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Master thesis

Compactification of $D = 11$ supergravity on $S^4 \times T^3$

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Abstract

We have a look at compactification as a special way of explaining why we only observe 4 spacetime dimensions although theories as string or M-theory require more. In particular, we treat compactification of 11-dimensional supergravity, which is the low energy limit of M-theory, on $S^4 \times T^3$. At first, we present the basic ideas behind those theories and compactification. Then we introduce important concepts of supersymmetry and supergravity. Afterwards, our particular case of compactification is treated, where we first review the results for the compactification on S^4 before we calculate the scalar potential from the reduction ansatz for a so-called twisted torus.

Sammanfattning

Vi betraktar kompaktifiering som ett särskilt sätt att förklara varför vi bara observerar 4 rumtidsdimensioner även fast teorier som sträng- eller M-teori kräver fler. I synnerhet behandlar vi kompaktifiering av 11-dimensionell supergravitation, vilken är lågenergigränsen för M-teori, på $S^4 \times T^3$. Först presenterar vi de grundläggande idéerna bakom dessa teorier och kompaktifiering. Sedan introducerar vi viktiga begrepp för supersymmetri och supergravitation. Därefter behandlas vårt särskilda fall av kompaktifiering, varvid vi först granskar resultaten av kompaktifiering på S^4 innan vi beräknar skalärpotentialen från reduktionsansatsen för en så kallad vriden torus.

Zusammenfassung

Wir betrachten Kompaktifizierung als eine besondere Möglichkeit, zu erklären weshalb wir nur 4 Raumzeitdimensionen beobachten, obwohl Theorien wie String- oder M-Theorie mehr benötigen. Insbesondere behandeln wir die Kompaktifizierung von 11-dimensionaler Supergravitation, die der niederenergetische Grenzfall von M-Theorie ist, auf $S^4 \times T^3$. Als erstes präsentieren wir die grundlegenden Ideen, die hinter diesen Theorien und Kompaktifizierung stehen. Anschließend wird unser spezifischer Fall von Kompaktifizierung behandelt, wobei wir zuerst die Ergebnisse der Kompaktifizierung auf S^4 wiedergeben, bevor wir das skalare Potential mit Hilfe des Reduktionsansatzes für einen sogenannten verdrehten Torus berechnen.

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1 Introduction

In this thesis we will have a look at compactification of the low energy limit of M-theory, which is known to be 11-dimensional supergravity. Compactification is a procedure to come from a higher-dimensional theory to a lower-dimensional theory by replacing some of the dimensions of the original theory by a small compact manifold. Therefore, it is an important tool for connecting theories as string or M-theory with phenomenology since it leads to explanations why only our four spacetime dimensions are directly observed. In this thesis we will investigate the compactification on $S^4 \times T^3$.

The current chapter provides a conceptual introduction to string theory, M-theory and compactification. Before we present them, we will motivate those theories by the desire for a consistent theory of quantum gravity. Further, we will give a short example for a compactification in string theory, i.e. Kaluza-Klein reduction of a bosonic string. T-duality will be mentioned as an interesting feature of string theory.

In the subsequent chapter we will learn about supergravity. As an preparation for this, we will first introduce important concepts of supersymmetry, i.e. superalgebras, spinors in arbitrary dimensions, on- and off-shell supersymmetry and supermultiplets. Afterwards, we will shortly present the first order formulation of general relativity, 11-dimensional supergravity, coset spaces and superspace.

Chapter 3 deals with our particular example of compactification. At first, we will review the Kaluza-Klein reduction of $D = 11$ supergravity on S^4. This means that we will state the ansätze for the linearised and the non-linearised cases. We will also look on the resulting 7-dimensional gauged supergravity. Following this, we will introduce the idea of a twisted torus before we perform the reduction of 7-dimensional supergravity on such a torus, i.e. on T^3. Our aim is to obtain the scalar potential from this reduction.

In the last chapter we will summarise our conclusions and dare an outlook.

1.1 Quantum gravity

The 20th century brought us two pillars of modern physics: Einstein's theory of gravity - the so-called general relativity - and quantum theory describing physics on atomic and subatomic level. Each of these theories offered new spectacular insights into nature and both have been tested experimentally very well. In most situations one can ignore either one of those because classical gravity is too weak to be relevant for basically all processes on quantum level while quantum physical effects are irrelevant if one wants to describe the behaviour of the majority of large and massive objects in the universe, e.g. stars or planets. Nevertheless, it is proposed that there could be objects that fall into the regime of both theories. Those would be infinitely small but accompanied by an infinite curvature of spacetime. Prominent examples are black holes and our universe at the moment of big bang (or big crunch). According to Einstein's theory, those would be spacetime singularities. This means that a very large amount of mass is gathered at one point in spacetime which leads to a local infinite mass density. Infinities in physics are usually considered to hint at a limitation of a theory. In the case of spacetime singularities, a quantum theory of gravity, which would be an UV-completion of general relativity, might be able to avoid their appearance. However, general relativity and quantum physics do not go hand in hand. In fact, until today there is no consistent framework that is proven to be a correct unification of both theories. Certainly, there are various attempts to quantum gravity but besides huge theoretical challenges, it is intricate to verify them experimentally. Direct quantum gravitational experiments seem to be impossible at all. Therefore, one tries to make indirect observations of quantum gravitational effects, e.g. in cosmology.

1.2 String theory

Currently, string theory is a leading approach to quantum gravity. It naturally includes the graviton which is proposed to be the force carrier of gravity. In addition, string theory also addresses open questions in the standard model of particle physics and could be a candidate for a theory unifying all known fundamental forces in nature. It is often appreciated for its usage of beautiful mathematics and led to new inspiration for several mathematical areas, e.g. topology. Since the turn of the millennium, string theoretical concepts additionally found application in the fields of superconductivity, early universe cosmology, fluid mechanics and quantum information [1].

In string theory one considers extended one-dimensional objects called strings instead of point-like particles. They are able to vibrate as for example a guitar string. Depending on its vibrational mode, we interpret a string as a particle with specific properties that we can observe, e.g. as an electron. Strings can be open or closed like a circle. Open strings belong to the Yang-Mills sector (includes gauge fields) and closed strings belong to the gravity sector. Due to the question of its mathematical consistency, string theory has to be formulated in more than our four well-known spacetime dimensions. The easiest example is the bosonic string theory in 26 dimensions ($D = 26$). However, it contains only bosons and in addition a hypothetical type of particle, the so-called tachyon, which is proposed to have a negative mass-squared. The latter is considered to be unnatural. Such issues are solved by incorporating supersymmetry in string theory. Supersymmetry assigns a fermionic (bosonic) 'superpartner' to each boson (fermion) of the standard model. In this way the tachyon is removed from the theory but, as desired, fermions appear. The resulting theories are named superstring theories and describe bosonic and fermionic particles. They are formulated in $D = 10$ dimensions.

1.3 M-theory

There are five known $D = 10$ superstring theories: [2]

- Type I,
- Type IIA,
- Type IIB,
- $E_8 \times E_8$ heterotic,
- $SO(32)$ heterotic.

Each of these theories is acceptable as a theory of quantum gravity and preference is only given in particular situations. For instance, the E-heterotic (abbreviation for $E_8 \times E_8$ heterotic) theory is preferred in attempts to make contact with particle physics [2]. Nonetheless, five theories of quantum gravity which are equally good in fulfilling their purpose would go over the top. Hence, one could expect that there is a more fundamental theory of which each superstring theory is a special limit. Indeed, nowadays most string researchers are convinced that this theory is existing. It is called M-theory, whereby the meaning of the M is ambiguous. M-theory is formulated in 11 spacetime dimensions and is still under construction since it is only known in bulk. So far one knows that its fundamental objects are so-called p-branes, which also appear in string theories. A p-brane is just a p-dimensional physical object with certain properties. For instance, a 1-brane is a fundamental string as we know it from string theory and a 2-brane is a 2-dimensional surface. In principle, branes are only generalisations of the concept of a point particle for arbitrary dimensions. M-theory is governed by 2-branes and 5-branes as their magnetic duals. Well-known are also the procedures to come from M-theory to type IIA or E-heterotic string theory. Dualities connect the other three types of superstring theories to those two. Furthermore, one knows that M-theory's low energy limit is 11-dimensional supergravity - a theory of gravity predicted by supersymmetry [3, 4].

1.4 Why compactify?

A striking property of string and M-theory is that they are usually formulated in more than our four spacetime dimensions. However, we do not encounter these extra dimensions in our daily life and so far we did not observe them in any experiment. So the questions arise where and how these additional dimensions are hiding. If string and M-theory claim to be theories of our real physical world, they are required to provide an answer to those questions. The most common answer is compactification of the extra dimensions. This means that, in contrast to our four spacetime dimensions, these do not have infinite extension and have the shape of a particular compact manifold, e.g. a sphere or a torus. Strings or branes can then be wrapped around them. Depending on the size of the compact dimension and the number of windings of a string or brane around it, we should be able to observe various quantised energy states which we can interpret as different particles in our experiments. In order for these states to correspond to particles we already know, the compact dimensions have to be in the order of the Planck length (around 10^{-35}m) [4]. This would elegantly explain why we did not directly observe any extra dimensions yet and why they do not even seem to play a role on the atomic level of our world.

To get an idea of a compactified dimension one can think about a theory proposed by Theodore Kaluza and Oscar Klein in the early 1920s [5, 6]. It was an attempt to unify gravity and electromagnetism by taking a fourth space dimension into account. This additional dimension was thought to be shaped as a circle with very small radius. One can imagine the resulting 5-dimensional spacetime by replacing all points in our ordinary 3-dimensional space by a circle. Since it is so small, we cannot observe the extra dimension directly. To imagine other compactifications one could just replace each single point in space with the respective compact manifold.

1.5 Kaluza-Klein reduction of bosonic string theory

As an example for a compactified theory, we will now have a look at the compactification of $D = 26$ closed bosonic string theory on a circle. We will follow [7] and [8]. In particular, we will mostly make use of the notation and elaboration in [8].

1.5.1 Kaluza-Klein reduction in field theory

At first, we discuss compactification in field theory as a good approximation since stringy behaviour does not show up if we consider a compact dimension with a size much larger than the string length scale $l_s := \sqrt{\alpha'}$.

We start considering a 26-dimensional field theoretical toy model of a massless scalar $\phi(x^0, ..., x^{25})$ on a manifold M_{26} and compactify it on a circle with radius $R \gg l_s$. As the compactified dimension we choose x^{25}. Since it is compactified on a circle, it fulfils $x^{25} \equiv x^{25} + 2\pi R$. Hence, we can Fourier expand ϕ in terms of x^{25}:

$$\phi(x^0, ..., x^{25}) = \sum_{k \in \mathbb{Z}} \exp(ikx^{25}/R)\phi_k(x^0, ..., x^{24}) \ , \tag{1.1}$$

where the 2π in the numerator in the exponent got cancelled out by the circumference of the circle.

In the 25-dimensional space M_{25} we would in principal observe each single field $\phi_k(x^0, ..., x^{24})$. However, we will now see that in the case of low energies only one field would be observable.

The momentum component of a field $\phi_k(x^0, ..., x^{24})$ in the compactified dimension is given by $p_{25} = \frac{k}{R}$. By using the mass-shell condition

$$0 = p^2 = p^2_{M_{25}} + p^2_{25} \ , \tag{1.2}$$

and $p^2_{M_{25}} = -m^2$, we find: $m_k = \frac{k}{R}$.

The observable mass of a scalar field is anti-proportional to the radius of the circle. This means the smaller the circle the larger the field's mass. Hence, if our experiments reach only energies $E \ll \frac{1}{R}$, then we cannot observe any massive fields of this theory. So only the field ϕ_0 would be observable for us because it is massless. In this way physics of 25 dimensions is reproduced for low energies by compactification of a 26-dimensional spacetime. This particular procedure is called Kaluza-Klein reduction.

1.5.2 Closed bosonic string

Now we finally go to bosonic string theory in 26 dimensions. We consider a closed bosonic string on a 26-dimensional manifold M_{26} and then compactify the theory on a circle such that we obtain a manifold $M_{25} \times K_R$, where K_R is a circle of radius R. We choose again the 25th space dimension as the compact dimension. Since the metric of the resulting space is still flat (a circle as a 1-torus is locally flat), the Polyakov worldsheet action remains the same after compactification:

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \ , \tag{1.3}$$

where $\eta_{\mu\nu}$ is the flat Minkowski metric, $X^\mu(\sigma, \tau)$ are worldsheet fields, $g^{\alpha\beta}$ is the worldsheet metric and $g := \det(g)$. The worldsheet is the 2-dimensional surface spanned by a string propagating in time.

Next, we want to investigate the mass spectrum of the bosonic string in the compactified theory. The hamiltonian of the uncompactified theory looks like:

$$H = \frac{l}{4\pi\alpha' p^+} \int_0^l d\sigma [2\pi\alpha' \Pi^i \Pi^i + \frac{1}{2\pi\alpha'} \partial_\sigma X^i \partial_\sigma X^i] \ , \tag{1.4}$$

where l is the length of the string, p^+ is a longitudinal momentum and Π^i is the momentum density conjugate to X^i.

Now we have to incorporate the boundary conditions of a closed string. For the dimensions $i = 1, ..., 24$ they are:

$$X^i(\sigma + l, \tau) = X^i(\sigma, \tau) \ . \tag{1.5}$$

Since we compactify the 25th space dimension on a circle, we have to take the periodicity into account by adding $2\pi Rw$ with $w \in \mathbb{Z}$ as a winding number. This number counts how often the string is winding around the circle. The more often a string is winded around, the higher is the energy of the corresponding state. We get:

$$X^{25}(\sigma + l, \tau) = X^{25}(\sigma, \tau) + 2\pi Rw \ . \tag{1.6}$$

By using the mode expansion for the boundary conditions for X^{25},

$$X^{25}(\sigma, \tau) = x^{25} + \tfrac{p_{25}}{p^+}\tau + \tfrac{2\pi Rw}{l}\sigma + i\sqrt{\tfrac{\alpha'}{2}} \sum_{n \in \mathbb{Z}\setminus\{0\}} \left[\tfrac{\alpha_n^i}{n} e^{-2\pi i n(\sigma+\tau)/l} + \tfrac{\tilde{\alpha}_n^i}{n} e^{2\pi i n(\sigma-\tau)/l}\right] ,$$

and using the result that p_{25} has to be quantised as $p_{25} = \frac{k}{R}$ with $k \in \mathbb{Z}$, one obtains as the hamiltonian for the compactified theory (after integration):

$$H = \sum_{i=2}^{24} \frac{p_i^2}{2p^+} + \frac{(k/R)^2}{2p^+} + \frac{R^2 w^2}{2\alpha'^2 p^+} + \frac{1}{\alpha' p^+}(N + \tilde{N} - 2) \ , \tag{1.7}$$

where the N, $\tilde{N}$ are the level numbers of the state.

The mass spectrum is calculated by the following formula:

$$m^2 = 2p^+ H - \sum_{i=2}^{24} p_i^2 \ . \tag{1.8}$$

For the uncompactified theory this gives:

$$m^2_{uncomp.} = \frac{2}{\alpha'}(N + \tilde{N} - 2) \ . \tag{1.9}$$

For the compactified theory we obtain for the mass in 25 dimensions:

$$m^2_{comp.} = \frac{k^2}{R^2} + \frac{R^2}{\alpha'^2} w^2 + \frac{2}{\alpha'}(N + \tilde{N} - 2) \ . \tag{1.10}$$

If we compare both mass spectra, then we see that they differ by a summand $\frac{k^2}{R^2} + \frac{R^2}{\alpha'^2} w^2$. This means that in addition to the states in the uncompactified case we have momentum (correspond to k) and winding states (correspond to w).

As we stated for the field theoretical Kaluza-Klein reduction, stringy effects only appear if we consider a radius in the order of the string scale l_s. In any other case we would not have any winding states in our spectrum since the compact dimension would be too large for the string to wind around it. We can see the effect of R's value if we have a look at the massless states. In the non-compact theory massless states correspond to $N = \tilde{N} = 1$ while in the compactified theory we could have more massless states coming from the winding in the case $k = N = \tilde{N} = 0$ with $w = \frac{2\sqrt{\alpha'}}{R}$. Here we observe that for $R > 2\sqrt{\alpha'}$ the condition $w \in \mathbb{Z}$ cannot be fulfilled anymore. Hence, the string cannot wind entirely around the compact dimension and the stringy effects disappear for large values of R.

1.6 T-duality

In the end of this chapter we want to have a short look at a remarkable feature of string theory - T-duality. A theory on a circle K_R is T-dual to a theory on a circle $K_{\alpha'/R}$ if both theories describe the same physics.

If we consider the mass spectrum we derived for the compactified theory, then we see that it is invariant under a T-duality transformation $R \to \frac{\alpha'}{R}$:

$$m^2_{comp.} \xrightarrow{T} \frac{R^2}{\alpha'^2} k^2 + \frac{w^2}{R^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2) \ . \tag{1.11}$$

k and w exchanged roles but both can take the same values since $k, w \in \mathbb{Z}$. Therefore, the spectrum is still the same after the transformation. We can just rename them, $w \leftrightarrow k$, and obtain exactly the same spectrum as before.

This result is quite interesting in the limit $R \to \infty$ (large circle in the original theory and small circle in the T-dual theory). If we consider pure momentum states (k-term is the only contribution) before the transformation, then we see that they become massless in this limit. This corresponds to a decompactification of the theory. This makes sense since the compactified theory becomes infinitely large in this limit. However, if we consider pure winding states of the T-dual theory in this limit, then we see that they become massless instead. By defining the new radius $R' := \alpha'/R$, we see that the circle in the T-dual theory becomes 0-dimensional. This is again a decompactification in that sense that the compactified dimension vanishes.

2 Supersymmetry and supergravity

In this section we want to take a more detailed look at supersymmetry (abbreviated as SUSY) and supergravity (abbreviated as SUGRA) because we will make heavy use of them in the rest of this thesis.

As references for this section we will use the introductions in [9, 11]. In particular, we will use their notations and we will always refer to them unless we explicitly give another reference. [9] are lecture notes by Horatiu Nastase. The content we used from those notes is based on the references [15] and [25] - [39].

A far more detailed introduction to SUSY and SUGRA can be found for example in [12].

2.1 Supersymmetry

Supersymmetry describes a symmetry between fermions and bosons. As already mentioned earlier in this thesis, it assigns a fermionic (bosonic) superpartner to each boson (fermion) of a supersymmetric theory (e.g. the SUSY-extended standard model). It is a powerful tool to simplify certain calculations and arose from considerations about the possible symmetries in particle physics.

There exists a special convention for naming superpartners. Bosonic superpartners of standard model fermions are named like these fermions but with an additional s as first letter (e.g. squarks for quarks or selectron for an electron). Fermionic superpartners of standard model bosons are named like these bosons but with -ino as suffix (e.g. gluino for a gluon or gravitino for a graviton). Following the rules for Italian masculine suffixes, we denote the plural of a fermionic superpartner with -ini as ending (e.g. gluini or gravitini).

In the following we will introduce the concepts of supersymmetry that are relevant for this thesis.

2.1.1 Superalgebra

At first, we introduce the concept of a superalgebra or graded Lie algebra. Such an algebra distinguishes between even and odd generators. While for example the Lorentz generators $M_{\mu\nu}$ of the $SO(1,3)$ Lorentz group, the generators of translation symmetries P_μ and the generators of internal symmetries of particle physics T_r are considered to be even, the novel supercharges Q_a^i (irreducible spinors) are considered to be odd. Using this classification, the generators of the graded Lie algebra fulfil:

$$[\text{even}, \text{even}] = \text{even}, \quad \{\text{odd}, \text{odd}\} = \text{even}, \quad [\text{even}, \text{odd}] = \text{odd} \ . \tag{2.1}$$

The supercharges Q_α^i are in the spinor representation of the Lorentz group, where α is a spinor index and $i \in \{1, ..., \mathcal{N}\}$ is an internal index. $\mathcal{N}$ denotes the number of different supercharges we can have in the theory. They fulfil the supersymmetry algebra: [10]

$$\{Q_\alpha^i, Q_\beta^j\} = 2(\gamma^\mu C)_{\alpha\beta}\delta^{ij}P_\mu \ , \tag{2.2}$$

where C is the charge conjugation matrix. Acting with Q_α^i on a boson field gives a spinor field and vice versa. More explicitly, Q_α^i relates two fields whose helicity differs by 1/2. In this way the supercharges give us supersymmetry as a symmetry between bosons and fermions:

$$\delta\text{boson} = \text{fermion} \ , \quad \delta\text{fermion} = \text{boson} \ . \tag{2.3}$$

The symmetry variation δ is given by

$$\delta_\epsilon = \epsilon^\alpha Q_\alpha \ , \tag{2.4}$$

where ϵ is the variation or symmetry law parameter.

2.1.2 Spinors in D dimensions

Here we will discuss spinors in various dimensions since later we will consider them in theories with $D > 4$. Many details about spinors in higher dimensions can be found in [16].

In every dimension there is always a so-called spinor representation (or Dirac representation) for the Lorentz group $SO(1, D-1)$, where the Lorentz generators are defined by

$$\gamma_{\mu\nu} := \frac{1}{2}[\gamma_\mu, \gamma_\nu] \ . \tag{2.5}$$

The Dirac matrices are satisfying the Clifford algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} \ , \tag{2.6}$$

where $g_{\mu\nu}$ is the $SO(1, D-1)$ invariant metric.

These matrices take spinors into spinors:

$$(\gamma_\mu)^\alpha{}_\beta \chi^\beta = \tilde{\chi}^\alpha \ . \tag{2.7}$$

Dirac spinors have $2^{[[D/2]]+1}$ real components, where $[[x]]$ is the integer part of x. However, the Dirac representation is reducible. Therefore, we must impose either the Weyl (chirality), the Majorana (reality) or even both conditions (Majorana-Weyl spinors) in order to obtain irreducible representations.

The Weyl condition is

$$\gamma_{D+1}\chi = \pm\chi \ , \tag{2.8}$$

where $+$ $(-)$ stands for right-(left-)handed spinors ,

while the Majorana condition is

$$\chi^C := \chi^T C = \bar{\chi} := \chi^\dagger i\gamma_0 \ , \tag{2.9}$$

where C is the charge conjugation matrix relating γ_m with γ_m^T.

The Weyl spinor condition exists only in even dimensions. If we cannot define a Majorana condition in $\mathcal{N} > 1$ cases, then we can at least define a symplectic Majorana condition for spinors containing $\mathcal{N} = 1$ spinors as components. We cannot define a usual Majorana condition in the case of pseudo-reality, which means that there is no complex conjugate of a spinor, i.e. $(\chi^*)^* = -\chi$.

Different from Dirac spinors, Majorana and chiral spinors have $2^{[[D/2]]}$ real components. Albeit, Majorana-Weyl spinors have even only $2^{[[D/2]]-1}$ real components. [16]

For convenience one usually uses Majorana spinors in supersymmetry.

The parameter of the SUSY transformation law ϵ^i_α (e.g. in $D = 2$ Wess-Zumino model $\delta\phi = \bar{\epsilon}_\alpha \psi^\alpha$) is also a spinor.

2.1.3 On- and off-shell SUSY

Supersymmetry is a symmetry between bosons and fermions. Therefore, the number of degrees of freedom of bosons has to be the same as for fermions. They can match on-shell or off-shell, which corresponds to on-shell supersymmetry or off-shell supersymmetry, respectively.

In general, supersymmetry is formulated on-shell. For example, in the simple $D = 2$ Wess-Zumino model we can consider one Majorana fermion ψ with one degree of freedom on-shell and one real scalar ϕ with as well one degree of freedem on-shell. Their respective SUSY variation is given by

$$\delta\psi = \partial\!\!\!/ \phi \epsilon \tag{2.10}$$

and

$$\delta\phi = \bar{\epsilon}\psi \ . \tag{2.11}$$

However, we can also consider the $D = 2$ Wess-Zumino model off-shell. In this case a Majorana fermion has two degrees of freedom off-shell, while a real scalar still has only one degree of freedom. To cure this problem we need to introduce an additional scalar field F. This field has to be auxiliary because we want to obtain our former result if we go back to the on-shell case. Auxiliary means that this field is non-dynamical and has no propagating degree of freedom. More concretely, it has no kinetic term in the Lagrangian. Its equation of motion is given by $F = 0$.

In the off-shell formulation, the SUSY variations are

$$\delta\psi = \bar{\partial}\phi\epsilon + F\epsilon \ , \tag{2.12}$$

$$\delta\phi = \bar{\epsilon}\psi \tag{2.13}$$

and

$$\delta F = \bar{\epsilon}\bar{\partial}\psi \ . \tag{2.14}$$

In more complex models, where the mismatch in the degrees of freedom off-shell could be larger, one might have to introduce more auxiliary fields, which of course could also be vectors or tensors. However, there are also theories with high $\mathcal{N}$ in which no off-shell supersymmetry exists.

2.1.4 Supermultiplets

Supermultiplets are the representations of supersymmetry that include the superpartners. More specifically, they are the irreducible representations of the super-Poincaré algebra (see 2.2.2 for more about this algebra). This means that the supermultiplets are closed under Lorentz and SUSY transformations.

These multiplets cannot contain any fields with spin higher than 2 since there cannot be an interacting quantum field theory containing such fields. This requirement restricts the number of possible supercharges in a supersymmetrical theory because supercharges are filling the supermultiplets with fields of increasing helicity [40](eigenvalues of helicity range from $-S$ to $+S$, where S denotes the spin of a field).

Since we are mostly interested in supergravities, we will now only consider the supermultiplets which usually appear in these theories. Those are gravity, vector, chiral, hyper- and tensor multiplets. The gravity multiplets contain all on-shell SUSY fields and in particular they are the smallest multiplets containing the graviton (and its superpartner gravitino). The vector multiplets contain only fields up to spin 1 and exist only for $\mathcal{N} \leq 4$. Chiral and hypermultiplets contain the same types of fields (spin 0 and 1/2). However, in 4 dimensions, chiral multiplets exist only in $\mathcal{N} = 1$ theories, while hypermultiplets are only existing in $\mathcal{N} = 2$ theories. The tensor multiplets contain antisymmetric tensors, which can be dualised to scalars (vectors) in 4 (5) dimensions. Therefore, tensor multiplets rather appear in higher dimensions.

In the following tables we summarise the field contents of the different multiplets in 4 dimensions. These tables are adapted from [12]. Be aware that the $\mathcal{N} = 7$ gravity multiplet and the $\mathcal{N} = 3$ vector multiplet are not mentioned since they do not exist in the sense that they are not consistent and imply $\mathcal{N} = 8/\mathcal{N} = 4$, respectively.

field	spin	$\mathcal{N} = 1$	$\mathcal{N} = 2$	$\mathcal{N} = 3$	$\mathcal{N} = 4$	$\mathcal{N} = 5$	$\mathcal{N} = 6$	$\mathcal{N} = 8$
$g_{\mu\nu}$	2	1	1	1	1	1	1	1
ψ_μ	3/2	1	2	3	4	5	6	8
A_μ	1	-	1	3	6	10	16	28
λ	1/2	-	-	1	4	11	26	56
ϕ	0	-	-	-	2	10	30	70

Table 1: *Field content of the gravity multiplets in $D = 4$*

field	spin	$\mathcal{N} = 1$	$\mathcal{N} = 2$	$\mathcal{N} = 4$
A_μ	1	1	1	1
λ	1/2	1	2	4
ϕ	0	-	2	6

Table 2: *Field content of the vector multiplets in $D = 4$*

field	spin	$\mathcal{N} = 1$	$\mathcal{N} = 2$
λ	1/2	1	2
ϕ	0	2	4

Table 3: *Field content of the chiral and hypermultiplets in $D = 4$*

2.2 Supergravity

Equipped with knowledge about supersymmetry, we can now look at our major theory of interest - supergravity. At first, we will introduce another formulation of general relativity which turns out to be highly relevant for SUGRA. Subsequently, we will see how supergravity can be defined and which is its highest possible dimension. Afterwards, we will shortly introduce the concepts of coset space and superspace. In the end, 11-dimensional supergravity will be presented.

2.2.1 First order formulation of general relativity

General relativity can be seen as a gauge theory, where the Christoffel symbols $\Gamma^{\mu}{}_{\nu\rho}(g)$, defined by the metric, act like gauge fields of gravity and the Riemann tensors $R^{\mu}{}_{\nu\rho\sigma}(\Gamma)$ are their field strengths. However, general relativity can also be formulated as a gauge theory in terms of other objects. Those objects are the vielbein e_{μ}^{a} and the spin connection ω_{μ}^{ab}. In the following, we will discuss these two objects.

Locally any space is flat on a scale much smaller than its curvature. So spacetime can be described in terms of locally inertial frames. This means that we have a local Lorentz invariance. Such inertial frames are described by vielbeine e_{μ}^{a} which are defined by

$$g_{\mu\nu}(x) = e_{\mu}^{a}(x)e_{\nu}^{b}(x)\eta_{ab} \ , \tag{2.15}$$

where η_{ab} is the Minkowski metric, μ is a curved index (general coordinate transformations) and a is a new so-called flat index (local Lorentz gauge transformations). Sometimes a is also referred to as internal index. The metric $g_{\mu\nu}$ and the vielbein e_{μ}^{a} have the same number of degrees of freedom and therefore descriptions in terms of any of these two objects are equivalent.

If we want to operate on objects with internal indices as a, we need to have a corresponding derivative. Therefore, we introduce the spin connection ω_{μ}^{ab} as connection for the action of the Lorentz group on spinors.

The covariant derivative on spinors is given by

$$D_{\mu}\psi = \partial_{\mu}\psi + \frac{1}{4}\omega_{\mu}^{ab}\gamma_{ab}\psi \ , \tag{2.16}$$

and the covariant derivative on objects with mixed indices is given by

$$D_\mu G^a_\nu = \partial_\mu G^a_\nu + \omega^{ab}_\mu G^b_\nu - \Gamma^\rho{}_{\mu\nu} G^a_\rho \ . \tag{2.17}$$

One usually demands $D_\mu e^a_\nu = 0$ (vielbein postulate).

We can construct a field strength for the spin connection:

$$R^{ab}_{\mu\nu}(\omega) = \partial_\mu \omega^{ab}_\nu - \partial_\nu \omega^{ab}_\mu + \omega^{ab}_\mu \omega^{bc}_\nu - \omega^{ac}_\nu \omega^{cb}_\mu \ . \tag{2.18}$$

It can be related to our well-known Riemann tensor by

$$R^{ab}_{\rho\sigma}(\omega(e)) = e^a_\mu e^{\nu b} R^\mu{}_{\nu\rho\sigma}(\Gamma(e)) \ , \tag{2.19}$$

which means that this new field strength is just the Riemann tensor with two flattened indices. $e^{\nu b}$ is the inverse vielbein with raised flat index b.

The local Lorentz transformations and translations generate the Poincaré algebra $\mathfrak{iso}(1,3)$

$$[M_{\mu\nu}, M^{\rho\sigma}] = -2\delta^{[\rho}_{[\mu} M_{\nu]}{}^{\sigma]} \ , \tag{2.20}$$

$$[P_\mu, M_{\nu\rho}] = \eta_{\mu[\nu} P_{\rho]} \ , \tag{2.21}$$

$$[P_\mu, P_\nu] = 0 \ , \tag{2.22}$$

where $M_{\mu\nu} = M_{[\mu\nu]}$ and P_μ are again the generators of the $SO(1,3)$ Lorentz transformations and the translations, respectively.

General relativity can be obtained in the so-called first order formulation as a gauge theory by gauging the $\mathfrak{iso}(1,3)$ algebra by using the vielbein and the spin connection as gauge fields [41, 42, 43]. First order formulation basically means that the action (here Einstein-Hilbert) becomes first order in derivatives of propagating fields.

2.2.2 From GR to SUGRA and degrees of freedom

There are two ways to define a supergravity: Either we supersymmetrise general relativity or we take a supersymmetric model and make its supersymmetry local.

To supersymmetrise general relativity we extend its local bosonic spacetime symmetries to superalgebras. In the case of the aforementioned $\mathfrak{iso}(1,3)$ algebra, we promote it to the super-Poincaré algebra by adding (2.2) and the following bracket relations:

$$[M_{\mu\nu}, Q^i{}_\alpha] = -\frac{1}{4}(\gamma_{\mu\nu})_\alpha{}^\beta Q^i{}_\beta \ , \tag{2.23}$$

$$[P_\mu, Q^i{}_\alpha] = 0 \ . \tag{2.24}$$

To make a supersymmetry local we have to make the transformation parameter ϵ^α local. Of course we also have to introduce a covariant derivative containing a new gauge field. It turns out that this gauge field is in fact $\psi_{\mu\alpha}$, which is the gravitino - the supersymmetric partner of the graviton. Here μ denotes a curved index and α is a flat index, i.e. a local Lorentz spinor index.

The gravitino is a spin 3/2 field. Since it is the supersymmetric partner of the graviton, there has to be a SUSY transformation of the kind $\psi_{\mu\alpha} = Q_\alpha(\text{graviton})$. Usually we interpret the metric as the graviton field. However, the index structure of this transformation urges us to conclude that we need an object with only one index instead. Since we know that descriptions in terms of the metric or the vielbein are equivalent, we can interpret the vielbein as the graviton field instead. For this reason, the first order formulation is quite important in supergravity.

As a supersymmetric theory SUGRA has to fulfil the condition that the number of degrees of freedom of its fermions matches the one of its bosons. We will now state the on-shell number of degrees of freedom for different types of SUGRA fields in D dimensions. The letter n denotes here the number of components of a spinor (see 2.1.2 for spinor components in D dimensions).

Scalar	1
Gauge field A_μ	$D-2$
Graviton $g_{\mu\nu}$	$(D-1)(D-2)/2-1$
$S=1/2$ spinor	$n/2$
Gravitino ψ_μ^α	$(D-3)n/2$
Antisymmetric tensor $A_{\mu_1\ldots\mu_r}$	$(D-2)\ldots(D-1-r)/(1\cdot 2\cdot\ldots\cdot r)$

Table 4: *Degrees of freedom for different fields in D dimensions*

2.2.3 The highest dimension of supergravity

In theories with local supersymmetry as supergravity, the maximal amount of supercharges (spinor components) is 32. Therefore, a supergravity with 32 supercharges is called maximal supergravity. A SUGRA with 16 supercharges is called half-maximal. Minimal supergravities are those which are $\mathcal{N} = 1$ and extended supergravities are those which are $\mathcal{N} > 1$.

The closure of the supermultiplets under SUSY transformations is the reason for the limitation to maximal 32 supercharge components in any theory with local SUSY. In 4 dimensions any supercharge changes the helicity by $1/2$. This means that the $\mathcal{N} = 8$ case, which corresponds to 32 supercharge spinor components, already covers the helicity range from -2 to $+2$. Additional supercharges would require fields with spin $S \geq 5/2$. However, for such fields there are no known consistent interactions. [16]

Supergravities are not formulated in dimensions higher than 11 because any higher dimension would lead after dimensional reduction on a torus to $\mathcal{N} \geq 8$ in 4 dimensions. This would be in contrast to the statement that the $\mathcal{N} = 8$ SUGRA is the maximal supergravity in 4 dimensions. A detailed explanation of this can be found in [12]. In short, it says that a toroidal dimensional reduction of $D \geq 12$ supergravity would lead to an extension of the gravity multiplet such that it also contains fields with spin of at least $5/2$, i.e. additional gravitons with higher spin would have to be introduced. However, as explained above, this is not possible. [12]

Therefore, $D = 11$ is the maximal dimension for supergravity. The $D = 11$ SUGRA is unique since its simplest (Majorana) spinor has 32 components and so only $\mathcal{N} = 1$ is permitted. In this way, it is simultaneously maximal and minimal.

2.2.4 $D = 11$ supergravity

Now we introduce $D = 11$ supergravity in more detail since it is the one we will dimensionally reduce in the next section.

Since it has a maximal supersymmetry, its field content is restricted to a single massless supermultiplet, which is the gravity multiplet. Therefore, 11-dimensional supergravity contains the vielbein (graviton) ${e_\mu}^m$ and a Majorana-gravitino $\psi_{\mu\alpha}$. However, the graviton has only 44 degrees of freedom while the gravitino has 128. This means that bosonic and fermionic degrees would not match, which cannot be. Hence, there has to be another field. We see that an antisymmetric 3-tensor field $A_{\mu\nu\rho}$ would close that gap because it has exactly 84 degrees of freedom (see table 4). So the field content of 11-dimensional supergravity is $\{{e_\mu}^m, A_{\mu\nu\rho}; \psi_{\mu\alpha}\}$.

We can define a field strength for the 3-tensor field

$$F_{\mu\nu\rho\sigma} = 24\partial_{[\mu}A_{\nu\rho\sigma]} := \partial_\mu A_{\nu\rho\sigma} + 23 \text{ terms} . \tag{2.25}$$

Here we used the anti-symmetrisation notation for tensor indices (see for example [13]). It is defined by

$$A_{[\mu_1 \ldots \mu_r]} := \frac{1}{r!} \sum_{P \in S_r} \operatorname{sgn}(P) A_{\mu_{P(1)} \ldots \mu_{P(r)}} , \tag{2.26}$$

where S_r is the symmetric group of order r with permutations P of subindices $1, ..., r$ as elements. The sign of P is defined by

$$\operatorname{sgn}(P) := \begin{cases} +1 & \text{if } P \text{ is even} \\ -1 & \text{if } P \text{ is odd} \end{cases} . \tag{2.27}$$

Likewise, we can also define a symmetrisation for tensor indices by

$$A_{(\mu_1 \ldots \mu_r)} := \frac{1}{r!} \sum_{P \in S_r} A_{\mu_{P(1)} \ldots \mu_{P(r)}} . \tag{2.28}$$

The Lagrangian of $D = 11$ SUGRA is given by

$$\begin{aligned} \mathcal{L} = &- \frac{e}{2k^2} R(e, \omega) - \frac{e}{2} \bar{\psi}_\mu \Gamma^{\mu\nu\rho} D_\rho \left(\frac{\omega + \hat{\omega}}{2} \right) - \frac{e}{48} F^2_{\mu\nu\rho\sigma} \\ &- \frac{\sqrt{2}}{384} k [\bar{\psi}_\mu {\Gamma^\mu}_{\alpha\beta\gamma\delta\nu} \psi^\nu + 12 \bar{\psi}^\alpha \Gamma^{\beta\gamma} \psi^\delta] (F_{\alpha\beta\gamma\delta} + \hat{F}_{\alpha\beta\gamma\delta}) \\ &- \frac{\sqrt{2}}{6 \cdot (24)^2} k \epsilon^{\mu_1 \ldots \mu_{11}} F_{\mu_1 \ldots \mu_4} F_{\mu_5 \ldots \mu_8} A_{\mu_9 \mu_{10} \mu_{11}} , \end{aligned} \tag{2.29}$$

where $e := \det({e_\mu}^m)$, k is the Newton constant, $\epsilon^{\mu_1 \ldots \mu_{11}}$ is the Levi-Civita tensor in 11 dimensions and Γ denotes a gamma matrix in 11 dimensions.

More precisely, the 11-dimensional gamma matrices are defined by (see [15])

$$\Gamma_a = \tau_a \otimes \gamma_5 \tag{2.30}$$

for $a = 0, ..., 6$ and

$$\Gamma_{6+m} = \mathbb{I} \otimes \gamma_m \tag{2.31}$$

for $m = 1, ..., 4$. τ is a 7-dimensional and γ a 4-dimensional "gamma" matrix.

Furthermore, we use

$$\Gamma_{M_1...M_r} := \Gamma_{[M_1}...\Gamma_{M_r]} \ . \tag{2.32}$$

A hat ˆ denotes a supercovariant object. Such an object's local SUSY transformation does not contain any $\partial_\mu \epsilon$ term.

The supercovariant extensions of $\omega(e)$ and $F_{\alpha\beta\gamma\delta}$ are given by

$$\hat{\omega}_{\mu mn}(e) = \omega_{\mu mn}(e) + \frac{1}{8}(\bar{\psi}^\alpha \Gamma_{\alpha\mu mn\beta}\psi^\beta) \tag{2.33}$$

and

$$\hat{F}_{\alpha\beta\gamma\delta} = 24 \left[\partial_{[\alpha} A_{\beta\gamma\delta]} + \frac{1}{16\sqrt{2}} \bar{\psi}_{[\alpha} \Gamma_{\beta\gamma} \psi_{\delta]} \right] \ . \tag{2.34}$$

2.2.5 Coset spaces

Certain manifolds can be expressed as a coset $G/H = \{g \in G \mid g \sim gh, h \in H\}$ of a continuous group G and a continuous subgroup $H \subset G$. Such a manifold is called coset manifold or coset space. By multiplying with a group element $g \in G$ we can move from one point on the manifold to another.

A famous example for a coset manifold is the n-sphere

$$S^n = \frac{SO(n+1)}{SO(n)} \ , \tag{2.35}$$

where $SO(n+1)$ is the group of invariances of the sphere and $SO(n)$ is the group of rotations that leave a point of the sphere invariant.

Definition 1 *A maximal compact subgroup $H_{\max}$ is a group with compact topology and fulfilment of the following maximality condition:*

$$H_{\max} \subseteq H \subseteq G \Rightarrow H_{\max} = H \vee H = G \ .$$

The fields of maximal and half-maximal supergravities transform in certain irreducible representations of their global symmetry group G. If $H_{\max}$ denotes a maximal compact subgroup of G, then all physical scalar degrees of freedom of those SUGRAs span the coset space $G/H_{\max}$. Those scalars are divided into dilatons and axions and their total number equals the dimension of the coset space. The number of dilatons is #dilatons $= 11 - D$, where D denotes as usual the number of dimensions of the supergravity theory. This means that after a reduction from the SUGRA in the highest possible dimension we have a dilaton for each reduced dimension. It further means that there are no dilatons in 11 dimensions. The number of axions is thus just #axions $= D_{coset} + D - 11$, where D_{coset} is the coset space dimension.

2.2.6 Superspace

In principle, supersymmetry and therefore also supergravity can be formulated by using the superspace formalism. We will not make use of it in our further treatment but since it is in general an important concept, we will quickly introduce it here. A far more detailed introduction can be found for example in [35].

The superspace formalism is a formalism with built-in supersymmetry. In practice this means that instead of fields that only depend on the position, we will consider fields in a space with fermionic coordinates θ^A besides the bosonic x^μ. This is done in such a way that SUSY is manifest and such fields are called superfields.

The coordinates θ^A are just ordinary Grassmann numbers, satisfying

$$\{\theta, \theta\} = 2\theta^2 = 0 \ , \tag{2.36}$$

which means that a Taylor expansion of an arbitrary function $f(\theta)$ is maximal of order θ:

$$f(\theta) = a + b\theta \ . \tag{2.37}$$

$D = 4$ superspace can be defined as a coset

$$\frac{\text{super-Poincaré group}}{\text{Lorentz group}} \tag{2.38}$$

because it is invariant under the super-Poincaré group and a Lorentz transformation does not change any of its points.

In the superspace formalism, two new objects called Supervielbein $E_M^A(x, \theta)$ and super-spin connection $\Omega_M^{AB}(x, \theta)$ appear, which are just the superspace analogues of vielbein and spin connection. Hence, they can also be used to define a covariant derivative on the superspace. The description of general relativity generalised to superspace by using the supervielbein and super-spin connection is named super-geometric approach.

3 Compactification on $S^4 \times T^3$

Here we will perform a compactification of 11-dimensional supergravity on $S^4 \times T^3$. We split our treatment into two parts. At first, we review the Kaluza-Klein reduction of $D = 11$ SUGRA on S^4 to the maximal $D = 7$ gauged supergravity done by Nastase, Vaman and van Nieuwenhuizen [14, 15]. Afterwards, we perform the twisted compactification of $D = 7$ SUGRA on T^3.

3.1 Kaluza-Klein reduction on $AdS_7 \times S^4$

In this subsection we will mostly follow [9] but also make use of the original papers [14, 15]. If we not denote it otherwise, we will always refer to one of those three references.

There are several maximally supersymmetric backgrounds for 11-dimensional supergravity. Most interesting for compactification are the cases $AdS_4 \times S^7$ and $AdS_7 \times S^4$. Until recently, the latter one was the only case with a full-known solution to the non-linear ansatz for Kaluza-Klein reduction of $D = 11$ SUGRA.

Our aim is to obtain the maximal gauged supergravity in 7 dimensions with gauge group $\mathrm{SO}(5)_\mathrm{g}$ by this reduction. In particular, we will write down ansätze with which we could reproduce properties of D=7 SUGRA as for example, its SUSY transformations. Already in the ungauged case (but also in the gauged theory) we have local composite $SO(5)_c$ symmetry. It is some kind of fake gauge symmetry since it has gauge fields that are not independent but compositions of the scalars in the theory.

Until further notice we will use the following conventions for indices (as in [14, 15]):

$\Lambda,\ \Pi,\ ... \in \{0, ..., 10\}$ are curved vector indices in 11 dimensions.

$M,\ N,\ ... \in \{0, ..., 10\}$ are flat vector indices in 11 dimensions.

$\alpha,\ \beta,\ ... \in \{0, ..., 6\}$ are curved vector indices in 7 dimensions.

$a,\ b,\ ... \in \{0, ..., 6\}$ are flat vector indices in 7 dimensions.

$\mu,\ \nu,\ ... \in \{1, 2, 3, 4\}$ are curved vector indices in 4 dimensions.

$m,\ n,\ ... \in \{1, 2, 3, 4\}$ are flat vector indices in 4 dimensions.

$I,\ J,\ ... \in \{1, 2, 3, 4\}$ are USp(4) indices.

3.1.1 First order formulation

Initially, we will give the first order formulation for the antisymmetric tensor field for 11-dimensional supergravity before we turn to the linear ansatz and the full non-linear reduction on S^4.

The Lagrangian in first order formulation of the 11-dimensional supergravity is given by

$$
\begin{aligned}
\mathcal{L} = & - \frac{E}{2k^2} R(E,\Omega) - \frac{E}{2}\bar{\Psi}_\Lambda \Gamma^{\Lambda\Pi\Sigma} D_\Pi \left(\frac{\Omega + \hat{\Omega}}{2} \right) \Psi_\Sigma \\
& + \frac{E}{48} (\mathcal{F}_{\Lambda\Pi\Sigma\Omega} \mathcal{F}^{\Lambda\Pi\Sigma\Omega} - 48 \mathcal{F}^{\Lambda\Pi\Sigma\Omega} \partial_\Lambda A_{\Pi\Sigma\Omega}) \\
& - \frac{k\sqrt{2}}{6} \epsilon^{\Lambda_0 \ldots \Lambda_{10}} \partial_{\Lambda_0} A_{\Lambda_1\Lambda_2\Lambda_3} \partial_{\Lambda_4} A_{\Lambda_5\Lambda_6\Lambda_7} A_{\Lambda_8\Lambda_9\Lambda_{10}} \\
& - \frac{k\sqrt{2}}{8} E [\bar{\Psi}_\Pi \Gamma^{\Pi\Lambda_1 \ldots \Lambda_4 \Sigma} \Psi_\Sigma + 12 \bar{\Psi}^{\Lambda_1} \Gamma^{\Lambda_2\Lambda_3} \Psi^{\Lambda_4}] \frac{1}{24} \left(\frac{F + \hat{F}}{2} \right)_{\Lambda_1 \ldots \Lambda_4} ,
\end{aligned}
\tag{3.1}
$$

where

$$E = \det({E_\Lambda}^M) \ , \tag{3.2}$$

$$R(E,\Omega) = {R_{\Lambda\Pi}}^{MN}(\Omega) E^M_\Lambda E^N_\Pi \ , \tag{3.3}$$

$$ {R_{\Lambda\Pi}}^{MN}(\Omega) = (\partial_\Lambda {\Omega_\Pi}^{MN} + {\Omega_\Lambda}^M{}_P {\Omega_\Pi}^{NP} - \partial_\Pi {\Omega_\Lambda}^{MN} - {\Omega_\Pi}^M{}_P {\Omega_\Lambda}^{NP}) \ , \tag{3.4}$$

$$D_\Pi \left(\frac{\Omega + \hat{\Omega}}{2} \right) \Psi_\Sigma = \partial_\Pi \Psi_\Sigma + \frac{1}{4} \left[\frac{\Omega_\Pi^{MN} + \hat{\Omega}_\Pi^{MN}}{2} \right] \Gamma_{MN} \Psi_\Sigma \ , \tag{3.5}$$

$$\tilde{F}_{\Lambda_1 \ldots \Lambda_4} = (\frac{F + \hat{F}}{2})_{\Lambda_1 \ldots \Lambda_4} = 24 \left[\partial_{[\Lambda_1} A_{\Lambda_2\Lambda_3\Lambda_4]} + \frac{1}{16\sqrt{2}} \bar{\Psi}_{[\Lambda_1} \Gamma_{\Lambda_2\Lambda_3} \Psi_{\Lambda_4]} \right], \tag{3.6}$$

$$F_{\Lambda\Pi\Sigma\Omega} := 24 \partial_{[\Lambda} A_{\Pi\Sigma\Omega]} \tag{3.7}$$

and $\mathcal{F}_{\Lambda\Pi\Sigma\Omega}$ fulfils the equation of motion $\mathcal{F}_{\Lambda\Pi\Sigma\Omega} = F_{\Lambda\Pi\Sigma\Omega}$.

For convenience we define:

$$\mathcal{F}_{\Lambda\Pi\Sigma\Omega} := F_{\Lambda\Pi\Sigma\Omega} + \frac{\mathcal{B}_{MNPQ} E^M_\Lambda E^N_\Pi E^P_\Sigma E^Q_\Omega}{\sqrt{E}} \ , \tag{3.8}$$

where $\mathcal{B}_{MNPQ}$ is an auxiliary field we introduced in the 11-dimensional space to guarantee gauge invariance.

Here we denoted a vielbein in 11 dimensions by a big letter E. Vielbeine in lower dimensions are still denoted by a small e.

Ω is the $D = 11$ spin connection and related to its supercovariant version by

$$\Omega_\Pi^{\ MN} = \hat{\Omega}_\Pi^{\ MN} - \frac{1}{8}\bar{\Psi}^\Lambda \Gamma_{\Lambda\Pi}{}^{MN}{}_\Sigma \Psi^\Sigma \ . \tag{3.9}$$

Using this, we can write down the following SUSY transformation rules:

$$\delta E_\Lambda^M = \frac{k}{2}\bar{\epsilon}\Gamma^M \Psi_\Lambda \ , \tag{3.10}$$

$$\begin{aligned}\delta\Psi_\Lambda = & \frac{D_\Lambda \hat{\Omega}\epsilon}{k} + \frac{\sqrt{2}}{12}(\Gamma^{\Lambda_1\ldots\Lambda_4}{}_\Lambda - 8\delta_\Lambda^{\Lambda_1}\Gamma^{\Lambda_2\Lambda_3\Lambda_4}\epsilon\frac{\hat{F}_{\Lambda_1\ldots\Lambda_4}}{24}) \\ & + \frac{1}{24}(b\Gamma_\Lambda^{\Lambda_1\ldots\Lambda_4}\frac{\mathcal{B}_{\Lambda_1\ldots\Lambda_4}}{\sqrt{E}} - a\Gamma^{\Lambda_1\Lambda_2\Lambda_3}\frac{\mathcal{B}_{\Lambda\Lambda_1\Lambda_2\Lambda_3}}{\sqrt{E}})\epsilon \ ,\end{aligned} \tag{3.11}$$

$$\delta A_{\Lambda_1\Lambda_2\Lambda_3} = -\frac{\sqrt{2}}{8}\bar{\epsilon}\Gamma_{[\Lambda_1\Lambda_2}\Psi_{\Lambda_3]} \ , \tag{3.12}$$

$$\delta\mathcal{B}_{MNPQ} = \sqrt{E}\bar{\epsilon}[a\Gamma_{MNP}E_Q^\Lambda R_\Lambda(\Psi) + b\Gamma_{MNPQ\Lambda}R^\Lambda(\Psi)] \ . \tag{3.13}$$

$R^\Lambda(\Psi)$ is the gravitino field equation, given by

$$R^\Lambda(\Psi) = -\Gamma^{\Lambda\Pi\Sigma}D_\Pi\Psi_\Sigma - \frac{\sqrt{2}}{4}k\frac{\hat{F}_{\Lambda_1\ldots\Lambda_4}}{24}\Gamma^{\Lambda\Lambda_1\ldots\Lambda_5}\Psi_{\Lambda_5} - 3\sqrt{2}k\frac{\hat{F}^{\Lambda\Pi\Sigma\Omega}}{k}\Gamma_{\Pi\Sigma}\Psi_\Omega \ . \tag{3.14}$$

The constants a, b are free parameters and can be fixed by requiring consistency of the reduction on S^4 to the values $a = -2/3$ and $b = 1/3$ (see [15]).

We obtain a description of the background by making use of the Freund-Rubin ansatz

$$F_{\mu\nu\rho\sigma} = \frac{3}{\sqrt{2}}\frac{1}{R_{S^4}}(\det(\overset{\circ}{e}{}_{\mu}^{m})(x))\epsilon_{\mu\nu\rho\sigma} \ , \tag{3.15}$$

where R_{S^4} denotes the radius of S^4.

A circle $\circ$ above a quantity denotes that this object belongs to the background (e.g. $\overset{\circ}{e}{}_{\mu}^{m}$ as a background vielbein).

In the $AdS_7 \times S^4$ background the Einstein equations are

$$R_{\mu\nu} - \frac{1}{2}\overset{\circ}{g}_{\mu\nu}R = \frac{1}{6}(F_{\mu\Lambda\Pi\Sigma}F_{\nu}^{\Lambda\Pi\Sigma} - \frac{1}{8}\overset{\circ}{g}_{\mu\nu}F^2) = -\frac{9}{4}\frac{1}{R_{S^4}^2}\overset{\circ}{g}_{\mu\nu} \ , \tag{3.16}$$

$$R_{\alpha\beta} - \frac{1}{2}\overset{\circ}{g}_{\alpha\beta}R = \frac{1}{48}\overset{\circ}{g}_{\alpha\beta}F^2 = \frac{9}{4}\frac{1}{R_{S^4}^2}\overset{\circ}{g}_{\alpha\beta} \tag{3.17}$$

with maximally symmetric solution

$$R_{\mu\nu}{}^{mn}(\overset{\circ}{e}{}^{(4)}) = \frac{1}{R_{S^4}^2}(\overset{\circ}{e}{}_{\mu}^{m}(x)\overset{\circ}{e}{}_{\nu}^{n}(x) - \overset{\circ}{e}{}_{\nu}^{m}(x)\overset{\circ}{e}{}_{\mu}^{n}(x)) \ , \tag{3.18}$$

$$R_{\alpha\beta}{}^{ab}(\overset{\circ}{e}{}^{(7)}) = -\frac{1}{4}\frac{1}{R_{S^4}^2}(\overset{\circ}{e}{}_{\alpha}^{a}(y)\overset{\circ}{e}{}_{\beta}^{b}(y) - \overset{\circ}{e}{}_{\beta}^{a}(y)\overset{\circ}{e}{}_{\alpha}^{b}(y)) \ . \tag{3.19}$$

3.1.2 Linear ansatz

Now we look at the linear ansatz since in the original work it was used as a starting point for the consistent non-linear reduction.

By x we denote compact and by y non-compact coordinates.

We expand every field of the 11-dimensional theory as follows:

$$\Phi_{\alpha\mu}(y,x) = \sum_I \phi_{\alpha,I}(y)Y_{\mu}^{I}(x) \qquad \text{(bosons)} \ , \tag{3.20}$$

$$\Psi^{Z}(y,x) = \sum_I \Psi_I^{z_1}(y)Y^{I,z_2}(x) \qquad \text{(fermions)} \ . \tag{3.21}$$

ϕ is a 7-dimensional field and Y is a spherical harmonic. The spinor index $Z \in \{1, ..., 32\}$ splits up into $Z = z_1 \oplus z_2$, where z_1 is a spinor index on AdS_7 and z_2 on S^4.

Now we write down ansätze for the metric, the gravitino, the antisymmetric tensor and the auxiliary field.

We split the metric into the background metric and a fluctuation

$$g_{\Lambda\Pi} = \overset{\circ}{g}_{\Lambda\Pi} + k h_{\Lambda\Pi} \ , \tag{3.22}$$

where the 7-dimensional fluctuation is given by the ansatz

$$h_{\alpha\beta}(y,x) = h_{\alpha\beta}(y) - \frac{\overset{\circ}{g}_{\alpha\beta}(y)}{5}(h_{\mu\nu}(y,x)\overset{\circ}{g}^{\mu\nu}(x)) \ , \tag{3.23}$$

($h_{\alpha\beta}(y)$ is the 7-dimensional graviton fluctuation),

the fluctuation with mixed indices is given by

$$h_{\mu\alpha}(y,x) = B_{\alpha,IJ}(y) V_\mu^{IJ}(x) \ , \tag{3.24}$$

($B_{\alpha,IJ}(y)$ is the gauge field and $V_\mu^{IJ}(x)$ are the 10 Killing vectors on S^4)

and the fluctuation with compact indices is given by

$$h_{\mu\nu}(y,x) = S_{IJKL}(y)\eta_{\mu\nu}^{IJKL}(x) \ , \tag{3.25}$$

($S_{IJKL}(y)$ are scalars in the **14** representation of $USp(4)$ and $\eta_{\mu\nu}^{IJKL}(x)$ is the corresponding spherical harmonic [15]).

Now we look at the gravitini. For the gravitino with compact index we use the ansatz

$$\Psi_\mu(y,x) = \lambda_{J,KL}(y)\gamma_5^{1/2}\eta_\mu^{JKL}(x) \ , \tag{3.26}$$

where $\gamma_5^{1/2} = \frac{i-1}{2}(1 + i\gamma_5)$.

The fermions $\lambda_{J,KL}(y)$ with $J, K, L \in \{1, 2, 3, 4\}$ are in the **16** representation of USp(4) and $\eta_\mu^{JKL}(x)$ is the corresponding spherical harmonic [15].

For the gravitino with non-compact index we use

$$\Psi_\alpha(y,x) = \psi_{\alpha I}(y)\gamma_5^{\pm 1/2}\eta^I(x) - \frac{1}{5}\tau_\alpha\gamma_5\gamma^\mu\Psi_\mu(y,x) \ , \tag{3.27}$$

where $\eta^I(x)$ is a Killing spinor (see 3.1.3), τ_α is a "gamma" matrix in 7 dimensions, γ^μ is a Dirac matrix in 4 dimensions and $\gamma_5^{-1/2} = \frac{1+i}{2}(i\gamma_5 - 1)$.

Next, we look at the ansatz for the antisymmetric tensor. With only compact indices it is given by

$$A_{\mu\nu\rho}(y,x) = \frac{\sqrt{2}}{40}\sqrt{\overset{\circ}{g}}\epsilon_{\mu\nu\rho\sigma}D^\sigma h_\lambda^\lambda \ , \tag{3.28}$$

(with $h_\lambda^\lambda = h_{\mu\nu}(y,x)\overset{\circ}{g}{}^{\mu\nu}(x)$),

with only one non-compact index it is given by

$$A_{\alpha\mu\nu}(y,x) = \frac{i}{12\sqrt{2}}B_{\alpha,IJ}(y)\bar{\eta}^I(x)\gamma_{\mu\nu}\gamma_5\eta^I(x) \ , \tag{3.29}$$

and with only non-compact indices by

$$A_{\alpha\beta\gamma}(y,x) = \frac{1}{6}A_{\alpha\beta\gamma,IJ}(y)\phi_5^{IJ}(x) \ , \tag{3.30}$$

($A_{\alpha\beta\gamma,IJ}(y)$ are antisymmetric, symplectic-traceless tensors in the **5** representation of USp(4) and $\phi_5^{IJ}(x) = \bar{\eta}^I\gamma_5\eta^J$ is the corresponding spherical harmonic [15]).

Since there is no independent field with two non-compact indices, this case is given by

$$A_{\alpha\beta\mu} = 0 \ . \tag{3.31}$$

Finally, the ansatz for the auxiliary field is

$$\mathcal{B}_{\alpha\beta\gamma,IJ} = \frac{1}{5}\left(S_{\alpha\beta\gamma,IJ} + \frac{1}{6}\epsilon_{\alpha\beta\gamma}{}^{\delta\epsilon\eta\zeta}D_\delta S_{\epsilon\eta\zeta,IJ}\right) \tag{3.32}$$

with the 7-dimensional supergravity field $S_{\alpha\beta\gamma,IJ}$.

3.1.3 Killing spinors and spherical harmonics

Here we will have a closer look at Killing spinors and spherical harmonics which are relevant for our elaboration. At first, we state what a Killing spinor is and write down some useful relations. Afterwards, we express the spherical harmonics in terms of Killing spinors.

Definition 2 *[17] A Killing spinor is a spinor field η on a Riemannian spin manifold that satisfies for all vector fields X*

$$\nabla_X \eta = \beta X \cdot \eta \ ,$$

where $\beta \in \mathbb{C}$.

The Killing spinor η^I fulfils the following relations

$$\mathring{D}_\mu \eta^I = \frac{i}{2}\gamma_\mu \eta^I \ , \tag{3.33}$$

$$\mathring{D}_\mu \bar{\eta}^I = -\frac{i}{2}\bar{\eta}^I \gamma_\mu \ , \tag{3.34}$$

$$\bar{\eta}^I \eta^J = \Omega^{IJ} \tag{3.35}$$

and

$$\eta^\alpha_J \bar{\eta}^J_\beta = -\delta^\alpha_\beta \ , \tag{3.36}$$

where Ω is the constant antisymmetric USp(4)-invariant metric, $\alpha, \beta \in \{1,2,3,4\}$ are spinor indices, $\eta^\alpha_J = \eta^{\alpha I}\Omega_{IJ}$ and $\eta^{\alpha I} = [\exp(\frac{-i}{2}x^\mu \delta^m_\mu \gamma^m)]^\alpha{}_\beta \Omega^{\beta I}$.

According to [18], Ω looks like

$$\Omega^{IJ} = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} \quad \text{with} \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ . \tag{3.37}$$

Now all spherical harmonics can be given as constructions out of Killing spinors.

The scalar spherical harmonics are given by

$$Y^A = \frac{1}{4}(\gamma^A)_{IJ}\phi_5^{IJ} \ , \tag{3.38}$$

where $\phi_5^{IJ} = \bar{\eta}^I\gamma_5\eta^J$ is the scalar field harmonic and Y^A satisfies

$$Y^A Y^A = 1 \ . \tag{3.39}$$

$A, ... \in \{1, ..., 5\}$ are vector indices of the $\mathrm{SO}(5)_g$ gauge group and the 5-dimensional Dirac matrices γ^A are given by

$$\gamma^A = \{i\gamma^\mu\gamma_5, \gamma_5\} \ . \tag{3.40}$$

What happens here is that the 4-dimensional matrices get embedded in 5 dimensions by splitting up the index A such that $A = \mu \oplus 5$. So the fifth 5-dimensional Dirac matrix is the 4-dimensional γ_5.

From this we can obtain the matrices $(\gamma^A)_{IJ}$ by using the lowering and raising rules for USp(4) indices

$$X_I = X^J\Omega_{JI} \ , \tag{3.41}$$

$$X^J = \tilde{\Omega}^{IJ}X_J \ . \tag{3.42}$$

One has $\tilde{\Omega}^{IJ}\Omega_{JK} = \delta^J_K$.

Further, we can write the Killing vectors in terms of the Killing spinors as

$$V_\mu^{IJ} = V_\mu^{JI} = \bar{\eta}^I\gamma_\mu\eta^J \ . \tag{3.43}$$

We can also introduce

$$V_\mu^{AB} = -\frac{i}{8}(\gamma^{AB})_{IJ}V_\mu^{IJ} \tag{3.44}$$

and

$$V_\mu^{IJ} = iV_\mu^{AB}(\gamma_{AB})^{IJ} \ . \tag{3.45}$$

They all satisfy the Killing equation

$$\overset{\circ}{D}_{(\mu}V_{\nu)} = 0 \ . \tag{3.46}$$

Next, conformal Killing vectors are given by

$$C_\mu^{IJ} = -C_\mu^{JI} = \bar{\eta}^I \gamma_\mu \gamma_5 \eta^J \tag{3.47}$$

and satisfy

$$C_\mu^{IJ} \Omega_{IJ} = 0 \ . \tag{3.48}$$

Again we can also introduce

$$C_\mu^A = \frac{i}{4} C_\mu^{IJ} (\gamma^A)_{IJ} = \overset{\circ}{D}_\mu Y^A = \partial_\mu Y^A \ . \tag{3.49}$$

They all satisfy the conformal Killing equation

$$\overset{\circ}{D}_{(\mu}C_{\nu)} = \frac{1}{4} \overset{\circ}{g}_{\mu\nu} (\overset{\circ}{D}{}^\rho C_\rho) \ . \tag{3.50}$$

The spherical harmonic for the metric fluctuation with compact indices is the sum of two spherical harmonics with different eigenvalues of the operator $\Box$:

$$\eta_{\mu\nu}^{IJKL} = \eta_{\mu\nu}^{IJKL}(-2) - \frac{1}{3} \eta_{\mu\nu}^{IJKL}(-10) \ , \tag{3.51}$$

where $\Box \eta_{\mu\nu}^{IJKL}(a) = a \eta_{\mu\nu}^{IJKL}(a)$.

We can rewrite them in terms of objects we defined earlier:

$$\eta_{\mu\nu}^{IJKL}(-2) = C_{(\mu}^{IJ} C_{\nu)}^{KL} - \frac{1}{4} \overset{\circ}{g}_{\mu\nu} C_\lambda^{IJ} C^{\lambda KL} \ , \tag{3.52}$$

$$\eta_{\mu\nu}^{IJKL}(-10) = \overset{\circ}{g}_{\mu\nu} \left(\phi_5^{IJ} \phi_5^{KL} + \frac{1}{4} C_\lambda^{IJ} C^{\lambda KL} \right) \ . \tag{3.53}$$

The spherical harmonic for the gravitino with compact index is also the sum of two spherical harmonics but they differ in their eigenvalues of the kinetic operator:

$$\eta_\mu^{JKL} = \eta_\mu^{JKL}(-2) + \eta_\mu^{JKL}(-6) \ , \tag{3.54}$$

where $\gamma^\nu \overset{\circ}{D}_\nu \eta_\mu^{JKL}(a) = a\eta_\mu^{JKL}(a)$.

Again we can rewrite them in terms of objects we defined earlier:

$$\eta_\mu^{JKL}(-2) = 3(\eta^J C_\mu^{KL} - \frac{1}{4}\gamma_\mu \gamma^\nu \eta^J C_\nu^{KL}) \ , \tag{3.55}$$

$$\eta_\mu^{JKL}(-6) = \gamma_\mu(\eta^J \phi_5^{KL} - \frac{1}{4}\gamma^\nu \eta^J C_\nu^{KL}) \ . \tag{3.56}$$

3.1.4 Non-linear ansatz

Finally, we look at the non-linear ansatz. Again $A,\ B,\ ... \in \{1, ..., 5\}$ are $\mathrm{SO}(5)_\mathrm{g}$ vector indices and $I,\ J,\ ... \in \{1, ..., 4\}$ are spinor indices. $i,\ j,\ ... \in \{1, ..., 5\}$ are $\mathrm{SO}(5)_\mathrm{c}$ vector indices and $I',\ J', ... \in \{1, ..., 4\}$ are spinor indices.

Now we are writing down ansätze for the vielbein, the gravitini, the SUSY parameter, the field strength of the antisymmetric tensor and the auxiliary field.

Instead of an ansatz for the metric (as in the linear ansatz) we have this ansatz for the vielbein

$$E_\alpha^a(y,x) = e_\alpha^a(y)\Delta^{-1/5}(y,x) \ , \tag{3.57}$$

where $\Delta(y,x) = \det(E_\mu^m)/\det(\overset{\circ}{e}{}_\mu^m)$.

Further, we make a gauge choice such that

$$E_\mu^a = 0 \ . \tag{3.58}$$

E_α^m is given by the ansatz

$$E_\alpha^m(y,x) = B_\alpha^\mu(y,x) E_\mu^m \tag{3.59}$$

with

$$B_\alpha^\mu(y,x) = -2B_\alpha^{AB} V^{\mu,AB} \ , \tag{3.60}$$

where $V^{\mu,AB}$ is the corresponding Killing vector.

The ansatz for E_μ^m is a bit more complicated:

$$E_\mu^m = \frac{1}{4}\Delta^{2/5}\Pi^i_A C^A_\mu C^{mB}\mathrm{tr}(U^{-1}\gamma^i U\gamma_B) \ . \tag{3.61}$$

Π^i_A are scalars parametrising the coset $\mathrm{SL}(5,\mathbb{R})/\mathrm{SO}(5)_\mathrm{c}$.

$U^{I'}{}_I$ is an USp(4) matrix that relates $\mathrm{SO}(5)_\mathrm{g}$ spinor indices with those of $\mathrm{SO}(5)_\mathrm{c}$. It fulfils

$$(\tilde{\Omega}\cdot U^T\cdot\Omega)^I{}_{I'} = -(U^{-1})^I{}_{I'} \ . \tag{3.62}$$

The 11-dimensional metric is given by

$$ds^2_{11} = \Delta^{-2/5}g_{\alpha\beta}dy^\alpha dy^\beta + \Delta^{4/5}T^{-1}_{AB}(dY^A + 2B^{AC}Y^C)(dY^B + 2B^{BD}Y^D) \ , \tag{3.63}$$

where $T^{AB} = (\Pi^{-1})^A_i(\Pi^{-1})^B_j\delta^{ij}$.

Next, we look at the gravitini ansätze. We divide the gravitini $\Psi_M := E^\Lambda_M\Psi_\Lambda$ into $\Psi_m := E^\Lambda_m\Psi_\Lambda$ and $\Psi_a := E^\Lambda_a\Psi_\Lambda$. They are given by

$$\Psi_a = \Delta^{1/10}(\gamma_5)^{-p}\psi_a - \frac{A}{5}\tau_a\gamma_5\gamma^m\Delta^{1/10}(\gamma_5)^q\psi_m \tag{3.64}$$

and

$$\Psi_m = \Delta^{1/10}(\gamma_5)^q\psi_m \ . \tag{3.65}$$

Following [15], it turns out that $A = 1$ and $p = q = -1/2$ are required.

The fields ψ_α and ψ_m can be written as

$$\psi_\alpha(y,x) = \psi_{\alpha I'}(y)U^{I'}{}_I(y,x)\eta^I(x) \tag{3.66}$$

and

$$\psi_m(y,x) = \lambda_{J'K'L'}(y)U^{J'}{}_J(y,x)U^{K'}{}_K(y,x)U^{L'}{}_L(y,x)\eta^{JKL}_m(x) \ . \tag{3.67}$$

The ansatz for the SUSY parameter ϵ is

$$\epsilon(y,x) = \Delta^{1/10}(\gamma_5)^{-1/2}\varepsilon(y,x) \tag{3.68}$$

with

$$\varepsilon(y,x) = \varepsilon_{I'}(y)U^{I'}{}_I(y,x)\eta^I(x)\ . \tag{3.69}$$

Now we turn to the field strength ansätze.

The ansatz for the field strength with only compact indices is

$$\begin{aligned}\frac{\sqrt{2}}{3}F_{\mu\nu\rho\sigma} = &\epsilon_{\mu\nu\rho\sigma}\sqrt{\det(\overset{\circ}{g})}\Bigg[1+\frac{1}{3}\left(\frac{T}{Y_AY_BT^{AB}}-5\right) \\ &-\frac{2}{3}\left(\frac{Y_A(T^2)^{AB}Y_B}{(Y_AT^{AB}Y_B)^2}-1\right)\Bigg]\ .\end{aligned} \tag{3.70}$$

With only one non-compact index it is

$$\begin{aligned}\frac{\sqrt{2}}{3}F_{\mu\nu\rho\alpha} = &\partial_{[\mu}\left(\epsilon_{ABCDE}B^{AB}_\alpha C^C_\nu C^D_{\rho]}\frac{T^{EF}Y_F}{Y\cdot T\cdot Y}\right) \\ &+\sqrt{\overset{\circ}{g}}\epsilon_{\mu\nu\rho\sigma}C^\sigma_A\frac{1}{3}\left(\frac{\partial_\alpha T^{AB}Y_B}{Y_AT^{AB}Y_B}-\frac{T^{AB}Y_B}{(Y_AT^{AB}Y_B)^2}(Y_C\partial_\alpha T^{CD}Y_D)\right)\ .\end{aligned} \tag{3.71}$$

With two non-compact indices it is

$$\begin{aligned}\frac{\sqrt{2}}{3}F_{\mu\nu\alpha\beta} = &\frac{2}{3}\Bigg[\partial_{[\alpha}\left(\epsilon_{ABCDE}B^{AB}_{\beta]}C^C_\mu C^D_\nu\frac{T^{EF}Y_F}{Y\cdot T\cdot Y}\right) \\ &+2\partial_{[\mu}\left(\epsilon_{ABCDE}B^{AF}_{[\alpha}Y_FB^{BC}_{\beta]}C^D_{\nu]}\frac{T^{EG}Y_G}{Y\cdot T\cdot Y}\right)\Bigg]\ .\end{aligned} \tag{3.72}$$

With three non-compact indices it is

$$\begin{aligned}\frac{\sqrt{2}}{3}F_{\mu\alpha\beta\gamma} = &\partial_\mu\mathcal{A}_{\alpha\beta\gamma} \\ &+\frac{4}{3}\partial_\mu\left(\epsilon_{ABCDE}B^{AB}_{[\alpha}B^{CF}_\beta Y_FB^{DG}_{\gamma]}Y_G\frac{T^{EH}Y_H}{Y\cdot T\cdot Y}\right) \\ &-2\partial_{[\alpha}\left(\epsilon_{ABCDE}B^{AB}_\beta B^{CF}_{\gamma]}Y_FC^D_\mu\frac{T^{EG}Y_G}{Y\cdot T\cdot Y}\right) \\ &+\partial_\mu\left(\epsilon_{ABCDE}(\partial_{[\alpha}B^{AB}_\beta+\frac{4}{3}B^{AF}_{[\alpha}B^{FB}_\beta)B^{CD}_{\gamma]}Y^E\right)\end{aligned} \tag{3.73}$$

and with only non-compact indices it is

$$\begin{aligned}\frac{\sqrt{2}}{3}F_{\alpha\beta\gamma\delta} =& 4\partial_{[\alpha}\mathcal{A}_{\beta\gamma\delta]} \\ &+ 4\partial_{[\alpha}\epsilon_{ABCDE}\left(\frac{4}{3}B^{AB}_{\beta}B^{CF}_{\gamma}Y_F B^{DG}_{\delta]}Y_G\frac{T^{EH}Y_H}{Y\cdot T\cdot Y}\right. \\ &\left.+ (\partial_\beta B^{AB}_{\gamma} + \frac{4}{3}B^{AF}_{\beta}B^{FB}_{\gamma})B^{CD}_{\delta]}Y^E\right) ,\end{aligned} \tag{3.74}$$

where

$$\mathcal{A}_{\alpha\beta\gamma} = \frac{8i}{\sqrt{3}}S_{\alpha\beta\gamma,B}Y^B \ . \tag{3.75}$$

We can also summarise these results in form language:

$$\begin{aligned}\frac{\sqrt{2}}{3}F_{(4)} = \epsilon_{ABCDE}&\left(-\frac{1}{3}DY^A DY^B DY^C DY^D\frac{(T\cdot Y)^E}{Y\cdot T\cdot Y}\right) \\ &+ \frac{4}{3}DY^A DY^B DY^C D[\frac{(T\cdot Y)^D}{Y\cdot T\cdot Y}]Y^E \\ &+ 2F^{AB}_{(2)}DY^C DY^D\frac{(T\cdot Y)^E}{Y\cdot T\cdot Y} + F^{AB}_{(2)}F^{CD}_{(2)}Y^E) + d(\mathcal{A}) \ ,\end{aligned} \tag{3.76}$$

where

$$F^{AB}_{(2)} = 2(dB^{AB} + 2(B\cdot B)^{AB}) \tag{3.77}$$

and

$$DY^A = dY^A + 2B^{AB}\cdot Y_B \ . \tag{3.78}$$

Finally, the ansatz for the auxiliary field is

$$\begin{aligned}\frac{\mathcal{B}_{\alpha\beta\gamma\delta}}{\sqrt{E}} =& -24\sqrt{3}i\nabla_{[\alpha}S_{\beta\gamma\delta],A}Y^A + \sqrt{3}i\epsilon_{\alpha\beta\gamma\delta}{}^{\epsilon\eta\zeta}T^{AB}S_{\epsilon\eta\zeta,B}Y_A \\ &+ 9\epsilon_{ABCDE}F^{BC}_{[\alpha\beta}F^{DE}_{\gamma\delta]}Y^A + 2-\text{fermi terms} \ .\end{aligned} \tag{3.79}$$

3.1.5 Maximal D=7 gauged supergravity

By using those reduction ansätze, we obtain the maximal $D = 7$ gauged supergravity, where the global symmetry group $SO(5)$ is gauged to $SO(5)_g$.

In 7 dimensions the maximal supergravity is $\mathcal{N} = 4$.

A special formalism to gauge a theory is the so-called embedding tensor formalism. It describes a constant tensor Θ, the embedding tensor, which completely specifies the gauged theory. Our short description of this formalism is entirely based on [11], whose corresponding treatment is based on [44, 45, 46], and [19]. More detailed dealings with this topic can be found there.

If V denotes the representation of $SO(5)$ in which the vector B_α^{AB} of the ungauged theory transforms, then the embedding tensor is a map

$$\Theta : V \to \mathfrak{so}(5) \ , \tag{3.80}$$

where $\mathfrak{so}(5)$ is the Lie algebra of $SO(5)$.

The embedding tensor defines the gauge-covariant derivative

$$D_\mu = \partial_\mu - g B_\mu^{AB} \Theta_{AB,C}{}^D t^C{}_D \ , \tag{3.81}$$

where g is a gauge coupling constant and $t^C{}_D$ are the generators of $\mathfrak{so}(5)$.

Further, Θ also explicitly finds the generators X_{AB} of $SO(5)_g$ among the generators of the global symmetry group by

$$X_{AB} = \Theta_{AB,C}{}^D t^C{}_D \ . \tag{3.82}$$

According to [19], for our special case ($D = 7$ SUGRA from S^4 reduction), the embedding tensor is given by

$$\Theta_{AB,C}{}^D = \delta^D_{[A} Y_{B]C} \ , \tag{3.83}$$

where Y_{BC} is, for the $SO(5)$ case, a 5-dimensional unit matrix. In general, this matrix could be more complicated.

After this short introduction to the embedding tensor formalism, we now turn towards other aspects of our theory.

The 7-dimensional maximal gauged supergravity model includes the following fields:

$$
\begin{aligned}
&e_\alpha^a && \text{(vielbein)} \ , \\
&\psi_\alpha^{I'} && \text{(gravitini)} \ , \\
&B_\alpha^{AB} = -B_\alpha^{BA} && (\mathrm{SO}(5)_{\mathrm{g}} \text{ vector}) \ , \\
&\Pi_A{}^i && \text{(scalars)} \ , \\
&S_{\alpha\beta\gamma,A} && \text{(antisymmetric tensor)} \ , \\
&\lambda_i^{I'} && \text{(spin 1/2 fields)} \ .
\end{aligned}
$$

Their SUSY transformations are given by

$$
\delta e_\alpha^a = \frac{1}{2}\bar{\epsilon}\tau^a\psi_a \ , \tag{3.84}
$$

$$
\Pi_A{}^i\Pi_B{}^j\delta B_\alpha^{AB} = \frac{1}{4}\bar{\epsilon}\gamma^{ij}\psi_\alpha + \frac{1}{8}\bar{\epsilon}\tau_\alpha\gamma^k\gamma^{ij}\lambda_k \ , \tag{3.85}
$$

$$
\begin{aligned}
\delta S_{\alpha\beta\gamma,A} =& -\frac{i\sqrt{3}}{8m}\Pi_A{}^i(2\bar{\epsilon}\gamma_{ijk}\psi_{[\alpha} + \bar{\epsilon}\tau_{[\alpha}\gamma^l\gamma_{ijk}\lambda_l)\Pi_B{}^j\Pi_C{}^k F_{\beta\gamma]}^{BC} \\
& - \frac{i\sqrt{3}}{4m}\delta_{ij}\Pi_A{}^j D_{[\alpha}(2\bar{\epsilon}\tau_\beta\gamma^i\psi_{\gamma]} + \bar{\epsilon}\tau_{\beta\gamma]}\lambda^i) \\
& + \frac{i\sqrt{3}}{12}\delta_{AB}\Pi^{-1}{}_i{}^B(3\bar{\epsilon}\tau_{[\alpha\beta}\gamma^i\psi_{\gamma]} - \bar{\epsilon}\tau_{\alpha\beta\gamma}\lambda^i) \ ,
\end{aligned} \tag{3.86}
$$

$$
\Pi^{-1}{}_i{}^A\delta\Pi_A{}^j = \frac{1}{4}(\bar{\epsilon}\gamma_i\lambda^j + \bar{\epsilon}\gamma^j\lambda_i) \ , \tag{3.87}
$$

$$
\begin{aligned}
\delta\psi_\alpha =& \nabla_\alpha\epsilon - \frac{1}{20}mT\tau_\alpha\epsilon - \frac{1}{40}(\tau_\alpha{}^{\beta\gamma} - 8\delta_\alpha^\beta\tau^\gamma)\gamma_{ij}\epsilon\Pi_A{}^i\Pi_B{}^j F_{\beta\gamma}^{AB} \\
& + \frac{im}{10\sqrt{3}}(\tau_\alpha{}^{\beta\gamma\delta} - \frac{9}{2}\delta_\alpha^\beta\tau^{\gamma\delta})\gamma^i\epsilon\Pi^{-1}{}_i{}^A S_{\beta\gamma\delta,A} \ ,
\end{aligned} \tag{3.88}
$$

$$
\begin{aligned}
\delta\lambda_i =& \frac{1}{16}\tau^{\alpha\beta}(\gamma_{kl}\gamma_i - \frac{1}{5}\gamma_i\gamma_{kl})\epsilon\Pi_A{}^k\Pi_B{}^l F_{\alpha\beta}^{AB} \\
& + \frac{im}{20\sqrt{3}}\tau^{\alpha\beta\gamma}(\gamma_i{}^j - 4\delta_i^j)\epsilon\Pi^{-1}{}_j{}^A S_{\alpha\beta\gamma,A} \\
& + \frac{1}{2}m(T_{ij} - \frac{1}{5}T)\gamma^j\epsilon + \frac{1}{2}\tau^\alpha\gamma^j P_{\alpha ij} \ ,
\end{aligned} \tag{3.89}
$$

where $m = 1/R_{S^4}$.

$P_{\alpha ij}$ and the composite connection $\nabla_\alpha = \partial_\alpha + Q_\alpha$ are given by

$$\Pi^{-1}{}_i{}^A(\delta_A^B\partial_\alpha + gB_{\alpha A}{}^B)\Pi_B{}^k\delta_{kj} = Q_{\alpha ij} + P_{\alpha ij} \ , \tag{3.90}$$

where $Q_{\alpha ij}$ is the antisymmetric and $P_{\alpha ij}$ is the symmetric part.

The Lagrangian is quite long and therefore we only give the two terms we are interested in:

$$\mathcal{L} = eR - \frac{1}{2}em^2(T^2 - 2T_{ij}T^{ij}) + ... \ , \tag{3.91}$$

where $T_{ij} = (\Pi^{-1})_i^A(\Pi^{-1})_j^B\delta_{AB}$.

A complete version of the Lagrangian can be found e.g. in [14] or [15].

As usual, the first term contains the 7-dimensional Ricci scalar. We will later use it for our next reduction step. The second term can be interpreted as $-2e\Lambda^{(7)}_{\text{eff.}}$ with an effective cosmological constant $\Lambda^{(7)}_{\text{eff.}}$ formed by scalars. The (7) indents that this constant belongs to the 7-dimensional theory. In the non-shown terms all the other fields appear but in this thesis we are only interested in gravity and the scalars.

3.2 Compactification on a twisted T^3

After we reviewed the Kaluza-Klein reduction of 11-dimensional SUGRA on S^4, we will now perform the compactification of maximal 7-dimensional gauged supergravity on a twisted 3-torus in order to obtain a supergravity in 4 spacetime dimensions. In particular, we want to obtain a scalar potential in our lower-dimensional theory.

As long as we do not denote it differently, our explanations will follow [11] whose treatment of this topic is based on [21].

3.2.1 What is a twisted torus?

A twisted torus is a torus which we promoted to a group manifold. This means that we are considering an object with the topology of a torus T^d but with the geometry of a d-dimensional group G.

Here we will clarify what a group manifold is and introduce important concepts which we will need to use this kind of objects for our work. Later we will see how the twisting of the torus appears in practice.

Definition 3 *[13] A Lie group is a differentiable manifold G which has a group structure such that the following group operations are differentiable:*

$$
\begin{aligned}
(1)\ \cdot \quad &: G \times G \to G\ , \\
&(g_1, g_2) \mapsto g_1 \cdot g_2\ , \\
\\
(2)\ {}^{-1} \quad &: G \to G\ , \\
&g \mapsto g^{-1}\ .
\end{aligned}
$$

The manifold G is also called a group manifold. An easy example for such a group manifold is

$$S^1 = \{p = \exp(i\theta) \mid \theta \in [0, 2\pi)\}\ . \tag{3.92}$$

Obviously, this manifold has the group structure of $U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$.

On a group manifold we can introduce a coordinate system $\{y^m\}_{m=1,\ldots,\dim(G)}$ in order to parametrise group elements $g(y) \in G$.

In addition, we can always define two types of translations on G:

Definition 4 *[13] Let $h, g \in G$. Then the diffeomorphisms $\Lambda_L(h)\ :\ G \to G$ (left-translation) and $\Lambda_R(h)\ :\ G \to G$ (right-translation) are defined by*

$$
\begin{aligned}
\Lambda_L(h)g &:= hg \\
\Lambda_R(h)g &:= gh\ .
\end{aligned}
$$

In principle, both kind of translations give equivalent theories [13]. Therefore, it is sufficient to just focus on one of them. Usually one chooses the left-translation.

The left-translation induces a map (see [13])

$$\Lambda_{L*}(h) \ : \ \mathrm{T_g G} \to \mathrm{T_{hg} G} \ . \tag{3.93}$$

By using this map we can define left-invariance:

Definition 5 *[13] Let X be a vector field on G. Then X is a left-invariant vector field if $\Lambda_{L*}(h)X|_g = X|_{hg}$.*

Now we can introduce a set of left-invariant 1-forms $\{\sigma^m\}$, which fulfil

$$T_m \sigma^m = g^{-1} dg \ , \tag{3.94}$$

where T_m are the generators of G and $d := dy^m \partial/\partial y^m$.

We can express these 1-forms as

$$\sigma^m = U^m{}_n(y) dy^n \ , \tag{3.95}$$

where $U^m{}_n(y)$ is a function on G and $\{\mathrm{dy^m}\}$ is a 1-form coordinate basis. Later we will refer to U as the twist matrix.

The y-independent structure constants of G can be written in terms of U as

$$f_{mn}{}^p = -2(U^{-1})^q{}_m (U^{-1})^r{}_n \partial_{[q} U^p{}_{r]} \ . \tag{3.96}$$

By the left-invariant 1-forms σ^m a class of metrics on G is defined for which the left-translation $\Lambda_L(h)$ is an isometry. Those metrics are of the following form:

$$\mathrm{ds_G^2} = g_{mn} \sigma^m \otimes \sigma^n \ . \tag{3.97}$$

$\{L_m\}$ denotes a basis of Killing vectors that generate the isometries defined by $\Lambda_L(h)$. Those Killing vectors fulfil

$$[L_m, L_n] = f_{mn}{}^p L_p \ . \tag{3.98}$$

3.2.2 Twisted toroidal reduction

In this part we will actually perform the compactification on a twisted torus. As before, we will denote the coordinates of the compact space with y and the coordinates of the spacetime with x.

Here we use the following index conventions:

μ, ν, ρ, σ are 4-dimensional curved indices.

$a, ..., f$ are 4-dimensional flat indices.

$m, ..., v$ are 3-dimensional curved indices.

i, j, k, l and in rare cases also g, h are 3-dimensional flat indices.

Whenever we run out of indices, we use the capital versions of those indices as well. An index with a hat ˆ or an object with such a hat like a metric or vielbein belongs to the 7-dimensional theory.

The usual torus reduction ansatz for the metric is

$$d\hat{s}^2 = \exp(2\alpha\phi)ds^2 + \exp(2\beta\phi)M_{mn}(dy^m + A^m_{\ \mu})(dy^n + A^n_{\ \nu}) \ , \qquad (3.99)$$

where α and β are constants, ds is the 4-dimensional spacetime line element, ϕ is a dilaton, A^m_μ are three new vectors and M_{mn} is a symmetric matrix.

A toroidal reduction is consistent. However, it does not give masses to the scalar fields as the dilaton ϕ. We want masses for those fields because otherwise we could not explain why we did not observe those fields in our experiments yet. One way to obtain masses is to compactify on a manifold that is not a torus. Unfortunately, one often encounters difficulties with the consistency of non-toroidal reductions. Compactification on a twisted torus is an interesting case because it is consistent but also creates a potential for the scalars as we will determine it later.

As previously described, we obtain a twisted torus by promoting a torus to a group manifold. This practically means that we replace dy^m by

$\sigma^m = U^m{}_n(y)dy^n$, which introduces a y-dependence in the metric and the vielbein. Of course we do not want this dependence in our 4-dimensional theory but luckily it will turn out that all $U^m{}_n(y)$ will be combined in a way such that they give us the y-independent structure constants of our group manifold as they are defined in (3.96).

The twisted toroidal reduction ansatz for the metric is

$$d\hat{s}^2 = \exp(2\alpha\phi)ds^2 + \exp(2\beta\phi)M_{mn}(\sigma^m + A^m{}_\mu)(\sigma^n + A^n{}_\nu) \qquad (3.100)$$

and the Lagrangian of the 7-dimensional theory is

$$\hat{\mathcal{L}} = \hat{e}\hat{R} - 2\hat{e}\Lambda^{(7)}_{\text{eff.}} + \ldots \quad , \qquad (3.101)$$

as seen before.

We want to perform the reduction by using the vielbein formalism instead of the metric. A suitable vielbein is obtained by following the respective treatment in [20].

Since the matrix M_{mn} is symmetric, we can write it as

$$M_{mn} = L^i_m L^j_n \delta_{ij} \ . \qquad (3.102)$$

This helps us to find our 7-dimensional vielbein, which we can now write as

$$\hat{e}_{\hat{\mu}}{}^{\hat{a}} = \begin{pmatrix} \exp(\alpha\phi)e_\mu{}^a & \exp(\beta\phi)L^i_m A^m{}_\mu \\ 0 & \exp(\beta\phi)L_n{}^i U^n{}_m \end{pmatrix} \ , \qquad (3.103)$$

where $e_\mu{}^a$ is the vielbein of the 4-dimensional spacetime.

We write the 7-dimensional Minkowski metric $\hat{\eta}_{\hat{a}\hat{b}}$ by using the 4-dimensional Minkowski metric η_{ab} as

$$\hat{\eta}_{\hat{a}\hat{b}} = \begin{pmatrix} \eta_{ab} & 0 \\ 0 & \delta_{ij} \end{pmatrix} \qquad (3.104)$$

and can now check whether our vielbein is correct.

If our vielbein is the correct choice, then

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \hat{e}_{\hat{\mu}}{}^{\hat{a}}\hat{e}_{\hat{\nu}}{}^{\hat{b}}\hat{\eta}_{\hat{a}\hat{b}} \qquad (3.105)$$

has to be fulfilled.

Indeed, after contracting the indices, we obtain

$$\begin{aligned}\hat{e}_{\hat{\mu}}{}^{\hat{a}}\hat{e}_{\hat{\nu}}{}^{\hat{b}}\hat{\eta}_{\hat{a}\hat{b}} &= \exp(2\alpha\phi)e_\mu{}^a e_\nu{}^b \eta_{ab} \\ &\quad + \exp(2\beta\phi)(L^i_m L^j_p A^m{}_\mu A^p{}_\nu \delta_{ij} + L^i_m L^j_q A^m{}_\mu U^q{}_p \delta_{ij} \\ &\quad + L^i_n L^j_p U^n{}_m A^p{}_\nu \delta_{ij} + L^i_n L^j_q U^n{}_m U^q{}_p \delta_{ij}) \\ &= \exp(2\alpha\phi)g_{\mu\nu} + \exp(2\beta\phi)M_{mn}(A^m{}_\mu A^n{}_\nu + A^m{}_\mu U^n{}_q \\ &\quad + U^m{}_p A^n{}_\nu + U^m{}_p U^n{}_q) \\ &= \hat{g}_{\hat{\mu}\hat{\nu}} \ ,\end{aligned}$$

which is exactly what we wanted.

We will also need the inverse vielbein, which we determine to be

$$\hat{e}_{\hat{b}}{}^{\hat{\mu}} = \begin{pmatrix} \exp(-\alpha\phi)e_b{}^\mu & -\exp(\alpha\phi)e_a{}^\mu A^q{}_\mu (U^{-1})^p{}_q \\ 0 & \exp(-\beta\phi)L^q_i (U^{-1})^p{}_q \end{pmatrix} \ , \tag{3.106}$$

where $L^m_i L^n_j \delta^{ij} = M^{mn}$ gives us the inverse of M_{mn}.

In addition, we want U and L to be constructed such that

$$\det(U) = \det(L) = 1 \ . \tag{3.107}$$

With this construction we can easily calculate the determinant of the vielbein and obtain

$$\hat{e} = e\exp((4\alpha + 3\beta)\phi) \ . \tag{3.108}$$

We are mainly interested in the scalar potential V which we want to determine from the reduction ansatz. Therefore, we can simplify our vielbein by setting the vectors A^m_μ to zero because they will anyway not appear in the interesting terms. This reduces possibly appearing covariant derivatives to partial derivatives and gives us the following vielbein

$$\hat{e}_{\hat{\mu}}{}^{\hat{a}} = \begin{pmatrix} \exp(\alpha\phi)e_\mu{}^a & 0 \\ 0 & \exp(\beta\phi)L_n{}^i U^n{}_m \end{pmatrix} \ . \tag{3.109}$$

Of course the determinant is unchanged by this simplification.

Now we want to calculate the Ricci scalar $\hat{R}$ by using the twisted reduction ansatz. We expect it to give us the 4-dimensional Ricci scalar R plus some additional terms. One of those extra terms should be our scalar potential.

The first step towards the Ricci scalar is the determination of the components of the connection one-form $\hat{\omega}_{\hat{r}\hat{s}}$. We denote the $\hat{t}$ component of $\hat{\omega}_{\hat{r}\hat{s}}$ by $\hat{\omega}_{\hat{t},\hat{r}\hat{s}}$. Note that those are the components with solely flat indices and that the connection one-form is antisymmetric in its indices:

$$\hat{\omega}_{\hat{r}\hat{s}} = -\hat{\omega}_{\hat{s}\hat{r}} \ . \tag{3.110}$$

Naturally, all components are antisymmetric in their last two indices as well.

Due to this antisymmetry, we just need to compute $\hat{\omega}_{c,ab}$, $\hat{\omega}_{c,aj}$, $\hat{\omega}_{c,ij}$, $\hat{\omega}_{k,ab}$, $\hat{\omega}_{k,aj}$ and $\hat{\omega}_{k,ij}$.

Following [21] we can determine the components by

$$\hat{\omega}_{\hat{t},\hat{r}\hat{s}} = -\hat{\Omega}_{\hat{t}\hat{r},\hat{s}} + \hat{\Omega}_{\hat{r}\hat{s},\hat{t}} - \hat{\Omega}_{\hat{s}\hat{t},\hat{r}} \ , \tag{3.111}$$

where

$$\hat{\Omega}_{\hat{r}\hat{s},\hat{t}} = \frac{1}{2}(\hat{e}_{\hat{r}}{}^{\hat{\mu}}\hat{e}_{\hat{s}}{}^{\hat{\nu}} - \hat{e}_{\hat{s}}{}^{\hat{\mu}}\hat{e}_{\hat{r}}{}^{\hat{\nu}})\partial_{\hat{\nu}}\hat{e}_{\hat{\mu}\hat{t}} \ . \tag{3.112}$$

$\hat{\Omega}_{\hat{r}\hat{s},\hat{t}}$ is obviously antisymmetric in its first two indices.

In the following, we list the relevant $\hat{\Omega}$ components:

$$\hat{\Omega}_{ab,c} = \exp(-\alpha\phi)(\Omega_{ab,c} + \frac{1}{2}\alpha(\partial_\nu\phi)(\eta_{ac}e_b{}^\nu - \eta_{bc}e_a{}^\nu)) \ , \tag{3.113}$$

$$\hat{\Omega}_{aj,c} = 0 \ , \tag{3.114}$$

$$\hat{\Omega}_{ca,j} = 0 \ , \tag{3.115}$$

$$\hat{\Omega}_{ij,c} = 0 \ , \tag{3.116}$$

$$\hat{\Omega}_{ci,j} = -\frac{1}{2}\exp(-\alpha\phi)e_c{}^{\nu}(\beta(\partial_\nu\phi)\delta_{ij} + L_i^p(\partial_\nu L_p^k)\delta_{kj}) \ , \tag{3.117}$$

$$\hat{\Omega}_{ij,k} = \frac{1}{4}\exp(-\beta\phi)f_{qp}{}^{r}L_r^l\delta_{lk}(L_i^q L_j^p - L_j^q L_i^p) \ . \tag{3.118}$$

Using these results, we obtain the following components of the connection one-form:

$$\hat{\omega}_{c,ab} = \exp(-\alpha\phi)(\omega_{c,ab} + \alpha(\partial_\nu\phi)(\eta_{ac}e_b{}^{\nu} - \eta_{bc}e_a{}^{\nu})) \ , \tag{3.119}$$

$$\hat{\omega}_{c,aj} = 0 \ , \tag{3.120}$$

$$\hat{\omega}_{c,ij} = \exp(-\alpha\phi)e_c{}^{\nu}(\partial_\nu L_p^k)L_{[i}^p\delta_{j]k} \ , \tag{3.121}$$

$$\hat{\omega}_{k,ab} = 0 \ , \tag{3.122}$$

$$\hat{\omega}_{k,aj} = -\frac{1}{2}\exp(-\alpha\phi)e_a{}^{\nu}(2\beta(\partial_\nu\phi)\delta_{kj} + (\partial_\nu L_p^l)(L_j^p\delta_{lk} + L_k^p\delta_{lj})) \ , \tag{3.123}$$

$$\hat{\omega}_{k,ij} = \frac{1}{4}f_{qp}{}^{r}\exp(-\beta\phi)L_r^l[2L_{[i}^q L_{j]}^p\delta_{lk} - 2L_{[k}^q L_{i]}^p\delta_{lj} - 2L_{[j}^q L_{k]}^p\delta_{li}] \ . \tag{3.124}$$

We will need the components of the connection one-form with one curved index. Those are obtained by

$$\hat{\omega}_{\hat{\mu},\hat{r}\hat{s}} = \hat{e}_{\hat{\mu}}{}^{\hat{t}}\hat{\omega}_{\hat{t},\hat{r}\hat{s}} \ . \tag{3.125}$$

The non-vanishing components are

$$\hat{\omega}_{\mu,ab} = e_\mu{}^{c}[\omega_{c,ab} + \alpha(\partial_\rho\phi)(\eta_{ac}e_b{}^{\rho} - \eta_{bc}e_a{}^{\rho})] \ , \tag{3.126}$$

$$\hat{\omega}_{\mu,ij} = (\partial_\mu L_p^k)L_{[i}^p\delta_{j]k} \ , \tag{3.127}$$

$$\hat{\omega}_{m,ij} = \frac{1}{4} f_{qp}{}^{r} U^{n}{}_{m} [2 L^{q}_{[i} L^{p}_{j]} M_{nr} - 4 \delta^{[q}_{n} L^{p]}_{[i} \delta_{j]l} L^{l}_{r}] \ , \tag{3.128}$$

$$\begin{aligned} \hat{\omega}_{n,aj} = -\frac{1}{2} \exp((\beta - \alpha)\phi) e_{a}{}^{\nu} U^{m}{}_{n} [2\beta(\partial_\nu \phi) \delta_{kj} L^{k}_{m} \\ + (\partial_\nu L^{l}_{p})(L^{p}_{j} L^{k}_{m} \delta_{lk} + h^{p}_{m} \delta_{lj})] \ , \end{aligned} \tag{3.129}$$

where h^p_m here denotes the 3-dimensional metric of the compact space.

In order to now obtain the Ricci scalar, we follow [21] again:

$$\hat{R} = \hat{e}^{\hat{\mu}\hat{r}} \hat{e}^{\hat{\nu}\hat{s}} \hat{R}_{\hat{\mu}\hat{\nu}\hat{r}\hat{s}} \ , \tag{3.130}$$

where the components of the Riemann tensor are given by

$$\hat{R}_{\hat{\mu}\hat{\nu}\hat{r}\hat{s}} = 2\partial_{[\hat{\mu}} \hat{\omega}_{\hat{\nu}],\hat{r}\hat{s}} + 2\hat{\omega}_{[\hat{\mu},\hat{r}}{}^{\hat{t}} \hat{\omega}_{\hat{\nu}],\hat{t}\hat{s}} \ . \tag{3.131}$$

We make use of the antisymmetry property of the components:

$$R_{\hat{\mu}\hat{\nu}\hat{r}\hat{s}} = -R_{\hat{\nu}\hat{\mu}\hat{r}\hat{s}} = -R_{\hat{\mu}\hat{\nu}\hat{s}\hat{r}} \ . \tag{3.132}$$

Due to this antisymmetry, we only need to care about 9 components. In terms of the connection one-form components we find:

$$\hat{R}_{\mu\nu ab} = 2\partial_{[\mu} \hat{\omega}_{\nu],ab} + 2\hat{\omega}_{[\mu,a}{}^{c} \hat{\omega}_{\nu],cb} \ , \tag{3.133}$$

$$\hat{R}_{\mu\nu aj} = 0 \ , \tag{3.134}$$

$$\hat{R}_{\mu\nu ij} = 2\partial_{[\mu} \hat{\omega}_{\nu],ij} + 2\hat{\omega}_{[\mu,i}{}^{k} \hat{\omega}_{\nu],kj} \ , \tag{3.135}$$

$$\hat{R}_{\mu nab} = 0 \ , \tag{3.136}$$

$$\hat{R}_{\mu naj} = \partial_{\mu} \hat{\omega}_{n,aj} + \hat{\omega}_{\mu,a}{}^{c} \hat{\omega}_{n,cj} \ , \tag{3.137}$$

$$\hat{R}_{\mu nij} = 2\partial_{[\mu} \hat{\omega}_{n],ij} + 2\hat{\omega}_{[\mu,i}{}^{k} \hat{\omega}_{n],kj} \ , \tag{3.138}$$

$$\hat{R}_{mnab} = 2\hat{\omega}_{[m,a}{}^{k}\hat{\omega}_{n],kb}\ , \tag{3.139}$$

$$\hat{R}_{mnaj} = 2\partial_{[m}\hat{\omega}_{n],aj} + 2\hat{\omega}_{[m,a}{}^{k}\hat{\omega}_{n],kj}\ , \tag{3.140}$$

$$\hat{R}_{mnij} = 2\partial_{[m}\hat{\omega}_{n],ij} + 2\hat{\omega}_{[m,i}{}^{k}\hat{\omega}_{n],kj} + 2\hat{\omega}_{[m,i}{}^{c}\hat{\omega}_{n],cj}\ , \tag{3.141}$$

where by, for example, $\hat{\omega}_{[\mu,a}{}^{c}\hat{\omega}_{\nu],cb}$ we mean an antisymmetrisation in μ and ν. This means that the indices on the other side of the comma are not affected by the antisymmetrisation.

As we can see, most of the Riemann tensor components are non-zero. However, if we want to compute the Ricci scalar, 4 of those terms must be contracted with vielbein components that are zero. Hence, our Ricci scalar is given by:

$$\begin{aligned} \hat{R} &= \hat{R}_{\mu\nu ab}\hat{e}^{\mu a}\hat{e}^{\nu b} + \hat{R}_{\mu naj}\hat{e}^{\mu a}\hat{e}^{nj} + \hat{R}_{n\mu ja}\hat{e}^{nj}\hat{e}^{\mu a} + \hat{R}_{mnij}\hat{e}^{mi}\hat{e}^{nj} \\ &= \hat{R}_{\mu\nu ab}\hat{e}^{\mu a}\hat{e}^{\nu b} + 2\hat{R}_{\mu naj}\hat{e}^{\mu a}\hat{e}^{nj} + \hat{R}_{mnij}\hat{e}^{mi}\hat{e}^{nj}\ . \end{aligned} \tag{3.142}$$

In terms of the connection one-form the Ricci scalar is

$$\begin{aligned} \hat{R} = & 2\partial_{[\mu}\hat{\omega}_{\nu],ab}\hat{e}^{\mu a}\hat{e}^{\nu b} + 2\hat{\omega}_{[\mu,a}{}^{c}\hat{\omega}_{\nu],cb}\hat{e}^{\mu a}\hat{e}^{\nu b} + 2\partial_{\mu}\hat{\omega}_{n,aj}\hat{e}^{\mu a}\hat{e}^{nj} \\ & + 2\hat{\omega}_{\mu,a}{}^{c}\hat{\omega}_{n,cj}\hat{e}^{\mu a}\hat{e}^{nj} + 2\partial_{[m}\hat{\omega}_{n],ij}\hat{e}^{mi}\hat{e}^{nj} + 2\hat{\omega}_{[m,i}{}^{k}\hat{\omega}_{n],kj}\hat{e}^{mi}\hat{e}^{nj} \\ & + 2\hat{\omega}_{[m,i}{}^{c}\hat{\omega}_{n],cj}\hat{e}^{mi}\hat{e}^{nj}\ . \end{aligned} \tag{3.143}$$

We split up our computation in parts and obtain for the first two terms

$$\begin{aligned} 2[\partial_{[\mu}\hat{\omega}_{\nu],ab} + \hat{\omega}_{[\mu,a}{}^{c}\hat{\omega}_{\nu],cb}]\hat{e}^{\mu a}\hat{e}^{\nu b} = & \exp(-2\alpha\phi)[R - 6\alpha(\partial_{\mu}\partial^{\mu}\phi) \\ & - 2\alpha(\partial_{\mu}e_{\nu}{}^{c})e_{c}{}^{\nu}(\partial^{\mu}\phi) - 4\alpha(\partial_{\mu}e_{a}{}^{\nu})e^{\mu a}(\partial^{\nu}\phi) \\ & - 4\alpha(\partial^{\nu}\phi)\omega_{f,}{}^{fc}e_{c}{}^{\nu} - 6\alpha^{2}(\partial\phi)^{2}]\ . \end{aligned} \tag{3.144}$$

The third term is

$$\begin{aligned} 2\partial_{\mu}\hat{\omega}_{n,aj}\hat{e}^{\mu a}\hat{e}^{nj} = & -6\beta\exp(-2\alpha\phi)[(\beta-\alpha)(\partial\phi)^{2} + (\partial_{\nu}\phi)(\partial_{\mu}e_{a}^{\nu})e^{\mu a} \\ & + (\partial_{\nu}\partial^{\nu}\phi)]\ . \end{aligned} \tag{3.145}$$

For the fourth term we get

$$2\hat{\omega}_{\mu,a}{}^{c}\hat{\omega}_{n,cj}\hat{e}^{\mu a}\hat{e}^{nj} = -6\beta\exp(-2\alpha\phi)(\partial_{\nu}\phi)[\omega_{e,}{}^{ec}e_{c}^{\nu} + 3\alpha(\partial^{\nu}\phi)]\ . \tag{3.146}$$

The fifth and sixth term are given by

$$2[\partial_{[m}\hat{\omega}_{n],ij} + \hat{\omega}_{[m,i}{}^{k}\hat{\omega}_{n],kj}]\hat{e}^{mi}\hat{e}^{nj} = -\frac{1}{4}\exp(-2\beta\phi)[2f_{qp}{}^{r}f_{rs}{}^{q}M^{ps} + f_{qp}{}^{r}f_{tu}{}^{s}M^{qt}M^{pu}M_{sr}] \ . \tag{3.147}$$

Finally, the last term is

$$2\hat{\omega}_{[m,i}{}^{c}\hat{\omega}_{n],cj}\hat{e}^{mi}\hat{e}^{nj} = \exp(-2\alpha)(\frac{1}{4}\mathrm{tr}(\partial_\mu M \partial^\mu M^{-1}) - 6\beta^2(\partial\phi)^2) \ . \tag{3.148}$$

We often made use of $(\partial_\nu L^l_p)(M^{pq}L^k_q\delta_{lk} + L^p_l) = 0$. Results for the terms before the contractions with the vielbeine can be found in the section Appendix.

Now we have everything together to write down our Ricci scalar or even the Lagrangian. However, we do not know anything about the constants α and β yet. When we look at (3.145), we can see that our 4-dimensional Ricci scalar has a prefactor $\exp(-2\alpha\phi)$, which becomes $e\exp((2\alpha + 3\beta)\phi)$ if we also consider the vielbein determinant. Our aim is to reproduce the usual 4-dimensional Einstein-Hilbert Lagrangian. Therefore, this prefactor has to be simply e. This requirement gives us the following relation between α and β:

$$\alpha = -\frac{3}{2}\beta \tag{3.149}$$

and tells us that the vielbein determinant is

$$\hat{e} = e\exp(2\alpha\phi) \ . \tag{3.150}$$

Plugging in all these results into (3.101), we obtain

$$\begin{aligned} e^{-1}\mathcal{L} = R &- \frac{10}{3}\alpha^2(\partial\phi)^2 + \frac{1}{4}\mathrm{tr}(\partial_\mu M\partial^\mu M^{-1}) + V - 2\exp(2\alpha\phi)\Lambda^{(7)}_{\text{eff.}} \\ &- 2\alpha[(\partial_\mu\partial^\mu\phi) + (\partial^\mu\phi)(\partial_\mu e_\nu{}^{c})e_c{}^{\nu}] + ... \end{aligned} \tag{3.151}$$

for the 4-dimensional Lagrangian. Hereby we identified

$$V = -\frac{1}{4}\exp(2(\alpha-\beta)\phi)[2f_{qp}{}^{r}f_{rs}{}^{q}M^{ps} + f_{qp}{}^{r}f_{tu}{}^{s}M^{qt}M^{pu}M_{sr}] \ . \tag{3.152}$$

Our total scalar potential is then given by

$$\mathcal{V} = -V + 2\exp(2\alpha\phi)\Lambda^{(7)}_{\text{eff.}} \ , \tag{3.153}$$

which we can interpret as a new effective cosmological constant $-2\Lambda_{\text{eff.}}^{(4)}$, where the (4) indicates that this constant belongs to our 4-dimensional spacetime.

Of course we also want that our kinetic term of the scalar field ϕ has the usual prefactor $-1/2$. This additional requirement leads us to

$$\alpha^2 = \frac{3}{20} \ . \tag{3.154}$$

The terms $-2\alpha[(\partial_\mu \partial^\mu \phi) + (\partial^\mu \phi)(\partial_\mu e_\nu{}^c) e_c{}^\nu]$ can be ignored since they will integrate out in the Einstein-Hilbert action.

All in all, our final result is

$$e^{-1}\mathcal{L} = R - \frac{1}{2}(\partial\phi)^2 + \frac{1}{4}\text{tr}(\partial_\mu M \partial^\mu M^{-1}) - 2\Lambda_{\text{eff.}}^{(4)} + \dots \ . \tag{3.155}$$

Before we close this chapter, we want to argue why the last two terms can be integrated out. In particular, we will show that those two terms actually are a covariant divergence. It is defined by (see e.g. [22]):

$$\nabla_\mu V^\mu(x) = \frac{1}{e}\partial_\mu(eV^\mu(x)) \tag{3.156}$$

for a vector field $V(x)$.

If we expand this definition, then we find

$$\nabla_\mu V^\mu(x) = \partial_\mu V^\mu(x) + \frac{1}{e}V^\mu(x)(\partial_\mu e) \ . \tag{3.157}$$

Obviously, we can identify the first term in this expansion with $(\partial_\mu \partial^\mu \phi)$ for $V^\mu = \partial^\mu \phi$. What remains to show is the identification $(\partial_\mu e_\nu{}^c) e_c{}^\nu = \frac{1}{e}(\partial_\mu e)$ for e being the determinant of our 4-dimensional vielbein.

For this we use the following formulas for the determinant of an n-dimensional vielbein and its inverse, which can be found, for example, in [23]:

$$\det(e_\mu{}^a) = \frac{1}{n!}\varepsilon^{\mu_1\mu_2\dots\mu_n}\varepsilon_{a_1a_2\dots a_n} e_{\mu_1}{}^{a_1} e_{\mu_2}{}^{a_2} \dots e_{\mu_n}{}^{a_n} \ , \tag{3.158}$$

$$\det(e_a{}^\mu) = \frac{1}{n!}\varepsilon_{\mu_1\mu_2...\mu_n}\varepsilon^{a_1a_2...a_n}e_{a_1}{}^{\mu_1}e_{a_2}{}^{\mu_2}...e_{a_n}{}^{\mu_n} \ . \tag{3.159}$$

Since we know that the following is true for an invertible matrix $\mathcal{M}$,

$$\det(\mathcal{M}^{-1}) = \frac{1}{\det(\mathcal{M})} \ , \tag{3.160}$$

we can write

$$\frac{1}{e} \cdot e = \det(e_\mu{}^a)\det(e_a{}^\mu) \tag{3.161}$$

for our $n = 4$-dimensional vielbein $e_\mu{}^a$.

Since $1/e \cdot e = 1$, we know

$$\varepsilon^{\mu_1...\mu_4}\varepsilon_{a_1...a_4}\varepsilon_{\nu_1...\nu_4}\varepsilon^{b_1...b_4}e_{\mu_1}{}^{a_1}...e_{\mu_4}{}^{a_4}e_{b_1}{}^{\nu_1}...e_{b_4}{}^{\nu_4} = (4!)^2 \ . \tag{3.162}$$

This $(4!)^2$ is the result of a sum of terms which are all proportional to 4 since each of them resulted from one to four factors like $\delta^c{}_c = 4$ which appear due to the contractions of vielbeine with inverse vielbeine. This means that, if for some reason, one contraction like $e_\rho{}^c e_c{}^\rho$ is taken out of each term, we get only $3! \cdot 4!$ as a result.

Keeping this in mind, we now look at the derivative of the determinant of a 4-dimensional vielbein, which turns out to be

$$\partial_\rho e = 4 \cdot \frac{1}{4!}\varepsilon^{\mu_1...\mu_4}\varepsilon_{a_1...a_4}(\partial_\rho e_{\mu_1}{}^{a_1})e_{\mu_2}{}^{a_2}e_{\mu_3}{}^{a_3}e_{\mu_4}{}^{a_4} \tag{3.163}$$

because we can exchange and rename indices such that we get the upper result.

If we now multiply this derivative with the determinant of the inverse vielbein, vielbeine and inverse vielbeine are of course contracted again. However, due to the derivative, one pair will not give us a 4 but $(\partial_\mu e_\rho{}^c)e_c{}^\rho$ instead. Therefore, we obtain

$$\frac{1}{e}(\partial_\mu e) = \frac{4}{4!} \cdot \frac{3! \cdot 4!}{4!}(\partial_\mu e_\rho{}^c)e_c{}^\rho \ , \tag{3.164}$$

which is exactly our desired identification and hence we conclude

$$(\partial_\mu \partial^\mu \phi) + (\partial^\mu \phi)(\partial_\mu e_\nu{}^c)e_c{}^\nu = \nabla_\mu(\partial^\mu \phi) \ . \tag{3.165}$$

If we now consider this term in the action, we see

$$\int e\nabla_\mu(\partial^\mu \phi)d^4x = \int \partial_\mu(e(\partial^\mu \phi))d^4x = 0 \tag{3.166}$$

because the scalar field ϕ and all its derivatives vanish at the boundaries.

4 Conclusions and outlook

In this final chapter of our thesis we want to have a look back at what we did and also state what more could be done.

This thesis dealt with the compactification of 11-dimensional supergravity on $S^4 \times T^3$. We wanted to describe a way to come from an 11-dimensional theory to a theory in our well-known 4 spacetime dimensions. Our final objective was the determination of a scalar potential which appears in the 4-dimensional theory due to the reduction on a twisted torus.

The first chapter was devoted to string and M-theory as well as to the concept of compactification. We explained why we need to compactify theories and went through a simple example for a compactification, i.e. the Kaluza-Klein reduction of a bosonic string.

In the second chapter we first introduced some relevant concepts of supersymmetry. Afterwards, we presented aspects of supergravity. Particularly important was the introduction of 11-dimensional supergravity since it was the theory we considered in the first part of chapter 3.

Initially, we reviewed the Kaluza-Klein reduction of $D = 11$ supergravity on S^4 to 7-dimensional $\mathcal{N} = 4$ gauged supergravity in the third chapter. After that, we shortly discussed this supergravity theory in 7 dimensions before we compactified it on a twisted torus to 4-dimensional supergravity. From this compactification we obtained

$$V = -\frac{1}{4}\exp(2(\alpha - \beta)\phi)[2f_{qp}{}^r f_{rs}{}^q M^{ps} + f_{qp}{}^r f_{tu}{}^s M^{qt} M^{pu} M_{sr}] \ ,$$

which resembles the result of [21] if we put in our choice for the vielbein. Also the treatment in [24] seems to indicate that our result is correct since they use a comparable reduction ansatz but it is important to note that they compactify a different theory.

Our total scalar potential was

$$\mathcal{V} = \frac{1}{4}\exp(2(\alpha-\beta)\phi)[2f_{qp}{}^{r}f_{rs}{}^{q}M^{ps} + f_{qp}{}^{r}f_{tu}{}^{s}M^{qt}M^{pu}M_{sr}] + 2\exp(2\alpha\phi)\Lambda^{(7)}_{\text{eff.}}\ ,$$

which was interpreted as a term including a new effective cosmological constant $\Lambda^{(4)}_{\text{eff.}}$.

For sure, there are many more things we could do. For example, we could perform our calculations for the twisted toroidal reduction without ignoring the vectors A^{m}_{μ}. However, this would not affect our scalar potential of course. Further, we could investigate different choices for our group manifold to which we promoted the torus. A more complex but interesting task would be to also derive the SUSY transformation rules from the twisted reduction. Certainly, it might also be interesting to look at compactifications on $S^4 \times M_3$, where M_3 is a compact manifold other than a twisted torus.

Appendix

Here we give the terms in the Ricci scalar (3.143) before we contracted them with the vielbeine:

$$
\begin{aligned}
2[\partial_{[\mu}\hat{\omega}_{\nu],ab} + \hat{\omega}_{[\mu,a}{}^{c}\hat{\omega}_{\nu],cb}] = & \, 2\partial_{[\mu}(\omega_{\nu],ab}) + 2\omega_{[\mu,a}{}^{c}\omega_{\nu],cb} \\
& + 4\alpha[(\partial_\rho\phi)(\partial_{[\mu}e_{\nu]}^{c})\eta_{c[a}e_{b]}{}^{\rho} + (\partial_\rho\partial_{[\mu}\phi)e_{\nu]}^{c}\eta_{c[a}e_{b]}{}^{\rho}] \\
& + 2\alpha(\partial_\rho\phi)(\eta_{ac}(\partial_{[\mu}e_b{}^{\rho}) - \eta_{bc}(\partial_{[\mu}e_a{}^{\rho}))e_{\nu]}^{c} \\
& + 4\alpha\eta^{dc}e_{[\mu}^{f}e_{\nu]}^{e}[\omega_{f,ad}(\partial_\rho\phi)\eta_{e[c}e_{b]}{}^{\rho} + \omega_{e,cb}(\partial_\rho\phi)\eta_{f[a}e_{d]}{}^{\rho}] \\
& + 8\alpha^2\eta^{dc}e_{[\mu}^{f}e_{\nu]}^{e}(\partial_\rho\phi)(\partial_\sigma\phi)\eta_{e[c}e_{b]}{}^{\sigma}\eta_{f[a}e_{d]}{}^{\rho} \ ,
\end{aligned}
\tag{A1}
$$

$$
\begin{aligned}
2\partial_\mu\hat{\omega}_{n,aj} = & - \exp((\beta-\alpha)\phi)U^{m}{}_{n}\{[(\beta-\alpha)(\partial_\mu\phi)e_a^\nu + \partial_\mu e_a^\nu] \\
& \cdot [2\beta(\partial_\nu\phi)\delta_{kj}L_m^k + (\partial_\nu L_p^l)(L_j^p L_m^k\delta_{lk} + h_m^p\delta_{lj})] \\
& + e_a^\nu[2\beta(\partial^2_{\mu\nu}\phi)\delta_{kj}L_m^k + 2\beta(\partial_\nu\phi)\delta_{kj}(\partial_\mu L_m^k) \\
& + (\partial^2_{\mu\nu}L_p^l)(L_j^p L_m^k\delta_{lk} + h_m^p\delta_{lj}) \\
& + (\partial_\nu L_p^l)((\partial_\mu L_j^p)L_m^k\delta_{lk} + L_j^p(\partial_\mu L_m^k)\delta_{lk})]\} \ ,
\end{aligned}
\tag{A2}
$$

$$
\begin{aligned}
2\hat{\omega}_{\mu,a}{}^{c}\hat{\omega}_{n,cj} = & - \exp((\beta-\alpha)\phi)e_\mu^e e_c^\nu U^{m}{}_{n}\eta^{dc} \\
& \cdot [\omega_{e,ad} + \alpha(\partial_\rho\phi)(\eta_{ae}e_d^\rho - \eta_{de}e_a^\rho)] \\
& \cdot [2\beta(\partial_\nu\phi)\delta_{kj}L_m^k + (\partial_\nu L_p^l)(L_j^p L_m^k\delta_{lk} + h_m^p\delta_{lj})] \ ,
\end{aligned}
\tag{A3}
$$

$$
\begin{aligned}
2[\partial_{[m}\hat{\omega}_{n],ij} + \hat{\omega}_{[m,i}{}^{k}\hat{\omega}_{n],kj}] = & \, f_{qp}{}^{r}(\partial_{[m}U^{s}{}_{n]})[L_{[i}^{q}L_{j]}^{p}M_{sr} - 2h_s^{[q}L_{[i}^{p]}\delta_{j]l}L_r^l] \\
& + \frac{1}{4}f_{qp}{}^{r}f_{tu}{}^{v}U^{s}{}_{[m}U^{w}{}_{n]}[L_{[i}^{q}L_{k]}^{p}M_{sr} - 2h_s^{[q}L_{[i}^{p]}\delta_{k]h}L_r^h] \\
& \cdot [L_{[l}^{t}L_{j]}^{u}M_{wv} - 2h_w^{[t}L_{[l}^{u]}\delta_{j]g}L_v^g]\delta^{kl}
\end{aligned}
\tag{A4}
$$

$$
\begin{aligned}
2\hat{\omega}_{[m,i}{}^{c}\hat{\omega}_{n],cj} = & - \frac{1}{2}\exp(2(\beta-\alpha))U^{q}{}_{[m}U^{r}{}_{n]}e_d^\nu e_c^\rho\eta^{dc} \\
& \cdot [2\beta(\partial_\nu\phi)\delta_{ki}L_q^k + (\partial_\nu L_p^l)(L_i^p L_q^k\delta_{lk} + h_q^p\delta_{li})] \\
& \cdot [2\beta(\partial_\rho\phi)\delta_{Kj}L_r^K + (\partial_\rho L_s^L)(L_j^s L_r^K\delta_{LK} + h_r^s\delta_{Lj})] \ ,
\end{aligned}
\tag{A5}
$$

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