Fermi gas close to unitarity: ϵ-expansion

Olof Ohlsson Sax

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Supervisor: Konstantin Zarembo

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Department of Theoretical Physics

UPPSALA
UNIVERSITET

Abstract

In this thesis I consider an attractively interacting, degenerate Fermi gas close to the unitary limit, where the two-particle scattering length is much larger than the inter-particle distance and the interaction range is vanishingly small. In this limit there are problems with traditional perturbation theory, since there is no small dimensionless parameter that can be used in an expansion. To overcome these problems, a new model was recently proposed by Y. Nishida and D. T. Son. By treating the gas in $d = 4 - \epsilon$ spatial dimensions, where ϵ is assumed to be a small quantity, it is possible to do a perturbative expansion in ϵ, in terms of Feynman diagrams. In this thesis this ϵ-expansion model is studied and used to calculate the density of the boson condensate, the fermion number density and the chemical potential of the fermion gas, in the vicinity of the unitary point. The results are then extrapolated to three spatial dimensions and compared to results from various experiments and quantum Monte Carlo calculation.

Contents

1 Introduction

At zero temperature, a Fermi gas with an attractive pairwise interaction of any strength will form a superfluid [1]. The nature of this superfluid is, however, highly dependent on the interaction strength. With a strong enough interaction, fermion pairs can form dimers, i.e. two-particle bound states. The Fermi gas then effectively turns into an interacting Bose gas. Already in 1925 Albert Einstein realized that at sufficiently low temperature, such a gas should condensate into a state where every particle is in the single dimer ground state. Such a state is today known as a Bose–Einstein condensate (BEC) (see Ref. [2] for a review).

If the attraction between the fermions on the other hand is weak, no localized bound states can form. But even though there are no dimer formation, fermions close to the Fermi surface tend to form pairs, as was shown by Bardeen, Cooper, and Schrieffer [3]. These pairs, commonly called *Cooper pairs,* consist of fermions of opposite momenta and antiparallell spins. Even though these pairs are not localized in space, but rather correlated in momentum space, they still can condensate in a way similar to the Bose–Einstein condensation of the dimers of the strong coupling regime — the pairs have total spin zero, and consequently behave as bosons. The BCS theory was published in 1957 as a, extraordinarily successful, microscopic theory of superconductivity and is reviewed in Ref. [4]. A central result of this theory is that the dispersion relation of the fermions around the Fermi surface develop a gap, i.e. there is a minimum excitation energy above the Fermi energy. Much insight to the physics of the gas can be gained through calculations of this gap.

Thus weakly and strongly interacting superfluid Fermi gases have long been separately well understood. Little was known, though, about the relation between them until the work by Legget [5] and later Nozières and Schmitt-Rink [6], who applied the gap equation from the BCS mean field theory, which is quantitatively valid only in a weak interaction and high density regime, to fermions with an arbitrary strong interaction. The result was a mean field model where the BCS state was treated as a variational ground state, which in a continuous fashion connected the two types of fermionic superfluids, thereby demonstrating that they are two regimes of the same phenomena.

1.1 The BCS–BEC crossover

The transition from the BCS side to the BEC side — the BCS–BEC crossover — occurs continuously without any phase transition [5, 6]. Thus the extended BCS theory can be used to get some qualitative results also on the BEC side. But while the gap equation correctly gives the binding energy for a pair of fermions, it fails to take into account the interaction between these dimers, and thus fails to describe physical properties of the BEC side [7], and one has to go beyond the BCS mean-field equation to do quantitative calculations.

Since there is no hope for an exact solution to the full equations of a large system of interacting fermions, the natural way to proceed to get quantitative results would be using a perturbation theory with an expansion in some small quantity. If one were to keep the BCS gap equation as a starting point, the natural quantity to expand in would be the strength of interaction between the dimers. But this is not small on the BEC side [7].

For a system of attractively interacting fermions with particle density n, there are three different intrinsical length scales, the range of the interaction potential r_0, the two-particle scattering length a, and the inter-particle distance $n^{-1/3}$.

As a system undergoes a BCS–BEC crossover, the two-particle scattering length a strongly varies in magnitude. The range of the interaction r_0, in contrast, remains fairly constant throughout the crossover. There are two principal cases which we need to differentiate between. Fermions with an interaction for which $r_0 \gg n^{1/3}$ are referred to as being in the narrow-resonance regime. It is then possible to build on the BCS mean field theory and construct simple models which are good approximations all through the BCS–BEC crossover, e.g. the two-channel model which we will discuss briefly at the end of this section [7, 8]. If, on the other hand, $r_0 \ll n^{1/3}$ and $r_0 \ll a$, then the fermions are said to be in the wide-resonance regime. Here the problems with the BCS gap equation is much worse. This regime is also more relevant to the recent experimental realization of these systems, which we will discuss below. Thus I will, in this thesis, from now on assume that we consider fermions in the wide-resonance regime.

I will also assume that the fermions interact exclusively through an s-wave channel, so that scattering particles always have a two-particle wave function with zero angular momentum. This is, at least at low energies, a good approximation since higher angular momentum scattering is suppressed by the centrifugal barrier.

At zero temperature an interacting Fermi gas in the wide-resonance regime is, across the BCS–BEC crossover, characterized by the dimensionless number $\eta = 1/(ap_{\mathrm{F}})$, where a is the two-particle scattering length and p_{F} is the Fermi momentum of the corresponding non-interacting Fermi gas. For a weakly interacting system the scattering length is small and negative, and hence η is a large negative number. The ground state of such a system is a BCS superfluid where the fermions are paired in momentum space. A strongly interacting system, on the other hand, has a small positive scattering length which means that η is large and positive. Now the fermions will form two-body bound states due to the strong attraction. In the ground state these bound states will Bose–Einstein condensate, and hence this corresponds to the BEC-side of the crossover. A system of interacting fermions with a diverging scattering length, i.e. $\eta \approx 0$ is commonly referred to as a unitary Fermi gas [9].

Given this, a natural expansion parameter is the so called gas parameter which is given by $an^{1/3}$. For a gas that is dilute enough, this is a small quantity in both the BCS and BEC limits. But as seen above the two-particle scattering amplitude diverges in the unitary limit. Thus this description fails to describe the physics of the crossover region itself [7].

The simplest kind of model is a one-channel model, where one only considers a gas of fermions with a pairwise interaction. In contrast to this it is possible to construct a two channel model, where a boson field, describing the bound states, is manually introduced in addition to the fermions [8]. The benefit of this is that the interaction width, which in a one-channel model always is vanishingly small, can now be treated as a free parameter. In the narrow-resonance regime, it is then possible to make an expansion in terms of the small quantity $\gamma = 1/(p_{\mathrm{F}} r_0)$. But as already stated, we are mainly interested in wide resonances, and then the parameter γ will not be a small quantity to make an expansion in. From

a fundamental point of view, a one-channel approach seems more attractive, since it more closely models the basic problem of a Fermi gas with an attractive interaction. But a two-channel model have a phenomenological appeal, since it more accurately describes the physics of a system with a resonant interaction, which is used in experimental setups.

1.2 The Feshbach resonance

In recent years, the problem of a Fermi gas in the BCS–BEC crossover has received much new interest. This is mainly due to the experimental success in producing degenerate atomic Fermi gases of ^{6}Li and ^{40}K [10, 11]. In these experiments, atoms are cooled down in a magnetic trap. The atoms are prepared to form a mixture of two different spin states, which are coupled by the hyperfine interaction. The interaction between two atoms depend on their total spin, in such a way that there is a possibility to form a quasibound state for one combination of spins, but not for the other. Thus atom can scatter through this intermediate dimer state — so called Feshbach resonant scattering [8]. By applying an external magnetic field the energy gap between the two spin states will shift due to the Zeeman splitting, making it possible to vary the interaction strength, and hence the scattering length, in a controlled manner. Using this technique of scattering through a Feshbach resonance, it has been possible to experimentally verify many of the properties of a Fermi gas that undergoes a BCS–BEC crossover. Some of these result will be discussed in section 4.8, where we will compare them to the results that we have calculated using an ϵ-expansion approach.

1.3 The unitary Fermi gas

The unitary Fermi gas is characterized by the limits $a \to \infty$ and $r_0 \to 0$. The only remaining quantity that has the physical dimension of length is the inter-particle distance, given by $n^{1/3}$, where n is the fermion particle number density. The energy scale of the system can then be given in terms of the Fermi energy of a non-interacting gas with the same density, $\varepsilon_{\mathrm{F}} = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}$. The only other available quantity with the dimension of energy is the chemical potential μ of the fermions. But this is also uniquely determined by fixing the number density n. Thus by considering only the physical dimensions in the problem, we expect that energy per particle of a Fermi gas in the unitary limit will be given by an expression of the form

$$\frac{E}{n} = \frac{3}{5}\xi_0 \varepsilon_{\mathrm{F}}, \tag{1}$$

where ξ_0 is a dimensionless constant depending only on the ratio $\mu/\varepsilon_{\mathrm{F}}$, which is assumed to be of the order of unity. The numerical prefactor is given for later convenience. When we generalize to a Fermi gas in an arbitrary number of dimensions d, this prefactor will turn out to depend on d, and so will the parameter ξ_0. A property like E that is determined using purely dimensional arguments, and thus independent of the specific details of the inter-particle interaction, is said to be *universal* [12].

Due to this independence of the microscopic details of the interaction, a Fermi gas at unitarity has been considered as a starting model for describing vastly different physical systems. One example of the usage of the unitary

Fermi gas is to treat it as an idealization of dilute nuclear matter. This idea has been useful in creating models of the physics of neutron stars [13, 14]. Furthermore Chen et al. have suggested that there is an important relation to high T_C superconductors [15]. Finally Nishida and Abuki have developed a relativistic generalization of the whole BCS–BEC crossover and suggested that there are some relevance to QCD, and more specifically to models of cold dense quark matter and of the quark-gluon plasma [16].

1.4 The ϵ-expansion

While the universal properties of the unitary Fermi gas makes it interesting as a model of various physical systems, they also complicate a theoretical description. As we have seen many properties of the gas can be expressed in terms of the parameter ξ_0. But this parameter is of the order of unity and thus not useful for a perturbative expansion. Because of this, much of the research on the subject has been limited to numerical Monte Carlo calculations. Recently, however, there have been a couple of generalizations to the problem, which have enabled an analytical description even at unitarity. Veillette et al. [17] considered an arbitrary number of spin-1/2 fermion flavors with a $Sp(2N)$ symmetry, noticing that the mean field theory has an exact solution for $N \to \infty$ and then deriving corrections in terms of the small parameter $1/N$. In the end the physical limit of $N \to 1$ is taken.

In this thesis I will describe another technique for introducing a small parameter that can be used for perturbation theory. I will consider a Fermi gas around the unitary limit in $d = 4 - \epsilon$ spatial dimensions, where ϵ is assumed to be perturbatively small, and then derive a perturbation theory i terms of ϵ. This technique of working in almost four spatial dimensions and then in the end extrapolate to the usual three was first used by Wilson and Fisher, in the calculation of critical exponents of generalized Ising and Heisenberg models [18, 19]. Recently Nishida and Son [9, 20] used it to describe a Fermi gas in the unitary limit. There have also been various elaborations of Nishida's and Son's original paper. Nishida [21] treated a unitary Fermi gas at a finite temperature; Arnold et al. [22] made the first next-to-next-leading-order calculation; Rupak et al. [23] treated a Fermi gas with a finite spin polarization; and Chen and Nakano [24] expanded this technique to be useful all along the BCS–BEC crossover.

Many of the results in this thesis were first derived in Ref. [9] or Ref. [24]. I will, however, be able to derive these results using a conceptually simpler model. I will also present the, sometimes rather involved, calculations in much more detail. Some of the differences between the treatments will be further discussed in section 5.

2 Two-particle scattering in the unitary limit

To better understand our problem, we start by considering two-particle scattering of fermions with a four-fermion point interaction. The Lagrangian of the system is given by

$$\mathcal{L} = \sum_{\sigma=\uparrow,\downarrow} \psi_\sigma^\dagger \left(i\partial_t + \frac{\nabla^2}{2m} \right) \psi_\sigma + c_0 \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\downarrow \psi_\uparrow, \tag{2}$$

where c_0 is a coupling constant. The fermions have a propagator

$$iG(p_0, \boldsymbol{p}) = \frac{i}{p_0 - \varepsilon_{\boldsymbol{p}} + i\delta}, \tag{3}$$

where $\varepsilon_{\boldsymbol{p}} = \frac{\boldsymbol{p}^2}{2m}$.

The T-matrix amplitude for two-particle scattering in vacuum is given by an infinite sum of bubble diagrams, which can be summed as a geometric series

$iT(p_0, \boldsymbol{p}) =$  (4)

$$= ic_0 \sum_{n=0}^{\infty} (i\mathcal{M})^n = \frac{ic_0}{1 - i\mathcal{M}}, \tag{5}$$

where $ic_0 \times i\mathcal{M}$ is the one-bubble diagram, i.e. the second diagram of the above expansion. Hence

$$i\mathcal{M} = ic_0 \int \frac{dk}{(2\pi)^{d+1}} \frac{i}{\frac{p_0}{2} + k_0 - \varepsilon_{\frac{\boldsymbol{p}}{2}+\boldsymbol{k}} + i\delta} \frac{i}{\frac{p_0}{2} - k_0 - \varepsilon_{\frac{\boldsymbol{p}}{2}-\boldsymbol{k}} + i\delta} \tag{6}$$

$$= ic_0 \int \frac{d\boldsymbol{k}}{(2\pi)^d} \int \frac{dk_0}{2\pi} \frac{1}{k_0 - \left(\varepsilon_{\frac{\boldsymbol{p}}{2}+\boldsymbol{k}} - \frac{p_0}{2} - i\delta\right)} \frac{1}{k_0 - \left(\frac{p_0}{2} - \varepsilon_{\frac{\boldsymbol{p}}{2}-\boldsymbol{k}} + i\delta\right)}. \tag{7}$$

Performing the k_0 integral by closing the contour around the pole in the upper half of the complex plane, this gives

$$i\mathcal{M} = -c_0 \int \frac{d\boldsymbol{k}}{(2\pi)^d} \frac{1}{\left(\frac{p_0}{2} - \varepsilon_{\frac{\boldsymbol{p}}{2}-\boldsymbol{k}} + i\delta\right) - \left(\varepsilon_{\frac{\boldsymbol{p}}{2}+\boldsymbol{k}} - \frac{p_0}{2} - i\delta\right)} \tag{8}$$

$$= c_0 \int \frac{d\boldsymbol{k}}{(2\pi)^d} \frac{1}{\varepsilon_{\frac{\boldsymbol{p}}{2}+\boldsymbol{k}} + \varepsilon_{\frac{\boldsymbol{p}}{2}-\boldsymbol{k}} - p_0 - i\delta} \tag{9}$$

$$= c_0 \int \frac{d\boldsymbol{k}}{(2\pi)^d} \frac{1}{2\varepsilon_{\boldsymbol{k}} - p_0 + \frac{\varepsilon_{\boldsymbol{k}}}{2} - i\delta}. \tag{10}$$

Hence

$$T(p_0, \boldsymbol{p})^{-1} = \frac{1}{c_0} - \int \frac{d\boldsymbol{k}}{(2\pi)^d} \frac{1}{2\varepsilon_{\boldsymbol{k}} - p_0 + \frac{\varepsilon_{\boldsymbol{p}}}{2} - i\delta}. \tag{11}$$

The integral in eq. (11) has an ultraviolet divergence but can be performed using dimensional regularization. Since we will need to calculate such integrals repeatedly in this thesis, there is, in appendix A, a collection of some formulas that are useful when performing integrals in an arbitrary number of dimensions d.

Letting $\Delta = m\left(\frac{\varepsilon_{\boldsymbol{p}}}{2} - p_0 - i\delta\right)$ and changing to spherical coordinates, eq. (11) can be written as

$$i\mathcal{M} = c_0 \int \frac{d\Omega}{(2\pi)^d} \int d|\boldsymbol{k}| \frac{m\boldsymbol{k}^{d-1}}{\boldsymbol{k}^2 + \Delta}, \tag{12}$$

where $d\Omega$ is a d-dimensional volume element. Changing variables of integration to $z = \boldsymbol{p}^2/\Delta$ gives

$$i\mathcal{M} = c_0 \frac{m}{2} \Delta^{\frac{d}{2}-1} \int \frac{d\Omega}{(2\pi)^d} \int dz \frac{z^{\frac{d}{2}-1}}{1+z}, \tag{13}$$

which can be integrated using the formulas (A2) and (A3), giving

$$i\mathcal{M} = c_0 \left(\frac{m}{4\pi}\right)^{\frac{d}{2}} \Gamma\left(1-\tfrac{d}{2}\right) \Gamma\left(\tfrac{d}{2}\right) \left(-p_0 + \frac{\varepsilon_{\boldsymbol{p}}}{2} - i\delta\right)^{\frac{d}{2}-1}. \tag{14}$$

Appendix B contains a short review of some relations involving the two-particle scattering length. Among other things, it is shown that at threshold, i.e. when $p_0 \to 0$ and $\boldsymbol{p} \to 0$, the two-particle scattering cross section is given by

$$\sigma = 4\pi a^2. \tag{B33}$$

But in the unitary limit the scattering amplitude a goes to infinity, and so does the cross section. The only contribution to the cross section that could diverge in this way is the scattering amplitude T. Thus $T(0,0)^{-1} = 0$ in the unitary limit. From eq. (14) and eq. (11) we see that $i\mathcal{M}$ will vanish at threshold and thus that, in this limit, $|c_0| \to \infty$. Putting $d = 4-\epsilon$ and using the approximation (see eq. (A4))

$$\Gamma(-1+\epsilon) = -\frac{1}{\epsilon} + \gamma - 1 + \mathcal{O}(\epsilon), \tag{15}$$

where $\gamma \approx 0.5772$ is the Euler-Mascheroni constant, we get,

$$iT(p_0, \boldsymbol{p}) \approx -\left(\frac{4\pi}{m}\right)^{2-\frac{\epsilon}{2}} \frac{1}{\Gamma\left(-1+\frac{\epsilon}{2}\right)} \frac{i}{\left(-p_0 + \frac{\varepsilon_{\boldsymbol{p}}}{2} - i\delta\right)^{1-\frac{\epsilon}{2}}} \tag{16}$$

$$= -\left(\frac{4\pi}{m}\right)^{2-\frac{\epsilon}{2}} \frac{-\frac{\epsilon}{2}}{1 - \frac{\epsilon}{2}\left(-\gamma + 1 + \mathcal{O}(\epsilon)\right)} \frac{i}{\left(-p_0 + \frac{\varepsilon_{\boldsymbol{p}}}{2} - i\delta\right)^{1-\frac{\epsilon}{2}}} \tag{17}$$

$$= -\frac{8\pi^2\epsilon}{m^2} \frac{i}{p_0 - \frac{\varepsilon_{\boldsymbol{p}}}{2} + i\delta} + \mathcal{O}(\epsilon^2). \tag{18}$$

This expression has a very interesting form. If we define

$$g^2 = \frac{8\pi^2\epsilon}{m^2} \tag{19}$$

and

$$D(p_0, \boldsymbol{p}) = \frac{1}{p_0 - \frac{\varepsilon_{\boldsymbol{p}}}{2} + i\delta}, \tag{20}$$

the T-matrix can, to leading order in ϵ, be written as

$$iT(p_0, \boldsymbol{p}) \approx (ig)^2 iD(p_0, \boldsymbol{p}). \tag{21}$$

$D(p_0, \boldsymbol{p})$ has the form of a propagator of a particle with mass $2m$, which will be interpreted as a bound state of two fermions. Eq. (21) means that we, close to four spatial dimensions, can treat the process of two-body scattering as mediated by an intermediate boson, formed by a resonance. The effective coupling between the fermions and the bound state is given by $g \sim \sqrt{\epsilon}$, which is small near four dimensions. Using this picture it should be possible to construct a perturbative expansion for the unitary Fermi gas close to four spatial dimensions.

2.1 Binding energy

We want to describe not only a Fermi gas in the unitary limit, but also a gas in the vicinity of this limit. A good way to access this neighborhood is to consider the binding energy of a bound pair of fermions. In appendix B it is shown that if two particles scatter at very low energy in an attractive potential which have a range much shorter than the scattering length, then the binding energy of a bound state of the two particles are given by

$$\varepsilon_{\mathrm{b}} \approx \frac{1}{ma^2}, \tag{22}$$

where m is the mass of each of the particles. This can be rewritten in terms of the dimensionless parameter $\eta = \frac{1}{ap_{\mathrm{F}}}$ as

$$\varepsilon_{\mathrm{b}} \approx \frac{\eta^2 p_{\mathrm{F}}^2}{m} = 2\eta^2 \varepsilon_{\mathrm{F}}, \tag{23}$$

where $\varepsilon_{\mathrm{F}} = \frac{p_{\mathrm{F}}^2}{2m}$ is the Fermi energy of the free Fermi gas.

Following Ref. [25], we will now calculate the binding energy ε_{b} by assuming that the two-particle scattering amplitude T has a pole at zero momentum transfer, i.e. $T^{-1}(-\varepsilon_{\mathrm{b}}, 0) = 0$. According to eq. (11) this will be given by

$$\frac{1}{c_0} = \Gamma\left(1 - \frac{d}{2}\right)\left(\frac{m}{4\pi}\right)^{\frac{d}{2}} \varepsilon_{\mathrm{b}}^{\frac{d}{2}-1}. \tag{24}$$

At $d = 4 - \epsilon$ this gives, to leading order in ϵ

$$\frac{1}{c_0} \approx -\frac{\varepsilon_{\mathrm{b}}}{2\epsilon}\left(\frac{m}{2\pi}\right)^2 = -\frac{\varepsilon_{\mathrm{b}}}{g^2}. \tag{25}$$

A true bound state, with a positive binding energy, can only form if $c_0 < 0$. But later we want to use the connection between ε_{b} and η to get access to both sides of the region around the unitary limit, and then we need to consider also a positive c_0. Hence we will use the above expressions also in the $c_0 > 0$ case, even though there is no interpretation in terms of a binding energy.

There is another sign problem that we need to consider. If we express the ε_{b}-factor in eq. (24) in terms of η, using eq. (23), we get a term of the form $(\eta^2)^{d/2-1}$. But when we in the end want to take the $d \to 3$ limit, this gives $|\eta|$. Thus we will not be able to distinguish between the BCS and BEC sides, where the sign of η differs. To accommodate this we will simply let $|\eta| \to \eta$ in three spatial dimensions. If one would want to treat this more rigorously, this treatment should correspond to the prescription in Ref. [24], where there is an assumed extra factor of $\mathrm{sgn}(a)$ in front of the expression for $1/c_0$ in eq. (24).

3 Perturbation theory using an ϵ-expansion

We now want to use the above observations to derive a perturbative expansion for an attractive Fermi gas close to unitarity. We start with a gas of fermions which have the same kind of four fermion interactions as those in eq. (2), but which also have a chemical potential μ. Thus the Lagrangian is given by

$$\hat{\mathcal{L}} = \sum_{\sigma=\uparrow,\downarrow} \psi_\sigma^\dagger \left(i\partial_t + \frac{\nabla^2}{2m} + \mu\right)\psi_\sigma + c_0 \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\downarrow \psi_\uparrow. \tag{26}$$

We want to rewrite $\hat{\mathcal{L}}$ so that all terms in it are quadratic in the ψ:s. This can be done using a Hubbard-Stratonovich transformation. But first we write $\hat{\mathcal{L}}$ in terms of the Nambu-Gorkov fields Ψ and $\Psi^\dagger$, which are given by[1]

$$\Psi(x) = \begin{pmatrix} \psi_\uparrow(x) \\ \psi_\downarrow^\dagger(x) \end{pmatrix}, \qquad \Psi^\dagger(x) = \begin{pmatrix} \psi_\uparrow^\dagger(x) & \psi_\downarrow(x) \end{pmatrix}. \tag{27}$$

Later we will also need to write these fields in momentum space. Fourier expansion gives the corresponding expressions

$$\Psi(p) = \begin{pmatrix} \psi_\uparrow(p) \\ \psi_\downarrow^\dagger(-p) \end{pmatrix}, \qquad \Psi^\dagger(p) = \begin{pmatrix} \psi_\uparrow^\dagger(p) & \psi_\downarrow(-p) \end{pmatrix}. \tag{28}$$

To understand how to introduce Ψ and $\Psi^\dagger$ into $\hat{\mathcal{L}}$, we consider the free Lagrangian

$$\hat{\mathcal{L}}_0 = \Psi^\dagger \left(i\partial_t + \left(\frac{\nabla^2}{2m} + \mu \right) \sigma_3 \right) \Psi, \tag{29}$$

where σ_i and $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$ are the Pauli matrices. Using partial integration and then fermion anti-commutativity we get

$$\hat{\mathcal{L}}_0 = \psi_\uparrow^\dagger \left(i\partial_t + \frac{\nabla^2}{2m} + \mu \right) \psi_\uparrow + \psi_\downarrow \left(i\partial_t - \frac{\nabla^2}{2m} - \mu \right) \psi_\downarrow^\dagger \tag{30}$$

$$= \psi_\uparrow^\dagger \left(i\partial_t + \frac{\nabla^2}{2m} + \mu \right) \psi_\uparrow - \left[\left(i\partial_t + \frac{\nabla^2}{2m} + \mu \right) \psi_\downarrow \right] \psi_\downarrow^\dagger \tag{31}$$

$$= \psi_\uparrow^\dagger \left(i\partial_t + \frac{\nabla^2}{2m} + \mu \right) \psi_\uparrow + \psi_\downarrow^\dagger \left(i\partial_t + \frac{\nabla^2}{2m} + \mu \right) \psi_\downarrow, \tag{32}$$

where a constant term from the anti-commutation relation has been left out. Moreover we note that

$$\Psi^\dagger \sigma_- \Psi = \begin{pmatrix} \psi_\uparrow^\dagger & \psi_\downarrow \end{pmatrix} \begin{pmatrix} 0 \\ \psi_\uparrow \end{pmatrix} = \psi_\downarrow \psi_\uparrow, \tag{33}$$

so that

$$\left(\Psi^\dagger \sigma_- \Psi \right)^\dagger \left(\Psi^\dagger \sigma_- \Psi \right) = \left(\Psi^\dagger \sigma_+ \Psi \right) \left(\Psi^\dagger \sigma_- \Psi \right) = \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\downarrow \psi_\uparrow. \tag{34}$$

Hence the Lagrangian of eq. (26) can be written as

$$\hat{\mathcal{L}} = \Psi^\dagger \left(i\partial_t + \left(\frac{\nabla^2}{2m} + \mu \right) \sigma_3 \right) \Psi + c_0 \left(\Psi^\dagger \sigma_+ \Psi \right) \left(\Psi^\dagger \sigma_- \Psi \right). \tag{35}$$

Now we can perform a Hubbard-Stratonovich transformation on eq. (35). To do this we introduce an auxiliary complex scalar field ϕ and consider

$$\Delta\hat{\mathcal{L}} = -\frac{1}{c_0} \left(\phi^* - c_0 \Psi^\dagger \sigma_+ \Psi \right) \left(\phi - c_0 \Psi^\dagger \sigma_- \Psi \right) \tag{36}$$

$$= -\frac{1}{c_0} \phi^* \phi + \Psi^\dagger \sigma_+ \Psi \phi + \Psi^\dagger \sigma_- \Psi \phi^* - c_0 \left(\Psi^\dagger \sigma_+ \Psi \right) \left(\Psi^\dagger \sigma_- \Psi \right). \tag{37}$$

[1] This formalism was independently introduced first by Gorkov [26] and then by Nambu [27] in the context of the BCS theory of superconductivity. For a review see Ref. [4].

In the path integral

$$\int D\phi D\phi^* e^{-\int dx \Delta\hat{\mathcal{L}}}, \tag{38}$$

the field ϕ can be integrated out since, for any constant c (with respect to ϕ), an integral of the type

$$\int D\phi D\phi^* e^{-\int dx (\phi^* - c^*)(\phi - c)} \tag{39}$$

can be calculated by shifting variables to $\phi' = \phi - c$, giving

$$\int D\phi' D\phi'^* e^{-\int dx \phi'^* \phi'} = 1. \tag{40}$$

Thus we can shift $\hat{\mathcal{L}}$ by $\Delta\hat{\mathcal{L}}$ without changing the physics it describes. We denote the shifted Lagrangian by $\mathcal{L} = \hat{\mathcal{L}} + \Delta\hat{\mathcal{L}}$,

$$\mathcal{L} = \Psi^\dagger \left(i\partial_t + \left(\frac{\nabla^2}{2m} + \mu \right) \sigma_3 \right) \Psi - \frac{1}{c_0} \phi^* \phi + \Psi^\dagger \sigma_+ \Psi \phi + \Psi^\dagger \sigma_- \Psi \phi^*. \tag{41}$$

The ground state of this system is a superfluid state where ϕ condenses. To describe this we expand ϕ around the vacuum expectation value $\langle \phi \rangle = \phi_0$, by introducing

$$\phi = \phi_0 + g\varphi. \tag{42}$$

The Lagrangian $\mathcal{L}$ is symmetric under the transformations

$$\Psi \to e^{i\frac{\alpha}{2}\sigma_3} \Psi, \qquad\qquad \phi \to e^{i\alpha} \phi. \tag{43}$$

Thus we can absorb the overall phase into α and choose ϕ_0 to be real. We also assume that the coupling constant g behave as $g \sim \sqrt{\epsilon}$. Later we will show that g is, essentially, the same constant as the one we met in section 2.

There is no propagator for the boson field in $\mathcal{L}$. To be able to describe interactions in terms of a dynamic boson field, we manually add a kinetic term to $\mathcal{L}$ and then later subtract the same term. Since a boson consists of two fermions it has mass $2m$ and chemical potential 2μ. Hence the extra term will have the form

$$\varphi^* \left(i\partial_t + \frac{\nabla^2}{4m} + 2\mu \right) \varphi.$$

With this addition we can write our Lagrangian as $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2$, where

$$\begin{aligned} \mathcal{L}_0 = \Psi^\dagger \left(i\partial_t + \left(\frac{\nabla^2}{2m} + \mu \right) \sigma_3 + \phi_0(\sigma_+ + \sigma_-) \right) \Psi \\ + \varphi^* \left(i\partial_t + \frac{\nabla^2}{4m} + \mu_B \right) \varphi - \frac{\phi_0^2}{c_0}, \end{aligned} \tag{44}$$

$$\mathcal{L}_1 = g\Psi^\dagger \sigma_+ \Psi \varphi + g\Psi^\dagger \sigma_- \Psi \varphi^*, \tag{45}$$

$$\mathcal{L}_2 = -\frac{g\phi_0}{c_0}(\varphi + \varphi^*) - \varphi^* \left(i\partial_t + \frac{\nabla^2}{4m} + 2\mu \right) \varphi, \tag{46}$$

and where $\mu_B = 2\mu - \frac{g^2}{c_0}$ is the effective chemical potential of the bosons.

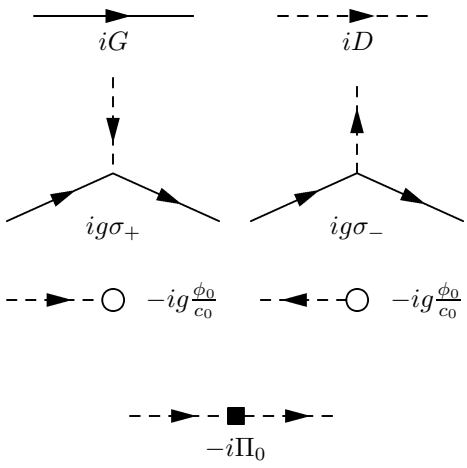


Figure 1: Feynman rules from the Lagrangian $\mathcal{L} = \mathcal{L}_0+\mathcal{L}_1+\mathcal{L}_2$. The propagators in the first row come from $\mathcal{L}_0$ while the vertices in the second row are given by $\mathcal{L}_1$ and the tadpoles and counter term of the last two rows come from $\mathcal{L}_2$. Solid lines denote fermion propagators while dashed lines denote boson propagators.

By formally treating $\mathcal{L}_1$ and $\mathcal{L}_2$ as small perturbations to $\mathcal{L}_0$ we can get a number of Feynman rules that can be used to understand our Fermi gas. These rules are described below and are summarized in figure 1.

The $\mathcal{L}_0$ term describe two types of non-interacting particles, a fermion quasi-particle and a boson. In momentum space the boson propagator is given by

$$D(p_0, \boldsymbol{p}) = \left(p_0 - \frac{\varepsilon_{\boldsymbol{p}}}{2} + \mu_B + i\delta\right)^{-1}. \tag{47}$$

We note the similarity between this expression and the boson propagator given previously in eq. (20). The fermion propagator is given by

$$G(p_0, \boldsymbol{p}) = \begin{pmatrix} p_0 - \varepsilon_{\boldsymbol{p}} + \mu & \phi_0 \\ \phi_0 & p_0 + \varepsilon_{\boldsymbol{p}} - \mu \end{pmatrix}^{-1} \tag{48}$$

$$= \frac{1}{p_0^2 - E_{\boldsymbol{p}}^2 + i\delta} \begin{pmatrix} p_0 + \varepsilon_{\boldsymbol{p}} - \mu & -\phi_0 \\ -\phi_0 & p_0 - \varepsilon_{\boldsymbol{p}} + \mu \end{pmatrix}, \tag{49}$$

where $E_{\boldsymbol{p}}$ is the energy of a fermion, i.e.

$$E_{\boldsymbol{p}} = \sqrt{(\varepsilon_{\boldsymbol{p}} - \mu)^2 + \phi_0^2}. \tag{50}$$

$\mathcal{L}_1$ gives two fermion–boson interaction vertices. These vertices are proportional to the coupling constant g which is small in the limit of small ϵ. Finally $\mathcal{L}_2$ gives two boson tadpoles and a counter term that are needed to avoid double counting of some diagrams that we get from $\mathcal{L}_0$ and $\mathcal{L}_1$. This last term cancels the manually added boson propagator in $\mathcal{L}_0$. The counter term is given by

$$-i\Pi_0(p_0, \boldsymbol{p}) = -i\left(p_0 - \frac{\varepsilon_{\boldsymbol{p}}}{2} + 2\mu\right). \tag{51}$$

The basis of the ϵ-expansion treatment of the unitary Fermi gas is the observation by Nussinov and Nussinov [28] that in four dimensions, the ground state of the gas is a gas of free bosons. We can see this by considering the wave function Φ of two bound fermions with zero binding energy. In d spatial dimensions this wave function behave as $\Phi(\boldsymbol{r}) \sim 1/r^{d-2}$. For $d \geq 4$ the normalization integral $\int d^d\boldsymbol{r}|\Phi(\boldsymbol{r})|^2$ has a singularity at $r \to 0$. Thus the bound states have zero size and cannot interact with each other.

At zero temperature such a boson gas will condense. But the chemical potential of a Bose–Einstein condensate vanishes [29]. Hence, in the unitary limit,

$$\mu_B = 2\mu - \frac{g^2}{c_0} = \mathcal{O}(\epsilon). \tag{52}$$

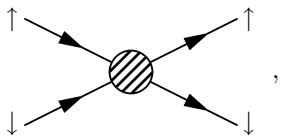
From section 2.1 we know that $-g^2/c_0 = \varepsilon_\mathrm{b}\,(1 + \mathcal{O}(\epsilon))$. Thus the above relation also can be written as

$$2\mu + \varepsilon_\mathrm{b} = \mathcal{O}(\epsilon). \tag{53}$$

Furthermore we will assume that $\mu \sim \mathcal{O}(\epsilon)$ while $\phi_0 \sim \mathcal{O}(1)$. These assumption will later be explicitly checked.

3.1 Two-particle scattering revisited

We now return to the problem of two-particle scattering in the unitary limit. In section 2 we saw that the amplitude for two-particle scattering, described by the diagram



,

where the shaded circle indicates the T-matrix, and the scattering particles carry momentum $\frac{\boldsymbol{p}}{2} \pm \boldsymbol{k}$ before scattering and $\frac{\boldsymbol{p}}{2} \pm \boldsymbol{k}'$ after scattering, is given by

$$iT(p) \times \psi_\uparrow^\dagger\left(\frac{\boldsymbol{p}}{2} + \boldsymbol{k}'\right)\psi_\downarrow^\dagger\left(\frac{\boldsymbol{p}}{2} - \boldsymbol{k}'\right)\psi_\downarrow\left(\frac{\boldsymbol{p}}{2} - \boldsymbol{k}\right)\psi_\uparrow\left(\frac{\boldsymbol{p}}{2} + \boldsymbol{k}\right). \tag{54}$$

In the unitary limit and close to four dimensions, the T-matrix is, as seen in section 2, given by

$$iT = -\frac{8\pi^2\epsilon}{m^2}\frac{i}{p_0 - \frac{\varepsilon_{\boldsymbol{p}}}{2} + i\delta} + \mathcal{O}(\epsilon^2). \tag{18}$$

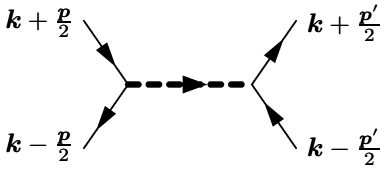
In terms of the Nambu-Gorkov fields Ψ and $\Psi^\dagger$, we can write

$$\begin{aligned}\Psi^\dagger\left(\boldsymbol{k}' + \frac{\boldsymbol{p}}{2}\right)\sigma_+\Psi\left(\boldsymbol{k}' - \frac{\boldsymbol{p}}{2}\right) &\\ = \begin{pmatrix}\psi_\uparrow^\dagger\left(\frac{\boldsymbol{p}}{2} + \boldsymbol{k}'\right) & \psi_\downarrow\left(\frac{\boldsymbol{p}}{2} - \boldsymbol{k}'\right)\end{pmatrix}\begin{pmatrix}\psi_\downarrow^\dagger\left(\frac{\boldsymbol{p}}{2} - \boldsymbol{k}'\right)\\ 0\end{pmatrix}&\\ = \psi_\uparrow^\dagger\left(\frac{\boldsymbol{p}}{2} + \boldsymbol{k}'\right)\psi_\downarrow^\dagger\left(\frac{\boldsymbol{p}}{2} - \boldsymbol{k}'\right)&\end{aligned}$$

and

$$\Psi^\dagger\left(\boldsymbol{k}-\frac{\boldsymbol{p}}{2}\right)\sigma_-\Psi\left(\boldsymbol{k}+\frac{\boldsymbol{p}}{2}\right)$$
$$=\begin{pmatrix}\psi_\uparrow^\dagger\left(\boldsymbol{k}-\frac{\boldsymbol{p}}{2}\right) & \psi_\downarrow\left(\frac{\boldsymbol{p}}{2}-\boldsymbol{k}\right)\end{pmatrix}\begin{pmatrix}0\\ \psi_\uparrow\left(\frac{\boldsymbol{p}}{2}+\boldsymbol{k}\right)\end{pmatrix}$$
$$=\psi_\downarrow\left(\frac{\boldsymbol{p}}{2}-\boldsymbol{k}\right)\psi_\uparrow\left(\frac{\boldsymbol{p}}{2}+\boldsymbol{k}\right).$$

Thus the corresponding two-particle scattering diagram is given by



where the thick dashed lines indicates the *exact* boson Green's function which we will denote by $\Delta(p)$. From this diagram we then get the scattering amplitude

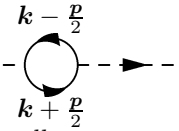
$$-ig^2\Delta(p)\times\Psi^\dagger\left(\boldsymbol{k}'+\frac{\boldsymbol{p}}{2}\right)\sigma_+\Psi\left(\boldsymbol{k}'-\frac{\boldsymbol{p}}{2}\right)\times\Psi^\dagger\left(\boldsymbol{k}-\frac{\boldsymbol{p}}{2}\right)\sigma_-\Psi\left(\boldsymbol{k}+\frac{\boldsymbol{p}}{2}\right)$$
$$=-ig^2\Delta(p)\times\psi_\uparrow^\dagger\left(\frac{\boldsymbol{p}}{2}+\boldsymbol{k}'\right)\psi_\downarrow^\dagger\left(\frac{\boldsymbol{p}}{2}-\boldsymbol{k}'\right)\psi_\downarrow\left(\frac{\boldsymbol{p}}{2}-\boldsymbol{k}\right)\psi_\uparrow\left(\frac{\boldsymbol{p}}{2}+\boldsymbol{k}\right). \quad (55)$$

3.2 The boson propagator

To be able to compare the two-particle scattering amplitudes of eq. (54) and eq. (55), we need to calculate the exact boson propagator $\Delta(p)$. To the lowest order in ϵ this consists of the plain boson propagator $D(p)$ together with the counter term from $\mathcal{L}_2$ and a boson self energy diagram. We start by calculating this diagram. To make some quite complex expressions a little more readable we introduce the following short hand notation:

$$\varepsilon_\pm=\varepsilon_{\boldsymbol{k}\pm\frac{\boldsymbol{p}}{2}}, \quad (56)$$
$$e_\pm=\sqrt{\varepsilon^2_{\boldsymbol{k}\pm\frac{\boldsymbol{p}}{2}}+\phi_0^2}, \quad (57)$$
$$E\pm=\sqrt{(\varepsilon_{\boldsymbol{k}\pm\frac{\boldsymbol{p}}{2}}-\mu)^2+\phi_0^2}. \quad (58)$$

The simplest boson self energy diagram that contributes is then given by

$-i\Pi_1=$ 

(59)

$$=-g^2\int\frac{dk}{(2\pi)^{d+1}}\,\mathrm{Tr}\left[\sigma_-G\left(k+\frac{p}{2}\right)\sigma_+G\left(k-\frac{p}{2}\right)\right] \quad (60)$$
$$=-g^2\int\frac{dk}{(2\pi)^{d+1}}\,\mathrm{Tr}\left[G_{11}\left(k+\frac{p}{2}\right)G_{22}\left(k-\frac{p}{2}\right)\right]. \quad (61)$$

On first sight this diagram seems to give an $\mathcal{O}(\epsilon)$ contribution to $\Delta(p)$, in which case it could be ignored for now, since we are only interested in calculating $\Delta(p)$

to the lowest order. However the above integral has an ultraviolet divergence, and when the integration is performed using dimensional regularization this gives an extra $\frac{1}{\epsilon}$ contribution to Π_1. Hence Π_1 is, in effect, of order $\mathcal{O}(1)$ and we need to take it's contribution into account. But, since this $\mathcal{O}(1)$ behavior is an effect of the ultraviolet behavior of the diagram, we only need to consider the large $\boldsymbol{p}$ limit.

Before taking this limit we perform the k_0 integral by closing the contour in the upper half of the complex plane.

$$-i\Pi_1 = -g^2 \int \frac{dk}{(2\pi)^{d+1}} \frac{k_0 + \frac{p_0}{2} + \varepsilon_+ - \mu}{\left(k_0 + \frac{p_0}{2}\right)^2 - E_+^2 + i\delta} \frac{k_0 - \frac{p_0}{2} - \varepsilon_- + \mu}{\left(k_0 - \frac{p_0}{2}\right)^2 - E_-^2 + i\delta} \tag{62}$$

$$\begin{aligned} &= ig^2 \int \frac{d\boldsymbol{k}}{(2\pi)^d} \left[\frac{1}{2E_+} \frac{E_+ - \varepsilon_+ + \mu}{p_0 - E_- + E_+} \frac{p_0 + E_+ + \varepsilon_- - \mu}{p_0 + E_- + E_+} \right. \\ &\quad \left. + \frac{1}{2E_-} \frac{E_- + \varepsilon_- - \mu}{p_0 + E_+ - E_-} \frac{p_0 - E_- + \varepsilon_+ - \mu}{p_0 - E_+ - E_-} \right]. \end{aligned} \tag{63}$$

We will later see that, in the unitary limit, μ is small compared to $\varepsilon_{\boldsymbol{k}\pm\frac{\boldsymbol{p}}{2}}$ and ϕ_0. Using this assumption we expand the above integral up to first order in μ, keeping in mind the μ dependence of $E_\pm$. The first term of $-i\Pi_1$ then gives

$$\begin{aligned} &\frac{1}{2e_+} \left(1 + \frac{\varepsilon_+}{e_+^2}\mu\right) \frac{e_+ - \varepsilon_+ + \left(1 - \frac{\varepsilon_+}{e_+}\right)\mu}{p_0 - e_- + e_+} \frac{p_0 + e_+ + \varepsilon_- - \left(1 + \frac{\varepsilon_+}{e_+}\right)\mu}{p_0 + e_+ + e_-} \\ &\times \left(1 + \left(\frac{\varepsilon_+}{e_+} - \frac{\varepsilon_-}{e_-}\right) \frac{\mu}{p_0 + e_+ - e_-}\right) \left(1 + \left(\frac{\varepsilon_+}{e_+} + \frac{\varepsilon_-}{e_-}\right) \frac{\mu}{p_0 + e_+ + e_-}\right), \end{aligned} \tag{64}$$

while the second term gives

$$\begin{aligned} &\frac{1}{2e_-} \left(1 + \frac{\varepsilon_-}{e_-^2}\mu\right) \frac{e_- + \varepsilon_- - \left(1 + \frac{\varepsilon_-}{e_-}\right)\mu}{p_0 - e_- + e_+} \frac{p_0 - e_- + \varepsilon_+ - \left(1 - \frac{\varepsilon_-}{e_-}\right)\mu}{p_0 - e_+ - e_-} \\ &\times \left(1 + \left(\frac{\varepsilon_+}{e_+} - \frac{\varepsilon_-}{e_-}\right) \frac{\mu}{p_0 + e_+ - e_-}\right) \left(1 - \left(\frac{\varepsilon_+}{e_+} + \frac{\varepsilon_-}{e_-}\right) \frac{\mu}{p_0 - e_+ - e_-}\right). \end{aligned} \tag{65}$$

As previously argued, we are mainly interested in the ultraviolet behavior of these terms, i.e. of the limit $e_\pm \to \varepsilon_\pm$. If we plug this into the above expressions we see that the first term vanishes while the second one gives

$$\begin{aligned} &\frac{1}{2\varepsilon_-} \left(1 + \frac{\mu}{\varepsilon_-}\right) \frac{2\varepsilon_- - 2\mu}{p_0 - \varepsilon_+ - \varepsilon_-} \left(1 - \frac{2\mu}{p_0 - \varepsilon_+ - \varepsilon_-}\right) \\ &\qquad = \frac{1}{p_0 - \varepsilon_+ - \varepsilon_-} - \frac{2\mu}{(p_0 - \varepsilon_+ - \varepsilon_-)^2}. \\ &\qquad = \frac{1}{p_0 - 2\varepsilon_{\boldsymbol{k}} + \frac{\varepsilon_{\boldsymbol{p}}}{2}} - \frac{2\mu}{\left(p_0 - 2\varepsilon_{\boldsymbol{k}} + \frac{\varepsilon_{\boldsymbol{p}}}{2}\right)^2}. \end{aligned} \tag{66}$$

Hence the ultraviolet behavior of $-i\Pi_1$ is given by

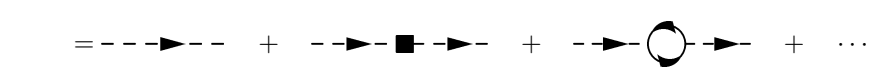
$$-i\Pi_1 \to ig^2 \int \frac{d\boldsymbol{k}}{(2\pi)^d} \frac{1}{p_0 - 2\varepsilon_{\boldsymbol{k}} - \frac{\varepsilon_{\boldsymbol{p}}}{2}} - ig^2 \int \frac{d\boldsymbol{k}}{(2\pi)^d} \frac{2\mu}{\left(p_0 - 2\varepsilon_{\boldsymbol{k}} - \frac{\varepsilon_{\boldsymbol{p}}}{2}\right)^2}. \tag{67}$$

The first of these integrals was calculated previously (see eq. (10)), and the second integral can be calculated using exactly the same method, again using some formulas from appendix A. The result for $d = 4 - \epsilon$ is

$$-i\Pi_1 = ig^2 \frac{m^2}{8\pi^2\epsilon} \left(p_0 - \frac{\varepsilon_{\boldsymbol{p}}}{2} + 2\mu\right) (1 + \mathcal{O}(\epsilon)) \,. \tag{68}$$

We can now write down the exact boson propagator $i\Delta(p)$ in the unitary limit to the leading order. As noted above it consists of the pure propagator $iD(p)$, the counter term $-i\Pi_0$ and the boson self energy amplitude $-i\Pi_1$. Together these give

$i\Delta(p) =$  (69)

$=$  (70)

$$= iD(p) - iD(p) \left(i\Pi_0 + i\Pi_1\right) iD(p) + \mathcal{O}(\epsilon) \tag{71}$$

$$= iD(p) + \left(g^2 \frac{m^2}{8\pi^2\epsilon} - 1\right) iD(p) \left(p_0 - \frac{\varepsilon_{\boldsymbol{p}}}{2} + 2\mu\right) D(p) + \mathcal{O}(\epsilon). \tag{72}$$

We also need to expand $D(p)$ to $\mathcal{O}(\epsilon)$.

$$D(p) = \left(p_0 - \frac{\varepsilon_{\boldsymbol{p}}}{2} + \mu_B + i\delta\right)^{-1} \tag{73}$$

$$\approx \frac{1}{p_0 - \frac{\varepsilon_{\boldsymbol{p}}}{2} - i\delta} - \frac{\mu_B}{\left(p_0 - \frac{\varepsilon_{\boldsymbol{p}}}{2} + i\delta\right)^2} + \mathcal{O}(\epsilon^2), \tag{74}$$

where we have used the assumptions that μ_B is of order ϵ.

At last we can compare our two expressions for the two-particle scattering amplitude, eq. (54) and eq. (55). For the $\mathcal{O}(1)$ terms to be equal the renormalized coupling constant g should be given by

$$g = \frac{\sqrt{8\pi^2\epsilon}}{m}. \tag{75}$$

With this value for g the $\mathcal{O}(1)$ contribution from Π_1 is exactly cancelled by the Π_0 counter term. Hence the exact propagator is given by

$$i\Delta(p) = iD(p) + \mathcal{O}(\epsilon). \tag{76}$$

We also note that this expression for g is the same as the one anticipated in section 2.

There is one final complication to the value of g — we need to consider it's physical dimension. Let M, L and T indicate dimensions of mass, length and time respectively. We use units with $\hbar = 1$, and hence $ML^2 = T$. In such units a non-relativistic field such as Ψ or φ should have dimensions $L^{-d/2}$, while the Lagrangian $\mathcal{L}$ should have dimensions $L^{-d}T^{-1}$. From $\mathcal{L}$ we then see that the dimensions of g then should be $L^{d/2-2}M^{-1}$. Thus eq. (75) gives the right

dimensions for d exactly equal to four. For other values of d we need to add an extra factor to get the units right. With this extra factor, g is given by

$$g = \frac{\sqrt{8\pi^2\epsilon}}{m}\left(\frac{m\phi_0}{2\pi}\right)^{\epsilon/4}, \tag{77}$$

where a factor of $(2\pi)^{-\epsilon/4}$ has been introduced for later convenience. Clearly this does not change the fact that the $\mathcal{O}(1)$ contributions to the boson self energy vanishes in the unitary limit.

3.3 Power counting

We have seen above that, as anticipated, the renormalized coupling constant g is proportional to $\sqrt{\epsilon}$ and hence small when we consider a system in close to four spatial dimensions. We now want to do a systematic expansion in ϵ in terms of Feynman diagrams. To do this we need to be able to calculate the proportionality factor of each diagram in terms of powers of ϵ. A given diagram will consist of a number of fermion–boson vertices, which each will bring a factor of $g \sim \sqrt{\epsilon}$. Hence a naïve expectation would be that a diagram would behave as $\epsilon^{N_g/2}$, where N_g is the number of fermion–boson vertices of the diagram. We may also use the assumptions that μ and μ_B are of order $\mathcal{O}(\epsilon)$ to further simplify the resulting expressions by expanding them to the appropriate order in ϵ.

There will however be exceptions to this simple rule for some diagrams that contain integrals that have ultraviolet divergences at $d = 4$. Such a divergence will, at $d = 4 - \epsilon$ give rise to inverse powers of ϵ, thereby lowering the ϵ-proportionality of the diagram compared to the naïve expectation.

In four dimensions, any loop integral will, in the ultraviolet region, behave as

$$\int dp \sim \int d\boldsymbol{p}\ \varepsilon_{\boldsymbol{p}} \sim p^6, \tag{78}$$

while each fermion or boson propagator behaves as $G(p) \sim p^{-2}$ or $D(p) \sim p^{-2}$. Thus a diagram containing L loops, P_F fermion propagators and P_B boson propagators, can be expected to diverge as p^D, where D is called the superficial degree of divergence, and is given by

$$D = 6L - 2P_F - 2P_B. \tag{79}$$

The number of loop integral in a diagram is

$$L = (P_F + P_B) - (N_g - 1), \tag{80}$$

since there is a momentum integral for each propagator and a delta function for each vertex, and since one of the delta functions is used for momentum conservation. If a diagram has E_F external fermions and E_B external bosons, then the number of vertices will be

$$N_g = \frac{2P_F + E_F}{2} = 2P_B + E_B, \tag{81}$$

since each vertex involves one boson and two fermions and each propagator has to be connected to two vertices. Together these relations give

$$D = 6 - 2(E_F + E_B). \tag{82}$$

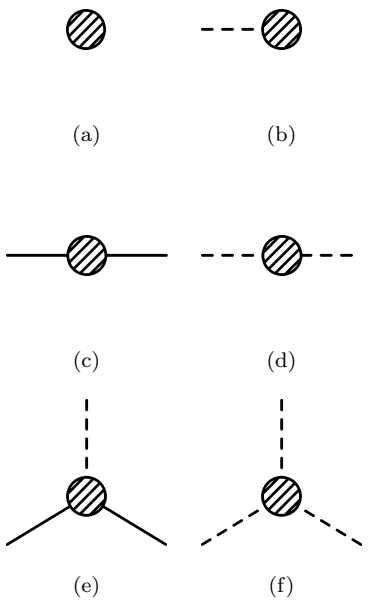


Figure 2: The six different types of diagram where an ultraviolet divergence may occur according to eq. (82).

Hence only diagrams with no more than three external lines may contain inverse powers of ϵ coming from ultraviolet divergence. There are only six different kinds of diagrams that potentially contain ultraviolet divergences. These are illustrated in figure 2. Note that the conservation of the number of fermions means that any diagram with an odd number of external fermions vanishes.

By now looking in more detail on the analytical properties of the two propagators, we will be able to reduce the number of diagrams that have an ultraviolet divergence even further. We start by splitting the propagators into advanced and retarded parts, $G(p) = G^A(p) + G^R(p)$, where $G^A(p)$ has poles only in the upper half of the complex plane, and $G^R(p)$ only in the lower half plane. We will then get

$$G^R(p_0, \boldsymbol{p}) = \frac{1}{2E_{\boldsymbol{p}}} \frac{1}{p_0 - E_{\boldsymbol{p}} + i\delta} \begin{pmatrix} p_0 + \varepsilon_{\boldsymbol{p}} - \mu & -\phi_0 \\ -\phi_0 & p_0 - \varepsilon_{\boldsymbol{p}} + \mu \end{pmatrix}, \tag{83}$$

$$G^A(p_0, \boldsymbol{p}) = -\frac{1}{2E_{\boldsymbol{p}}} \frac{1}{p_0 + E_{\boldsymbol{p}} - i\delta} \begin{pmatrix} p_0 + \varepsilon_{\boldsymbol{p}} - \mu & -\phi_0 \\ -\phi_0 & p_0 - \varepsilon_{\boldsymbol{p}} + \mu \end{pmatrix}, \tag{84}$$

$$D^R(p_0, \boldsymbol{p}) = \frac{1}{p_0 - \frac{\varepsilon_{\boldsymbol{p}}}{2} - \mu_B + i\delta}, \tag{85}$$

$$D^A(p_0, \boldsymbol{p}) = 0. \tag{86}$$

We want to know how these propagators behave in the ultraviolet limit. To get the behavior of G_{11} and G_{22} we examine the residues at the different poles

in p_0 in the limit $\varepsilon_{\boldsymbol{p}} \gg \phi_0, \mu$. For G^R these residues are

$$\operatorname{Res} G^R_{11} = \frac{E_{\boldsymbol{p}} + \varepsilon_{\boldsymbol{p}}}{2E_{\boldsymbol{p}}} \approx 1 \tag{87}$$

$$\operatorname{Res} G^R_{22} = \frac{E_{\boldsymbol{p}} - \varepsilon_{\boldsymbol{p}}}{2E_{\boldsymbol{p}}} = \frac{\sqrt{(\varepsilon_{\boldsymbol{p}} - \mu)^2 + \phi_0^2} - \varepsilon_{\boldsymbol{p}}}{2E_{\boldsymbol{p}}} \tag{88}$$

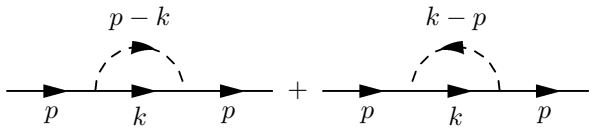
$$\approx \frac{\varepsilon_{\boldsymbol{p}} - \mu + \frac{1}{2}\frac{\mu^2 + \phi_0^2}{\varepsilon_{\boldsymbol{p}}} - \varepsilon_{\boldsymbol{p}}}{2\varepsilon_{\boldsymbol{p}}} \approx \frac{1}{4}\frac{\phi_0^2}{\varepsilon_{\boldsymbol{p}}^2}, \tag{89}$$

with the ultraviolet behavior $\operatorname{Res} G^R_{11} \sim \mathcal{O}(1)$ and $\operatorname{Res} G^R_{22} \sim p^{-4}$. Close to the pole, these residues will be multiplied by a factor $p_0^{-1} \sim p^{-2}$. G^A can be treated in an exactly equivalent way. Now we can write down the behavior of all the propagators in the ultraviolet limit.

$$\begin{aligned} &G^R_{11} \sim G^A_{22} \sim D^R \sim p^{-2} \\ &G^R_{22} \sim G^A_{11} \sim p^{-6} \\ &G_{12} \sim G_{21} \sim p^{-4} \end{aligned} \tag{90}$$

To find if there any divergent diagrams for which we need to adjust our power counting rules, we will consider skeleton diagrams of the different types in figure 2 and look for divergences.

Fermion self energy. The first diagram we look at is the fermion self energy, figure 2(c). The one-loop order corrections are given by

$\Sigma =$ 

(91)

$$= g^2 \int \frac{dk}{(2\pi)^{d+1}} \left(\sigma_+ G(k) \sigma_- D(p-k) + \sigma_- G(k) \sigma_+ D(k-p)\right) \tag{92}$$

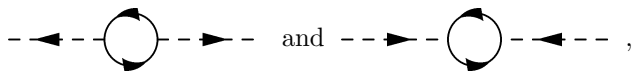
The non-zero components of this expression are

$$\Sigma_{11} = g^2 \int \frac{dk}{(2\pi)^{d+1}} G_{22}(k) D(p-k) \tag{93}$$

$$\Sigma_{22} = g^2 \int \frac{dk}{(2\pi)^{d+1}} G_{11}(k) D(k-p). \tag{94}$$

The first k_0-integral can be performed by closing the contour in the upper half plane and the second one by closing it in the lower half plane. Then both integrands will behave as k^{-8} at large k and there will be no ultraviolet divergences.

Boson self energy. There are a number of one-loop diagrams that contribute to the boson self energy, figure 2(d). First we have the diagrams



,

which both give the same kind of integrand,

$$\mathrm{Tr}\,\sigma_- G\left(k+\frac{p}{2}\right)\sigma_- G\left(k-\frac{p}{2}\right) \;=\; G_{12}\left(k+\frac{p}{2}\right)G_{12}\left(k-\frac{p}{2}\right) \;\sim\; k^{-8}. \quad (95)$$

Hence neither of these diagrams possess any ultraviolet divergence.

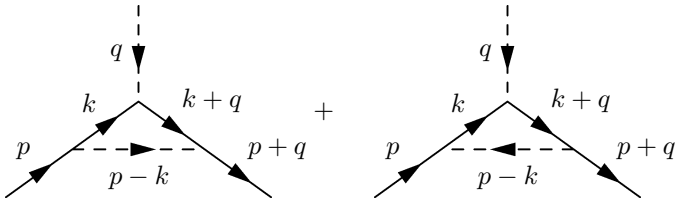
We also have the diagram

which was already considered in section 3.1. There we saw that this diagram contains an ultraviolet divergence. This divergence makes the contribution to the boson self energy of order $\mathcal{O}(1)$ instead of the expected $\mathcal{O}(\epsilon)$. In the same section we also saw that this divergence was cancelled exactly by the counter term from $\mathcal{L}_2$. Taking this into account, the behavior of this diagram is, as expected,

$$= \mathcal{O}(\epsilon). \quad (96)$$

Fermion–boson vertices. Now let us look at the fermion–boson vertex of figure 2(e). In the case of in-going boson the one-loop diagrams contributing are

$\Gamma =$



(97)

$$= -g^2 \int \frac{dk}{(2\pi)^{d+1}} [\sigma_+ G(k+q)\sigma_+ G(k)\sigma_- D(p-k) + \sigma_- G(k+q)\sigma_+ G(k)\sigma_+ D(k-p)]. \quad (98)$$

Again we get two non-vanishing components,

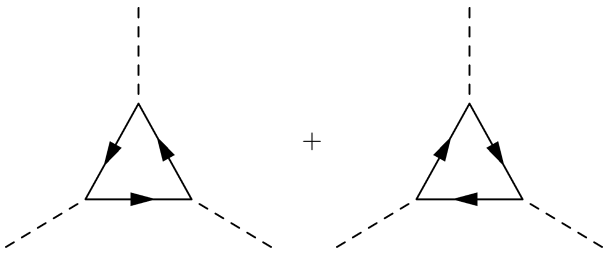
$$\Gamma_{11} = -g^2 \int \frac{dk}{(2\pi)^{d+1}} G_{21}(k+q)G_{22}(k)D(p-k) \quad (99)$$

$$\Gamma_{22} = -g^2 \int \frac{dk}{(2\pi)^{d+1}} G_{11}(k+q)G_{21}(k)D(k-p). \quad (100)$$

Using the large k behavior from eqs. (90) we see that neither of these terms will diverge in the ultraviolet. The two diagram which we get by reversing the direction of the external boson in eq. (97) will result in very similar expressions and will not give any divergences.

Three boson vertices. There is a number of different three boson vertices of the type of figure 2(f) which differ in the directions of the external bosons.

In one-loop order such a diagram will give



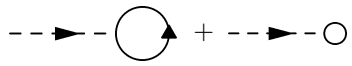
(101)

These two diagrams will give an integrand of the form

$$\mathrm{Tr}[\sigma_\pm G(p_1)\sigma_\pm G(p_2)\sigma_\pm G(p_3) + \sigma_\pm G(-p_1)\sigma_\pm G(-p_2)\sigma_\pm G(-p_3)]. \tag{102}$$

By considering the ultraviolet behavior of expressions of the forms $\sigma_\pm G(p)\sigma_\pm$ and $G(p)\sigma_\pm G(k)$ one easily sees that non of these terms can contain any divergence.

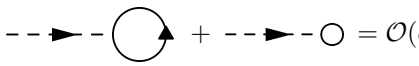
Boson tadpole To the leading order the boson tadpole, figure 2(b), is given by the diagrams



where the second term is one of the boson tadpoles of $\mathcal{L}_2$, giving a contribution of $-i\frac{g\phi_0}{c_0}$. The first diagram is given by the integral

$$g\int \frac{dk}{(2\pi)^{d+1}}\,\mathrm{Tr}\,[\sigma_+ G(k)] = g\int \frac{dk}{(2\pi)^{d+1}} G_{21}(k). \tag{103}$$

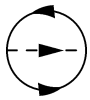
The total boson tadpole should vanish in all orders of ϵ, since the vacuum expectation value of the original boson field ϕ is, by definition, ϕ_0. This requirement will lead us to an equation for ϕ_0. In the next section we will derive a similar equation by instead demanding that ϕ_0 minimizes the effective. We can then check that if ϕ_0 is a solution to the gap equation corresponding to first order in ϵ, the boson tadpole do vanish to the same order. This is explicitly done in section 4.3. We thus get



(104)

This agrees with our expectations.

Vacuum bubble. The only remaining type of diagram is the vacuum bubble, figure 2(a). Since the advanced boson propagator vanishes, so does any boson vacuum bubble. The simplest fermion bubble is given by the diagram



which is finite. We postpone explicit calculation of it's value to the next section.

In this section we have shown that there only are two kinds of diagrams that do not behave according to our naïve expectation due to ultraviolet divergences. We also saw that these divergences can be cancelled by the use of the counter term and tadpoles from the $\mathcal{L}_2$ term of the Lagrangian. Thus, as long as we remember to add the counter terms whenever we encounter one of these diagrams, we can perform a perturbative expansion by counting the ϵ-factors from the coupling constant g. In the unitary limit, where, as previously mentioned, $\mu, \mu_B \sim \mathcal{O}(\epsilon)$, we also need to expand in these quantities to really get an expression which contain terms only up to a given order in ϵ.

4 Thermodynamic relations

Now that we know the Feynman rules and the rules for counting powers of ϵ in the unitary Fermi gas, we will use them to calculate some thermodynamic quantities that describe the physics of the gas. We begin by looking at the effective potential.

4.1 The effective potential

Consider the partition function, which is given by

$$Z[\phi_0] = \int D\Psi D\Psi^\dagger D\varphi D\varphi^* \exp\left[i \int dx \mathcal{L}\right]. \tag{105}$$

This can be calculated by writing

$$Z[\phi_0] = \int D\Psi D\Psi^\dagger D\varphi D\varphi^* \exp\left[i \int dx (\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2)\right] \tag{106}$$

$$\begin{aligned} &\approx \int D\Psi D\Psi^\dagger D\varphi D\varphi^* \exp\left[i \int dx \mathcal{L}_0\right] \\ &\quad \times \left(1 + i \int dx (\mathcal{L}_1 + \mathcal{L}_2) + \cdots\right) \end{aligned} \tag{107}$$

The constant part of $\mathcal{L}_0$ can be taken out of the integral. Each integration over the two terms quadratic in Ψ and φ respectively will give a functional determinant factor. The sum over terms involving interactions can then be represented by a sum over Feynman diagrams. Since the above expression does not involve any external fields, each diagram in this sum will be a connected vacuum diagram. Hence

$$\begin{aligned} Z[\phi_0] &= \exp\left(i \int dx \mathcal{L}_0[\phi_0]\right) \det(\tilde{G}^{-1})[\det(\tilde{D}^{-1})]^{-1} \\ &\quad \times \left(1 + \int dx \sum \text{conn. diag.} + \cdots\right) \end{aligned} \tag{108}$$

$$\begin{aligned} &\approx \exp\left(i \int dx \mathcal{L}_0[\phi_0]\right) \det(\tilde{G}^{-1})[\det(\tilde{D}^{-1})]^{-1} \\ &\quad \times \exp\left(\int dx \sum \text{conn. diag.}\right), \end{aligned} \tag{109}$$

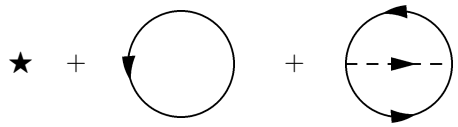


Figure 3: Vacuum diagrams that contribute to the effective potential up to order $\mathcal{O}(\epsilon)$. ★ denotes the constant contribution $\frac{\phi_0^2}{c_0}$. The one-loop diagram is of order $\mathcal{O}(1)$ while the two-loop diagram is of order $\mathcal{O}(\epsilon)$.

where $\tilde{G}^{-1}$ and $\tilde{D}^{-1}$ represents the coefficients of the quadratic terms of $\mathcal{L}_0$. Now we can define the effective action $\Gamma[\phi_0]$ satisfying

$$Z[\phi_0] = e^{i\Gamma[\phi_0]}. \tag{110}$$

The effective action is then given by

$$\Gamma[\phi_0] = -i \log Z[\phi_0] \tag{111}$$

$$\begin{aligned} &= -\frac{\phi_0^2}{c_0} \int dx - i \log\det(\tilde{G}^{-1}) + i \log\det(\tilde{D}^{-1}) \\ &\quad - i \int dx \sum \text{conn. diag.} \end{aligned} \tag{112}$$

All of the terms in our expression for Γ contain a volume factor (VT), from the integration over all of space-time. We can now finally define the effective potential V_{eff} by

$$\Gamma[\phi_0] = -(VT) V_{\text{eff}}(\phi_0). \tag{113}$$

$$\begin{aligned} V_{\text{eff}} = \frac{\phi_0^2}{c_0} &+ i \int \frac{dp}{(2\pi)^{d+1}} \log\det G^{-1}(p) \\ &- i \int \frac{dp}{(2\pi)^{d+1}} \log\det D^{-1}(p) + i \sum \text{conn. diag.} \end{aligned} \tag{114}$$

When we have calculated V_{eff} we can use it to calculate ϕ_0. We have assumed ϕ_0 to be the vacuum expectation value of the boson field ϕ. Thus the effective potential $V_{\text{eff}}(\phi_0) = V_0(\phi_0) + V_1(\phi_0) + V_2(\phi_0) + \mathcal{O}(\epsilon^2)$ will have a minimum at some $\phi_0 = \bar{\phi}_0$, i.e.

$$\left.\frac{\partial V_{\text{eff}}}{\partial \phi_0}\right|_{\phi_0 = \bar{\phi}_0} = 0. \tag{115}$$

The effective potential also gives the fermion particle number n:

$$n = -\left.\frac{\partial V_{\text{eff}}}{d\mu}\right|_{\phi_0 = \bar{\phi}_0}. \tag{116}$$

In this thesis we will calculate the effective potential up to next-to-leading order. The diagrams that contributes to this order are shown in figure 3. As already seen, the first term is given by

$$V_0(\phi_0) = \frac{\phi_0^2}{c_0}. \tag{117}$$

The one-loop diagram gives the determinant term

$$V_1(\phi_0) = i\int \frac{dp}{(2\pi)^{d+1}} \log\det G^{-1}(p) - i\int \frac{dp}{(2\pi)^{d+1}} \log\det D^{-1}(p). \quad (118)$$

The p_0 in the first integral can be calculated by considering

$$\frac{\partial}{\partial E_{\boldsymbol{p}}} \int \frac{dp_0}{2\pi} \log(p_0^2 - E_{\boldsymbol{p}}^2 + i\delta) = \int \frac{dp_0}{2\pi} \frac{2E_{\boldsymbol{p}}}{p_0^2 - E_{\boldsymbol{p}}^2 + i\delta} = i. \quad (119)$$

An integration over $E_{\boldsymbol{p}}$ gives

$$V_1(\phi_0) = -\int \frac{d\boldsymbol{p}}{(2\pi)^d} E_{\boldsymbol{p}}, \quad (120)$$

where an infinite term independent of $E_{\boldsymbol{p}}$, and hence of ϕ_0, has been ignored.

Using the same procedure to calculate the second integral in eq. (118) gives

$$\frac{\partial}{\partial\left(\frac{\varepsilon_{\boldsymbol{p}}}{2} + \mu_B\right)} \int \frac{dp_0}{2\pi} \log\left(p_0 - \frac{\varepsilon_{\boldsymbol{p}}}{2} - \mu_B + i\delta\right) = \int \frac{dp_0}{2\pi} \frac{1}{p_0 - \frac{\varepsilon_{\boldsymbol{p}}}{2} - \mu_B + i\delta} = 0, \quad (121)$$

since the contour integral can be integrated in the upper half plane.

Since $\mu \sim \epsilon$ and ϕ_0 is $\mathcal{O}(1)$ we expand the integrand of V_1 in μ:

$$E_{\boldsymbol{p}} = \sqrt{(\varepsilon_{\boldsymbol{p}} - \mu)^2 + \phi_0^2} \approx \sqrt{\varepsilon_{\boldsymbol{p}}^2 + \phi_0^2} - \mu \frac{\varepsilon_{\boldsymbol{p}}}{\sqrt{\varepsilon_{\boldsymbol{p}}^2 + \phi_0^2}} + \mu^2 \frac{1}{\sqrt{\varepsilon_{\boldsymbol{p}}^2 + \phi_0^2}} + \cdots. \quad (122)$$

The first term will give a $\mathcal{O}(1)$ contribution to V_{eff}. The integral over the second term has an ultraviolet divergence, hence it will be of order $\mathcal{O}(1)$ instead of $\mathcal{O}(\epsilon)$. The integral over the third term is finite and will thus give an $\mathcal{O}(\epsilon^2)$ contribution which can be neglected.

Using the same method as previously, first changing variables of integration to spherical coordinates and then to $z = (\varepsilon_{\boldsymbol{p}}/\phi_0)^2$, these integrals can be performed. The result is

$$V_1(\phi_0) = -\frac{1}{2\Gamma\left(\frac{d}{2}\right)} \left(\frac{m\phi_0}{2\pi}\right)^{\frac{d}{2}} \times \left(\frac{\Gamma\left(\frac{d}{4}\right)\Gamma\left(-\frac{1}{2} - \frac{d}{4}\right)}{\Gamma\left(-\frac{1}{2}\right)} \phi_0 - \frac{\Gamma\left(\frac{d}{4} + \frac{1}{2}\right)\Gamma\left(-\frac{d}{4}\right)}{\Gamma\left(\frac{1}{2}\right)} \mu\right). \quad (123)$$

This should be evaluated at $d = 4 - \epsilon$. We want to calculate V_{eff} to next-to-leading order, hence we have to evaluate the above expression to $\mathcal{O}(\epsilon)$. To do this we need to expand the Γ factors. There is a singularity in the second term, making it $\mathcal{O}(1)$ instead of $\mathcal{O}(\epsilon)$. This singular term can be expanded as previously using eq. (A4). The convergent Γ functions can be Taylor expanded using the digamma function $\psi(x) = \frac{d}{dx}\log\Gamma(x)$, see appendix A. The result

will be

$$V_1(\phi_0) = \frac{\phi_0}{3}\left(1 + \frac{7 - 3(\gamma + \log 2)}{6}\epsilon\right)\left(\frac{m\phi_0}{2\pi}\right)^{\frac{d}{2}} - \frac{\mu}{\epsilon}\left(1 + \frac{1 - 2(\gamma - \log 2)}{4}\epsilon\right)\left(\frac{m\phi_0}{2\pi}\right)^{\frac{d}{2}} + \mathcal{O}(\epsilon^2). \quad (124)$$

The two-loop diagram from figure 3 gives the contribution

$$V_2(\phi_0) = g^2 \int \frac{dp\ dq}{(2\pi)^{2d+2}} \operatorname{Tr}\left[\sigma_- G(p)\sigma_+ G(q)\right] D(p-q) \quad (125)$$

$$= g^2 \int \frac{dp\ dq}{(2\pi)^{2d+2}} G_{11}(p) G_{22}(q) D(p-q) \quad (126)$$

$$= g^2 \int \frac{dp\ dq}{(2\pi)^{2d+2}} \frac{p_0 + \varepsilon_{\boldsymbol{p}} - \mu}{p_0^2 - E_{\boldsymbol{p}}^2 + i\delta} \frac{q_0 - \varepsilon_{\boldsymbol{q}} + \mu}{q_0^2 - E_{\boldsymbol{q}}^2 + i\delta} \times \frac{1}{p_0 - q_0 - \frac{\varepsilon_{\boldsymbol{p}-\boldsymbol{q}}}{2} + \mu_B + i\delta}. \quad (127)$$

We can perform the p_0 and q_0 integrals by closing the first contour in the upper half plane and the second one in the lower half plane. The remaining integral is

$$V_2(\phi_0) = -\frac{g^2}{4} \int \frac{d\boldsymbol{p}\ d\boldsymbol{q}}{(2\pi)^{2d}} \frac{\left(E_{\boldsymbol{p}} - \varepsilon_{\boldsymbol{p}} + \mu\right)\left(E_{\boldsymbol{q}} - \varepsilon_{\boldsymbol{q}} + \mu\right)}{E_{\boldsymbol{p}} E_{\boldsymbol{q}} \left(E_{\boldsymbol{p}} + E_{\boldsymbol{q}} + \frac{\varepsilon_{\boldsymbol{p}-\boldsymbol{q}}}{2} + \mu_B\right)}. \quad (128)$$

This integral converges in four dimensions. Together with the factor of g^2 in front of it, this means that the contribution from V_2 to V_{eff} will be at least of order ϵ. Hence we only need to consider the constant contribution from the integral and thus can calculate it at $d = 4$. Since $\mu \sim \epsilon$ we can also ignore any μ in V_2. If we introduce

$$e_{\boldsymbol{p}} = \sqrt{\varepsilon_{\boldsymbol{p}}^2 + \phi_0^2}, \quad (129)$$

we can write V_2 as

$$V_2(\phi_0) = -\frac{g^2}{4} \int \frac{d\boldsymbol{p}\ d\boldsymbol{q}}{(2\pi)^{2d}} \frac{\left(e_{\boldsymbol{p}} - \varepsilon_{\boldsymbol{p}}\right)\left(e_{\boldsymbol{q}} - \varepsilon_{\boldsymbol{q}}\right)}{e_{\boldsymbol{p}} e_{\boldsymbol{q}} \left(e_{\boldsymbol{p}} + e_{\boldsymbol{q}} + \frac{\varepsilon_{\boldsymbol{p}-\boldsymbol{q}}}{2}\right)}. \quad (130)$$

To simplify this we can change to spherical coordinates. In four dimensions the spherical volume element is given by

$$d\boldsymbol{p} = p^3 dp\ \sin^2\theta d\theta\ \sin\omega d\omega\ d\phi. \quad (131)$$

Since the only angular dependence in eq. (130) is in the term

$$\frac{\varepsilon_{\boldsymbol{p}-\boldsymbol{q}}}{2} = \frac{(\boldsymbol{p}-\boldsymbol{q})^2}{4m} = \frac{\varepsilon_{\boldsymbol{p}}}{2} + \frac{\varepsilon_{\boldsymbol{q}}}{2} - \frac{\boldsymbol{p}\cdot\boldsymbol{q}}{2m} = \frac{\varepsilon_{\boldsymbol{p}}}{2} + \frac{\varepsilon_{\boldsymbol{q}}}{2} - \sqrt{\varepsilon_{\boldsymbol{p}}\varepsilon_{\boldsymbol{q}}}\cos\theta, \quad (132)$$

all integrals over the angular variables except one are trivial. By changing the radial variables to $x = \varepsilon_{\boldsymbol{p}}/\phi_0$ and $y = \varepsilon_{\boldsymbol{q}}/\phi_0$ and, for convenience, introducing the functions $f(x) = \sqrt{x^2+1}$ and $g(x,y) = f(x) + f(y) + \frac{x+y}{2}$, we get

$$V_2(\phi_0) = -\epsilon\left(\frac{m\phi_0}{2\pi}\right)^{d/2} \frac{\phi_0}{\pi} \int_0^\infty dx \int_0^\infty dy \int_0^\pi d\theta \times \frac{\left(f(x) - x\right)\left(f(y) - y\right)}{f(x)f(y)} \frac{xy\sin^2\theta}{g(x,y) - \sqrt{xy}\cos\theta}. \quad (133)$$

This integral has to be calculated using numerical methods. The result can be written as

$$V_2(\phi_0) = -C\epsilon \left(\frac{m\phi_0}{2\pi}\right)^{d/2} \phi_0, \tag{134}$$

where the constant C can be calculated to (see Ref. [9])

$$C \approx 0.14424. \tag{135}$$

We can now write down the full effective potential to next-to-leading order in the unitary limit.

$$V_{\text{eff}}(\phi_0) = V_0(\phi_0) + V_1(\phi_0) + V_2(\phi_0) \tag{136}$$

$$\begin{aligned} &= \frac{\phi_0^2}{c_0} + \left[\frac{\phi_0}{3}\left(1 + \frac{7 - 3(\gamma + \log 2)}{6}\epsilon - 3C\epsilon\right)\right. \\ &\qquad \left. - \frac{\mu}{\epsilon}\left(1 + \frac{1 - 2(\gamma - \log 2)}{4}\epsilon\right)\right]\left(\frac{m\phi_0}{2\pi}\right)^{\frac{d}{2}} + \mathcal{O}(\epsilon^2). \end{aligned} \tag{137}$$

4.2 The gap equation

Now that we have calculated the effective potential we can use it to find an equation for the gap ϕ_0. We get this equations by minimizing V_{eff} with respect to ϕ_0, i.e. we want to find $\bar{\phi}_0$ such that

$$\left.\frac{\partial V_{\text{eff}}}{\partial \phi_0}\right|_{\phi_0 = \bar{\phi}_0} = 0. \tag{116}$$

In the unitary limit this condition gives the equation

$$\begin{aligned} \phi_0 = &\frac{2\mu}{\epsilon}\left[1 + (3C - 1 + \log 2)\epsilon\right] \\ &- \frac{2}{c_0}\phi_0^{\epsilon/2}\left(\frac{2\pi}{m}\right)^{d/2}\left[1 + \left(3C - 1 + \frac{\gamma + \log 2}{2}\right)\epsilon\right] + \mathcal{O}(\epsilon^2). \end{aligned} \tag{138}$$

From eq. (24) we can expand $1/c_0$ to next-to-leading order in ϵ, getting

$$\frac{1}{c_0} = -\frac{1}{2\epsilon}\left(1 + \frac{1 - \gamma + \log 2}{2}\epsilon\right)\left(\frac{m}{2\pi}\right)^{d/2} \varepsilon_{\text{b}}^{1-\epsilon/2} + \mathcal{O}(\epsilon^2). \tag{139}$$

Inserting this into eq. (138) we get

$$\begin{aligned} \phi_0 = &\frac{2\mu}{\epsilon}\Big[1 + (3C - 1 + \log 2)\epsilon\Big] \\ &+ \frac{\varepsilon_{\text{b}}}{\epsilon}\left(\frac{\phi_0}{\varepsilon_{\text{b}}}\right)^{\epsilon/2}\left[1 + \left(3C - \frac{1}{2} + \log 2\right)\epsilon\right] + \mathcal{O}(\epsilon^2). \end{aligned} \tag{140}$$

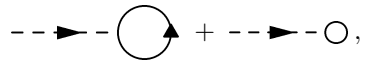
The leading order solution is thus given by

$$\phi_0 = \frac{2\mu + \varepsilon_{\text{b}}}{\epsilon} + \mathcal{O}(\epsilon). \tag{141}$$

Since ϕ_0 is $\mathcal{O}(1)$ we see that the solution to the gap equation confirms our previous assumption that $\mu \sim \epsilon$.

4.3 The boson tadpole

Before we continue our calculation of thermodynamic quantities for the unitary Fermi gas, we return to the problem of the vanishing of the boson tadpole. As we have already seen, the leading order contribution to the boson tadpole is given by the diagrams



which gives

$$\Xi = g \int \frac{dp}{(2\pi)^{d+1}} G_{21}(p) - ig\frac{\phi_0}{c_0} \tag{142}$$

$$= ig\frac{\phi_0}{2} \int \frac{d\boldsymbol{p}}{(2\pi)^d} \frac{1}{E_{\boldsymbol{p}}} - ig\frac{\phi_0}{c_0}. \tag{143}$$

By expanding the integral to first order in μ we can write this as

$$\Xi = ig\frac{\phi_0}{2} \int \frac{d\boldsymbol{p}}{(2\pi)^d} \left[\frac{1}{\sqrt{\varepsilon_{\boldsymbol{p}}^2 + \phi_0^2}} + \mu \frac{\varepsilon_{\boldsymbol{p}}}{\left(\varepsilon_{\boldsymbol{p}}^2 + \phi_0^2\right)^{3/2}} \right] - ig\frac{\phi_0}{c_0} \tag{144}$$

$$= -\frac{ig}{2} \left[\left(\frac{m\phi_0}{2\pi} \right)^2 - \mu \frac{m^2\phi_0}{2\pi^2\epsilon} \right] - ig\frac{\phi_0}{c_0} + \mathcal{O}(\epsilon^{3/2}) \tag{145}$$

$$= -\frac{ig}{2} \left(\frac{m\phi_0}{2\pi} \right)^2 + 2i\frac{\mu\phi_0}{g} - ig\frac{\phi_0}{c_0} + \mathcal{O}(\epsilon^{3/2}) \tag{146}$$

$$= -i\frac{\phi_0}{g} \left[\frac{g^2m^2}{8\pi^2}\phi_0 - 2\mu + \frac{g^2}{c_0} \right] + \mathcal{O}(\epsilon^{3/2}) \tag{147}$$

$$= -i\frac{\phi_0}{g} \left[\epsilon\phi_0 - 2\mu + \frac{g^2}{c_0} \right] + \mathcal{O}(\epsilon^{3/2}). \tag{148}$$

Substituting in the leading order gap equation solution, $\phi_0 = (2\mu + \varepsilon_{\mathrm{b}})/\epsilon$, and using the fact that $\varepsilon_{\mathrm{b}} = -g^2/c_0 + \mathcal{O}(\epsilon^2)$, we see that to the leading order the boson tadpole vanishes, as anticipated.

4.4 The fermion number density

If we now solve eq. (140) for μ/ϵ and substitute the result into the expression for V_{eff}, and at the same time use eq. (139) to rewrite the ϕ_0^2/c_0 term, we get the following expression for V_{eff} at the minimizing point ϕ_0:

$$V_{\mathrm{eff}} = -\frac{\phi_0}{6} \left[1 + \left(\frac{17}{12} - \frac{\gamma + \log 2}{2} - 3C \right) \epsilon - \frac{3}{4} \left(\frac{\varepsilon_{\mathrm{b}}}{\phi_0} \right)^{\frac{d}{2}-1} \right] \left(\frac{m\phi_0}{2\pi} \right)^{d/2}. \tag{149}$$

Using this solution we can also calculate the fermion particle number density n. Differentiating our expression for V_{eff} with respect to the chemical potential,

we get

$$\begin{aligned} n &= -\frac{\partial V_{\text{eff}}}{\partial \mu} \\ &= \frac{1}{2}\left\{\left[1+\left(\frac{5}{4}-\frac{\gamma+\log 2}{2}-3C\right)\epsilon\right]-\frac{1}{2}\left(\frac{\varepsilon_{\text{b}}}{\phi_0}\right)^{\frac{d}{2}-1}\right\}\left(\frac{m\phi_0}{2\pi}\right)^{d/2}\frac{\partial\phi_0}{\partial\mu}. \end{aligned} \tag{150}$$

We now need to be somewhat careful to get the order of the different terms right. As we soon will see $\partial\phi_0/\partial\mu$ is of order $\mathcal{O}(\epsilon^{-1})$. We also consider $\varepsilon_{\text{b}}/\phi_0 \sim \mathcal{O}(\epsilon)$. Thus the first term in the above expression is of order $\mathcal{O}(\epsilon^{-1})$, while the second one is of order $\mathcal{O}(1)$. From the solution to the gap equation, eq. (140), we can calculate the derivative

$$\frac{\partial\phi_0}{\partial\mu} = \frac{2}{\epsilon}\left[1+(3C-1+\log 2)\,\epsilon\right]+\frac{1}{2}\left(\frac{\varepsilon_{\text{b}}}{\phi_0}\right)^{\frac{d}{2}-1}\frac{\partial\phi_0}{\partial\mu}+\mathcal{O}(\epsilon) \tag{151}$$

$$= \frac{2}{\epsilon}\left[1+\frac{1}{2}\left(\frac{\varepsilon_{\text{b}}}{\phi_0}\right)^{\frac{d}{2}-1}+(3C-1+\log 2)\,\epsilon\right]+\mathcal{O}(\epsilon). \tag{152}$$

Insertion of this into our previous expression for n, keeping only terms up to $\mathcal{O}(1)$, now yields

$$n = \frac{1}{\epsilon}\left(1+\frac{1+2\log 2-2\gamma}{4}\epsilon\right)\left(\frac{m\phi_0}{2\pi}\right)^{d/2}+\mathcal{O}(\epsilon). \tag{153}$$

Using the definition of the coupling constant g we can rewrite the expression for n as

$$n = \left(1+\frac{1+2\log 2-2\gamma}{4}\epsilon\right)\frac{2\phi_0^2}{g^2}. \tag{154}$$

The quantity $n_b = \phi_0^2/g^2$ can be interpreted as the number density of the bosons in the condensate. The vacuum expectation value ϕ_0 thus is rescaled by a factor of $1/g$, in analogy with the rescaling of the dynamical field φ. In the $\epsilon \to 0$ limit, the above expression for n approaches $2\phi_0^2/g^2 = 2n_b$. Since each boson consists of two fermions, this is a confirmation of the remark made by Nussinov and Nussinov [28] — in four dimensions the unitary Fermi gas becomes a gas of condensated bosons.

In calculating n we got an $\mathcal{O}(1)$ cancellation between terms in $\varepsilon_{\text{b}}/\phi_0$ from the derivative $\partial V_{\text{eff}}/\partial\mu$ and the inner derivative $\partial\phi_0/\partial\mu$, making the final result dependent only on the boson density n_b and on the dimension of space, through ϵ. Thus we cannot use this result to observe the evolution of e.g. the relative boson condensate density n_b/n as we move away from unitarity. This indicates that in $d = 4$, the boson condensate density is already maximized in the unitary limit, again in accordance with the remark by Nussinov and Nussinov. Hence this cancellation is not expected in higher orders in ϵ. This is also confirmed by the next-to-next-to-leading-order calculation by Arnold et al. [22].

To compare our interacting Fermi gas with a non-interacting one, we need to fix the energy scale. To do this we use the Fermi energy ε_{F} of the free gas with the corresponding particle density. The density n of the free gas can

be calculated by integrating over all filled states. Remembering that we are interested in fermions with two spin states, we get, in d dimensions,

$$n = 2 \int\limits_{|\boldsymbol{p}|<p_\mathrm{F}} \frac{d\boldsymbol{p}}{(2\pi)^d} = 2 \int \frac{d\Omega}{(2\pi)^d} \int_0^{p_\mathrm{F}} dp p^{d-1} \tag{155}$$

$$= \frac{2}{(4\pi)^{\frac{d}{2}}} \frac{p_\mathrm{F}^d}{\Gamma\left(\frac{d}{2}+1\right)}. \tag{156}$$

Hence the Fermi energy of the free Fermi gas, $\varepsilon_\mathrm{F} = \frac{p_\mathrm{F}^2}{2m}$, is given by

$$\varepsilon_\mathrm{F} = \frac{2\pi}{m} \left(\frac{1}{2} \Gamma\left(\frac{d}{2}+1\right) n \right)^{\frac{2}{d}}. \tag{157}$$

Substituting in our previous expression for n in the unitary regime, eq. (153), we get

$$\varepsilon_\mathrm{F} = \frac{\phi_0}{\epsilon^{2/d}} \left(1 - \frac{1 - 1\log 2}{4} \epsilon \right) (1 + \mathcal{O}(\epsilon)). \tag{158}$$

4.5 The chemical potential

We now want to derive an expression for the chemical potential μ of the Fermi gas close to unitarity, in terms of the Fermi energy ε_F of the corresponding free fermion gas. To do this we again solve eq. (140), the gap equation solution, for μ. If we then divide this by the above expression for ε_F, we get

$$\frac{\mu}{\varepsilon_\mathrm{F}} = \frac{\epsilon^{1+2/d}}{2} \left[1 - \left(3C - \frac{5}{4}(1 - \log 2) \right) \epsilon \right] - \frac{1}{2} \frac{\varepsilon_\mathrm{b}}{\varepsilon_\mathrm{F}} \left(\frac{\varepsilon_\mathrm{b}}{\phi_0} \right)^{-\epsilon/2} \left(1 + \frac{\epsilon}{2} \right) + \mathcal{O}(\epsilon^{7/2}), \tag{159}$$

or, by a second usage of eq. (158),

$$\frac{\mu}{\varepsilon_\mathrm{F}} = \frac{\epsilon^{1+2/d}}{2} \left[1 - \left(3C - \frac{5}{4}(1 - \log 2) \right) \epsilon \right] - \frac{1}{2} \left(\frac{\varepsilon_\mathrm{b}}{\varepsilon_\mathrm{F}} \right)^{1-\epsilon/2} \left(1 + \frac{\epsilon}{2} + \frac{\epsilon}{4} \log \epsilon \right) + \mathcal{O}(\epsilon^{7/2}). \tag{160}$$

We have to be a bit careful in the last term. If we just were to expand the $\varepsilon_\mathrm{b}/\varepsilon_\mathrm{F}$ factor to the next-to-leading order in ϵ we would get

$$\left(\frac{\varepsilon_\mathrm{b}}{\varepsilon_\mathrm{F}} \right)^{1-\epsilon/2} = \left(\frac{\varepsilon_\mathrm{b}}{\varepsilon_\mathrm{F}} \right) \left(1 - \frac{\epsilon}{2} \log \frac{\varepsilon_\mathrm{b}}{\varepsilon_\mathrm{F}} \right) + \mathcal{O}(\epsilon^{7/2}). \tag{161}$$

This would reproduce the result of Nishida and Son [9]. But we are interested in the explicit variation of this quantity around unitarity and this expression is problematic in the $\epsilon \to 1$, $|\varepsilon_\mathrm{b}/\varepsilon_\mathrm{F}| \ll 1$ limit. To get around this problem, we use a prescription inspired by Chen and Nakano [24]. First we define

$$Q(\varepsilon_\mathrm{b}) = \frac{1}{\epsilon} \left[\left(\frac{\varepsilon_\mathrm{b}}{\varepsilon_\mathrm{F}} \right)^{-\epsilon/2} - 1 \right]. \tag{162}$$

Taylor expansion of $Q(\varepsilon_\mathrm{b})$ gives

$$Q(\varepsilon_\mathrm{b}) \approx \frac{1}{\epsilon}\left(1 - \frac{\epsilon}{2}\log\frac{\varepsilon_\mathrm{b}}{\varepsilon_\mathrm{F}} + \cdots - 1\right), \tag{163}$$

which we will treat as an $\mathcal{O}(1)$ expression, even though it contains higher order terms as well. Thus we can write

$$\frac{1}{2}\left(\frac{\varepsilon_\mathrm{b}}{\varepsilon_\mathrm{F}}\right)^{1-\epsilon/2}\left(1+\frac{\epsilon}{2}\right) = \frac{1}{2}\frac{\varepsilon_\mathrm{b}}{\varepsilon_\mathrm{F}}\left[1+\epsilon Q(\varepsilon_\mathrm{b})\right]\left(1+\frac{\epsilon}{2}\right) + \mathcal{O}(\epsilon^{7/2}) \tag{164}$$

$$= \frac{1}{2}\frac{\varepsilon_\mathrm{b}}{\varepsilon_\mathrm{F}}\left[1+\epsilon Q(\varepsilon_\mathrm{b}) + \frac{\epsilon}{2}\right] + \mathcal{O}(\epsilon^{7/2}) \tag{165}$$

$$= \frac{1}{2}\frac{\varepsilon_\mathrm{b}}{\varepsilon_\mathrm{F}}\left[\left(\frac{\varepsilon_\mathrm{b}}{\varepsilon_\mathrm{F}}\right)^{-\epsilon/2} + \frac{\epsilon}{2}\right] + \mathcal{O}(\epsilon^{7/2}). \tag{166}$$

Eq. (160) then becomes

$$\frac{\mu}{\varepsilon_\mathrm{F}} = \frac{\epsilon^{1+2/d}}{2}\left[1 - \left(3C - \frac{5}{4}(1-\log 2)\right)\epsilon\right] - \frac{1}{2}\frac{\varepsilon_\mathrm{b}}{\varepsilon_\mathrm{F}}\left[\left(\frac{\varepsilon_\mathrm{b}}{\varepsilon_\mathrm{F}}\right)^{-\epsilon/2} + \frac{\epsilon}{2} + \frac{\epsilon}{4}\log\epsilon\right] + \mathcal{O}(\epsilon^{7/2}). \tag{167}$$

The result in eq. (167) then agrees with that of Ref. [24].

4.6 Other thermodynamic quantities

For a free Fermi gas in d spatial dimensions, we can calculate the total energy density by integrating

$$E = 2\int\limits_{|\boldsymbol{p}|<p_\mathrm{F}} \frac{\boldsymbol{p}}{(2\pi)^d}\frac{\boldsymbol{p}^2}{2m} \tag{168}$$

$$= \frac{2}{m}\frac{1}{(4\pi)^{d/2}}\frac{1}{\Gamma\left(\frac{d}{2}\right)}\frac{p_\mathrm{F}^{d+2}}{d+2}. \tag{169}$$

Using the previously integrated expression for n in terms of p_F, eq. (156), this gives

$$E = \frac{p_\mathrm{F}^2}{2m}\frac{d}{d+2}n = \frac{d}{d+2}n\varepsilon_\mathrm{F}. \tag{170}$$

But E is also given by the thermodynamic relation

$$E = \mu n - P, \tag{171}$$

and, at zero temperature, the chemical potential of a free Fermi gas is given by $\mu = \varepsilon_\mathrm{F}$. Thus we also get that

$$P = \left(1 - \frac{d}{d+2}\right)n\varepsilon_\mathrm{F} = \frac{2}{d+2}n\varepsilon_\mathrm{F}. \tag{172}$$

In the unitary limit, where $\varepsilon_\mathrm{b}/\varepsilon_\mathrm{F} \to 0$, there are only two quantities with the dimension of energy remaining, ε_F and μ. But both of these are uniquely given

by the particle density and the dimensionality of space. Thus we can define the universal parameter ξ_0 such that $\mu \to \xi_0 \varepsilon_\mathrm{F}$ when $\varepsilon_\mathrm{b} \to 0$. By purely dimensional arguments, we would then expect it to be possible to write the energy density and pressure per particle in the form

$$\frac{E}{n} = \frac{d}{d+2}\alpha(\xi_0)\varepsilon_\mathrm{F}, \qquad \frac{P}{n} = \frac{2}{d+2}\beta(\xi_0)\varepsilon_\mathrm{F}, \tag{173}$$

where $\alpha(\xi_0)$ and $\beta(\xi_0)$ are some analytical functions. We are now going to check that we can get relations of this form from our previous calculations, and calculate corrections for a Fermi gas in the vicinity of unitarity.

From eq. (167) we can write ξ_0 as

$$\xi_0 = \frac{\epsilon^{1+2/d}}{2}\left[1 - \left(3C - \frac{5}{4}(1-\log 2)\right)\epsilon\right] + \mathcal{O}(\epsilon^{7/2}). \tag{174}$$

We also note that, for $d = 4 - \epsilon$,

$$\frac{d}{d+2} = \frac{2}{3}\left(1 - \frac{\epsilon}{12}\right) + \mathcal{O}(\epsilon^2), \qquad \frac{2}{d+2} = \frac{1}{3}\left(1 + \frac{\epsilon}{6}\right) + \mathcal{O}(\epsilon^2). \tag{175}$$

We start by calculating the pressure. This is given by the minimized effective potential as $P = -V_\mathrm{eff}(\phi_0)$. Dividing expression (149) for the effective potential with the particle density from eq. (153) and the Fermi energy gives

$$\frac{P}{n\varepsilon_\mathrm{F}} = \frac{\epsilon}{6}\frac{\phi_0}{\varepsilon_\mathrm{F}}\left[1 - \left(\frac{7}{6} - \log 2 - 3C\right) - \frac{3}{4}\left(\frac{\varepsilon_\mathrm{b}}{\phi_0}\right)^{d/2-1}\right] \tag{176}$$

$$= \frac{\epsilon^{1+2/d}}{6}\left[1 + \left(\frac{17}{12} - \frac{5}{4}\log 2 - 3C\right)\epsilon\right] - \frac{\epsilon}{8}\frac{\phi_0}{\varepsilon_\mathrm{F}}\left(\frac{\varepsilon_\mathrm{b}}{\phi_0}\right)^{d/2-1} \tag{177}$$

$$= \frac{1}{3}\left(1 + \frac{\epsilon}{6}\right)\xi_0 - \frac{\epsilon}{8}\left(\frac{\varepsilon_\mathrm{b}}{\varepsilon_\mathrm{F}}\right)^{d/2-1} \tag{178}$$

$$= \frac{2}{d+2}\xi_0 - \frac{\epsilon}{8}\left(\frac{\varepsilon_\mathrm{b}}{\varepsilon_\mathrm{F}}\right)^{d/2-1}, \tag{179}$$

which has the anticipated form, and where we again refrain from expanding the $\left(\frac{\varepsilon_\mathrm{b}}{\varepsilon_\mathrm{F}}\right)^{d/2-1}$ term.

To calculate E we start from eq. (171) and use eqs. (167), (174) and (179) to get

$$\frac{E}{n\varepsilon_\mathrm{F}} = \frac{\mu}{\varepsilon_\mathrm{F}} - \frac{P}{n\varepsilon_\mathrm{F}} \tag{180}$$

$$= \left(1 - \frac{2}{d+2}\right)\xi_0 - \frac{1}{2}\frac{\varepsilon_\mathrm{b}}{\varepsilon_\mathrm{F}}\left[\left(\frac{\varepsilon_\mathrm{b}}{\varepsilon_\mathrm{F}}\right)^{-\epsilon/2} + \frac{\epsilon}{2} + \frac{\epsilon}{4}\log\epsilon\right] + \frac{\epsilon}{8}\frac{\varepsilon_\mathrm{b}}{\varepsilon_\mathrm{F}} \tag{181}$$

$$= \frac{d}{d+2}\xi_0 - \frac{1}{4}\frac{\varepsilon_\mathrm{b}}{\varepsilon_\mathrm{F}}\left[\frac{3}{2}\left(\frac{\varepsilon_\mathrm{b}}{\varepsilon_\mathrm{F}}\right)^{-\epsilon/2} + \epsilon + \frac{\epsilon}{2}\log\epsilon\right]. \tag{182}$$

Thus we have seen that both the pressure and the energy can be expressed in the form of eq. (173), with both the unknown coefficients given simply by ξ_0, so

that the energy scale of the unitary Fermi gas is set by the universal parameter ξ_0 and given by the expression $\xi_0\varepsilon_\mathrm{F}$, as stated in section 1.3.

Now we can define the parameter ξ, which is the ratio between the energy per particle of a Fermi gas close to unitarity, at a given μ and ε_b, to the energy per particle of the corresponding free Fermi gas. By dimensional arguments we already saw that $\xi|_{\varepsilon_\mathrm{b}=0} = \xi_0 = \mu/\varepsilon_\mathrm{F}$. Since $E = \frac{d}{d+2}\varepsilon_\mathrm{F}$ for a free gas, we get

$$\xi = \frac{d+2}{d}\frac{E}{n\varepsilon_\mathrm{F}} \tag{183}$$

$$= \xi_0 - \frac{d+2}{4d}\frac{\varepsilon_\mathrm{b}}{\varepsilon_\mathrm{F}}\left[\frac{3}{2}\left(\frac{\varepsilon_\mathrm{b}}{\varepsilon_\mathrm{F}}\right)^{-\epsilon/2} + \epsilon + \frac{\epsilon}{2}\log\epsilon\right]. \tag{184}$$

4.7 Extrapolation to three spatial dimensions

The above results are obtained in a theory where $\epsilon = 4-d$ is a small number. But we would want to have results for the physical case of three spatial dimensions, which corresponds to setting $\epsilon = 1$. Looking at our results for $\mu/\varepsilon_\mathrm{F}$, eq. (167), we see that the order ϵ corrections in the first term is small compared to the lowest order term, even at $\epsilon = 1$. The second term of this equation is proportional to $\varepsilon_\mathrm{b}/\varepsilon_\mathrm{F}$ which is also small in the vicinity of unitarity. Thus one could hope that this expression is useful even in this limit.

To get the explicit expressions at $d = 3$ we just have to let $\epsilon \to 1$ in eq. (167). This gives

$$\frac{\mu}{\varepsilon_\mathrm{F}} \;\to\; \frac{9 - 12C - 5\log 2}{8} - \frac{1}{2}\sqrt{\frac{\varepsilon_\mathrm{b}}{\varepsilon_\mathrm{F}}} - \frac{1}{4}\frac{\varepsilon_\mathrm{b}}{\varepsilon_\mathrm{F}}. \tag{185}$$

This can be expressed in terms of the parameter η using the relation $\varepsilon_\mathrm{b} = 2\eta^2\varepsilon_\mathrm{F}$ from eq. (23). We then get[2]

$$\frac{\mu}{\varepsilon_\mathrm{F}} \;\to\; 0.475 - 0.707\eta - 0.5\eta^2. \tag{186}$$

In the unitary limit, $\eta \to 0$, this gives the universal parameter

$$\xi_0 \approx 0.475. \qquad (d = 3) \tag{187}$$

Thus, by taking the $\epsilon \to 1$ limit of eq. (184), we get the following result for the universal parameter in the vicinity of unitarity:

$$\xi \;\to\; 0.475 - 0.884\eta - 0.833\eta^2. \tag{188}$$

There are, however, doubt in the validity of this naïve extrapolation to three spatial dimensions. The smallness of the next-to-leading-order (NLO) correction to ξ_0 gave us hope of a meaningful result at $\epsilon = 1$. But without knowledge of higher order terms, there is no reason to expect that the expansion is convergent in this limit. Thus the next-to-next-to-leading-order (NNLO) results by Arnold et al. [22] are very interesting. For the parameter ξ_0 in the unitary limit the result is discouraging — the $\mathcal{O}(\epsilon^2)$ is almost as big as the leading term, and

[2]Naïvely the second term should be written as $-0.707|\eta|$. As commented on in section 2.1, this is replaced by a linear term to accommodate the loss of the sign of the scattering length a (and hence of η) in our four-dimensional treatment.

has the opposite sign. This emphasizes the statement by the authors that in a simple minded application of the ϵ-expansion it is important to quit when one is ahead. Still it is possible to better the situation by taking into consideration also the behavior of the theory close to two spatial dimensions This is done by Nishida and Son [9], who, in addition to the four-dimensional case, consider a unitary Fermi gas for $d = 2 + \epsilon$. The resulting expression for ξ_0 is given by $\xi_0 = 1 - \epsilon + \mathcal{O}(\epsilon^2)$. By using this as a boundary condition at $d = 2$ and applying a Padé approximation to perform a Borel summation, taking into account terms up to NLO from the ϵ-expansion in four dimensions, they get the result $\xi_0 \approx 0.378 \pm 0.013$. Including also the corrections from NNLO, Arnold et al. [22] get $\xi_0 \approx 0.367 \pm 0.010$. In the next section we will compare our results to those of various experimental measurements and Monte Carlo calculations. We will then see that neither the results of Ref. [9], nor those of Ref. [22], are obvious improvements of the result of naïve extrapolation to $\epsilon = 1$.

4.8 Comparison with experiments and simulations

As explained in the introduction, there has, in recent years, been a number of experiments where fermionic atom gases have successfully been condensed into superfluids, and where it has been possible to control the interaction strength in such a way that a unitary Fermi gas is created [30–35]. In many of these experiments, one of the goals has been to determine the value of the universal parameter at the unitary point, ξ_0. There has also been a number of projects that do theoretical calculations on the unitary Fermi gas by means of quantum Monte Carlo methods [36–42]. In figure 4 our result for ξ_0 from eq. (187) is compared to the results of various experiments, Monte Carlo simulations and analytical calculations. The measured and calculated values used to create this figure are tabulated in table 1.

It is hard to extract results on the variation of ξ in the vicinity of unitarity from published experimental data. Furthermore most Monte Carlo studies mainly focus on other physical aspects of Fermi gases in the unitary limit, such as non-zero temperature effects and spin-polarized gases. There are, however, two articles where explicit results from Monte Carlo calculations of the energy per particle in a wider region of the BCS–BEC crossover are given [36, 37]. In figure 5 these results are compared to the $\epsilon \to 1$ extrapolation of the ξ results from the next-to-leading-order ϵ-expansion, eq. (188).

5 Discussion

Although we have the same starting point as, and arrive at the same expressions for ξ_0 and ξ as Chen and Nakano [24], and Nishida and Son [9], there are some differences in the formalisms that I want to comment on.[3] One difference is in the way the Lagrangian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2$ is divided. In this thesis we keep the terms in the chemical potentials μ and μ_B together with the kinetic terms in $\mathcal{L}_0$. Since we only really consider the neighborhood of unitarity, where, as we have

[3] As noted in section 4 there are some places where the results of Ref. [24] and Ref. [9] differ, and where we are able to reproduce both results, but choose one of them in order to make a better comparison with data from other calculations for the BCS–BEC crossover, see e.g. the discussion leading to eq. (167).

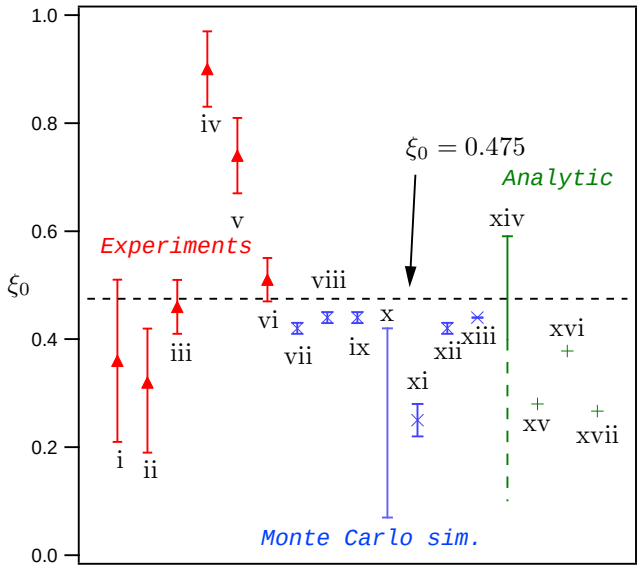


Figure 4: Different results for the value of ξ_0, the universal parameter in the unitary limit. The dashed line indicates the result $\xi_0 \approx 0.475$ from the ϵ-expansion. This is compared with the results from various experimental measurements (▲), Monte Carlo calculations ($\times$) and analytical calculations ($+$). For references and details of the different values see table 1(a) for the experimental data (points i–vi), table 1(b) for the results from Monte Carlo calculations (points vii–xiii) and table 1(c) for results from analytical calculations (points xiv–xvii).

Table 1: Compilation of results of the measurements/calculations of the parameter ξ_0, as reported by a number of different sources. These are to be compared to the value $\xi_0 \approx 0.475$ given by the ϵ-expansion when we extrapolate to three spatial dimensions. The roman numerals indicates the corresponding point in figure 4.

(a) Experimental measurements

	Author(s)	ξ_0
i	Bourdel et al. [34]	0.36 ± 0.15
ii	Bartenstein et al. [33]	$0.32^{+0.10}_{-0.13}$
iii	Partridge et al. [32]	0.46 ± 0.05
iv	O'Hara et al. [31]	0.90 ± 0.07
v	Gehm et al. [30]	0.74 ± 0.07
vi	Kinast et al. [35]	0.51 ± 0.04

(b) Monte Carlo calculations

	Author(s)	ξ_0
vii	Astrakharchik et al. [36]	0.42 ± 0.01
viii	Chang et al. [37]	0.44 ± 0.01
ix	Carlson et al. [41]	0.44 ± 0.01
x	Lee and Schäfer [40]	0.07 – 0.42
xi	Lee [39]	0.25 ± 0.03
xii	Carlson and Reddy [38]	0.42 ± 0.01
xiii	Bulgac et al. [42]	0.44

(c) Analytical calculations

	Method	Author(s)	ξ_0
xiv	BCS mean field	see Ref. [17]	< 0.5906
xv	Large-N expansion	Veillette et al. [17]	0.28
xvi	ϵ-exp. (NLO) (Padé)	Nishida and Son [9]	0.378 ± 0.013
xvii	ϵ-exp. (NNLO) (Padé)	Arnold et al. [22]	0.267 ± 0.010
	ϵ-exp. (NLO) ($\epsilon = 1$)	This thesis and Ref. [9]	0.475

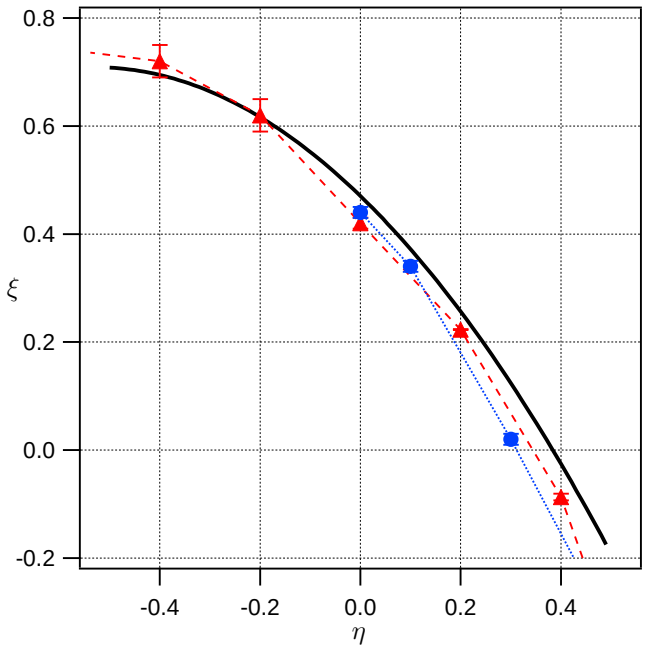
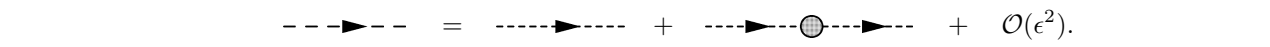


Figure 5: The universal parameter ξ as an function of η in the vicinity of the universal point. The result from eq. (188) is given by the full thick line. The markers indicate the calculations by Astrakharchik et al. [36] (▲), and Chang et al. [37] (■).

seen, both μ/ϕ_0 and μ_B/ϕ_0 can be treated as $\mathcal{O}(\epsilon)$, we could have put these terms in $\mathcal{L}_1$ together with the fermion–boson interactions, which are of order $\mathcal{O}(\epsilon^{1/2})$. This is what Nishida and Son do. Instead we choose to make use of the smallness of μ and μ_B by doing Taylor expansions of the amplitudes. One advantage of this is that the Feynman rules become somewhat simpler. If we were to treat μ and μ_B as perturbations we would get two more vertices — a μ insertions to the fermion propagator and a μ_B insertion to the boson propagator. This difference can be illustrated with the following equation, where the lines with short dashes on the right hand side represent boson propagators on at $\mu_B = 0$ and the gray circle in the propagator represent an insertion of a factor of μ_B.



Moreover there would be more divergent diagrams, such as a boson self energy one-fermion-loop diagram with a μ insertion. To cancel these we would have to split the counter term into two and use the different terms to cancel different divergences. All of this is automatically taken care of in our formalism, at the expense of manual Taylor expansion of amplitudes.

Another advantage of keeping the chemical potentials in the propagators is that it should lead to an easier generalization to the full BCS–BEC crossover, where μ and μ_B can not be assumed small. Chen and Nakano do treat the full BCS–BEC crossover in the ϵ-expansion using a scheme much like the one in this thesis, but they split the Lagrangians in different ways on the BEC and unitary sides, where at least μ_B can be expected to be small, compared to the BCS side where neither μ nor μ_B can be treated perturbatively. This means that they have to redo much of the basic calculations separately in the two cases. Instead it should be possible to use the same Lagrangian and do most of the checking of the power counting rules once, and only consider the different limits separately in the explicit calculation of V_{eff} — where one has to use that ε_{b} is small in the unitary limit, ϕ_0 is small on the BCS side, and μ_B in the BEC limit. The calculations required to do this may be a bit more complicated, but there is an arguable conceptual advantage in sticking to one case as far as possible.

There is also a difference between the formalism in the treatment of the renormalized coupling constant g. Nishida and Son [9] simply take as an ansatz the expression for g and then show that the boson self energy renormalization vanishes. That this second step works out seems somewhat coincidental. Their different splitting of $\mathcal{L}$, which gives two different $\mathcal{O}(\epsilon)$ diagrams that has to cancel against different counter terms, enhances this feeling even more. In this thesis we instead saw that the same expression for g is uniquely determined by consistency requirements for the two-particle scattering amplitude and considerations of physical dimensions.

6 Conclusions

In this thesis we have seen that it is possible to perform a perturbative expansion of a Fermi gas in the vicinity of unitarity using an ϵ-expansion. This approach was based on the observation by Nussinov and Nussinov that in four spatial dimensions, the ground state of the unitary Fermi gas is a gas of non-interacting bosons.

We started by noticing that in $d = 4 - \epsilon$ spatial dimensions, the full two-particle scattering amplitude T, in the unitary limit have the form of a boson propagator times the square of a coupling constant, which is proportional to $\sqrt{\epsilon}$ and thus small when considered close to four dimensions. This gives us a possibility to interpret the scattering of two particles in vacuum as mediated by an intermediate resonance, which we treat as a virtual boson. The strong fermion–fermion point interaction hence renormalizes into a weak interaction over a distance, mediated by a scalar boson field.

Inspired by this observation we then considered a gas at finite density and derived a number of Feynman rules that enabled us to perform perturbative expansions. We also showed that the different terms in such an expansion behave as expected when it comes to factors of ϵ, with two exceptions — there is one boson self energy diagram and one type of boson tadpole diagram that, due to ultraviolet divergences, give one factor of ϵ less than expected. But among our Feynman rules there were two types of counter terms, and using these it was possible to cancel the divergent terms. Thus it is possible to perform a perturbative expansion by writing down a number of diagrams, and calculations will work out as expected as long as the counter terms are used when needed.

As a demonstration of this perturbative technique we calculated the effective potential to the next-to-leading-order in ϵ. This result could then be used to calculate the boson and fermion number densities, and the pressure and ground state energy of the gas. By dimensional arguments we saw that, in the unitary limit, the energy per particle could be expressed as $\xi_0 \varepsilon_\mathrm{F}$, where ε_F was the Fermi energy of a free Fermi gas of the same density, and $\xi_0 = \mu/\varepsilon_\mathrm{F}$ was a dimensionless constant, which we were able to calculate as a function of the number of spatial dimensions d only.

For a Fermi gas in the vicinity of unitarity, we derived corrections to the energy per particle $\xi\varepsilon_\mathrm{F}$, in terms of the binding energy of a bound pair of fermions. This could in turn be related to the parameter $\eta = 1/(ap_\mathrm{F})$, which can be used to characterize a Fermi gas in the BCS–BEC crossover.

To be able to draw conclusions for the physical case of three spatial dimensions, we performed a naïve extrapolation to $\epsilon = 1$, giving us an expression for the universal parameter ξ in terms of only η. Thus we were able to compare our result to experimental measurements as well as quantum Monte Carlo calculations. In the unitary limit, a variety of different figures are reported for the value of ξ_0, but the general agreement with our result of $\xi_0 \approx 0.475$ is quite good. Also our expression for ξ in the region agreed well with the results of Monte Carlo calculations in the region $-0.5 \leq \eta \leq 0.5$.

We also compared our results with those of Nishida and Son [9] — who in addition to the four-dimensional side also took into account the behavior at $d = 2$ as a boundary condition — and to the results of the next-to-next-to-leading-order calculation by Arnold et al. [22]. However neither of these elaborations of the naïve next-to-leading-order results made an obvious improvement when compared to data from experiments and Monte Carlo calculations.

A Some useful formulas

This appendix presents some formulas that frequently turn out to be useful in dimensional regularization, where one has to calculate integrals in an arbitrary number of dimensions, d. The relations listed here are derived or referenced in Ref. [43].

The d-dimensional volume element $d\boldsymbol{k}$ can be written, using spherical coordinates, as

$$\frac{d\boldsymbol{k}}{(2\pi)^d} = \frac{d\Omega_d}{(2\pi)^d} d|\boldsymbol{k}||\boldsymbol{k}|^{d-1}, \tag{A1}$$

where $d\Omega_d$ is the solid angle element. Often the integrand has a spherical symmetry. The angular part of the integral can then be computed using the following formula for the area of a unit sphere in d dimensions:

$$\int \frac{d\Omega_d}{(2\pi)^d} = \frac{2\pi^{\frac{d}{2}}}{(2\pi)^d} \frac{1}{\Gamma(\frac{d}{2})} = \frac{2}{(4\pi)^{\frac{d}{2}}} \frac{1}{\Gamma(\frac{d}{2})}. \tag{A2}$$

To perform the radial integration, we can often perform a change of variables and then use the formula

$$\int_0^\infty dz \frac{z^{m-1}}{(1+z)^{m+n}} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}. \tag{A3}$$

When using the above formulas we have to evaluate the Γ function in various points. This function has singularities at $x = 0, -1, -2, \ldots$. Close to these poles Γ can be expanded as

$$\Gamma(-n+x) = \frac{(-1)^n}{n!} \left(\frac{1}{x} - \gamma + 1 + \cdots + \frac{1}{n} + \mathcal{O}(x) \right), \tag{A4}$$

where $\gamma \approx 0.5772$ is the Euler-Mascheroni constant.

At points that are not singularities of Γ we can use an ordinary Taylor expansion. In doing this we can calculate the derivative of $\Gamma(x)$ as [44]

$$\frac{d}{dx}\Gamma(x) = \psi(x)\Gamma(x), \tag{A5}$$

where $\psi(x)$ is the digamma function which satisfies the recurrence relation

$$\psi(x+1) = \psi(x) + \frac{1}{x}, \tag{A6}$$

and takes the values

$$\psi(1) = -\gamma, \tag{A7}$$

$$\psi\left(\tfrac{1}{2}\right) = -\gamma - 2\log 2. \tag{A8}$$

As an example of using these formulas we calculate the following integral, which is encountered many times in this thesis:

$$i\mathcal{M} = \int \frac{d\boldsymbol{k}}{(2\pi)^d} \frac{1}{(\varepsilon_{\boldsymbol{k}} + \Delta)^n}. \tag{A9}$$

This integral has an ultraviolet divergence if $n \leq 2$. We are now going to calculate $i\mathcal{M}$ to leading order in ϵ for this case. First we change to spherical coordinates in accordance with eq. (A1).

$$i\mathcal{M} = \int \frac{d\Omega}{(2\pi)^d} \int d|\boldsymbol{k}| \frac{\boldsymbol{k}^{d-1}}{(\varepsilon_{\boldsymbol{k}} + \Delta)^n}. \tag{A10}$$

By changing variables of integration to $z = \frac{\varepsilon_{\boldsymbol{k}}}{\Delta}$ this becomes

$$i\mathcal{M} = \frac{(2m)^{\frac{d}{2}}}{2} \Delta^{\frac{d}{2}-n} \int \frac{d\boldsymbol{k}}{(2\pi)^d} \int dz \frac{z^{\frac{d}{2}-1}}{(z+1)^n}. \tag{A11}$$

Now we can use eq. (A3) and eq. (A2) to perform the integrals. This gives

$$i\mathcal{M} = \left(\frac{m}{2\pi}\right)^{\frac{d}{2}} \Delta^{\frac{d}{2}-n} \frac{\Gamma\left(n - \frac{d}{2}\right)}{\Gamma(n)}. \tag{A12}$$

In $d = 4 - \epsilon$ spatial dimensions, we can now finally use the approximation of eq. (A4) to write $i\mathcal{M}$ to the first order in ϵ as

$$i\mathcal{M} = \frac{m^2}{4\pi^2} \Delta^{2-n} \Gamma\left(n - 2 + \frac{\epsilon}{2}\right) \tag{A13}$$

$$= \frac{(-1)^n}{(2-n)!} \frac{m^2}{2\pi^2 \epsilon} \Delta^{2-n} \left(1 + \mathcal{O}(\epsilon)\right). \tag{A14}$$

This result can now directly be used to calculate e.g. the following integral from the thesis:

$$ig^2 \int \frac{d\boldsymbol{k}}{(2\pi)^d} \frac{1}{p_0 - 2\varepsilon_{\boldsymbol{k}} - \frac{\varepsilon_{\boldsymbol{p}}}{2} + 2\mu} = -i\frac{g^2}{2} \int \frac{d\boldsymbol{k}}{(2\pi)^d} \frac{1}{\varepsilon_{\boldsymbol{k}} + \frac{1}{2}\left(-p_0 + \frac{\varepsilon_{\boldsymbol{p}}}{2} - 2\mu\right)} \tag{A15}$$

$$= ig^2 \frac{m^2}{8\pi^2 \epsilon} \left(p_0 - \frac{\varepsilon_{\boldsymbol{p}}}{2} + 2\mu\right) \left(1 + \mathcal{O}(\epsilon)\right). \tag{A16}$$

B The two-particle scattering length

This section presents a simple derivation of the relation between the scattering length and the binding energy of a pair of interacting particles. The subject is treated in much more depth in e.g. Ref [45].

Consider a particle scattering against a spherically symmetric potential V of range r_0. Far outside the potential, the incident wave function is a plane wave e^{ikz}, where $k = \sqrt{2\mu E}$ and μ is the reduced mass of the system. To find the scattering amplitude we need to solve the Schrödinger equation

$$\left[-\frac{\nabla^2}{2\mu} + V(r)\right]\psi(\mathbf{r}) = -E\psi(\mathbf{r}). \tag{B17}$$

This equation is separable and spherically symmetric. Hence we can expand the total wave function as

$$\psi(\mathbf{r}) = \sum_l \frac{u_l(r)}{r} P_l(\cos\theta), \tag{B18}$$

where P_l are spherical Legendre functions. The radial part of the Schrödinger equation is then given by

$$\left[-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + 2\mu V(r) - k^2\right] u_l(r) = 0. \tag{B19}$$

For $r > r_0$ the solutions to eq. (B19) can be written in terms of spherical Bessel and Neumann functions. In the region $r \to \infty$, we can use the asymptotic behavior of these functions to write the radial wave functions as

$$\frac{u_l(r)}{r} \to a_l \frac{\sin\left(kr - \frac{\pi}{2}l + \delta_l\right)}{kr}, \tag{B20}$$

where δ_l are real valued phase shifts, and a_l constants.

The total wave function should, at least for big r, be a superposition of the incoming plane wave, and an outgoing scattered spherical wave,

$$\psi(\boldsymbol{r}) = e^{ikz} + f(\theta)\frac{e^{ikr}}{r}. \tag{B21}$$

The plane wave part can be expanded into angular momentum eigenstates,

$$e^{ikz} = e^{ikr\cos\theta} = \sum_l (2l+1) i^l j_l(kr) P_l(\cos\theta), \tag{B22}$$

where $j_l(z)$ are spherical Bessel functions and P_l are Legendre polynomials. For large r this becomes

$$\sum_l (2l+1) i^l \frac{\sin\left(kr - \frac{\pi}{2}l\right)}{kr} P_l(\cos\theta). \tag{B23}$$

By comparing eqs. (B21) and (B23) with eqs. (B18) and (B20) we can identify the coefficients a_l. This gives the radial wave function

$$\frac{u_l(r)}{r} = (2l+1) i^l e^{i\delta_l} \frac{\sin\left(kr - \frac{\pi}{2}l + \delta_l\right)}{kr}, \tag{B24}$$

and the scattering amplitude

$$f(\theta) = \frac{1}{k}\sum_l (2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\theta). \tag{B25}$$

Integrating $\frac{d\sigma}{d\Omega} = |f(\theta)|^2$, we get the total cross-section

$$\sigma = \frac{4\pi}{k^2}\sum_l (2l+1)\sin^2\delta_l. \tag{B26}$$

Now we consider scattering at very low energy, i.e. $k \approx 0$. In this limit only the S-wave ($l = 0$) part of the wave function will contribute. For $r > r_0$ the radial Schrödinger equation will then simply be

$$\frac{d^2u}{dr^2} = 0, \tag{B27}$$

which has the solution

$$u_0(r) = A(r-a), \tag{B28}$$

where A is an unimportant constant and a is a constant of dimension length called the *scattering length*.

In addition to eq. (B28), $u_0(r)$ is, according to the previous treatment of scattering against a potential, also given by

$$u_0(r) = \lim_{k\to 0}\frac{1}{k}\sin(kr+\delta_0). \tag{B29}$$

Using these two expressions for $u_0(r)$, we can write the logarithmic derivative of $u_0(r)$ in two different ways:

$$\frac{d}{dr}\log u_0 = \lim_{k\to 0} k\cot(kr+\delta_0) = \frac{1}{r-a}. \tag{B30}$$

Even though this expression gives the true wave function only for $r > r_0$, we can take the $r \to 0$ limit, which gives

$$\lim_{k\to 0} k\cot(\delta_0) = -\frac{1}{a}. \tag{B31}$$

By the previous expression for the scattering amplitude we now have

$$f(\theta) = \frac{e^{i\delta_l}\sin\delta_l}{k} = \frac{1}{k\cot\delta_l - ik}, \tag{B32}$$

and hence

$$\sigma = 4\pi\lim_{k\to 0}\left|\frac{1}{k\cot\delta_l - ik}\right|^2 = 4\pi a^2. \tag{B33}$$

From eq. (B28) we see that the scattering length gives the zero-intersection of the exterior radial wave function. If the potential is repulsive $a > 0$, and roughly of order r_0. An example of this is given in figure 6(a). For an attractive potential, the scattering length can, on the other hand, be arbitrary large, and of both signs. In the case of a weak attractive potential, the scattering length is negative, as seen in figure 6(b). If the interaction strength is increased there

will be a change of sign, as indicated in figure 6(c), which shows the case of a strong attractive potential. Thus, for an attractive potential, $|a|$ gets small in both the strong and the weak attraction limits. The change of sign is related to the development of a bound state, as intuitively can be seen from the plots in figure 6. The unitary limit, where $|a| \to \infty$, thus corresponds to the threshold limit for bound state formation. To see the relation between the scattering length and the possibility to form bound states more clearly, we will now derive a relation between the scattering length and the binding energy of such a state.

Assume that we have an attractive potential well of depth $V_0 < 0$ and range r_0m and that we can form a bound state with an energy E that is small ($|E| \ll |V_0|$) but negative. For $r > r_0$ the wave function is given by $e^{-\kappa r}$, with $\kappa = \sqrt{2\mu|E|} \ll 1/r_0$. Expanding this we get $e^{-\kappa r} \approx 1 - \kappa r + \ldots$, which is of the same form as the scattering wave function $u_0(r)$, for a very large a.

Now consider the wave function inside the well. Both in the case of a bound state at energy $E < 0$ and in the case of scattering at $E > 0$ this will, as long as $|E| \ll |V_0|$, be given by $\sin k'r$, where

$$\frac{k'^2}{2\mu} = E - V_0 \approx |V_0|. \tag{B34}$$

Since the inside wave functions in the two cases are essentially equal, we can equate the logarithmic derivatives of the outside wave functions at the boundary,

$$-\frac{\kappa e^{-\kappa r}}{e^{-\kappa r}}\bigg|_{r=r_0} = \frac{1}{r-a}\bigg|_{r=r_0}, \tag{B35}$$

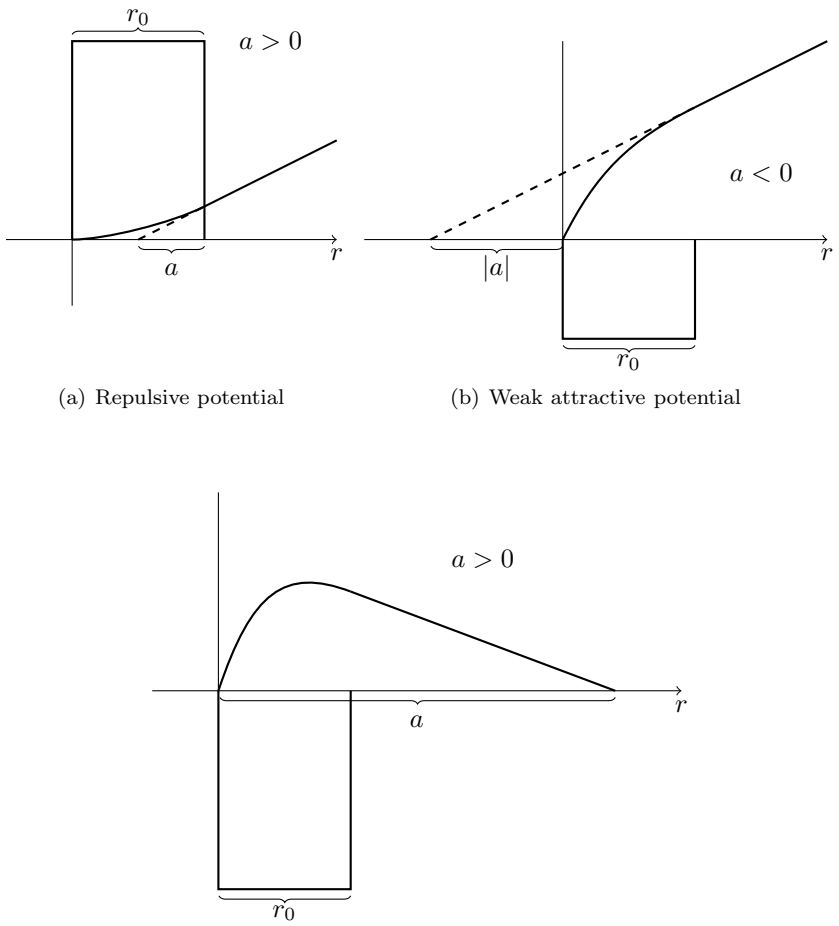
or, for $r_0 \ll a$,

$$\kappa \approx \frac{1}{a}. \tag{B36}$$

Thus the binding energy ε_b of the bound state is given by

$$\varepsilon_\mathrm{b} = \frac{\kappa^2}{2\mu} \approx \frac{1}{2\mu a^2}. \tag{B37}$$

Hence we can calculate the binding energy of a loosely bound state, by measuring the cross-section for scattering at very low energies, provided that a is big compared to the range of the potential, r_0.



(a) Repulsive potential

(b) Weak attractive potential

(c) Strong attractive potential

Figure 6: Plot of $u_0(r)$ versus r for three different potential wells. The dashed lines show the extrapolation of the linear exterior wave function. Intuitively the zero-intersection of this line gives the scattering length a, as indicated in the figure.

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