal structure* is the property characterizing a physical state of the metric. The notion of causality introduces a relation $a < b$ between two events corresponding to a caused b. The metric however may more properly be thought of as describing the possibility of a to cause b or the *causal connectibility*, denoted $a \lesssim b$.

There are several causality conditions that are more or less restrictive on the dynamics of the theory. A metric is said to satisfy the *chronology condition* if there are no closed non-spacelike curves.*Future (past) distinguishability*If for all$p \in M$ and all sufficiently small open sets O containing p, no future (past) directed timelike curve that begins at p and leaves O ever returns to O. *Strong causality*If for all$p \in M$ and all sufficiently small open sets O containing p, no future directed timelike curve that begins in O and leaves O ever returns to O.

Strong causality thus implies both future and past distinguishability and each of these conditions imply chronology. However the converse is not true. Further we say that (M, g_{ab}) satisfies the *stable causality condition* if and only if there is a global function $T : M \to \mathbb{R}$ such that $\nabla T \equiv t^a$ is everywhere timelike. Stable causality implies strong causality and is thus the strongest requirement. Stable causality also implies that we can continuously divide all non-spacelike vectors k^a into two classes, based on whether the satisfy $g_{ab}k^a t^a \leq 0$ or $g_{ab}k^a t^a \geq 0$ which we label past- and future-directed vectors respectively. We denote the sets of all future and past directed vectors on M $\theta^+(M)$ and $\theta^-(M)$, respectively. A spacetime where this division is possible is called *time orientable*. Similarly a spacetime is said to be *spaceorientble* if it is possible to continuously divide bases consisting of three space-like vectors into right-handed and left handed.

Now let us consider two time oriented spacetimes (M, g_{ab}) and $(\tilde{M}, \tilde{g}_{ab})$. We then say that a bijection $\Phi : M \to \tilde{M}$ is a *causal isomorphism* iff for all $a, b \in M$

$$a \lesssim b \iff \Phi(a) \lesssim \Phi(b) \tag{1}$$

It now turns out that if (M, g_{ab}) and $(\tilde{M}, \tilde{g_{ab}})$ are time-orientable and also past and future distinguishing, then Φ is a diffeomorphism and $\Phi_* g_{ab}$ is conformally equivalent to $\tilde{g}_{ab}$[1], that is Φ is a *conformal transformation*. Further for a time orientable (M, g_{ab}) every diffeomorphism $\Phi : M \rightarrow M$ such that that $\Phi_*(\theta^{\pm}(M)) \subset \theta^{\pm}(M)$ and $(\Phi^{-1})_*(\theta^{\pm}(M)) \subset \theta^{\pm}(M)$ is a conformal transformation and conversely every conformal transformation C is such that $C_*(\theta^{\pm}(M)) \subset \theta^{\pm}(M)$ and $(C^{-1})_*(\theta^{\pm}(M)) \subset \theta^{\pm}(M)$[2].

Thus if we only allow (M, g_{ab}) that are time-orientable the group of diffeomorphisms that preserve the time orientation is identical to $Conf(M)$ and hence the equivalence classes under conformal transformations are characterized by their causal structure. A class of conformally equivalent metrics on M is called a *conformal structure*.

3.3 Causal structure

Let us consider the causal structure of spacetimes a little further. For a time orientable (M, g_{ab}) we define a *future directed timelike curve* λ to be a curve such that for every $p \in \lambda$ the tangent is a future directed timelike vector. Similarly a curve λ is *causal* if for every $p \in \lambda$ the tangent is a future directed non-spacelike vector. Using these definitions we can define the *chronological future* $I^+(S)$ and *causal future* $J^+(S)$of a set S as

$$I^+(S) \equiv q \in M : \exists \text{ future dir. timelike curve } \lambda(t),\ \lambda(0) = p, \lambda(1) = q \text{ for a } p \in S \quad (2)$$

$$J^+(S) \equiv q \in M : \exists \text{ future dir. non-spacelike curve } \lambda(t),\ \lambda(0) = p, \lambda(1) = q \text{ for a } p \in S \quad (3)$$

A set S is said to be *achronal* if there does not exist $p, q \in S$ such that $q \in I^+(p)$ For a closed achronal set S we can define the *past domain of dependence* $D^-(S)$ and *future domain of dependence* $D^+(S)$ as

$$D^{\pm}(S) \equiv \{p \in M : \text{every past/future inext. causal curve through } p \text{ intersects } S\} \quad (4)$$

The *domain of dependence* $D(S)$ is then defined as $D(S) = D^+(S) \cup D^-(S)$. A closed achronal set Σ for which $D(\Sigma) = M$ is called a *cauchy surface* and a spacetime with a cauchy surface is said to be *globally hyperbolic*.

A globally hyperbolic spacetime is stably causal and hence time orientable. A 4-dimensional globally hyperbolic spacetime M^4 is thus isomorphic to $\mathrm{R} \times M^3$ where M^3 is a 3-dimensional spacetime with with euclidean metric.

3.4 Foliation of spacetime

As mentioned above stable causality allows the foliation of spacetime of spacelike hypersurfaces defined by $T = constant$. We now construct such a foliation more explicitly. To do this we choose a timelike vector field t^a and a time-function t such that $t^a \nabla_a t = 1$. For each t we also introduce an embedding $T_t : \tilde{\Sigma} \rightarrow M$

of a manifold $\tilde{\Sigma}$ which is topologically R^3 into M, such that the $t = constant$ hypersurfaces are diffeomorphic to the embedded surfaces Σ_t. Thus the choice of t defines a one-parameter family of embeddings, a *foliation*. We can decompose t^a as

$$t^a = Nn^a + N^a \tag{5}$$

where n^a is the unit normal to Σ_t and N^a is tangential to Σ_t. $N = -g_{ab}t^a n^b$ is called the lapse function and $N^a = \gamma^a_b t^b$, where $\gamma_{ab} = g_{ab} + n^a n^b$ is the induced metric on Σ_t, the shift vector. With this notation the metric g_{ab} thus decomposes as

$$g_{ab} = \gamma_{ab} - n_a n_b \tag{6}$$

We can introduce an adapted chart $(\bar{x}, t)$ for M such that $\bar{x}$ is constant on each integral curve of the vector field. Generally the induced metric will be time dependent.

3.5 Isometries

Let us now consider the isometry group $Isom(M)$ of (M, g_{ab}). If $Isom(M)$ contains a one parameter subgroup $\Lambda_t : \mathbb{R} \times M \to M$, we can associate to it a vector field. The action of Λ_t on a point p defines a curve on M and is called the orbit of Λ_t and the collection of orbits define a congruence on M. The tangent vectors of this congruence defines a vector field ξ which can be thought of as the generators of this isometry and is called a *Killing field*. A nessecary and sufficient condition for a field ξ to be Killing is

$$\nabla_a \xi_b + \nabla_b \xi_a = 0 \tag{7}$$

Since Λ_t is an isometry group we have for the associated Lie derivative of the metric $\mathcal{L}_\xi g_{ab} = 0$. The existence of a killing field implies the existence of a conserved current $J^a_K \equiv K^{ab}\xi_b$ associated to a conserved symmetric tensor K_{ab} since

$$\nabla_a J^a_K = K^{ab}\nabla_a \xi_b + \nabla_a K^{ab}\xi_b = 0 \tag{8}$$

We can use the existence of killing fields to classify a (M, g_{ab}) or equivalently the diffeomorphism equivalence class of which (M, g_{ab}) is a representative. A $(M,_{ab})$ is said to be *stationary* if there is a one parameter group of isometries Λ_t whose corresponding Killing field is timelike. If there additionally exists a spacelike hypersurface Σ which is orthogonal to the orbits of Λ_t ,$(M,_{ab})$ is said to be *static*. Further if $Isom(M)$ contains a subgroup isomorphic to $SO(3)$ and the orbits of this subgroup are two-spheres.

3.6 Asymptotically flat spacetime

In many applications, such as the description of black holes below, one wishes to refer to properties of an infinite spacetime M "at infinity". One method commonly used is to map M onto an open subset of a another spacetime $\tilde{M}$

called the *unphysical spacetime* and study the properties of this instead. Let us now go through a particular example[3] of this construction that will be of use later on.

A (M, g_{ab}) is called *asymptotically flat at null and spatial infinity* if there exists a $(\tilde{M}, \tilde{g}_{ab})$ with $\tilde{g}_{ab}$ C^{∞} everywhere except possibly at a point i^0 where it is $C^{>0}$, and a conformal transformation $\Lambda : M \to \Lambda(M) \subset \tilde{M}$ such that $\tilde{g}_{ab} = \Omega^2 \Lambda^* g_{ab}$, that satisfies the following conditions

(1) $\overline{J^+(i^0)} \cup \overline{J^-(i^0)} = \tilde{M} - \Lambda(M)$ Hence i^0 is spacelike related to all points of $\Lambda(M)$ and $\partial M = \mathcal{I}^+ \cup \mathcal{I}^- \cup i^0$ where $\mathcal{I}^+ \equiv \partial J^+(i^0) - i^0$,$\mathcal{I}^- \equiv \partial J^-(i^0) - i^0$

(2) There is an open neighborhood U of ∂M such that (U, g_{ab}) is strongly causal.

(3) Ω can be extended as a function to all of $\tilde{M}$ that is C^2 at i^0 and C^{∞} elsewhere.

(4) On $\mathcal{I}^+$ and $\mathcal{I}^-$, $\Omega = 0$ and $\tilde{\nabla}_a \Omega \neq 0$. Further $\Omega(i^0) = 0$, $lim_{i^0} \tilde{\nabla}_a \Omega = 0$ and $lim_{i^0} \tilde{\nabla}_a \tilde{\nabla}_b \Omega = 2\tilde{g}_{ab}(i^0)$

(5) The map of null directions at i^0 into the space of integral curves of $n^a \equiv \tilde{g}_{ab} \tilde{\nabla}_a \Omega$ on $\mathcal{I}^+$ and $\mathcal{I}^-$ is a diffeomorphism. For any smooth function f on $\tilde{M} - i^0$ such that $f > 0$ on $M \cup \mathcal{I}^+ \cup \mathcal{I}^-$ and $\tilde{\nabla}_a(f^4 n^a) = 0$ on $\mathcal{I}^+ \cup \mathcal{I}^-$ the vector field $f^{-1} n^a$ is *complete* on $\mathcal{I}^+ \cup \mathcal{I}^-$, that is it generates a flow with domain $\mathbb{R} \times \mathcal{I}^+ \cup \mathcal{I}^-$.

3.7 Geodesics and Synges world function

Before proceeding let us consider *geodesics*. A curve λ is said to be a geodesic if there exists a parameterisation of λ such that the tangent t^a satisfies $t^a \nabla_a t^b = 0$. Such a parameterization is called *affine*. Further we find that $t^a \nabla_a (g_{cb} t^c t^b) = 0$ which implies that $g_{cb} t^c t^b$ is a constant along the geodesic. Geodesics are not generally mapped to themselves by diffeomorphisms but the image of a geodesic is still a geodesic. A diffeomorphism that do preserve geodesics is called a *geodesic map*, and the geodesic maps is thus a subset of the conformal transformations. If a geodesic is inextendible in at least one direction but have only a finite range of affine parameter we say that it is incomplete. The *normal convex neighborhood* of a point x is the set of points y connected to x by a unique geodesic. In the normal convex neighborhood of x we can define *Synges world function* $\sigma(x, y)$ by

$$\sigma(x, y) = \frac{1}{2}(\lambda_1 - \lambda_0) \int_{\lambda_0}^{\lambda_1} g_{ab} t^a t^b d\lambda \tag{9}$$

where λ_0 and λ_1 are the affine parameters of the unique geodesic at x and y respectively and t^a is the tangent of the geodesic. We note that since $g_{ab} t^a t^b = \epsilon$, where ϵ is a constant, along the geodesic $\sigma(x, y) = \frac{\epsilon}{2}(\lambda_1 - \lambda_0)^2$. Naturally since geodesics are not preserved by active diffeomorphisms neither is Synges world function.

3.8 Singularities and geodesic incompleteness

A *singularity* is not a precisely defined concept but it is generally used to denote something that is considered pathological, such as an unbounded curvature tensor.

A common notion of what constitutes a singular spacetime with metric is a (M, g_{ab}) which is timelike or null geodesically incomplete. For time oriented spacetimes we may naturally divide the spacetimes in categories based on whether there are geodesics that are future, past or future and past incomplete. We now adopt a language where we call singularities such that past directed causal geodesics are incomplete, singularities with future. Similarly singularities such that future directed geodesics are incomplete are called singularities with past. We could imagine three principal situations, singularities with past but no future, future but no past and singularities with both past and future.A singularity with future will be referred to as *naked*.

A singular spacetime is globally hyperbolic only if there exists a closed achronal set Π such that all singularities with future but no past are contained in $J^-(\Pi)$ and all singularities with past and no future are contained in $J^+(\Pi)$. Thus in particular the appearance of singularities with both past and future will always render the spacetime non-hyperbolic.

In the non-hyperbolic case while conditions on a single spacelike hypersurface is not enough to determine the entire history of the spacetime conditions on a set of surfaces such that together they intersect all inextendible causal curves is, provided it exists. The specification of conditions on a spacelike hypersurface Π such that $\Upsilon \subset J^-(\Pi)$ for a singularity Υ is equivalent to the specification of boundary conditions on Υ.

Suppose we have a spacelike hypersurface Γ, which no non-spacelike curve intersects more than once, but not necessarily a cauchy surface, then we say that spacetime is *asymptotically predictable* from Γ if $\mathscr{I}^+ \subset D(\Gamma)$. An asymptotically flat spacetime which fails to be asymptotically predictable from a particular spacelike hypersurface Γ must possess a naked singularity not contained in $J^-(\Gamma)$.

3.9 Boundaries and partially relational spacetimes

Let us now consider spacetimes with boundaries and partially relational spacetimes and especially the causal structure. A spacetime with a partially timelike boundary is not globally hyperbolic since causal curves can begin and end at the boundary without passing a specific spacelike hypersurface. In the case of a spacetime with background metric one may only consider fields of compact support and thus a globally hyperbolic region of the spacetime. When we consider a spacetime M where the metric is a dynamic field this is obviously not possible and the boundary ∂M must be taken into account. However one might conceive such a spacetime as embedded in a spacetime $\tilde{M}$ with background metric. If the metric is supposed to be smooth this imposes as mentioned above boundary conditions on M reducing the symmetry group.

3.10 Dynamics of the metric and matter fields

As discussed previously active diffeomorphisms preserve tensor relations. Therefore it is natural to formulate relations between dynamical quantities in terms of tensor equations. The dynamical equations of general relativity, the einstein equations are

$$R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = T_{ab} \tag{10}$$

In the lagrangian formulation of general relativity on a boundaryless manifold M this equation follows from the variation of an action $S = S_{g_{ab}} + S_M$ where $S_{g_{ab}} = \int_M \sqrt{-g} k(R - 2\Lambda)$ and S_M is the matter action. More specifically we say that the dynamics of the system is give by the condition $\delta S = 0$. We let Ψ^a be the collection of tensor fields where a denotes all tensor indices, that is in this case g^{ab} and all matterfields. Then if there exist a C^∞ tensor field E_a such that for all $\delta\Phi^{ab}$ at Φ_0^{ab}

$$\delta S[\Phi^a] = \int_M E_a \delta \Phi^a \tag{11}$$

we say that S is functionally differentiable at Ψ_0^a and E_a is the functional derivative of S which we denote as

$$E_a \equiv \frac{\delta S}{\delta \Psi^a} |_{\Psi_0^a} \tag{12}$$

We then say that the local equations of motion are given by $E_a = 0$. However when M is a manifold with a piecewise smooth boundary this action does not produce Einsteins equations since there are boundary terms. Variation of the the action in this case gives

$$\begin{aligned} \delta(S_{g_{ab}} + S_M)[g^{ab}] &= \int_M \sqrt{-g}(R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} - T_{ab}) \delta g^{ab} + \\ &\quad + \int_M \sqrt{-g} \nabla_a (g_{cb} \nabla^a \delta g^{cb} - \nabla_b \delta g^{ab}) \\ &= \int_M \sqrt{-g}(R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} - T_{ab}) \delta g^{ab} + \\ &\quad + \int_{\partial M} \sqrt{-g}(g_{cb} \nabla^a \delta g^{cb} - \nabla_b \delta g^{ab}) n_a \end{aligned} \tag{13}$$

thus in this case there is no functional derivative and thus we cannot define local equations of motion as above. If we insist on having local equations of motion we can construct an action producing einsteins equation by adding a term to cancel the surface term

$$S_H^{mod} = \int_M \sqrt{-g} R + \int_{\partial M} K \tag{14}$$

where $K = 2g^{ac}(g_{cb} - n_c n_b)\nabla_a n^b$.

The above action describing the dynamics of the metric and matterfields is only a special case that one can construct. It was shown by Iyer,Wald[4] that the most general diffeomorphism invariant lagrangian is of the form

$$\mathbf{L} = \mathbf{L}(g_{ab}, R_{bcde}, \nabla_a R_{bcde}, \ldots, \nabla_{(a_1} \ldots \nabla_{a_m)} R_{bcde}, \Psi, \nabla_a \Psi, \ldots, \nabla_{(a_1} \ldots \nabla_{a_m)} \Psi) \tag{15}$$

where Ψ again is a collection of tensor fields.

All the above formulations of dynamics assume the field theoretical description and are thus limited to spacetimes described by manifolds.When we consider partially relational spacetimes where the metric is a background field outside a subset $H \subset M$ and a dynamical field on H, we must use different forms of the action for the different regions.

3.11 Matter conditions

So far we have not restricted the dynamics of the metric in any sense. A number of restrictions can be put on the matterfields which then constrict the possible metrics. We first consider the conditions of *local causality* and *local energy momentum conservation*

Local causality The matterfield equations must be such that if U is a neighborhood of points p and q, then a signal can be sent between p and q if and only if p and q can be joined by a C^1 curve whose tangent is everywhere non-zero and non-spacelike.

Local energy momentum conservation The matterfield equations must be such that there exists a symmetric tensor T_{ab} depending on the fields, there covariant derivatives and the metric, and satisfies
I T_{ab} vanishes on an open set U if and only if the matter fields vanish on U.
II T_{ab} satisfies the equation $\nabla^b T_{ab} = 0$

Next we consider a set of possible restrictions on the energy tensor that will be used later on.

Weak energy condition The weak energy condition states that for all timelike vectors t^a we must have $T_{ab}t^a t^b \geq 0$

Null energy condition The corresponding condition for null-vectors is the null energy condition where for all null vectors l^a we have $T_{ab}l^a l^b \geq 0$

Dominant energy condition The dominant energy condition is the weak energy condition, $T_{ab}t^a t^b \geq 0$ for all timelike t^a, together with the requirement that $T_{ab}t^a$ is a non-spacelike vector, that is $T_{cb}T^b_a t^a t^c \leq 0$.

Strong energy condition The strong energy condition is that $T_{ab}t^at^b \geq \frac{1}{2}T^c_c t^e t_e$ for all timelike vectors t^a.

As can be seen the dominant energy condition implies the weak energy condition, but except for this the conditions are independent.

4 Gravitational collapse

The central concern for this treatment is naturally the properties of spacetimes corresponding to gravitational collapse and the appearance of singularities. To investigate these we first have to consider some properties of geodesic congruences.

4.1 Geodesic congruences

Let O be an open subset of M. A congruence in O is a family of curves such that for each $p \in O$ there passes precisely one curve in this family. Thus the tangents to the congruence is a vector field in O. Let us consider a congruence of null-geodesics with corresponding null tangent vectors k^a. For a $p \in O$ the tangent vectors v^a in T_pM that satisfy $v_a k^a = 0$ defines a 3-dimensional vector space V_p. Further we identify vectors v_1^a and v_2^a if there is a $c \in \mathbb{R}$ such that $v_1^a - v_2^a = ck^a$. The equivalence classes defined by this identification forms a 2-dimensional vectorspace $\tilde{V}_p$. A tensor $T^{a_1 a_2 \ldots a_k}_{b_1 b_2 \ldots b_l}$ over T_pM gives rise to a tensor over $\tilde{V}_p$ iff its contraction with k^a or k_a over one of the indices and vectors and dual vectors that give rise to elements of $\tilde{V}_p$ and $\tilde{V}_p^*$ is always zero. The metric g_{ab} and $B_{ab} = \nabla_b k_a$ both satisfy this property and gives rise to tensors $\tilde{g}_{ab}$ and $\tilde{B}_{ab}$. We can decompose $\tilde{B}_{ab}$ as

$$\tilde{B}_{ab} = \frac{1}{2}\theta\tilde{g}_{ab} + \tilde{\sigma}_{ab} + \tilde{\omega}_{ab} \tag{16}$$

where θ, $\tilde{\sigma}_{ab}$ and $\tilde{\omega}_{ab}$ are, respectively, the expansion, shear and twist of the congruence.

4.1.1 Cosmic censorship

It has been conjectured in many ways that gravitational collapse should not produce naked singularities given some requirements on the properties of the Einstein equation and matterfields. These conjectures are referred to as cosmic censorship conjectures. The following version is due to Penrose, Geroch and Horowitz[3]

Cosmic censorship conjecture Let M be a spacetime and $(\Sigma, \gamma_{ab}, K_{ab})$ be the initial data for einsteins equation with (Σ, γ_{ab}) a complete Riemannian manifold and with an einstein-matter equation that is a quasilinear, diagonal, second order hyperbolic system, that is an equation on the form

$$g^{ab}(x,\Phi_j,\check{\nabla}_c\Phi_j)\check{\nabla}_a\check{\nabla}_b\Phi_i = F_i(x,\Phi_j,\check{\nabla}_c\Phi_j) \tag{17}$$

where $\check{\nabla}_a$ is any derivative operator, g_{ab} is a smooth lorentz metric $\{\Phi_i\}$ are functions on M and $\{F_i\}$ are smooth functions of its variables. Further let T_{ab} satisfy the dominant energy condition.
Then if the maximal cauchy development $D^+(\Sigma)$ is extendible, for each point on the cauchy horizon $p \in H^+(\Sigma)$ in any extension, either strong causality is violated at p or $\overline{I^-(p)} \cap \Sigma$ is noncompact.

4.2 Black holes

To define what we mean by a black hole we restrict ourselves to the case of an asymptotically flat spacetime. The Black hole $\mathscr{B}$ can then be defined as the region of the spacetime M which is not in the chronological past of future null infinity $\mathscr{I}^+$,or equivalently $\mathscr{B} = M - I^-(\mathscr{I}^+)$. The boundary h^+ of $\mathscr{B}$ is called the future event horizon, and is by its definition a null surface. The black hole region can thus only be defined when the entire history of the spacetime is known, so in particular the location of the event horizon can not be determined by a local observer. Black holes are considered as a possible result of gravitational collapse in general relativity.

A black hole is said to be predictable if for a spacelike surface Σ, $I^+(\Sigma) \cap I^-(\mathscr{I}^+) \subset D(\Sigma)$ and $h^+ \subset D(\Sigma)$. This simply means that we require everything to the future of Σ that is not the interior of the black hole to be predictable from Σ but do not require anything for the black hole interior. For a black hole spacetime that contains a singularity Υ with future such that $I^+(\Upsilon) \cap (I^+(\Sigma) \cap I^-(\mathscr{I}^+)) \neq \emptyset$ the black hole will not be predictable.

An important result for the dynamics of the event horizon was found by Hawking (1971)[5]

Area theorem Let (M, g_{ab}) be a strongly asymptotically predictable spacetime satisfying $R_{ab}l^al^b \geq 0$ for all null l^a.Also let Σ_1 and Σ_2 be spacelike cauchy-surfaces for the globally hyperbolic region such that $\Sigma_2 \subset I^+(\Sigma_1)$ and let $\mathfrak{h}_1 \equiv h^+ \cap \Sigma_1$,$\mathfrak{h}_2 \equiv h^+ \cap \Sigma_2$.Then the area of $\mathfrak{h}_2$ is greater than or equal to the area of $\mathfrak{h}_1$.

If einsteins equation is valid $R_{ab}l^al^b \geq 0$ implies that $8\pi T_{ab}l^al^b \geq 0$ that is the null energy condition. The theorem is thus invalidated if the null energy condition is not satisfied.

4.2.1 Trapped surfaces and the apparent horizon

A 2-dimensional compact smooth spacelike submanifold T is called a trapped surface if the expansion θ of both sets of future directed null-geodesics orthogonal to T are everywhere negative. If we instead of requiring negativity of θ only require non-positivity we say the region is marginally trapped. A 3-dimensional spacelike surface C is called a trapped region if ∂C is a marginally trapped surface. Let Σ be an asymptotically flat cauchy surface. We define the totally trapped region $\mathscr{T}$ of Σ to be the closure of the union of all trapped regions C on Σ. The boundary $\mathscr{A} \equiv \partial\mathscr{T}$ of $\mathscr{T}$ is called the *apparent* horizon on Σ. Further we call the region of all trapped surfaces the *apparent black hole*. In a strongly asymptotically predictable spacetime where $R_{ab}l^a l^b \geq 0$ for all null l^a we have that $\mathscr{T} \subset \mathscr{B} \cap \Sigma$, and thus the apparent black hole is contained in $\mathscr{B}$.

4.2.2 Stationary black holes and killing horizons

A black hole $\mathscr{B}$ in a asymptotically flat spacetime M is said to be stationary if there exist a one-parameter group of isometries on M generated by a killing field ξ^a which is unit timelike at infinity.Further the black hole is said to be static if ξ^a is hypersurface orthogonal, and axi-symmetric if there in addition exists a one-parameter family of isometries that corresponds to rotations at infinity generated by a killing field φ^a. An axi-symmetric black hole spacetime where the 2-surfaces spanned by ξ^a and φ^a are orthogonal to a family of 2-surfaces, is said to be $\xi - \varphi$ *orthogonal*. This property holds for all axisymmetric black hole vacuum solutions to the einstein equation.

A null-surface where a killing field ς^a is null is called a *killing horizon* $\mathfrak{H}_\varsigma$. it can be shown that the event horizon of a stationary spacetime is a killing horizon.

For a stationary black hole spactime which is $\xi - \varphi$ orthogonal there exist a killing field of the form

$$\chi = \xi + \Omega\varphi \tag{18}$$

which is normal to the event horizon. The constant Ω is called the angular velocity of the horizon. Let $\mathfrak{H}_\chi$ be a killing horizon with normal killing field χ^a. Since the vector field $\nabla^a(\chi^b\chi_b)$ is also normal to $\mathfrak{H}_\chi$, there must be a relation

$$\nabla^a(\chi^b\chi_b) = -2\kappa\chi^a \tag{19}$$

where κ is called the *surface gravity* of $\mathfrak{H}_\chi$.

4.3 Singularity theorem

So far we have only discussed the possible end states of gravitational collapse, and so it is time to consider under what assumptions gravitational collapse might result in singularities. As defined above a spacetime is said to be geodesically complete if every geodesic can be extended to arbitrary values of its affine parameter. Hawking and Penrose[5] showed that a spacetime with metric (M, g_{ab}) cannot be null and timelike geodesically complete if:

(1) $R_{ab}k^a k^b \geq 0$ for all non-spacelike vectors k^b

(2) Every non-spacelike geodesic with tangent k^a contains a point where $k_{[a}R_{b]cd[e}k_{f]}k^c k^d \neq 0$

(3) There are no closed timelike curves

(4) There exists at least one of the following I a compact achronal set without edge II a closed trapped surface III a point $p \in M$ such that the expansion θ of the future directed null geodesics emanating from p becomes negative.

If einsteins equation is valid we find that $R_{ab}k^a k^b = 8\pi(T_{ab} - \frac{1}{2}Tg_{ab})k^a k^b$ and hence $R_{ab}k^a k^b \geq 0$ implies $(T_{ab} - \frac{1}{2}Tg_{ab})k^a k^b \geq 0$. For timelike k^a this is thus equal to the strong energy condition, and for null k^a it equals the null energy condition.

5 Quantum field theory and spacetime

Quantum theory is fundamentally different from classical theories in that the observer and the act of observation enters the theoretical construction very explicitly. The standard Dirac-von Neumann quantum theory can be stated as follows

Hilbert space axiom The possible states $\mid \psi >$ of a system can be represented as unit vectors in a Hilbert space $\mathscr{H}$.

Observables An observable is represented by a hermitean operator O on $\mathscr{H}$, and the eigenvalues of the operator are the possible results of a measurement corresponding to this observable. The expectation value of an observable O in the state $\mid \psi >$ is given by $< \psi \mid O \mid \psi >$, where $< \psi \mid \psi >$ is the inner product in $\mathscr{H}$.

Determinate properties If the system S is in an eigenstate to the operator O corresponding to a particular eigenvalue, a subsequent measurement corresponding to the same operator will give the same result.

Dynamics The system has two different modes of evolution. When no measurement is made the system S evolves according to the linear deterministic equations of motion. Upon measurement of an observable with associated operator O the state of S is reduced to the eigenstate of O corresponding to the eigenvalue the measurement resulted in.

The observer has a special role in the Dirac-von Neumann quantum theory since it the observer that causes the reduction of the state. While there is no specification of what constitutes an observer, states are defined relative to the observer and different observers generally have different state spaces, as illustrated by the Schrödingers cat thought experiment. As a consequence it is not meaningful to use concepts such as absolute state of a system, absolute values of observables or absolute events.

5.1 Unitarity and probability preservation

As mentioned above quantum mechanics is about relating the preparation of system to the measurement on the same system. However the preparation procedure may be quite different from the measurement procedure which would motivate the description of prepared states and measured states by different hilbert spaces. Let us call the hilbert space of states that a system can be prepared in H_1 , and a hilbert space of states it can be measured in H_2. The dynamical evolution of the system will then be described by a map between H_1 and H_2. Such a map is called a *unitary transformation* $U : H_1 \to H_2$ if it is bijective and satisfies

$$UU^* = U^*U = I \tag{20}$$

This property holds if and only if U preserves the inner product structure $<,>$ in the sense that for all vectors $X, Y \in H_1$

$$< UX \mid UY >_{H_2} = < X \mid Y >_{H_1} \tag{21}$$

Thus a unitary transformation is an isomorphism. In the case when $H_1 = H_2$ and U is an automorphism it is also called a *unitary operator*. Similarly a antiunitary transformation $A : H_1 \to H_2$ is a bijective map such that

$$< UX \mid UY >_{H_2} = < X \mid Y >^*_{H_1} \tag{22}$$

To see the physical significance of such maps consider the following. Since the probability of finding a system in an eigenstate $X_O \in H_1$ to an observable O when the system is in state Y is given by $\mid < X_O \mid Y > \mid^2$, then the probability of finding a system prepared in the state X_O imidiately after preparation in the same state is $\mid < X_O \mid X_O > \mid^2 = 1$ that is the probability of the state X_O being prepared is one. Let then $Z_i \in H_2$ be the eigenstates of an observable $\tilde{O}$ and use this as the basis of H_2. Let UX_O be the state H_2 in the and the total probability of being measured in any state is $\sum_i \mid < UX_O \mid Z_i > \mid^2 = \sum_i < UX_O \mid Z_i >< Z_i \mid UX_O > = < UX_O \mid UX_O >$. Thus if the probability of the system being prepared is to equal the probability of being measured, the map must preserve the scalar product between identical states, and hence be either unitary or antiunitary.

5.2 Pure and mixed states

A pure quantum state is a state that can be described by a single vector in the hilbert space $\mathscr{H}$. A mixed quantum state is a classical statistical distribution of pure states. A density operator is a positive semidefinite hermitean operator of trace 1 operating on $\mathscr{H}$. A system is pure iff the density operator is idempotent, that is $\rho = \rho^2$.

Since different observers construct different state spaces and in general do not even study the same system, the notion of pureness of a state is also relative to the observer and can only be established by the preparation of the system by the observer.

5.3 Entropy of entanglement

The von Neumann entropy $S_N(\rho)$ of a quantum state is defined as

$$S_N = -tr(\rho ln \rho) \tag{23}$$

$S_N(\rho)$ is only zero for pure states. Then consider a system AB with a Hilbert space $\mathscr{H}_{AB}$ in a pure state with density matrix ρ. Suppose further that AB is divided into two sub systems A and B with hilbertspaces $\mathscr{H}_A$ and $\mathscr{H}_B$, and that $\mathscr{H}_{AB} = \mathscr{H}_A \otimes \mathscr{H}_B$. The density matrix of a subsystem is obtained by taking the partial trace of ρ over the other subsystem. Thus

$$\rho_A = tr_B \rho \tag{24}$$

$$\rho_B = tr_A \rho \tag{25}$$

and the entropies of the two subsystems are

$$S_A = -tr(\rho_A ln(\rho_A)) \tag{26}$$

$$S_B = -tr(\rho_B ln(\rho_B)) \tag{27}$$

5.4 Quantum theory in spacetime

When constructing a quantum theory one must decide what constitutes observables and states. Different choices will generally lead to different predictions even when attempting to describe the "same" physical situation. A particular class of quantum theories are the *general quantum field theories* which can be said to be quantum theories of systems with infinite number of degrees of freedom. When formulating a theory including a spacetime in terms of manifolds and tensor fields one must decide whether topology is fixed or not and in the case of fixed topology whether the metric should be a fixed background, partially background and partially dynamic, or fully dynamic.

In *conventional* quantum field theories the topology of the manifold M is fixed and the metric is a background field. Usually the topology is that of $\mathbb{R}^4$ and the background is the Minkowski metric η_{ab}. The dynamical entities are a set of fields defined on M and the states are the field configurations modulo the internal gauge groups of the fields. It is assumed that observables can be associated to points or alternatively to finite regions of spacetime and thus the observables are represented by operators that are either associated to spacetime points or are operator valued distributions defined on M. The application of this construction to general spacetimes is usually reffered to as *quantum field theory in curved spacetime* and will be treated below. In this picture, and if the operators corresponding to observables are supposed to correspond to operations performed by the observer it means the observer should have the ability to perform measurements at any point of the spacetime. Alternatively since the states are defined relative to the observer and since the they are also defined relative to the background metric, the background metric must be defined by the observer. This reasoning leads to complications when considering background metrics corresponding to black holes as we shall see below.

Models where the metric is a fixed background field can naturally not include gravitational interaction. Several constructions has been attempted where interaction between matterfields and metric is introduced while a background structure is still retained.

One attempt, the *covariant perturbation method* has been to split the metric as $g_{ab} = b_{ab} + h_{ab}$ where b_{ab} is a a fixed background field which determines a fixed causal structure and a dynamical field h_{ab}. The field h_{ab} is treated as an ordinary quantum field in the fixed background and thus this approach may be motivated when modeling weak gravitational effects where a background can be operationally established, such as weak gravitational waves in a detector.

Another attempt is the so called *semi-classical gravity.* Here spacetime is described as a manifold M with a classical background metric g_{ab} and quantum matter fields ψ. The background metric is then dynamically coupled to the quantum fields by the *semi-classical Einstein-equation*

$$R_{ab} - \frac{1}{2}g_{ab}R = 8\pi < \psi \mid T_{ab} \mid \psi > \tag{28}$$

However since different matter field states are supported on different background metrics, the superposition principle does not apply for the quantum states, and it is thus not a proper quantum theory.

Yet another attempt is to apply the *path integral method* to gravity. One then constructs a formal expression as

$$< \gamma_{ab}, \psi \mid \gamma \equiv \int_{\Sigma_1}^{\Sigma_2} \mathcal{D}[g_{ab}, \psi] e^{iS[g_{ab},\psi]} \tag{29}$$

In the spirit of the idea that only preparation and measurement events are operationally defined and all other events thus are gauge one might argue that one should sum over all possible metrics and matterfield configurations that agree on the operationally defined events. Furthermore one might argue that the sum should also be made over all possible topologies consistent with the same preparation and measurement.

5.5 Conventional quantum field theory in Minkowski and general spacetimes

Conventional quantum field theory does as mentioned above describe the spacetime M and metric g_{ab} as fixed background structures. Further it is a "quantization" of a classical field theory by application a quantization procedure such as canonical quantization or covariant quantization. Thus in particular it assumes a classical description of a system by a collection of fields $\Phi : M \to N$ where the dynamics is given by an action S expressible as $S = \int_M \mathcal{L}$. Both the canonical and covariant quantization procedures requires that M is stably causal so that there exist a foliation of M by spacelike hypersurfaces Σ_t and a corresponding timefunction t, but while the canonical procedure makes explicit use of the foliation by choosing a hamiltonian, the covariant procedure does not.

5.5.1 Covariant quantization of a scalar field

In a Minkowski background spacetime the equation of motion for a local and lorentz covariant scalar field Φ is the Klein-Gordon equation $(\eta^{ab}\nabla_a\nabla_b + m^2)\Phi = 0$. Such a field has a well posed initial value formulation for the initial data $(\phi, n^a\nabla_a\phi)$ on any smooth spacelike cauchy-surface Σ with unit normal n^a. If we seek an equation for a geometrically covariant local scalar field with well posed initial value formulation in a general globally hyperbolic background (M, g_{ab}) we must assume equations of motion of the form

$$\nabla^2\phi + A^a\nabla_a\phi + B\phi + C = 0 \tag{30}$$

where ∇_a is any derivative operator, A^a is any smooth vector field and B and C are arbitrary smooth scalar functions. Such an equation is called *hyperbolic*. A specific and popular choice is $\nabla^2\phi - (m^2 + \xi R)\phi = 0$ where ξ is a constant. The reason is that when $m = 0$ choosing $\xi = \frac{n-2}{4(n-1)}$ renders the equation of motion invariant under Weyl rescalings, and is referred to as conformal coupling. Setting $\xi = 0$ on the other hand is referred to as minimal coupling, but is equal to conformal coupling when $dim(M) = 2$.

To covariantly quantize the field we introduce the scalar product

$$(\phi_1, \phi_2) \equiv -i\int_\Sigma \sqrt{\gamma}(\phi_1\nabla_a\phi_2^* - \phi_2\nabla_a\phi_1^*)n^a \tag{31}$$

where Σ is a spacelike hypersurface and n^a is the future directed unit normal to Σ. Now let $\{u_i\}$ be a complete set of mode solutions to 30 ,where the index i denotes the set of quantities necessary to label the modes, which are orthonormal in the scalar product, that is satisfying

$$(u_i, u_j) = \delta_{ij} \qquad (u_i^*, u_j^*) = -\delta_{ij} \qquad (u_i, u_j^*) = 0 \tag{32}$$

The field ϕ may now be expressed as

$$\phi(x) = \sum_i (a_i u_i(x) + a_i^\dagger u_i^*(x)) \tag{33}$$

The covariant quantization then consists of giving a_i and $a_i^\dagger$ operator status and imposing the commutation relations

$$[a_i, a_j^\dagger] = \delta_{ij} \qquad [a_i^\dagger, a_j^\dagger] = 0 \qquad [a_i, a_j] = 0 \tag{34}$$

There is however a complete arbitrariness in the choice of complete set of mode solutions and hence in the operators. Let us therefore consider a different family $\{v_i\}$. Since both families are complete the modes v_i can be expanded in terms of the u_i

$$v_i = \sum_j (\alpha_{ij} u_j + \beta_{ij} u_j^*) \tag{35}$$

and conversely

$$u_j = \sum_j (\alpha_{ji}^* v_i + \beta_{ji}^* v_i^*) \tag{36}$$

These relations are called Bogolubov transformations and the matrices α_{ij} and β_{ij} are called Bogolubov coefficients. Since both families are complete it follows that

$$\sum_k (\alpha_{ik}\alpha_{jk}^* - \beta_{ik}\beta_{jk}^*) = \delta_{ij} \tag{37}$$

and

$$\sum_k (\alpha_{ik}\beta_{jk} - \beta_{ik}\alpha_{jk}) = 0 \tag{38}$$

The corresponding relations for the operators $a_i, a_i^\dagger$ corresponding to $\{u_i\}$ and $b_i, b_i^\dagger$ corresponding to $\{v_i\}$ are

$$a_i = \sum_j (\alpha_{ji} b_j + \beta_{ji}^* b_j^\dagger) \tag{39}$$

and

$$b_j = \sum_j (\alpha_{ji}^* a_i + \beta_{ji} a_j^\dagger) \tag{40}$$

To each complete family of modes there is a special state called the *vacuum state* being the state annihilated by all the corresponding annihilation operators. Thus for $\{u_i\}$ the vacuum state $\mid 0_u >$ is thus defined by $a_i \mid 0_u >= 0 \forall i$, and the vacuum state $\mid 0_v >$ corresponding to $\{v_i\}$ is defined by $b_i \mid 0_v >= 0 \forall i$. the different vacua will be different if $\beta_{ij} \neq o$ since

$$a_i \mid 0_v >= \sum_j \beta_{ji}^* b_j^\dagger \mid 0_v > \neq 0 \tag{41}$$

And thus the expectation value of the number operator $N_i = a_i^\dagger a_i$ in the state $\mid 0_v >$ is

$$< 0_v \mid N_i \mid 0_v >= \sum_j \mid \beta_{ji} \mid^2 \tag{42}$$

5.5.2 Canonical quantization

In the canonical approach one takes the starting point in an action formulation of the classical field theory and defines a hamiltonian description of dynamics, that is it defines a Schrödinger picture description. This makes explicit use of time and therefore requires a specific choice of foliation of M. Given a foliation of hypersurfaces Σ_t one defines a canonical momentum density $\pi(x,t)$ on each Σ_t by

$$\pi(x,t) = \frac{\delta\mathcal{L}}{\delta\dot{\Phi}} \tag{43}$$

where $\dot{\Phi} = t^a\nabla_a\Phi = Nn^a\nabla_a\Phi + N^a\nabla_a\Phi$. The hamiltonian function $H_t(\phi,\pi)$ on a hypersurface Σ_t is defined by

$$H_t \equiv \pi\dot{\phi} - \mathcal{L}(\phi,\pi) \tag{44}$$

and the time evolution of phase space data is governed by the Hamilton-Jacobi equations

$$\dot{\phi} = t^a\nabla_a\phi = \frac{\delta H_t}{\delta\pi} \qquad \dot{\pi} = t^a\nabla_a\pi - \frac{\delta H_t}{\delta\phi} \tag{45}$$

The pair $[\phi(x,t),\pi(x,t)]$ on a spacelike hypersurface Σ_t is the cauchy-data of the field and is assumed to uniquely determine the field in $D(\Sigma_t)$. Restrictions on the possible cauchy-data can be made such as requiring ϕ and π to be smooth functions of compact support.

The phase space Γ_t corresponding to Σ_t can now be defined as

$$\Gamma_t \equiv ([\phi,\pi], \phi:\Sigma_t\to\mathbb{R}, \pi:\Sigma_t\to\mathbb{R}; \phi,\pi\in C_0^\infty(\Sigma_t)) \tag{46}$$

Thus it is implicitly assumed that there is no symmetries of the fields.

The quantization is then imposed by giving $\phi(x)$ and $\pi(x)$ operator status and requiring the canonical commutation relations

$$\begin{aligned}
&[\phi(x),\phi(y)] = 0\\
&[\pi(x),\pi(y)] = 0\\
&[\phi(x),\pi(y)] = i\delta(x-y)
\end{aligned} \tag{47}$$

where $\delta(x-y)$ is a function on Σ satisfying $\int_\Sigma dx\delta(x-y) = 1$

The evolution of the state is given by the Schrödinger equation

$$i\frac{\partial|\,\psi(t)>}{\partial t} = \hat{H}\mid\psi(t)> \tag{48}$$

where $\hat{H}$ is the hermitean operator obtained from the classical hamiltonian by substituting $\phi(x)$, $\pi(x)$ with $\hat{\phi}(x)$,$\hat{\pi}(x)$.

The canonical quantization procedure is not mathematically well defined since $\phi(x)$ is not an operator but an operator valued distribution. The classical hamiltonian usually contains products such as $\phi\phi$,and since it is a local function the product must be of fields at the same point. The canonically constructed hamiltonian contains products of distributions in the same point which is generally not defined. Therefore it usually appears divergences which are removed using renormalization techniques. In the case of minkowski background metric the preferred technique is normal ordering of operators.

5.6 Scattering theory

Since quantum theory is about the relation between preparation and measurement of a system one is naturally led to a description of a physical process as a relation between an initial state $\mid \Phi >_{in}$ and a final state $\mid \Phi >_{out}$ while being oblivious of any intermediates. In the case where preparation and measurement are made in different locations this approach is realized as *scattering theory*. Naturally the applicability of this approach depends on the specific assumptions made about the spacetime. When considering quantum fields defined relative a background metric it is essential that the spacetime is globally hyperbolic. On a minkowski background the scattering process is formulated as a relation between initial and final states corresponding to a free field theory. This generalizes to a general globally hyperbolic background where initial and final states can be either schrödinger states associated to cauchy surfaces Σ_{in} and Σ_{out}, or heisenberg states given by covariant quantization. Similarly for partially relational spacetimes where preparation and measurement take place in the relational part this approach can be used. Let H_{in} and H_{out} be the hilbert spaces of in and out states respectively.

Let us consider the case where $H_{in} = H_{out} = H$. If we assume that probabilities are preserved there must be an automorphism S on H such that $S(\mid \Phi >_{in}) = \mid \Phi >_{out}$ for all $\mid \Phi >_{in}$ and $\mid \Phi >_{out}$, which is the *S-matrix*. More generally if H_{in} and H_{out} are not the same hilbert space, the conservation of probability in the scattering event implies as discussed above the existence of an isomorphism $S : H_{in} \to H_{out}$, which we will call the S-operator. The assumption that H_{in} and H_{out} are isomorphic is called the *Asymptotic completeness axiom* Furthermore to consider more general scattering situations that include mixed states we consider the statespace $\mathfrak{G}_{in}$ consisting of density matrices on H_{in}, and $\mathfrak{G}_{out}$ consisting of density matrices on H_{out}. The map $\mathfrak{S} : \mathfrak{G}_{in} \to \mathfrak{G}_{out}$ relating the in states and out states is called the superscattering operator.

If we want $\mathfrak{S}$ to conserve probabilities this imposes the condition

$$tr(\mathfrak{S}\rho) = tr(\rho) \tag{49}$$

Furthermore one may require non-interference of probabilities, which is the same thing as linearity of $\mathfrak{S}$, that is

$$\mathfrak{S}(a\rho_1 + b\rho_2) = a\mathfrak{S}\rho_1 + b\mathfrak{S}\rho_2 \tag{50}$$

5.7 CPT-invariance

Consider the case that we have a notion of charges, time-orientation as well as space-orientation. If then we assume that the scattering process is symmetric under CPT inversion, there should be anti-unitary maps $\mathcal{C} : H_{in} \to H_{out}$ and $\mathcal{C}^{-1} : H_{out} \to H_{in}$ corresponding to CPT inversion. These maps induces maps $\mathfrak{C} : \mathfrak{G}_{in} \to \mathfrak{G}_{out}$ and $\mathfrak{C}^{-1} : \mathfrak{G}_{out} \to \mathfrak{G}_{in}$ through $\mathfrak{C}\rho = \mathcal{C}\rho\bar{\mathcal{C}}$. The CPT inversion of the pair $[\rho, \mathfrak{S}\rho]$ will be $[\rho^{'}, \mathfrak{S}\rho^{'}] = [\mathfrak{C}^{-1}\mathfrak{S}\rho, \mathfrak{C}\rho]$ for all $\rho \in H_{in}$.

Such a CPT-symmetry exists if and only if $\mathfrak{S}$ is invertible, since[6]

$$\mathfrak{C}\rho \equiv \mathfrak{S}\rho^{'} \equiv \mathfrak{S}\mathfrak{C}^{-1}\mathfrak{S}\rho \tag{51}$$

and thus

$$\mathfrak{C} = \mathfrak{S}\mathfrak{C}^{-1}\mathfrak{S} \Rightarrow \mathfrak{C}^{-1}\mathfrak{S}\mathfrak{C}^{-1} = \mathfrak{S}^{-1} \tag{52}$$

5.8 Pure to mixed states

If the $\mathfrak{S}$-operator is bijective and linear it must map pure states in $\mathfrak{G}_{in}$ to pure states in $\mathfrak{G}_{out}$. This can be shown by contradiction. Consider a pure state $\rho_p =\mid \Phi >< \Phi \mid$ where $\Phi \in H_{in}$ and consider a mixed state $\rho_m = \sum_i k_i \mid \xi_{p_i} >< \xi_{p_i} \mid$ where all $\xi_{p_i} \in H_{out}$. Then consider a mapping $\mathfrak{S}$ such that $\mathfrak{S}\rho_p = \rho_m$. If $\mathfrak{S}$ is bijective there must be a mapping $\mathfrak{S}^{-1}$ such that $\rho_p = \mathfrak{S}^{-1}\rho_m$ and if $\mathfrak{S}^{-1}$ is linear $\rho_p = \sum_i k_i \mathfrak{S}^{-1}(\mid \xi_{p_i} >< \xi_{p_i} \mid)$. Then for all states $\Psi \in H_{in}$ orthogonal to Φ we have $\sum_i k_i < \Psi \mid \mathfrak{S}^{-1}(\mid \xi_{p_i} >< \xi_{p_i} \mid) \mid \Psi >= 0$ Thus every state $\mathfrak{S}^{-1}(\mid \xi_{p_i} >< \xi_{p_i}) \mid)$ must be orthogonal to all states $\Psi \in H_{in}$ orthogonal to Φ and hence $\mathfrak{S}^{-1}(\mid \xi_{p_i} >< \xi_{p_i} \mid) =\mid \Phi >< \Phi \mid$, that is every pure state $\mid \xi_{p_i} >< \xi_{p_i}) \mid$ is mapped to the same pure state $\mid \Phi >< \Phi \mid$ which contradicts the assumption that $\mathfrak{S}$ is bijective.

If the scattering is CPT-symmetric, and $\mathfrak{S}$ thus bijective, and if $\mathfrak{S}$ conserves probability there must be a unitary or antiunitary map S such that

$$\mathfrak{S}(\mid \Psi >< \Psi \mid) = S \mid \Psi >< \Psi \mid S^{*} \tag{53}$$

we call this map the S'-operator.If the S'-operator is unitary it defines a isomorphism between H_{in} and H_{out} and can be identified with the S-operator.

Let us now suppose that the $\mathscr{H}_{in}$ and $\mathscr{H}_{out}$ spaces are not a good description of the scattering event and that there is another hilbert space $\mathscr{H}$ corresponding to an adequate description of the system.

Then assume that there are morphisms $U : \mathscr{H}_{in} \to \mathscr{H}$, $U^{-1} : \mathscr{H} \to \mathscr{H}_{in}$, $V : \mathscr{H}_{out} \to \mathscr{H}$ and $V^{-1} : \mathscr{H} \to \mathscr{H}_{out}$. If the S-operator is unitary we would then have to conclude that$U$ and V are epimorphisms, U^{-1} and V^{-1} are monomorphisms and that

$$UV^{-1} = S, VU^{-1} = S^{-1} \tag{54}$$

$$U^{-1}U = id_{\mathscr{H}_{in}}, V^{-1}V = id_{\mathscr{H}_{out}} \tag{55}$$

5.9 Hawking radiation

One application of the covariantly quantized conventional quantum field theory is Hawkings treatment of a massless minimally coupled scalar field in a spherical collapse spacetime[7][8]. The equations of motion for such a field is thus

$$\nabla^2\phi = 0 \tag{56}$$

The metric outside the collapsing body will be static and spherically symmetric. The general spherically symmetric static metric is given by

$$ds^2 = -C(r)dt^2 + C(r)^{-1}dr^2 + r^2 d\Omega^2 \tag{57}$$

Inside the collapsing body the metric will not be static and hence have the general form

$$ds^2 = A(r,\tau)(-d\tau^2 + dr^2) + r^2 d\Omega^2 \tag{58}$$

We can change to tortoise coordinates and express the exterior metric as

$$ds^2 = C(r)dudv + r^2 d\Omega^2 \tag{59}$$

where the tortoise coordinates are defined as

$$\begin{aligned} u &= t - \tilde{r} + \tilde{R}_0 \\ v &= t + \tilde{r} - \tilde{R}_0 \end{aligned} \tag{60}$$

$$\tilde{r} = \int C^{-1} dr \tag{61}$$

Here $\tilde{R}_0$ is a constant. The event horizon is given by $C(r) = 0$. For example in the Schwarzschild case $C(r) = (1 - \frac{2M}{r})$ but we do not choose an explicit form for now. And similarly the interior metric can be reexpressed as

$$ds^2 = A(U,V)dUdV + r^2 d\Omega^2 \tag{62}$$

where $A(U,V)$ is an arbitrary non-singular function and

$$\begin{aligned} U &= \tau - r + R_0 \\ V &= \tau + r - R_0 \end{aligned} \tag{63}$$

We denote the transformation between interior and exterior coordinates by

$$U = \alpha(u) \quad \text{and} \quad v = \beta(V) \tag{64}$$

Then we again conside the equation of motion

$$\begin{aligned} 0 = \nabla^2\phi &= \frac{1}{C(r)}\nabla_u\nabla_v\phi \quad \text{in exterior region} \\ 0 = \nabla^2\phi &= A^{-1}(u,v)\frac{dv}{dV}\frac{du}{dU}\nabla_u\nabla_v\phi \quad \text{in interior region} \end{aligned} \tag{65}$$

To simplify the analysis we suppress the angular variables which then reduces to a two dimensional problem. We then restrict our treatment to $r \geq 0$ and require null rays to reflect at $r = 0$. This can be achieved by imposing the boundary condition $\phi = 0$ at $r = 0$. At $r = 0$, $V = U - 2R_0$ and thus $v = \beta(U - 2R_0) = \beta(\alpha u - 2R_0)$. The general solution is $\phi = f(v) - f(\beta(\alpha u - 2R_0))$.

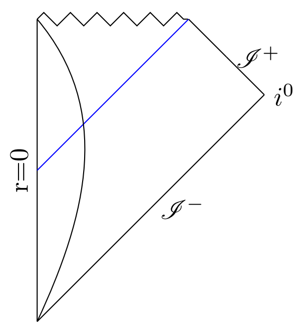
The next step is to choose families of modes and thus also the respective vacuum states. We then choose a family of solutions $\{s_\omega\}$ given by

$$s_\omega = i(4\pi\omega)^{-\frac{1}{2}}(e^{-i\omega v} - e^{-i\omega\beta(\alpha(u)-2R_0)}) \tag{66}$$

With this choice incoming waves $e^{-i\omega v}$ are converted into the possibly complicated function $e^{-i\omega\beta(\alpha(u)-2R_0)}$. We can also consider modes $\{t_\omega\}$ where the outgoing wave is simple and the incoming complicated.

$$t_\omega = i(4\pi\omega)^{-\frac{1}{2}}(e^{-i\omega u} - e^{-i\omega\alpha^{-1}(\beta^{-1}(v)-2R_0)}) \tag{67}$$

However it must be emphazised that $u \to \infty$ on the event horizon and hence $e^{-i\omega u}$ is not well defined here. To review hawkings construction let us consider the corresponding unphysical spacetime.



collapse spacetime

Here we can consider families of solutions $\{p_\omega\}$ and $\{q_\omega\}$ that have zero cauchy data on the event horizon and zero cauchy data on $\mathcal{I}^+$ respectively. The modes of $\{p_\omega\}$ will be characterized by being zero on $\mathcal{I}^+$ for $v > v_0$ where $v = v_0$ is the null ray that will reflect at $r = 0$ to form the event horizon, and the modes of $\{q_\omega\}$ will be characterized by being zero for $v < v_0$. Neither of theses two families are of course complete. The field ϕ can now be expanded either in terms of $\{s_\omega\}$ or $\{p_\omega\}$ and $\{q_\omega\}$.

$$\phi = \sum_i (a_i s_i + a_i^\dagger s_i^*) = \sum_i (b_i p_i + b_i^\dagger p_i^* + c_i q_i + \dagger c_i q_i^*) \tag{68}$$

we can expand p_i and q_i as

$$p_i = \sum_j (\alpha_{ij} s_j + \beta_{ij} s_j^*) \tag{69}$$

$$q_i = \sum_j (\gamma_{ij} s_j + \eta_{ij} s_j^*) \tag{70}$$

where $\alpha_{ij} = (p_i, s_j)$,$\beta_{ij} = (p_i, s_j^*)$,$\gamma_{ij} = (q_i, s_j)$ and $\eta_{ij} = (q_i, s_j^*)$. which leads to the following relationship between operators

$$a_i = \sum_j (\alpha_{ij} b_j + \beta_{ij}^* b_j^\dagger + \gamma_{ij} c_j + \eta_{ij}^* c_j^\dagger) \tag{71}$$

$$a_i^\dagger = \sum_j (\beta_{ij} b_j + \alpha_{ij}^* b_j^\dagger + \eta_{ij} c_j + \gamma_{ij}^* c_j^\dagger) \tag{72}$$

We may now consider a vacuum state defined by $a_i \mid 0_s >$. The expectation value of the number operator $N_{p_i} = b_i^\dagger b_i$ in this state is

$$< 0_s \mid b_i^\dagger b_i \mid 0_s >= \sum_j \mid \beta_{ij} \mid^2 \tag{73}$$

It was argued by hawking that a local observer at $\mathcal{I}^+$ will only measure the p_i modes and thus conclude that the $\mid 0_s >$ contains a non-zero number of p_i modes provided the p_i are chosen such that $\beta_{ij} \neq 0$. Strictly speaking such an observer could not be a observer moving on timelike paths trough the physical spacetime M since only null-paths can end on $\mathcal{I}^+$. It must also be remarked that the states defined by the covariant quantization procedure are global and as mentioned above are defined relative to the background metric. Since no local observer can know the global structure of the metric, and much less operationally define it, the states are not defined relative to any local observer but to an entity called the *superobserver*. It is nevertheless assumed that a local observer can perform local operations corresponding to the field operators constructed from creation and annihilation operators acting on the global states. We shall return to this later but first let us go through the reasoning leading to the hawking radiation.

We can construct a family of modes with zero cauchy data on the event horizon where the outgoing modes are simple as

$$p_\omega = i(4\pi\omega)^{-\frac{1}{2}}(e^{-i\omega u} - e^{-i\omega\alpha^{-1}(\beta^{-1}(v) - 2R_0)}) \tag{74}$$

for $v < v_0$ where v_0 is the null ray that will form the event horizon upon being reflected at $r = 0$, and

$$p_\omega = i(4\pi\omega)^{-\frac{1}{2}} e^{-i\omega u} \tag{75}$$

for $v > v_0$ and outside the event horizon. We then seek $\beta_{ij} = (p_i, s_j^*)$, which we can evaluate for example on a hypersurface Σ given by $t = 0$ chosen such that the it ends at $r = 0$ and intersects the event horizon in the future of the point where the surface of the collapsing body intersects the horizon. Then $n^u = n^v = C(r)^{-\frac{1}{2}}$, and the determinant of the induced metric γ_Σ is $C(r)$, and hence $\sqrt{\gamma_\Sigma} n^u = \sqrt{\gamma_\Sigma} n^v = 1$. We can thus calculate

$$
\begin{aligned}
\beta_{\omega\tilde{\omega}} =& i \int_{\Sigma_{out}} \frac{1}{4\pi} (\tilde{\omega}\omega)^{-\frac{1}{2}} (e^{-i\omega u} \partial_u (e^{i\tilde{\omega} v} - e^{i\tilde{\omega}\beta(\alpha(u)-2R_0)}) \\
& - \partial_u e^{-i\omega u} (e^{i\tilde{\omega} v} - e^{i\tilde{\omega}\beta(\alpha(u)-2R_0)}) \\
& + e^{-i\omega u} \partial_v (e^{i\tilde{\omega} v} - e^{i\tilde{\omega}\beta(\alpha(u)-2R_0)})) = \\
=& i \int_{\Sigma_{out}} \frac{1}{4\pi} (\tilde{\omega}\omega)^{-\frac{1}{2}} (e^{-i\omega u} (-i\tilde{\omega} \frac{\partial \beta(\alpha(u)-2R_0)}{\partial u} e^{i\tilde{\omega}\beta(\alpha(u)-2R_0)}) \\
& - i\omega e^{i\tilde{\omega}\beta(\alpha(u)-2R_0)} e^{-i\omega u} + i\omega e^{-i\tilde{\omega} v} e^{-i\omega u} + i\tilde{\omega} e^{-i\omega u} e^{+i\tilde{\omega} v}) = \\
=& i \int_{-\infty}^{+\infty} \frac{1}{4\pi} (\tilde{\omega}\omega)^{-\frac{1}{2}} (e^{-i\omega u} (i\tilde{\omega} \frac{\partial \beta(\alpha(u)-2R_0)}{\partial u} e^{i\tilde{\omega}\beta(\alpha(u)-2R_0)}) \\
& - i\omega e^{i\tilde{\omega}\beta(\alpha(u)-2R_0)} e^{-i\omega u} + i(\omega - \tilde{\omega}) e^{i(\tilde{\omega}-\omega)u}) = \\
=& i \int_{-\infty}^{+\infty} \frac{1}{4\pi} (\tilde{\omega}\omega)^{-\frac{1}{2}} (e^{-i\omega u} (-i\tilde{\omega} \frac{\partial \beta(\alpha(u)-2R_0)}{\partial u} e^{i\tilde{\omega}\beta(\alpha(u)-2R_0)}) \\
& - i\omega e^{i\tilde{\omega}\beta(\alpha(u)-2R_0)} e^{-i\omega u}) = \\
=& \frac{1}{4\pi} (\tilde{\omega}\omega)^{-\frac{1}{2}} (-e^{-i\omega u} e^{i\tilde{\omega}\beta(\alpha(u)-2R_0)}) |_{-\infty}^{+\infty} - 2 \int_{-\infty}^{+\infty} i\omega e^{i\tilde{\omega}\beta(\alpha(u)-2R_0)} e^{-i\omega u}
\end{aligned}
\tag{76}
$$

where Σ_{out} is the part of the hypersurface outside the horizon. This result is dependent on the detailed behavior of the collapsing body through α and β. To find hawkings result we need to make certain approximations as we shall see below. To get there consider the following. We wish to find the approximate functional form of the modes in the region close to i^+.

Differentiating the relations $U = \alpha(u)$ and $v = \beta(V)$ gives

$$
\frac{dU}{du} = \frac{d\tau}{dt} + C(r)(1 - \frac{d\tau}{dr}) - \frac{dr}{dt} \tag{77}
$$

$$
\frac{dv}{dV} = \frac{dt}{d\tau} + \frac{dt}{dr} + C^{-1}(r)(1 + \frac{dr}{d\tau}) \tag{78}
$$

The surface of the collapsing body is given by $R(\tau) = r$. Now when $R(\tau) = r$ the interior and exterior metric must match that is

$$
A(R(\tau), \tau)(-d\tau^2 + dr^2) = -C(R(\tau))dt^2 + C(R(\tau))^{-1} dr^2 \tag{79}
$$

This means at the boundary $R(\tau) = r$

$$
\frac{dt^2}{d\tau^2} = \frac{1}{C^2(R(\tau))} (A(R(\tau), \tau) C(R(\tau))(1 - \frac{dr^2}{d\tau^2}) + \frac{dr^2}{d\tau^2}) \tag{80}
$$

Thus on the boundary

$$
\frac{dU}{du} = (1 - \frac{dr}{d\tau}) C(R(\tau)) (A(R(\tau), \tau) C(R(\tau))(1 - \frac{dr^2}{d\tau^2}) + \frac{dr^2}{d\tau^2})^{-\frac{1}{2}} + C(r)(1 - \frac{d\tau}{dr}) \tag{81}
$$

$$\frac{dv}{dV} = \frac{(1+\frac{d\tau}{dr})}{C(R(\tau))}(A(R(\tau),\tau)C(R(\tau))(1-\frac{dr^2}{d\tau^2})+\frac{dr^2}{d\tau^2})^{\frac{1}{2}}+C^{-1}(R(\tau))(1+\frac{dr}{d\tau}) \quad (82)$$

Close to the event horizon where $C = 0$ we can approximate the expressions as

$$\frac{dU}{du} \approxeq (\frac{dr}{d\tau} - 1)C(R(\tau))(2\frac{dr}{d\tau})^{-1} \quad (83)$$

$$\frac{dv}{dV} \approxeq A(1 - \frac{dr}{d\tau})(\frac{dr}{d\tau})^{-1} \quad (84)$$

We then consider C as a function of U expand the function $C(U)$ around $C(U_h)$ as

$$C(r) = C(U_h) + \frac{\partial C}{\partial U} \mid_{U=U_h} (U - U_h) + \mathcal{O}((U - U_h)^2) \quad (85)$$

where $U_h = \tau_h - R(\tau_h) + R_0$ and τ_h is the argument for which $R(\tau) = r$ intersects the horizon. Thus $C(U_h) = C(R(\tau_h)) = 0$ with this additional approximation we have

$$\frac{dU}{du} \approxeq (\frac{dr}{d\tau} - 1)(2\frac{dr}{d\tau})^{-1}\frac{\partial C}{\partial U} \mid_{U=U_h} (U - U_h) \quad (86)$$

which implies

$$\frac{dU}{(U - U_h)} =\approxeq (\frac{dr}{d\tau} - 1)(2\frac{dr}{d\tau})^{-1}\frac{1}{2}\frac{\partial C}{\partial U} \mid_{U=U_h} du = -\frac{1}{2}\frac{\partial C}{\partial r} \mid_{r=R(\tau_h)} du \quad (87)$$

which integrates to

$$-\,ln(U - U_h) = -ln(U - \tau_h + R(\tau_h) - R_0) \approxeq u \quad (88)$$

where $\kappa = \frac{1}{2}\frac{\partial C}{\partial r} \mid_{r=R(\tau_h)}$ is the surface gravity. We see that when $U \to \tau_h - R(\tau_h) + R_0$, $u \to \infty$. Inverting 88 gives

$$U \approxeq e^{-\kappa u} + c \quad (89)$$

where $c = constant$. Thus we have found the approximate form of αu when u is large. If we consider a region with only a narrow range of values for v and V we may approximately treat A as constant. In this is the case and C is close to zero and we treat $\frac{dr}{d\tau}$ as constant we can integrate to

$$v \approxeq AV(1 - \frac{dr}{d\tau})(\frac{dr}{d\tau})^{-1} \quad (90)$$

If we assume that $\beta(v)$ is such that the late time asymptotic region of $\mathcal{I}^+$ only has a narrow range of values for V for a narrow range of v we may use this approximation there. With these two approximations the s modes in this region can be expressed as

$$s_\omega \approxeq i(4\pi\omega)^{-\frac{1}{2}}(e^{-i\omega v} - e^{-i\omega(e^{c\kappa u}+d)}) \quad (91)$$

where $c = A(1 - \frac{dr}{d\tau})(\frac{dr}{d\tau})^{-1}$ and d are constants.

Hawkings result now follows if we replace the original s_ω with this approximation.

$$\beta_{\omega\tilde{\omega}} = \frac{1}{4\pi}(\tilde{\omega}\omega)^{-\frac{1}{2}}(-e^{-i\omega u}e^{i\tilde{\omega}(e^{c\kappa u}+d)}) \mid_{-\infty}^{+\infty} -2\int_{-\infty}^{+\infty} i\omega e^{i\tilde{\omega}(e^{c\kappa u}+d)}e^{-i\omega u} \quad (92)$$

Then with a change of variables $x = (e^{c\kappa u} + d)$ this becomes

$$\beta_{\omega\tilde{\omega}} = \frac{1}{4\pi}(\tilde{\omega}\omega)^{-\frac{1}{2}}(-(x-d)^{-i\frac{\omega}{c\kappa}}e^{i\tilde{\omega}x}) \mid_{d}^{+\infty} -2\int_{d}^{+\infty} i\frac{\omega}{c\kappa}e^{i\tilde{\omega}x}(x-d)^{-i\frac{\omega}{c\kappa}-1} \quad (93)$$

which we can express in terms of upper incomplete gamma functions

$$\beta_{\omega\tilde{\omega}} = \frac{1}{4\pi}(\tilde{\omega}\omega)^{-\frac{1}{2}}(-(x-d)^{-i\frac{\omega}{c\kappa}}e^{i\tilde{\omega}x}) \mid_{d}^{+\infty} -2i\frac{\omega}{c\kappa}\Gamma(-i\frac{\omega}{c\kappa}, d)(i\tilde{\omega})^{+i\frac{\omega}{c\kappa}}e^{-i\tilde{\omega}d}) \quad (94)$$

Then if we further approximate by discarding the first term and approximating the incomplete gamma function by a complete gamma function, by setting $d = 0$ we have

$$\beta_{\omega\tilde{\omega}} = \frac{1}{2\pi}(\frac{\tilde{\omega}}{\omega})^{\frac{1}{2}}\Gamma(1 + i\frac{\omega}{c\kappa})(-i\tilde{\omega})^{-1-i\frac{\omega}{c\kappa}} \quad (95)$$

but $(-i\tilde{\omega})^{-1-i\frac{\omega}{c\kappa}} = e^{(-1-i\frac{\omega}{c\kappa})(ln|\tilde{\omega}|-i\frac{\pi}{2})} = i(\tilde{\omega})^{-1-i\frac{\omega}{c\kappa}}e^{-\frac{\pi\omega}{2c\kappa}}$. Further $\Gamma(1 + x)\Gamma(1-x) = \frac{\pi x}{sin(\pi x)}$, and noting this we are ready to compute $\mid \beta_{\omega\tilde{\omega}} \mid^2$

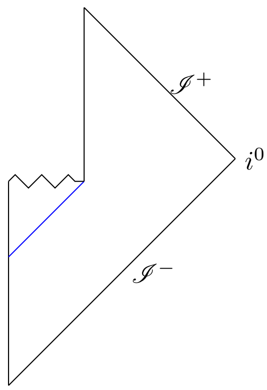
$$\mid \beta_{\omega\tilde{\omega}} \mid^2 = \frac{1}{2\pi\tilde{\omega}c\kappa}\frac{1}{e^{\frac{2\pi\omega}{c\kappa}} - 1} \quad (96)$$

Thus in this approximation the the expectation value has the form of the spectrum of a black body of temperature $T = \frac{c\kappa}{2\pi k}$, where k is boltzmans constant.

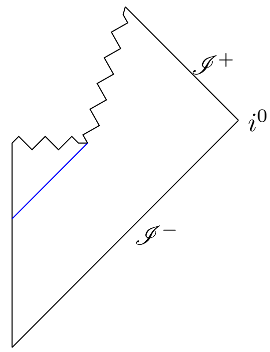
5.10 Evaporating black hole spacetimes

It was conjectured by Hawking[9] that the behavior of quantum fields in the collapse metric would cause the black hole to *evaporate*, that is lose mass until it disappeared if back reaction was included. Provided that einsteins equation is valid the area theorem implies that the area of the event horizon can decrease only if the null energy theorem is not satisfied.

We then turn to the question of what the causal structure of such an evaporation spacetime would be. One possibility is that the singularity vanishes without intersecting the horizon, and thus has no future. We call this case a type 0 spacetime. Assuming that a classical black hole in a asymptotically flat spacetime evolves in such a way that the singularity do meet the event horizon, there is still the question of how the spacetime evolves following this event. There are a few a priori possibilities:



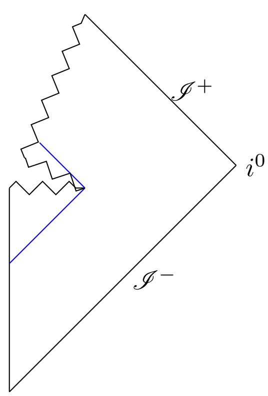
Type I



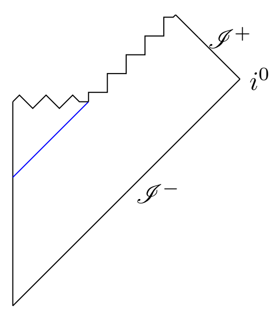
Type II

Type I The singularity ends a the point where it meets the event horizon. In this case the endpoint of the singularity has a future as well as a past, thus rendering the spacetime non-globally hyperbolic. In this case $I^-(\mathscr{I}^+) \subset I^+(\mathscr{I}^-)$

Type II The singularity extends further past the intersection with the event horizon being non-spacelike, having an endpoint or extending all the way to $\mathscr{I}^+$, but being only partially null. This spacetime to is non-globally hyperbolic and $I^-(\mathscr{I}^+) \subset I^+(\mathscr{I}^-)$.



Type III

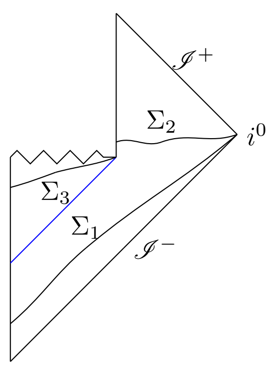


Type IV

Type III The singularity extends past the intersection with the event horizon, but is at least partially spacelike in such a way that $I^-(\mathscr{I}^+) \cap I^+(\mathscr{I}^-) \neq I^+(\mathscr{I}^-)$.

Type IV The singularity extends past the intersection with the event horizon but is null and extends all the way to $\mathscr{I}^+$. This spacetime still globally hyperbolic.

We now turn to the issue of information loss in the application of the covariantly quantized quantum field theory on the background collapse metric. We consider massive fields as well as massless. Let us consider a spacelike hypersurface Σ_1 that intersects the collapsing body outside the event horizon and the corresponding hilbert space H_1 of data on Σ_1. Similarly we introduce a hypersurface Σ_2 in the future of the endpoint of evaporation and the corresponding hilbert space H_2, and then a hypersurface Σ_3 in the black hole region such that all causal curves through the horizon also intersect Σ_3 and the corresponding hilbert space H_3. Let us first consider the case where the singularity has no future, that is a type 0 spacetime.



Type 0

There should then exist a S-matrix relating a state described by cauchy data on Σ_1 to states on Σ_2 and Σ_3, hence there is no unitary mapping from Σ_1 and Σ_2. Information is thus lost in the sense that given only knowledge about the state of H_2 the state of H_1 cannot be derived. It has been conjectured that the state on Σ_2 should be derived from the state in $H_2 \otimes H_3$ by partially tracing over H_3. The state of Σ_2 is thus generally mixed.The source of this information loss is that states of the covariantly quantized theory are defined relative to the superobserver and thus are global, but it is conjectured that there exist local observers that can determine the state through local operators, but only in causally connected regions. When we consider spacetimes of type I to III where the singularity has a future, boundary conditions on the singularity would have to be included and the above analysis is invalidated. In a spacetime of type IV a spacelike hypersurface could always be chosen such that it was a cauchy surface for the spacetime, and thus one need not make the construction with two separate hypersurfaces.

Likewise the derivation of hawking radiation is also invalidated by the presence of naked singularities and the form of the bogolubov coefficients is impossible to determine since the boundary conditions on the naked singularity are unknown.

5.11 Nice slices

While the above discussion is largely framed in the heisenberg picture covariantly quantized construction we may frame similar arguments in the schrödinger picture.

Given a stably causal classical spacetime there always exist a foliation Σ_t. The assumption that physics of a general curved spacetime should be describable by quantum fields defined on spacelike hypersurfaces of a foliation chosen such that curvature is everywhere small on the hypersurfaces sometimes goes under the name of the *nice slice* assumption.

In the nice slice description of a quantum field in non-evaporating black hole background spacetime M we introduce a foliation of M by a family of cauchy surfaces. The surfaces should avoid regions of strong curvature and cut through the infalling matter as well as the assumed outgoing hawking radiation. Further the surfaces should be everywhere smooth and have small extrinsic curvature. Provided such a family exists it is assumed that one can introduce a vector field v orthogonal to the nice slices, and a generator of motion along this vector field. This generator, the nice-slice hamiltonian H_{NS} is supposed to map the state $\mid \psi >$ of the system on one slice to the state on another slice by the schrödinger equation.

$$i\partial \mid \psi >= H_{NS} \mid \psi > \tag{97}$$

If the spacetime has a global timelike killing field ξ_a and the foliation is chosen such that this field is orthogonal to the hypersurfaces one can construct a conserved current $J_b = \xi^a_T ab$ and provided either there are no boundaries of M except those defined hypersurfaces of the foliation or that J_b vanishes on any other boundaries than those that are part of the foliation, a classical hamiltonian according to

$$H = \int_{\Sigma_t} \xi^a T^b_b n_a \tag{98}$$

where n_a is normal to Σ. The quantized version of H could then serve as the nice-slice hamiltonian. However in general there is no global killing field and H is time dependent. Thus and there is no conserved nice-slice hamiltonian in general.

5.11.1 Foliations of evaporation spacetimes

If one assumes that the black hole is evaporating and the singularity meets the event horizon in a point, there will not be any cauchy surfaces, as is the case for spacetimes of type I-III. Further a foliation of such a spacetime by spacelike surfaces must include surfaces entering the region of large curvature where the singularity intersects the event horizon, hence it is impossible to foliate spacetime by nice-slices. One can still arguably construct a foliation such that all the surfaces up to those cutting through the end stages of evaporation would be nice-slices. However assuming that the state space consists of field configurations of compact support whose dynamics are given by local equations of motion, and assuming that field configurations propagating into a singularity cannot reappear and/or that naked singularities Υ give rise to field configurations propagating into $J^+(\Upsilon)$ the model will lose predictability in globally non-hyperbolic spacetimes. In particular type $I - III$ evaporation spacetimes predictability will be lost. However let us consider the consequences of assuming that the naked singularity will not affect the evolution of the states directly, that is no field configurations will appear from Υ. We shall call such a naked singularity *inert*. But first let us consider the following theorem.

The no quantum Xerox principle The *no quantum Xerox principle*[10] states that the information of a quantum state cannot be replicated to another quantum state by a linear operator. Suppose that the hilbert space of prepared states is $\mathscr{H}_A$ and that the hilbert space of measured states factorizes as $\mathscr{H} = \mathscr{H}_B \otimes \mathscr{H}_C$.Then suppose there exist a linear operator X that replicates the information in the state $\mid \psi_A >\in \mathscr{H}_A$ to a state $\mid \psi_B >\in \mathscr{H}_B$, and also to a state $\mid \psi_C >\in \mathscr{H}_C$ according to the rule

$$X(\mid \psi_A >) = \mid \psi_B > \otimes \mid \psi_C > \tag{99}$$

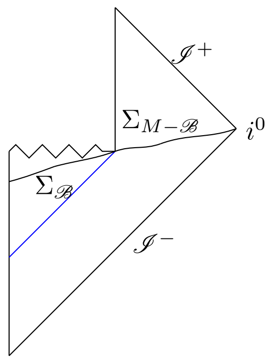
We can always write $\mid \psi_A > = \mid \alpha_A > + \mid \beta_A >$ where $\mid \alpha_A >$ and $\mid \beta_A > \in \mathscr{H}_A$. But then since X is linear

$$\begin{aligned} X(\mid \psi_A >) = X(\mid \alpha_A >) + X(\mid \beta_A >) = \\ = \mid \alpha_B > \otimes \mid \alpha_C > + \mid \beta_C > \otimes \mid \beta_C > \neq \mid \psi_B > \otimes \mid \psi_C > \end{aligned} \tag{100}$$

Thus such an operator X cannot exist. Now let us proceed.

5.12 Non-unitarity in the evaporating black hole spacetime

An argument for information loss in the nice-slice description in type 0 evaporating black hole spacetime is given in[11]. However the argument generalizes to type I,II and III, once again provided the naked singularities are inert. Let us assume that we can define a quantum field on each hypersurface of a foliation and that it is local so that field operators will commute when their arguments are spacelike separated. Next consider a late hypersurface Σ_P near the point where the singularity intersects the horizon, and partition it in the part inside the black hole region $\Sigma_{\mathscr{B}}$ and the part outside the black hole region $\Sigma_{M-\mathscr{B}}$.



Since the field operators commute we can form a complete set of commuting observables using local fields defined on $\Sigma_{\mathscr{B}}$ and $\Sigma_{M-\mathscr{B}}$, and thus the hilbert space of states on the slice Σ_P factorizes into a tensor product of functionals on $\Sigma_{\mathscr{B}}$ and $\Sigma_{M-\mathscr{B}}$ respectively.

$$\mathscr{H} = \mathscr{H}(\Sigma_{\mathscr{B}}) \otimes \mathscr{H}(\Sigma_{M-\mathscr{B}}) \tag{101}$$

Let us assume that the hamiltonian defines a linear map from states in the Hilbert space on an initial slice Σ_{IN} to states in $\mathscr{H}$. If we then assume that the states of $\mathscr{H}(\mathscr{B})$ contain information and use the no quantum xerox principle we find that linear evolution of the states cannot replicate the information in the states of $\mathscr{H}(\Sigma_{\mathscr{B}})$ to the states of $\mathscr{H}(\Sigma_{M-\mathscr{B}})$.

Next follows the question of how to describe evolution beyond Σ_P. Let ρ be the density matrix corresponding to a state $\mid \psi >\in \mathscr{H}$. Then we can define the density matrix for $\Sigma_{M-\mathscr{B}}$ as $\rho_{\Sigma_{M-\mathscr{B}}} = Tr_{\Sigma_{\mathscr{B}}}\rho$ If we assume that states on future hypersurfaces should be evolved from the state given by $\rho_{\Sigma_{M-\mathscr{B}}}$, then in general information will be lost and evolution non-unitary.

There are however special cases when unitarity is preserved. Consider a state $\mid \psi(\Sigma_{IN}) >$ on Σ_{IN} that is pure. Then the state $\mid \psi(\Sigma_P) >$ on Σ_P must also be pure. Further if a post-evaporation state $\mid \psi(\Sigma_F) >$ on a slice Σ_F is to be pure and if we assume it has evolved linearly from a state on $\Sigma_{M-\mathscr{B}}$ we would have to assume that we could express $\mid \psi(\Sigma_P) >$ as a product state

$$\mid \psi(\Sigma_P) >=\mid \varphi(\Sigma_{\mathscr{B}}) > \otimes \mid \xi(\Sigma_{M-\mathscr{B}}) > \tag{102}$$

In other words there would be no correlations between the state in the interior of $\mathscr{B}$ and the state exterior to it.We thus require an evolution of the kind

$$\mid \psi_i(\Sigma_{IN}) >\rightarrow \mid \varphi_i(\Sigma_{\mathscr{B}}) > \otimes \mid \xi_i(\Sigma_{M-\mathscr{B}}) > \tag{103}$$

for any initial state $\mid \psi_i(\Sigma_{IN}) >$. Now since evolution is assumed to be linear we must have

$$\mid \psi_1(\Sigma_{IN}) > + \mid \psi_2(\Sigma_{IN}) >\rightarrow$$
$$\rightarrow \mid \varphi_1(\Sigma_{\mathscr{B}}) > \otimes \mid \xi_1(\Sigma_{M-\mathscr{B}}) > + \mid \varphi_2(\Sigma_{\mathscr{B}}) > \otimes \mid \xi_2(\Sigma_{M-\mathscr{B}}) >$$

This state however has uncorrelated interior and exterior only if either $\mid \varphi_1(\Sigma_{\mathscr{B}}) >=\mid \varphi_2(\Sigma_{\mathscr{B}}) >$ or $\mid \xi_1(\Sigma_{M-\mathscr{B}}) >=\mid \xi_2(\Sigma_{M-\mathscr{B}}) >$ The first option would be saying that the exterior state would be the same regardless of initial conditions which might be considered an undesirable property. The second option is thus that the interior state is the same for all initial states. This kind of evolution is however hard to reconcile with the idea that the equations of motion are local and that the event horizon has no special local properties.Further even though evolution is unitary information is only preserved if in the second case the interior state contain no information.

The nice slice description, while different has similar ontological features to the covariantly quantized description. States are defined with reference to spacelike hypersurfaces and are thus globally defined on each slice, but field operators are local. The observer would thus have to operationally define the nice slice but it is not possible for a local observer to define the nice slices that enter the black hole region. Likewise the arguments above refer to correlations of field operators inside and outside the black hole region, which are also operationally impossible for a local observer to measure.

6 Weyl rescalings

Before we proceed let us consider some technical things that will be of use later on. A local rescaling of the metric giving rise to a new metric $\tilde{g}_{ab}$ defined by

$$\tilde{g}_{ab} \equiv \Omega^2 g_{ab} \tag{104}$$

where Ω is a smooth strictly positive function, is called a *Weyl rescaling*. The metrics g_{ab} and $\tilde{g}_{ab}$ are said to be *conformally related*, and this defines an equivalence relation on the set of metrics on M. The set of Weyl-rescalings forms a group $Weyl(M)$. Since $\tilde{g}_{ab}$ and g_{ab} are related by the positive scalar Ω^2 a vector that is spacelike, null or timelike in g_{ab} will be spacelike, null or timelike respectively in $\tilde{g}_{ab}$ as well. An equation for a field Φ is said to be conformally invariant if there exist a number $s \in \mathbb{R}$ such that ϕ is a solution to the equations of motion when the metric is g_{ab} iff $\tilde{\phi} \equiv \Omega^s \phi$ is a solution to the equations of motion when the metric is $\tilde{g}_{ab} \equiv \Omega^2 g_{ab}$.

6.1 Conformal frame

Let us now consider how different quantities change under Weyl rescalings. Let M be a n-dimensional manifold and let g_{ab} and $\tilde{g}_{ab}$ be two metrics on M related by a Weyl-rescaling given by $\tilde{g}_{ab} \equiv \Omega^2 g_{ab}$ and let $\tilde{R}^a_{bcd}$, $\tilde{R}_{ab}$, $\tilde{R}$, $\tilde{\nabla}_a$,$\tilde{\nabla}^2$ denote the quantities associated to $\tilde{g}_{ab}$ while R^a_{bcd}, R_{ab}, R,∇_a,∇^2 are the quantities associated to g_{ab}. Then it can be shown that

$$\begin{aligned}\tilde{R}^a_{bcd} =& R^a_{bcd} - 2(\delta^a_{[c}\delta^\sigma_{d]}\delta^\rho_b - g_{b[c}\delta^\sigma_{c]}g^{a\rho})\Omega^{-1}\nabla_\sigma\nabla_\rho\Omega \\ &+2(2\delta^a_{[c}\delta^\sigma_{d]}\delta^\rho_b - 2g_{b[c}\delta^\sigma_{c]}g^{a\rho} + g_{b[c}\delta^a_{c]}g^{\sigma\rho})\Omega^{-2}\nabla_\rho\Omega\nabla_\sigma\Omega\end{aligned}$$

$$R^a_{bcd} = \tilde{R}^a_{bcd} - 2(\delta^a_{[c}\delta^\sigma_{d]}\delta^\rho_b - \tilde{g}_{b[c}\delta^\sigma_{c]}\tilde{g}^{a\rho})\Omega^{-1}\tilde{\nabla}_\sigma\tilde{\nabla}_\rho\Omega$$
$$+2\tilde{g}_{b[c}\delta^a_{c]}\tilde{g}^{\sigma\rho}\Omega^{-2}\tilde{\nabla}_\rho\Omega\tilde{\nabla}_\sigma\Omega$$

$$\tilde{R}_{ab} = R_{ab} - ((n-2)\delta^\rho_a\delta^\sigma_b - g_{ab}g^{\rho\sigma})\Omega^{-1}\nabla_\sigma\nabla_\rho\Omega + (2(n-2)\delta^\rho_a\delta^\sigma_b - (n-3)g_{ab}g^{\rho\sigma})\Omega^{-2}\nabla_\rho\Omega\nabla_\sigma\Omega$$
$$R_{ab} = \tilde{R}_{ab} - ((n-2)\delta^\rho_a\delta^\sigma_b - \tilde{g}_{ab}\tilde{g}^{\rho\sigma})\Omega^{-1}\tilde{\nabla}_\sigma\tilde{\nabla}_\rho\Omega + (2(n-1)\tilde{g}_{ab}\tilde{g}^{\rho\sigma}\Omega^{-2}\tilde{\nabla}_\rho\Omega\tilde{\nabla}_\sigma\Omega$$
$$\tilde{R} = \Omega^{-2}R - 2(n-1)g^{\rho\sigma}\Omega^{-3}\nabla_\sigma\nabla_\rho\Omega - (n-1)(n-4)g^{\rho\sigma}\Omega{-}4\nabla_\rho\Omega\nabla_\sigma\Omega \quad (105)$$
$$R = \Omega^2\tilde{R} - 2(n-1)\tilde{g}^{\rho\sigma}\Omega\tilde{\nabla}_\sigma\tilde{\nabla}_\rho\Omega - n(n-1)\tilde{g}^{\rho\sigma}\Omega{-}1\tilde{\nabla}_\rho\Omega\tilde{\nabla}_\sigma\Omega$$

If we introduce a scalar field Φ we find that

$$\tilde{\nabla}_a\Phi = \nabla_a\Phi$$
$$\tilde{\nabla}_a\tilde{\nabla}_b\Phi = \nabla_a\nabla_b\Phi - (\delta^\rho_a\delta^\sigma_b + \delta^\sigma_a\delta^\rho_b - g_{ab}g^{\rho\sigma})\Omega^{-1}\nabla_\rho\Omega\nabla_\sigma\Phi$$
$$\nabla_a\nabla_b\Phi = \tilde{\nabla}_a\tilde{\nabla}_b\Phi + (\delta^\rho_a\delta^\sigma_b + \delta^\sigma_a\delta^\rho_b - \tilde{g}_{ab}\tilde{g}^{\rho\sigma})\Omega^{-1}\tilde{\nabla}_\rho\Omega\tilde{\nabla}_\sigma\Phi$$
$$\tilde{\nabla}^2\Phi = \Omega^{-2}\nabla^2\Phi + (n-2)g^{\rho\sigma}\Omega{-}3\nabla_\rho\Omega\nabla_\sigma\Phi$$
$$\nabla^2\Phi = \Omega^2\tilde{\nabla}^2\Phi - (n-2)\tilde{g}^{\rho\sigma}\Omega\tilde{\nabla}_\rho\Omega\tilde{\nabla}_\sigma\Phi$$

6.2 Conformal gauge

We say that a metric g_{ab} is conformally flat if there is a conformally related metric $\tilde{g}_{ab}$ with everywhere vanishing R. Let us introduce a field ρ such that $\tilde{g}_{ab} \equiv e^{-2\rho}g_{ab}$. Then from105

$$\tilde{R} = e^{2\rho}R + 2g^{ab}e^{2\rho}\nabla_a\nabla_b\rho$$

and thus g_{ab} is conformally flat if there is a field ρ such that

$$R = -2g^{ab}\nabla_a\nabla_b\rho$$

and thus

$$\tilde{R} = 0$$

In the new metric $\tilde{g}_{ab}$ R is given by

$$R = -2\tilde{g}^{ab}e^{-2\rho}\tilde{\nabla}_a\tilde{\nabla}_b\rho = -2e^{-2\rho}\tilde{\nabla}^2\rho \quad (106)$$

In 2D all metrics are conformally flat and we call such reformulation of the metric the conformal gauge. Since $Conf(M)$ is a subgroup of the diffeomorphism group, and the action preserves time orientation, any diffeomorphism invariant 2D metric is equivalent to the conformally related metrics.

In conformal gauge apart from 106 we also have

$$R_{ab} = \tilde{R_{ab}} - \tilde{g_{ab}}(\tilde{\nabla}\rho)^2$$
$$\nabla_a\phi = \tilde{\nabla}_a\phi$$

$$(\nabla\phi)^2 = e^{-2\rho}(\tilde{\nabla}\phi)^2$$
$$\nabla_a \nabla b\phi = \tilde{\nabla}_a \tilde{\nabla}_b \phi - (\delta_a^\mu \delta_b^\nu + \delta_a^\nu \delta_b^\mu - \tilde{g}_{ab}\tilde{g}^{\mu\nu})\tilde{\nabla}_{nu}\rho\tilde{\nabla}_{mu}\phi$$
$$\nabla^2\phi = e^{-2\rho}\tilde{\nabla}^2$$
$$\sqrt{-g} = e^{2\rho}\sqrt{-\tilde{g}}$$

In the conformal gauge we can introduce lightcone coordinates given by

$$g_{++} = g_{--} = 0 \qquad g_{+-} = g_{-+} = -\frac{1}{2} \qquad g^{+-} = g^{-+} = -2$$

With this choice

$$\tilde{\nabla}^2\Phi = -4\partial_+\partial_-\Phi \qquad (\tilde{\nabla}\Phi)^2 = -4\partial_+\Phi\partial_-\Phi$$
$$\sqrt{-\tilde{g}} = \frac{1}{2}$$

7 Stress tensor

We now turn our attention to the subject of how to define the quantum mechanical stress tensor. The precise definition is important both when quantum fields are defined on a fixed background metric and in the semi classical treatments.

The classical stress tensor $T_{ab}(x)$ for a field ψ at a point x is as already mentioned related to the matter action as $T_{ab} \equiv \frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{ab}}$. For example the classical expression for the stress tensor of a Klein-Gordon field is $T_{ab} = \nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}((\nabla\phi)^2 + m^2\phi^2) + \xi(g_{ab}\nabla^2\phi^2 - \nabla_a\nabla_b\phi^2 + (R_{ab} - \frac{1}{2}g_{ab}R)\phi^2)$. As in the case of the Klein-Gordon field, the stress tensor generally contains terms quadratic in ϕ. Thus if one tries to construct a quantum operator associated to a point by giving the field ϕ operator status it will be ill defined.

However constructions of operators corresponding to the classical stress tensor has been carried out by making ad hoc modifications to remove divergences. In Minkowski spacetime the normal ordering procedure is usually employed but for general spacetimes one must develop other techniques. Such techniques include point-splitting, dimensional, and zeta-function regularization, however all these methods have some ad hoc features.

7.1 Axioms for the stress tensor

In the absence of any obvious way to define a stress-energy operator associated to a point one can try to determine which characteristics $< \psi \mid T_{ab} \mid \psi >$ should have provided it satisfies certain conditions one might consider desirable. One such prescription was proposed by Wald[12][13] where the stress tensor is described by a map $T_{ab} : H_\psi \times \bar{H}_\psi \to \mathscr{T}(2,0)$ where H_ψ is the hilbert space of the field ψ, $\bar{H}_\psi$ is the dual space of H_ψ and $\mathscr{T}(2,0)$ is the vector space of 2-covariant symmetric tensor fields on M. On such a map several conditions for a T_{ab} in an asymptotically flat background spacetime M with metric g_{ab} are imposed in the so called axiomatic approach.

Covariant conservation Since $\nabla_a T^a_b = 0$ for classical fields the expectation value is required to be covariantly conserved, that is that is for any state $\mid \Psi >$,$\nabla_a < \psi \mid T^a_b \mid \psi >= 0$

Causality For variations of g_{ab} which vanish in the causal past $J^-(x)$ of x and are sufficiently well behaved to make possible the identification of the new and old state spaces, such that one can meaningfully identify a particular state on the altered and unaltered metrics, the stress tensor $< \psi \mid T_{ab} \mid \psi >$ at x for a particular "in-state" is unchanged. Similarly for a particular "out-state", $< \psi_{out} \mid T_{ab} \mid \psi_{out} >$ at a point $x \in M$ depends only on the metric g_{ab} in the causal future $J^+(x)$ of x.

Consistency The formal expression for T_{ab} , that is the one derived from the classical action should be valid for calculating $< A \mid T_{ab} \mid B >$ whenever $< A \mid B >= 0$.

Reduce to normal ordering in Minkowski space The expression for $< \psi \mid T_{ab} \mid \psi >$ should agree with the normal ordered expression in Minkowski space.

No higher order terms Consider a sequence $\{(g_{ab})_i\}$ of C^∞ metrics that agree outside a fixed compact region R, and are such that the components of $(g_{ab})_i$ and the derivatives of theses components up to forth order in a fixed chart converge uniformly to a C^∞ metric $\tilde{g}_{ab}$ and its derivatives up to forth order respectively.For such a sequence we require that for fixed in- or out-states $< \psi_{in} \mid T_{ab} \mid \psi_{in} >$ or $\{< \psi_{out} \mid T_{ab} \mid \psi_{out} >_i\}$ and its derivatives up to third order converges pointwise up to third order respectively.

To see how these conditions constrain the form of $< \psi \mid T_{ab} \mid \psi >$, we consider two different stress energy operators T_{ab} and $\tilde{T}_{ab}$ and consider the difference U_{ab} between them

$$U_{ab} \equiv T_{ab} - \tilde{T}_{ab} \tag{107}$$

Then if both T_{ab} and $\tilde{T}_{ab}$ satisfy the consistency axiom their matrix elements between two orthogonal states must agree. However this implies that

$$U_{ab} = c_{ab} I \tag{108}$$

where c_{ab} is a c-number and I is the identity operator. Then if they both satisfy the causality axiom $c_{ab}(x)$ must be a functional of only the metric in $J^-(x)$ but simultaneously be a functional only of the metric in $J^+(x)$, thus it can only depend on the metric at x. Further if both stress tensors are covariantly conserved their difference is too.

$$\nabla^a c_{ab} = 0 \tag{109}$$

If the stress tensors reduce to the normal ordered expressions when curvature vanishes, $c_{ab} = 0$ must be zero when the curvature is zero. Thus a two stress energy operators satisfying the first four axioms can differ by at most a conserved local curvature term.

7.2 The covariant point-splitting method

Let us now consider the covariant point splitting renormalization method as we shall use some objects constructed here later. Instead of the classical stress tensor $T_{ab}(x)$, one considers the corresponding bi-tensor $T_{ab}(x,y)$ where the products in the same point are replaced by symmetric products in two different points. That is for example $T_{ab}(x) = \nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}((\nabla\phi)^2$ is replaced by

$$\begin{aligned} T_{ab}(x,y) = &\frac{1}{2}(\nabla_a\phi(x)\nabla_b\phi(y) + \nabla_a\phi(y)\nabla_b\phi(x)) \\ &-\frac{1}{4}g_{ab}(\nabla_a\phi(x)\nabla_b\phi(y) + \nabla_a\phi(y)\nabla_b\phi(x)) \end{aligned} \tag{110}$$

The bi-tensor transforms as a tensor in both x an y. The Covariant point-splitting method was developed by Christensen[14]. It takes it starting point in the stress-bi tensor and to construct the quantum operator from this bi-distribution we first consider

$$G(x,y) \equiv \frac{< out \mid \phi(x)\phi(y) + \phi(y)\phi(x) \mid in >}{< out \mid in >} \tag{111}$$

$G(x,y)$ is a solution to the equation of motion in each variable. From this one then constructs the formal expression for the stress bi-tensor as

$$\begin{aligned} T_{ab}(x,y) = &\frac{1}{2}(\nabla^x_a\nabla^y_b G(x,y) + \nabla^y_a\nabla^x_b G(x,y)) \\ &-\frac{1}{4}g_{ab}(\nabla^x_a\nabla^y_b G(x,y) + \nabla^y_a\nabla^x_b G(x,y)) \equiv \\ \equiv &\frac{1}{2}\frac{< out \mid \nabla_a\phi(x)\nabla_b\phi(y) + \nabla_a\phi(y)\nabla_b\phi(x) \mid in >}{< out \mid in >} \\ -&\frac{1}{4}g_{ab}\frac{< out \mid \nabla_a\phi(x)\nabla_b\phi(y) + \nabla_a\phi(y)\nabla_b\phi(x) \mid in >}{< out \mid in >} \end{aligned} \tag{112}$$

The limit $lim_{x\to y}T_{ab}(x,y)$ does however not exist and must be renormalized. The prescription of christensen however yields direction dependent terms, which must be removed, for example by averaging over all directions, if one seeks to recover an object corresponding to the classical stress tensor.

7.3 Trace anomaly

An important issue related to our treatment of black hole evaporation, as we shall see below, is the trace of the stress tensor. To find its relation to the renormalized stress tensor we again consider the bi-distribution $G(x,y)$ and the following conjecture.

Conjecture on singularity structure The bi-distribution $G(x,y)$ defined by

$$G(x,y) =< \phi(x)\phi(y) + \phi(y)\phi(x) > \tag{113}$$

can, whenever x and y are not null separated and y belongs to the normal convex neighborhood of x, be realized as a function with the singularity structure as $x \to y$ of the form

$$G(x,y) = \frac{2}{(4\pi)^2} z(\frac{2}{\sigma} + vln(\sigma) + w) \tag{114}$$

where z,v and w are smooth functions of x and y and σ is Synges world function.

G(x,y) is a solution to the equation of motion in both x and y. The bi-distribution $G(x,y)$ for a conformally invariant scalar field in minkowski space which is given by $G^M(x,y) = \frac{1}{4\pi^2\sigma}$. Thus the above conjecture is true for conformally invariant fields in Minkowski space. A function that is a solution to the wave equation in x but not necessarily in y and has the singularity structure above will be called a *Hadamard elementary solution*. The functions $z(x,y)$ and $v(x,y)$ in a Hadamard solution are symmetric in x and y, but $w(x,y)$ need not be. v and w can be expanded in powers of σ as

$$v(x,y) = \Sigma_{n=0}^{\infty} v_n(x,y)\sigma^n \tag{115}$$

$$w(x,y) = \Sigma_{n=0}^{\infty} w_n(x,y)\sigma^n \tag{116}$$

Wald[15] gives a construction of a renormalized stress tensor satisfying the first four axioms for a scalar field ϕ satisfying the equation $\nabla^2\phi - \frac{1}{\xi}R\phi = 0$. Consider $G(x,y)$ for a conformally invariant field in a general curved spacetime and define

$$G^R = G(x,y) - G^L(x,y) \tag{117}$$

where $G^L(x,y)$ is the Hadamard solution defined by setting w_0 equal to what it is for $G(x,y)$ in minkowski space, that is zero. The corresponding stress bi tensor

$$\begin{aligned} T_{ab}(x,y) =& \frac{1}{2}(\nabla_a^x \nabla_b^y G^R(x,y) + \nabla_a^y \nabla_b^x G^R(x,y)) - \frac{1}{2} g_{ab}((\nabla^x)^c \nabla_c^y G^R(x,y)) \\ & - \frac{1}{2\xi}(\frac{1}{2} g_{ab} R + R_{ab}) G^R(x,y) + \frac{1}{2\xi} g_{ab}((\nabla^x)^2 G^R(x,y) + (\nabla^y)^2 G^R(x,y)) \\ & - \frac{1}{2\xi}(\nabla_a^x \nabla_b^x G^R(x,y) + \nabla_a^y \nabla_b^y G^R(x,y)) \end{aligned} \tag{118}$$

will then have a well defined coincidence limit $T_{ab}(x) = lim_{y\to x} T_{ab}(x,y)$. This construction satisfies the consistency, causality and reduction to normal ordering axioms however it will not be covariantly conserved. To see this we use Synges theorem[14] to get

$$\nabla_a^x T_b^a(x) = lim_{y\to x}(\nabla_a^x T_b^a(x,y) + \nabla_a^y T_b^a(x,y)) \tag{119}$$

using that $G^R(y,x)$ is a solution to the equation of motion in x we find that

$$\nabla_a^x T_b^a(x) = lim_{y\to x}(\nabla_b^x (\nabla^y)^2 G^R(x,y) - \frac{1}{\xi} R(y) G^R(x,y)) \tag{120}$$

Thus $T_{ab}(x)$ is conserved only if $G^R(x,y)$ satisfies the equation of motion in y. But since $(\nabla^y)^2 G(x,y) - \frac{1}{\xi}R(y)G(x,y) = 0$ and $((\nabla^y)^2 - \frac{1}{\xi}R(y))(G^L(x,y) - \frac{2}{(4\pi)^2}zw) = -((\nabla^x)^2 - \frac{1}{\xi}R(x))(\frac{2}{(4\pi)^2}zw)$ due to symmetry and the G^L being a solution to the equations of motion in x, we have $((\nabla^y)^2 - \frac{1}{\xi}R(y))G^R(x,y) = -((\nabla^y)^2 - \frac{1}{\xi}R(y))G^L(x,y) = \frac{2}{(4\pi)^2}(((\nabla^x)^2 - \frac{1}{\xi}R(x))zw - ((\nabla^y)^2 - \frac{1}{\xi}R(y))zw)$. Now

$$\nabla^x_a T^a_b(x) = \frac{1}{2(4\pi)^2} lim_{y\to x} \nabla^x_b((\nabla^x)^2 zw - (\nabla^y)^2 zw) \tag{121}$$

We then consider the expansion $w = w_1\sigma + \mathcal{O}(\sigma^2)$. The coincidence limits of the derivatives of σ are

$$[\nabla^x_a \sigma(x,y)]_c = [\nabla^y_a \sigma(x,y)]_c = 0$$

$$[\nabla^x_b \nabla^x_a \sigma(x,y)]_c = [\nabla^y_b \nabla^y_a \sigma(x,y)]_c = g_{ab}$$

$$[\nabla^y_b \nabla^x_a \sigma(x,y)]_c = [\nabla^x_b \nabla^y_a \sigma(x,y)]_c = -g_{ab}$$

$$[\nabla^x_c \nabla^x_b \nabla^x_a \sigma(x,y)]_c = [\nabla^y_c \nabla^x_b \nabla^x_a \sigma(x,y)]_c = [\nabla^y_c \nabla^y_b \nabla^x_a \sigma(x,y)]_c = [\nabla^y_c \nabla^y_b \nabla^y_a \sigma(x,y)]_c = 0 \tag{122}$$

Thus the only terms of the expansion that are nonzero after differentiation are those where two derivatives act on $\sigma(x,y)$ and further $[z]_c = 1$ and $[\nabla_a z]_c = 0$, so

$$\nabla^x_a T^a_b(x) = \frac{3}{(4\pi)^2}[\nabla^y_b w_1] - [\nabla^x_b w_1] \tag{123}$$

Synges theorem gives $\nabla^x_b w_1 = [\nabla^y_b w_1] + [\nabla^x_b w_1]$ and thus

$$\nabla^x_a T^a_b(x) = \frac{3}{(4\pi)^2}[\nabla^x_a [w_1] - 2[\nabla^x_a w_1] \tag{124}$$

the coincidence limits of w_1 and v_1 are $[w_1]_c = -\frac{3}{2}[v_1]_c$ and $[\nabla^x_b w_1]_c = -\frac{4}{3}[\nabla^x_b v_1]_c$ and therefore

$$\nabla^x_a T^a_b(x) = \frac{3}{(4\pi)^2}(-\frac{3}{2}\nabla^x_b[v_1]_c + \frac{8}{3}[\nabla^x_b w_1]_c) \tag{125}$$

but v_1 is symmetric in x and y and therefore $2[\nabla^x_b v_1]_c = \nabla^x_b[v_1]_c$ and thus

$$\nabla^x_a T^a_b(x) = -\frac{1}{2(4\pi)^2}\nabla^x_b[v_1]_c \tag{126}$$

The coincidence limit $[v_1]_c = \frac{1}{360}(C^{abcd}C_{abcd} + R^{ab}R_{ab} - \frac{1}{3}R^2 + \nabla^2 R)$ and hence T_{ab} is not covariantly conserved. We can however add the term $\frac{1}{2(4\pi)^2}g_{ab}[v_1]_c$ to the previous construction to make it covaraintly conserved. In the case of a conformally invariant field, that is $\xi = 6$ the original construction is traceless, and the last correction thus gives the it a trace.

$$T^a_a(x) = \frac{2}{(4\pi)^2}[v_1]_c = \frac{1}{2880\pi^2}(C^{abcd}C_{abcd} + R^{ab}R_{ab} - \frac{1}{3}R^2 + \nabla^2 R) \tag{127}$$

By the above discussion any renormalization prescription satisfying the first four axioms must therefore have this trace up to the addition of the trace of a local conserved curvature term. The question is then whether there is such a local conserved curvature term that cancels the trace of $T_{ab}(x)$. While lacking in a definite answer we can consider the following argument. Upon a weyl rescaling of the metric $g_{ab} \to \Omega^2 g_{ab}$ the trace scales as Ω^{-4}. If we assume that a local conserved curvature term must vary analytically with Ω^{-1}, the coefficients of each power of Ω^{-1} must be separately conserved, and thus we may limit our search to those with dimension $(length)^{-4}$. While we do not know all such local conserved curvature terms we can consider the two known A and B obtained as

$$A_{ab} = \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}} (\sqrt{-g} R_{ab} R^{ab}) = -\frac{1}{2} g_{ab} (R_{cd} R^{cd} + \nabla^2 R) + \nabla_a \nabla_b R - \nabla^2 R_{ab} + 2 R^{cd} R_{cadb} \tag{128}$$

$$B_{ab} = \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}} (\sqrt{-g} R^2) = -\frac{1}{2} g_{ab} R^2 + 2 R R_{ab} - 2 g_{ab} \nabla^2 R + 2 \nabla_a \nabla_b R \tag{129}$$

These have the trace $B^a_a = 3A^a_a = -6\nabla^2 R$, and can therefore only cancel the $\nabla^2 R$ term of 130. The trace, provided the above argument is correct, is therefore on the form

$$T^a_a(x) = \frac{2}{(4\pi)^2} [v_1]_c = \frac{1}{2880\pi^2} (C^{abcd} C_{abcd} + R^{ab} R_{ab} - \frac{1}{3} R^2 + C \nabla^2 R) \tag{130}$$

where C is an arbitrary constant. The stress tensor will be compatible with the no higher order term axiom only if $C = 0$.

The form of the trace anomaly in 2 dimensions was motivated by Davies[16] by the following argument. Assume the metric is of the form

$$ds^2 = C(u,v) du dv \tag{131}$$

Then the only non-vanishing christoffel symbols are

$$\Gamma^u_{uu} = C^{-1} \partial_u C \qquad \Gamma^v_{vv} = C^{-1} \partial_v C \tag{132}$$

In this case the covariant conservation equation for the stress tensor is

$$\nabla^\mu T_{\mu v} = 0 \Rightarrow \partial_v T_{uu} + \frac{1}{4} C \partial_u T^\mu_\mu = 0 \tag{133}$$

Then if we assume that T^μ_μ is a non-vanishing local scalar function of dimension $(length)^{-2}$, it must consist of terms quadratic in derivatives of C, and if we assume there is no length scale in the theory T^μ_μ must be of degree -1 in C, and hence

$$T^\mu_\mu = a C^{-2} \partial_u \partial_v C + b C^{-3} \partial_u C \partial_v C \tag{134}$$

Then in the 2 dimensional Milne spacetime which is a reparameterization of Minkowski space $C = e^{u+v}$, and since we wish the trace to vanish here we set

$a = -b$. Thus $T^\mu_\mu = aC^{-2}\partial_u\partial_v C - aC^{-3}\partial_u C\partial_v C$. Frome section 6 we had that $R = -C^{-2}\partial_u\partial_v C + C^{-3}\partial_u C\partial_v C$, and hence T^μ_μ is proportional to R. Further $\frac{1}{4}C\partial_u T^\mu_\mu = -\frac{1}{2}a\partial_v(C^{\frac{1}{2}}\partial_u^2 C^{-\frac{1}{2}})$ and then the covariant conservation equation can be integrated to

$$T_{uu} = -\frac{1}{2}aC^{\frac{1}{2}}\partial_u^2 C^{-\frac{1}{2}} + f(u) \tag{135}$$

Here $f(u)$ is an arbitrary function of u. By applying the point splitting arguments to a conformally invariant field in two dimensions Davies, Fulling and Unruh[17] derived the trace of the stress tensor in this case to be

$$T^\mu_\mu = \frac{1}{24\pi}R \tag{136}$$

Hence this determines the constant a.

7.4 The stress tensor and hawking radiation

Christensen and Fulling[18] related the hawking effect to the stress tensor in the following way. Once again we consider the static spherically symmetric 2-dimensional metric

$$ds^2 = -C(r)dt^2 + C(r)^{-1}dr^2 \tag{137}$$

The crucial assumption made is that $< T_{ab} >$ is time independent. Then the covariant conservation $\nabla_a < T^a_b >$ gives

$$\begin{aligned} \nabla_a < T^a_r >= \partial_a < T^a_r > + \Gamma^b_{ar} < T^a_b >= \\ \partial_r < T^r_r > + \Gamma^r_{rr} < T^r_r > + \Gamma^t_{rt} < T^t_t >= \\ \partial_r < T^r_r > + \frac{1}{2}\frac{\partial C}{\partial r}C(r)^{-1}(< T^r_r > - < T^t_t >) = 0 \end{aligned} \tag{138}$$

which implies since $< T^a_a >=< T^r_r > + < T^t_t >$

$$\partial_r(C(r) < T^r_r >) = \frac{1}{2}\frac{\partial C}{\partial r}(< T^a_a >) = 0 \tag{139}$$

and this integrates to

$$C(r) < T^r_r >= \frac{1}{2}\int_{r_0}^r \frac{\partial C}{\partial r} < T^a_a > + Q = 0 \tag{140}$$

Denoting $H(r) \equiv \frac{1}{2}\int_{r_0}^r \frac{\partial C}{\partial r}(< T^a_a >)$ we then have

$$< T^r_r >= \frac{H(r)}{C(r)} + \frac{Q}{C(r)} \tag{141}$$

$$< T^t_t >=< T^a_a > - \frac{H(r)}{C(r)} - \frac{Q}{C(r)} \tag{142}$$

further

$$\nabla_a < T^a_t >= \partial_a < T^a_t > + \Gamma^b_{at} < T^a_b >=$$
$$\partial_r < T^r_t > + \Gamma^r_{tt} < T^t_r > + \Gamma^t_{rt} < T^r_t >=$$
$$\partial_r < T^r_t > + \frac{1}{2}\frac{\partial C}{\partial r}(C(r)^{-1} < T^r_t > - C(r) < T^t_r >) = 0 \tag{143}$$

but since $< T^r_t >= C(r)^2 < T^t_r >$, we have

$$\partial_r < T^r_t >= 0 \Rightarrow < T^r_t >= K \tag{144}$$

where K is a constant. We then change to the new radial coordinate $r^* = \int^r C(r)^{-1} dr$, where $< T^{r^*}_{r^*} >=< T^r_r >$ and $< T^{r^*}_t >= - < T^t_{r^*} >= C(r)^{-1} < T^r_t >$. We can thus express the stress tensor in r^*, t coordinates as

$$T^a_b = \begin{pmatrix} < T^a_a > - \frac{H(r)}{C(r)} - \frac{Q}{C(r)} & -C(r)^{-1}K \\ C(r)^{-1}K & \frac{H(r)}{C(r)} + \frac{Q}{C(r)} \end{pmatrix}$$

The next assumption is that the stress tensor should be given by $< T^{r^*}_{r^*} >= - < T^t_t >=< T^{r^*}_t >= - < T^t_{r^*} >= \frac{\pi}{12}(kT)^2$, corresponding to blackbody radiation.

$$T^a_b = \frac{\pi}{12}(kT)^2 \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$$

Then if $< T^a_a >= 0$ everywhere the only way to have the desired form is to choose $Q = K$, however with this choice $T_{uu} = \frac{1}{4}(< T_{r^*r^*} > + < T_{tt} > -2 < T_{r^*t} >) = -2C(r)^{-1}K$ and $lim_{r \to r_h} C^{-2}(r) \mid T_{uu}(r) \mid = \infty$ where $u = t - r^*$. Now if we allow $< T^a_a > \neq 0$ we find that for the stress tensor to have the form 7.4 it is required that $< T^a_a > - \frac{H(r)}{C(r)} - \frac{Q}{C(r)} = -C(r)^{-1}K$ and $\frac{H(r)}{C(r)} + \frac{Q}{C(r)} = C(r)^{-1}K$, thus the trace must again be zero. Looking at the limit of $r \to \infty$ where $T^t_t \to < T^a_a > (\infty) - \frac{H(\infty)}{C(\infty)} - \frac{Q}{C(\infty)}$ and $T^t_{r^*} \to -\frac{K}{C(\infty)}$ we may require the stress tensor to be of the form 7.4 only at infinity. Then provided $C(\infty) \neq 0$, $H(\infty) = \frac{1}{2}\int_{r_0}^{\infty} \frac{\partial C}{\partial r}(< T^a_a >) = K - Q$. Further we must have that $H(\infty) + Q = K = \frac{\pi}{12}(kT)^2 C(\infty)$. We now change the view and assume the form of the trace is $< T^a_a >= \frac{1}{24\pi} R$, where in this case $R = -\frac{\partial^2 C(r)}{\partial r^2}$ we then find

$$< T^a_a >= -\frac{1}{24\pi}\frac{\partial^2 C(r)}{\partial r^2} \tag{145}$$

and thus

$$H(r) = -\frac{1}{96\pi}((\frac{\partial C}{\partial r}(r))^2 - (\frac{\partial C}{\partial r}(r_0))^2) \tag{146}$$

then assuming $\frac{\partial C}{\partial r}(\infty) = 0$, as in the schwarschild case, $H(\infty) = \frac{1}{96\pi}(\frac{\partial C}{\partial r}(r_0))^2$. So if we assume the trace anomaly of Davies, Fulling and Unruh we find that

$$\frac{1}{96\pi}(\frac{\partial C}{\partial r}(r_0))^2 + Q = \frac{\pi}{12}(kT)^2 C(\infty) \tag{147}$$

Now if we assume that $Q = 0$ and $r_0 = r_h$, r_h being the radial coordinate of the event horizon, we find that

$$\frac{1}{2\pi}\frac{\kappa}{\sqrt{C(\infty)}} = (kT) \tag{148}$$

which is of the same functional form as the temperature of the hawking radiation derived above.

7.5 Connection between Weyl invariance and trace of the stress tensor

Let us consider a classical field theory where the action $S(g_{ab})$ is invariant under Weyl rescalings $g_{ab} \to \tilde{g}_{ab} = \Omega^2 g_{ab}$. Then functionally differentiating $S(\tilde{g}_{ab})$ with respect to $\tilde{g}_{ab}$ around g_{ab} gives

$$S[\tilde{g}_{ab}] = S[g_{ab}] + \int_M \frac{\delta S[\tilde{g}_{ab}]}{\delta \tilde{g}^{ab}} \delta \tilde{g}_{ab} \tag{149}$$

but since $\delta \tilde{g}_{ab} = -2\tilde{g}_{ab}\Omega^{-1}\delta\Omega$ this yields

$$S[\tilde{g}_{ab}] = S[g_{ab}] + \int_M \sqrt{-\tilde{g}} T^a_a[\tilde{g}_{ab}]\Omega^{-1}\delta\Omega \tag{150}$$

and hence

$$T^a_a[\tilde{g}_{ab}] = -\frac{\Omega}{\sqrt{-\tilde{g}}}\frac{\delta S[\tilde{g}_{ab}]}{\delta\Omega} |_{\Omega=1} \tag{151}$$

Thus if the classical action is invariant under weyl rescalings the classical stress tensor is traceless. Therefore the trace anomaly is sometimes called the weyl anomaly.

7.6 The effective action for the trace anomaly

One approach to the issue of back reaction has been to create a classical theory associated to the particular quantum theory where the classical stress tensor is taken to equal the expectation value of the quantum mechanical stress tensor. The action corresponding to this classical tensor is called the *effective action* W and is defined by

$$\frac{2}{\sqrt{-g}}\frac{\delta W}{\delta g^{ab}} \equiv T_{ab} \tag{152}$$

We now consider the effective action associated to the trace anomaly for N conformally invariant scalar fields in two dimensions $< T^a_a >= \frac{N}{24\pi}\sqrt{-g}(R(x) + const)$. We then wish to find an action S_P that reproduces this trace.

$$-g^{ab}\frac{2}{\sqrt{-g}}\frac{\delta S_P}{\delta g^{ab}} = g^{ab} < T_{ab} >= \frac{N}{24\pi}(R(x) + const) \tag{153}$$

It is shown [see Appendix] that the action S_P constructed by Polyakov[19][20] given by

$$S_P = \frac{N}{96\pi}\int d^2x d^2y \sqrt{-g(x)}\sqrt{-g(y)}R(x)R(y)G(x,y) + const\int d^2x\sqrt{-g(x)} \quad (154)$$

where $G(x,y)$ is a Green's function for ∇^2, gives this result. We shall call it the *polyakov action.* The variation of this action with respect to the metric is

$$\delta S_P[g_{ab}] = \frac{N}{96\pi}\int_M \sqrt{-g}(-\frac{1}{2}g_{ab}(\nabla\varphi)^2 + \nabla_a\varphi\nabla_b\varphi + 2\nabla_a\nabla_b\varphi - 2g_{ab}\nabla^2\varphi - const\frac{1}{2}g_{ab})\delta g^{ab} \quad (155)$$

where $\varphi = \int_M \sqrt{-g}R(y)G(x,y)$

The green function $G(x,y)$ can be decomposed into two parts $G_0(x,y)$ and $G_H(x,y)$, the *fundamental solution* $G_0(x,y)$ satisfies $\nabla^2 G_0(x,y) = \delta(x,y)$ and the *homogeneous* part $G_H(x,y)$ satisfying $(\nabla^y)^2 G_H(x,y) = (\nabla^x)^2 G_H(x,y) = 0$. The fundamental solution in a general curved spacetime is not known but it is often conjectured that it has the form of a symmetric Hadamard elementary solution. Further the *homogeneous* part $G_H(x,y)$ is only determined up to a function $f(x,y)$ satisfying $(\nabla^y)^2 f(x,y) = (\nabla^x)^2 f(x,y) = 0$, that is any harmonic function, unless we impose boundary conditions. From above it follows that the Polyakov action is not invariant under Weyl rescalings. Now let us examine φ a little further. In conformal gauge we can reexpress it as

$$\varphi = \int_M -2\sqrt{-\tilde{g}}\tilde{\nabla}^2\rho G(x,y) \quad (156)$$

And then by Greens identity

$$\int_M \sqrt{-\tilde{g}}\tilde{\nabla}^2\rho G = \rho - \int_{\partial M}(\rho\tilde{\nabla}_a G - G\tilde{\nabla}_a\rho)n^a \quad (157)$$

To simplify notation we introduce $A_\partial \equiv \int_{\partial M}(\rho\tilde{\nabla}_a G - G\tilde{\nabla}_a\rho)n^a$ and thus

$$\varphi = -2\rho + 2A_\partial \quad (158)$$

We note that $\tilde{\nabla}^2 A_\partial = 0$ for all $x \notin \partial M$

We now look at a second action S_c that also reproduces the trace anomaly . We can motivate it in the following way. In conformal gauge the trace anomaly is $T^a_a = \frac{N}{24\pi}\sqrt{-g}R = -\frac{N}{12\pi}\sqrt{-\tilde{g}}\tilde{\nabla}^2\rho$. Then we seek the effective action S_c for which

$$\delta S_c = -\frac{N}{12\pi}\int_M \sqrt{-\tilde{g}}\tilde{\nabla}^2\rho\delta\rho \quad (159)$$

and find that

$$S_c = \frac{N}{24\pi}\int_M \sqrt{-\tilde{g}}(\tilde{\nabla}\rho)^2 \quad (160)$$

We then want to compare it to the polyakov action. We therefore reexpress it by inserting $\int_M \sqrt{-g(\tilde{y})}\tilde{\nabla}^2 G(x-y) = \int_M \delta(x-y) = 1$ and partially integrating, which yields

$$\begin{aligned}
S_c =& \frac{N}{24\pi}\int_M \sqrt{-\tilde{g}}(\tilde{\nabla}\rho)^2 \int_M \delta(x-y) = \\
&-\frac{N}{24\pi}\int_M \sqrt{-\tilde{g}}\tilde{\nabla}^2\rho \int_M \delta(x-y)\rho(y) + \frac{N}{24\pi}\int_{\partial M}\sqrt{-\tilde{g}}\rho\tilde{\nabla}_a\rho n^a = \\
&= -\frac{N}{24\pi}\int_M \sqrt{-\tilde{g}}\tilde{\nabla}^2\rho \int_M \sqrt{-g(\tilde{y})}G(x-y)\tilde{\nabla}^2\rho(y) \\
&+ \frac{N}{24\pi}\int_M \sqrt{-\tilde{g}}\tilde{\nabla}^2\rho A_\partial + \frac{N}{24\pi}\int_{\partial M}\sqrt{-\tilde{g}}\rho\tilde{\nabla}_a\rho n^a \\
&= -\frac{N}{96\pi}\int_M \sqrt{-\tilde{g}}\varphi\tilde{\nabla}^2\varphi + \frac{N}{24\pi}\int_M \sqrt{-\tilde{g}}\tilde{\nabla}^2\rho A_\partial + \frac{N}{24\pi}\int_{\partial M}\sqrt{-\tilde{g}}\rho\tilde{\nabla}_a\rho n^a
\end{aligned} \tag{161}$$

Comparing to the polyakov action in conformal gauge

$$S_P = -\frac{N}{96\pi}\int_M \sqrt{-\tilde{g}}\varphi\tilde{\nabla}^2\varphi \tag{162}$$

we see that apart from boundary terms the difference is exactly the terms coming from he non-local function A_∂, and thus the two actions are identical if $\partial M = 0$

8 The CGHS-model

The most general spherically symmetric metric can be expressed as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + \frac{1}{\lambda}e^{-2\phi}d\Omega^2 \tag{163}$$

where ϕ is a scalar and $\frac{1}{\lambda}e^{-2\phi}$ is the area of the two-sphere defined by constant x^μ and x^ν. When we limit dynamics to such spherically symmetric metrics einsteins field equations for $g_{\mu\nu}$ can be reformulated as

$$G_{\mu\nu} = 2\nabla_\mu\nabla_\nu\phi - 2\nabla_\mu\phi\nabla_\nu\phi + 3g_{\mu\nu}(\nabla\phi)^2 - 2g_{\mu\nu}\nabla^2\phi - \lambda^2 g_{\mu\nu}e^{2\phi} \tag{164}$$

$$G_{\Theta\Theta} = sin^2(\Theta)G_{\Phi\Phi} = \frac{1}{\lambda^2}e^{-2\phi}((\nabla\phi)^2 - \nabla^2\phi - \frac{1}{2}R) \tag{165}$$

where Θ and Φ are angular coordinates For a boundaryless spacetime M these equations can be derived by variation of $g_{\mu\nu}$ and ϕ from the two-dimensional action

$$\frac{1}{2\pi}\int_M \sqrt{-g}e^{-2\phi}(R + 2(\nabla\phi)^2 + 2\lambda^2 e^{2\phi}) \tag{166}$$

The scalar ϕ is often referred to as the *dilaton*. A number of dilaton gravity theories has been constructed where the relation between the two-dimensional metric and the dilaton field is different. The one we shall investigate here is the CGHS-model, which is an exactly solvable 2-dimensional dilation gravity model originally constructed by Callans,Giddings,Harvey and Strominger[21]. The action is given by

$$S_{CGHS} = \frac{1}{2\pi}\int_M \sqrt{-g}(e^{-2\phi}(R + 4(\nabla\phi)^2 + 4\lambda^2) - \frac{1}{2}\sum_{i=1}^{N}(\nabla f_i)^2) \tag{167}$$

where g_{ab} is the metric ϕ a dilaton, f_i matterfields and λ is a constant. The gauge group is assumed to be $Diff(M)$.

The equations of motion are, assuming $\delta g_{ab} = 0$, $\delta\phi = 0$, $\delta f_i = 0$ and $\nabla_c \delta g_{ab} = 0$ at ∂M

$$\begin{aligned}\frac{\delta S_{CGHS}}{\delta g_{ab}} =& \frac{1}{2\pi}\sqrt{-g}(e^{-2\phi}(2g_{ab}((\nabla\phi)^2 - \lambda^2 - \nabla^2\phi) + 2\nabla_a\nabla_b\phi) \\ &+ \frac{1}{4}\sum_{i=1}^{N}(g_{ab}(\nabla f_i)^2 - 2\nabla_a f_i \nabla_b f_i) = 0\end{aligned} \tag{168}$$

$$\frac{\delta S_{CGHS}}{\delta\phi} = \frac{1}{2\pi}\sqrt{-g}(-8e^{-2\phi}(\frac{1}{4}R + \lambda^2 - (\nabla\phi)^2 + \nabla^2\phi)) = 0 \tag{169}$$

$$\frac{\delta S_{CGHS}}{\delta f_i} = \frac{1}{2\pi}\sqrt{-g}\nabla^2 f_i = 0 \tag{170}$$

We reexpress the equations of motion in conformal gauge

$$\begin{aligned}\frac{\delta S_{CGHS}}{\delta g_{ab}} =& \frac{1}{2\pi}e^{2\rho}\sqrt{-\tilde{g}}(e^{-2\phi}(2\tilde{g}_{ab}((\tilde{\nabla}\phi)^2 - \tilde{\nabla}^2\phi) - 2\tilde{g}_{ab}e^{2\rho}\lambda^2) + 2\tilde{\nabla}_a\tilde{\nabla}_b\phi \\ &-2(\delta_a^\mu\delta_b^\nu + \delta_a^\nu\delta_b^\mu - \tilde{g}_{ab}\tilde{g}^{\mu\nu})\tilde{\nabla}_\nu\rho\tilde{\nabla}_\mu\phi) + T_{ab} = 0\end{aligned} \tag{171}$$

$$\frac{\delta S_{CGHS}}{\delta\phi} = \frac{1}{2\pi}\sqrt{-\tilde{g}}(-8e^{-2\phi}(-\frac{1}{2}\tilde{\nabla}^2\rho + \lambda^2 e^{2\rho} - (\tilde{\nabla}\phi)^2 + \tilde{\nabla}^2\phi)) = 0 \tag{172}$$

$$\frac{\delta S_{CGHS}}{\delta f_i} = \frac{1}{2\pi}e^{2\rho}\sqrt{-\tilde{g}}\tilde{\nabla}^2 f_i \tag{173}$$

where $T_{ab} = -\frac{1}{4}\sum_{i=1}^{N}(\tilde{g}_{ab}(\tilde{\nabla}f_i)^2 - 2\tilde{\nabla}_a f_i \tilde{\nabla}_b f_i)$. In conformal gauge light cone coordinates

$$\frac{\delta S_{CGHS}}{\delta g_{+-}} = \frac{1}{4\pi}e^{2\rho}((\partial_+\partial_- e^{-2\phi} - e^{2(\rho-\phi)}\lambda^2) = 0 \tag{174}$$

$$\frac{\delta S_{CGHS}}{\delta g_{\pm\pm}} = \frac{1}{4\pi}e^{2\rho}(e^{-2\phi}(2\partial_\pm^2\phi - 4\partial_\pm\rho\partial_\pm\phi) - T_{\pm\pm}) = 0 \tag{175}$$

$$\frac{\delta S_{CGHS}}{\delta\phi} = -\frac{2}{\pi}(-2e^{-2\phi}\partial_+\partial_-\rho - \lambda^2 e^{2(\rho-\phi)} + \partial_+\partial_- e^{-2\phi} + e^{-2\phi}2\partial_+\partial_-\phi) = 0 \tag{176}$$

$$\frac{\delta S_{CGHS}}{\delta f_i} = -\frac{1}{\pi}e^{2\rho}\partial_+\partial_- f_i = 0 \tag{177}$$

8.1 Conserved currents and conformal invariance of the CGHS-model

The CGHS-action in conformal gauge is

$$S_{CGHS} = \frac{1}{2\pi}\int_M \sqrt{-\tilde{g}}(e^{-2\phi}(-2\tilde{\nabla}^2\rho + 4(\tilde{\nabla}\phi)^2 + 4e^{2\rho}\lambda^2) - \frac{1}{2}\sum_{i=1}^{N}(\tilde{\nabla}f_i)^2) \tag{178}$$

partial integration yields

$$S_{CGHS} = \frac{1}{2\pi}\int_M \sqrt{-\tilde{g}}(e^{-2\phi}(-4\tilde{g}^{\mu\nu}\tilde{\nabla}_\nu\phi\tilde{\nabla}_\mu\rho + 4\tilde{g}^{\mu\nu}\tilde{\nabla}_\nu\phi\tilde{\nabla}_\mu\phi + 4e^{2\rho}\lambda^2) - \frac{1}{2}\sum_{i=1}^{N}(\tilde{\nabla}f_i)^2) + \frac{1}{2\pi}\int_{\partial M}\sqrt{-\tilde{g}}e^{-2\phi}\tilde{\nabla}_\mu\rho n^\mu \tag{179}$$

where n^μ is the normal to ∂M. Then if the spacetime has no boundary or if $\tilde{\nabla}_\mu\rho = 0$ on ∂M, so that there is a functional derivative of the action, and we consider the variation $\delta\rho = \delta\phi = \epsilon(x)e^{2\phi}$ we see

$$\delta S_{CGHS} = \int_M \frac{\partial L_{CGHS}}{\partial\phi}\epsilon(x)e^{2\phi} + \frac{\partial L_{CGHS}}{\partial\tilde{\nabla}_\mu\phi}\tilde{\nabla}_\mu(\epsilon(x)e^{2\phi}) + \frac{\partial L_{CGHS}}{\partial\rho}\epsilon(x)e^{2\phi} + \frac{\partial L_{CGHS}}{\partial\tilde{\nabla}_\mu\rho}\tilde{\nabla}_\mu(\epsilon(x)e^{2\phi}) = 0 \tag{180}$$

where $L_{CGHS} = \sqrt{-\tilde{g}}(e^{-2\phi}(-4\tilde{g}^{\mu\nu}\tilde{\nabla}_\nu\phi\tilde{\nabla}_\mu\rho + 4\tilde{g}^{\mu\nu}\tilde{\nabla}_\nu\phi\tilde{\nabla}_\mu\phi + 4e^{2\rho}\lambda^2) - \frac{1}{2}\sum_{i=1}^{N}(\tilde{\nabla}f_i)^2)$ but since by the equations of motion

$$\delta S_{CGHS} = \int_M \frac{\partial L_{CGHS}}{\partial\phi}e^{2\phi} + \frac{\partial L_{CGHS}}{\partial\tilde{\nabla}_\mu\phi}\tilde{\nabla}_\mu(e^{2\phi}) = 0 \tag{181}$$

and

$$\delta S_{CGHS} = \int_M \frac{\partial L_{CGHS}}{\partial\rho}e^{2\phi} + \frac{\partial L_{CGHS}}{\partial\tilde{\nabla}_\mu\rho}\tilde{\nabla}_\mu(e^{2\phi}) = 0 \tag{182}$$

we find that

$$\delta S_{CGHS} = \int_M \frac{\partial L_{CGHS}}{\partial\tilde{\nabla}_\mu\phi}e^{2\phi}\tilde{\nabla}_\mu\epsilon(x) + \frac{\partial L_{CGHS}}{\partial\tilde{\nabla}_\mu\rho}e^{2\phi}\tilde{\nabla}_\mu\epsilon(x) \equiv \equiv \int_M j^\mu\tilde{\nabla}_\mu\epsilon(x) = 0 \tag{183}$$

where

$$j^\mu = (\frac{\partial L_{CGHS}}{\partial\tilde{\nabla}_\nu\phi} + \frac{\partial L_{CGHS}}{\partial\tilde{\nabla}_\nu\rho})e^{2\phi} = (\tilde{\nabla}^\mu(\phi - \rho)) \tag{184}$$

partially integrating 183 assuming no boundary or $\epsilon = 0$ on ∂M we find

$$\int_M \tilde{\nabla}_\mu j^\mu \tilde{\nabla}_\mu \epsilon(x) = 0 \tag{185}$$

which must be true for arbitrary ϵ and therefore $\tilde{\nabla}_\mu j^\mu = 0$. Hence $\tilde{\nabla}^2(\phi - \rho) = 0$. We find the same relation if we combine 206 with 208, that is

$$\partial_+ \partial_- (\phi - \rho) = 0 \tag{186}$$

Thus $\phi - \rho = f(x^+) + g(x^-)$ However since $Conf(M)$ is a gauge group of the system we can gauge fix the solutions by choosing $\phi = \rho$. Then the equations of motion reduce to

$$\partial_+ \partial_- e^{-2\phi} + \lambda^2 = 0 \tag{187}$$

$$\partial_\pm^2 e^{-2\phi} + T_{\pm\pm} = 0 \tag{188}$$

$$\partial_+ \partial_- f_i = 0 \tag{189}$$

The solutions are thus

$$\begin{aligned} e^{-2\phi} = & -\lambda^2 (x^+ + c)(x^- + d) + \int_0^{x^+} \int_0^u T_{++}(y) dy + \int_0^{x^-} \int_0^u T_{--}(y) dy + e = \\ & -\lambda^2 (x^+ + c)(x^- + d) - x^+ P(x^+) + M(x^+) - x^- P(x^-) + M(x^-) + e \end{aligned} \tag{190}$$

where c, d and e are constants.

$$M(x^\pm) = \int_0^{x^\pm} du (u^\pm T_{\pm\pm}(u^\pm) \tag{191}$$

$$P(x^\pm) = \int_0^{x^\pm} du T_{\pm\pm}(u^\pm) \tag{192}$$

The curvature is given by $R = 8e^{-2\phi}\partial_+\partial_-\phi = 4(\lambda^2 + e^{2\phi}\partial_+ e^{-2\phi}\partial_- e^{-2\phi})$ and hence curvature will be bounded unless $\partial_+ e^{-2\phi}$, $\partial_- e^{-2\phi}$ are unbounded when $e^{-2\phi} \neq 0$. On the other hand when $e^{-2\phi} = 0$ a necessary condition for bounded curvature is that $\partial_+ e^{-2\phi}\partial_- e^{-2\phi} = 0$. Using the general form of the solutions we can express curvature as

$$R = \frac{(\lambda^2 (M(x^+) + M(x^-) - cP(x^+) - dP(x^-) - e) - P(x^+)P(x^-)}{-\lambda^2 (x^+ + c)(x^- + d) - x^+ P(x^+) + M(x^+) - x^- P(x^-) + M(x^-) + e} \tag{193}$$

We see that the condition $e^{-2\phi} = 0$ can be expressed $\lambda^2(x^+ + c)(x^- + d) = -x^+ P(x^+) + M(x^+) - x^- P(x^-) + M(x^-) + e$ and for $f = 0$ we find the family of solutions $e^{-2\phi} = -\lambda^2 (x^+ + c)(x^- + d) + e$ which gives us singular curvature for $\lambda^2(x^+ + c)(x^- + d) = e$ unless $e = 0$. These solutions constitute a one parameter family of black holes with future event horizon given by $x^+ = -c, x^- > -d$ and

$x^- = -d, x^x > -c$ and past event horizon given by $x^+ = -c, x^- < -d$ and $x^- = -d, x^x < -c$. The $e = 0$ solution has everywhere zero curvature. We return to a more detailed analysis later.

The CGHS-action in conformal gauge and lightcone coordinates is

$$\begin{aligned} S_{CGHS} =& \frac{1}{\pi} \int_M (e^{-2\phi}(2\partial_+\partial_-\rho - 4\partial_+\phi\partial_-\phi + e^{2\rho}\lambda^2) + \frac{1}{2}\sum_{i=1}^{N}(\partial_+ f_i \partial_- f_i)) \\ & - \frac{2\kappa}{\pi} \int_M \partial_+\partial_-\rho \int_M \partial_+\partial_-\rho(y) G(x,y) \end{aligned} \tag{194}$$

The curve were $e^{-2\phi} = 0$ is given by

$$-\lambda^2(x^+ + c)(x^- + d) + x^+ P(x^+) - M(x^+) + x^- P(x^-) - M(x^-) + e = 0 \tag{195}$$

Thus when we consider this model as a dimensionally reduced spherically symmetric spacetime where $e^{-2\phi}$ is the area of a two sphere, we can impose the reflecting boundary condition $f = 0$ on the $e^{-2\phi} = 0$ curve and disregard the rest of the solution.

8.2 Inclusion of a polyakov term

We now want to modify the classical theory by including terms related to the trace anomaly in the action. Callan, Giddings, Harvey and Strominger[21] argued that the action should be supplemented by the term

$$S_c = \frac{\kappa}{\pi} \int_M \partial_+\rho\partial_-\rho \tag{196}$$

which, in the case of vanishing boundaries, is equivalent to the polyakov term

$$S_P = -\frac{\kappa}{8\pi} \int_M \sqrt{-g(x)} R(x) \int_M \sqrt{-g(y)} R(y) G(x,y) \tag{197}$$

where $G(x, y)$ is a Green function for ∇^2 and $\kappa = \frac{N}{12}$, N being the number of matterfields. We might note that while S_P is nonlocal S_c is local. We shall use the polyakov action in our derivations. The additions to the equations of motion are then assuming that the spacetime M is a manifold with piecewise smooth boundary ∂M and that $\delta g_{ab} = 0$ and $\nabla_c g_{ab} = 0$ at ∂M

$$\frac{\delta S_P}{\delta g_{ab}} = \frac{N}{96\pi} \sqrt{-g}(-\frac{1}{2} g_{ab}(\nabla\varphi)^2 + \nabla_a\varphi\nabla_b\varphi + 2\nabla_a\nabla_b\varphi - 2g_{ab}\nabla^2\varphi) \tag{198}$$

and in conformal gauge

$$\frac{\delta S_P}{\delta g_{ab}} = \frac{N}{96\pi} e^{2\rho}\sqrt{-\tilde{g}}(-\frac{1}{2}\tilde{g}_{ab}(\tilde{\nabla}\varphi)^2 + \tilde{\nabla}_a\varphi\tilde{\nabla}_b\varphi + 2\tilde{\nabla}_a\tilde{\nabla}_b\varphi - 2(\delta^\mu_a\delta^\nu_b + \delta^\nu_a\delta^\mu_b - \tilde{g}_{ab}\tilde{g}^{\mu\nu})\tilde{\nabla}_\nu\rho\tilde{\nabla}_\mu\varphi - 2\tilde{g}_{ab}\tilde{\nabla}^2\varphi) \tag{199}$$

and in lightcone coordinates given by

$$g_{++} = g_{--} = 0 \qquad g_{+-} = g_{-+} = -\frac{1}{2} \qquad g^{+-} = g^{-+} = -2 \tag{200}$$

$$\frac{\delta S_P}{\delta g_{+-}} = -\frac{N}{96\pi} e^{2\rho} \partial_+ \partial_- \varphi \tag{201}$$

$$\frac{\delta S_P}{\delta g_{\pm\pm}} = \frac{N}{192\pi} e^{2\rho} (+\partial_\pm \varphi \partial_\pm \varphi + 2\partial_\pm^2 \varphi - 4\partial_\pm \rho \partial_\pm \varphi) \tag{202}$$

Then using $\varphi = -2\rho + 2A_\partial$

$$\frac{\delta S_P}{\delta g_{+-}} = \frac{N}{48\pi} e^{2\rho} \partial_+ \partial_- \rho \tag{203}$$

$$\frac{\delta S_P}{\delta g_{\pm\pm}} = \frac{N}{48\pi} e^{2\rho} (-\partial_\pm^2 \rho + \partial_\pm \rho \partial_\pm \rho + \partial_\pm^2 A_\partial - \partial_\pm A_\partial \partial_\pm A_\partial) \tag{204}$$

Now we introduce the notation

$$t_{\pm\pm} \equiv \partial_\pm^2 A_\partial - \partial_\pm A_\partial \partial_\pm A_\partial \tag{205}$$

The function $t_{\pm\pm}$ does only appear if we consider a spacetime with boundary, and is determined by our choice of G and the boundary conditions for ρ at ∂M. Since $\partial_+ \partial_- G(x, y) = 0$ for $x \neq y$ we find that A_∂ will be a harmonic function for $x \neq y$ and thus $\partial_\pm A_\partial$ must be a function of only $x^\pm$ when $x \neq y$. Therefore $t_{\pm\pm}$ is a function of $x^\pm$. If curvature is zero everywhere, $A_\partial = \rho$ and thus $t_{\pm\pm} = \partial_\pm^2 \rho - \partial_\pm \rho \partial_\pm \rho$.

And the equations of motion are

$$\begin{aligned} \frac{\delta S_{CGHS} + S_P}{\delta g_{+-}} = & \frac{1}{4\pi} e^{2\rho} ((\partial_+ \partial_- e^{-2\phi} - e^{2(\rho - \phi)} \lambda^2) \\ & + \frac{\kappa}{4\pi} (e^{2\rho} \partial_+ \partial_- \rho) = 0 \end{aligned} \tag{206}$$

$$\begin{aligned} \frac{\delta S_{CGHS} + S_P}{\delta g_{\pm\pm}} = & \frac{1}{4\pi} e^{2\rho} (e^{-2\phi} (2\partial_\pm^2 \phi - 4\partial_\pm \rho \partial_\pm \phi) + T_{\pm\pm}) \\ & + \frac{\kappa}{4\pi} e^{2\rho} (-\partial_\pm^2 \rho + \partial_\pm \rho \partial_\pm \rho + t_{\pm\pm}) \end{aligned} \tag{207}$$

$$\frac{\delta S_{CGHS} + S_P}{\delta \phi} = -\frac{2}{\pi} (-2e^{-2\phi} \partial_+ \partial_- \rho - \lambda^2 e^{2(\rho - \phi)} + \partial_+ \partial_- e^{-2\phi} + e^{-2\phi} 2\partial_+ \partial_- \phi) = 0 \tag{208}$$

$$\frac{\delta S_{CGHS} + S_P}{\delta f_i} = -\frac{1}{\pi} e^{2\rho} \partial_+ \partial_- f_i = 0 \tag{209}$$

Here it has been used that $T_{+-} = \frac{1}{4}\sum_{i=1}^{N}(2\partial_- f_i \partial_+ f_i - 2\partial_+ f_i \partial_- f_i) = 0$

The tensor corresponding to the renormalized stress tensor

$$\begin{aligned} T^R_{ab} =& -\frac{1}{4}\sum_{i=1}^{N}(g_{ab}(\nabla f_i)^2 - 2\nabla_a f_i \nabla_b f_i) - \frac{2\pi}{\sqrt{-g}}\frac{\delta S_P}{\delta g_{ab}} \\ =& -\frac{1}{4}\sum_{i=1}^{N}(g_{ab}(\nabla f_i)^2 - 2\nabla_a f_i \nabla_b f_i) - \frac{\kappa}{4}(g_{ab}R - (\nabla_a \nabla_b \int_M \sqrt{-g(y)}R(y)G(x,y))) \end{aligned} \tag{210}$$

can be reexpressed as

$$T^R_{\pm\pm} = T_{\pm\pm} - \frac{\kappa}{2}(\partial^2_\pm \rho - \partial_\pm \rho \partial_\pm \rho + t_{\pm\pm}) \tag{211}$$

The action in conformal gauge is

$$\begin{aligned} S_{CGHS} + S_P =& \frac{1}{2\pi}\int_M \sqrt{-\tilde{g}}(e^{-2\phi}(-2\tilde{\nabla}^2\rho + 4(\tilde{\nabla}\phi)^2 + 4e^{2\rho}\lambda^2) - \frac{1}{2}\sum_{i=1}^{N}(\tilde{\nabla} f_i)^2) \\ & -\frac{\kappa}{2\pi}\int_M \sqrt{-\tilde{g}}\tilde{\nabla}^2\rho \int_M \sqrt{-\tilde{g}(y)}\tilde{\nabla}^2\rho(y)G(x,y) \end{aligned} \tag{212}$$

partial integration yields

$$\begin{aligned} S_{CGHS} =& \frac{1}{2\pi}\int_M \sqrt{-\tilde{g}}(e^{-2\phi}(-4\tilde{g}^{\mu\nu}\tilde{\nabla}_\nu\phi\tilde{\nabla}_\mu\rho + 4\tilde{g}^{\mu\nu}\tilde{\nabla}_\nu\phi\tilde{\nabla}_\mu\phi + 4e^{2\rho}\lambda^2) - \quad \frac{1}{2}\sum_{i=1}^{N}(\tilde{\nabla} f_i)^2 + \kappa\tilde{g}^{\mu\nu}\tilde{\nabla}_\nu\rho\tilde{\nabla}_\mu\rho) \\ & -\frac{\kappa}{2\pi}\int_{\partial M}\sqrt{-\tilde{g}}\rho\tilde{\nabla}_\mu\rho n^\mu + \frac{1}{2\pi}\int_{\partial M}\sqrt{-\tilde{g}}e^{-2\phi}\tilde{\nabla}_\mu\rho n^\mu \\ & -\frac{\kappa}{2\pi}\int_M \sqrt{-\tilde{g}}\tilde{\nabla}^2\rho \int_{\partial M}\sqrt{-\tilde{g}(y)}(G(x,y)\tilde{\nabla}_\mu\rho n^\mu - \rho\tilde{\nabla}_\mu G(x,y)n^\mu) \end{aligned} \tag{213}$$

where n^μ is the normal to ∂M. Thus provided the spacetime has no boundary or $\tilde{\nabla}_\mu\rho = 0$ and $\rho\tilde{\nabla}_\mu G(x,y) = 0$ on ∂M , and we again consider the variation $\delta\phi = \delta\rho = \epsilon(x)e^{-2\phi}$, we find the conserved current

$$j^\mu = \nabla^\mu(\phi - \rho) + e^{2\phi}\frac{\kappa}{2}\nabla^\mu\rho \tag{214}$$

and hence $\tilde{\nabla}^2(\phi - \rho) + e^{2\phi}\frac{\kappa}{2}\tilde{\nabla}^2\rho + e^{2\phi}\kappa\tilde{\nabla}_c\phi\tilde{\nabla}^c\rho = 0$ However adding the $+-$ and ϕ equations gives

$$e^{-2\phi}(\partial_+\partial_-\phi - \partial_+\partial_-\rho) - \frac{\kappa}{2}\partial_+\partial_-\rho = 0 \tag{215}$$

8.3 RST and BPP

Since the CGHS-model with the added polyakov term is not easily solved several modified models has been constructed by adding local counter terms to S_{CGHS}. Russo, Susskind and Thorlacius[22][23][24] modified the model by including the term S_{RST} in the action, given by

$$S_{RST} = -\frac{\xi}{4\pi}\int_M \sqrt{-g}\phi R \tag{216}$$

Here $\xi = \eta\frac{N}{12}$ where η is a constant added to allow us to distinguish and scale terms originating from this action. Bose, Parker and Peleg[25] made a similar modification by adding the term S_{BPP} given by

$$S_{BPP} = \frac{\chi}{2\pi}\int_M \sqrt{-g}((\nabla\phi)^2 - \phi R) \tag{217}$$

where $\chi = \epsilon\frac{N}{12}$ and ϵ is a constant added to allow us to distinguish and scale terms. We now wish to study the contributions from these additions.

If we assume that the spacetime M be a manifold with piecewise smooth boundary ∂M, the variation of S_{RST} and S_{BPP} assuming that $\delta g_{ab} = 0$, $\delta\phi = 0$, $\delta f_i = 0$ and $\nabla_c g_{ab} = 0$ at ∂M gives the following equations of motion.

$$\frac{\delta S_{RST}}{\delta g_{ab}} = -\frac{\xi}{4\pi}\sqrt{-g}(g_{ab}\nabla^2\phi - \nabla_a\nabla_b\phi) = 0 \tag{218}$$

$$\frac{\delta S_{RST}}{\delta\phi} = -\frac{\xi}{4\pi}\sqrt{-g}R = 0 \tag{219}$$

$$\frac{\delta S_{BPP}}{\delta g_{ab}} = \frac{\chi}{2\pi}\sqrt{-g}(-\frac{1}{2}g_{ab}(\nabla\phi)^2 + \nabla_a\phi\nabla_b\phi - g_{ab}\nabla^2\phi + \nabla_a\nabla_b\phi) = 0 \tag{220}$$

$$\frac{\delta S_{BPP}}{\delta\phi} = \frac{\chi}{2\pi}\sqrt{-g}(-2\nabla^2\phi - R) = 0 \tag{221}$$

reexpressing in conformal gauge

$$\begin{aligned}\frac{\delta S_{RST}}{\delta g_{ab}} = &-\frac{\xi}{4\pi}e^{2\rho}\sqrt{-\tilde{g}}(\tilde{g}_{ab}\tilde{\nabla}^2\phi - \tilde{\nabla}_a\tilde{\nabla}_b\phi + \\ &(\delta^\mu_a\delta^\nu_b + \delta^\nu_a\delta^\mu_b - \tilde{g}_{ab}\tilde{g}^{\mu\nu})\tilde{\nabla}_\nu\rho\tilde{\nabla}_\mu\phi = 0\end{aligned} \tag{222}$$

$$\frac{\delta S_{RST}}{\delta\phi} = \frac{\xi}{2\pi}\sqrt{-\tilde{g}}\tilde{\nabla}^2\rho = 0 \tag{223}$$

$$\begin{aligned}\frac{\delta S_{BPP}}{\delta g_{ab}} = &\frac{2\chi}{\pi}e^{2\rho}\sqrt{-\tilde{g}}(-\frac{1}{2}\tilde{g}_{ab}(\tilde{\nabla}\phi)^2 + \tilde{\nabla}_a\phi\tilde{\nabla}_b\phi - \tilde{g}_{ab}\tilde{\nabla}^2 + \tilde{\nabla}_a\tilde{\nabla}_b\phi \\ &-(\delta^\mu_a\delta^\nu_b + \delta^\nu_a\delta^\mu_b - \tilde{g}_{ab}\tilde{g}^{\mu\nu})\tilde{\nabla}_\nu\rho\tilde{\nabla}_\mu\phi) = 0\end{aligned} \tag{224}$$

$$\frac{\delta S_{BPP}}{\delta \phi} = \frac{\chi}{\pi}\sqrt{-\tilde{g}}(-\tilde{\nabla}^2\phi + \tilde{\nabla}^2\rho) = 0 \tag{225}$$

and in lightcone coordinates

$$\frac{\delta S_{RST}}{\delta g_{+-}} = -\frac{\xi}{8\pi}e^{2\rho}(\partial_+\partial_-\phi) = 0 \tag{226}$$

$$\frac{\delta S_{RST}}{\delta g_{\pm\pm}} = -\frac{\xi}{2\pi}e^{2\rho}\frac{1}{4}(-\partial_\pm\partial_\pm\phi + 2\partial_\pm\rho\partial_\pm\phi) \tag{227}$$

$$\frac{\delta S_{RST}}{\delta \phi} = -\frac{\xi}{\pi}\partial_+\partial_-\rho = 0 \tag{228}$$

$$\frac{\delta S_{BPP}}{\delta g_{+-}} = \frac{\chi}{4\pi}e^{2\rho}(-\partial_+\partial_-\phi) = 0 \tag{229}$$

$$\frac{\delta S_{BPP}}{\delta g_{\pm\pm}} = \frac{\chi}{4\pi}e^{2\rho}(\partial_\pm\phi\partial_\pm\phi + \partial_\pm^2\phi - 2\partial_\pm\rho\partial_\pm\phi) = 0 \tag{230}$$

$$\frac{\delta S_{BPP}}{\delta \phi} = \frac{2\chi}{\pi}(\partial_+\partial_-\phi - \partial_+\partial_-\rho) = 0 \tag{231}$$

With the RST addition the equations of motion are

$$\begin{aligned}\frac{\delta S_{CGHS} + S_P + S_{RST}}{\delta g_{+-}} =& \frac{1}{4\pi}e^{2\rho}(\partial_+\partial_- e^{-2\phi} - e^{2(\rho-\phi)}\lambda^2) \\ &+ \frac{\kappa}{4\pi}(e^{2\rho}\partial_+\partial_-\rho) - \frac{\xi}{8\pi}e^{2\rho}(\partial_+\partial_-\phi) = 0\end{aligned} \tag{232}$$

$$\begin{aligned}\frac{\delta S_{CGHS} + S_P + S_{RST}}{\delta g_{\pm\pm}} =& \frac{1}{4\pi}e^{2\rho}(e^{-2\phi}(2\partial_\pm^2\phi - 4\partial_\pm\rho\partial_\pm\phi) + T_{\pm\pm}) \\ &+ \frac{\kappa}{4\pi}e^{2\rho}(\partial_\pm^2\rho - \partial_\pm\rho\partial_\pm\rho - t_{\pm\pm}) \\ &- \frac{\xi}{8\pi}e^{2\rho}(-\partial_\pm^2\phi + 2\partial_\pm\rho\partial_\pm\phi) = 0\end{aligned} \tag{233}$$

$$\begin{aligned}\frac{\delta S_{CGHS} + S_P + S_{RST}}{\delta \phi} =& -\frac{1}{\pi}(-2e^{-2\phi}\partial_+\partial_-\rho - \lambda^2 e^{2(\rho-\phi)} + \partial_+\partial_- e^{-2\phi} + e^{-2\phi}2\partial_+\partial_-\phi) \\ &- \frac{\xi}{2\pi}\partial_+\partial_-\rho = 0\end{aligned} \tag{234}$$

$$\frac{\delta S_{CGHS} + S_P + S_{RST}}{\delta f_i} = -\frac{1}{\pi}e^{2\rho}\partial_+\partial_- f_i = 0 \tag{235}$$

And with the BPP addition the equations of motion are

$$\frac{\delta S_{CGHS}+S_P+S_{BPP}}{\delta g_{+-}} = \frac{1}{4\pi}e^{2\rho}(\partial_+\partial_- e^{-2\phi} - e^{2(\rho-\phi)}\lambda^2) + \\ + \frac{\kappa}{4\pi}(e^{2\rho}\partial_+\partial_-\rho) - \frac{\chi}{4\pi}e^{2\rho}(\partial_+\partial_-\phi) = 0 \tag{236}$$

$$\frac{\delta S_{CGHS}+S_P+S_{BPP}}{\delta g_{\pm\pm}} = \frac{1}{4\pi}e^{2\rho}(e^{-2\phi}(2\partial_\pm^2\phi - 4\partial_\pm\rho\partial_\pm\phi) + T_{\pm\pm}) \\ + \frac{\kappa}{4\pi}e^{2\rho}(\partial_\pm^2\rho - \partial_\pm\rho\partial_\pm\rho - t_{\pm\pm}) \\ + \frac{\chi}{4\pi}e^{2\rho}(\partial_\pm\phi\partial_\pm\phi + \partial_\pm^2\phi - 2\partial_\pm\rho\partial_\pm\phi) = 0 \tag{237}$$

$$\frac{\delta S_{CGHS}+S_P+S_{BPP}}{\delta \phi} = -\frac{1}{\pi}(-2e^{-2\phi}\partial_+\partial_-\rho - \lambda^2 e^{2(\rho-\phi)} + \partial_+\partial_- e^{-2\phi} + e^{-2\phi}2\partial_+\partial_-\phi) \\ + \frac{\chi}{\pi}(\partial_+\partial_-\phi - \partial_+\partial_-\rho) = 0 \tag{238}$$

$$\frac{\delta S_{CGHS}+S_P+S_{BPP}}{\delta f_i} = -\frac{1}{2\pi}e^{2\rho}\partial_+\partial_- f_i = 0 \tag{239}$$

8.4 Conserved currents

The CGHS+P+RST action in conformal gauge is

$$S_{CGHS}+S_P+S_{RST} = \frac{1}{2\pi}\int_M \sqrt{-\tilde{g}}(e^{-2\phi}(-2\tilde{\nabla}^2\rho + 4(\tilde{\nabla}\phi)^2 + 4e^{2\rho}\lambda^2) - \frac{1}{2}\sum_{i=1}^{N}(\tilde{\nabla}f_i)^2) \\ - \frac{\kappa}{2\pi}\int_M \sqrt{-\tilde{g}}\tilde{\nabla}^2\rho \int_M \sqrt{-\tilde{g}(y)}\tilde{\nabla}^2\rho(y)G(x,y) + \frac{\xi}{2\pi}\int_M \sqrt{-\tilde{g}}\phi\tilde{\nabla}^2\rho \tag{240}$$

partial integration yields

$$S_{CGHS}+S_P+S_{RST} = \frac{1}{2\pi}\int_M \sqrt{-\tilde{g}}(e^{-2\phi}(-4\tilde{\nabla}^\mu\phi\tilde{\nabla}_\mu\rho + 4\tilde{\nabla}^\mu\phi\tilde{\nabla}_\mu\phi + 4e^{2\rho}\lambda^2) - \frac{1}{2}\sum_{i=1}^{N}(\tilde{\nabla}f_i)^2) \\ - \frac{1}{2\pi}\int_M \kappa\tilde{\nabla}^\mu\rho\tilde{\nabla}_\mu\rho + \xi\tilde{\nabla}^\mu\phi\tilde{\nabla}_\mu\rho \\ + \frac{1}{2\pi}\int_{\partial M} \sqrt{-\tilde{g}}(e^{-2\phi}\tilde{\nabla}_\mu\rho + \kappa\rho\tilde{\nabla}_\mu\rho + \xi\phi\tilde{\nabla}_{mu}\rho)n^\mu \\ - \frac{\kappa}{2\pi}\int_M \sqrt{-\tilde{g}}\tilde{\nabla}^2\rho \int_{\partial M} \sqrt{-\tilde{g}(y)}(G(x,y)\tilde{\nabla}_\mu\rho - \rho\tilde{\nabla}_\mu G(x,y))n^\mu \tag{241}$$

where n^μ is the normal to ∂M. Thus provided the spacetime has no boundary or $\tilde{\nabla}_\mu \rho = 0$ and $\rho \tilde{\nabla}_\mu G(x,y) = 0$ on ∂M, and $\xi = \kappa$, there is a conserved current

$$j^\mu = \frac{\partial L_{CGHS} + L_P + L_{RST}}{\partial \tilde{\nabla}_\nu \phi} \frac{1}{e^{-2\phi} + \frac{\kappa}{4}} = \tilde{\nabla}^\mu (\phi - \rho) \tag{242}$$

and thus as in the CGHS case $\tilde{\nabla}^2(\phi - \rho) = 0$

8.5 Solving the equations of motion

First we consider the equations in which the RST- term has been added and set $\xi = \kappa$.

$$(\partial_+ \partial_- e^{-2\phi} - e^{2(\rho-\phi)} \lambda^2) + \kappa(\partial_+ \partial_- \rho) - \frac{\kappa}{2}(\partial_+ \partial_- \phi) = 0 \tag{243}$$

$$(e^{-2\phi}(2\partial_\pm^2 \phi - 4\partial_\pm \rho \partial_\pm \phi) + T_{\pm\pm}) + \kappa(\partial_\pm^2 \rho - \partial_\pm \rho \partial_\pm \rho - t_{\pm\pm}) - \frac{\kappa}{2}(-\partial_\pm^2 \phi + 2\partial_\pm \rho \partial_\pm \phi) = 0 \tag{244}$$

$$-(-2e^{-2\phi} \partial_+ \partial_- \rho - \lambda^2 e^{2(\rho-\phi)} + \partial_+ \partial_- e^{-2\phi} + 2e^{-2\phi} \partial_+ \partial_- \phi) - \frac{\kappa}{2} \partial_+ \partial_- \rho = 0 \tag{245}$$

$$\partial_+ \partial_- f_i = 0 \tag{246}$$

with $\Omega = e^{-2\phi} + \frac{\kappa}{2}\phi$ and $\Xi = e^{-2\phi} + \kappa(\rho - \frac{1}{2}\phi)$ the equations are

A

$$\partial_+ \partial_- \Xi - e^{\frac{2}{\kappa}(\Xi - \Omega)} \lambda^2 = 0 \tag{247}$$

B

$$\frac{1}{\kappa}((\partial_\pm \Omega)^2 - (\partial_\pm \Xi)^2) + \partial_\pm^2 \Xi + T_{\pm\pm} + \kappa t_{\pm\pm} = 0 \tag{248}$$

and by adding the first and third equation above

C

$$\frac{1}{\kappa} \frac{\partial \Omega}{\partial \phi}(\partial_+ \partial_- (\Xi - \Omega)) = 0 \tag{249}$$

D

$$\partial_+ \partial_- f_i = 0 \tag{250}$$

For $e^{-2\phi} \neq \frac{\kappa}{4}$ C implies $\Xi - \Omega = f_1(x^+) + f_2(x^-)$, and then A implies $\Xi = \lambda^2 \int^{x^+} e^{\frac{2}{\kappa} f_1(x^+)} dx^+ \int^{x^-} e^{\frac{2}{\kappa} f_2(x^-)} dx^- + g(x^-) + h(x^+)$ and $\Xi = \lambda^2 \int^{x^+} e^{\frac{2}{\kappa} f(x^+)} dx^+ \int^{x^-} e^{\frac{2}{\kappa} g(x^-)} dx^- + h(x^+) + g(x^-) - (f_1(x^+) + f_2(x^-))$. and then B implies $\frac{1}{\kappa}(\partial_+ f_1(x^+))^2 + \partial_+^2 h(x^+) = -T_{++} - \kappa t_{++}$. In the case we had chosen S_c action the $t_{\pm\pm}$ would of course not be there.

Next adding the first and third equation we again find

$$(e^{-2\phi} - \frac{\kappa}{4})(\partial_+ \partial_- (\phi - \rho)) = 0 \tag{251}$$

Thus when $e^{-2\phi} \neq \frac{\kappa}{4}$

$$\partial_+\partial_-(\phi - \rho) = 0 \tag{252}$$

This implies that

$$\phi = \rho + f(x^+) + g(x^-) \tag{253}$$

Let us set $\phi = \rho$, which would be justified if $Conf(M)$ is the gauge group. We also use the reformulation of the nonlocal parts from above. Thus the equations of motion are further simplified to

$$\partial_+\partial_-(e^{-2\phi} + \frac{\kappa}{2}) + \lambda^2 = 0 \tag{254}$$

$$\partial_\pm^2(e^{-2\phi} + \frac{\kappa}{2}) + T_{\pm\pm} - \kappa t_\pm = 0 \tag{255}$$

$$\partial_-\partial_+ f_i = 0 \tag{256}$$

Let us then introduce the function $\Omega \equiv e^{-2\phi} + \frac{\kappa}{2}\phi$. Using Ω the equations of motion can be recasted as

A

$$\partial_+\partial_-\Omega + \lambda^2 = 0 \tag{257}$$

B

$$\partial_\pm^2\Omega + T_{\pm\pm} - \kappa t_\pm = 0 \tag{258}$$

C

$$\partial_-\partial_+ f_i = 0 \tag{259}$$

The structure of B implies that given only a solution Ω there is an arbitrariness in the decomposition of $\partial_\pm^2\Omega$ into $T_{\pm\pm}$ and $\kappa t_{\pm\pm}$. Now let us look at A. The solution of this equation yields

$$\Omega = -\lambda^2(x^+ - d)(x^- - c) + F_+(x^+) + F_-(x^-) \tag{260}$$

where $F_+(x^+)$ and $F_-(x^-)$ are arbitrary functions. This implies that $\partial_\pm^2\Omega$ is a function of only $x^\pm$, as before.

The most general solution to the equations of motion A and B is

$$\begin{aligned}\Omega = -\lambda^2(x^+ - d)(x^- - c) + x^+(Q(x^+) - P(x^+)) + x^-(Q(x^-) - P(x^-))\\ +M(x^+) + M(x^-) - N(x^+) - N(x^-) + F\end{aligned} \tag{261}$$

where

$$M(x^\pm) = \int_0^{x^\pm} du u^\pm T_{\pm\pm}(u^\pm) \tag{262}$$

$$N(x^\pm) = \int_0^{x^\pm} du\kappa u^\pm t_{\pm\pm}(u^\pm) \tag{263}$$

$$P(x^{\pm}) = \int_0^{x^{\pm}} du T_{\pm\pm}(u^{\pm}) \tag{264}$$

$$Q(x^{\pm}) = \int_0^{x^{\pm}} du\kappa t_{\pm\pm}(u^{\pm}) \tag{265}$$

Thus in particular if we chose to consider spacetimes without boundary, $N(x^+) - N(x^-) + x^+Q(x^+) + x^-Q(x^-) = 0$ The general solution of C is

$$f_i = f_i^+(x^+) + f_i^-(x^-) \tag{266}$$

Next we consider the equations of motion with the BPP-term. Setting $\chi = \kappa$ and combining Ia and II we find

$$e^{-2\phi}\partial_+\partial-(\rho - \phi) = 0 \tag{267}$$

. We set $\phi = \rho$ as before. The equations of motion are then

A

$$\partial_+\partial_- A - \lambda^2 = 0 \tag{268}$$

B

$$\partial_{\pm}^2 A + T_{\pm\pm} - \kappa t_{\pm\pm} = 0 \tag{269}$$

C

$$\partial_+\partial_- f_i = 0 \tag{270}$$

where $A = e^{-2\phi}$. The structure of the equations are the same as the RST case with Ω replaced by A and therefore the equations of motion are.

$$\begin{aligned} A = -\lambda^2(x^+ - d)(x^- - c) + x^+(Q(x^+) - P(x^+)) + x^-(Q(x^-) - P(x^-)) \\ +M(x^+) + M(x^-) - N(x^+) - N(x^-) + F \end{aligned} \tag{271}$$

where

$$M(x^{\pm}) = \int_0^{x^{\pm}} du u^{\pm} T_{\pm\pm}(u^{\pm}) \tag{272}$$

$$N(x^{\pm}) = \int_0^{x^{\pm}} du\kappa u^{\pm} t_{\pm\pm}(u^{\pm}) \tag{273}$$

$$P(x^{\pm}) = \int_0^{x^{\pm}} du T_{\pm\pm}(u^{\pm}) \tag{274}$$

$$Q(x^{\pm}) = \int_0^{x^{\pm}} du\kappa t_{\pm\pm}(u^{\pm}) \tag{275}$$

Thus again if we chose to consider spacetimes without boundary, $N(x^+) - N(x^-) + x^+Q(x^+) + x^-Q(x^-) = 0$, and thus in this case the equations of motion are identical to the equations of motion in the original CGHS-model. The general solution for the matterfield equation is

$$f_i = f_i^+(x^+) + f_i^-(x^-) \tag{276}$$

8.6 Analysis of solutions

Since the RST and BPP equations can be given such similar structure many features of the analysis of the solutions will apply to both cases. We therefore introduce Ψ which is either Ω or A depending on which case we study.

Let us consider the partial derivatives of Ψ

$$\partial_+\Psi = Q(x^+) - P(x^+) - \lambda^2(x^- - c) \tag{277}$$

$$\partial_-\Psi = Q(x^-) - P(x^-) - \lambda^2(x^+ - d) \tag{278}$$

We see that for fixed x^+, $\partial_+\Psi$ is a linear function of x^-. Likewise for fixed x^-, $\partial_-\Psi$ is a linear function of x^+. Thus the values of $\partial_+\Psi$ or $\partial_-\Psi$ are completely determined everywhere provided they are known in at least one point for each x^+ or x^+ respectively. Further if in addition to this Ψ is known in at least one point, Ψ is determined everywhere.

Let us again consider $\partial_+\Psi$ on a line of constant x^+. Along this line $\partial_+\Psi$ is linearly decreasing, in positive x^- direction. Similarly $\partial_-\Psi$ decreases linearly in positive x^+ direction along lines of constant x^-. Setting $\partial_+\Psi = 0$ and $\partial_-\Psi = 0$ then defines two lines γ_{h+} and γ_{h-} respectively given by

$$\gamma_{h+} = (x^+, \frac{Q(x^+) - P(x^+)}{\lambda^2} + c) \tag{279}$$

$$\gamma_{h-} = (\frac{Q(x^-) - P(x^-)}{\lambda^2} + d, x^-) \tag{280}$$

We note that the intersections of γ_{h+} and γ_{h-} are local minima of Ψ and all local minima of Ψ are intersections of γ_{h+} and γ_{h-}. We now look at the tangents of γ_{h+} and γ_{h-}. We see that

$$\partial_+ x^-_{h+}(x^+) = \kappa t_{++}(x^+) - T_{++}(x^+) \tag{281}$$

$$\partial_- x^+_{h-}(x^-) = \kappa t_{--}(x^-) - T_{--}(x^-) \tag{282}$$

Thus γ_{h+} is spacelike when $\kappa t_{++}(x^+) - T_{++}(x^+) < 0$, null when $\kappa t_{++}(x^+) - T_{++}(x^+) = 0$ and timelike when $\kappa t_{++}(x^+) - T_{++}(x^+) > 0$. Similarly γ_{h-} is spacelike when $\kappa t_{--}(x^-) - T_{--}(x^-) < 0$, null when $\kappa t_{--}(x^-) - T_{--}(x^-) = 0$ and when $\kappa t_{--}(x^-) - T_{--}(x^-) > 0$, it is timelike. As mentioned before the original CGHS equations of motion are identical to those of BPP if $t_{\pm\pm} = 0$, and hence the analysis of the BPP model applies to CGHS as well when $t_{\pm\pm} = 0$.

8.7 Ill defined solutions and the critical curve

Let us first consider the RST case and analyse Ω as a function of ϕ.We see that

$$\frac{\partial\Omega}{\partial\phi} = -2e^{-2\phi} + \frac{\kappa}{2} \tag{283}$$

Thus Ω has a minimum when 283 is zero, that is when $\phi = -\frac{1}{2}ln(\frac{\kappa}{4})$ and $\Omega = \frac{\kappa}{4} - \frac{\kappa}{4}ln(\frac{\kappa}{4})$. The curve where $\Omega = \frac{\kappa}{4} - \frac{\kappa}{4}ln(\frac{\kappa}{4})$ will be called the critical curve γ_{cr}, and it is along this curve curvature can become unbounded. To see this let us reexpress the curvature as

$$R = 8e^{-2\phi}\partial_-\partial_+\phi = \frac{8e^{-2\phi}}{\frac{\partial\Omega}{\partial\phi}}(\partial_-\partial_+\Omega - \frac{4e^{-2\phi}}{(\frac{\partial\Omega}{\partial\phi})^2}\partial_+\Omega\partial_-\Omega) \qquad (284)$$

or equivalently

$$R = \frac{-4e^{-2\phi}}{(\frac{\kappa}{4} - e^{-2\phi})}(\lambda^2 + 4e^{-2\phi}\partial_+\phi\partial_-\phi) \qquad (285)$$

Thus when $\frac{\kappa}{4} \neq e^{-2\phi}$ the curvature can be singular only if $\partial_+\phi\partial_-\phi$ is singular. On the other hand when $\frac{\kappa}{4} = e^{-2\phi}$, that is on the curve γ_{cr}, the curvature is singular unless

$$\partial_+\phi\partial_-\phi = -\frac{\lambda^2}{\kappa} \qquad (286)$$

Assuming that 286 holds and that neither of $\partial_+\phi$ and $\partial_-\phi$ is singular we must conclude both are non-zero and this implies

$$\partial_+\Omega|_{\gamma_{cr}} = (\frac{\kappa}{2} - 2e^{-2\phi})\partial_+\phi|_{\gamma_{cr}} = 0 \qquad (287)$$

$$\partial_-\Omega|_{\gamma_{cr}} = (\frac{\kappa}{2} - 2e^{-2\phi})\partial_-\phi|_{\gamma_{cr}} = 0 \qquad (288)$$

Thus $\partial_+\Omega|_{\gamma_{cr}} = \partial_-\Omega|_{\gamma_{cr}} = 0$ is a necessary condition for bounded R on γ_{cr} when $\partial_+\phi$ and $\partial_-\phi$ are bounded. In other words curvature is bounded on γ_{cr} only there is a local minima of Ω, that is if γ_{h+} and γ_{h-} intersect γ_{cr}.

Further since $\frac{\kappa}{4} - \frac{\kappa}{4}ln(\frac{\kappa}{4})$ is the smallest value Ω can have for real ϕ, there is no real ϕ corresponding to Ω for $\Omega < \frac{\kappa}{4} - \frac{\kappa}{4}ln(\frac{\kappa}{4})$. Since we have assumed real ϕ this means the solution is ill defined in this region. We will call the region Γ_{cr}. Provided this region exists it must be everywhere bounded by γ_{cr} which implies that if Γ_{cr} is a compact region γ_{cr} must bifurcate and can not be expressed as a function of either x^+ or x^-. We note that a bifurcation point of γ_{cr} is a point where Ω is constant in at least two different direction which, implies that it is a local extremal point of Ω and this in turn implies that it is an intersection of γ_{h-} and γ_{h+}. Further as argued above γ_{h+} together with γ_{h-} and γ_{cr} completely determines Ω.

If we instead consider A as a function of ϕ we note that it has no local minima but is monotonically decreasing, however A cannot be smaller than zero for any real ϕ. In terms of A he curvature can be reexpressed as

$$R = 8e^{-2\phi}\partial_-\partial_+\phi = 4(\lambda^2 + \frac{1}{A}\partial_+A\partial_-A) \qquad (289)$$

or equivalently

$$R = 4(\lambda^2 + 4A\partial_+\phi\partial_-\phi) \qquad (290)$$

We see that curvature when $A \neq 0$ curvature is singular only if one or both of ∂_+A and ∂_-A are singular. When $A = 0$ on the other hand a necessary condition for zero curvature is that one of $\partial_+ A$ and $\partial_- A$ are zero. However when γ_0 is non-null this implies that both $\partial_+ A$ and $\partial_- A$ are zero. As in the RST case, if γ_{h+} and γ_{h-} do not coincide with γ_{cr} there will be a region where $A < 0$ which and thus the solution is ill defined, and as before this implies the bifurcation of γ_{cr}. Thus the RST and BPP cases are analogous.

8.8 Bounded curvature condition

Let us first consider the RST case. The condition for bounded curvature on γ_{cr}, $\partial_+\Omega|_{\gamma_{cr}} = \partial_-\Omega|_{\gamma_{cr}} = 0$ implies that γ_{cr} must coincide with γ_{h+} and γ_{h-}, and thus is the same as that curvature is bounded only where γ_{cr} is a local extremum. However since γ_{cr} is defined as the curve where Ω has the smallest possible value compatible with a real valued dilaton, unless we want solutions that are everywhere ill defined except on γ_{cr}, γ_{cr} must be a local minima where it coincides with γ_{h+} and γ_{h-}. This implies that γ_{cr} must be timelike when curvature is bounded, since if γ_{h+} and γ_{h-} coincide when spacelike they will define a local maximum. Now if we parameterise γ_{cr} by s where it coincides with γ_{h+} and γ_{h-} we find

$$Q(x^-(s)) - P(x^-(s)) = \lambda^2(x^+(s) - d) \tag{291}$$

$$Q(x^+(s)) - P(x^+(s)) = \lambda^2(x^-(s) - c) \tag{292}$$

and differentiating

$$\frac{\partial x^+}{\partial s}(\kappa t_{++}(x^+(s)) - T_{++}(x^+(s))) = \lambda^2 \frac{\partial x^-}{\partial s} \tag{293}$$

$$\frac{\partial x^-}{\partial s}(\kappa t_{--}(x^-(s)) - T_{--}(x^-(s))) = \lambda^2 \frac{\partial x^+}{\partial s} \tag{294}$$

where if we require γ_{cr} to be timelike $\kappa t_{\pm\pm}(x^\pm(s)) - T_{\pm\pm}(x^\pm(s)) > 0$ which also follows from the properties of γ_{h+} and γ_{h-}. Hence in the case where $\partial M = 0$ a requirement for bounded curvature and well defined solutions is that $T_{\pm\pm} < 0$.

Further 293 and 294 implies

$$(\frac{\partial x^-}{\partial s})^2(t_{--}(x^-(s)) - T_{--}(x^-(s))) = (\frac{\partial x^+}{\partial s})^2(t_{++}(x^+(s)) - T_{++}(x^+(s))) \tag{295}$$

Furthermore if γ_{h+} and γ_{h-} coincides for an extended section this means Ω is constant on this section and hence if γ_{cr} intersects this section it must coincide entirely with it.

In the BPP case we have a similar situation. The curvature on γ_0, when it is timelike, is as mentioned before bounded only if $\partial_+ A$ and $\partial_- A$ are zero. Hence

$$Q(x^+(t)) - P(x^+(t)) = \lambda^2(x^-(t) - c) \tag{296}$$

$$Q(x^-(t)) - P(x^-(t)) = \lambda^2(x^+(t) - d) \tag{297}$$

and by differentiation

$$\frac{\partial x^+}{\partial t}(t_{++}(x^+(t)) - T_{++}(x^+(t))) = \lambda^2 \frac{\partial x^-}{\partial t} \tag{298}$$

$$\frac{\partial x^-}{\partial t}(t_{--}(x^-(t)) - T_{--}(x^-(t))) = \lambda^2 \frac{\partial x^+}{\partial t} \tag{299}$$

which implies

$$(\frac{\partial x^-}{\partial t})^2(t_{--}(x^-(t)) - T_{--}(x^-(t))) = (\frac{\partial x^+}{\partial t})^2(t_{++}(x^+(t)) - T_{++}(x^+(t))) \tag{300}$$

8.9 Trapped surfaces and singularities

We now turn to the issue of trapped surfaces in the two models. If we interprate $e^{-2\phi}$ as the area of a two-sphere we find that if the area function is decreasing in both future null directions, that is if $\partial_\pm e^{-2\phi} < 0$, the two-sphere is a trapped surface. First we look at the RST case where we divide the spacetime into the three regions Γ_{ext}, Γ_{int} and Γ_{cr}, where Γ_{ext} is the region where $\phi < -\frac{1}{2}ln(\frac{\kappa}{4})$ and Γ_{int} is the region where $\phi > -\frac{1}{2}ln(\frac{\kappa}{4})$. The boundaries between the three regions are defined by γ_{cr}. Here we see that in Γ_{ext} the two-sphere corresponding to a point is trapped if

$$\partial_+\phi > 0 \Rightarrow \partial_+\Omega < 0 \tag{301}$$

and

$$\partial_-\phi > 0 \Rightarrow \partial_-\Omega < 0 \tag{302}$$

and in the region Γ_{int} a two sphere is trapped if

$$\partial_+\phi > 0 \Rightarrow \partial_+\Omega > 0 \tag{303}$$

and

$$\partial_-\phi > 0 \Rightarrow \partial_-\Omega > 0 \tag{304}$$

Thus the appearance of trapped regions is connected to the behavior of γ_{h+} and γ_{h-}. Let us denote the region interior to γ_{h+} where $\partial_+\Omega < 0$ by Γ_+, and the region interior to γ_{h-} where $\partial_-\Omega > 0$ by Γ_-. With this terminology we note that

$$((\Gamma_+ \cap \overline{\Gamma_-}) \cap \Gamma_{ext}) \cup ((\overline{\Gamma_+} \cap \Gamma_-) \cap \Gamma_{int}) \tag{305}$$

is the trapped region. Let us now characterize the different possibilities of behavior of the critical curve γ_{cr}. As discussed above curvature is singular where it does not coincide with γ_{h+} and γ_{h-}. Furthermore since γ_{cr} is a curve where Ω is constant, when

$$\gamma_{cr} \subset (\Gamma_+ \cap \overline{\Gamma_-}) \cup (\overline{\Gamma_+} \cap \Gamma_-) \tag{306}$$

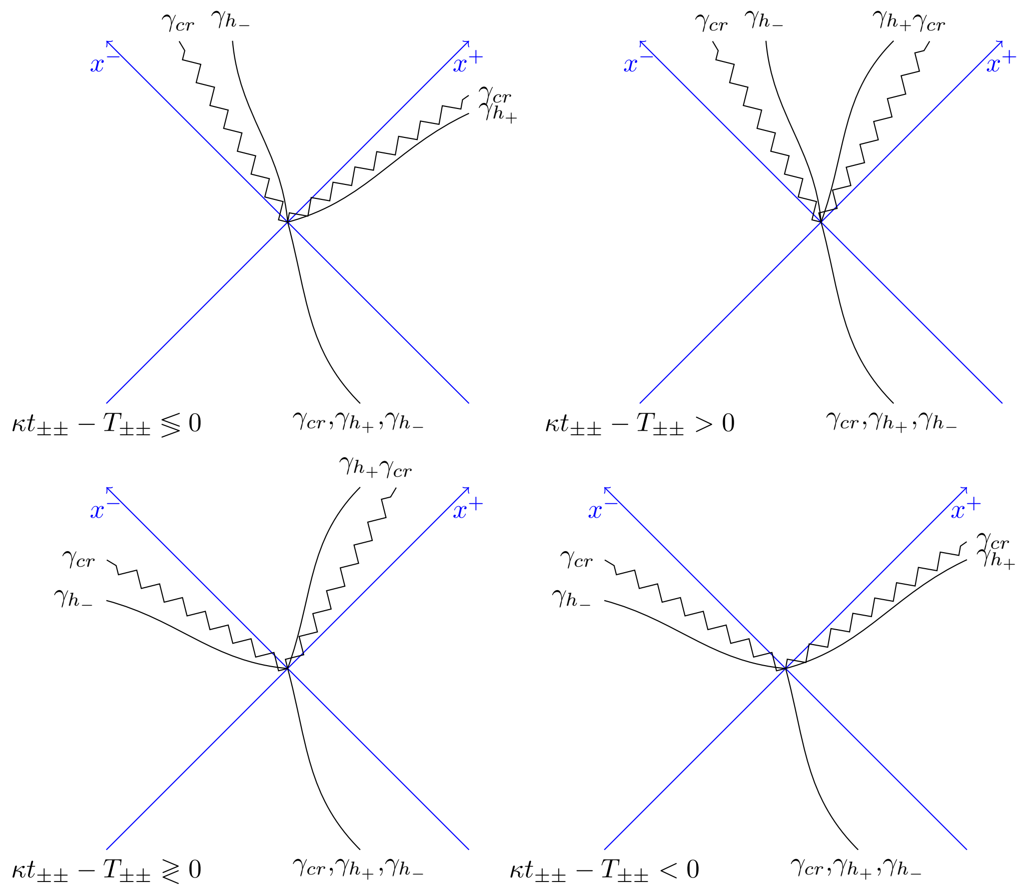
the tangent cannot be non-spacelike since all non-spacelike directions are directions of either decreasing or increasing Ω, hence γ_{cr} must be spacelike. By the same reasoning when

$$\gamma_{cr} \subset (\Gamma_+ \cap \Gamma_-) \cup (\overline{\Gamma_+} \cap \overline{\Gamma_-}) \tag{307}$$

γ_{cr} must be non-spacelike since all spacelike directions are directions of increasing or decreasing Ω. Further γ_{cr} can only be null when crossing γ_{h+} or γ_{h-}. Let us now assume that γ_{h+}, γ_{h-} and γ_{cr} initially coincides and thus curvature is bounded ant that they are timelike when coinciding, but that γ_{h+} and γ_{h-} diverge causing γ_{cr} to bifurcate. The there are a few different possibilities. If either γ_{h+} or γ_{h-} is spacelike,$\Gamma_+ \cap \overline{\Gamma_-}$ must be nonzero since γ_{h+} and γ_{h-} are functions of x^+ and x^- respectively. So if γ_{h+} is spacelike after the point of divergence, γ_{cr} must be inside $\Gamma_+ \cap \overline{\Gamma_-}$ and since γ_{cr} is null upon crossing γ_{h+}, it will remain inside unless γ_{h+} turns timelike, or possibly come to coincide with γ_{h+} if γ_{h+} turns null. If however γ_{h+} do turn timelike and remain so γ_{cr} must cross it and become timelike. If γ_{h+} is initially timelike then γ_{cr} must also be initially timelike, and naturally if γ_{h+} then turns spacelike and remain so γ_{cr} must cross it and become spacelike. Analogously if γ_{h+} is timelike after the point of divergence γ_{cr} must be outside $(\Gamma_+ \cap \overline{\Gamma_-}) \cup (\overline{\Gamma_+} \cap \Gamma_-)$ and remain so unless γ_{h+} turn spacelike or null and remain so.

The behavior of γ_{h+} is, as discussed before, determined by $\kappa t_{++}(x^+) - T_{++}(x^+)$. Thus in particular if $\kappa t_{++}(x^+) = 0$, the singularity being the boundary of Γ_{ext} will initially be inside the trapped region if and only if initially $T_{++} > 0$. If T_{++} would go to zero, and remain so, the singularity would eventually coincide with γ_{h+}, and if T_{++} became negative and remained so the singularity would eventually become timelike. If initially $T_{++} < 0$ the singularity would be timelike initially. Since the stress tensor T_{++} is always positive, it is the t_{++} term that may or may not cause evaporation of the black hole.

Likewise the behavior of γ_{h-} determines the qualitative behavior of the branch of the singular γ_{cr} bounding Γ_{int}. If γ_{h-} is initially timelike, γ_{cr} must be initially timelike and in Γ_- and will remain so unless γ_{h-} turn spacelike and remain so in which case it will cross γ_{h-} and become spacelike, or if γ_{h-} turn null and remain so in which case γ_{cr} will ultimately coincide with it. If γ_{h-} is initially spacelike γ_{cr} must also be initially spacelike and in $\overline{\Gamma_-}$, and will remain so unless γ_{h-} turn timelike and remain so in which case γ_{cr} will cross and become timelike , or if γ_{h-} turns null and remain so in which case γ_{cr} will ultimately coincide with it. And further we know that in the case of $\kappa t_{--}(x^-) = 0$, γ_{h-} is timelike if $T_{--}(x^-) < 0$, null if $T_{--}(x^-) = 0$ and spacelike if $T_{--}(x^-) > 0$. The four principal cases of initial behavior where both γ_{h-} and γ_{h+} are non-null are displayed below.



In the BPP model a two sphere is trapped if

$$\partial_+ \phi > 0 \Rightarrow \partial_+ A < 0 \tag{308}$$

and

$$\partial_- \phi > 0 \Rightarrow \partial_- A < 0 \tag{309}$$

Thus the region interior to γ_{h+} and exterior to γ_{h-} is the trapped region, and no trapped region exists if γ_{h+} is entirely in the interior of γ_{h-} or if γ_{h+} and γ_{h-} coincide. The analysis of the behavior of γ_{cr} is completely analogous to the analysis in the RST-case, except that we may possibly want to ignore the region interior to γ_{cr} if we consider $A = 0$ to be the boundary of spacetime. The above analysis applies to the original CGHS model as well, except that $\kappa t_{\pm\pm} = 0$. Hence in this case the singularity will always be behind or coincide with an apparent horizon if $T_{\pm\pm} \geq 0$

8.10 Zero curvature condition

Let us consider the case when the scalar curvature vanishes, that is $R = 8e^{-2\phi}\partial_-\partial_+\phi = 0 \Rightarrow \partial_-\partial_+\phi = 0$. In this case we find that

$$\phi = F(x^+) + G(x^-) \tag{310}$$

and thus in the RST case

$$\Omega = e^{-2(F(x^+)+G(x^-))}+\frac{\kappa}{2}(F(x^+)+G(x^-)) = H(x^+)K(x^-)-\frac{\kappa}{4}ln(H(x^+)K(x^-)) \tag{311}$$

where $e^{-2F(x^+)} = H(x^+)$ and $e^{-2G(x^-)} = K(x^-)$. Comparing this to the general form of the solutions 261 we see that

$$H(x^+)K(x^-) = \lambda^2(x^- - c)(d - x^+) \tag{312}$$

$$F = -cd \tag{313}$$

and thus

$$\Omega = \lambda^2(x^- - c)(d - x^+) - \frac{\kappa}{4}ln(\lambda^2(x^- - c)(d - x^+)) \tag{314}$$

which implies that

$$x^+(Q(x^+) - P(x^+)) + M(x^+) - N(x^+) = -\frac{\kappa}{4}ln(C_1(d - x^+)) \tag{315}$$

$$x^-(Q(x^-) - P(x^-)) + M(x^-) - N(x^-) = -\frac{\kappa}{4}ln(C_2(x^- - c)) \tag{316}$$

where C_1,C_2 are constants and $C_1C_2 = \lambda^2$ Differentiating 315 and 316 we find

$$\kappa t_{++} - T_{++} = \frac{\kappa}{4}\frac{1}{(x^+ - d)^2} \tag{317}$$

$$\kappa t_{--} - T_{--} = \frac{\kappa}{4}\frac{1}{(x^- - c)^2} \tag{318}$$

We note that in the case when $\partial M = 0$ and $t_{\pm\pm} = 0$ there are no zero curvature solutions with vanishing stress energy tensor. Further for curvature to be zero it is required that $T_{\pm\pm} < 0$ which is not possible if the scalar fields f_i is gives the only contribution to the tensor as above.

In the case of nonzero ∂M, and hence nonzero $t_{\pm\pm}$, the condition that $\phi = -\frac{1}{2}ln(\lambda^2(x^- - c)(d - x^+))$ is consistent with the requirement that when curvature is everywhere vanishing

$$t_{\pm\pm} = \partial_\pm^2\phi - \partial_\pm\phi\partial_\pm\phi \tag{319}$$

only if $T_{\pm\pm} = 0$. Thus in particular curvature cannot be everywhere zero if somewhere $T_{\pm\pm} \neq 0$. We may wish to impose zero curvature as an initial condition, for example require vanishing curvature for $x^+ < x_1^+$. If we also require vanishing stress tensors in this region that condition determines $t_{\pm\pm}$ for $x^+ < x_1^+$. It will also determine t_{--} for all x^- in the span of the initial region, however it will not determine t_{++} for $x^+ > x_1^+$. We may note that the zero curvature condition always forces γ_{h+} and γ_{h-} to be timelike.

Alternatively in the BPP case

$$A = e^{-2(F(x^+)+G(x^-))} = H(x^+)K(x^-) \tag{320}$$

where $e^{-2F(x^+)} = H(x^+)$ and $e^{-2G(x^-)} = K(x^-)$. Comparing to the equations of motion

$$A = -\lambda^2(x^- - c)(x^+ - d) \tag{321}$$

Hence in this case

$$x^+(Q(x^+) - P(x^+)) + M(x^+) - N(x^+) = 0 \tag{322}$$

$$x^-(Q(x^-) - P(x^-)) + M(x^-) - N(x^-) = 0 \tag{323}$$

and thus

$$\kappa t_{\pm\pm} = T_{\pm\pm} \tag{324}$$

Hence in the case of nonzero ∂M curvature can be zero only if $-\partial_\pm^2\phi + \partial_\pm\phi\partial_\pm\phi = T_{\pm\pm}$. In the case of $\partial M = 0$ or equivalently in the original CGHS model, curvature is zero only if $T_{\pm\pm} = 0$

8.11 Reflection at curves

Let us once again consider the bounded curvature conditions 295,300. These can be interpreted as a kind of reflection conditions on γ_0 and γ_{cr} respectively. However let us first consider the reflection condition imposed by $f|_{\gamma_0} = 0$ or $f|_{\gamma_{cr}} = 0$ respectively. This implies $f^+|_{\gamma_0} = -f^-|_{\gamma_0}$ or $f^+|_{\gamma_{cr}} = -f^-|_{\gamma_{cr}}$ and parameterizing γ_0 or γ_{cr} by t we find

$$\partial_t f = \frac{\partial x^+}{\partial t}\partial_+ f^+ + \frac{\partial x^-}{\partial t}\partial_- f^- = 0 \tag{325}$$

and therefore

$$(\frac{\partial x^+}{\partial t})^2 T_{++} = (\frac{\partial x^-}{\partial t})^2 T_{--} \tag{326}$$

Imposing this condition separately would thus mean that the curvature would be bounded only if

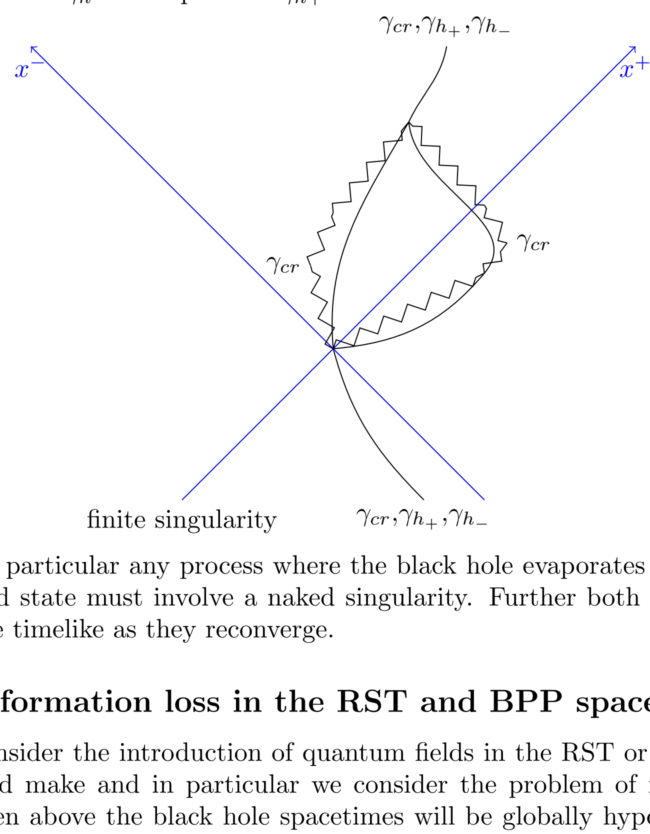
$$(\frac{\partial x^-}{\partial t})^2 t_{--}(x^-(t)) = (\frac{\partial x^+}{\partial t})^2 t_{++}(x^+(t)) \tag{327}$$

Thus in particular if $\partial M = 0$ the bounded curvature condition is identical to the reflection condition for the matter fields.

Let us once again consider the dilaton to correspond to the area A of a two-sphere by $e^{-2\phi} = A$. Similarly to the gravitational collapse treated earlier we may consider imposing the reflection condition on γ_0, and consider it as the boundary of M. The equations of motion we have used so far are derived for the interior of M, and on the boundary one must take into account the inhomogeneous part of the green function G and then we can no longer assume $t_{\pm\pm}$ to be a function of $x^{\pm}$ alone. Imposing the reflection condition on γ_{cr} however would be teleological since T_{--} is a function of x^- and hence T_{--} would have to appear in advance of the reflection.

8.12 Extension of singular curves

We now turn to some general properties of the singular parts of γ_{cr} and γ_0 in the RST and BPP models respectively. If we require derivatives $\partial_\pm\Omega$ and $\partial_\pm A$ respectively to be continuous, the curves γ_{h+} and γ_{h-} will also be continuous. Further since γ_{h+} and γ_{h-} can be realized as functions of x^+ and x^- respectively, and if we assume that the two curves coincide initially and finally, the two points where γ_{h+} and γ_{h-} diverge and reconverge respectively must be separated by a pure timelike interval, if γ_{h+} and γ_{h-} are timelike where they coincide. In the case where γ_{h+} and γ_{h-} are spacelike where they coincide, the divergence and reconvergence points must be separated by a pure spacelike interval. Thus if we consider solutions where the non-singular parts of γ_{cr} and γ_0 are timelike the singular sections cannot be everywhere spacelike, that if there is singularities there must be naked singularities. For example we can consider the case of an initially timelike γ_{h-} and spacelike γ_{h+}.



Thus in particular any process where the black hole evaporates into a non-singular end state must involve a naked singularity. Further both branches of γ_{cr} must be timelike as they reconverge.

8.13 Information loss in the RST and BPP spacetimes

We now consider the introduction of quantum fields in the RST or BPP backgrounds and make and in particular we consider the problem of information loss. As seen above the black hole spacetimes will be globally hyperbolic only if $t_{\pm\pm} - T_{\pm\pm} \leq 0$ after the point of divergence of γ_{h+} and γ_{h-} and thus both branches of γ_{cr} or γ_0 have the characteristics of either an eternal black hole or spacetime IV. The black hole spacetimes may however be predictable even if $t_{\pm\pm} - T_{\pm\pm} \leq 0$ for bounded intervals. In these cases the argument for information loss can be constructed like in the preceeding sections. In the case where the black holes evaporate and the end state is non-singular, the appearance of a naked singularity is unavoidable and thus no derivation of bogolubov coefficients of analysis of information loss is possible unless assumptions of boundary conditions on the naked singularity are made. One such a priori possible assumption would of course be that the information falling into the hidden singularity reappears from the naked singularity, thus the possibility of information retention is not excluded assuming that the singularity should be interpreted as a breakdown of our specific model rather than as an actual end of spacetime.

9 Discussion

The argument for information loss is built on the assumption that it is physically meaningful to define global quantum states relative to a background metric. However a black hole metric cannot be operationally defined by the observer, and this appears to be the source of the paradox. Hence one might argue that the construction in which the paradox appears does not correspond to any realizable experiment, and thus a different model for the evolution of quantum fields in the setting of gravitational collapse is necessary. Accepting the assumptions made in the argument for information loss we can still seek a resolution.

One resolution within a framework where quantum states are defined relative a background metric where the background is unchanged would be to modify the dynamics of the fields such that all information in a quantum state is duplicated with one copy outside and one copy inside the black hole, or that no information ever enters the black hole. This however is not possible if the evolution operator is linear. The argument is further complicated if the background contains naked singularities which causes the spacetime to become globally non-hyperbolic and makes the question of information loss impossible to answer even in the absence of black holes. A naked singularity is one of the a priori possible outcomes of the evaporation process, and in the RST-model a nessecary outcome if the end state is nonsingular. Thus the question of information loss in the approach where quantum fields are defined relative a classical RST background can not be answered, however the possibility of information retention can not be excluded.

Another way to resolve the problem, if not of the paradox itself, would be that gravitational collapse did not result in black holes, or singularities at all. In the RST model and BPP models alike such solutions exist, but the formation of singularities is determined by the behavior of γ_{h+} and γ_{h-} which depends on the specific choice of greens function in the polyakov action.

Another issue is the fact that we have used the renormalized stress tensor defined relative a background spacetime and then implanted the effective action corresponding to the correction terms of this tensor in a diffeomorphism invariant theory. It is unclear if this makes physical sense.
A further interesting point to make is that the arguments leading to hawking radiation and its relation to the trace anomaly cannot be made in the evaporation background spacetimes that result from the inclusion of the polyakov term in the RST and BPP models. Hence while it was conjectured that hawking radiation would cause black holes to evaporate, in these particular models the argument for hawking radiation is lost when evaporation occurs.

To reach a better understanding of the application of quantum field theory one would need a careful analysis of the connection between preparation and measurement procedures, the operational construction of background structures, and the concepts of quantum field theory.

At this point with with no empirical evidence of the end states of gravitational collapse and much less their properties any attempt to describe such a process whether classical or quantum mechanical must be largely conjectural and speculative.

10 Appendix A: Hilbert-like actions

In the CGHS, RST, and BPP models we come across terms in the actions of the form $\int_M \sqrt{-g}AR$ where A is not a function of the metric g_{ab}. We therefore need to consider the variation with respect to the metric for theses terms. The case $A = const$ is of course the ordinary hilbert action but if A is nonconstant there are additional terms. We consider

$$\int_M A\delta\sqrt{-g}R = \int_M A\sqrt{-g}(-\frac{1}{2}g_{ab}R + R_{ab})\delta g^{ab} + \int_M A\sqrt{-g}(\nabla_c(g_{ab}\nabla^c\delta g_{ab} - \nabla_b\delta g^{bc}) \tag{328}$$

Then in 2-dimensions $-\frac{1}{2}g_{ab}R = R_{ab}$ and thus

$$\int_M A\delta\sqrt{-g}R = \int_M A\sqrt{-g}(\nabla_c(g_{ab}\nabla^c\delta g_{ab} - \nabla_b\delta g^{bc}) \tag{329}$$

and partial integration yields

$$\begin{aligned}\int_M \sqrt{-g}A\nabla_c(g_{ab}\nabla^c\delta g_{ab} - \nabla_b\delta g^{bc}) = \\ \int_{\partial M} \sqrt{-g}A(g_{ab}\nabla^c\delta g_{ab} - \nabla_b\delta g^{bc})n_c - \int_M \sqrt{-g}(\nabla_c A)(g_{ab}\nabla^c\delta g^{ab} - \nabla_b\delta g^{bc}) = \\ \int_{\partial M} \sqrt{-g}A(g_{ab}\nabla^c\delta g_{ab} - \nabla_b\delta g^{bc})n_c - \int_{\partial M} \sqrt{-g}(\nabla_c A)(g_{ab}\delta g^{ab})n^c \\ + \int_{\partial M} \sqrt{-g}(\nabla_c A)(\delta g^{cb})n_b + \int_M \sqrt{-g}(g_{ab}\nabla^2 A - \nabla_a\nabla_b A)\delta g^{ab}\end{aligned} \tag{330}$$

Assuming $\delta g_{ab} = 0$ at ∂M we find

$$\int_M A\delta(\sqrt{-g}R) = \int_{\partial M} \sqrt{-g}A(g_{ab}\nabla^c\delta g_{ab} - \nabla_b\delta g^{bc})n_c + \int_M \sqrt{-g}(g_{ab}\nabla^2 A - \nabla_a\nabla_b A)\delta g^{ab} \tag{331}$$

Thus in particular from in the CGHS case

$$\int_M e^{-2\phi}\delta(\sqrt{-g}R) = \int_{\partial M} \sqrt{-g}e^{-2\phi}(g_{ab}\nabla^c\delta g_{ab} - \nabla_b\delta g^{bc})n_c + \int_M \sqrt{-g}(g_{ab}\nabla^2 e^{-2\phi} - \nabla_a\nabla_b e^{-2\phi})\delta g^{ab} \tag{332}$$

11 Appendix B: CGHS action

The variation of the CGHS-action assuming that $\delta g_{ab} = 0$, $\delta\phi = 0$, $\delta f_i = 0$ at ∂M gives the following result

$$\begin{aligned}\delta S_{CGHS}[g_{ab}] =& \frac{1}{2\pi}\int_M (\sqrt{-g}(e^{-2\phi}(-\frac{1}{2}g_{ab}R + 2g_{ab}(\nabla\phi)^2 - 2g_{ab}\lambda^2 + R_{ab} - 2g_{ab}\nabla^2\phi + 2\nabla_a\nabla_b\phi) \\ &+ \frac{1}{4}\sum_{i=1}^{N}(g_{ab}(\nabla f_i)^2 - 2\nabla_a f_i\nabla_b f_i))\delta g^{ab} + \int_{\partial M} e^{-2\phi}(g_{ab}\nabla^c\delta g_{ab} - \nabla_b\delta g^{bc})n_c\end{aligned} \tag{333}$$

$$\delta S_{CGHS}[\phi] = \frac{1}{2\pi}\int_M \sqrt{-g}(-8e^{-2\phi}(\frac{1}{4}R + \lambda^2 - (\nabla\phi)^2 + \nabla^2\phi))\delta\phi \quad (334)$$

$$\delta S_{CGHS}[f_i] = \frac{1}{2\pi}\int_M \sqrt{-g}\nabla^2 f_i \delta f_i \quad (335)$$

If we assume that $\nabla_c \delta g_{ab} = 0$ at ∂M or alternatively if we add appropriate terms to the action to cancel the surface terms we can identify the functional derivatives of the action as

$$\frac{\delta S_{CGHS}}{\delta g_{ab}} = \sqrt{-g}(e^{-2\phi}(-\frac{1}{2}g_{ab}R + 2g_{ab}(\nabla\phi)^2 - 2g_{ab}\lambda^2 + R_{ab} - 2g_{ab}\nabla^2\phi + 2\nabla_a\nabla_b\phi)$$
$$+\frac{1}{4}\sum_{i=1}^{N}(g_{ab}(\nabla f_i)^2 - 2\nabla_a f_i \nabla_b f_i) \quad (336)$$

$$\frac{\delta S_{CGHS}}{\delta\phi} = \frac{1}{2\pi}\sqrt{-g}(-8e^{-2\phi}(\frac{1}{4}R + \lambda^2 - (\nabla\phi)^2 + \nabla^2\phi)) \quad (337)$$

$$\frac{\delta S_{CGHS}}{\delta f_i} = \frac{1}{2\pi}\sqrt{-g}\nabla^2 f_i \quad (338)$$

Finally using that $R_{ab} = \frac{1}{2}g_{ab}R$ in 2-dimensions

$$\frac{\delta S_{CGHS}}{\delta g_{ab}} = \frac{1}{2\pi}\sqrt{-g}(e^{-2\phi}(2g_{ab}((\nabla\phi)^2 - \lambda^2 - \nabla^2\phi) + 2\nabla_a\nabla_b\phi)$$
$$+\frac{1}{4}\sum_{i=1}^{N}(g_{ab}(\nabla f_i)^2 - 2\nabla_a f_i \nabla_b f_i) \quad (339)$$

$$\frac{\delta S_{CGHS}}{\delta\phi} = \frac{1}{2\pi}\sqrt{-g}(-8e^{-2\phi}(\frac{1}{4}R + \lambda^2 - (\nabla\phi)^2 + \nabla^2\phi)) \quad (340)$$

$$\frac{\delta S_{CGHS}}{\delta f_i} = \frac{1}{2\pi}\sqrt{-g}\nabla^2 f_i \quad (341)$$

12 Appendix C: RST action

Variation of S_{RST} assuming that $\delta g_{ab} = 0$ and $\delta\phi = 0$ at ∂M gives the following result

$$\delta S_{RST}[g_{ab}] = -\frac{N}{48\pi}\int_M \sqrt{-g}(-\frac{1}{2}g_{ab}\phi R + R_{ab}\phi + g_{ab}\nabla^2\phi - \nabla_a\nabla_b\phi)\delta g_{ab}$$
$$+\int_{\partial M}\phi(g_{ab}\nabla^c\delta g_{ab} - \nabla_b\delta g^{bc})n_c \quad (342)$$

$$\delta S_{RST}[\phi] = -\frac{N}{48\pi}\int_M \sqrt{-g}R\delta\phi \quad (343)$$

If we assume that $\nabla_c \delta g_{ab} = 0$ at ∂M or alternatively if we add appropriate terms to the action to cancel the surface terms we can identify the functional derivatives of the action as

$$\frac{\delta S_{RST}}{\delta g_{ab}} = -\frac{N}{48\pi}\sqrt{-g}(-\frac{1}{2}g_{ab}\phi R + R_{ab}\phi + g_{ab}\nabla^2\phi - \nabla_a\nabla_b\phi) \tag{344}$$

$$\frac{\delta S_{RST}}{\delta \phi} = -\frac{N}{48\pi}\sqrt{-g}R \tag{345}$$

and then if we use that $R_{ab} = \frac{1}{2}g_{ab}R$ in 2-dimensions

$$\frac{\delta S_{RST}}{\delta g_{ab}} = -\frac{N}{48\pi}\sqrt{-g}(g_{ab}\nabla^2\phi - \nabla_a\nabla_b\phi) \tag{346}$$

$$\frac{\delta S_{RST}}{\delta \phi} = -\frac{N}{48\pi}\sqrt{-g}R \tag{347}$$

13 Appendix D: BPP action

Variation of S_{BPP} assuming that $\delta g_{ab} = 0$ and $\delta\phi = 0$ at ∂M gives the following result

$$\delta S_{BPP}[g_{ab}] = \frac{N}{24\pi}\int_M \sqrt{-g}(-\frac{1}{2}g_{ab}(\nabla\phi)^2 + \nabla_a\phi\nabla_b\phi + \frac{1}{2}g_{ab}\phi R - R_{ab}\phi - g_{ab}\nabla^2\phi + \nabla_a\nabla_b\phi)\delta g_{ab}$$
$$- \frac{N}{24\pi}\int_{\partial M}\phi(g_{ab}\nabla^c\delta g_{ab} - \nabla_b\delta g^{bc})n_c$$

$$\delta S_{BPP}[\phi] = \frac{N}{24\pi}\int_M \sqrt{-g}(-\nabla^2\phi - R\delta\phi) \tag{348}$$

If we assume that $\nabla_c \delta g_{ab} = 0$ at ∂M or alternatively if we add appropriate terms to the action to cancel the surface terms we can identify the functional derivatives of the action as

$$\frac{\delta S_{BPP}}{\delta g_{ab}} = \frac{N}{24\pi}\sqrt{-g}(-\frac{1}{2}g_{ab}(\nabla\phi)^2 + \nabla_a\phi\nabla_b\phi + \frac{1}{2}g_{ab}\phi R - R_{ab}\phi - g_{ab}\nabla^2\phi + \nabla_a\nabla_b\phi) \tag{349}$$

$$\frac{\delta S_{BPP}}{\delta \phi} = \frac{N}{24\pi}\sqrt{-g}(-\nabla^2\phi - R) \tag{350}$$

and then if we use that $R_{ab} = \frac{1}{2}g_{ab}R$ in 2-dimensions

$$\frac{\delta S_{BPP}}{\delta g_{ab}} = \frac{N}{24\pi}\sqrt{-g}(-\frac{1}{2}g_{ab}(\nabla\phi)^2 + \nabla_a\phi\nabla_b\phi - g_{ab}\nabla^2\phi + \nabla_a\nabla_b\phi) \tag{351}$$

$$\frac{\delta S_{BPP}}{\delta \phi} = \frac{N}{24\pi}\sqrt{-g}(-\nabla^2\phi - R) \tag{352}$$

14 Appendix E: Polyakov action

Let us now consider the problem of varying the polyakov action

$$S_P = -\frac{N}{96\pi}\int_M \sqrt{-g}R\int_M \sqrt{-g(y)}R(y)G(x,y) + const\int_M \sqrt{-g(x)} \tag{353}$$

First we introduce the notation $\varphi = \int_M \sqrt{-g}R(y)G(x,y)$ Then the action is

$$S_P = -\frac{N}{96\pi}\int_M \sqrt{-g}\varphi\nabla^2\varphi + const\int_M \sqrt{-g(x)} \tag{354}$$

partial integration yields

$$S_P = \frac{N}{96\pi}\int_M \sqrt{-g}(\nabla\varphi)^2 - \int_{\partial M}\varphi\nabla_a\varphi n^a + const\int_M \sqrt{-g(x)} \tag{355}$$

Then we do a variation with respect to the metric g^{ab} of the first term

$$\begin{aligned}\delta S_P[g_{ab}] =& \frac{N}{96\pi}\int_M \sqrt{-g}(-\frac{1}{2}g_{ab}(\nabla\varphi)^2 + \nabla_a\varphi\nabla_b\varphi - 2\nabla^2\varphi\frac{\delta\varphi}{\delta g^{ab}})\delta g^{ab} \\ &+ \int_{\partial M}\frac{\delta\varphi}{\delta g^{ab}}\delta g^{ab}\nabla_a\varphi n^a\end{aligned} \tag{356}$$

where

$$\int_M \frac{\delta\varphi}{\delta g^{ab}}\delta g^{ab} = \int_M \sqrt{-g}(g_{ab}\nabla^2\delta g^{ab} - \nabla_a\nabla_b\delta g^{ab})G(x,y) \tag{357}$$

and then

$$\begin{aligned}&\int_M \sqrt{-g}2\nabla^2\varphi\frac{\delta\varphi}{\delta g^{ab}}\delta g^{ab} = -\int_M 2\sqrt{-g}\nabla_a\nabla_b\varphi\delta g^{ab} + \int_M 2\sqrt{-g}g_{ab}\nabla^2\varphi\delta g^{ab} \\ &- 2\int_{\partial M}((\int_M \nabla_a\nabla_b\delta g^{ab}G(x,y))\nabla_c\varphi - \varphi\nabla_c(\int_M \nabla_a\nabla_b\delta g^{ab}G(x,y)))n^c \\ &+ \int_M \sqrt{-g}2\nabla^2\varphi\int_{\partial M} g_{ab}(G(x,y)\nabla_c\delta g^{ab} - \delta g^{ab}\nabla_c G(x,y))n^c - \\ &- 2\int_{\partial M}\varphi\nabla_b\delta g^{cb}n_b - \nabla_b(\varphi\delta g^{cb})n_b\end{aligned} \tag{358}$$

Thus provided that $\delta g^{ab} = 0$ and $\nabla_c\delta g^{ab} = 0$ on ∂M

$$\delta S_P[g_{ab}] = \frac{N}{96\pi}\int_M \sqrt{-g}(-\frac{1}{2}g_{ab}(\nabla\varphi)^2 + \nabla_a\varphi\nabla_b\varphi + 2\nabla_a\nabla_b\varphi - 2g_{ab}\nabla^2\varphi - const\sqrt{-g}\frac{1}{2}g_{ab})\delta g^{ab} \tag{359}$$

And hence

$$g^{ab}\frac{\delta S_P}{\delta g^{ab}} = -\frac{N}{48\pi}\sqrt{-g}\nabla^2\varphi - const\sqrt{-g} = -\frac{N}{48\pi}\sqrt{-g}R - const\sqrt{-g} \tag{360}$$

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