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Equivariant Cohomology and Localisation

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Abstract

Equivariant localisation is based on exploiting certain symmetries of some systems, generally represented by a non-free action of a Lie group on a manifold, to reduce the dimensionality of integral calculations that commonly appear in theoretical physics. In this work we present Cartan's model of equivariant cohomology in different scenarios, such as differential manifolds, symplectic manifolds or vector bundles and we reproduce the main corresponding localisation results.

Contents

Chapter 1

Introduction

Symmetries have been largely used during the last decades to simplify systems in mathematics and theoretical physics, especially in the study of dynamical systems. In the beginning of the 1980s, the Dutch mathematicians Hans Duistermaat and Gert Heckman observed that some integral operations common in symplectic geometry could be simplified when certain symmetry conditions where met. More precisely, they proved that some oscillatory integrals over compact symplectic manifolds could be reduced to easier operations by establishing that the semi-classical approximation was, in fact, exact. Some months later, Michael Atiyah and Raoul Bott showed that those results could be understood in terms of the so-called *equivariant cohomology*, a concept introduced by Henri Cartan in the 1950s, but it was the French mathematicians Nicole Berline and Michèle Vergne who in 1982 and 1983 derived the first general integration formulae valid for Killing vector fields on general compact Riemannian manifolds. This was the birth of what now is called *equivariant localisation*, the object of the present work.

Equivariant localisation is based on exploiting certain symmetries of some systems, generally represented by a non-free action of a Lie group on a manifold, to reduce the dimensionality of integral calculations. Particularly interesting are the cases when this reduction, known as localisation, allows a finite-dimensional integral to be expressed as a sum of a finite number of elements. Equivariant localisation can also be used to reduce path integral calculations -which appear very often in theoretical physics even though they are not completely well defined from a mathematical point of view- to more familiar and better defined finite-dimensional integrals, although this case lies out of the scope of this work. Localisation properties can be explained in different mathematical languages. In this work we will use Cartan's equivariant cohomology terms, which are based on the creation of similar elements to the de Rahm complex but taking into account the action of a Lie group on the manifold. To perform calculations, we will introduce the concept of anti-commutative variables and do operations in the exterior bundle of the manifold understanding it as a supermanifold. This will allow us to establish an invariance under coordinate changes and proof the exactness of saddle-point approximations on our integral calculations.

The present work is structured as follows. On Chapter 2 we do a review of necessary mathematical concepts, including differential geometry, Lie groups and algebras and the basics about supergeometry and supermanifolds. It contains reproductions of the main references of this work [1, 2], as well as [7]. On chapter 3 we study equivariant cohomology in different scenarios: manifolds, vector bundles, symplectic manifolds and supermanifolds, following and reproducing mostly [1, 3]. Finally on Chapter 4 we develop localisation results on these scenarios, like the Berline-Vergne and the Duistermaat-Heckman theorems. This chapter contains all the original work, providing more extended and detailed proof of the results of [1, 19], illustrating them with examples and showing how the techniques can be used as a prescription to non-compact scenarios.

Chapter 2

Mathematical Preliminaries

In this chapter we will review some necessary mathematical concepts of differential geometry, vector bundles, Lie groups and an introduction to the concept of supermanifolds. If the reader is already familiar with these concepts, we suggest to skip this chapter.

2.1 Review of Differential Geometry

2.1.1 Calculus on Differential Manifolds

Vectors, differential forms and tensors

We start by a short review on the calculus on differential manifolds following mostly [1, 2]. Let $\mathcal{M}$ be a $\mathcal{C}^\infty$ n-dimensional manifold, i.e. a paracompact Hausdorff topological space with an open covering $\mathcal{M} = \bigcup_i U_i$, where each U_i is homeomorphic to $\mathbb{R}^n$ and the local homeomorphisms induce $\mathcal{C}^\infty$-coordinate transformations on the overlapping regions. This means that locally the manifold can be treated as $\mathbb{R}^n$ even though the global properties of the manifold can be very different from the Euclidean space. In particular, around each point $p \in \mathcal{M}$ we can choose a neighbourhood $U \simeq \mathbb{R}^n$ and define local coordinates $x = (x^1, x^2, ..., x^n)$ using the local homeomorphisms. This allows to define tangent vectors of the form

$$V = V^\mu \frac{\partial}{\partial x^\mu} \tag{2.1}$$

that act on any $f \in \mathcal{C}^\infty(U)$ as $V(f) := V^\mu \partial_\mu f(p)$. The $\{\partial/\partial x^\mu\}$ span an n-dimensional $\mathbb{R}$-vector space called the *tangent space at* p, $T_p\mathcal{M}$. Generalising this concept, we can define *vector fields* $V = V^\mu(x)\partial/\partial x^\mu$, where $V^\mu(x) \in \mathcal{C}^\infty(\mathcal{M})$. They are defined in the *tangent bundle of* $\mathcal{M}$, the disjoint union of the tangent spaces of all points in $\mathcal{M}$

$$T\mathcal{M} := \bigsqcup_{x \in \mathcal{M}} T_x\mathcal{M}. \tag{2.2}$$

Any given $\mathbb{R}$-vector space W has a dual $\mathbb{R}$-vector space $W^* := \mathrm{Hom}_{\mathbb{R}}(W, \mathbb{R})$, the space of linear functionals on W. The dual space of the tangent space at a point p is called the *cotangent space at* p, $T_p^*\mathcal{M}$. Its *dual basis* $\{dx^\mu\}$ is defined by the relation

$$dx^\mu \left(\frac{\partial}{\partial x^\nu} \right) = \delta^\mu{}_\nu, \tag{2.3}$$

and the disjoint union of all cotangent spaces forms the *cotangent bundle*

$$T^*\mathcal{M} := \bigsqcup_{x \in \mathcal{M}} T_x^*\mathcal{M}. \tag{2.4}$$

These notions can be easily generalised using tensor products to define new spaces:

- The space of *rank-$(k,0)$ tensors at x*, $(T_x^*\mathcal{M})^{\bigotimes k} := \{T : T_x\mathcal{M} \otimes \overset{k}{\cdots} \otimes T_x\mathcal{M} \to \mathbb{R},\ T \text{ multilinear}\}$, with elements
$$T = T_{\mu_1 \dots \mu_k} dx^{\mu_1} \otimes \dots \otimes dx^{\mu_k} \tag{2.5}$$
- The space of *rank-$(0,l)$ tensors at x*, $(T_x\mathcal{M})^{\bigotimes l}$, with elements
$$T = T^{\mu_1 \dots \mu_l} \frac{\partial}{\partial x^{\mu_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\mu_l}} \tag{2.6}$$
- The space of *rank-(k,l) tensors at x*, $(T_x^*\mathcal{M})^{\bigotimes k} \otimes (T_x\mathcal{M})^{\bigotimes l}$, with elements
$$T = T^{\nu_1 \dots \nu_l}_{\mu_1 \dots \mu_k} \frac{\partial}{\partial x^{\nu_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\nu_l}} \otimes dx^{\mu_1} \otimes \dots \otimes dx^{\mu_k} \tag{2.7}$$

and their corresponding *tensor fields*, where the components $T^{\nu_1 \dots \nu_l}_{\mu_1 \dots \mu_k}(x)$ are $\mathcal{C}^\infty$ functions on $\mathcal{M}$.

The notation of partial derivatives and differentials is especially useful when it comes to $\mathcal{C}^\infty$ coordinate transformations $x' = x'(x)$, since the chain rule applies in the usual form
$$\frac{\partial}{\partial x^\mu} = \frac{\partial x'^\lambda}{\partial x^\mu} \frac{\partial}{\partial x'^\lambda}, \qquad dx^\mu = \frac{\partial x^\mu}{\partial x'^\lambda} dx'^\lambda. \tag{2.8}$$
Moreover, since the tensor fields are defined globally in $\mathcal{M}$, they must be coordinate-independent objects, so the local coordinates must transform as
$$T'^{\lambda_1 \dots \lambda_l}_{\rho_1 \dots \rho_k}(x') = \frac{\partial x'^{\lambda_1}}{\partial x^{\mu_1}} \dots \frac{\partial x'^{\lambda_l}}{\partial x^{\mu_l}} \frac{\partial x^{\nu_1}}{\partial x'^{\rho_1}} \dots \frac{\partial x^{\nu_k}}{\partial x'^{\rho_k}} T^{\mu_1 \dots \mu_l}_{\nu_1 \dots \nu_k}(x) \tag{2.9}$$
in order to have $T' = T$.

Differential forms and the de Rahm Complex

Some of the most common objects in differential geometry are the *differential forms*. To be able to define them, however, we must first define the *exterior product*, $\wedge$, which is the multilinear antisymmetric multiplication of elements of the cotangent space at $x \in \mathcal{M}$,
$$dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k} = \sum_{P \in S_k} \epsilon(P)\ dx^{\mu_{P(1)}} \otimes \dots \otimes dx^{\mu_{P(k)}}, \tag{2.10}$$
where S_k is the permutation group of k elements and $\epsilon(P)$ denotes the sign of the permutation p. So for example $dx \wedge dy = dx \otimes dy - dy \otimes dx$. The linear combinations of all these elements form the *antisymmetrisation of the k-th tensor power of the cotangent space at x*,
$$\mathcal{A}(T_x^*\mathcal{M})^{\bigotimes k} := \left\{ \alpha = \frac{1}{k!} \alpha_{\mu_1 \dots \mu_k} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k} \right\}. \tag{2.11}$$
If we think again the components $\alpha_{\mu_1 \dots \mu_k}$ as $\mathcal{C}^\infty$ functions on $\mathcal{M}$, we get the disjoint union of the antisymmetrisation spaces at each $x \in \mathcal{M}$, called the *k-th exterior power*,
$$\Lambda^k \mathcal{M} := \bigsqcup_{x \in \mathcal{M}} \mathcal{A}(T_x^*\mathcal{M})^{\bigotimes k}. \tag{2.12}$$

Its elements are called differential k-forms, whose components are totally antisymmetric in their indices $\mu_1, ..., \mu_k$. As particular cases, we have $\Lambda^0\mathcal{M} = \mathcal{C}^\infty(\mathcal{M})$, $\Lambda^1\mathcal{M} = T^*\mathcal{M}$ and $\Lambda^k\mathcal{M} = \{0\}$ for $k > n$. The direct sum of all the $k-$th exterior powers,

$$\Lambda\mathcal{M} = \bigoplus_{k=0}^{n} \Lambda^k\mathcal{M}, \tag{2.13}$$

can be made into a graded-commutative algebra, called *exterior algebra of* $\mathcal{M}$, by considering the exterior product $\wedge$. Given $\alpha \in \Lambda^p\mathcal{M}$ and $\beta \in \Lambda^q\mathcal{M}$, we have $\alpha \wedge \beta \in \Lambda^{p+q}\mathcal{M}$, where

$$\alpha \wedge \beta = \frac{1}{(p+q)!}(\alpha \wedge \beta)_{\mu_1 \dots \mu_{p+q}}(x)\; dx^{\mu_1} \wedge ... \wedge dx^{\mu_{p+1}}, \tag{2.14}$$

$$(\alpha \wedge \beta)_{\mu_1 \dots \mu_{p+q}}(x) = \sum_{P \in S_{p+q}} \epsilon(P)\alpha_{\mu_{P(1)} \dots \mu_{P(p)}}(x)\beta_{\mu_{P(p+1)} \dots \mu_{P(p+q)}}(x). \tag{2.15}$$

Being graded-commutative means that $\beta \wedge \alpha = (-1)^{pq}\beta \wedge \alpha$.

Finally, we would like to briefly introduce the *de Rahm complex.* To do so, we have to first introduce the linear operator

$$d : \Lambda^k\mathcal{M} \to \Lambda^{k+1}\mathcal{M} \tag{2.16}$$

locally defined by

$$d\alpha = \frac{1}{(k+1)!}(d\alpha)_{\mu_1 \dots \mu_{k+1}}(x)\; dx^{\mu_1} \wedge ... \wedge dx^{\mu_{k+1}} \tag{2.17}$$

$$(d\alpha)_{\mu_1 \dots \mu_{k+1}}(x) = \sum_{P \in S_{k+1}} \epsilon(P)\frac{\partial}{\partial x^{\mu_{P(1)}}}\alpha_{\mu_{P(2)} \dots \mu_{P(k+1)}}(x). \tag{2.18}$$

This operator is called *exterior derivative* and it is a graded derivation, meaning that it satisfies the graded Leibniz property

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p\alpha \wedge d\beta, \qquad \alpha \in \Lambda^p\mathcal{M}, \beta \in \Lambda^q\mathcal{M}. \tag{2.19}$$

The exterior derivative has another very important property: it is nilpotent, $d^2 = 0$. This property allows us to build the so-called *de Rahm complex* $(\mathcal{M}, \Lambda\mathcal{M}, d)$ in the following way. We start by defining two important subspaces of $\Lambda\mathcal{M}$:

- ker $d = \{\alpha \in \Lambda\mathcal{M} : d\alpha = 0\}$, whose elements are called *closed.*
- im d $= \{\alpha \in \Lambda\mathcal{M} : \exists\beta \in \Lambda\mathcal{M}$ such that $\alpha = d\beta\}$, whose elements are called *exact.*

From the nilpotency of the exterior derivative it easily follows that every exact form is closed. Therefore we can define the so-called *k-th cohomology group*

$$H^k(\mathcal{M}, \mathbb{R}) := \frac{\ker\, d_{|\Lambda^k\mathcal{M}}}{\mathrm{im}\; d_{|\Lambda^{k-1}\mathcal{M}}}, \tag{2.20}$$

so that $\alpha, \beta \in [\alpha]$ if and only if $\alpha - \beta = d\gamma$ for some $\gamma \in \Lambda^{k-1}\mathcal{M}$.

The de Rahm cohomology has many applications within theoretical physics. Two important results that we shall not prove -but can be found in many textbooks like [6]- are the following:

Lemma 2.1 (Poincaré's lemma)**.** *If* $d\omega = 0$ *in a star-shaped region* $S \subseteq \mathcal{M}$*, then* $\omega = d\theta$ *for some* θ *in* S*. This is only true globally in* $\mathcal{M}$ *if* $\omega \in [0]$.

Theorem 2.2 (Stoke's theorem)**.**

$$\int_{\mathcal{M}} d\omega = \oint_{\partial\mathcal{M}} \omega. \tag{2.21}$$

In particular, this implies that if $\partial\mathcal{M} = \emptyset$*, then integrals depend only on the cohomology class.*

2.1.2 Vector Bundles, Connections and Curvature

Definition of vector bundle

We briefly review the concept of *vector bundle*. For a more detailed explanation, we recommend [2, 6]. A vector bundle $(E, \pi, \mathcal{M})$ of rank n consists of two differentiable manifolds $E, \mathcal{M}$ called the *total space* and *base space* respectively, and a differentiable map $\pi : E \to \mathcal{M}$, so that each $E_x := \pi^{-1}(x)$ has the structure of an n-dimensional $\mathbb{R}$-vector space $\forall x \in \mathcal{M}$ and the *local triviality condition* is satisfied: for each $x \in \mathcal{M}$ there is a neighbourhood U of x and a diffeomorphism $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ so that $\forall y \in U$, $\varphi_y := \varphi|_{E_y} : E_y \to \{y\} \times \mathbb{R}^n$ is a vector space isomorphism. This condition basically says that locally the vector bundle looks like the product $\mathcal{M} \times \mathbb{R}^n$ even though its global properties might be very different. In the particular case $E = \mathcal{M} \times \mathbb{R}^n$, the bundle is called *trivial*.

The pair (U, φ) is normally called a *bundle chart* and φ is the *local trivialisation*. If $\{U_i\}$ is an open covering of $\mathcal{M}$ such that on each U_i the bundle is trivial and $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^n$ are the corresponding local trivialisations, then for each $U_i \cap U_j \neq \emptyset$ we define the *transition maps* $\varphi_{ji} : U_i \cap U_j \to GL(n, \mathbb{R})$ by the expression

$$\varphi_j \circ \varphi_i^{-1}(x, v) = (x, \varphi_{ji}(x)v), \quad \forall x \in U_j \cap U_j, \quad v \in \mathbb{R}^n, \tag{2.22}$$

where $GL(n, \mathbb{R})$ is the general linear group of $\mathbb{R}^n$. As a consequence of the vector bundle definition, the transition maps satisfy three important properties:

$$\varphi_{ii}(x) = \mathrm{id}_{\mathbb{R}^n} \quad \text{for} \quad x \in U_i \tag{2.23}$$

$$\varphi_{ij}(x)\varphi_{ji}(x) = \mathrm{id}_{\mathbb{R}^n} \quad \text{for} \quad x \in U_i \cap U_j \tag{2.24}$$

$$\varphi_{ik}(x)\varphi_{kj}(x)\varphi_{ji}(x) = \mathrm{id}_{\mathbb{R}^n} \quad \text{for} \quad x \in U_i \cap U_j \cap U_k. \tag{2.25}$$

Given a vector bundle $(E, \pi, \mathcal{M})$, we define a *section* to be a differentiable map $s : \mathcal{M} \to E$ such that $\pi \circ s = \mathrm{id}_{\mathcal{M}}$. The space of sections of E is denoted by $\Gamma(E)$. The tangent bundle $T\mathcal{M}$ is an example of a vector bundle and vector fields are examples of sections.

Linear connections

An important object to be defined on a vector bundle is a *(linear) connection*. Let $(E, \pi, \mathcal{M})$ be a vector bundle and consider a map

$$\nabla : \Gamma(E) \to \Gamma(E) \otimes \Gamma(T_x^*\mathcal{M}), \tag{2.26}$$

where the notation $\nabla_V \sigma := \nabla\sigma(V)$, $\sigma \in \Gamma(E)$, $V \in \Gamma(T\mathcal{M})$ is commonly used. We say that ∇ is a *(linear) connection* or *covariant derivative* if it satisfies the following properties:

- ∇ is tensorial in V:

$$\nabla_{V+W}\sigma = \nabla_V\sigma + \nabla_W\sigma, \qquad V, W \in T_x\mathcal{M},\ \sigma \in \Gamma(E), \tag{2.27}$$
$$\nabla_{fV}\sigma = f\nabla_V\sigma, \qquad f \in \mathcal{C}^\infty(\mathcal{M}),\ V \in \Gamma(T\mathcal{M}),\ \sigma \in \Gamma(E). \tag{2.28}$$

- ∇ is $\mathbb{R}$-linear in σ:

$$\nabla_V(\sigma + \tau) = \nabla_V\sigma + \nabla_V\tau, \qquad V \in T_x\mathcal{M},\ \sigma, \tau \in \Gamma(E). \tag{2.29}$$

- The product rule is satisfied

$$\nabla_V(f\sigma) = V(f)\sigma + f\nabla_V\sigma, \qquad f \in \mathcal{C}^\infty(\mathcal{M}),\ V \in T_x\mathcal{M},\ \sigma \in \Gamma(E). \tag{2.30}$$

It is interesting to describe ∇ in terms of local coordinates. To this end, consider a point $x_0 \in \mathcal{M}$, a neighbourhood U and the corresponding coordinate vector fields $\partial/\partial x^i$. We can assume $E|_U \simeq U \times \mathbb{R}^n$ and from the basis of $\mathbb{R}^n$ obtain a basis $\mu_1, ..., \mu_n$ of $\Gamma(E|_U)$. Then we can locally characterise a linear connection ∇ using the so-called *Christoffel symbols* Γ^i_{jk} defined by

$$\nabla_{\frac{\partial}{\partial x_i}}\mu_j =: \Gamma^k_{ij}\,\mu_k, \tag{2.31}$$

where naturally $i = 1, ..., d =$dim $\mathcal{M}$ and $j, k = 1, ..., n$.

Consider now $\mu \in \Gamma(E|_U)$, which we can always write as $\mu(y) = a^k(y)\mu_k(y)$, and a smooth curve $c(t)$ in U. Writing $\mu(t) := \mu(c(t))$ -which becomes a section along $c(t)$- and $V(t) = \dot{c}(t) = \dot{c}^i(t)\partial/\partial x^i$, from the definition of linear connection and Christoffel symbols we obtain:

$$\begin{aligned}\nabla_{V(t)}\mu(t) &= \nabla_{\dot{c}^i(t)\frac{\partial}{\partial x^i}}\left(a^k(t)\mu_k(c(t))\right) = \dot{c}^i(t)\nabla_{\frac{\partial}{\partial x^i}}\left(a^k(t)\mu_k(c(t))\right)\\ &= \dot{c}^i(t)\frac{\partial a^k}{\partial x^i}(t)\mu_k(c(t)) + \dot{c}^i(t)a^k(t)\Gamma^j_{ik}(c(t))\mu_j(c(t))\\ &= \dot{a}^k(t)\mu_k(c(t)) + \dot{c}^i(t)a^k(t)\Gamma^j_{ik}(c(t))\mu_j(c(t)),\end{aligned} \tag{2.32}$$

which looks like a linear system of first degree ODEs for the coefficients $a^1(t), ..., a^n(t)$. Therefore, given an initial value $\mu(0) \in E_{c(0)}$ there is a unique solution for $\mu(t)$ such that

$$\nabla_{\dot{c}(t)}\mu(t) = 0. \tag{2.33}$$

This solution is called the *parallel transport* of $\mu(0)$ along the curve $c(t)$.

The concept of parallel transport can help us give a geometric intuition to linear connections. Consider a point $\psi \in E$ and the corresponding tangent space $T_\psi E$. There is a naturally distiguished subspace, called the *vertical subspace*, $V_\psi := E_x \subseteq T_\psi E$, where $x = \pi(\psi)$, but there is no distiguished complementary subspace H_ψ to V_ψ in the sense that $T_\psi E = V_\psi \oplus H_\psi$. However, given a linear connection ∇, we can parallel transport ψ in the direction X for every $X \in T_x\mathcal{M}$ using a curve with $c(0) = x$, $\dot{c}(0) = X$, obtaining a curve $\psi(t)$ in each case. Then the space spanned by all vectors

$$\frac{d}{dt}\psi(t)|_{t=0} = 0 \tag{2.34}$$

is complementary to V_ψ, obtaining the desired horizontal space. In this sense, we can say that ∇ gives a rule on how the fibres of neighbouring points are connected with each other, hence the name *connection.*

It is also interesting to observe that from (2.32) we can write in general

$$\nabla(a^k\mu_k) = da^k\,\mu_k + a^j A\mu_k, \tag{2.35}$$

where $A \in \Gamma(\mathfrak{gl}(n,\mathbb{R}) \otimes T^*\mathcal{M}|_U)$ is the $(n \times n)-$matrix-valued 1-form on U defined by $A^j_k = \Gamma^j_{lk}dx^l$. We write $\nabla = d + A$.

Once we have $\nabla : \Lambda^p(E) \longrightarrow \Lambda^{p+1}(E)$, we can define its extension to product or dual bundles. Let us consider E_1, E_2 vector bundles over $\mathcal{M}$ with their respective connections ∇_1, ∇_2. Then the induced connection ∇ on the product bundle $E_1 \otimes E_2$ is defined by the requirement

$$\nabla(\mu_1 \otimes \mu_2) = \nabla_1\mu_1 \otimes \mu_2 + \mu_1 \otimes \nabla_2\mu_2, \qquad \mu_i \in \Gamma(E_i). \tag{2.36}$$

In a similar manner, if E is a vector bundle and E^* its dual bundle -i.e. the bundle over $\mathcal{M}$ with E^*_x as fibre space-, then if ∇ is a connection on E, the induced connection ∇^* on E^* is defined by the requirement

$$d\nu^*(\mu) = \nu^*(\nabla\mu) + (\nabla^*\nu^*)(\mu), \qquad \mu \in E, \nu^* \in E^*. \tag{2.37}$$

If we think it coordinate-wise, from $\nabla = d + A$,

$$0 = d\mu^*_i(\mu_j) = \mu^*_j(A^k_i\mu_k) + (A^{*l}_j\mu^*_l)(\mu_i) = A^j_i + A^{*i}_j, \tag{2.38}$$

from which we conclude that $A^* = -A^T$. A particular case that we will use often is the bundle End(E). If ∇ is a connection on E, then the induced connection ∇_E in End(E)= $E \otimes E^*$ will be

$$\begin{aligned}\nabla_E\sigma = \nabla_E(\sigma^i_j\mu_i \otimes \mu^*_j) &= (d\sigma^i_j)\mu_i \otimes \mu^*_j + \sigma^i_j\nabla_E(\mu_i \otimes \mu^*_j)\\ &= (d\sigma^i_j)\mu_i \otimes \mu^*_j + \sigma^i_j(\nabla\mu_i) \otimes \mu^*_j + \sigma^i_j\mu_i \otimes (\nabla^*\mu^*_j)\\ &= (d\sigma^i_j)\mu_i \otimes \mu^*_j + \sigma^i_jA^k_i\mu_k \otimes \mu^*_j + \sigma^i_jA^{*k}_j\mu_i \otimes \mu^*_k\\ &= (d\sigma^i_j)\mu_i \otimes \mu^*_j + \sigma^i_jA^k_i\mu_k \otimes \mu^*_j - \sigma^i_jA^j_k\mu_i \otimes \mu^*_k \qquad (2.39)\\ &= d\sigma + [A,\sigma], \qquad (2.40)\end{aligned}$$

where $\sigma = \sigma^i_j\mu_i \otimes \mu^*_j \in \Gamma(E \otimes E^*) = \Gamma(\mathrm{End}(E))$.

The curvature of a vector bundle

It is useful to extend ∇ from $\Gamma(E)$ to $\Gamma(E) \otimes \Lambda^p(\mathcal{M})$ by imposing

$$\nabla(\mu \otimes \alpha) := \nabla\mu \wedge \alpha + \mu \otimes d\alpha, \qquad \mu \in \Gamma(E),\ \alpha \in \Lambda^p(\mathcal{M}). \tag{2.41}$$

It is customary to write $\Lambda^p(E) := \Gamma(E) \otimes \Lambda^p\mathcal{M}$. In these terms, we have now $\nabla : \Lambda^p(E) \longrightarrow \Lambda^{p+1}(E)$, which reminds us to $d : \Lambda^p\mathcal{M} \to \Lambda^{p+1}\mathcal{M}$. We would be temptated to assume that ∇ is also nilpotent, but this is not true in general. The failure of ∇ to be nilpotent, and therefore to build a complex, is measured by the *curvature* of E,

$$F := \nabla \circ \nabla : \Lambda^0(E) \to \Lambda^2(E). \tag{2.42}$$

If $F = 0$ we say that the bundle is *flat*. Using the matrix A defined previously we can give a local expression for F. Given $\mu \in \Lambda^0(E) = \Gamma(E)$ we have

$$F(\mu) = (d + A) \circ (d + A)\mu = (d + A)(d\mu + A\mu) = d^2\mu + d(A\mu) + A\,d\mu + A \wedge A\mu \tag{2.43}$$
$$= (dA)\mu - A\,d\mu + A\,d\mu + A \wedge A\mu = (dA)\mu + A \wedge A\mu, \tag{2.44}$$

where the minus sign comes from the fact that A is a 1-form. We can thus compactly write $F = dA + A \wedge A$.

We can also think F in (2.42) as $F \in \Lambda^2(E) \otimes (\Lambda^0(E))^* = \Lambda^2(\mathrm{End}(E))$. This allows us to get the following result:

Theorem 2.3 (Second Bianchi identity). *If F is the curvature of a connection ∇ of a bundle E, then $\nabla_E F = 0$.*

The proof is straightforward considering $F = dA + A \wedge A$ and (2.40):

$$\nabla_E F = dF + [A, F] = d(dA + A \wedge A) + [A, dA + A \wedge A] =$$
$$= d^2 A + dA \wedge A - A \wedge dA + A \wedge dA + A \wedge A \wedge A - dA \wedge A - A \wedge A \wedge A = 0 \tag{2.45}$$

The Levi-Civita connection

A manifold $\mathcal{M}$ is said to be *Riemannian* if it has a globally-defined non-degenerate tensor

$$g = g_{\mu\nu}(x)\, dx^\mu \otimes dx^\nu, \tag{2.46}$$

which is called *metric tensor* or simply the *metric*. Consider now a vector bundle E on $\mathcal{M}$ with a connection ∇. The connection is said to be *metric* if

$$d\,g(\mu, \nu) = g(\nabla\mu, \nu) + g(\mu, \nabla\nu), \qquad \mu, \nu \in \Gamma(E). \tag{2.47}$$

Take now the concrete case $E = T\mathcal{M}$, i.e. the tangent bundle. ∇ is said to be *torsion-free* if

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0, \qquad X, Y \in T\mathcal{M}. \tag{2.48}$$

This condition is equivalent to saying that the Christoffel symbols are symmetric with regard to the two lower indices. It is easy to proof (see [2]) that there exists one and only one connection on $T\mathcal{M}$ which is metric and torsion-free. This connection is referred to as the *Levi-Civita connection* and its Christoffel symbols are given by

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}). \tag{2.49}$$

It is easy to see that $\Gamma^k_{ij} = \Gamma^k_{ji}$, so it is indeed torsion-free.

The curvature R of this connection is normally given in terms of the *Riemannian curvature tensor* coefficients $R^k{}_{lij}$ as follows

$$R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \frac{\partial}{\partial x^l} = R^k{}_{lij}\, \frac{\partial}{\partial x^k}. \tag{2.50}$$

The first index can be lowered as $R_{klij} := g_{km} R^m{}_{lij}$, i.e.

$$R_{klij} = g\left(R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^k}\right). \tag{2.51}$$

These coefficients present very useful symmetries, namely

$$R_{klij} = -R_{lkij} \tag{2.52}$$
$$R_{klij} = -R_{klji} \tag{2.53}$$
$$R_{klij} = R_{ijkl} \tag{2.54}$$
$$R_{klij} + R_{kijl} + R_{kjli} = 0, \tag{2.55}$$

where (2.55) is commonly referred to as the *first Bianchi identity*. The coefficients can moreover be explicitely calculated in terms of the Christoffel symbols

$$R^k{}_{lij} = \partial_i \Gamma^k_{jl} - \partial_j \Gamma^k_{il} + \Gamma^k_{im} \Gamma^m jl - \Gamma^k_{jm} \Gamma^m_{il}. \tag{2.56}$$

Characteristic classes

We will now define the notion of *characteristic class*, which is a cohomology class that does not depend on the choice of the connection. Let H be a Lie group and $\mathfrak{h}$ its Lie algebra. We say that a polynomial $P(z)$ is invariant on $\mathfrak{h}$ if it is invariant under the adjoint action of H on $\mathfrak{h}$, i.e.

$$P(h^{-1}Yh) = P(Y), \qquad h \in H, Y \in \mathfrak{h}. \tag{2.57}$$

In particular, we can consider P to be a polynomial function on $\mathfrak{h}$-valued 2-forms on $\mathcal{M}$. Then the H-invariance implies

$$dP(\alpha) = rP(\nabla\alpha), \qquad \alpha \in \Lambda^2\mathcal{M} \otimes \mathfrak{h}, \tag{2.58}$$

where r is the degree of P. A common choice is to take α to be the curvature F, so that then $dP(F) = 0$ because of the second Bianchi identity. Then $P(F)$ defines a cohomological class of $\mathcal{M}$. The most interesting property of this cohomological class is that it does not depend on the choice of ∇. Consider the particular example of $P(\alpha) = \text{Tr } \alpha^n$ and a continuous family of connections ∇_t with curvatures $F_t = {\nabla_t}^2$. Then from

$$\frac{d}{dt} F_t = \left[\nabla_t, \frac{d}{dt}\nabla_t\right] \tag{2.59}$$

we find

$$\frac{d}{dt}\text{Tr } {F_t}^n = n\text{Tr } \left(\frac{d}{dt}F_t\right) {F_t}^{n-1} = n\text{Tr } \left[\nabla_t, \left(\frac{d}{dt}\nabla_t\right) {F_t}^{n-1}\right] = d\text{Tr } \left(\frac{d}{dt}\nabla_t\right) {F_t}^{n-1}, \tag{2.60}$$

so the difference between $P(F)$ of two different connections is an exact form and therefore belongs to the same cohomological class.

2.2 Lie Groups and Lie Algebras

We shall review the concepts of Lie groups and Lie algebras. More detailed explanations can be found for example in [2, 7].

Adjoint representations of Lie groups and algebras

A *Lie group* is a manifold G equipped with group operations

$$\begin{array}{ccccc} G \times G & \longrightarrow & G & \text{(multiplication)} \\ (g,h) & \longmapsto & gh & \end{array} \qquad \begin{array}{ccccc} G & \longrightarrow & G & \text{(inverse)} \\ g & \longmapsto & g^{-1} & \end{array} \tag{2.61}$$

which are also differentiable maps. The general linear group of linear isomorphisms $GL(n,\mathbb{R})$ with the matrix product is an example of a Lie group. Other important examples of Lie groups are the closed subsets of $GL(n,\mathbb{R})$, which are collectively refered to as *linear Lie groups*.

A *Lie algebra* is an $\mathbb{R}$-vector space $\mathcal{A}$ together with a bilinear map $[\cdot,\cdot] : \mathcal{A}\times\mathcal{A} \to \mathcal{A}$, called the *Lie bracket*, satisfying

$$[a,a] = 0, \qquad \forall a \in \mathcal{A}, \tag{2.62}$$

$$[a,[b,c]] + [c,[a,b]] + [b,[c,a]] = 0, \qquad \forall a,b,c \in \mathcal{A}, \tag{2.63}$$

where (2.63) is called the *Jacobi identity*. From these properties follows immediately that the Lie bracket is anticommutative: $[b,a] = -[a,b]$. The tangent space to a linear Lie group G at the identity element e, $\mathfrak{g} = T_eG$, together with the Lie bracket $[X,Y] = XY - YX$, is an important example of a Lie algebra.

Given a Lie group G, we can define the following three operations:

$$\begin{array}{rccc} L_a : & G & \longrightarrow & G \\ & b & \longmapsto & ab \end{array} \qquad \begin{array}{rccc} R_a : & G & \longrightarrow & G \\ & b & \longmapsto & ba \end{array} \qquad \begin{array}{rccc} \mathbf{Ad}_a : & G & \longrightarrow & G \\ & b & \longmapsto & aba^{-1}, \end{array} \tag{2.64}$$

which are called *left translation*, *right translation* and *conjugation* respectively. Clearly, $\mathbf{Ad}_a = R_{a^{-1}} \circ L_a$. The three operations are diffeomorphisms and the last one is also an automorphism. It is particularly interesting to differentiate the conjugation at the identity, since $d(R_{a^{-1}} \circ L_a)_e : T_eG \longrightarrow T_eG$, commonly written as $Ad_a : \mathfrak{g} \longrightarrow \mathfrak{g}$, is an isomorphism of Lie algebras, i.e. a linear isomorphism that preserves the Lie bracket. The map

$$\begin{array}{rccc} Ad : & G & \longrightarrow & GL(\mathfrak{g}) \\ & a & \longmapsto & Ad_a, \end{array} \tag{2.65}$$

where $GL(\mathfrak{g})$ denotes the bijective linear maps on $\mathfrak{g}$, is called the *adjoint representation of* G. In the case of a linear Lie group, it takes the simple form $Ad(a)(X) = Ad_a(X) = aXa^{-1}$, for $a \in G$, $X \in \mathfrak{g}$. One can also differentiate the adjoint representation of G at the identity, $dAd_e : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, where $\mathfrak{gl}(\mathfrak{g})$ is the vector space $End(\mathfrak{g},\mathfrak{g})$ of linear endomorphisms. This map, commonly written as $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a map of Lie algebras and it is called the *adjoint representation of* $\mathfrak{g}$. If G is a linear Lie group, then $ad(A)(B) = [A,B] = AB - BA$, for $A, B \in \mathfrak{g}$, and since ad preserves the Lie bracket: $ad([u,v]) = [ad(u), ad(v)]$, for $u, v \in \mathfrak{g}$.

Left-invariant and right-invariant vector fields

Let G be a Lie group and $X \in \Gamma(G)$. We say that X is a *left-invariant vector field* (resp. *right-invariant*) if

$$d(L_a)_b(X(b)) = X(L_a(b)) = X(ab), \qquad \forall a,b \in G, \tag{2.66}$$

(resp.)

$$d(R_a)_b(X(b)) = X(R_a(b)) = X(ba), \qquad \forall a,b \in G. \tag{2.67}$$

Since the two cases are totally analog, we will focus only on the left-invariant one. X being left-invariant is equivalent to saying that the diagram

$$\begin{tikzcd} TG \arrow[r, "d(L_a)"] & TG \\ G \arrow[u, "X"] \arrow[r, "L_a"'] & G \arrow[u, "X"'] \end{tikzcd} \tag{2.68}$$

is commutative. An interesting property of left-invariant vector fields is that taking $b = e$ in (2.66) we get

$$X(a) = d(L_a)_e(X(e)), \tag{2.69}$$

so X is determined by just one value, $X(e) \in \mathfrak{g}$. Conversely, given any $v \in \mathfrak{g}$ we can define a vector field

$$v^L(a) := d(L_a)_e(v), \qquad \forall a \in G, \tag{2.70}$$

which is left-invariant because of the chain rule,

$$v^L(ab) = d(L_{ab})_e(v) = d(L_a \circ L_b)_e(v) = d(L_a)_b(v^L(b)) \tag{2.71}$$

and $v^L(e) = v$. Therefore, we conclude that the map $X \mapsto X(1)$ is an isomorphism between the set of left-invariant vector spaces on G and $\mathfrak{g}$. Moreover, the map

$$\begin{array}{ccc} G \times \mathfrak{g} & \longrightarrow & TG \\ (a, v) & \longmapsto & v^L(a) \end{array} \tag{2.72}$$

is an isomorphism. Following the same procedure with right-invariant vector fields a similar isomorphism is defined.

Relation between Lie groups and Lie Algebras

We have seen that the tangent space to a Lie group G at the identity element e, $\mathfrak{g} = T_eG$ is a Lie algebra. The natural question arises of whether from $\mathfrak{g}$ we can recover G. Even though it cannot be always done in a surjective way, there is a map that relates $\mathfrak{g}$ to G (cf. [2, 7]), namely the exponential map, which for linear Lie groups corresponds to the usual matrix exponentiation

$$e^X := I + X + \frac{1}{2}X^2 + \frac{1}{3!}X^3 + ..., \qquad X \in \mathfrak{g}, \tag{2.73}$$

with its usual properties as group homeomorphism:

- $e^{(s+t)X} = e^{sX}e^{tX}$.
- $e^0 = I$.
- $e^X e^{-X} = I$.

2.3 Supergeometry and supermanifolds

In this subsection we will review some basic notions of superanalysis and supergeometry. A more detailed description of these concepts can be found in [8, 9, 10, 11, 12]. We will first introduce Grassmann algebras and the corresponding rules of integration and derivation from a purely algebraic point of view. Afterwards we will review the definition of supermanifold and some basic differential structures. We will finish by considering some examples and giving them a geometrical meaning.

Grassmann algebras

Consider a set of anticommutative variables θ^i, $i = 1, ..., n$ satisfying

$$\{\theta^i, \theta^j\} = 0, \tag{2.74}$$

where $\{\cdot, \cdot\}$ denotes the anticommutator. This means that $\theta^i\theta^j = -\theta^j\theta^i$ and $(\theta^i)^2 = 0$. The real algebra Ξ_n generated by these elements is called the *Grassmann algebra* of n elements. The set $\{1, \theta^i, \theta^i\theta^j, \theta^i\theta^j\theta^k ...\}$, where the indices are never repeated, form a basis of the corresponding vector space, so its dimension is $2n$. A general element $z \in \Xi_n$, called sometimes *supernumber*, can always be decomposed in $z = z_B + z_S$, where $z_B \in \mathbb{R}$ is called the *body* and

$$z_S = \sum_{k=1}^{n} \frac{1}{k!} f_{i_1 ... i_k} \theta^{i_1} ... \theta^{i_k} \tag{2.75}$$

is called the *soul*. Note that the soul is always nilpotent: $(z_S)^{n+1} = 0$. The coefficients $f_{i_1...i_k} \in \mathbb{R}$ are not unique in general, but we can impose conditions to make them so. The most common one, and the one that we will use, is to demand that they are completely antisymmetric in their indices.

We can also try to think of analytic functions $f = f(\theta)$, but since the θ^i are nilpotent, all Taylor series will be finite and therefore of the form

$$f(\theta) = f(\theta^1, ..., \theta^n) = f_0 + f_i\theta^i + \frac{1}{2} f_{ij}\theta^i\theta^j + \frac{1}{3!} f_{ijk}\theta^i\theta^j\theta^k + ... \tag{2.76}$$

which is exactly the same expression as for supernumbers. An interesting property is that if two functions $f_1(\theta), f_2(\theta)$ have only components of even degree -that is, involving an even number of anticommutative variables-, then they commute: $f_1(\theta)f_2(\theta) = f_2(\theta)f_1(\theta)$.

We can introduce the concept of derivation and integration in Grassmann algebras. The (left) derivative with respect to θ^i is defined by the expression

$$\left\{ \frac{\partial}{\partial \theta^i}, \theta^j \right\} = \delta_i^j. \tag{2.77}$$

In a similar fashion, we introduce the *Berezin rules of integration*:

$$\int d\theta^i \, \theta^i = 1, \qquad \int d\theta^i \, 1 = 0, \tag{2.78}$$

so that if $d^n\theta := d\theta^n d\theta^{n-1} ... d\theta^1$ then

$$\int d^n\theta \;\; \theta^1\theta^2 ... \theta^n = 1. \tag{2.79}$$

In particular we see that only top-degree components are relevant.

Consider now a linear variable change $\varphi^j = A^j{}_i\theta^i$. Then imposing (2.79) on φ^j we see that

$$\begin{aligned} 1 \overset{!}{=} \int d^n\varphi \;\; \varphi^1\varphi^2 ... \varphi^n &= \int d^n\varphi \, (A^1_{\nu_1}\theta^{\nu_1})(A^2_{\nu_2}\theta^{\nu_1}) ... (A^n_{\nu_n}\theta^{\nu_n}) = A^1_{\nu_1}A^2_{\nu_2} ... A^n_{\nu_n} \int d^n\varphi \, \theta^{\nu_1}\theta^{\nu_2} ... \theta^{\nu_n} \\ &= A^1_{\nu_1}A^2_{\nu_2} ... A^n_{\nu_n} \varepsilon_{\nu_1\nu_2...\nu_n} \int d^n\varphi \, \theta^1\theta^2 ... \theta^n = (\det A) \int d^n\varphi \, \theta^1\theta^2 ... \theta^n, \end{aligned} \tag{2.80}$$

where $\varepsilon_{\nu_1\nu_2\ldots\nu_n}$ is the Levi-Civita symbol and takes into account the sign of the permutation of the (θ^i). From here we see that by imposing (2.79) to hold on (θ^i), we get $d^n\varphi = (\det A)^{-1} d^n\theta$. Observe that the integration measure transforms in the opposite way of the usual measure dx^n under a variable change for (x^i), where we would have the determinant and not its inverse.

This fact has many interesting consequences. For example,

$$
\begin{aligned}
\int d^n\varphi\, d^n\theta\, e^{-\varphi^i M_{ij}\theta^j} &= (\det M)\int d^n\varphi\, d^n\theta'\, e^{-\varphi^i\theta'_i} = (\det M)\int d^n\varphi\, d^n\theta'\, e^{-\varphi^1\theta'_1}\ldots e^{-\varphi^n\theta'_n} \\
&= (\det M)\left[\int d\varphi\, d\theta' e^{-\varphi\theta'}\right]^n = (\det M)\left[\int d\varphi\, d\theta'(1-\varphi\theta')\right]^n \\
&= (\det M)\left[\int d\varphi\, d\theta'\, \theta'\varphi\right]^n = \det M,
\end{aligned}
\tag{2.81}
$$

where we introduced the new variables $\theta'_i = M_{ij}\theta^j$ and used the fact that even functions commute. A similar result, which will turn out to be very useful in this work, is the following: consider a skew-symmetric matrix $B \in M_{2n\times 2n}$, $B^T = -B$, which can always be diagonalised using a unitary matrix U as $D = U^T B U$, where D is of the form

$$
D = \begin{pmatrix} 0 & \lambda_1 & & & \\ -\lambda_1 & 0 & & & \\ & & 0 & \lambda_2 & \\ & & -\lambda_2 & 0 & \\ & & & & \ddots \end{pmatrix}.
\tag{2.82}
$$

Then, using variable change $\varphi^j = U^j{}_i\theta^i$ we have

$$
\begin{aligned}
\int d^{2n}\theta\, e^{\frac{1}{2}\theta^i B_{ij}\theta^j} &= \int d^{2n}\varphi\, e^{\frac{1}{2}\varphi^i D_{ij}\varphi^j} = \int d^{2n}\varphi \prod_{k=1}^{n} e^{\lambda_k\varphi^{2k-1}\varphi^{2k}} \\
&= \int d^{2n}\varphi \prod_{k=1}^{n}\lambda_k\varphi^{2k-1}\varphi^{2k} = \prod_{k=1}^{n}\lambda_k = \mathrm{Pfaff}(B) = (\det B)^{\frac{1}{2}},
\end{aligned}
\tag{2.83}
$$

where $\mathrm{Pfaff}(B)$ denotes the Pfaffian of B. Here we used that $\det(U)$=$\det(U^{-1})$=1.

Supermanifolds

Grassman algebras are examples of vector superspaces. A *vector superspace* $\mathbb{V}$ is a vector space together with a choice of two subspaces $\mathbb{V}_0, \mathbb{V}_1 \subseteq \mathbb{V}$ such that

$$
\mathbb{V} = \mathbb{V}_0 \oplus \mathbb{V}_1,
\tag{2.84}
$$

i.e. a $\mathbb{Z}_2$-graded vector space. Elements of the subspace $\mathbb{V}_0$ are said to be *even* and elements of $\mathbb{V}_1$ are said to be *odd*. If dim $\mathbb{V}_0 = m$ and dim $\mathbb{V}_1 = n$, we say that $\mathbb{V}$ is $(m|n)-$dimensional. An important example is when $\mathbb{V}_0 = \mathbb{R}^m$ and $\mathbb{V}_1 = \mathbb{R}^n$, in which we normally write $\mathbb{V} = \mathbb{R}^{m|n}$.

We defined a manifold to be an object obtained by gluing together open subsets of $\mathbb{R}^n$ by means of smooth transformations. This can be formulated in a purely algebraic way by identifying $U \subseteq \mathbb{R}^n$ with the algebra $\mathcal{C}^\infty(U)$ and maps $U \to V$ with $Hom_{\mathbb{R}}(\mathcal{C}^\infty(U), \mathcal{C}^\infty(V))$. We will try to generalise this notion using vector superspaces to get to the concept of *supermanifold*. To start with, identify an open subset $U_n \subseteq \mathbb{R}^{m|n}$ with the $\mathbb{Z}_2$-graded algebra $\mathcal{C}^\infty(U) \otimes \Xi_n$, where $U \subseteq \mathbb{R}^m$. If $U = \mathbb{R}^m$, then we get $U_n = \mathbb{R}^{m|n}$. Elements of $\mathcal{C}^\infty(U) \otimes \Xi_n$ are of the form

$$F = \sum_k \frac{1}{k!} f_{i_1 \ldots i_k}(x) \theta^{i_1} \ldots \theta^{i_k}, \qquad i = 1, ..., n, \tag{2.85}$$

where $f_{i_1 \ldots i_k}$ are smooth functions on U totally antisymmetric in their indices. We can think of them as functions depending on the commuting variables $(x^1, ..., x^m) \in U$ and the anticommuting variables $\theta^1, ..., \theta^n \in \Xi_n$. In these terms, a map $U_n \to V_{n'}$, where $U_n, V_{n'}$ are superdomains, is specified by

$$\tilde{x}^i = a^i(x^1, ..., x^m, \theta^1, ..., \theta^n), \qquad i = 1, ..., m', \tag{2.86}$$

$$\tilde{\theta}^j = \alpha^j(x^1, ..., x^m, \theta^1, ..., \theta^n), \qquad j = 1, ..., n', \tag{2.87}$$

where $(x^1, ..., x^m, \theta^1, ..., \theta^n)$ are the coordinates in U_n and $(\tilde{x}^1, ..., \tilde{x}^{m'}, \tilde{\theta}^1, ..., \tilde{\theta}^{n'})$ are the coordinates in $V_{n'}$. With all these elements, we can define a *supermanifold* as a set of open subsets of $\mathbb{R}^{m|n}$ glued together by means of invertible maps. Note that this definition is strictly algebraic.

Using the Berezin rules of integration (2.78), we can calculate the integral of a function F as in (2.85),

$$\int_{U_n} d^n x d^n \theta \; F = n! \int_U d^n x \; \frac{1}{n!} f_{i_1 \ldots i_n}(x) = \int_U d^n x \; f_{i_1 \ldots i_n}(x) \tag{2.88}$$

There are other concepts of differential geometry of manifolds that can be extended to supermanifolds. Given U_n with coordinates $(x^1, ..., x^m) \in U$, $\theta^1, ..., \theta^n \in \Xi_n$, we can define differential k-forms on U_n to be functions of commuting variables $x^1, ..., x^m, \tilde{\theta}^1, ..., \tilde{\theta}^n$ and anticommuting variables $\theta^1, ..., \theta^n, \tilde{x}^1, ..., \tilde{x}^m$ such that they are polynomial of degree k with respect to the variables $\tilde{x}^1, ..., \tilde{x}^m, \tilde{\theta}^1, ..., \tilde{\theta}^n$. These last variables can be identified with the differentials $dx^i, d\theta^j$ respectively. Note that their parity is the opposite of the parity of the corresponding variable.

In these terms, we can generalise the *de Rahm differential* to supermanifolds:

$$d = \tilde{x}^i \frac{\partial}{\partial x^i} + \tilde{\theta}^j \frac{\partial}{\partial \theta^j}. \tag{2.89}$$

The odd tangent bundle

Given a supermanifold $\mathcal{M}$, we can also introduce the notions of tangent bundle $T\mathcal{M}$ and cotangent bundle $T^*\mathcal{M}$. We can also define the *odd tangent bundle* and *odd cotangent bundle*, $\Pi T\mathcal{M}$ and $\Pi T^*\mathcal{M}$ by taking the tangent and cotangent bundles and reversing the parity of the fibres. A section of $T\mathcal{M}$ or $\Pi T\mathcal{M}$ can be understood as a vector field on $\mathcal{M}$ and a section of $T^*\mathcal{M}$ or $\Pi T\mathcal{M}$ and $\Pi T^*\mathcal{M}$ can be understood as a differential 1-form, in a similar way as it happens with ordinary manifolds.

Let us now consider a concrete example that will be of particular importance in this work. Take an ordinary manifold $\mathcal{M}$ with local coordinates $(x^1, ..., x^m)$ and think of it as a $(m|0)$-dimensional supermanifold. Consider now its odd tangent bundle $\Pi T\mathcal{M}$, which will have local coordinates $(x^1, ..., x^m, \theta^1, ..., \theta^m)$, where $x^1, ..., x^m$ are commuting variables and $\theta^1, ..., \theta^m$ are anticommuting variables. This suggests the interesting identification $\theta^i \sim dx^i$, where the product $\theta^i \theta^j$ would correspond to the exterior product $dx^i \wedge dx^j$. This establishes a correspondance between the algebra of smooth functions on $\Pi T\mathcal{M}$ and the exterior algebra of $\mathcal{M}$, i.e. $\mathcal{C}^\infty(\Pi T\mathcal{M}) \simeq \Lambda\mathcal{M}$, in which differential forms on $\mathcal{M}$ are now understood as functions on the odd tangent bundle.

If we think of $\Pi T\mathcal{M}$ as an $(m|m)-$dimensional supermanifold, we can define integration on it. A remarkable property of this supermanifold is that its integration measure is invariant under a smooth variable change $x' = f(x)$ of the commutative variables. This is proven as follows:

- Under the change $x' = f(x)$, the integration measure of the commutative part transforms as usual $d^m x' = \det\left(\frac{\partial f}{\partial x}\right) d^m x$.

- From the identification $\theta^i \sim dx^i$ and the chain rule (2.8) we have

$$\theta'^{\mu} = \frac{\partial f^{\mu}}{\partial x^{\nu}} \theta^{\nu}. \tag{2.90}$$

  In particular, if we fix $x \in \mathcal{M}$, (2.90) is a linear variable change in the θ^i and therefore as we saw in (2.80), the integration measure of the anticommutative part transforms as $d^m\theta' = \det\left(\frac{\partial f}{\partial x}\right)^{-1} d^m\theta$.

- If we combine both transformations we have

$$\int_{\Pi T\mathcal{M}} d^m x\, d^m\theta = \int_{\Pi T\mathcal{M}} d^m x' d^m\theta' \det\left(\frac{\partial f}{\partial x}\right) \det\left(\frac{\partial f}{\partial x}\right)^{-1} = \int_{\Pi T\mathcal{M}} d^m x' d^m\theta', \tag{2.91}$$

  so the integration measure is independent of the variable change. This corresponds geometrically to the fact that integration of differential forms over a manifold is well-defined -i.e. it is independent of the choice of local coordinates-.

Chapter 3

Equivariant Cohomology

In this chapter we introduce the concept of Cartan's equivariant cohomology in different scenarios. We will start by introducing an action of a Lie group on a manifold and afterwards we will build the equivariant complex in manifold, vector bundles, symplectic manifolds and supermanifolds.

3.1 Lie group actions on a manifold

Actions and orbits

We want now to generalise the concept of cohomology to manifolds on which a Lie group is acting. To this end, consider a compact differentiable manifold $\mathcal{M}$ with $\partial\mathcal{M} = \emptyset$ and a Lie group G with Lie algebra $\mathfrak{g}$. We say that G *acts* on $\mathcal{M}$ if there is a smooth map

$$\begin{array}{ccc} G \times \mathcal{M} & \longrightarrow & \mathcal{M} \\ (g, x) & \longmapsto & g \cdot x \end{array} \tag{3.1}$$

such that $e \cdot x = x$ for all $x \in \mathcal{M}$ and $g_1 \cdot (g_2 \cdot x) = (g_1 \cdot g_2) \cdot x$ for all $g_1, g_2 \in G$. We already saw two examples that can be understood as group actions,

$$\begin{array}{ccc} G \times G & \longrightarrow & G \\ (a, g) & \longmapsto & \mathbf{Ad}_a(g) = aga^{-1} \end{array} \qquad \begin{array}{ccc} G \times \mathfrak{g} & \longrightarrow & G \\ (a, X) & \longmapsto & Ad_a(X) = aXa^{-1}, \end{array} \tag{3.2}$$

where in the second case we assumed G to be a linear Lie group. Another common action is the so-called *natural coadjoint action*

$$\begin{array}{cccc} Ad^* : & G \times \mathfrak{g}^* & \longrightarrow & \mathfrak{g}^* \\ & (a, \alpha) & \longmapsto & Ad^*_a \alpha, \end{array} \tag{3.3}$$

where $(Ad^*_a\alpha)(X) := \alpha(a^{-1}Xa)$. It satisfies the defining property $(Ad^*_a\alpha)(Ad_a(X)) = \alpha(X)$.

Given an action $G \times \mathcal{M} \to \mathcal{M}$, we denote by $\mathcal{M}^G$ the set of invariant elements by G, that is,

$$\mathcal{M}^G := \{x \in \mathcal{M} \mid \forall g \in G,\ g \cdot x = x\}. \tag{3.4}$$

Similarly, given two manifolds $\mathcal{M}_1, \mathcal{M}_2$ with G-actions, we say that a map $f : \mathcal{M}_1 \to \mathcal{M}_2$ is *equivariant* if

$$f(g \cdot x) = g \cdot f(x), \qquad \forall x \in \mathcal{M}_1. \tag{3.5}$$

Another natural concept to be defined is the orbit of $x \in \mathcal{M}$,

$$G \cdot x := \{g \cdot x \mid g \in G\}. \tag{3.6}$$

Representations of G

We saw in section §2.2 that the elements of a Lie group G are represented in terms of elements of $\mathfrak{g}$ by the exponential map $\exp : \mathfrak{g} \to G$. In particular, each element $g \in G$ can be written as

$$g = e^{c^a X^a}, \tag{3.7}$$

where c^a are constants and X^a are the generators of $\mathfrak{g}$, which obey the relations $[X^a, X^b] = f^{abc}X^c$. f^{abc} are called the antisymmetric *structure constants* of the Lie algebra. In these terms, the space in which the constants c^a lie defines the Lie group manifold G. Also, if we think of the Lie algebra as the tangent space of G at the unity, $\mathfrak{g} \simeq T_eG$, then we have

$$X^a = \left.\frac{\partial}{\partial c^a} g\right|_{c^a=0}. \tag{3.8}$$

These elements allow us to define some representations of G. The first one is the so-called *adjoint representation* of G in terms of Hermitian matrices $n \times n$, where n =dim G, whit the generators as

$$(ad\ X^a)_{bc} := i f^{abc}. \tag{3.9}$$

Consider now a path g_t through G starting at the origin, $g_0 = e$. Since the action of G on $\mathcal{M}$ is smooth, the path g_t creates a continuous flow on the manifold, locally represented by $g_t \cdot x = x(t)$, for $t \in \mathbb{R}^+$. If we think of this flow as a coordinate change, using the rule (2.9), g_t also generates an action on $\Lambda\mathcal{M}$ by using the pullback,

$$(g_t \cdot \alpha)(x) = \alpha(x(t)). \tag{3.10}$$

A simple special case is $\Lambda^0\mathcal{M} = \mathcal{C}^\infty(\mathcal{M})$, where

$$(g_t \cdot f)(x) = f(x(t)) = e^{tV(x(t))} f(x), \qquad f \in \mathcal{C}^\infty(\mathcal{M}). \tag{3.11}$$

In this expression, the vector field $V(x) = V^\mu(x)\partial/\partial x^\mu$ is representing a Lie algebra element. It is related to the flow by the expression $\dot{x}^\mu(t) = V^\mu(x(t))$. If V^a is the vector field corresponding to X^a of $\mathfrak{g}$, then there is a representation of the Lie algebra in the space of $\mathcal{C}^\infty$ functions by

$$[V^a, V^b](h) = f^{abc}V^c(h), \qquad \forall h \in \mathcal{C}^\infty(\mathcal{M}). \tag{3.12}$$

We can also understand this as a representation of the Lie group G in $T\mathcal{M}$.

The Lie derivative

It is convenient to introduce the *interior multiplication* operator $i_V : \Lambda^k\mathcal{M} \to \Lambda^{k-1}\mathcal{M}$, defined locally by

$$i_V\alpha = \frac{1}{(k-1)!} V^{\mu_1}(x)\alpha_{\mu_1\mu_2\ldots\mu_k}(x)\, dx^{\mu_2} \wedge dx^{\mu_3} \wedge \ldots \wedge dx^{\mu_k}, \tag{3.13}$$

or using anticommutative variables,

$$i_V = V^\mu(x)\frac{\partial}{\partial\theta^\mu}. \tag{3.14}$$

This operator is a graded derivation. For a general tensor T, i_VT represents its component along the V direction. It is also useful to study the infinitessimal behaviour of path actions, that is, when $t \to 0$. This behaviour is given by the *Lie derivative* along V,

$$\mathcal{L}_V : \Lambda^k\mathcal{M} \longrightarrow \Lambda^k\mathcal{M}, \tag{3.15}$$

defined by

$$\mathcal{L}_V \alpha(x(0)) = \frac{d}{dt}\alpha(x(t))\bigg|_{t=0}. \tag{3.16}$$

It can be shown that this is equivalent to saying

$$\mathcal{L}_V = d\,i_v + i_v d, \tag{3.17}$$

so using anticommutative variables,

$$\begin{aligned}\mathcal{L}_V = d\,i_v + i_v d &= \left(\theta^\mu \frac{\partial}{\partial x^\mu}\right)\left(V^\nu \frac{\partial}{\partial \theta^\nu}\right) + \left(V^\nu \frac{\partial}{\partial \theta^\nu}\right)\left(\theta^\mu \frac{\partial}{\partial x^\mu}\right) \\ &= \theta^\mu \left(\frac{\partial V^\nu}{\partial x^\mu}\right)\frac{\partial}{\partial \theta^\nu} + \theta^\mu \theta^\nu \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial \theta^\nu} + V^\nu \left(\frac{\partial \theta^\mu}{\partial \theta^\nu}\right)\frac{\partial}{\partial x^\mu} - V^\nu \theta^\mu \frac{\partial}{\partial \theta^\nu}\frac{\partial}{\partial x^\mu} \\ &= \theta^\mu (\partial_\mu V^\nu)\frac{\partial}{\partial \theta^\nu} + V^\mu \frac{\partial}{\partial x^\mu}.\end{aligned} \tag{3.18}$$

Lie derivatives also constitute a representation of G, in this case in $\Lambda\mathcal{M}$:

$$[\mathcal{L}_{V^a}, \mathcal{L}_{V^b}](\alpha) = f^{abc}\mathcal{L}_{V^c}(\alpha), \qquad \alpha \in \Lambda\mathcal{M}. \tag{3.19}$$

Moreover, it can be shown that its action on contractions is $[i_{V^a}, \mathcal{L}_{V^b}](\alpha) = f^{abc} i_{V^c}(\alpha)$.

3.2 Equivariant Cohomology on Manifolds

Equivariant differential forms

Our first goal is to extend the notion of cohomology to the space of orbits, which can be understood as $\mathcal{M}$ modulo the action of G. A simple case is when the action is *free*, i.e. when for every $x \in \mathcal{M}$, $g \cdot x = x$ implies $g = e$. In this case the space of orbits, $\mathcal{M}/G$, with the inherited topology becomes a differential manifold of dimension dim $(\mathcal{M}/G)$ = dim $\mathcal{M}$ - dim G and we can simply define the $G-$equivariant cohomology of $\mathcal{M}$ to be the usual cohomology of $\mathcal{M}/G$,

$$H^k_G(\mathcal{M}) := H^k(\mathcal{M}/G). \tag{3.20}$$

Nonetheless, if the G-action is not free, i.e. it has fixed points, then $\mathcal{M}/G$ can become singular. Given $x \in \mathcal{M}$, its orbit has dimension dim $G \cdot x$ = dim G - dim G_x, where $G_x := \{g \in G \mid g \cdot x = x\}$ is the isotropy subgroup of x and might not be trivial for a non-free action. In this case, there would be no smooth concept of dimensionality and therefore $\mathcal{M}/G$ cannot be a manifold, so we cannot simply use (3.20) but more sophisticated methods should be introduced instead.

Consider the symmetric algebra over the dual vector space $\mathfrak{g}^*$, $S(\mathfrak{g}^*)$, composed by all symmetric polynomial functions of $\mathfrak{g}$. Let $S(\mathfrak{g}^*) \otimes \Lambda\mathcal{M}$ be the set of symmetric polynomials of $\mathfrak{g}$ that take values in the exterior algebra of $\mathcal{M}$. Then the action of G on $S(\mathfrak{g}^*) \otimes \Lambda\mathcal{M}$ is given by

$$\begin{array}{rlcl} act: & G \times S(\mathfrak{g}^*) \otimes \Lambda\mathcal{M} & \longrightarrow & S(\mathfrak{g}^*) \otimes \Lambda\mathcal{M} \\ & (g, \alpha) & \longmapsto & act_g \alpha, \end{array} \tag{3.21}$$

where $(act_g\alpha)(X) := g \cdot (\alpha(g^{-1}Xg))$ for $g \in G$ and $X \in \mathfrak{g}$. We denote by

$$\Lambda_G\mathcal{M} = (S(\mathfrak{g}^*) \otimes \Lambda\mathcal{M})^G \tag{3.22}$$

the $G-$invariant subalgebra, i.e. $\alpha \in \Lambda_G\mathcal{M}$ if $act_g\alpha = \alpha$. Let us now think $\alpha \in S(\mathfrak{g}^*) \otimes \Lambda\mathcal{M}$ as a map $\alpha : \mathfrak{g} \to \Lambda\mathcal{M}$. Then α is $G-$equivariant if

$$\alpha(gXg^{-1}) = g \cdot \alpha(X) \quad \Longleftrightarrow \quad \alpha(X) = g \cdot (\alpha(g^{-1}Xg)) \quad \Longleftrightarrow \quad \alpha(X) = (act_g\alpha)(X), \tag{3.23}$$

which is equivalent to say that $\alpha \in \Lambda_G\mathcal{M}$. This means that the so-called *equivariant differential forms* are exactly the elements of the $G-$invariant subalgebra $\Lambda_G\mathcal{M}$.

We can assign a $\mathbb{Z}$-grading to $P \otimes \alpha \in S(\mathfrak{g}^*) \otimes \Lambda\mathcal{M}$ with the formula

$$\deg(P \otimes \alpha) := 2\deg(P) + \deg(\alpha), \qquad P \in S(\mathfrak{g}^*),\ \alpha \in \Lambda\mathcal{M}, \tag{3.24}$$

so twice its degree as polynomial plus its degree as differential form. This allows us to write as well

$$\Lambda_G\mathcal{M} = \bigoplus_{k=0}^{\infty} \Lambda_G^k\mathcal{M}. \tag{3.25}$$

It is now time to introduce the *equivariant exterior derivative*, $d_{\mathfrak{g}}$, defined by the expression

$$(d_{\mathfrak{g}}\alpha)(X) = (d + i_V)(\alpha(X)), \qquad X \in \mathfrak{g}, \alpha \in S(\mathfrak{g}^*) \otimes \Lambda\mathcal{M}, \tag{3.26}$$

where $V = c^aV^a$ is the vector field on $\mathcal{M}$ corresponding to $X = c^aX^a \in \mathfrak{g}$. Using anticommutative variables this operator can be written as

$$(d_{\mathfrak{g}}\alpha)(X) = (d + i_V)(\alpha(X)) = \left(\theta^\mu \frac{\partial}{\partial x^\mu} + V^\mu \frac{\partial}{\partial \theta^\mu}\right)(\alpha(X)). \tag{3.27}$$

The first thing we have to check is that if $\alpha \in \Lambda_G M$, then $d_{\mathfrak{g}}\alpha \in \Lambda_G M$:

$$\begin{aligned} g \cdot (d_{\mathfrak{g}}\alpha)(X) &= g \cdot (d\alpha)(X) + g \cdot (i_V\alpha)(X) = d(g \cdot \alpha)(X) + i_{gVg^{-1}}\alpha(X) \\ &= d\alpha(gXg^{-1}) + i_{gVg^{-1}}\alpha(gXg^{-1}) = (d_{\mathfrak{g}}\alpha)(gXg^{-1}), \end{aligned} \tag{3.28}$$

so the condition (3.23) is satisfied.

It is easy to see that actually $d_{\mathfrak{g}}$ increases the degree of equivariant differential forms by one:

$$d_{\mathfrak{g}} : \Lambda_G^k\mathcal{M} \longrightarrow \Lambda_G^{k+1}\mathcal{M}. \tag{3.29}$$

Let $\{\xi^a\}$ be the basis of $\mathfrak{g}^*$ dual to $\{X^a\}$, i.e. $\xi^a(X^b) = \delta^{ab}$. By linearity of $d_{\mathfrak{g}}$ it is enough to consider elements of the form $P^i(\xi) \otimes \alpha^{(j)}$, where $P^i \in S(\mathfrak{g}^*)$ is a homogeneous polynomial of degree i, $\alpha^{(j)} \in \Lambda^j\mathcal{M}$ is a differential $j-$form and $2i + j = k$. Then we can write

$$d_{\mathfrak{g}}(P^i(\xi) \otimes \alpha^{(j)}) = P^i(\xi) \otimes d\alpha^{(j)} + \xi^a P^i(\xi) \otimes i_{V^a}\alpha^{(j)}, \tag{3.30}$$

because

$$d(P^i(\xi) \otimes \alpha^{(j)})(X) = (P^i(\xi) \otimes d\alpha^{(j)})(X) \tag{3.31}$$

and

$$\begin{aligned} i_V(P^i(\xi) \otimes \alpha^{(j)})(X) &= P^i(\xi)(X) \otimes i_V\alpha^{(j)}(X) = P^i(\xi)(X) \otimes i_{c^aV^a}\alpha^{(j)}(X) \\ &= P^i(\xi)(X) \otimes i_{\xi^a(X)V^a}\alpha^{(j)}(X) = \xi^a(X)P^i(\xi)(X) \otimes i_{V^a}\alpha^{(j)}(X), \end{aligned} \tag{3.32}$$

where we used that $\xi^a(X) = c^a$. Finally, if we compute the degrees

$$\begin{aligned} &\deg(P^i(\xi) \otimes d\alpha^{(j)}) = 2i + (j+1) = k+1 \\ &\deg(\xi^a P^i(\xi) \otimes i_{V^a}\alpha^{(j)}) = 2(i+1) + (j-1) = k+1, \end{aligned} \tag{3.33}$$

so $d_{\mathfrak{g}}\alpha$ has degree $k+1$.

Moreover, as it happened with the de Rahm exterior differential, $d_{\mathfrak{g}}$ is nilpotent in $\Lambda_G M$. We have:

$$\begin{aligned} d_{\mathfrak{g}}{}^2 &= \left(\theta^\mu \frac{\partial}{\partial x^\mu} + V^\mu \frac{\partial}{\partial \theta^\mu}\right)\left(\theta^\nu \frac{\partial}{\partial x^\nu} + V^\nu \frac{\partial}{\partial \theta^\nu}\right) = \theta^\mu \left(\frac{\partial V^\nu}{\partial x^\mu}\right)\frac{\partial}{\partial \theta^\nu} + V^\mu \left(\frac{\partial \theta^\nu}{\partial \theta^\mu}\right)\frac{\partial}{\partial x^\nu} \\ &= \theta^\mu(\partial_\mu V^\nu)\frac{\partial}{\partial \theta^\nu} + V^\mu \delta^\nu_\mu \frac{\partial}{\partial x^\nu} = \theta^\mu(\partial_\mu V^\nu)\frac{\partial}{\partial\ \theta^\nu} + V^\mu \frac{\partial}{\partial x^\mu} = \mathcal{L}_V, \end{aligned} \tag{3.34}$$

where we used the commutativity and anticommutativity properties of the variables. Since the equivariant differential forms are precisely those that are $G-$invariant, if $\alpha \in \Lambda_G\mathcal{M}$ then $\mathcal{L}_V\alpha = 0$ for any V corresponding to a $X \in \mathfrak{g}$ and therefore $(d_{\mathfrak{g}}, \Lambda^k_G\mathcal{M})$ builds a complex, called the *equivariant complex.*

Following the same structure as in the de Rahm cohomology, we define

- equivariantly closed forms α: $d_{\mathfrak{g}}\alpha = 0$.
- equivariantly exact forms α: $\exists\beta$ such that $\alpha = d_{\mathfrak{g}}\beta$.
- G-equivariant cohomology groups on $\mathcal{M}$:

$$H^k_G(\mathcal{M}) := \frac{\ker d_{\mathfrak{g}|\Lambda^k_G\mathcal{M}}}{\text{im } d_{\mathfrak{g}|\Lambda^{k-1}_G\mathcal{M}}}, \tag{3.35}$$

and with this we managed to define a cohomology for $\mathcal{M}$ taking into account the action of G, which was our initial goal. We can still make some final observations. For example, if $G = \{e\}$, then $V \equiv 0$ and the G-equivariant cohomology reduces to the usual de Rahm one. If we take $\mathcal{M} = \{pt\}$, then $H_G(pt) = S(\mathfrak{g}^*)^G$, since there are no differential forms on a point. Finally, it is worth observing that if $\alpha \in \Lambda_G\mathcal{M}$ is equivariantly exact, then its top component $\alpha^{(n)} \in \Lambda^n\mathcal{M}$ is exact in the de Rahm sense.

3.3 Equivariant Cohomology on Fibre Bundles

Consider now a vector bundle $(E, \pi, \mathcal{M})$. Assume E and $\mathcal{M}$ have G-actions that are compatible,

$$g \cdot \pi(x) = \pi(g \cdot x), \qquad \forall x \in E,\ \forall g \in G, \tag{3.36}$$

that is, π is an equivariant map. If moreover the G-action on E satisfies that the map $\gamma : E_x \to E_{\gamma\cdot x}$ is linear, we say that $(E, \pi, \mathcal{M})$ is a *G-equivariant vector bundle.* These G-action can be extended to $\Lambda(E)$ by the Lie derivatives, as we did before. Following the same steps, we can also define the *equivariant E-valued differential forms*:

$$\Lambda_G(E) := (S(\mathfrak{g}^*) \otimes \Lambda(E))^G \tag{3.37}$$

and assign the same $\mathbb{Z}$-grading. As usual we write

$$\Lambda_G(E) = \bigoplus_{k=0}^{\infty} \Lambda^k_G(E). \tag{3.38}$$

G also acts on the sections $s \in \Gamma(E)$ as

$$(g \cdot s)(x) = \gamma \cdot s(g^{-1} \cdot x). \tag{3.39}$$

Let now ∇ be a linear connection on E that commutes with the action of G on $\Lambda(E)$, i.e. $[\nabla, \mathcal{L}_V] = 0$ for all V corresponding to an $X \in \mathfrak{g}$. We say that such ∇ is a *G-invariant connection.* With it we can define the so-called *equivariant covariant derivative* $\nabla_\mathfrak{g}$ as

$$(\nabla_\mathfrak{g}\alpha)(X) = (\nabla + i_V)(\alpha(X)), \qquad X \in \mathfrak{g},\ \alpha \in S(\mathfrak{g}^*) \otimes \Lambda(E), \tag{3.40}$$

where V is again the vector field corresponding to X. As before, it can also be considered as an operator $\nabla_\mathfrak{g} : \Lambda_G^k(E) \longrightarrow \Lambda_G^{k+1}(E)$. Parallel to the bundle curvature, we can also define the *equivariant curvature* $F_\mathfrak{g}$, by the expression

$$(F_\mathfrak{g}\alpha)(X) = (\nabla_\mathfrak{g}\alpha)^2 + (\mathcal{L}_V\alpha)(X). \tag{3.41}$$

As a consequence of the usual Bianchi identity and that ∇ is G-invariant, the equivariant curvature satisfies the *equivariant Bianchi identity*:

$$[\nabla_\mathfrak{g}, F_\mathfrak{g}] = 0. \tag{3.42}$$

The difference between the equivariant curvature $F_\mathfrak{g}$ and the usual bundle curvature F is normally referred to as the *moment map* μ_V,

$$(F_\mathfrak{g}\alpha)(X) = (F\alpha)(X) + \mu_V(\alpha(X)), \tag{3.43}$$

where

$$\mu_V = \mathcal{L}_V - [i_V, \nabla]. \tag{3.44}$$

If we take a local trivialisation $U \times W$ of E, where $U \subseteq \mathcal{M}$, then we can regard the moment map as a function $\mu : \Lambda U \otimes W \to \mathfrak{g}^*$. It is also important to observe that from the equivariant Bianchi identity follows

$$\nabla\mu_V = i_V F, \tag{3.45}$$

or in other words, that a non-trivial moment map produces a non-zero vertical component of the connection ∇.

Equivariant characteristic classes

The notion of characteristic class that we saw in §2.1.2 can be generalised to equivariant bundles. Let now P be a G-invariant polynomial and take the equivariant curvature $\nabla_\mathfrak{g}$ and the equivariant exterior differential instead of the usual ones. Equation (2.58) generalises to

$$d_\mathfrak{g} P(F_\mathfrak{g}) = rP(\nabla_\mathfrak{g} F_\mathfrak{g}) = 0, \tag{3.46}$$

so that $P(F_\mathfrak{g})$ is an equivariant cohomology class of $\Lambda_G\mathcal{M}$ elements. We can evaluate $P(F_\mathfrak{g})$ on an element $X \in \mathfrak{g}$: $P(F_)(X) = P(F_V) =: P_V(F)$, where V is the vector field corresponding to X. One of the most common examples of an equivariant characteristic class is the *equivariant Euler class*,

$$E_\mathfrak{g} = \mathrm{Pfaff}(F_\mathfrak{g}). \tag{3.47}$$

3.4 Equivariant Cohomology on Symplectic Manifolds

Symplectic manifolds

We want now to apply the notions of equivariant cohomology on vector bundles to the particular case of sympletic manifolds. A *symplectic manifold* is an even dimensional differential manifold $\mathcal{M}$, dim $\mathcal{M} = 2n$, together with a globally defined non-degenerate closed 2-form ω, called the *symplectic form.* In local coordinates,

$$\omega = \frac{1}{2}\omega_{\mu\nu}(x)dx^\mu \wedge dx^\nu. \tag{3.48}$$

The closedness condition $d\omega = 0$ becomes in local coordinates,

$$\partial_\mu\omega_{\nu\lambda} + \partial_\nu\omega_{\lambda\mu} + \partial_\lambda\omega_{\mu\nu} = 0, \tag{3.49}$$

and the non-degeneracy condition simply means that if we think of $\omega(x)$ as a matrix, then

$$\det\omega(x) \neq 0, \qquad \forall x \in \mathcal{M}. \tag{3.50}$$

The symplectic form defines a non-trivial de Rahm cohomology class $[\omega] \in H^2(\mathcal{M};\mathbb{R})$ and by the Poincaré lemma, we can always find locally a 1-form $\vartheta = \vartheta_\mu(x)dx^\mu$ such that $\omega = d\vartheta$, or in local coordinates,

$$\omega_{\mu\nu} = \partial_\mu\vartheta_\nu - \partial_\nu\vartheta_\mu. \tag{3.51}$$

Such a 1-form is called *symplectic potential.* In case ϑ can be defined globally, we say that the system is *integrable.*

A key concept in symplectic geometry are the so-called *canonical* or *symplectic transformations*, that is, diffeomorphisms of $\mathcal{M}$ that leave ω invariant. They are determined by functions $F \in \mathcal{C}^\infty(\mathcal{M})$ in the following way

$$\begin{array}{lcl} \vartheta & \longrightarrow & \vartheta_F := \vartheta + dF \\ \vartheta_\mu(x) & \longrightarrow & \vartheta_{F,\mu}(x) = \vartheta_\mu(x) + \partial_\mu F(x) \end{array} \tag{3.52}$$

so that by nilpotency

$$\omega_F = d\vartheta_F = d\vartheta + d^2F = d\vartheta = \omega. \tag{3.53}$$

$F(x)$ is called the *generating function* of the canonical transformation.

The existence of ω is important because it allows us, among other things, to define the following bilinear function

$$\begin{array}{rccl} \{\cdot,\cdot\}_\omega : & \Lambda^0\mathcal{M}\otimes\Lambda^0\mathcal{M} & \longmapsto & \Lambda^0\mathcal{M} \\ & (f,g) & \longmapsto & \{f,g\}_\omega = \omega^{-1}(df,dg) \end{array} \tag{3.54}$$

called the *Poisson bracket.* In local coordinates,

$$\{f,g\}_\omega = \omega^{\mu\nu}(x)\,\partial_\mu f(x)\,\partial_\nu g(x), \tag{3.55}$$

where $\omega^{\mu\nu}$ is the inverse matrix of $\omega_{\mu\nu}$. The Poisson bracket satisfies the following properties:

- Antisymmetry: $\{f,g\}_\omega = -\{g,f\}_\omega$.
- Leibniz property: $\{f,gh\}_\omega = g\{f,h\}_\omega + h\{f,g\}_\omega$.
- Jacobi identity: $\{f,\{g,h\}_\omega\}_\omega + \{g,\{h,f\}_\omega\}_\omega + \{h,\{f,g\}_\omega\}_\omega = 0$. This is a consequence of ω being closed.

These properties make $\{\cdot,\cdot\}_\omega$ a Lie bracket and $(\Lambda^0\mathcal{M}, \{\cdot,\cdot\}_\omega)$ a Lie algebra, called the *Poisson algebra* of $(\mathcal{M}, \omega)$. The relation of symplectic manifolds with the usual classical Hamiltonian mechanics is given by Darboux's theorem (see [13]), which states that locally there exists a system of coordinates $(p_\mu, q^\mu)_{\mu=1}^n$ on $\mathcal{M}$, called Darboux coordinates, such that the symplectic 2-form looks like

$$\omega = dp_\mu \wedge dq^\mu. \tag{3.56}$$

In these terms,

$$\{p_\mu, p_\nu\}_\omega = \{q^\mu, q^\nu\}_\omega = 0, \qquad \{p_\mu, q^\nu\}_\omega = \delta_\mu^\nu, \tag{3.57}$$

and the symplectic potential can be written as $\vartheta = p_\mu dq^\mu$. A canonical transformation

$$\vartheta = p_\mu dq^\mu \to \vartheta_F = \vartheta + dF = P_\mu dQ^\mu, \tag{3.58}$$

where (P_μ, Q^μ) are also Darboux coordinates. Therefore we have

$$p_\mu dq^\mu - P_\mu dQ^\mu = dF. \tag{3.59}$$

The exterior products of ω with itself determine non-trivial $2k$-forms on $\mathcal{M}$. Especially important is the $2n$-form

$$d\mu_L = \frac{\omega^n}{n!} = \sqrt{\det \omega(x)}\; d^{2n}x, \tag{3.60}$$

which is a natural volume element invariant under canonical transformations, commonly referred to as *Liouville measure.*

Symplectic line bundles

Assume now that there is a connected Lie group G acting on $\mathcal{M}$, generating vector fields V^a and satisfying $[V^a, V^b](h) = f^{abc}V^c(h)$. We assume also that this G-action is symplectic, meaning that it preserves the symplectic structure

$$\mathcal{L}_{V^a}\omega = 0, \tag{3.61}$$

or in other words, G acts by symplectic transformations. Since $d\omega = 0$, this condition is equivalent to say $di_{V^a}\omega = 0$. Following the constructions of the last subchapter, define a complex line bundle $\pi : L \to \mathcal{M}$ with connection 1-form the symplectic potential ϑ. If, moreover, $\mathcal{L}_{V^a}\vartheta = 0$, then the associated covariant derivative $\nabla = d + \vartheta$ is G-invariant and therefore L is an equivariant vector bundle. The associated momentum map $H : \mathcal{M} \to \mathfrak{g}^*$, evaluated on $X \in \mathfrak{g}$ corresponding to V, is called the *Hamiltonian* corresponding to V:

$$H_V = H(X) = \mathcal{L}_V - [i_V, \nabla] = i_V\vartheta = V^\mu\vartheta_\mu, \tag{3.62}$$

which in particular implies that

$$dH_V = -i_V\omega, \tag{3.63}$$

or in local coordinates,

$$\partial_\mu H_V(x) = V^\nu(x)\omega_{\mu\nu}(x). \tag{3.64}$$

In general we will assume $\mathcal{L}_{V^a}\vartheta = 0$ instead of (3.61). If we can find a Hamiltonian that is globally defined as a $\mathcal{C}^\infty$ function, then we say that the group action is *Hamiltonian.* In these terms, a vector field V that satisfies (3.63) is said to be the Hamiltonian vector field associated with H_V, and we shall call the triple $(\mathcal{M}, \omega, H_V)$ a *Hamiltonian system* or *dynamical system.*

The integral curves of a Hamiltonian system $(\mathcal{M}, \omega, H_V)$, that is, the solutions $x(t)$ such that $\dot{x}(t) = V(x(t))$, define the usual *Hamilton equations of motion*,

$$\dot{x}^\mu(t) = \omega^{\mu\nu}(x(t))\, \partial_\nu H_V(x(t)) = \{x^\mu, H_V\}_\omega. \tag{3.65}$$

This last expression, the Poisson bracket of a function f with the Hamiltonian, actually determines the infinitessimal variation of any f along the integral curve $x(t)$ of the dynamical system:

$$\{f, H_V\}_\omega = \mathcal{L}_V f = \frac{d}{dt} f(x(t))\bigg|_{t=0}. \tag{3.66}$$

If we express (3.65) in the Darboux coordinates we get the usual form of the Hamilton equations of motion,

$$\dot{q}^\mu = \frac{\partial H}{\partial p_\mu}, \qquad \dot{p}_\mu = -\frac{\partial H}{\partial q^\mu}. \tag{3.67}$$

If we now calculate the curvature of the equivariant bundle $\pi : L \to \mathcal{M}$ as in (3.41), we have

$$\omega_{\mathfrak{g}} = \omega + H_V \tag{3.68}$$

and evaluating it at a $X \in \mathfrak{g}$ we get

$$(d_{\mathfrak{g}}\omega_{\mathfrak{g}})(X) = (d + i_V)(\omega + H_V) = 0, \tag{3.69}$$

so $\omega_{\mathfrak{g}}$ is equivariantly closed. We usually say that $\omega_{\mathfrak{g}}$ is the equivariant extension of the symplectic form ω, since it is the only extension from a closed 2-form to an equivariantly closed one.

3.5 Equivariant Cohomology on Supermanifolds

Superalgebras and superbundles

We will provide some basic definitions of supergeometry to start with. A *superspace* E is a $\mathbb{Z}_2-$graded vector space, $E = E^+ \oplus E^-$. If this space is provided with a product that respects the $\mathbb{Z}_2-$grading, i.e. $A^i \cdot A^j \subset A^{i+j}$, where $i, j = 0, 1 \pmod 2$, then we say that it is a *superalgebra*. Two common examples of a superalgebra are the following

- The exterior algebra of a vector space V,

$$\Lambda^\pm V := \sum_{(-1)^i = \pm 1} \Lambda^i V. \tag{3.70}$$

- The algebra $\mathrm{End}(E)$ of endomorphisms of a superspace E,

$$\begin{aligned} \mathrm{End}^+ &:= \mathrm{Hom}(E^+, E^+) \oplus \mathrm{Hom}(E^-, E^-) \\ \mathrm{End}^- &:= \mathrm{Hom}(E^+, E^-) \oplus \mathrm{Hom}(E^-, E^+). \end{aligned} \tag{3.71}$$

The *supercommutator* of a pair of elements of a superalgebra is defined as

$$[a, b] := ab - (-1)^{|a|\cdot|b|} ba, \tag{3.72}$$

which satisfies the axioms of a *Lie superalgebra*:

- $[a,b] + (-1)^{|a|\cdot|b|}[b,a] = 0$
- $[a,[b,c]] = [[a,b],c] + (-1)^{|a|\cdot|b|}[b,[a,c]]$

We say that a superalgebra is *supercommutative* if $[\cdot,\cdot] \equiv 0$. ΛV is an example of a supercommutative superalgebra.

A *superbundle* on $\mathcal{M}$ is a vector bundle $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$, where $\mathcal{E}^+, \mathcal{E}^-$ are vector bundles on $\mathcal{M}$. In other words, it is a bundle whose fibres are superspaces. Let $\Lambda(\mathcal{E}) = \Gamma(\mathcal{E}) \otimes \Lambda\mathcal{M}$ be the space of $\mathcal{E}$-valued differential forms on $\mathcal{M}$. This space has an inherent $\mathbb{Z}-$grading -the degree of the forms-, and we can also define a $\mathbb{Z}_2-$grading defined by $\Lambda(\mathcal{E}) = \Lambda^+(\mathcal{E}) \oplus \Lambda^-(\mathcal{E})$, where

$$\Lambda^{\pm}(\mathcal{E}) := \sum_i \Lambda^{2i}(\mathcal{E}^{\pm}) \oplus \sum_i \Lambda^{2i+1}(\mathcal{E}^{\mp}). \tag{3.73}$$

As a simple example, $\Gamma(\mathcal{E}^{\pm}) \subset \Lambda^{\pm}(\mathcal{E})$.

If the fibres of a superbundle have a Lie superalgebra structure, then we can talk about a bundle of Lie superalgebras $\mathfrak{G}$. Then $\Lambda(\mathfrak{G})$ is itself a Lie superalgebra with respect to the Lie superbracket

$$[\alpha_1 \otimes X_1, \alpha_2 \otimes X_2] = (-1)^{|X_1|\cdot|\alpha_2|}(\alpha_1 \wedge \alpha_2) \otimes [X_1, X_2], \tag{3.74}$$

where $\alpha_1, \alpha_2 \in \Lambda\mathcal{M}$ and $X_1, X_2 \in \mathfrak{G}_x$.

A slightly more sophisticated structure is the following: consider a module E over a superalgebra $\mathfrak{g}$, provided with an action $\rho : \mathfrak{g} \to \mathrm{Diff}(E)$. Generalising this, we can consider a superbundle $\mathcal{E}$ on a manifold $\mathcal{M}$, where each fibre is a supermodule E_x over a superalgebra $\mathfrak{G}_x$ with an action $\rho_x : \mathfrak{G}_x \to \mathrm{Diff}(\mathcal{E}_x)$. We can also talk about a global action $\rho : \mathfrak{G} \to \mathrm{Diff}(\mathcal{E})$ by joining all the fibre actions. Finally we can construct $\Lambda(\mathcal{E})$ as a supermodule -which is defined in the same way as a superspace- for $\Lambda(\mathfrak{G})$ with respect to the action $\rho : \Lambda(\mathfrak{G}) \to \mathrm{Diff}(\Lambda(\mathcal{E}))$, defined as

$$\rho(\alpha \otimes X)(\beta \otimes v) = (-1)^{|X|\cdot|\beta|}(\alpha \wedge \beta) \otimes (\rho(X)v), \tag{3.75}$$

where $\alpha, \beta \in \Lambda\mathcal{M}$, $X \in \mathfrak{G}_x$ and $v \in \mathcal{E}_x$. An example of this structure is the bundle of Lie superalgebras $\mathrm{End}(\mathcal{E})$ of a superbundle $\mathcal{E}$ on $\mathcal{M}$, since $\Lambda\mathcal{M}\otimes\mathcal{E}$ is a bundle of modules for the superalgebra $\Lambda\mathcal{M}\otimes\mathrm{End}(\mathcal{E})$. In other words, $\Lambda(\mathrm{End}(\mathcal{E}))$ is a superalgebra that has $\Lambda(\mathcal{E})$ as a supermodule.

It can be shown that a any differential operator on $\Lambda(\mathcal{E})$ which supercommutes with the action of $\Lambda\mathcal{M}$ is given by the action of an element of $\Lambda(\mathrm{End}(\mathcal{E}))$. Such operators are called *local*.

Superconnections

Given a superbundle $\mathcal{E}$ over $\mathcal{M}$, a *superconnection* is an odd-parity first-order differential operator

$$\mathbb{A} : \Lambda^{\pm}(\mathcal{E}) \to \Lambda^{\mp}(\mathcal{E}) \tag{3.76}$$

which satisfies the $\mathbb{Z}_2$-Leibniz's rule,

$$\mathbb{A}(\alpha \wedge \vartheta) = d\alpha \wedge \vartheta + (-1)^{|\alpha|}\alpha \wedge \mathbb{A}\vartheta, \tag{3.77}$$

where $\alpha \in \Lambda\mathcal{M}$ and $\vartheta \in \Lambda(\mathcal{E})$. If $\mathbb{A}$ is a superconnection on $\mathcal{E}$, then $\mathbb{A}$ is extended to act on $\Lambda(\mathrm{End}(\mathcal{E}))$ in a way consistent with the $\mathbb{Z}_2$-Leibniz's rule,

$$\mathbb{A}\alpha = [\mathbb{A}, \alpha], \qquad \alpha \in \Lambda(\mathrm{End}(\mathcal{E})). \tag{3.78}$$

We observe that since $[\mathbb{A}, \alpha]$ is an operator that commutes with the exterior multiplication by any $\beta \in \Lambda\mathcal{M}$, $\mathbb{A}\alpha \in \Lambda(\mathrm{End}(\mathcal{E}))$.

In a similar way to bundle connections, the *curvature* of a superconnection $\mathbb{A}$ is defined as the operator $F := \mathbb{A}^2$ on $\Lambda(\mathcal{E})$. The curvature is an example of a local operator, i.e. it is given by the action of a differential form $\omega_F \in \Lambda(\mathrm{End}(\mathcal{E}))$, with even total degree and satisfying the Bianchi identity $\mathbb{A}\omega_F = 0$. To see that its local, it is enough to see that it supercommutes with the multiplication by any $\alpha \in \Lambda\mathcal{M}$:

$$[\mathbb{A}^2, \alpha \wedge \cdot] = [\mathbb{A}, [\mathbb{A}, \alpha \wedge \cdot]] = (d^2\alpha) \wedge \cdot = 0. \tag{3.79}$$

And the Bianchi identity follows from the obvious calculation $\mathbb{A}\omega_F = [\mathbb{A}, \mathbb{A}^2] = 0$. If $\mathbb{A}$ is a connection in itself, then both curvatures coincide.

A superconnection $\mathbb{A}$ is entirely determined by its restriction to $\Gamma(\mathcal{E})$, which is an operator $\mathbb{A} : \Gamma(E) \longrightarrow \Lambda^{\pm}(\mathcal{E})$ that satisfies

$$\mathbb{A}(fs) = df \otimes s + f\mathbb{A}s, \qquad \forall f \in \mathcal{C}^{\infty}(\mathcal{M}),\ s \in \Gamma(\mathcal{E}). \tag{3.80}$$

From here, we just have to extend the operator to all $\Lambda(\mathcal{E})$ by the expression

$$\mathbb{A}(\alpha \otimes s) = d\alpha \otimes s + (-1)^{|\alpha|}\alpha \wedge \mathbb{A}s, \qquad \forall \alpha \in \Lambda\mathcal{M},\ s \in \Gamma(\mathcal{E}). \tag{3.81}$$

The expression (3.80) can help us get a better insight of the action of the superconnection. We can break $\mathbb{A}$, acting on $\Gamma(\mathcal{E})$, into its homogeneous components $\mathbb{A}_i : \Gamma(\mathcal{E}) \to \Lambda^i(\mathcal{E})$

$$\mathbb{A} = \mathbb{A}_0 + \mathbb{A}_1 + \mathbb{A}_2 + ... \tag{3.82}$$

so that (3.80) becomes

$$\mathbb{A}(fs) = \sum_{i=0}^{n} \mathbb{A}_i(fs) = df \otimes s + f\sum_{i=0}^{n} \mathbb{A}_i s. \tag{3.83}$$

If we decompose this equation degree-wise, we see that $\mathbb{A}_1(fs) = df \otimes s + f\mathbb{A}_1 s$, so $\mathbb{A}_1$ is a covariant derivative on $\mathcal{E}$. Moreover, since $\mathbb{A}_1$ has odd total degree, it preserves the $\mathbb{Z}_2-$grading of $\mathcal{E}$, meaning that it is a direct sum of covariant derivatives on the bundles $\mathcal{E}^{\pm}$ and that therefore they are both preserved by $\mathbb{A}_1$. For $i \neq 1$, $\mathbb{A}_i(fs) = f\mathbb{A}_i s$, i.e. $\mathbb{A}_i$ is given by the action of a differential form $\omega_i \in \Lambda(\mathrm{End}(\mathcal{E}))$, with ω_i being of odd total degree, following that $\mathbb{A}_i$ are local operators.

Chapter 4

Finite-dimensional Localisation

In this chapter we present the main results of localisation on the different scenarios considered in the previous chapter and prove them using the language of equivariant cohomology.

4.1 Localisation Principle and the Berline-Vergne Theorem

Geometrical setting

One of the biggest applications of equivariant cohomology is the simplification of integral calculations that appear in theoretical physics through so-called *localisation* processes. In this section we will analise the most simple example of such localisation results. Let $\mathcal{M}$ be an even-dimensional compact orientable manifold without boundary, $\partial\mathcal{M} = \emptyset$, and G a Lie group acting on the manifold $G \times \mathcal{M} \longrightarrow \mathcal{M}$. For the sake of simplicity, we will first assume that $G = U(1) \simeq S^1$. Let $V \in \Gamma(T\mathcal{M})$ be the vector field on $\mathcal{M}$ corresponding to the G-action. Our first step is to describe the equivariant cohomology corresponding to this setting. Since dim G=1, $\mathfrak{u}(1) = \mathbb{R}$ and therefore the role of $\phi \in S(\mathfrak{u}(1)^*)$ is not relevant. It is customary to set $\phi = 1$, even though the final results can be proven to be independent of this choice. In these terms, the equivariant exterior differential becomes

$$d_V := d_{\mathfrak{u}(1)} = d + i_V = \theta^\mu \frac{\partial}{\partial x^\mu} + V^\mu \frac{\partial}{\partial \theta^\mu}, \tag{4.1}$$

where $V = V^\mu \partial/\partial x^\mu$ is the expression of the vector field in local coordinates (x^μ). The operator d_V acts on the space of G-equivariant forms (i.e. G-invariant forms), $\Lambda_V\mathcal{M} = \{\alpha \in \Lambda\mathcal{M} \mid \mathcal{L}_V\alpha = 0\}$. Define also the set

$$\mathcal{M}_V := \{x \in \mathcal{M} \mid V(x) = 0\}, \tag{4.2}$$

often called the *fixed point locus* of the G-action. For our purposes, in this chapter we will assume that $\mathcal{M}_V$ consists of the union of isolated points. Assume further that $\mathcal{M}$ has a $G-$invariant Riemannian structure metric tensor g such that

$$\mathcal{L}_V g = 0. \tag{4.3}$$

Equation (4.3) translates in local coordinates to

$$g_{\mu\lambda}\partial_\nu V^\lambda + g_{\nu\lambda}\partial_\mu V^\lambda + V^\lambda \partial_\lambda g_{\mu\nu} = 0. \tag{4.4}$$

In this section we will work with the *Levi-Civita connection* on $T\mathcal{M}$ that we introduced in §2.1.2. The Levi-Civita connection acts on vector fields $V = V^\mu \partial/\partial x^\mu$ as

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\nu\lambda} V^\lambda. \tag{4.5}$$

It is interesting to see that the condition (4.3) can also be expressed as

$$g_{\nu\lambda}\nabla_\mu V^\lambda + g_{\mu\lambda}\nabla_\nu V^\lambda = 0. \tag{4.6}$$

This can be easily proven:

$$\begin{aligned}
g_{\nu\lambda}\nabla_\mu V^\lambda + g_{\mu\lambda}\nabla_\nu V^\lambda &= g_{\nu\lambda}\partial_\mu V^\lambda + g_{\nu\lambda}\Gamma^\lambda_{\mu\rho}V^\rho + g_{\mu\lambda}\partial_\nu V^\lambda + g_{\mu\lambda}\Gamma^\lambda_{\nu\rho}V^\rho \\
&= g_{\nu\lambda}\partial_\mu V^\lambda + \frac{1}{2}V^\rho\partial_\mu g_{\rho\nu} + \frac{1}{2}V^\rho\partial_\rho g_{\mu\nu} - \frac{1}{2}V^\rho\partial_\upsilon g_{\mu\rho} \\
&+ g_{\mu\lambda}\partial_\nu V^\lambda + \frac{1}{2}V^\rho\partial_\nu g_{\rho\mu} + \frac{1}{2}V^\rho\partial_\rho g_{\nu\mu} - \frac{1}{2}V^\rho\partial_\mu g_{\nu\rho} \\
&= g_{\nu\lambda}\partial_\mu V^\lambda + g_{\mu\lambda}\partial_\nu V^\lambda + V^\rho\partial_\rho g_{\mu\nu} = 0,
\end{aligned} \tag{4.7}$$

where we used (4.4) in the last step. In terms of Riemannian geometry, we say that these equations express the fact that V is a Killing vector of the metric g. It is noteworthy that if $\mathcal{M}$ and G are compact, then we can always find such a g so that V is a Killing vector (by averaging over the Haar measure). Let us observe that the existence of a Riemannian metric g allows us to construct an isomorphism between the space of vector fields and the space of 1-forms:

$$\begin{array}{ccl}
T\mathcal{M} & \longrightarrow & T^*\mathcal{M} \\
V & \longmapsto & \beta \equiv g(V,\cdot) = g_{\mu\nu}(x)V^\mu(x)dx^\nu,
\end{array} \tag{4.8}$$

which is an isomorphism because det $g(x) \neq 0$. In the particular case of V being the vector field corresponding to the G-action, $\beta = g_{\mu\nu}V^\mu\theta^\nu$ presents the interesting property

$$\begin{aligned}
\mathcal{L}_V\beta &= \left(\theta^\mu(\partial_\mu V^\nu)\frac{\partial}{\partial\theta^\nu} + V^\mu\frac{\partial}{\partial x^\mu}\right)g_{\lambda\rho}V^\lambda\theta^\rho \\
&= (\partial_\mu V^\nu)g_{\lambda\rho}V^\lambda\theta^\mu\delta^\rho_\nu + V^\mu V^\lambda(\partial_\mu g_{\lambda\rho})\theta^\rho + V^\mu g_{\lambda\rho}(\partial_\mu V^\lambda)\theta^\rho \\
&= (\partial_\mu V^\rho)g_{\lambda\rho}V^\lambda\theta^\mu \quad + V^\mu V^\lambda(\partial_\mu g_{\lambda\rho})\theta^\rho + V^\mu g_{\lambda\rho}(\partial_\mu V^\lambda)\theta^\rho \\
&= \left[g_{\lambda\mu}(\partial_\rho V^\mu) + V^\mu(\partial_\mu g_{\lambda\rho}) + g_{\mu\rho}(\partial_\lambda V^\mu)\right]V^\lambda\theta^\rho = 0,
\end{aligned} \tag{4.9}$$

where in the second last step we relabelled the first term ($\mu \leftrightarrow \rho$) and the last term ($\mu \leftrightarrow \lambda$) and in the last step we used (4.4). We can also calculate

$$\begin{aligned}
d_V\beta &= \left(\theta^\mu\frac{\partial}{\partial x^\mu} + V^\mu\frac{\partial}{\partial\theta^\mu}\right)g_{\nu\lambda}V^\nu\theta^\lambda \\
&= (\partial_\mu g_{\nu\lambda})V^\nu\theta^\mu\theta^\lambda + g_{\nu\lambda}(\partial_\mu V^\nu)\theta^\mu\theta^\lambda + g_{\mu\nu}V^\mu V^\nu =: \Omega_V + K_V,
\end{aligned} \tag{4.10}$$

where

$$\Omega_V = d\beta = \frac{1}{2}(\Omega_V)_{\mu\lambda}\theta^\mu\theta^\lambda \tag{4.11}$$

is the 2-form with local components

$$(\Omega_V)_{\mu\lambda} = 2\left((\partial_\mu g_{\nu\lambda})V^\nu + g_{\nu\lambda}(\partial_\mu V^\nu)\right) \tag{4.12}$$

and K_V is the $\mathcal{C}^\infty$ function

$$K_V = g_{\mu\nu}V^\mu V^\nu. \tag{4.13}$$

The Berline-Vergne Theorem

Consider now an equivariantly-closed form $\alpha \in \Lambda_G\mathcal{M}$, $d_V\alpha = 0$. Our goal is to calculate $\int_{\mathcal{M}} \alpha$ and show that actually its value is determined by the values of α on $\mathcal{M}_V$. This fact is known as the *localisation principle.* To start with, consider first a closed form ω in the de Rahm sense, $d\omega = 0$. Then, $\forall \lambda \in \Lambda\mathcal{M}$,

$$\int_{\mathcal{M}} \omega + d\lambda = \int_{\mathcal{M}} \omega, \tag{4.14}$$

because $\partial\mathcal{M} = \emptyset$ and by Stokes' theorem $\int_{\mathcal{M}} d\lambda = \int_{\partial M} \lambda = 0$. In other words, the value of $\int_{\mathcal{M}} \omega$ depends only on the cohomological class of ω. In general, we can descend the integral map

$$\int_{\mathcal{M}} : \begin{array}{ccc} \Lambda^k\mathcal{M} & \longrightarrow & \Lambda^{n-k}(pt) = \delta^{nk}\mathbb{R} \\ \omega & \longmapsto & \int_{\mathcal{M}} \omega, \end{array} \tag{4.15}$$

to a linear homomorphism of the cohomology groups

$$\int_{\mathcal{M}} : H^n(\mathcal{M};\mathbb{R}) \to H^0(pt;\mathbb{R}) = \mathbb{R}. \tag{4.16}$$

Consider now an equivariantly-closed form λ. The condition $d_V\lambda = 0$ implies that its top component is closed in the de Rahm sense, so the equivariant version of (4.14) is also true:

$$\int_{\mathcal{M}} \alpha + d_V\lambda = \int_{\mathcal{M}} \alpha. \tag{4.17}$$

Therefore, we can think the integration map as $\displaystyle\int_{\mathcal{M}} : H_G(\mathcal{M}) \to H_G(pt) = S(\mathfrak{g}^*)^G$ by using the identification

$$\left(\int_{\mathcal{M}} \alpha\right)(X) := \int_{\mathcal{M}} \alpha(X), \qquad X \in \mathfrak{g}. \tag{4.18}$$

As it happened in the de Rahm case, $\int_{\mathcal{M}} \alpha$ only depends on the equivariant cohomology class of α and not of the particular representant, so we can always choose the one that is most convenient for us.

We have now enough elements to introduce the first localisation result:

Theorem 4.1 (**Berline-Vergne Theorem**)**.** *Let $\mathcal{M}$ be an even-dimensional compact orientable manifold without boundary, $\partial\mathcal{M} = \emptyset$, with an action $U(1) \times \mathcal{M} \longrightarrow \mathcal{M}$. Let $V \in \Gamma(T\mathcal{M})$ be the vector field on $\mathcal{M}$ corresponding to the $U(1)$-action and assume $\mathcal{M}_V = \{x \in \mathcal{M} \mid V(x) = 0\}$ consists only of isolated points. Then if α is an equivariantly-closed form, $d_V\alpha = 0$, we have*

$$\int_{\mathcal{M}} \alpha = \sum_{x_i \in M_V} (-2\pi)^{n/2} \frac{\alpha^{(0)}(x_i)}{\text{Pfaff } \partial V(x_i)}, \tag{4.19}$$

where $\alpha^{(0)}$ is the 0-th component of α.

This theorem already expresses how powerful localisation results are. In this case, a finite-dimensional integral is reduced to a simple sum. To prove the theorem, consider now the integral

$$\mathcal{Z}(s) = \int_{\mathcal{M}} \alpha\, e^{-s d_V \beta} = \int_{\mathcal{M}} \alpha\, e^{-s(\Omega_V + K_V)}, \qquad s \in \mathbb{R}^+ \tag{4.20}$$

and think of it as a function of the parameter s. Assume this function is regular and its limits $s \to 0^+$ and $s \to +\infty$ exist. It is clear that the first one is the value we are interested in. Since deg $\beta = 1$,

$$d_V(\beta\, e^{-sd_v\beta}) = (d_V\beta)\, e^{-sd_v\beta} - \beta\, d_V(e^{-sd_V\beta}). \tag{4.21}$$

Reordering terms and multiplying by α:

$$\alpha\,(d_V\beta)\, e^{-sd_V\beta} = \alpha\, d_V(\beta\, e^{-sd_V\beta}) + \alpha\,\beta\, d_V(e^{-sd_V\beta}) \tag{4.22}$$

and using the fact that $d_V\alpha = 0$ we have

$$\alpha\,(d_V\beta)\, e^{-sd_V\beta} = d_V(\alpha\beta\, e^{-sd_V\beta}) + \beta\, d_V(\alpha\, e^{-sd_V\beta}) \tag{4.23}$$

Using (4.23) we get

$$\frac{d}{ds}\mathcal{Z}(s) = \frac{d}{ds}\int_{\mathcal{M}} \alpha\, e^{-sd_V\beta} = -\int_{\mathcal{M}} \alpha\,(d_V\beta)\, e^{-sd_V\beta} = -\int_{\mathcal{M}} \left\{d_V(\alpha\,\beta\, e^{-sd_V\beta}) + \beta\, d_V(\alpha\, e^{-sd_V\beta})\right\} \tag{4.24}$$

but since the first term is the integral of the total equivariant differential of a form, it vanishes. Regarding the second term and using the fact that $d_V^2\beta = \mathcal{L}_V\beta = 0$ we get

$$\frac{d}{ds}\mathcal{Z}(s) = -\int_{\mathcal{M}} \beta\, d_V(\alpha\, e^{-sd_V\beta}) = -\int_{\mathcal{M}} \beta\alpha d_V(e^{-sd_V\beta}) = s\int_{\mathcal{M}} \beta\,\alpha\,(d_V^2\beta)\, e^{-sd_V\beta} = 0. \tag{4.25}$$

Therefore the value of the integral does not depend on the parameter s, i.e. $\alpha\, e^{-sd_V\beta} \in [\alpha]$. The limit $s \to +\infty$ will be especially interesting for us, as it will make the localisation manifest and through the Laplace approximation we will get the exact value of the integral. Let $x_i \in \mathcal{M}_V$ be a fixed point of the vector field V. We can expand $V(x)$ and $g(x)$ around this point

$$V^\mu(x) = V^\mu{}_{|_{x_i}} + (\partial_\rho V^\mu)_{|_{x_i}}(x-x_i)^\rho + \mathcal{O}(x^2) = (\partial_\rho V^\mu)_{|_{x_i}}(x-x_i)^\rho + \mathcal{O}(x^2) \tag{4.26}$$

$$g_{\mu\nu}(x) = g_{\mu\nu}{}_{|_{x_i}} + (\partial_\rho g_{\mu\nu})_{|_{x_i}}(x-x_i)^\rho + \mathcal{O}(x^2) \tag{4.27}$$

and from this we can expand Ω_V and K_V to the first non-vanishing order

$$\begin{aligned}(\Omega_V)_{\mu\lambda}(x) &= (\partial_\mu g_{\lambda\nu})(x)V^\nu(x) + g_{\lambda\nu}(x)(\partial_\mu V^\nu)(x) - (\partial_\lambda g_{\mu\nu})(x)V^\nu(x) - g_{\mu\nu}(x)(\partial_\lambda V^\nu)(x)\\ &= g_{\lambda\nu}{}_{|_{x_i}}(\partial_\mu V^\nu)_{|_{x_i}} - g_{\mu\nu}{}_{|_{x_i}}(\partial_\lambda V^\nu)_{|_{x_i}} + \mathcal{O}(x)\end{aligned} \tag{4.28}$$

$$\begin{aligned}\Omega_V(x) &= \frac{1}{2}(\Omega_V)_{\mu\lambda}(x)\theta^\mu\theta^\lambda = \frac{1}{2}\left(g_{\lambda\nu}{}_{|_{x_i}}(\partial_\mu V^\nu)_{|_{x_i}} - g_{\mu\nu}{}_{|_{x_i}}(\partial_\lambda V^\nu)_{|_{x_i}}\right) + \mathcal{O}(x)\\ &= g_{\lambda\nu}{}_{|_{x_i}}(\partial_\mu V^\nu)_{|_{x_i}}\theta^\mu\theta^\lambda + \mathcal{O}(x)\end{aligned} \tag{4.29}$$

$$K_V(x) = g_{\mu\nu}(x)V^\mu(x)V^\nu(x) = g_{\mu\nu}{}_{|_{x_i}}(\partial_\rho V^\mu)_{|_{x_i}}(\partial_\sigma V^\nu)_{|_{x_i}}(x-x_i)^\rho(x-x_i)^\sigma + \mathcal{O}(x^3). \tag{4.30}$$

We can now start the final calculation

$$\begin{aligned}\int_{\mathcal{M}}\alpha &= \lim_{s\to\infty}\int_{\mathcal{M}}\alpha\, e^{-sd_V\beta} = \lim_{s\to\infty}\int_{\mathcal{M}}\alpha\, e^{-s(K_V+\Omega_V)}\\ &= \lim_{s\to\infty}\int_{\mathcal{M}\otimes\Lambda^1\mathcal{M}} d^nx\, d^n\theta\, \alpha(x,\theta)\, e^{-s[K_V(x)+\frac{1}{2}(\Omega_V)_{\mu\lambda}(x)\theta^\mu\theta^\lambda]}\end{aligned} \tag{4.31}$$

but in the limit $s \to \infty$ the integrand localises around $\mathcal{M}_V$, so we can use the expansions (4.30) and sum over all points in $\mathcal{M}_V$, extending the integration domain to the whole $\mathbb{R}^n$ as the function vanishes anyway away from the origin and it will be more convenient

$$\begin{aligned}&= \lim_{s\to\infty}\sum_{x_i\in\mathcal{M}_V}\int_{\mathbb{R}^n\otimes\Lambda^1\mathbb{R}^n} d^nx\, d^n\theta\, \alpha^{(0)}(x_i)\, \exp\left(-sg_{\nu\lambda}{}_{|_{x_i}}(\partial_\mu V^\nu)_{|_{x_i}}\theta^\mu\theta^\lambda\right)\\ &\qquad\times\exp\left(-sg_{\mu\nu}{}_{|_{x_i}}(\partial_\rho V^\mu)_{|_{x_i}}(\partial_\sigma V^\nu)_{|_{x_i}}(x-x_i)^\rho(x-x_i)^\sigma\right).\end{aligned} \tag{4.32}$$

But this is nothing else than a Gaussian integral times a Grassman Gaussian integral, as in (2.83), so we can evaluate it exactly,

$$\begin{aligned}
&= \sum_{x_i\in\mathcal{M}_V} \alpha^{(0)}(x_i)\,\mathrm{Pfaff}\left(-2\,s\,g_{\nu\lambda|_{x_i}}(\partial_\mu V^\nu)\right)\sqrt{\frac{(2\pi)^n}{\det\left(2\,s\,g_{\mu\nu|_{x_i}}(\partial_\rho V^\mu)_{|x_i}(\partial_\sigma V^\nu)_{|x_i}\right)}}\\
&= \sum_{x_i\in\mathcal{M}_V} \alpha^{(0)}(x_i)\,\sqrt{(-2s)^n}\,\mathrm{Pfaff}\left(g_{\nu\lambda|_{x_i}}(\partial_\mu V^\nu)_{|x_i}\right)\sqrt{\frac{(2\pi)^n}{(2s)^n\det(g_{\mu\nu})_{|x_i}\det\left(\partial_\rho V^\nu\right)_{|x_i}\det\left(\partial_\sigma V^\nu\right)_{|x_i}}}\\
&= \sum_{x_i\in M_V}(-2\pi)^{n/2}\frac{\alpha^{(0)}(x_i)}{\mathrm{Pfaff}\,\partial V(x_i)}.
\end{aligned}\tag{4.33}$$

which is what we wanted to see.

It is important to observe that the invariance of the integration measure under linear variable changes is the reason why the quadratic expansion is actually exact. To see this, consider the variable change $x\to x/\sqrt{s}$, $\theta\to\theta/\sqrt{s}$. Looking at (4.30), we can see that in the limit $s\to\infty$ of $s(\Omega_V+K_V)$, the only terms that will not depend on a negative power of s are the ones explicitly written.

Interpretation of the denominator

We can try to give a geometrical interpretation to the denominator Pfaff $\partial V(x_i)$. We will need the fact that the Lie derivative $\mathcal{L}_V$ of a vector field V is not only defined on $\Lambda\mathcal{M}$, but can be extended to any tensor (see [2]). In particular, it can be extended to vector fields $W=W^\mu\partial/\partial x^\mu$:

$$\mathcal{L}_V W=[V,W]=V^\mu\frac{\partial W^\nu}{\partial x^\mu}\frac{\partial}{\partial x^\nu}-W^\mu\frac{\partial V^\nu}{\partial x^\mu}\frac{\partial}{\partial x^\nu}=\left(V^\mu\frac{\partial W^\nu}{\partial x^\mu}-W^\mu\frac{\partial V^\nu}{\partial x^\mu}\right)\frac{\partial}{\partial x^\nu}.\tag{4.34}$$

Let us now consider a point $x_i\in\mathcal{M}_V$. Since $V(x_i)=0$, we can write the restriction of $\mathcal{L}_V$ to the space $T_{x_i}\mathcal{M}$ as

$$\begin{array}{rccl} L_{x_i}: & T_{x_i}\mathcal{M} & \longrightarrow & T_{x_i}\mathcal{M}\\ & W^\mu & \longmapsto & -(\partial_\mu V^\nu)_{|x_i}W^\mu, \end{array}\tag{4.35}$$

which is a linear transformation. L_{x_i} is moreover invertible. To see this, let us assume that $\exists W\in T_{x_i}\mathcal{M}$, $W\neq 0$ such that $L_{x_i}W=0$. Then consider the integral curve $x(t)$ such that $x(0)=x_i$ and $\dot{x}(0)=W$. All the points of this curve would be invariant under the action of V, which would be a contradiction to x_i being isolated, so L_{x_i} is indeed invertible. Another interesting property is that since L_{x_i} is the Lie derivative of an action of a compact Lie group, it has only imaginary eigenvalues. Therefore, $\exists\{e_\mu\}$ oriented basis of $T_{x_i}\mathcal{M}$ such that

$$L_{x_i}=\begin{pmatrix} 0 & -\lambda_1 & & & & & \\ \lambda_1 & 0 & & & & & \\ & & 0 & -\lambda_2 & & & \\ & & \lambda_2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & -\lambda_l \\ & & & & & \lambda_l & 0 \end{pmatrix},\tag{4.36}$$

$l=n/2$, i.e.

$$\begin{aligned} L_{x_i}e_{2j-1} &= \lambda_j e_{2j}\\ L_{x_i}e_{2j} &= -\lambda_j e_{2j-1} \end{aligned}\tag{4.37}$$

In these terms, the denominator becomes

$$\text{Pfaff } \partial V(x_i) = \text{Pfaff } (-L_{x_i}) = \sqrt{\det(-L_{x_i})} = \lambda_1 ... \lambda_l. \tag{4.38}$$

Example: Equivariant localisation on the sphere

We will illustrate the Berline-Vergne theorem and its proof using a concrete example: the calculation of the volume of the unit sphere S^2. We will first remind how the volume is usually calculated. After that, we will follow all the steps of the proof of Berline-Vergne's theorem with our example and finally we will apply the main result of the theorem directly. Naturally, the same result is expected in the three cases.

Let us parametrise S^2 using spherical coordinates (φ, ϕ), where $0 \leq \varphi \leq 2\pi$ and $0 \leq \phi \leq \pi$. The round metric induced from its embedding in $\mathbb{R}^3$ is

$$g = \begin{pmatrix} \sin^2\phi & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.39}$$

so $g_{\varphi\varphi} = \sin^2\phi$, $g_{\varphi\phi} = 0$ and $g_{\phi\phi} = 1$. The usual Riemannian volume form is

$$\omega = \sqrt{\det g}\, dx^1 \wedge ... \wedge dx^n = \sin\phi\, d\varphi \wedge d\phi \tag{4.40}$$

and integrating it over the manifold we get the total volume

$$Vol = \int_{S^2} \omega = \int_{S^2} d\varphi\, d\phi\, \sin\phi = \int_0^{2\pi} d\varphi \int_0^{\pi} d\phi\, \sin\phi = 4\pi. \tag{4.41}$$

Let us express the setting in the language of equivariant localisation. We have a manifold $\mathcal{M} = S^2$ and consider a $U(1) = S^1$ action on S^2 consisting on turning the sphere in the φ direction, i.e. around its vertical axis. This action keeps the volume form invariant, since it does not depend on the coordinate φ, so we can apply the equivariant localisation argument. The corresponding vector field to this action is simply

$$V = \frac{\partial}{\partial\varphi}, \tag{4.42}$$

so $V^\varphi = 1$ and $V^\phi = 0$. In ΠTS^2 the volume form can be expressed as $\omega = \sin\phi\, \theta^\varphi\theta^\phi$. The equivariant exterior differential is

$$d_V = \theta^\mu \frac{\partial}{\partial x^\mu} + V^\mu \frac{\partial}{\partial\theta^\mu} = \theta^\varphi \frac{\partial}{\partial\varphi} + \theta^\phi \frac{\partial}{\partial\phi} + \frac{\partial}{\partial\theta^\varphi} \tag{4.43}$$

We observe that the volume form ω is not equivariantly closed,

$$d_V\omega = \left(\theta^\varphi \frac{\partial}{\partial\varphi} + \theta^\phi \frac{\partial}{\partial\phi} + \frac{\partial}{\partial\theta^\varphi}\right) \sin\phi\, \theta^\varphi\theta^\phi = \sin\phi\, \theta^\phi, \tag{4.44}$$

where we used the anticommutativity of the variables and the derivatives. In order to apply the Berline-Vergne theorem, we must find an equivariant extension of ω, i.e. an equivariantly closed form α such that its top component coincides with ω. Take

$$\alpha = \omega + \cos\phi = \sin\phi\, \theta^\varphi\theta^\phi + \cos\phi, \tag{4.45}$$

which is equivariantly closed. Now consider the 1-form

$$\beta = g_{\mu\nu} V^\mu \theta^\nu = g_{\varphi\varphi} V^\varphi \theta^\varphi = \sin^2\phi\, \theta^\varphi \tag{4.46}$$

with

$$d_V\beta = \left(\theta^\varphi \frac{\partial}{\partial\varphi} + \theta^\phi \frac{\partial}{\partial\phi} + \frac{\partial}{\partial\theta^\varphi}\right) \sin^2\phi\,\theta^\varphi = -2\sin\phi\,\cos\phi\,\theta^\varphi\theta^\phi + \sin^2\phi. \tag{4.47}$$

Here we easily recognise the 2-form $\Omega_V = -2\sin\phi\cos\phi\,\theta^\varphi\theta^\phi$ and the function $K_V = \sin^2\phi$. We will also need the exponential

$$e^{-sd_V\beta} = e^{-s(\Omega_V + K_V)} = e^{2s\sin\phi\cos\phi\,\theta^\varphi\theta^\phi} e^{-s\sin^2\phi} = \left(1 + 2s\sin\phi\cos\phi\,\theta^\varphi\theta^\phi\right) e^{-s\sin^2\phi}. \tag{4.48}$$

We can now reproduce the calculation of the proof of Berline-Vergne's theorem:

$$\begin{aligned}
Vol &= \int_{\Pi TS^2} \omega = \int_{\Pi TS^2} \alpha = \lim_{s\to\infty} \int_{\Pi TS^2} \alpha e^{-sd_V\beta} \\
&= \lim_{s\to\infty} \int_{\Pi TS^2} d\phi\, d\theta^\phi d\theta^\varphi (\sin\phi\,\theta^\varphi\theta^\phi + \cos\phi) \left(1 + 2s\sin\phi\cos\phi\,\theta^\varphi\theta^\phi\right) e^{-s\sin^2\phi} \\
&= \lim_{s\to\infty} \int_{\Pi TS^2} d\varphi\, d\phi\, d\theta^\phi d\theta^\varphi \left\{\sin\phi\,\theta^\varphi\theta^\phi + \cos\phi + 2s\sin^2\phi\cos\phi\,\theta^\varphi\theta^\phi\theta^\varphi\theta^\phi + 2s\sin\phi\cos^2\phi\,\theta^\varphi\theta^\phi\right\} e^{-s\sin^2\phi} \\
&= \lim_{s\to\infty} \int_{S^2} d\varphi\, d\phi \left(\sin\phi + 2s\sin\phi\cos^2\phi\right) e^{-s\sin^2\phi} \\
&= -\lim_{s\to\infty} \int_0^{2\pi} d\varphi \int_1^{-1} dx(1-2sx^2)e^{-s(1-x^2)} = 2\pi \lim_{s\to\infty} e^{-s} \int_{-1}^1 dx(1+2sx^2)e^{sx^2} \\
&= 2\pi \lim_{s\to\infty} \left\{ e^{-s}\int_{-1}^1 dx e^{sx^2} + 2se^{-s}\left[\frac{x}{2s}e^{sx^2}\right]_{-1}^1 - 2se^{-s}\int_{-1}^1 dx\frac{1}{2s}e^{sx^2} \right\} \\
&= 4\pi \lim_{s\to\infty} e^{-s}(e^s) = 4\pi,
\end{aligned} \tag{4.49}$$

where in the fifth line we introduced the variable change $x = \cos\phi$ and in the sixth we integrated by parts. As expected, (4.41) and (4.49) coincide.

Finally, we will apply equation (4.19) to our problem. The fixed points of the $U(1)$ action are the north pole x_N and the south pole x_S Since the coordinate system (ϕ, φ) is not well defined in these points, and we are now concerned with the local behaviour of the field around the fixed points, we will consider neighbourhoods U_N and U_S of x_N, x_S respectively.

For U_N we introduce the local coordinates (x, y), defined simply as $x = \cos\varphi$ and $y = \sin\varphi$. The vector field V is expressed in these terms as

$$V = \frac{\partial}{\partial\varphi} = \frac{\partial x}{\partial\varphi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\varphi}\frac{\partial}{\partial y} = -\sin\varphi\frac{\partial}{\partial x} + \cos\varphi\frac{\partial}{\partial y} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}. \tag{4.50}$$

The Jacobian of V is

$$\partial V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{4.51}$$

so Pfaff $\partial V(x_N) = -1$.

In a similar fashion, for U_S we introduce the local coordinates (u, v), defined as $u = \cos\varphi$ and $v = -\sin\varphi$. Note that they have the opposite orientation as in U_N. This is because we are now in the southern hemisphere and we parametrise the manifold as "seen from the outside". The field in these coordinates is

$$V = \frac{\partial}{\partial\varphi} = \frac{\partial u}{\partial\varphi}\frac{\partial}{\partial u} + \frac{\partial v}{\partial\varphi}\frac{\partial}{\partial v} = -\sin\varphi\frac{\partial}{\partial u} - \cos\varphi\frac{\partial}{\partial v} = v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}. \tag{4.52}$$

The Jacobian of V is

$$\partial V = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{4.53}$$

so Pfaff $\partial V(x_S) = 1$.

Using $\alpha^{(0)} = \cos\phi$ we find

$$\begin{aligned} Vol &= \sum_{x_i \in M_V} (-2\pi)^{n/2} \frac{\alpha^{(0)}(x_i)}{\text{Pfaff } \partial V(x_i)} = (-2\pi)^{n/2} \frac{\alpha^{(0)}(x_N)}{\text{Pfaff } \partial V(x_N)} + (-2\pi)^{n/2} \frac{\alpha^{(0)}(x_S)}{\text{Pfaff } \partial V(x_S)} \\ &= -2\pi \frac{1}{-1} - 2\pi \frac{-1}{1} = 4\pi, \end{aligned} \tag{4.54}$$

which coincides with (4.41) and (4.49) as expected.

Prescription: Equivariant localisation on a paraboloid

Besides thinking of concrete examples, as we just did with the sphere, we can also show that the calculations that appear in the proof of the Berline-Vergne theorem can be applied to other situations. In this case we will consider a situation in which all initial requirements are satisfied but one: the manifold is not compact. Therefore, we cannot talk about an *example* of the theorem but rather a *prescription* of the technique used in the proof. In particular, it makes no sense to calculate the volume of the manifold, since it is infinite.

Let us consider our manifold $\mathcal{M}$ to be the paraboloid $x^2 + y^2 - z = 0$ embedded in $\mathbb{R}^3$. We will use polar coordinates (r, φ), which relate to the coordinates in $\mathbb{R}^3$ as:

$$\begin{cases} X^1 = r\cos\varphi \\ X^2 = r\sin\varphi \\ X^3 = r^2, \end{cases} \tag{4.55}$$

which results in the induced metric

$$g_{rr} = \delta_{\alpha\beta} \frac{\partial X^\alpha}{\partial r} \frac{\partial X^\beta}{\partial r} = \cos^2\varphi + \sin^2\varphi + 4r^2 = 1 + 4r^2 \tag{4.56}$$

$$g_{r\varphi} = \delta_{\alpha\beta} \frac{\partial X^\alpha}{\partial r} \frac{\partial X^\beta}{\partial \varphi} = \cos\varphi(-r\sin\varphi) + r\cos\varphi\sin\varphi = 0 \tag{4.57}$$

$$g_{\varphi\varphi} = \delta_{\alpha\beta} \frac{\partial X^\alpha}{\partial \varphi} \frac{\partial X^\beta}{\partial \varphi} = r^2\sin^2\varphi + r^2\cos^2\varphi = r^2. \tag{4.58}$$

$$\tag{4.59}$$

The form that would play the role of the volume form is then

$$\omega = \sqrt{\det g}\, dx^1 \wedge \ldots \wedge dx^n = r\sqrt{1+4r^2}\, dr \wedge d\varphi = r\sqrt{1+4r^2}\,\theta^r\theta^\varphi. \tag{4.60}$$

Consider the $U(1)$ action given by rotation around the vertical axis, i.e.

$$V = \frac{\partial}{\partial\varphi}. \tag{4.61}$$

The equivariant exterior differential is then

$$d_V = \theta^\mu \frac{\partial}{\partial x^\mu} + V^\mu \frac{\partial}{\partial \theta^\mu} = \theta^r \frac{\partial}{\partial r} + \theta^\varphi \frac{\partial}{\partial \varphi} + \frac{\partial}{\partial \theta^\varphi}. \tag{4.62}$$

As it happened with the sphere, in this case the volume form is not equivariantly closed either,

$$d_V\omega = \left(\theta^r \frac{\partial}{\partial r} + \theta^\varphi \frac{\partial}{\partial \varphi} + \frac{\partial}{\partial \theta^\varphi}\right) r\sqrt{1+4r^2}\theta^r\theta^\varphi = -r\sqrt{1+4r^2}\,\theta^r, \tag{4.63}$$

so we must consider the equivariant extension

$$\alpha = \omega + \frac{1}{12}(1+4r^2)^{3/2} = r\sqrt{1+4r^2}\theta^r\theta^\varphi + \frac{1}{12}(1+4r^2)^{3/2}. \tag{4.64}$$

Now consider the 1-form

$$\beta = g_{\mu\nu}V^\mu\theta^\nu = g_{\varphi\varphi}V^\varphi\theta^\varphi = r^2\theta^\varphi, \tag{4.65}$$

its equivariant derivative

$$d_V\beta = \left(\theta^r \frac{\partial}{\partial r} + \theta^\varphi \frac{\partial}{\partial \varphi} + \frac{\partial}{\partial \theta^\varphi}\right) r^2\theta^\varphi = 2r\theta^r\theta^\varphi + r^2, \tag{4.66}$$

and the exponential

$$e^{-sd_V\beta} = e^{-2sr\theta^r\theta^\varphi}e^{-sr^2} = (1-2sr\theta^r\theta^\varphi)e^{-sr^2}. \tag{4.67}$$

The so-called *equivariant volume* -since we cannot talk about the volume of a paraboloid in a literal sense- will be then

$$\begin{aligned}
Vol_{eq} &= \int_{\Pi T\mathcal{M}} \omega = \int_{\Pi T\mathcal{M}} \alpha = \lim_{s\to\infty} \int_{\Pi T\mathcal{M}} \alpha e^{-sd_V\beta} \\
&= \lim_{s\to\infty} \int_{\Pi T\mathcal{M}} drd\varphi d\theta^\varphi d\theta^r \left(r\sqrt{1+4r^2}\theta^r\theta^\varphi + \frac{1}{12}(1+4r^2)^{3/2}\right)(1-2sr\theta^r\theta^\varphi)e^{-sr^2} \\
&= \lim_{s\to\infty} \int_{\Pi T\mathcal{M}} drd\varphi d\theta^\varphi d\theta^r \left\{r\sqrt{1+4r^2}\theta^r\theta^\varphi + \frac{1}{12}(1+4r^2)^{3/2} - 2sr^2\sqrt{1+4r^2}\theta^r\theta^\varphi\theta^r\theta^\varphi \right. \\
&\qquad\qquad \left. -\frac{1}{6}sr(1+4r^2)^{3/2}\theta^r\theta^\varphi\right\} e^{-sr^2} \\
&= \lim_{s\to\infty} \int_{\mathcal{M}} drd\varphi \left(r\sqrt{1+4r^2} - \frac{1}{6}sr(1+4r^2)^{3/2}\right) e^{-sr^2} \\
&= 2\pi \lim_{s\to\infty} \left\{\int_0^\infty drr\sqrt{1+4r^2}e^{-sr^2} - \frac{1}{6}\int_0^\infty drsr(1+4r^2)^{3/2}e^{-sr^2}\right\} \\
&= 2\pi \lim_{s\to\infty} \left\{\int_0^\infty drr\sqrt{1+4r^2}e^{-sr^2} + \frac{1}{6}\left[\frac{1}{2}(1+4r^2)^{3/2}e^{-sr^2}\right]_0^\infty - \frac{1}{6}\int_0^\infty dr6r\sqrt{1+4r^2}e^{-sr^2}\right\} \\
&= 2\pi \lim_{s\to\infty} \left\{\int_0^\infty drr\sqrt{1+4r^2}e^{-sr^2} - \frac{1}{12} - \int_0^\infty drr\sqrt{1+4r^2}e^{-sr^2}\right\} = -\frac{\pi}{6},
\end{aligned} \tag{4.68}$$

where we integrated by parts in the seventh line. We can now compare this result with a direct application of (4.19). The local behaviour of the vector field around the single fixed point -the origin x_0- can be expressed in the same coordinates and terms as the south pole of the sphere with the neighbourhood U_S. Therefore, Pfaff $\partial V(x_0) = 1$.

Using $\alpha^{(0)} = (1/12)(1+4r^2)^{3/2}$ and we have

$$Vol_{eq} = \sum_{x_i \in M_V} (-2\pi)^{n/2} \frac{\alpha^{(0)}(x_i)}{\text{Pfaff } \partial V(x_i)} = -2\pi \frac{\alpha^{(0)}(x_0)}{\text{Pfaff } \partial V(x_0)} = -2\pi \frac{1}{12}\frac{1}{1} = -\frac{\pi}{6}, \tag{4.69}$$

which coincides with (4.68) as expected.

4.2 Localisation for Dynamical Systems: the Duistermaat-Heckman Theorem

We will now use the localisation techniques to solve a family of integrals that usually appear when dealing with dynamical systems. To do so, assume that the Hamiltonian function H_V defined on a $2n$-dimensional manifold $\mathcal{M}$ -as defined in §3.3- is a Morse function, i.e. that all its critical points x_i, $dH_V(x_i) = 0$ are isolated and that the Hessian matrix of H on those points is non-degenerate,

$$\det \mathcal{H}_{\mathcal{V}}(x_i) \neq 0, \qquad \mathcal{H}_{\mathcal{V}}(x) = \left(\frac{\partial^2 H_V(x)}{\partial x^\mu \partial x^\nu}\right). \tag{4.70}$$

Notice that the critical points of H_V coincide with the set $\mathcal{M}_V$, because of (3.64). In statistical mechanics there is an important quantity of interest called the *classical partition function* (see [17] for a detailed description), which is the function

$$Z(T) = \int_{\mathcal{M}} \frac{\omega^n}{n!} e^{iTH_V} = \int_{\mathcal{M}} d^{2n}x \sqrt{\det \omega(x)}\, e^{-TH_V(x)}, \tag{4.71}$$

where for our purposes T will be a real parameter. While it is very seldom that an exact expression can be found for (4.71), it is common to approximate its value for $T \to \infty$ using the so-called *stationary-phase approximation* (cf. [5]). This approximation is based on the following observations:

- For $T \to \infty$, the integrant tends to damp to 0.
- This implies that $Z(T)$ can be assymptotically expanded in powers of $1/T$.
- The larger T gets, the more the integrand of $Z(T)$ tends to localise around the critical points of H_V, i.e. around $\mathcal{M}_V$. To evaluate these contributions, we could repeat the same procedure as in the previous section: expand both H_V and ω around each $x_i \in \mathcal{M}_V$ and express $Z(T)$ as an infinite series of Gaussian moment integrals. If we only take into account the lowest order contribution and sum over all $x_i \in \mathcal{M}_V$, we get the standard lowest-order stationary-phase approximation to the integral (4.71),

$$Z(T) = \left(\frac{2\pi}{T}\right)^n \sum_{x_i \in \mathcal{M}_V} e^{-TH_V(x_i)} \sqrt{\frac{\det \omega(x_i)}{\det \mathcal{H}(x_i)}} + \mathcal{O}(1/T^{n+1}). \tag{4.72}$$

This expression was actually the one that gave rise to the equivariant localisation theory in 1982, when Duistermaat and Heckman (see [15]) found a general class of Hamiltonian systems for which the stationary-phase approximation gives the exact value of $Z(T)$, i.e. all $\mathcal{O}(1/T^{n+1})$ terms in (4.72) vanish. If we consider our usual setting in which $\mathcal{M}_V$ consists of isolated points, then the *equivariant Darboux theorem* (see [5]) says that not only we can find local Darboux coordinates (p_μ, q^μ) in which the symplectic 2-form ω looks like (3.56), but also we can locate the origin of the coordinates $(p_\mu, q^\mu) = (0,0)$

on the fixed point x_i. This means that the $U(1)$-action on $\mathcal{M}$ can be locally expressed as a set of linear rotations, one for each (p_μ, q^μ) plane:

$$V = \sum_{j=1}^{n} \varepsilon_j \left(q_j \frac{\partial}{\partial p^j} + p^j \frac{\partial}{\partial q_j} \right), \tag{4.73}$$

where ε_j are some weights yet to be specified. From (3.64) we have

$$H(x) = H(x_i) + \sum_{j=1}^{n} \frac{\varepsilon_j}{2} (p_j^2 + {q^j}^2), \tag{4.74}$$

and therefore the solutions of (3.67) are simply circles around the critical points, $p_j(t), q^j(t) \simeq e^{i\varepsilon_j t}$. This gives a very graphical representation of both the $U(1)$-action on $\mathcal{M}$ and of the fact that the action preserves the Darboux coordinates. We see that in this case, the Hamiltonian function is actually quadratic and therefore the stationary-phase approximation is actually exact.

We can give an interpretation of this fact by using equivariant cohomology, as Atiyah and Bott pointed out in 1984 (see [16]). Using the same setting as in §3.3, we can observe that $Z(T)$ can be written as

$$Z(T) = \int_{\mathcal{M}} \alpha, \tag{4.75}$$

where α now is the inhomogeneous form

$$\alpha = \frac{1}{T^n} e^{-T(\omega + H_V)} = \frac{1}{T^n} e^{-TH_V} \sum_{j=0}^{n} \frac{(-T)^j}{j!} \omega^j, \tag{4.76}$$

whose $2j$-component is

$$\alpha_{2j} = e^{-TH_V} \frac{\omega^j}{(-T)^{n-j} j!}. \tag{4.77}$$

Recalling that $d_V \omega_V = d_V(\omega + H_V) = 0$, we can find $d_V \alpha = 0$ and therefore a simple application of the Berline-Vergne theorem (4.19) gives us

$$Z(T) = \left(-\frac{2\pi}{T} \right)^n \sum_{x_i \in \mathcal{M}_V} \frac{e^{-TH_V(x_i)}}{\text{Pfaff } \partial V(x_i)}. \tag{4.78}$$

Choosing the right sign for the denominator can be sometimes difficult. Assume we are using the equivariant Darboux coordiates in which $\omega(x_i)$ is skew-diagonal with skew-eigenvalues 1 and the Hessian $\mathcal{H}(p)$ is diagonal with eigenvalues $\varepsilon_j(x_i)$. Then from (3.64) we have $(\partial V)_{|x_i} = \omega^{-1}(x_i)\mathcal{H}(x_i)$, so $(\partial V)_{|x_i}$ is of the form

$$\begin{pmatrix} \varepsilon_1(x_i) & & & & \\ & -\varepsilon_1(x_i) & & & \\ & & \ddots & & \\ & & & \varepsilon_n(x_i) & \\ & & & & -\varepsilon_n(x_i) \end{pmatrix} \tag{4.79}$$

so Pfaff $(x_i) = (-1)^n \varepsilon_1(x_i) \ldots \varepsilon_n(x_i)$. We can therefore write directly

$$Z(T) = \left(\frac{2\pi}{T} \right)^n \sum_{x_i \in \mathcal{M}_V} e^{-TH(x_i)} \sqrt{\frac{\det \omega(x_i)}{\det \mathcal{H}(x_i)}}. \tag{4.80}$$

This result is sometimes referred to as the *Duistermaat-Heckman theorem* or *Duistermaat-Heckman integration formula.*

Prescription: Symplectic localisation in the plane

As we did in §4.1, we are going to consider a particular case in which the manifold $\mathcal{M}$ is not compact and therefore it cannot be considered an example of the theorem but rather a prescription of the techniques used in the proof. One of the most simple examples of a symplectic manifold is the plane $\mathcal{M} = \mathbb{R}^2$ with coordinates (p, q), together with the symplectic form

$$\omega = dp \wedge dq = \theta^p \theta^q. \tag{4.81}$$

These coordinates are already Darboux coordinates. Consider now the usual $U(1)$ action on $\mathbb{R}^2$ and its corresponding vector field

$$V = p\frac{\partial}{\partial q} - q\frac{\partial}{\partial p}. \tag{4.82}$$

The only fixed point is the origin $(p, q) = (0, 0)$ so comparing to (4.73), $\lambda_1(0) = -i$ and from (4.74) the Hamiltonian will be

$$H_V = \frac{p^2 + q^2}{2}. \tag{4.83}$$

We will first compute the integral $Z(T)$ in the classical way, i.e., as in (4.71):

$$Z(T) = \int_{T\mathbb{R}^2} \omega e^{-TH_V} = \int_{T\mathbb{R}^2} \theta^p\theta^q e^{-\frac{T}{2}(p^2+q^2)} = \int_0^{2\pi} d\varphi \int_0^{\infty} dr\, r\, e^{-\frac{T}{2}r^2} = 2\pi \int_0^{\infty} dx e^{-Tx} = \frac{2\pi}{T}. \tag{4.84}$$

Let us now express the system in equivariant terms. We first find the equivariant exterior differential

$$d_V = d + i_V = \theta^p \frac{\partial}{\partial p} + \theta^q \frac{\partial}{\partial q} + p\frac{\partial}{\partial \theta^q} - q\frac{\partial}{\partial \theta^p}, \tag{4.85}$$

and we check that $\omega_V = \omega + H_V$ is equivariantly closed

$$d_V\omega_V = \left(\theta^p \frac{\partial}{\partial p} + \theta^q \frac{\partial}{\partial q} + p\frac{\partial}{\partial \theta^q} - q\frac{\partial}{\partial \theta^p}\right)\left(\theta^p\theta^q + \frac{p^2+q^2}{2}\right) = p\theta^p + q\theta^q - p\theta^p - q\theta^q = 0. \tag{4.86}$$

The form α will be in this case

$$\alpha = \frac{1}{T^n} e^{-T(\omega + H_V)} = \frac{1}{T} e^{-T\left(\theta^p\theta^q + \frac{1}{2}(p^2+q^2)\right)} = \frac{1}{T}(1 - T\theta^p\theta^q) e^{-\frac{T}{2}(p^2+q^2)}, \tag{4.87}$$

and it is clear that its top component coincides with the integrand of (4.84). We can see that α is also equivariantly closed

$$d_V\alpha = \frac{1}{T}\left\{-Tp\theta^p - Tq\theta^q + Tq\theta^q + Tp\theta^p\right\} e^{-\frac{T}{2}(p^2+q^2)} = 0. \tag{4.88}$$

Regarding the field V, its action is the same as in the neighbourhood U_N in the sphere example, so applying the Berline-Vergne theorem:

$$Z(T) = -2\pi \frac{\alpha^{(0)}(0)}{\mathrm{Pfaff}\ \partial V(0)} = -2\pi \frac{1}{T}\frac{1}{-1} = \frac{2\pi}{T}, \tag{4.89}$$

which is the same result as in (4.84).

Finally, we will evaluate Duistermaat-Heckman integration formula directly. We have, as matrices

$$\omega(0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathcal{H}(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.90}$$

so $\det \omega(0) = \det \mathcal{H}(0) = 1$ and $\lambda(0) = 0$. We finally have

$$Z(T) = \frac{2\pi}{T} e^{-TH_V(0)} \sqrt{\frac{\det \omega(0)}{\det \mathcal{H}(0)}} = \frac{2\pi}{T}, \tag{4.91}$$

which also gives the desired result.

4.3 Localisation for Degenerate Systems

So far we have been assuming that the vector field V corresponding to the action of the Lie group G on $\mathcal{M}$ is non-degenerate, i.e. that the set of fixed points $\mathcal{M}_V$ consists of isolated points. In this section we will drop this assumption and consider the case in which the vector field can possibly be degenerate. In this case, the set of fixed points $\mathcal{M}_V$ becomes a sub-manifold of $\mathcal{M}$ (see [1, 2, 18] for more details) and while the use of localisation techniques is still possible, we have to do some changes in the procedure.

The normal bundle

We will assume that V is non-degenerate in the normal directions to $\mathcal{M}_V$, which will allow to define the so-called normal bundle $\mathcal{N}_V$ over $\mathcal{M}_V$ ([18]). Changing the order of the coordinates if necessary, we can take local coordinates in a neighbourhood of $\mathcal{M}_V$ so that a point $x \in \mathcal{M}$ can be written as $(x^1, ..., x^m, x^{m+1}, ..., x^n)$, where $m = \dim \mathcal{M}_V$ and $n = \dim \mathcal{M}$, in such a way that the points of $\mathcal{M}_V$ are the ones with $x^{m+1} = x^{m+2} = ... = x^n = 0$. We will use a triple index notation, where the latin indices $i, j, k...$ go from $1, ..., m$, the first greek indices $\alpha, \beta, \gamma...$ go from $m+1, ..., n$ and the middle greek indices $\lambda, \mu, \nu...$ combine the previous ones, i.e.

$$x = (x^\mu) = (x^i, x^\alpha), \qquad x \in \mathcal{M}_V \text{ iff } x^\alpha = 0. \tag{4.92}$$

The tangent space $T_x\mathcal{M}$ at $x \in \mathcal{M}_V$, spanned by $\{\partial/\partial x^\mu\}$, can be decomposed in the direct sum

$$T_x\mathcal{M} = T_x\mathcal{M}_V \oplus (T_x\mathcal{M}_V)^\perp, \tag{4.93}$$

where $T_x\mathcal{M}_V$ is spanned by $\{\partial/\partial x^i\}$ and $(T_x\mathcal{M}_V)^\perp$ is spanned by $\{\partial/\partial x^\alpha\}$. The normal bundle $\mathcal{N}_V$ is defined as the vector bundle over $\mathcal{M}_V$ whose fibres are $(T_x\mathcal{M}_V)^\perp$ and has the inherited metric connection from $T\mathcal{M}$. In this sense,

$$T\mathcal{M}_{|\mathcal{M}_V} = T\mathcal{M}_V \oplus \mathcal{N}_V \tag{4.94}$$

as vector bundles. We will write $\mathcal{N}_x = (T_x\mathcal{M}_V)^\perp$ to make notation easier.

By construction, the vector field $V = V^\mu(x^i, x^\alpha)\partial/\partial x^\mu$ satisfies

$$V^\mu(x^i, 0) = 0 \qquad \text{and} \qquad \nabla_i V^\mu(x^i, 0) = \partial_i V^\mu(x^i, 0) = 0. \tag{4.95}$$

If we consider the transformation L_x for $x \in \mathcal{M}_V$, as we did in (4.34),(4.35), and use the property (4.95), we conclude that the linear transformation will have the form

$$L_x = \begin{array}{c c} & \begin{array}{cc} i & \alpha \end{array} \\ \begin{array}{c} i \\ \alpha \end{array} & \begin{pmatrix} 0 & 0 \\ 0 & \bar{L}_x \end{pmatrix} \end{array}, \tag{4.96}$$

where $\bar{L}_x : \mathcal{N}_x \longrightarrow \mathcal{N}_x$ is antisymmetric and non-degenerate, so in particular it can be considered as an isomorphism. Using a linear transformation if necessary, we can assume that $\bar{L}_x$ has the form (4.36).

More generally, $\mathcal{L}_V$ preserves the splitting $T_x\mathcal{M} = T_x\mathcal{M}_V \oplus \mathcal{N}_x$, so it can be considered as a fibre map of the normal bundle.

We can also work out some of the basic elements of the normal bundle $\mathcal{N}_V$ using coordinates. For example, the sections of this bundle are simply the normal vector fields over $\mathcal{M}_V$,

$$W = W^\alpha(x^i)\frac{\partial}{\partial x^\alpha}, \tag{4.97}$$

the induced covariant derivative is simply the restriction of the covariant derivative of $T\mathcal{M}$,

$$\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^\alpha} = \Gamma^\beta_{\alpha i}\frac{\partial}{\partial x^\beta} \tag{4.98}$$

and the Riemannian curvature tensor is the restriction of the one in $T\mathcal{M}$,

$$F\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^\alpha} = R^\beta{}_{\alpha ij}\frac{\partial}{\partial x^\beta}, \tag{4.99}$$

where $R^\beta{}_{\alpha ij}$ are the coefficients of the usual Riemannian curvature tensor of $T\mathcal{M}$. It is important to observe that for $\mathcal{N}_V$ we only take into consideration the components with two latin and two greek indices.

Normal coordinates

In order to simplify our calculations, let $x \in \mathcal{M}_V$ and (x^i, x^α) be normal coordinates around x (see [2] for a detailed definition). The main properties of normal coordinates are that $g_{\mu\nu}(x) = \delta_{\mu\nu}$ and that the first derivatives of the metric vanish at x. The second order terms can be expressed in terms of the Riemmanian curvature tensor in the following form:

$$g_{\mu\nu}(x) = \delta_{\mu\nu} - \frac{1}{3}R_{\mu\rho\nu\sigma}x^\rho x^\sigma + \mathcal{O}(x^3). \tag{4.100}$$

Using this, equation (4.4) can be approximated by

$$\delta_{\mu\lambda}(\partial_\nu V^\lambda) - \frac{1}{3}R_{\mu\rho\lambda\sigma}(\partial_\nu V^\lambda)x^\rho x^\sigma + \delta_{\nu\lambda}(\partial_\mu V^\lambda) - \frac{1}{3}R_{\nu\rho\lambda\sigma}(\partial_\mu V^\lambda)x^\rho x^\sigma \tag{4.101}$$

$$- \frac{1}{3}R_{\mu\lambda\nu\sigma}(\partial_\rho V^\lambda)x^\rho x^\sigma - \frac{1}{3}R_{\mu\sigma\nu\lambda}(\partial_\rho V^\lambda)x^\rho x^\sigma = 0. \tag{4.102}$$

If we take the terms of zeroth order we get

$$\delta_{\mu\lambda}(\partial_\nu V^\lambda) + \delta_{\nu\lambda}(\partial_\mu V^\lambda) = 0, \tag{4.103}$$

which only states that (∂V) is an antisymmetric matrix. More interesting is to take the second order terms,

$$R_{\mu\rho\lambda\sigma}(\partial_\nu V^\lambda)x^\rho x^\sigma + R_{\nu\rho\lambda\sigma}(\partial_\mu V^\lambda)x^\rho x^\sigma + R_{\mu\lambda\nu\sigma}(\partial_\rho V^\lambda)x^\rho x^\sigma + R_{\mu\sigma\nu\lambda}(\partial_\rho V^\lambda)x^\rho x^\sigma = 0. \tag{4.104}$$

We recall that $(\partial_\mu V^\lambda) = 0$ if any of the two indices is latin. Consider the particular case $\mu = i, \nu = j$, then the only non-vanishing terms are

$$R_{i\gamma j\sigma}(\partial_\varepsilon V^\gamma)x^\varepsilon x^\sigma + R_{i\sigma j\gamma}(\partial_\varepsilon V^\gamma)x^\varepsilon x^\sigma = 0, \tag{4.105}$$

and since this is true for arbitrary value of the coordinates, we get

$$(R_{\gamma ij\sigma} + R_{\gamma ji\sigma})(\partial_\varepsilon V^\gamma) = 0. \tag{4.106}$$

Expansions of the forms

In a similar fashion as in the proof of the Berline-Vergne theorem (§4.1), we can take advantage of the invariance of the integration measure to rescale the variables and make higher order terms vanish in the limit $s \to \infty$. In this case, though, we will only rescale the variables

$$x^\alpha \longrightarrow \frac{1}{\sqrt{s}} x^\alpha, \qquad \theta^\alpha \longrightarrow \frac{1}{\sqrt{s}} \theta^\alpha. \tag{4.107}$$

We can now look for the expansions of Ω_V and K_V around $x \in \mathcal{M}_V$ and in terms of s. We start with the expansion of (Ω_V):

$$\begin{aligned}(\Omega_V)(x) = \frac{1}{2}(\Omega_V)_{\mu\nu}(x)\theta^\mu\theta^\nu &= \frac{1}{2}(\Omega_V)_{ij}(x)\theta^i\theta^j + \frac{1}{2\sqrt{s}}(\Omega_V)_{\alpha i}(x)\theta^\alpha\theta^i \\ &+ \frac{1}{2\sqrt{s}}(\Omega_V)_{i\alpha}(x)\theta^i\theta^\alpha + \frac{1}{2s}(\Omega_V)_{\alpha\beta}(x)\theta^\alpha\theta^\beta. \end{aligned} \tag{4.108}$$

Let's look at (4.108) term by term. We can write the first term as:

$$\begin{aligned}&\frac{1}{2}(\Omega_V)_{ij}(x)\theta^i\theta^j = \frac{1}{2}\left[\partial_i(g_{j\lambda}V^\lambda) - \partial_j(g_{i\lambda}V^\lambda)\right]\theta^i\theta^j \\ &= \frac{1}{2}\left[(\partial_i g_{j\lambda})V^\lambda + g_{j\lambda}(\partial_i V^\lambda) - (\partial_j g_{i\lambda})V^\lambda - g_{i\lambda}(\partial_j V^\lambda))\right]\theta^i\theta^j \\ &= \frac{1}{2}\left[(\partial_i g_{j\gamma})V^\gamma - (\partial_j g_{i\gamma})V^\gamma\right]\theta^i\theta^j \\ &= -\frac{1}{6\sqrt{s}}\left[(R_{ji\gamma k} + R_{jk\gamma i} - R_{ij\gamma k} - R_{ik\gamma j})(\partial_\varepsilon V^\gamma)x^\varepsilon x^k\right]\theta^i\theta^j \\ &- \frac{1}{6s}[(R_{ji\gamma\delta} + R_{j\delta\gamma i} - R_{ij\gamma\delta} - R_{i\delta\gamma j})(\partial_\varepsilon V^\gamma)x^\varepsilon x^\delta]\theta^i\theta^j. \end{aligned} \tag{4.109}$$

We focus first on the part proportional to $s^{-1/2}$. Writing the relation (4.106) as

$$(R_{\gamma kji} + R_{\gamma jki})(\partial_\varepsilon V^\gamma) = 0, \tag{4.110}$$

we have

$$-\frac{1}{6\sqrt{s}}\left[(R_{ji\gamma k} + R_{jk\gamma i} - R_{ij\gamma k} - R_{ik\gamma j})(\partial_\varepsilon V^\gamma)x^\varepsilon x^k\right]\theta^i\theta^j = 0 \tag{4.111}$$

because

$$\begin{aligned}&-\frac{1}{6\sqrt{s}}\left[(R_{ji\gamma k} + R_{jk\gamma i} - R_{ij\gamma k} - R_{ik\gamma j})(\partial_\varepsilon V^\gamma)x^\varepsilon x^k\right]\theta^i\theta^j \\ &= -\frac{1}{6\sqrt{s}}\left[(R_{\gamma kji} + R_{\gamma ijk} - R_{\gamma kij} - R_{\gamma jik})(\partial_\varepsilon V^\gamma)x^\varepsilon x^k\right]\theta^i\theta^j \\ &= -\frac{1}{6\sqrt{s}}\left[(R_{\gamma kji} - R_{\gamma jik} + R_{\gamma kji} + R_{\gamma jki})(\partial_\varepsilon V^\gamma)x^\varepsilon x^k\right]\theta^i\theta^j \\ &= -\frac{1}{6\sqrt{s}}\left[(R_{\gamma kji} + R_{\gamma jki} + R_{\gamma kji} + R_{\gamma jki})(\partial_\varepsilon V^\gamma)x^\varepsilon x^k\right]\theta^i\theta^j = 0, \end{aligned} \tag{4.112}$$

so this part vanishes. We focus now on the part proportional to s^{-1}, writing the relation (4.106) as

$$(R_{\gamma ij\delta} + R_{\gamma ji\delta})(\partial_\varepsilon V^\gamma) = 0. \tag{4.113}$$

we have

$$\begin{aligned}
&-\frac{1}{6s}[(R_{ji\gamma\delta}+R_{j\delta\gamma i}-R_{ij\gamma\delta}-R_{i\delta\gamma j})(\partial_\varepsilon V^\gamma)x^\varepsilon x^\delta]\theta^i\theta^j\\
&=-\frac{1}{6s}[(R_{\gamma\delta ji}+R_{\gamma ij\delta}-R_{\gamma\delta ij}-R_{\gamma ji\delta})(\partial_\varepsilon V^\gamma)x^\varepsilon x^\delta]\theta^i\theta^j\\
&=-\frac{1}{6s}\left[(R_{\gamma\delta ji}-\frac{1}{2}(R_{\gamma j\delta i}+R_{\gamma ji\delta})+R_{\gamma\delta ji}-\frac{1}{2}(R_{\gamma j\delta i}+R_{\gamma ji\delta}))(\partial_\varepsilon V^\gamma)x^\varepsilon x^\delta\right]\theta^i\theta^j\\
&=-\frac{1}{6s}\left[(R_{\gamma\delta ji}-\frac{1}{2}R_{\gamma\delta ji}+R_{\gamma\delta ji}-\frac{1}{2}R_{\gamma\delta ji})(\partial_\varepsilon V^\gamma)x^\varepsilon x^\delta\right]\theta^i\theta^j\\
&=-\frac{1}{2s}R_{\gamma\delta ji}(\partial_\varepsilon V^\gamma)x^\varepsilon x^\delta\theta^i\theta^j=\frac{1}{2s}R_{\gamma\delta ij}(\partial_\varepsilon V^\gamma)x^\varepsilon x^\delta\theta^i\theta^j.
\end{aligned}\tag{4.114}$$

We look now at the second and third terms of (4.108):

$$\begin{aligned}
&\frac{1}{2\sqrt{s}}(\Omega_V)_{\alpha i}(x)\theta^\alpha\theta^i+\frac{1}{2\sqrt{s}}(\Omega_V)_{i\alpha}(x)\theta^i\theta^\alpha=\frac{1}{2\sqrt{s}}\left[(\Omega_V)_{\alpha i}(x)-(\Omega_V)_{i\alpha}(x)\right]\theta^\alpha\theta^i\\
&=\frac{1}{2\sqrt{s}}\left[(\partial_\alpha g_{i\gamma})V^\gamma+g_{i\gamma}(\partial_\alpha V^\gamma)-(\partial_i g_{\alpha\gamma})V^\gamma-g_{\alpha\gamma}(\partial_i V^\gamma)\right.\\
&\left.-(\partial_i g_{\alpha\gamma})V^\gamma-g_{\alpha\gamma}(\partial_i V^\gamma))+(\partial_\alpha g_{i\gamma})V^\gamma+g_{i\gamma}(\partial_\alpha V^\gamma)\right]\theta^\alpha\theta^i\\
&=\frac{1}{\sqrt{s}}\left[(\partial_\alpha g_{i\gamma})V^\gamma+g_{i\gamma}(\partial_\alpha V^\gamma)-(\partial_i g_{\alpha\gamma})V^\gamma\right]\theta^\alpha\theta^i\\
&=\frac{1}{\sqrt{s}}\left[(\delta_{i\gamma}-\frac{1}{3}R_{ik\gamma l}x^kx^l)(\partial_\alpha V^\gamma)\right]\theta^\alpha\theta^i\\
&-\frac{1}{3s}[(R_{i\alpha\gamma k}+R_{ik\gamma\alpha})(\partial_\varepsilon V^\gamma)x^kx^\varepsilon+(R_{i\varepsilon\gamma k}+R_{ik\gamma\varepsilon})(\partial_\alpha V^\gamma)x^kx^\varepsilon\\
&-(R_{\alpha i\gamma k}+R_{\alpha k\gamma i})(\partial_\varepsilon V^\gamma)x^kx^\varepsilon]\theta^\alpha\theta^i+\mathcal{O}(s^{-3/2})
\end{aligned}\tag{4.115}$$

Regarding the part proportional to $s^{-1/2}$, it vanishes because $\delta_{i\gamma}(\partial_\alpha V^\gamma)=(\partial_\alpha V^i)=0$ and

$$R_{ik\gamma l}(\partial_\alpha V^\gamma)x^kx^l=R_{\gamma lik}(\partial_\alpha V^\gamma)x^kx^l=-R_{\gamma ilk}(\partial_\alpha V^\gamma)x^kx^l=0,\tag{4.116}$$

where we used identity (4.110) and the antisymmetry properties of R. The part proportional to s^{-1} vanishes as well. To see it, first take relation (4.104) and consider the particular case $\mu=\alpha,\nu=i$,

$$R_{\alpha\rho\lambda\sigma}(\partial_i V^\lambda)x^\rho x^\sigma+R_{i\rho\lambda\sigma}(\partial_\alpha V^\lambda)x^\rho x^\sigma+R_{\alpha\lambda i\sigma}(\partial_\rho V^\lambda)x^\rho x^\sigma+R_{\alpha\sigma i\lambda}(\partial_\rho V^\lambda)x^\rho x^\sigma=0.\tag{4.117}$$

Using again the properties of $(\partial_\rho V^\lambda)$ we get

$$R_{i\rho\gamma\sigma}(\partial_\alpha V^\gamma)x^\rho x^\sigma+\frac{1}{\sqrt{s}}R_{\alpha\gamma i\sigma}(\partial_\varepsilon V^\gamma)x^\varepsilon x^\sigma+\frac{1}{\sqrt{s}}R_{\alpha\sigma i\gamma}(\partial_\varepsilon V^\gamma)x^\varepsilon x^\sigma=0,\tag{4.118}$$

or more explicitely

$$R_{ik\gamma l}(\partial_\alpha V^\gamma)x^kx^l+\frac{1}{\sqrt{s}}(R_{ik\gamma\varepsilon}+R_{i\varepsilon\gamma k})(\partial_\alpha V^\gamma)x^kx^\varepsilon+\frac{1}{\sqrt{s}}(R_{\alpha\gamma ik}+R_{\alpha ki\gamma})(\partial_\varepsilon V^\gamma)+\mathcal{O}(s^{-1})=0.\tag{4.119}$$

We can forget about the first term because using (4.110)

$$R_{ik\gamma l}(\partial_\alpha V^\gamma)x^kx^l=R_{\gamma lik}(\partial_\alpha V^\gamma)x^kx^l=-R_{\gamma ilk}(\partial_\alpha V^\gamma)x^kx^l=0.\tag{4.120}$$

Rewriting $R_{\alpha\gamma ik} + R_{\alpha ki\gamma} = -R_{ik\gamma\alpha} - R_{\alpha ik\gamma} = -R_{ik\gamma\alpha} - R_{i\alpha\gamma k}$ we can see that equation (4.115) can be rewritten as

$$-\frac{1}{3s}[2(R_{i\varepsilon\gamma k} + R_{ik\gamma\varepsilon})(\partial_\alpha V^\gamma)x^k x^\varepsilon - (R_{\alpha i\gamma k} + R_{\alpha k\gamma i})(\partial_\varepsilon V^\gamma)x^k x^\varepsilon]\theta^\alpha\theta^i + \mathcal{O}(s^{-3/2}). \tag{4.121}$$

Now, from

$$\begin{aligned}
&(R_{i\varepsilon\gamma k} + R_{ik\gamma\varepsilon})(\partial_\alpha V^\gamma)x^k x^\varepsilon = (R_{\gamma ki\varepsilon} + R_{\gamma\varepsilon ik})(\partial_\alpha V^\gamma)x^k x^\varepsilon \\
&= (R_{\gamma ki\varepsilon} - R_{\gamma k\varepsilon i} - R_{\gamma ik\varepsilon})(\partial_\alpha V^\gamma)x^k x^\varepsilon \\
&= (R_{\gamma ki\varepsilon} - R_{\gamma ki\varepsilon} - R_{\gamma ik\varepsilon})(\partial_\alpha V^\gamma)x^k x^\varepsilon = -R_{\gamma ik\varepsilon}(\partial_\alpha V^\gamma)x^k x^\varepsilon = 0,
\end{aligned} \tag{4.122}$$

and since

$$-(R_{\alpha i\gamma k} + R_{\alpha k\gamma i})(\partial_\varepsilon V^\gamma)x^k x^\varepsilon = (R_{\alpha ik\gamma} + R_{\alpha ki\gamma})(\partial_\varepsilon V^\gamma)x^k x^\varepsilon = 0 \tag{4.123}$$

we conclude that the second and third term of (4.108) vanish.

Regarding the forth term of (4.108),

$$\begin{aligned}
&\frac{1}{2s}(\Omega_V)_{\alpha\beta}(x)\theta^\alpha\theta^\beta = \frac{1}{2s}\left[\partial_\alpha(g_{\beta\lambda}V^\lambda) - \partial_\beta(g_{\alpha\lambda}V^\lambda)\right]\theta^\alpha\theta^\beta \\
&= \frac{1}{2s}\left[(\partial_\alpha g_{\beta\lambda})V^\lambda + g_{\beta\lambda}(\partial_\alpha V^\lambda) - (\partial_\beta g_{\alpha\lambda})V^\lambda - g_{\alpha\lambda}(\partial_\beta V^\lambda))\right]\theta^\alpha\theta^\beta.
\end{aligned}$$

In this case there are no parts proportional to $s^{-1/2}$ and the parts proportional to s^{-1} are:

$$\frac{1}{2s}\left[g_{\beta\lambda}(\partial_\alpha V^\lambda) - g_{\alpha\lambda}(\partial_\beta V^\lambda))\right]\theta^\alpha\theta^\beta = \frac{1}{2s}(\Omega_V)_{\alpha\beta}\theta^\alpha\theta^\beta,$$

where $(\Omega_V)_{\alpha\beta}$ is here evaluated at x.

In a similar fashion, we can expand the function K_V:

$$\begin{aligned}
K_V &= g_{\mu\nu}(x^i, x^\alpha)V^\mu(x^i, x^\alpha)V^\nu(x^i, x^\alpha) = g_{\gamma\delta}(x^i, x^\alpha)V^\gamma(x^i, x^\alpha)V^\delta(x^i, x^\alpha) \\
&= g_{\gamma\delta}(x^i, 0) - \frac{1}{3}\left(\frac{1}{\sqrt{s}}R_{\gamma i\delta\beta}x^i x^\beta + \frac{1}{\sqrt{s}}R_{\gamma\alpha\delta j}x^\alpha x^j + \frac{1}{s}R_{\gamma\alpha\delta\beta}x^\alpha x^\beta\right)\left(\frac{1}{\sqrt{s}}(\partial_\varepsilon V^\gamma)x^\varepsilon\right)\times \\
&\times\left(\frac{1}{\sqrt{s}}(\partial_\zeta V^\gamma)x^\zeta\right) \\
&= \frac{1}{s}g_{\gamma\delta}(x^i, 0)(\partial_\varepsilon V^\gamma)(\partial_\zeta V^\delta)x^\varepsilon x^\zeta = \frac{1}{2s}(\Omega_V)_{\zeta\gamma}(\partial_\varepsilon V^\gamma)x^\varepsilon x^\zeta
\end{aligned} \tag{4.124}$$

As a conclusion, we see that

$$s(K_V + \Omega_V) \xrightarrow[s\to\infty]{} \frac{1}{2}(\Omega_V)_{\zeta\gamma}(\partial_\varepsilon V^\gamma)x^\varepsilon x^\zeta + \frac{1}{2}R_{\gamma\delta ij}(\partial_\varepsilon V^\gamma)x^\varepsilon x^\delta\theta^i\theta^j + \frac{1}{2}(\Omega_V)_{\alpha\beta}\theta^\alpha\theta^\beta. \tag{4.125}$$

This has two main implications: on the one hand, all non-quadratic terms vanish and on the other hand, we see that the only curvature coefficients that appear contain two greek and two latin indices, so the integration can be reduced to the normal bundle. More concretely,

$$\begin{aligned}
\int_{\mathcal{M}} \alpha &= \lim_{s\to\infty} \int_{\mathcal{M}} \alpha e^{-s d_V \beta} = \lim_{s\to\infty} \int_{\mathcal{M}\otimes\Lambda^1\mathcal{M}} d^n x d^n \theta \alpha(x,\theta) e^{-s(K_V+\Omega_V)} \\
&= \lim_{s\to\infty} \int_{\mathcal{M}\otimes\Lambda^1\mathcal{M}} d^n x^i d^n x^\alpha d^n \theta^\alpha d^n \theta^i \alpha(x,\theta) \exp\left[-s g_{\mu\nu}(x^i,x^\alpha) V^\mu(x^i,x^\alpha) V^\nu(x^i,x^\alpha) - \frac{s}{2}(\Omega_V)_{\mu\nu}(x^i,x^\alpha)\theta^\mu\theta^\nu\right] \\
&= \int_{\mathcal{M}\otimes\Lambda^1\mathcal{M}} d^n x^i d^n x^\alpha d^n \theta^\alpha d^n \theta^i \alpha(x,\theta) \exp\left[-\frac{1}{2}(\Omega_V)_{\zeta\gamma}(\partial_\varepsilon V^\gamma) x^\varepsilon x^\zeta\right] \exp\left[-\frac{1}{2}(\Omega_V)_{\alpha\beta}\theta^\alpha\theta^\beta\right] \times \\
&\times \exp\left[-\frac{1}{2} R_{\gamma\delta ij}(\partial_\varepsilon V^\gamma) x^\varepsilon x^\delta \theta^i \theta^j\right] \\
&= \int_{\mathcal{M}_V\otimes\Lambda^1\mathcal{M}_V} d^n x^i d^n \theta^i \alpha(x^i,\theta^i) \sqrt{\frac{(-2\pi)^L}{\det[(\Omega_V)_{\zeta\gamma}]\det[\partial_\varepsilon V^\gamma]}} (-1)^{L/2} \mathrm{Pfaff}[(\Omega_V)_{\alpha\beta}] \times \\
&\times \sqrt{\frac{(2\pi)^L}{\det[R_{\gamma\delta ij}\theta^i\theta^j]\det[\partial_\varepsilon V^\gamma]}} = \\
&= \int_{\mathcal{M}_V\otimes\Lambda^1\mathcal{M}_V} d^n x^i d^n \theta^i (-2\pi)^{L/2} \frac{\alpha(x^i,\theta^i)}{\mathrm{Pfaff}\,[(\Omega_V)_{\zeta\gamma} + R_{\zeta\gamma ij}\theta^i\theta^j]}
\end{aligned} \tag{4.126}$$

where L is the codimension of each component $\mathcal{M}_l$ of $\mathcal{M}_V$. Being more explicit and recognising the denominator as the equivariant Euler class, we can write

$$\int_{\mathcal{M}} \alpha = \sum_l \int_{\mathcal{M}_l\otimes\Lambda^1\mathcal{M}_l} d^n x^i d^n \theta^i (-2\pi)^{rk(\mathcal{M}_l)/2} \frac{\alpha(x^i,\theta^i)}{E_V(R)_{|\mathcal{N}_l}}, \tag{4.127}$$

where $\mathcal{N}_l$ is the normal bundle of $\mathcal{M}_l$.

4.4 Localisation for Torus Group Actions

So far we have always considered the action of a unidimensional Lie group, $G = U(1)$. In this section we will drop this assumption and consider the action $G \times \mathcal{M} \to \mathcal{M}$ of an arbitrary commutative Lie group. An important property of such groups is that they can always be written as a torus $\mathbb{T}^d = (S^1)^d$, the product of $d > 1$ circles, where $d = \dim G$.

Let $\{X^a\}_{a=1}^d$ be the basis of the Lie algebra $\mathfrak{g}$ of G and let $\{\phi^a\}_{a=1}^d$ be the dual basis of $\mathfrak{g}^*$, so

$$\phi^a(X^b) = \delta^{ab}. \tag{4.128}$$

Let $V^a \in T\mathcal{M}$ be the vector fields corresponding to $X^a \in \mathfrak{g}$, which are expressed in the basis of local coordinates as

$$V^a = V^{a,\mu}(x)\frac{\partial}{\partial x^\mu}, \quad a = 1, ..., d. \tag{4.129}$$

Then we can write the exterior equivariant derivative as

$$d_{\mathfrak{g}} = \mathbb{1} \otimes \theta^\mu \frac{\partial}{\partial x^\mu} + V^{a,\mu}\phi^a \otimes \frac{\partial}{\partial \theta^\mu} \tag{4.130}$$

and

$${d_{\mathfrak{g}}}^2 = \phi^a \otimes (d i_{V^a} + i_{V^a} d) = \phi^a \otimes \left(\theta^\mu(\partial_\mu V^{a,\nu})\frac{\partial}{\partial\theta^\nu} + V^{a,\mu}\frac{\partial}{\partial x^\mu}\right) = \phi^a \otimes \mathcal{L}_{V^a}. \tag{4.131}$$

The proof of the Berline-Vergne theorem is essentially the same as for the unidimensional case (§4.1), but with the substitution $V^\mu(x) = \phi^a V^{a,\mu}(x)$. The matrix (4.36) will look like

$$\begin{pmatrix} 0 & -\phi^1\lambda_{1,1} & & & & & & & & \\ \phi^1\lambda_{1,1} & 0 & & & & & & & & \\ & & \ddots & & & & & & & \\ & & & 0 & -\phi^1\lambda_{l_1,1} & & & & & \\ & & & \phi^1\lambda_{l_1,1} & 0 & & & & & \\ & & & & & \ddots & & & & \\ & & & & & & 0 & -\phi^d\lambda_{1,d} & & \\ & & & & & & \phi^d\lambda_{1,d} & 0 & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & 0 & -\phi^d\lambda_{l_d,d} \\ & & & & & & & & & \phi^d\lambda_{l_d,d} & 0 \end{pmatrix}, \tag{4.132}$$

so

$$\text{Pfaff}\,(\partial V) = (-1)^n \prod_{a=1}^{\dim G} \prod_{j=1}^{l_a} \phi^a \lambda_{a,j}. \tag{4.133}$$

It is important to note how the result of the integration is not a number but an element of $\mathfrak{g}^*$, as noted in (4.18).

Prescription: Torus action in $\mathbb{R}^{2n}$

We will now consider the multidimensional equivalent of the prescription in §4.2. As we pointed out before, this is not an example of the theorem, but rather a prescription on how to apply the calculations of its proof to the non-compact case. Let us consider $\mathcal{M} = \mathbb{R}^{2n}$ with coordinates (p_μ, q^μ). The most simple symplectic form for this case is

$$\omega = \sum_{j=1}^n dp_j \wedge dq^j = \sum_{j=1}^n \theta^{p_j}\theta^{q^j}. \tag{4.134}$$

Consider the $\mathbb{T}^n$-action where each X^a corresponds to

$$V^a = p_a \frac{\partial}{\partial q^a} - q^a \frac{\partial}{\partial p_a}. \tag{4.135}$$

The only fixed point is the origin $(p_j, q^j) = (0,0)$. The corresponding Hamiltonians will be:

$$H^a = \frac{p_a^2 + q^{a2}}{2} \tag{4.136}$$

The equivariant exterior differential will be

$$d_\mathfrak{g} = \sum_{a=1}^n \left(\theta^{p_a} \frac{\partial}{\partial p_a} + \theta^{q^a} \frac{\partial}{\partial q^a} + \phi^a p_a \frac{\partial}{\partial \theta^{q^a}} - \phi^a q^a \frac{\partial}{\partial \theta^{p_a}} \right) \tag{4.137}$$

and when applied to the equivariant symplectic form $\omega_\mathfrak{g} = \omega + H = \omega + \phi^a H^a$ we get

$$\begin{aligned} d_\mathfrak{g}\omega_\mathfrak{g} &= d_\mathfrak{g}\left[\sum_{a=1}^n \left(\theta^{p_a}\theta^{q^a} + \phi^a \frac{{p_a}^2 + q^{a2}}{2} \right) \right] \\ &= \sum_{a=1}^n \left(+\phi^a p_a \theta^{p_a} + \phi^a q^a \theta^{q^a} - \phi^a p_a \theta^{p_a} - \phi^a q^a \theta^{q^a} \right) = 0. \end{aligned} \tag{4.138}$$

In the first place, we will consider the form α as defined in (4.76) and we will apply the Berline-Vergne theorem. The Jacobian of the vector field is

$$\partial V = \begin{pmatrix} 0 & \phi^1 & & & \\ -\phi^1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \phi^n \\ & & & -\phi^n & 0 \end{pmatrix}, \tag{4.139}$$

because in this case $l_1 = ... = l_n = 1$. Therefore Pfaff $\partial V = (-1)^n \phi^1 ... \phi^n$, so we will finally have

$$Z(T) = (-2\pi)^n \frac{\alpha^{(0)}(0)}{\text{Pfaff } \partial V(0)} = (-2\pi)^n \frac{1}{T^n} \frac{1}{(-1)^n \phi^1 ... \phi^n} = \left(\frac{2\pi}{T}\right)^n \frac{1}{\phi^1 ... \phi^n}. \tag{4.140}$$

In the second place, we will apply the Duistermaat-Heckman integration formula (4.80) directly. To do so, we observe that

$$\omega(0) = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}, \qquad \mathcal{H}(0) = \begin{pmatrix} \phi^1 & 0 & & & \\ 0 & \phi^1 & & & \\ & & \ddots & & \\ & & & \phi^n & 0 \\ & & & 0 & \phi^n \end{pmatrix}, \tag{4.141}$$

so $\det \omega(0) = 1$ and $\det \mathcal{H}(0) = {\phi^1}^2 ... {\phi^n}^2$. Finally we have

$$Z(T) = \left(\frac{2\pi}{T}\right)^n e^{-TH(0)} \sqrt{\frac{\det \omega(0)}{\det \mathcal{H}(0)}} = \left(\frac{2\pi}{T}\right)^n \sqrt{\frac{1}{{\phi^1}^2 ... {\phi^n}^2}} = \left(\frac{2\pi}{T}\right)^n \frac{1}{\phi^1 ... \phi^n}, \tag{4.142}$$

which coincides with (4.140). It is worth observing that for $n = 1$ and $phi^1 = 1$ we recover (4.91).

Chapter 5

Outlook

In this work we have discussed how the presence of symmetries of a physical system represented as actions of a Lie group on manifolds allow to define the *equivariant cohomology.* We have considered different scenarios, such as ordinary differential manifolds, fibre bundles, symplectic manifolds and supermanifolds, reproducing the approach of R. J. Szabo, N. Berline *e. al.* [1, 3], namely the so-called *Cartan's model* of equivariant cohomology. We have also presented the main localisation results in these scenarios, based on the Berline-Vergne and Duistermaat-Heckman theorems, developing the proofs in an extended manner and illustrating the procedures both through examples and also through prescriptions on the use of the techniques in non-compact scenarios.

Nonetheless, the localisation techniques presented in this work can be formally extended to more general contexts and in particular to derive localisation formulae for path integrals in special cases by using BRST quantisation techniques. Equivariant localisation is a topic that still attracts interest of both mathematicians and physicists for the relations that it establishes between geometry and topology on one side and quantum and topological field theories on the other. It should be observed, however, that in these cases one must take into account that some problems are not entirely well defined from a mathematical point of view and require further research in the topic. We refer the curious reader to [1, 3] for more information.

Sammanfattning på Svenska

Symmetrier har i stor utsträckning använts under de senaste årtiondena för att förenkla systemen i matematik och teoretisk fysik, speciellt i studier av dynamiska system. I början av 1980-talet konstaterades att vissa integralberäkningar vanliga i symplektisk geometri skulle kunna förenklas när vissa symmetrivillkor uppfylldes. Med användning av ekvivariantkohomologi, ett begrepp som infördes av Henri Cartan på 1950-talet, generaliserades detta resultat några år senare till fallet med Killingvektorfält på generella kompakta Riemannmångfalder. I synnerhet visades att den stationära fasapproximationen faktiskt var exakt. Detta var födelsen av det som nu kallas ekvivariant lokalisering, föremålet för föreliggande arbete.

Ekvivariant lokalisering bygger på att utnyttja vissa symmetrier av system, som vi representerar genom en icke-fri Liegruppverkan på en mångfald, för att minska dimensionerna av integralberäkningar. Särskilt intressanta är de fall när denna reduktion, känd som lokalisering, tillåter en ändligdimensionell integral att uttryckas som en summa av ett ändligt antal element. Ekvivariant lokalisering kan också användas för att förenkla sökvägen från integralberäkningar till mer kända och bättre definierade ändligdimensionella integraler, men detta ligger utanför ramen för detta arbete.

Lokaliseringsegenskaper kan uttryckas på olika matematiska språk. I detta arbete använder vi Cartans modell för ekvivariantkohomologi, som bygger på att skapa element liknande de till de Rham-komplexet men med avseende på en Liegruppverkan på mångfalden. För att utföra beräkningar, introducerar vi begreppet anti-kommutativa variabler och gör verksamheten i mångfaldens yttre knippe, som vi förstår som en supermångfald. På så sätt kan vi etablera en invarians under koordinattransformationer och ge bevis på exakthet av approximationer via sadelpunkter på våra integrerade beräkningar.

Detta arbete inleds med en genomgång av nödvändiga matematiska begrepp, inklusive differentialgeometri, Liegrupper och algebror och grunderna om supergeometri och supermångfalder. Efter detta studerar vi ekvivariantkohomologi i olika scenarier: mångfalder, vektorknippen, symplektiska mångfalder och supermångfalder. Slutligen utvecklar vi de viktigaste lokaliseringsresultaten på dessa scenarier, som Berline-Vergnes och Duistermaat-Heckmans satser. Vi skriver ned bevisen för dessa resultat på ett detaljerad och utökad sätt, illustrerar dem med exempel och visar hur dessa tekniker kan användas som ett recept för icke-kompakta scenarier.

Resum en Català

Les simetries s'han fet servir extensament durant les darreres dècades per a simplificar sistemes dins de les matemàtiques i la física teòrica, especialment en l'estudi de sistemes dinàmics. Al principi dels anys 80 es va observar que algunes operacions integrals comuns dins la geometria simplèctica podien ésser simplificades quan es complien certes condicions. Fent servir el llenguatge de la cohomologia equivariant, un concepte introduït per Henri Caran als anys 50, es van generalitzar aquests resultats durant els anys següents al cas de camps vectorials de Killing sobre varietats de Riemann generals. En particular, es va probar que l'aproximació de la fase estacionària era, de fet, exacta. Això va marcar el naixement del que avui es coneix com a localització equivariant, l'objecte d'aquest treball.

La localització equivariant es basa en fer servir simetries de certs sistemes, representades com a una acció no lliure d'un grup de Lie sobre una varietat, per a reduir la dimensionalitat dels càlculs integrals. Especialment interessants són els casos en què aquesta reducció, coneguda com a localització, permet expressar una integral de dimensió finita com a una suma d'un nombre finit d'elements. La localització equivariant també pot ser usada per a reduir integrals de camí a integrals de dimensió finita, molt més conegudes i més ben definides, tot i que això queda fora de l'abast d'aquest treball.

Les propietats de la localització poden ser expressades en diversos llenguatges matemàtics. En aquest treball farem servir el model de Cartan per a la cohomologia equivariant, basat en la creació d'elements similars al complex de de Rahm però tenint en compte l'acció d'un grup de Lie sobre la varietat. Per a fer els càlculs, introduïm el concepte de variables anticommutatives i fem les operacions al feix fibrat exterior de la varietat, entenent-lo com una supervarietat. Això ens permet establir una invariància sota canvis de coordenades i provar l'exactitud de l'aproximació del punt de sella en els nostres càlculs.

Aquest treball comença amb una revisió dels conceptes matemàtics necessaris, incloent la geometria diferencial, els grups i àlgebres de Lie i nocions bàsiques de supergeometria i supervarietats. Després d'això estudiem la cohomologia equivariant en diversos escenaris: varietats diferencials, feixos fibrats, varietats simplèctiques i supervarietats. Finalment desenvolupem els resultats de localització en aquests escenaris, com els teoremes de Berline-Vergne o de Duistermaat-Heckman. Escrivim les proves d'aquests resultats de manera detallada i extensa, il·lustrant-los amb exemples i mostrant com aquestes tècniques també poden ser usades com a prescripció per a escenaris no compactes.

Resumen en Español

Las simetrías se han usado extensamente durante las últimas décadas para simplificar sistemas dentro de las matemáticas y la física teórica, especialmente en el estudio de sistemas dinámicos. Al principio de los años 80 se observó que algunas operaciones integrales comunes dentro de la geometría simpléctica podían ser simplificadas cuando se cumplían ciertas condiciones. Usando el lenguaje de la cohomología equivariante, un concepto introducido por Henri Cartan en los años 50, se generalizaron estos resultados durante los siguientes años para el caso de campos vectoriales de Killing sobre variedades de Riemann generales. En particular, se probó que la aproximación de la fase estacionaria era, de hecho, exacta. Esto marcó el nacimiento de lo que hoy se conoce como localización equivariante, el objeto del presente trabajo.

La localización equivariante se basa en el uso de las simetrías de ciertos sistemas, representadas como una acción no libre de un grupo de Lie sobre una variedad, para reducir la dimensionalidad de los cálculos integrales. Especialmente interesantes son los casos en que esta reducción, conocida como localización, permite expresar una integral de dimensión finita como una suma de un número finito de elementos. La localización equivariante también puede ser usada para reducir integrales de camino a integrales de dimensión finita, mucho más conocidas y mejor definidas, aunque esto yace fuera del alcance de este trabajo.

Las propiedades de localización pueden ser expresadas en distintos lenguajes matemáticos. En este trabajo usamos el modelo de Cartan para la cohomología equivariante, basado en la creación de elementos similares al complejo de de Rahm pero teniendo en cuenta la acción de un grupo de Lie sobre la variedad. Para realizar los cálculos, introducimos el concepto de variables anticonmutativas y hacemos las operaciones en el haz fibrado exterior de la variedad, entendiéndolo como una supervariedad. Esto nos permite establecer una invariancia bajo cambios de coordenadas y probar la exactitud de la aproximación del punto de sella en nuestros cálculos.

El presente trabajo empieza con una revisión de los conceptos matemáticos necesarios, incluyendo la geometría diferencial, los grupos y álgebras de Lie y nociones básicas de supergeometría y supervariedades. Después estudiamos la cohomología equivariante en distintos escenarios: variedades diferenciales, haces fibrados, variedades simplécticas y supervariedades. Finalmente desarrollamos los resultados de localización en estos escenarios, como los teoremas de Berline-Vergne o de Duistermaat-Heckman. Escribimos las demostracionesde de Rahm de estos resultados de forma detallada y extensa, ilustrándolos con ejemplos y mostrando cómo estas técnicas también pueden ser usadas como una prescripción para escenarios no compactos.

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