The geometry of supersymmetric non-linear sigma models in D $\leq$ 2 dimensions

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Abstract

After a review of the two-dimensional supersymmetric non-linear sigma models and the geometric constraints they put on the target space, I focus on sigma models in one dimension. The mathematical framework in terms of supersymmetry and complex geometry will also be studied and reviewed.

The geometric constraints arising in $D = 1$ are more general than in $D = 2$, and can only after some assumptions be reduced to the well known geometries arising in the two dimensional case.

Contents

1 Introduction

Non-linear sigma models provide a link between supersymmetry and complex geometry. The number of supersymmetries imposed on the sigma model determine the geometry of the target space, as was first realized in [3] and developed in [4], [5], [7], among others. In dimension $D = 2$, which is the dimension central for string theory, one supersymmetry implies no restriction on the target manifold, whereas two supersymmetires require a Kähler manifold and four supersymmetries require hyper-Kähler geometry. This has been described in detail in [7], [20], [31], [32] and will be studied and reviewed in section 6. The mathematical framework in terms of generalized complex structures was developed in [24] and [25].

Under the two assumptions, that the kinetic part of the Lagrangian depends only on the metric as $\sim g_{\mu\nu}(\phi)\partial_a\phi^\mu\partial^a\phi^\nu$, and that the fields ϕ^μ are functions of time and at least one spatial coordinate, three classes of supersymmetric sigma models are known: generic, Kähler and hyper-Kähler [29]. The first assumption can be extended by introducing an anti-symmetric B-field in addition to the metric, which was realized in [7] and will be reviewed in section 2.1 and 6. Obviously, the second assumption is automaticly relaxed when studying one-dimensional sigma models, since the fields don't depend on spatial coordinates per definition. Therefore, in $D = 1$, even a larger variety of supersymmetric sigma models can be constructed, as we will see in section 4.2.

The bosonic sigma model is derived in section 2 and its supersymmetric extension in section 4. Supersymmetry, (generalized) complex geometry and further mathematical framework needed for the study of non-linear sigma models are reviewed in section 3 and 5.

In section 7, I focus on the geometry of target space arising from supersymmetric non-linear sigma models in dimension $D = 1$. After a review of what is known in the area, I discuss some of these results in more detail in section 7.1 and in section 7.2 I explicitly construct a one-dimensional sigma model by dimensional reduction from two-dimensional sigma models. In section 7.3, the geometry arising on the target space by supersymmetric sigma models in one dimension is compared with higher dimensional cases. The manifolds of the one dimensional sigma models have a more complicated structure, and can only after certain assumptions be reduced to the well-known geometries that appear in dimension $D = 2$ [12]. Also, in $D = 1$, there is some flexibility when deriving the constraints imposed on the target space [22]. In one dimension, there is less space-time symmetry, which implies that one can construct more general Lagrangians than in higher dimensions. Some supersymmetric $D = 1$ sigma models feature target space geometries which cannot be reproduced by direct dimensional reduction from higher dimensional models, a fact which make them an interesting subject to study.

The one dimensional sigma models have many applications, such as describing the geodesic motion in the moduli space of black holes [17] and being the model for supersymmetric quantum mechanics, which arises in the light cone quantization of supersymmetric field theories.

For clarity, most of the longer calculations have been omitted or relegated to the appendix.

2 Sigma models

A sigma model is a set of maps $X^\mu : \sum \to \mathcal{T}$, where $\xi^i \in \sum, i = 1, ..., D$ are the coordinates on the D-dimensional parameter space $\sum$ and X^μ, $\mu = 0, ..., d-1$ are the coordinates in the d-dimensional target space $\mathcal{T}$, and an action giving the dynamics of the model.

2.1 The bosonic sigma model in $D = 2$

Although in no way fundamental, it is interesting that the action describing the 2-dimensional bosonic sigma model can be derived from a classical string. The potential energy of the string depends on its tension T, and setting $c = 1$, the mass density is equal to the tension and we get an action of the form

$$S = -T \int dA \tag{2.1}$$

Denote the Minkowski metric of the target space by $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, ..., 1)$ and let γ_{ab} be the induced metric on the world surface,

$$\gamma_{ab} = \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} \eta_{\mu\nu} = \begin{pmatrix} \left(\frac{\partial X}{d\xi^1}\right)^2 & \frac{\partial X^\mu}{d\xi^2} \frac{\partial X_\mu}{d\xi^1} \\ \frac{\partial X^\mu}{d\xi^1} \frac{\partial X_\mu}{d\xi^2} & \left(\frac{\partial X}{d\xi^2}\right)^2 \end{pmatrix}. \tag{2.2}$$

We require the action of the theory to be invariant under diffeomorphisms. Under a coordinate transformation, the invariant volume element is in general given by the so called *proper volume* $dV = d^p\xi\sqrt{-\det \gamma_{ab}}$. To see this, we note that under a coordinate transformation, writing $\gamma := \det \gamma_{ab}$ and the Jacobian matrix $\Lambda := \left[\frac{\partial \xi'^m}{\partial \xi^\mu}\right]$,

$$d^p\xi \mapsto \det \Lambda d^p\xi \quad \text{and} \quad \sqrt{-\gamma} \mapsto \sqrt{-\det((\Lambda^{-1})^T \gamma_{ab} \Lambda^{-1})} = \sqrt{-\gamma(\det \Lambda^{-1})^2} = \frac{\sqrt{-\gamma}}{\det \Lambda}. \tag{2.3}$$

Hence, the area element on the world sheet is given by $dA = d^2\xi\sqrt{-\det \gamma_{ab}}$ and we get the *Nambu-Goto action* [26]

$$S = -T \int d^2\xi \sqrt{-\det \gamma_{ab}} = -T \int d^2\xi \sqrt{(\partial_a X^\mu \partial_b X_\mu)^2 - (\partial_a X)^2 (\partial_b X)^2}, \tag{2.4}$$

where $a, b \in \{1, 2\}$ are the indices for the parameters ξ^a on the world surface. The difficulties of quantizing this action motivates the introduction of the classically equivalent *Polyakov action* [1], [2]

$$S = -\frac{T}{2} \int d^2\xi \sqrt{-h} h^{ab} \gamma_{ab}, \tag{2.5}$$

where $h := \det h_{ab}$, h_{ab} being defined as the independent metric of the world sheet. The fact that the Polyakov action is equivalent with the Nambu-Goto action can be seen by varying the action (2.5) with respect to h^{ab}:

$$\delta S = -\frac{T}{2} \int d^2\xi \sqrt{-h} \big[\gamma_{ab} - \frac{1}{2} h_{ab} h^{cd} \gamma_{cd}\big] \delta h^{ab}. \tag{2.6}$$

Requiring this to be zero gives $2\gamma_{ab} = h_{ab}h^{cd}\gamma_{cd}$ which in turn implies $2\sqrt{-\gamma} = h^{cd}\gamma_{cd}\sqrt{-h}$. Inserting this into the Polyakov action (2.5) recovers the Nambu-Goto action (2.4).

By a theorem by Hilbert, for any 2-dimensional surface with metric h_{ab} we can choose conformal coordinates in which the metric takes the diagonal form $h_{12} = h_{21} = 0, h_{11} = -h_{22}$ so that $\sqrt{-\det h_{ab}} = h_{11}$. In this gauge the Polyakov action (2.5) takes the simplified form

$$\begin{aligned} S &= -\frac{T}{2}\int d^2\xi\sqrt{-h}h^{ab}\gamma_{ab} \\ &= -\frac{T}{2}\int d^2\xi h_{11}(h^{11}\gamma_{11} + 0 + 0 - h^{11}\gamma_{22}) \\ &= -\frac{T}{2}\int d^2\xi\Big(\eta_{\mu\nu}\frac{\partial X^\mu}{\partial\xi^1}\frac{\partial X^\nu}{\partial\xi^1} - \eta_{\mu\nu}\frac{\partial X^\mu}{\partial\xi^2}\frac{\partial X^\nu}{\partial\xi^2}\Big) \\ &= \frac{T}{2}\int d^2\xi\eta_{\mu\nu}\partial_a X^\mu\partial^a X^\nu. \end{aligned} \tag{2.7}$$

For a target space with curvature, the Minkowski metric $\eta_{\mu\nu}$ is replaced by a general metric $G_{\mu\nu}$, and finally we arrive at the bosonic non-linear sigma model action

$$S = \frac{T}{2}\int d\tau d\sigma\partial_a X^\mu\partial^a X^\nu G_{\mu\nu}(X). \tag{2.8}$$

In $D = 2$, we can include an anti-symmetric tensor $B_{\mu\nu}$ in the background. Using light-cone coordinates, $x^{\pm\!\!\!\pm} = \frac{1}{\sqrt{2}}(\xi^1 \pm \xi^2)$ the action thus takes the simple form

$$S = \frac{1}{2}\int d^2x\Big[\partial_a X^\mu\eta^{ab}\partial_b X^\nu G_{\mu\nu}(X) + \epsilon^{ab}\partial_a X^\mu\partial_b X^\nu B_{\mu\nu}\Big] = \int d^2x\partial_{+\!\!\!+} X^\mu E_{\mu\nu}\partial_= X^\nu. \tag{2.9}$$

where $E_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}$ and we for simplicity skipped the factor T. The field equations are obtained from $\delta S = 0$:

$$\partial_{+\!\!\!+}\partial_= X^\mu + (\Gamma^{(0)\mu}_{\nu\tau} + T^\mu_{\nu\tau})\partial_{+\!\!\!+} X^\nu\partial_= X^\tau = 0, \tag{2.10}$$

or in shorter notation,

$$\nabla^{(+)}_{+\!\!\!+}\partial_= X^\mu = 0. \tag{2.11}$$

From these field equations one can see that the geometry of the target space involves torsion T.

2.2 The bosonic sigma model in $D = 1$

The geodesic equation for a free massive particle is, in accordance with the two dimensional case (2.1), given by extremizing the action

$$S = -m\int d\tau = -m\int\sqrt{-ds^2} = -m\int d\lambda\sqrt{-g_{\mu\nu}\frac{dX^\mu}{d\lambda}\frac{dX^\nu}{d\lambda}}, \tag{2.12}$$

where λ is a parameter proportional to the arc length. This is the one-dimensional analogue of the Nambu-Goto action (2.4). X^μ maps from the one-dimensional parameter space $t \in \Sigma$

to the target space $\mathcal{T}$ and can be viewed as the world line for a propagating particle. The geodesic equations resulting from $\delta S = 0$ read

$$\frac{d^2 X^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta} \frac{dX^\alpha}{d\lambda} \frac{dX^\beta}{d\lambda} = 0. \tag{2.13}$$

Since time t can be chosen as the parameter, the geodesic equations are obviously equivalent to the equations of motion $\nabla_t \dot{X}^\mu = \ddot{X}^\mu + \Gamma^\mu_{\alpha\beta} \dot{X}^\alpha \dot{X}^\beta = 0$ arising from the Euler-Lagrange equations for the Lagrangian for a free massive particle,

$$L = \frac{mv^2}{2} = \frac{m}{2}\, g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu. \tag{2.14}$$

The action for the one-dimensional sigma model can be derived in a manner similar to the two-dimensional case. The analogue of the Polyakov action (2.5) in one dimension is given by [1]

$$S = \frac{1}{2} \int dt \Big[\frac{1}{e}\, g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu - em^2\Big], \tag{2.15}$$

where $e = e(t)$ is the equivalent of the world-sheet metric h_{ab} in the two-dimensional case. Varying this action with respect to e gives the equations of motion

$$e = \frac{1}{m} \sqrt{-g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu}. \tag{2.16}$$

Eliminating e in (2.15) by inserting these equations of motion recovers the Nambu-Goto analogue (2.12) and shows the equivalence between the two actions. In the limit where $e = 1,\ m = 0$, the one-dimensional bosonic sigma model

$$S = \int dt\, L = \frac{1}{2} \int dt\, g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu \tag{2.17}$$

is finally recovered.

3 Supersymmetry and superfields

Supersymmetry is a symmetry relating bosons and fermions. It does so by combining integer and half-integer spin-states in one multiplet. The non-linear sigma model studied in the previous sections is valid only for bosons, and so fermions have to be included in the theory. Imposing supersymmetry simplifies the equations and relates the bosonic and fermionic fields in a way that has many far-reaching consequences. Supersymmetry is central in the recent understanding of non-perturbative physics [19] and it appears in most versions of string theory. Supersymmetry removes the tachyon out of string theory, and is a promising key ingredience for extending the standard model. Also, it relates physics and mathematics in an elegant way, as we will see in the following chapters.

3.1 The supersymmetry algebra

We first concentrate on $D = 4$ dimensions. The symmetries of quantum field theory can be divided into internal symmetries and the Poincaré group, i.e. the 10 dimensional symmetry group containing the 6 dimensional Lorentz transformations (boosts and rotations) and 4 dimensional translations. The attempts to find a larger symmetry group containing both the Poincaré group and the internal symmetry group came to a halt in 1967, after Coleman and Mandula proved the no-go theorem, saying that any larger symmetry group containing the Poincaré group and an internal symmetry group must be a direct product of the both. In other words, it is impossible to combine the Poincaré group and internal symmetries to a larger group in a non-trivial way.

The Coleman-Mandula theorem is based on the axioms of relativistic quantum field theory and the assumption, that all symmetries can be written in terms of Lie groups. Haag *et al* showed in the 70's that the no-go theorem can be circumvented by relaxing this last assumption, assuming instead that the infinitesimal generators of the symmetry obey a *graded Lie algebra*, or *superalgebra*. In a superalgebra, some of the generators are fermionic, which means they obey anti-commutation rules instead of commutation rules. This $\mathbb{Z}_2$ grading can for the bosonic (even) and fermionic (odd) infinitesimal generators be stated as the (anti-)commutation rules

$$\begin{aligned} [\text{even}, \text{even}] &= \text{even} \\ [\text{even}, \text{odd}] &= \text{odd} \\ \{\text{odd}, \text{odd}\} &= \text{even}. \end{aligned} \tag{3.1}$$

With B and F denoting even and odd generators, respectively, the generalized Jacobi identities are given by [9]

$$\begin{aligned} \big[[B_1, B_2], B_3\big] + \big[[B_3, B_1], B_2\big] + \big[[B_2, B_3], B_1\big] &= 0 \\ \big[[B_1, B_2], F_3\big] + \big[[F_3, B_1], B_2\big] + \big[[B_2, F_3], B_1\big] &= 0 \\ \big\{[B_1, F_2], F_3\big\} + \big\{[B_1, F_3], F_2\big\} + \big[\{F_2, F_3\}, B_1\big] &= 0 \\ \big[\{F_1, F_2\}, F_3\big] + \big[\{F_1, B_3\}, F_2\big] + \big[\{F_2, F_3\}, F_1\big] &= 0. \end{aligned} \tag{3.2}$$

Using the rules for the $\mathbb{Z}_2$ grading (3.1) and the generalized Jacobi identities, the supersymmetric algebra can be derived. For a more comprehensive derivation than the one given here, I refer to one of the textbooks [6], [9] or [14].

First, the supersymmetric group must contain the Poincaré group P, with generators for translations P_μ and for Lorentz transformations $M_{\mu\nu}$ fulfilling the algebra

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \\ [P_\mu, M_{\nu\tau}] &= \eta_{\mu[\tau}P_{\nu]} \\ [M_{\mu\nu}, M_{\tau\sigma}] &= \eta_{\tau[\mu}M_{\nu]\sigma} - \eta_{\sigma[\mu}M_{\nu]\tau}. \end{aligned} \tag{3.3}$$

Secondly, it may contain an internal symmetry group G, where the generators $B \in G$ fulfills its Lie algebra and commutes with the Poincaré generators

$$\begin{aligned} [B_I, B_J] &= f_{IJ}{}^K B_K \\ [P_\nu, B_I] &= 0 \\ [M_{\mu\nu}, B_I] &= 0, \end{aligned} \tag{3.4}$$

where $f_{IJ}{}^K$ is the structure constant for the Lie algebra of G. These six equations represent the equation for even generators in the $\mathbb{Z}_2$ graded algebra (3.1).

Now introducing N fermionic (odd) generators $Q^1_\alpha, Q^2_\alpha, ..., Q^N_\alpha$ will give N-extended Super-Poincaré algebra. Since Q are the only odd generators, the $\mathbb{Z}_2$ grading give the commutation rules between the even and odd generators as

$$\begin{aligned} [Q^i_\alpha, P_\mu] &= (a_\mu)^\beta_\alpha Q^i_\beta \\ [Q^i_\alpha, M_{\mu\nu}] &= (b_{\mu\nu})^\beta_\alpha Q^i_\beta \\ [Q^i_\alpha, B_I] &= (c_I)^{\beta i}_{\alpha j} Q^j_\beta, \end{aligned} \tag{3.5}$$

where a, b and c are yet undetermined. Inserting these relations in the generalized Jacobi identities and choosing the Q^i_α to be in the $(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$-representation of the Lorentz group yields

$$\begin{aligned} [Q^i_\alpha, P_\mu] &= 0 \\ [Q^i_\alpha, M_{\mu\nu}] &= \frac{1}{2}(\sigma_{\mu\nu})^\beta_\alpha Q^i_\beta \\ [Q^i_\alpha, B_I] &= (B_I)^i_j Q^j_\beta. \end{aligned} \tag{3.6}$$

The anti-commutation rule of the $\mathbb{Z}_2$-grading between the odd generators in equation (3.1) has not yet been considered. The fermionic generators must anti-commute to an even generator, which in its most general form is given by the linear combination

$$\{Q^i_\alpha, Q^j_\beta\} = r(\gamma^\mu C)_{\alpha\beta} P_\mu \delta^{ij} + s(\sigma^{\mu\nu}C)_{\alpha\beta} M_{\mu\nu}\delta^{ij} + C_{\alpha\beta}Z^{ij} + (\gamma_5 C)_{\alpha\beta} Y^{ij}, \tag{3.7}$$

where $C_{\alpha\beta} = -C_{\beta\alpha}$ is the charge conjugation matrix and Z^{ij}, Y^{ij} are the central charges. The central charges exist only in extended supersymmetry $N > 1$ [6], and are called central because they commute with all generators $\mathcal{O}$

$$[Z, \mathcal{O}] = [Y, \mathcal{O}] = 0. \tag{3.8}$$

Inserting equation (3.7) into the generalized Jacobi identities and normalizing P_μ by setting $r = 2$ finally gives

$$\{Q^i_\alpha, Q^j_\beta\} = 2(\gamma^\mu C)_{\alpha\beta} P_\mu \delta^{ij} + C_{\alpha\beta}Z^{ij} + (\gamma_5 C)_{\alpha\beta} Y^{ij}. \tag{3.9}$$

The full N-extended Super-Poincaré algebra in $D = 4$ is now given by the equations (3.3), (3.4), (3.6), (3.8) and (3.9).

The algebra can equivalently be written in Weyl representation using 2-component Weyl spinors. The equation (3.9) then take the form

$$\begin{aligned} \{Q^i_\alpha, \bar{Q}^j_{\dot{\alpha}}\} &= 2P_{\alpha\dot{\alpha}}\delta^{ij} \\ \{Q^i_\alpha, Q^j_\beta\} &= \varepsilon_{\alpha\beta}(Z^{ij} + Y^{ij}), \end{aligned} \tag{3.10}$$

where $P_{\alpha\dot{\alpha}} := (\sigma^\mu)_{\alpha\dot{\alpha}} P_\mu$ is a useful way of representing vector indices as pairs of spinor indices, and $\varepsilon_{\alpha\beta} = \varepsilon_{\dot{\alpha}\dot{\beta}} = -\varepsilon^{\alpha\beta} = -\varepsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

The algebra is greatly simplified in $N = 1$ supersymmetry and lower dimensions. In $D = 2$, the relevant part of the $N = (1, 1)$ superalgebra is given by

$$\begin{aligned} \{Q_\pm, Q_\pm\} &= 2i\partial_{\pm\pm} = 2P_{\pm\pm} \\ \{Q_\pm, Q_\mp\} &= 0, \end{aligned} \tag{3.11}$$

where $x^{++}, x^{=}$ are light-cone coordinates and the spinor index $\alpha = +, -$. Correspondingly, the $N = 1$ superalgebra in $D = 1$ dimensions is given by

$$\{Q, Q\} = 2i\partial_t = 2P. \tag{3.12}$$

We introduce covariant derivatives D as odd differential operators defined to anti-commute with the supersymmetry generators, $\{D, Q\} = 0$. Their explicit form and algebra in $D = 1$ and $D = 2$ can be taken to be

$$\begin{array}{lll} D = 1 & N = 1 & N = 2 \\ & D = \frac{\partial}{\partial\theta} + i\theta\frac{\partial}{\partial t} & D = \frac{\partial}{\partial\theta} + i\bar{\theta}\frac{\partial}{\partial t},\ \bar{D} = \frac{\partial}{\partial\bar{\theta}} + i\theta\frac{\partial}{\partial t} \\ & D^2 = i\partial_t & D^2 = \bar{D}^2 = 0,\ \{D, \bar{D}\} = 2i\partial_t \\ & & \\ D = 2 & N = (1,1) & N = (2,2) \\ & D_\pm = \frac{\partial}{\partial\theta^\pm} + i\theta^\pm\partial_{\pm\pm} & \mathbb{D}_\pm = \frac{\partial}{\partial\theta^\pm} + i\bar{\theta}^\pm\partial_{\pm\pm},\ \bar{\mathbb{D}}_\pm = \frac{\partial}{\partial\bar{\theta}^\pm} + i\theta^\pm\partial_{\pm\pm} \\ & D^2_\pm = i\partial_{\pm\pm} & \mathbb{D}^2 = \bar{\mathbb{D}}^2 = 0,\ \{\mathbb{D}, \bar{\mathbb{D}}\} = 2i\partial_{\pm\pm}. \end{array} \tag{3.13}$$

3.2 Superspace and superfields

In the same manner as the Minkowski space is defined as the coset of the Poincarégroup and the Lorentzgroup, $ISO(d-1,1)/SO(d-1,1)$, the superspace is defined as the coset of the Super-Poincarégroup and the Lorentzgroup, $SISO(d-1,1)/SO(d-1,1)$. The parameters of superspace are (x, θ) and relative to any origin, an element in the superspace is parametrized as

$$h(x, \theta) = e^{i(xP + \theta Q)}, \tag{3.14}$$

where xP and θQ is short-hand notation for $x^i P_i, i = 1, ..., D$ and $\theta^\alpha Q_\alpha, \alpha = 1, 2$.

A *superfield* is a function defined on the superspace, $\phi = \phi(x, \theta)$. Since θ are Grassmann variables, a Taylor expansion of the superfield in these parameters will terminate after a finite amount of terms. A superfield can thus be viewed as a collection of ordinary fields over the Minkowski space. In $D = 1$, the $N = 1$ and $N = 2$ superfields have 2 and 4 terms, respectively:

$$\begin{aligned}\phi^\mu(t,\theta) &= \phi^\mu| + \theta\phi^\mu| =: X^\mu(t) + \theta\lambda^\mu(t)\\ \phi^\mu(t,\theta,\bar\theta) &= \phi^\mu| + \theta D\phi^\mu| + \bar\theta\bar D\bar\phi^\mu| + \theta\bar\theta[D,\bar D]\phi|\\ &=: X^\mu(t) + \theta\lambda^\mu(t) + \bar\theta\bar\lambda^\mu(t) + \theta\bar\theta F^\mu(t),\end{aligned} \tag{3.15}$$

where $|$ is short-hand notation for $|_{\theta=\bar\theta=0}$. The component fields $(X^\mu, \lambda^\mu, F^\mu)$ are often referred to as a *multiplet*. The leading component X^μ is a bosonic scalar field, whereas the fields λ are fermionic (odd), since the covariant derivatives D are odd. F is an auxiliary field in the sense that its equations of motion are purely algebraic (i.e. contain no derivatives), and can so be used to eliminate F. Nevertheless, the presence of F will make it possible to write the supersymmetry transformations for the component fields which close off-shell.

In $D = 2$, the $N = (1,1)$ and $N = (2,2)$ superfields have, correspondingly, 4 and 16 terms:

$$\begin{aligned}\phi^\mu(x,\theta) &= X^\mu(x) + \theta^+\psi^\mu_+(x) + \theta^-\psi^\mu_-(x) + \theta^+\theta^-F^\mu(x),\\ \phi^\mu(x,\theta,\bar\theta) &= X^\mu(x) + \theta^+\psi^\mu_+(x) + \theta^-\psi^\mu_-(x) + \bar\theta^{\dot+}\bar\psi^\mu_{\dot+}(x) + \bar\theta^{\dot-}\bar\psi^\mu_{\dot-}(x)\\ &\quad +\theta^2M(x) + \bar\theta^2\bar M(x) + \theta^+\bar\theta^{\dot+}A_{+\dot+} + \theta^+\bar\theta^{\dot-}A_{+\dot-} + \theta^-\bar\theta^{\dot+}A_{-\dot+} + \theta^-\bar\theta^{\dot-}A_{-\dot-}\\ &\quad +\bar\theta^2\theta^+\lambda_+(x) + \bar\theta^2\theta^-\lambda_-(x) + \theta^2\bar\theta^{\dot+}\bar\lambda_{\dot+}(x) + \theta^2\bar\theta^{\dot-}\bar\lambda_{\dot-}(x) + \theta^2\bar\theta^2B(x).\end{aligned} \tag{3.16}$$

An infinitesimal supersymmetry transformation of a scalar superfield reads

$$\begin{aligned}\delta\phi &= \phi' - \phi\\ &= e^{i\epsilon Q}\phi(x,\theta)e^{-i\epsilon Q} - \phi(x,\theta)\\ &= i[\epsilon Q, \phi(x,\theta)],\end{aligned} \tag{3.17}$$

where in the last step the Baker-Hausdorff formula was used and all infinitesimal terms $\leq \epsilon^2$ were skipped.

4 Supersymmetric sigma models

4.1 Supersymmetric sigma models in $D = 2$

Starting from the bosonic sigma model (2.9), the $N = (1,1)$ supersymmetric sigma model is achieved by replacing bosonic fields by superfields and space-time derivatives by the spinorial covariant derivatives,

$$\left.\begin{array}{ccc} X^\mu(x) & \to & \phi^\mu(x,\theta) \\ \partial_{++}, \partial_{=} & \to & D_+, D_- \end{array}\right\} \Rightarrow S = \int d^2x d^2\theta D_+\phi^\mu E_{\mu\nu} D_-\phi^\nu. \tag{4.1}$$

The superfields $\phi = X + \theta^+\psi_+ + \theta^-\psi_- + \theta^+\theta^- F$ contain as we have seen the bosonic fields as lowest component in the Taylorexpansion in θ, $\phi^\mu(x,\theta)| = X^\mu(x)$, where $|$ denotes 'the θ-independent part of' as before. Using this and the properties of the Berezin integral, we see that the bosonic action is contained in the supersymmetric action.

$$\begin{aligned} S &= \int d^2x d^2\theta D_+\phi^\mu E_{\mu\nu} D_-\phi^\nu \\ &= \int d^2x D_+ D_- (D_+\phi^\mu E_{\mu\nu} D_-\phi^\nu)\Big| \\ &= \int d^2x \Big(- D_+^2\phi^\mu E_{\mu\nu}(\phi) D_-^2\phi^\nu + D_-D_+\phi^\mu E_{\mu\nu} D_+D_-\phi^\nu + 8 \text{ fermionic terms}\Big)\Big| \\ &= \int d^2x \Big(\underbrace{\partial_{++}X^\mu E_{\mu\nu}(X)\partial_{=}X^\nu}_{\text{the bosonic action}} - F^\mu E_{\mu\nu} F^\nu + 8 \text{ fermionic terms}\Big), \end{aligned} \tag{4.2}$$

where in the last step the algebra for the $N = (1,1)$ covariant derivatives in $D = 2$ were used, that is $D_+^2 = i\partial_{++}, D_-^2 = i\partial_{=}$.

An action written in terms of $N = (1,1)$-superfields $\phi^\mu(x,\theta)$ is manifestly invariant under $N = (1,1)$-transformations $\delta\phi^\mu = i\epsilon^+ Q_+\phi^\mu + i\epsilon^- Q_-\phi^\mu$ (see appendix A.2).

4.2 Supersymmetric sigma models in $D = 1$

In one dimension, the supersymmetric sigma model is constructed in the same manner as in the previous section for $D = 2$,

$$\left.\begin{array}{ccc} X^\mu(t) & \to & \phi^\mu(t,\theta) \\ \partial_t & \to & D \end{array}\right\} \Rightarrow S = -\frac{i}{2}\int dt d\theta g_{\mu\nu} D\phi^\mu \dot{\phi}^\nu. \tag{4.3}$$

As in the $D = 2$ case, the bosonic action is contained in this supersymmetric action, which can be seen by expanding the action in the components of the superfields ϕ^μ,

$$\begin{aligned} S &= -\frac{i}{2}\int dt\, d\theta\, g_{\mu\nu} D\phi^\mu \dot{\phi}^\nu \\ &= -\frac{i}{2}\int dt\, D\, g_{\mu\nu} D\phi^\mu \dot{\phi}^\nu\Big| \\ &= -\frac{i}{2}\int dt \Big(g_{\mu\nu} D^2\phi^\mu\dot{\phi}^\nu - g_{\mu\nu} D\phi^\mu D\dot{\phi}^\nu + g_{\mu\nu,\tau} D\phi^\tau D\phi^\mu \dot{\phi}^\nu\Big)\Big| \\ &= \int dt \Big(\underbrace{\frac{1}{2} g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu}_{\text{the bosonic action}} + \frac{i}{2} g_{\mu\nu}\lambda^\mu\dot{\lambda}^\nu - \frac{i}{2} g_{\mu\nu,\tau}\lambda^\tau\lambda^\mu\dot{X}^\nu\Big). \end{aligned} \tag{4.4}$$

In addition to the bosonic superfield ϕ^μ, one can introduce a fermionic superfield ψ^a with components

$$\psi^a\big| =: \lambda^a, \quad \nabla\psi^a\big| =: F^a, \tag{4.5}$$

where $\nabla\psi^a$ is defined introducing also a connection A as $\nabla\psi^a = D\psi^a + D\phi^\mu (A_\mu)^a_b \psi^b$. Comparing the component expansion of the bosonic field in equation (3.15), we see that the lowest component in ψ is a fermion λ and the second lowest an auxiliary field F. The introduction of a fermionic superfield ψ is necessary for the addition of a scalar potential in sigma models with $N = 1$ supersymmetry [22]. Attaching mass dimension zero to ϕ and $\frac{1}{2}$ to ψ, dimensional analysis shows that the most general $N = 1$ action with dimensionless couplings is given by [12]

$$\begin{aligned} S \;=\; & \int dt\, d\theta \Big(-\frac{i}{2}\, g_{\mu\nu} D\phi^\mu \dot{\phi}^\nu + \frac{1}{3!} h_{\mu\nu\tau} D\phi^\mu D\phi^\nu D\phi^\tau - \frac{1}{2} h_{ab}\psi^a \nabla\psi^b \\ & + \frac{1}{3!} I_{abc}\psi^a\psi^b\psi^c - i f_{\mu a}\dot{\phi}^\mu\psi^a + \frac{1}{2} m_{\mu ab}\psi^a\psi^b D\phi^\mu + \frac{1}{2} n_{\mu\nu a} D\phi^\mu D\phi^\nu \psi^a \Big). \end{aligned} \tag{4.6}$$

The model can be extended to include the coupling to a magnetic field and a scalar potential by adding to the action the two terms [22]

$$S = \int dt d\theta \Big(... + A_\mu D\phi^\mu + m s_a \psi^a \Big). \tag{4.7}$$

For many purposes, it is necessary only to consider special cases of this action. For example, the geometry of the moduli space of black holes is determined by a multiplet with a real scalar X^μ and its real fermionic partner λ^μ. The action of such a model is in components written as [17]

$$S = \frac{1}{2} \int dt \big[g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu + i g_{\mu\nu} \lambda^\mu \nabla_t^{(+)} \lambda^\nu - \frac{1}{3!}\, \partial_{[\mu} h_{\nu\tau\sigma]} \lambda^\mu \lambda^\nu \lambda^\tau \lambda^\sigma \big], \tag{4.8}$$

where $\nabla^{(+)}$ is a connection involving torsion h. This action corresponds to the $N = (1, 0)$ supersymmetric sigma model in $D = 2$, but in one dimension, the torsion h need not necessarily be a *closed* 3-form. For the case when h *is* closed, this action is obtained by direct dimensional reduction of the two-dimensional $N = (1, 0)$ action. In superspace formalism

$$\phi\big| = X, \quad D\phi\big| = \lambda, \quad D^2 = i\partial_t, \tag{4.9}$$

the action (4.8) reads

$$S = -\frac{1}{2} \int dt\, d\theta \, \big[i g_{\mu\nu} D\phi^\mu \dot{\phi}^\nu + \frac{1}{3!}\, h_{\mu\nu\tau} D\phi^\mu D\phi^\nu D\phi^\tau \big]. \tag{4.10}$$

5 Complex geometry

5.1 Complex structures

A *complex manifold* is defined as a topological space M with an atlas of charts to $\mathbb{C}^n$, so that the change of coordinates between the charts are holomorphic. In other words, every neighbourhood of the manifold looks like $\mathbb{C}^n$ in a coherent way. A complex n-dimensional manifold with complex vector fields $Z = X + iY$ can be viewed as a real 2n-dimensional manifold with real vector fields X, Y and a *complex structure* J which tells us how the two vector fields relate to one another, and which differential equations they have to fulfil in order for the change of coordinates between the complex vector fields $Z = X + iY$ to be holomorphic. The complex structure represents multiplication with i:

$$iZ = iX - Y \quad \Leftrightarrow \quad (X, Y) \overset{J}{\mapsto} (-Y, X). \tag{5.1}$$

Applying this map twice gives $J^2 = -1$. Any map fulfilling this condition is called an *almost complex structure.* Any almost complex structure $J : T_pM \to T_pM, \quad J^2 = -1$ has two eigenvalues $\pm i$. This implies that the tangent space of the manifold can be divided into two disjunct vector spaces $T_pM = T_pM^+ \oplus T_pM^-$, where $T_pM^\pm = \{Z \in T_pM : JZ = \pm iZ\}$. The distribution $T_pM^\pm$ is called *integrable* if and only if

$$X, Y \in T_pM^\pm \quad \Rightarrow [X, Y] \in T_pM^\pm, \tag{5.2}$$

where $[\cdot,\cdot]$ denotes the usual Lie bracket. A *complex structure* is an almost complex structure defining integrable subspaces. This condition for integrability can be rewritten using the projection operators $P^\pm := \frac{1}{2}(1 \mp iJ)$ as

$$P^\mp[P^\pm X, P^\pm Y] = 0 \quad \text{for } X, Y \in T_pM. \tag{5.3}$$

Defining the Nijenhuis tensor for J as $N(X,Y) := [X,Y]+J[JX,Y]+J[X,JY]-[JX,JY]$, this integrability condition can again be equivalently stated as the vanishing of the Nijenhuis tensor,

$$N(X, Y) = 0. \tag{5.4}$$

In other words, the condition $J^2 = -1$ is not sufficient for the change of coordinates to be holomorphic. The theorem by Newlander-Nirenberg says, that a sufficient condition for this is that the Nijenhuis tensor for J vanishes, $N(X,Y) = 0$. A structure J fulfilling the two conditions $J^2 = -1$ and $N(X,Y) = 0$ is called a *complex structure*, and a real manifold with a complex structure is called a complex manifold.

A Riemannian metric g of a complex manifold is called *hermitian* if $J^tgJ = g$, i.e. the complex structure J preserves the metric. The hermitian metric $ds^2 = g_{\mu\nu}dZ^\mu d\bar{Z}^\nu$ is called *Kähler* if the corresponding Kähler form $\omega = 2ig_{\mu\nu}dZ^\mu \wedge d\bar{Z}^\nu$ is closed, $d\omega = 0$. This implies the existence of a Kähler potential $K(Z, \bar{Z})$, so that the metric can be written locally as [3]

$$g_{\mu\nu} = \frac{\partial^2 K}{\partial Z^\mu \partial \bar{Z}^\nu}. \tag{5.5}$$

Denoting the coordinates of the real 2n dimensional manifold by X^i, $i = 1, ..., 2n$ and relating them to the complex coordinates by $Z^\mu = X^i + iX^{n+i}$, the Kähler form can be written in terms of the complex structure J by

$$\omega = 2ig_{\mu\nu}dZ^{\mu} \wedge d\bar{Z}^{\nu} = J_i^j g_{jk} dX^j \wedge dX^k. \tag{5.6}$$

The condition that the Kähler form is closed, $d\omega = 0$, is equivalent with the vanishing of the Levi-Civita covariant derivative of the complex structure

$$\nabla_i J_j^k = 0. \tag{5.7}$$

Conversely, $\nabla J = 0$ implies the vanishing of the Nijenhuis tensor $N(X,Y) = 0$ and the existence of a Kähler potential such that $g = \partial\bar{\partial}K$ [23].

Let us include torsion H in the connection,

$$\nabla^{(\pm)} = \nabla \pm g^{-1}H, \tag{5.8}$$

where ∇ is the Levi-Civita connection. If g is hermitian with respect to two complex structures, $J^{(\pm)t}gJ^{(\pm)} = g$, and the complex structures preserve the torsion, $J^{(\pm)t}HJ^{(\pm)} = H$, then a manifold for which the complex structure J is covariantly constant with respect to this connection,

$$\nabla^{(\pm)}J^{(\pm)} = 0 \tag{5.9}$$

is called a bihermitian complex manifold. A new interpretation of this geometry in terms of generalized complex geometry was given in [24] and [25].

5.2 Generalized complex structures

In the previous section, we saw that a complex structure is a map $J : TM \to TM$ with $J^2 = -1$ and whose Nijenhuis tensor vanishes. Complex structures can be generalized by substituting the tangent bundle by the direct sum of the tangent bundle and the cotangent bundle

$$TM \to TM \oplus T^*M, \tag{5.10}$$

and the Lie bracket by the Courant bracket

$$[X,Y] = XY - YX \to [X+\xi, Y+\eta]_C = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi), \tag{5.11}$$

where $X + \xi \in TM \oplus T^*M$, $\mathcal{L}_X$ denotes the Lie derivative along X, d the outer derivative and i_X the inner product. A *H-twisted* Courant bracket has an additional term including a closed 3-form H

$$[X+\xi, Y+\eta]_H = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi) + i_X i_Y H. \tag{5.12}$$

An important property of the Courant bracket, is that it allows an extra symmetry in addition to diffeomorphisms, namely b-field transformations involving a closed 2-form b acting as

$$X + \xi \mapsto X + \xi + i_X b. \tag{5.13}$$

The natural pairing $\mathcal{I}$ on $TM \oplus T^*M$ is given by $\Big\langle X+\xi, Y+\eta \Big\rangle = i_X\eta + i_Y\xi$. An almost *generalized* complex structure is thus, in accordance with the previous section, defined as an automorphism

$$\mathcal{J} : TM \oplus T^*M \to TM \oplus T^*M \tag{5.14}$$

which squares to minus one and preserves the natural pairing,

$$\mathcal{J}^2 = -1, \quad \mathcal{J}^t \mathcal{I} \mathcal{J} = \mathcal{I}. \tag{5.15}$$

The integrability condition is defined analogously as for complex structures. With projection operators defined as $\Pi_\pm := \frac{1}{2}(1 \mp i\mathcal{J})$, it can be written as

$$\Pi_\mp \left[\Pi_\pm(X+\xi), \Pi_\pm(Y+\eta)\right]_C = 0. \tag{5.16}$$

A map $\mathcal{J}$ fulfilling the conditions above is called a *generalized complex structure*, in accordance with the previous section. The generalized complex structure and the natural pairing can be written in local coordinates as [31]

$$\mathcal{J} = \begin{pmatrix} J & P \\ L & K \end{pmatrix}, \quad \mathcal{I} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5.17}$$

A *generalized Kähler geometry* is defined as a pair of two commuting generalized complex structures $\mathcal{J}_1, \mathcal{J}_2$ for which $\mathcal{G} = -\mathcal{J}_1\mathcal{J}_2$ defines a positive definite metric on $TM \oplus T^*M$. If (J, g, ω) is a Kähler form and we define two generalized complex structures by

$$\mathcal{J}_1 = \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix}, \quad \mathcal{J}_2 = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}, \tag{5.18}$$

then

$$\mathcal{G} = -\mathcal{J}_1\mathcal{J}_2 = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \tag{5.19}$$

defines a generalized Kähler geometry where the metric $\mathcal{G}$ is constructed from the Kähler metric g [27]. More generally, given a bihermitian structure $(g, B, J_\pm)$ with corresponding forms $\omega_\pm = gJ_\pm$, a generalized Kähler structure can be defined by the two generalized complex structures [25]

$$\mathcal{J}_{1,2} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}\begin{pmatrix} J_+ \pm J_- & -[\omega_+^{-1} \mp \omega_-^{-1}] \\ \omega_+ \mp \omega_- & -[J_+^t \pm J_-^t] \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}. \tag{5.20}$$

The inverse is true up to the symmetries of the Courant bracket; b-transforms and diffeomorphisms [33]. This is the explicit map between bihermitian geometry given by (g, B, J_+, J_-) and generalized Kähler geometry.

6 Geometry of supersymmetric sigma models in D = 2

Adding one supersymmetry to the sigma model does not result in any requirements on the geometry of the target space; we achieve the field equations

$$\nabla_{+}^{(+)} D_{-}\phi^{\mu} = 0. \tag{6.1}$$

(See appendix A.1.) This can be compared with the field equations achieved for the non-supersymmetric sigma model,

$$\nabla_{\mp}^{(+)} \partial_{=} X^{\mu} = 0. \tag{6.2}$$

The field equations (6.1) tell us, as in the bosonic case, that the target space is Riemannian with torsion. In order to get more conditions on the geometry of the target space, an extra supersymmetry has to be added to the model. This can be done in two ways; either by starting with a manifest $N = (1,1)$ sigma model and making an ansatz for an extra (non-manifest) supersymmetry, or by reducing the manifest $N = (2,2)$ sigma model to a manifest $N = (1,1)$ sigma model with one extra supersymmetry. These two methods will be studied in section 6.1.

In recent years, the concepts of complex structures have been generalized [24] [25], as reviewed in section 5.2. It is an interesting question to ask, whether the geometry arising from supersymmetric sigma models can be incorporated in this broader mathematical framework. Indeed, this question has been asked, and it has been found that sigma models do encompass a more general geometry. This will be studied in section 6.2.

6.1 Complex geometry realized in $D = 2$ sigma models

The manifest $N = (1,1)$ sigma model (4.1) $S = \int d^2x d^2\theta D_{+}\phi^{\mu} E_{\mu\nu}(\phi) D_{-}\phi^{\nu}$, where $E_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}$ can be extended to a non-manifest $N = (2,2)$ sigma model by making an ansatz for a second supersymmetry

$$\delta_2 \phi^{\mu} = \epsilon^{+} D_{+}\phi^{\nu} J_{\nu}^{(+)\mu} + \epsilon^{-} D_{-}\phi^{\nu} J_{\nu}^{(-)\mu}. \tag{6.3}$$

This ansatz is unique, as can be shown by dimensional analysis. The second supersymmetry should fulfill the same algebra as the $N = (1,1)$ supersymmetry algebra, $[\delta_2^{\pm}(\epsilon_1^{\pm}), \delta_2^{\pm}(\epsilon_2^{\pm})] = -2i\epsilon_1^{\pm}\epsilon_2^{\pm}\partial_{\pm\pm}$. Further, the new supersymmetry must commute with the first, $[\delta_1, \delta_2] = 0$, and the transformation in the left- and right-going direction must commute, $[\delta_2^{\pm}(\epsilon_1^{\pm}), \delta_2^{\mp}(\epsilon_2^{\mp})] = 0$. Under these assumptions, one can show that the $N = (1,1)$ action is invariant under the extended supersymmetry, if and only if the tensors $J_{\nu}^{(\pm)\mu}$ are covariantly constant complex structures, i.e. they fulfil the conditions [7]

- $J^{(\pm)}$ are almost complex structures, $J^{(\pm)2} = -1$
- $J^{(\pm)}$ leaves the metric invariant, $J^{(\pm)T} G J^{(\pm)} = G$ or with other words, the metric is hermitian with respect to $J^{(+)}$ and $J^{(-)}$,
- $J^{(\pm)}$ leaves the torsion invariant, $J_{[\lambda}^{(\pm)\mu} J_{\rho}^{(\pm)\nu} H_{|\mu\nu|\tau]} = H_{\lambda\rho\tau}$,

- The Nijenhuistensor vanish, $N^{(\pm)\tau}_{\mu\nu} = J^{(\pm)\sigma}_{\mu}\partial_{[\sigma}J^{(\pm)\tau}_{\nu]} - (\mu \Leftrightarrow \nu) = 0$

- $\nabla^{(\pm)}_{\tau}J^{(\pm)\mu}_{\nu} = 0$ with respect to the connection involving torsion, $\nabla^{(\pm)} = \nabla^{(0)} + G^{-1}dB$.

Hence, the manifest $N = (1,1)$ sigma model can be extended to non-manifest $N = (2,2)$ supersymmetry if and only if the target manifold is bi-hermitian. Letting the B-field be zero, the torsion $T = G^{-1}dB$ vanishes, and the covariant derivative reduces to the ordinary Levi-Civita connection. In this case, the target manifold is Kähler, according to the definition of a Kähler manifold in section 5.1.

In order to make the algebra close, in general, the field equations (6.1) had to be used. In other words, the algebra closes on-shell and it will not be possible to rewrite the action in a manifest $N = (2,2)$ invariant way. On the other hand, if the two complex structures commute, $[J^{(+)}, J^{(-)}] = 0$, the algebra *does* close off-shell, i.e. without using the field equations. If we want the algebra to close off-shell even in the case when the two complex structures don't commute, additional auxiliary spinorial $N = (1,1)$ fields have to be included in the Lagrangian. This will be studied in section 6.2.

As mentioned in the beginning of this chapter, the geometry of the target space can also be studied starting from a manifest $N = (2,2)$ sigma model $S = \int d^2xd^2\theta d^2\bar{\theta}K(\phi,\bar{\phi})$ and reduce it to a $N = (1,1)$ model with an additional non-manifest supersymmetry. This is done in detail in appendix A.4. The $N = (2,2)$ action is reduced to

$$S = -2\int d^2x\, d^2\theta \frac{\partial^2 K}{\partial\phi^\mu\partial\bar{\phi}^\nu} D^\alpha\phi^\mu D_\alpha\bar{\phi}^\nu \Big|, \tag{6.4}$$

where the Kähler metric can now be identified in terms of the Kähler potential as $g_{\mu\nu} = \frac{\partial^2 K}{\partial\phi^\mu\partial\bar{\phi}^\nu}$. In complex canonical coordinates, the second supersymmetry of this $N = (1,1)$ action is given by

$$\delta_2\phi^\mu = \epsilon^\alpha D_\alpha\phi^\nu J^\mu_\nu, \quad J^\mu_\nu = \begin{pmatrix} i\delta^i_j & 0 \\ 0 & -i\delta^i_j \end{pmatrix}. \tag{6.5}$$

J^μ_ν squares to minus one and it's Nijenhuis tensor vanishes. In other words, J^μ_ν is a complex structure. The same result as previously is achieved, namely that the $N = (1,1)$ supersymmetric sigma model with zero B-field admits extended supersymmetry if the target manifold is Kähler.

6.2 Generalized complex geometry realized in $D = 2$ sigma models

A seen in the previous section, the algebra for the $N = (2,2)$ supersymmetry close off-shell only when the two complex structures commute, $[J^{(+)}, J^{(-)}] = 0$. In the more general case when $[J^{(+)}, J^{(-)}] \neq 0$, new fields have to be introduced to make the algebra close. Since we want the new sigma model to possess the same physical degrees of freedom as the original one, the fields have to be auxiliary [30], [33]. The auxiliary fields transform in the cotangent space T^*M, which generalizes the geometry.

The fact that a $N = (2,2)$ model written in terms of (anti) semi-chiral fields $\mathbb{X}, \bar{\mathbb{X}}$ will give rise to such auxiliary fields when reduced to $N = (1,1)$, gave a hint how to construct a manifest $N = (2,2)$ sigma model. In [37] it was shown that chiral, twisted chiral and semi-chiral superfields are sufficient for the off-shell formulation of the most general manifest $N = (2, 2)$ sigma model with non-commuting complex structures. The underlying geometry of this model is generalized Kähler geometry [32]. Further, it was found that the generalized Kähler geometry has a potential K which determines the metric and the B-field.

Using the $N = (2, 2)$ covariant derivatives (3.13), we can define (anti) chiral fields ϕ ($\bar{\phi}$) by

$$\bar{\mathbb{D}}_{\pm}\phi = \mathbb{D}_{\pm}\bar{\phi} = 0, \tag{6.6}$$

twisted (anti) chiral χ ($\bar{\chi}$) fields by

$$\mathbb{D}_{+}\chi = \bar{\mathbb{D}}_{-}\chi = \bar{\mathbb{D}}_{+}\bar{\chi} = \mathbb{D}_{-}\bar{\chi} = 0 \tag{6.7}$$

and left or right (anti) semi-chiral fields by

$$\bar{\mathbb{D}}_{+}\mathbb{X}_L = \mathbb{D}_{+}\bar{\mathbb{X}}_L = 0, \quad \mathbb{D}_{-}\bar{\mathbb{X}}_R = \bar{\mathbb{D}}_{-}\mathbb{X}_R = 0. \tag{6.8}$$

With these fields, the most general $N = (2, 2)$ action is then given by [37]

$$S = \int d^2x d^2\theta d^2\bar{\theta}\, K(\phi, \bar{\phi}, \chi, \bar{\chi}, \mathbb{X}_L, \bar{\mathbb{X}}_L, \mathbb{X}_R, \bar{\mathbb{X}}_R) \tag{6.9}$$

describing the full generalized Kähler geometry. This $N = (2, 2)$ model can be reduced to $N = (1, 1)$ supersymmetry by writing the Lagrangian as

$$D^2Q^2K(\phi, \bar{\phi}, \chi, \bar{\chi}, \mathbb{X}_L, \bar{\mathbb{X}}_L, \mathbb{X}_R, \bar{\mathbb{X}}_R)\big| = D^2\widehat{K}(\phi_i, \chi_i, X_{L,R}, \psi_{L-}, \psi_{R+}). \tag{6.10}$$

In order to recover the original $N = (1, 1)$ sigma model (4.1), the fields $\phi_i, \chi_i, X_{L,R}$ are identified with the scalar fields in the $N = (1, 1)$ model, and the auxiliary spinorial fields ψ_{L-}, ψ_{R+} are integrated out using their field equations [32].

7 Geometry of supersymmetric sigma models in D = 1

The conditions imposed on the target space geometry by supersymmetry for non-linear sigma models in one dimension were first explored in [12]. It was found, that supersymmetry implies less constraints on the geometry and that there is no clear geometric interpretation of the constraints, as compared to the situation in higher dimensions. In the same paper, a $N = 3$ model was constructed explicitly, showing one of the differences between sigma models in one and higher dimensions. This will be discussed in section 7.3.

Supersymmetric topological sigma models were examined in [13]. There it was shown, that a one-dimensional topological model with Lagrangian $L_\alpha = b_{\mu\alpha}\dot{\phi}^\mu - \omega_{\mu\nu\alpha}\psi^\mu\psi^\nu$, where $\psi = \psi(t)$ are real fermions and $\omega = db$ is a closed non-degenerate 2-form admits off-shell closure of $N = 2$ supersymmetry if and only if the target manifold is an almost complex manifold and ω is (1,1) with respect to the almost complex structure. The action can be written in terms of $N = 1$ superfields as the Chern-Simons action

$$S = -i \int dtd\theta b_\mu D\phi^\mu. \tag{7.1}$$

$N = 4$ requires a hypercomplex target manifold. Further, the $D = 1$ sigma model (4.10) was examined in the limit where the kinetic term is set to zero, but no generic correspondence to the quantum theory of the topological model was found.

Spinning particles with $N = 1$ supersymmetry were studied in [15]. Conditions for the model to allow $N = 2$ supersymmetry were formulated in terms of a Yano Killing-tensor, i.e. instead of requiring a covariantly constant complex structure on the target manifold, a sufficient condition is the existence of a Yano Killing-tensor, satisfying $\nabla_{(\mu} I^\tau_{\nu)} = 0$. These results will be studied in section 7.1. In [16], they were used to find a new type of supersymmetry in the particle-like behaviour (centre of mass approximation) of strings.

The fact that the bosonic sector of the one-dimensional sigma models describe the geodesic motion in the moduli space of black holes was the motivation for examining the geometry of a point-particle model with extended world-line supersymmetry in [17]. Two basic kinds of $N = 2$ models in $D = 1$ were identified and studied. The two models, classified as $N = 2a$ and $N = 2b$ models, were found to be the reduction of two-dimensional models with $N = (1, 1)$ and $N = (2, 0)$ supersymmetry, respectively. These results will be used when constructing a $N = 1$ supersymmetric sigma model in $D = 1$ in section 7.2. Also $N = 4$ and $N = 8$ supersymmetry was studied in the paper, with the same result that the two basic kinds can be obtained by dimensional reduction of $N = (2, 2)$ ($N = (4, 4)$) or $N = (4, 0)$ ($N = (8, 0)$) models. $N = (1, 0)$ and $N = (2, 0)$ models in two dimensions had earlier been studied in [10] and [11].

In [18], the above results from [17] were used to study the bosonic sector of one-dimensional sigma models in the general case of N supersymmetries. For conventional supersymmetries there must exist $N - 1$ complex structures satisfying a Clifford algebra. When the complex structures are simultanously integrable, the action can be written in an extended superspace formulation. In this case, the geometry for $N = 2$ is given by a 2-form, for $N = 3$ by a 1-form and for $N = 4$ by a scalar potential. For higher supersymmetries, the metric is determined by a scalar potential satisfying differential constraints.

The method of obtaining one-dimensional supersymmetric sigma models by dimensional reduction of higher dimensional models was compared with the method of discrete light-cone quantization (DLCQ) in [19]. DLCQ takes a quantum field theory in D dimensions to quantum mechanics in $D-2$ spatial dimensions, whereas dimensional reduction takes quantum field theory in D dimensions to quantum field theory in one time-dimension, with the fields reinterpretated as coordinates.

In [21] and [22], conformal and superconformal quantum mechanics was studied using one-dimensional sigma models. Extension from the conformal symmetry $SL(2,\mathbb{R})$ to $SU(1,1|1)$ is possible if there exists a complex structure I and a holomorphic $U(1)$ isometry generated by $D^a I_a^b$. Conditions for the action to possess conformal and superconformal symmetry was derived in [22].

In [28], it was shown that a lot of off-shell $N=4$ multiplets in one dimension with irreducibility constraints of first order in spinor derivatives can be derived from non-linear realizations of the $N=4$, $D=1$ superconformal group $D(2,1;\alpha)$. In the paper, all known, as well as two new off-shell $N=4$ supermultiplets in one dimension were derived.

The relations between different multiplets were interpreted geometrically and clarified in [29]. In [34], different $N=4$ multiplets in $D=1$ were constructed by reducion from the multiplet $(4,4,0)$, where the notation $(\cdot,\cdot,\cdot)$ stands for the number of bosonic, fermionic and auxiliary fields. This multiplet was termed the "root" multiplet, since the metric of the bosonic manifold must depend on all four bosons in $(4,4,0)$, whereas other supermultiplets with fewer physical bosons have a metric which depends only on the physical bosons left after the reduction from $(4,4,0)$. All known $N=4$ superconformal actions as well as their interactions can be derived by reducion of the free action from this root multiplet. $N=4$ supersymmetry in $D=1$ sigma models was further investigated in [36]. It was shown that the tensor multiplet $(3,4,1)$ may be dualized into new non-linear supermultiplets with four bosonic and four fermionic superfields. The constraints imposed on the metric defines a hyper-Kähler geometry in the bosonic sector of the dualized system.

Some of the above results will be explored in more detail in section 7.1 and the explicit construction of a $N=1$ model in $D=1$ by dimensional reduction will be studied in section 7.2. The differences between one dimensional and higher dimensional supersymmetric sigma models will be discussed in section 7.3.

7.1 Geometry of $D=1$ supersymmetric sigma models

The most general $N=1$ supersymmetric sigma model is given in equation (4.6) and reads

$$\begin{aligned} S &= \int dt\, D\Big(-\frac{i}{2}\, g_{\mu\nu} D\phi^\mu \dot{\phi}^\nu + \frac{1}{3!} h_{\mu\nu\tau} D\phi^\mu D\phi^\nu D\phi^\tau - \frac{1}{2} h_{ab}\psi^a \nabla\psi^b \\ &+ \frac{1}{3!} I_{abc}\psi^a\psi^b\psi^c - i f_{\mu a}\dot{\phi}^\mu \psi^a + \frac{1}{2} m_{\mu ab}\psi^a\psi^b D\phi^\mu + \frac{1}{2} n_{\mu\nu a} D\phi^\mu D\phi^\nu \psi^a\Big). \end{aligned}$$

The ansatz for $N=2$ supersymmetry can be found by dimensional analysis to take the form [12]

$$\begin{aligned} \delta\phi^\mu &= \epsilon I^\mu_\nu D\phi^\nu + \epsilon e^\mu_a \psi^a \\ \delta\psi^a &= \epsilon I^a_b \nabla\psi^b - (A_\mu)^a_b \delta\phi^\mu \psi^b + i\epsilon e^a_\mu \dot{\phi}^\mu + \epsilon E^a_{\mu\nu} D\phi^\mu D\phi^\nu \\ &+ \epsilon M^a_{bc}\psi^b\psi^c + \epsilon F^a_{b\mu}\psi^b D\phi^\mu, \end{aligned} \tag{7.2}$$

i.e. the transformations can take a more general form than in higher dimensions. A complete list of the close to thirty different constraints put on the target space in order for the algebra to close and the action to be invariant under these transformations was given in [12].

The geometry of the moduli space of black holes is determined by a special case of the general one dimensional sigma model above, namely by the sector where the fermionic superfields vanish. Restricting to the bosonic superfields of the above action, the simplest $N = 1$ sigma model in $D = 1$ is given in equation (4.10) as

$$S = -\frac{1}{2}\int dt\, d\theta\, \big[ig_{\mu\nu}D\phi^{\mu}\dot{\phi}^{\nu} + \frac{1}{3!}h_{\mu\nu\tau}D\phi^{\mu}D\phi^{\nu}D\phi^{\tau}\big].$$

This action is found to possess an additional off-shell $N = 2$ supersymmetry $\delta\phi^{\mu} = \epsilon I^{\mu}_{\nu}D\phi^{\nu}$ provided that I is a complex structure preserving the metric and fulfilling the two conditions [17]

$$\begin{aligned} &\nabla^{(+)}_{(\mu}I^{\nu}_{\ \tau)} = 0, \\ &\partial_{[\mu}(I^{\tau}_{\nu}h_{|\tau|\sigma\lambda]}) - 2I^{\tau}_{[\mu}\partial_{[\tau}h_{\nu\sigma\lambda]]} = 0. \end{aligned} \tag{7.3}$$

These two requirements guarantee the invariance of the action (4.10) under the additional supersymmetry transformation. The second condition does not have a direct geometrical interpretation, but it can be rewritten using the inner derivative with respect to I as

$$\iota_I dh - \tfrac{2}{3}\, d\iota_I h = 0. \tag{7.4}$$

The requirement that I is a complex structure, i.e.

$$I^2 = -1, \quad N^{\tau}_{\mu\nu}(I) := I^{\lambda}_{\mu}I^{\tau}_{[\nu,\lambda]} - I^{\lambda}_{\nu}I^{\tau}_{[\mu,\lambda]} = 0 \tag{7.5}$$

ensures that the superalgebra closes to the right form [18].

Let us relax for a moment the requirement that I is a complex structure. If the torsion h vanishes, then the first condition becomes the Yano tensor condition $\nabla_{(\mu}I^{\tau}_{\nu)} = 0$ implying that $I_{\mu\nu}$ is a Yano Killing-tensor, and so the first condition can be interpreted as a generalized Yano condition for a connection with torsion. This condition makes it possible to write the Nijenhuis tensor as [17]

$$N^{\tau}_{\mu\nu}(I) = -I^{\tau}_{\lambda}\nabla_{\mu}I^{\lambda}_{\nu}. \tag{7.6}$$

Then if in fact I is a complex structure, $\nabla I = 0$ and the space is Kähler [18].

If h is closed and the complex structure is covariantly constant with respect to the connection with torsion, $\nabla^{(+)}_{\mu}I^{\nu}_{\tau} = 0$, then the two conditions (7.3) are satisfied and the bosonic part of the one dimensional sigma model (4.10) can be obtained by dimensional reduction of the $N = (2, 0)$ model in two dimensions. These are much stronger conditions, though, and hence there exist many geometries that allow $N = 2$ models in one dimension, but not $N = (2, 0)$ models in two dimensions [18].

As in the $D = 2$ case, the complex structure allows us to introduce complex coordinates $\varphi, \bar{\varphi}$ so that $\phi^{\mu} = (\varphi^i, \bar{\varphi}^j), i, j = 1, ..., \frac{d}{2}$ and $ds^2 = 2g_{ij}d\varphi^i d\bar{\varphi}^j$. In these coordinates, the complex structure takes the simple form

$$I^{\mu}_{\nu} = \begin{pmatrix} i\delta^i_j & 0 \\ 0 & -i\delta^i_j \end{pmatrix}. \tag{7.7}$$

7.2 Constructing $N = 1$ model in $D = 1$ by dimensional reduction

Reducing a $N = (1, 1)$ model in two dimensions directly to a $N = 1$ model in one dimension recovers only a small part of the most general $N = 1, D = 1$ action (4.6), as will be studied in subsection 7.2.1. What is the most general form of a $N = 1$ supersymmetric sigma model in one dimension that can be obtained by dimensional reduction from sigma models in two dimensions? To answer this question, the reduction from a $N = (2, 0)$ model in two dimensions must also be analyzed, as will be done in subsection 7.2.3. As mentioned in the beginning of this chapter, there are two basic kinds of manifest $N = 2$ models in one dimension, referred to as $N = 2a$ and $N = 2b$ models [17]. They can be obtained by dimensional reduction from the $N = (1, 1)$ and the $N = (2, 0)$ models, respectively. But not even the reduction from a manifest $N = 2$ model in one dimension recovers the most general $N = 1, D = 1$ action, as seen in subsecion 7.2.2.

7.2.1 Reduction directly from a $N = (1, 1)$ model

Let us study in detail how to construct a $N = 1$ sigma model in one dimension by dimensional reduction from a $N = (1, 1)$ sigma model in two dimensions.

The manifest $N = (1, 1)$ sigma model in two dimensions is given by (4.1)

$$S = \int d^2x d^2\theta D_+\phi^\mu E_{\mu\nu} D_-\phi^\nu,$$

where $\phi^\mu = \phi^\mu(x^{\pm\!\!\!+}, x^{=}, \theta^+, \theta^-)$ is a $N = (1, 1)$ superfield with component expansion given in (3.16). Let all fields be independent of the spatial variable σ, so that

$$\partial_{\pm\!\!\pm} = \partial_t \pm \partial_\sigma = \partial_t. \tag{7.8}$$

Given the covariant derivatives D_+, D_- and the Grassmann coordinates θ^+, θ^-, define new $N = 1$ supersymmetry generators D, Q and new Grassmann coordinates $\theta, \widetilde{\theta}$ as

$$\begin{array}{lll} \theta & := \frac{1}{\sqrt{2}}(\theta^+ + \theta^-) \\ \widetilde{\theta} & := \frac{1}{\sqrt{2}}(\theta^+ - \theta^-) \end{array}, \quad \begin{array}{lll} D & := \frac{1}{\sqrt{2}}(D_+ + D_-) = \frac{\partial}{\partial\theta} + i\theta\partial_t \\ Q & := \frac{1}{\sqrt{2}}(D_+ - D_-) = \frac{\partial}{\partial\widetilde{\theta}} + i\widetilde{\theta}\partial_t. \end{array} \tag{7.9}$$

Under the substitution $(\theta^+, \theta^-) \to (\theta, \widetilde{\theta})$, the multiplet $(X^\mu, \psi_+^\mu, \psi_-^\mu, F^\mu)$ is changed to a new multiplet $(X^\mu, \psi^\mu, \widetilde{\psi}^\mu, F^\mu)$ as

$$\begin{aligned} \phi^\mu(t, \theta^+, \theta^-) &= X^\mu(t) + \theta^+\psi_+^\mu(t) + \theta^-\psi_-^\mu(t) + \theta^+\theta^- F(t) \\ &= X^\mu(t) + \tfrac{1}{\sqrt{2}}(\theta + \widetilde{\theta})\psi_+^\mu(t) + \tfrac{1}{\sqrt{2}}(\theta - \widetilde{\theta})\psi_-^\mu(t) - \theta\widetilde{\theta}F(t) \\ &= X^\mu(t) + \theta\psi^\mu(t) + \widetilde{\theta}\widetilde{\psi}^\mu(t) - \theta\widetilde{\theta}F(t) \\ &=: \hat{\phi}^\mu(t, \theta, \widetilde{\theta}), \end{aligned} \tag{7.10}$$

with $\psi^\mu := \frac{1}{\sqrt{2}}(\psi_+^\mu + \psi_-^\mu)$ and $\widetilde{\psi}^\mu := \frac{1}{\sqrt{2}}(\psi_+^\mu - \psi_-^\mu)$. The goal is to reduce one of the two supersymmetries. Let us therefore study the components of the new multiplet $\hat{\phi}$ when we set $\widetilde{\theta} = 0$ (with other words, let $\theta^+ = \theta^-$).

$$\begin{array}{lll} \hat{\phi}\Big|_{\widetilde{\theta}=0} & = X^\mu(t) + \theta\psi^\mu(t) & =: \hat{X}^\mu(t,\theta) \\ D\hat{\phi}\Big|_{\widetilde{\theta}=0} & = \psi^\mu(t) & = D\hat{X}^\mu(t,\theta) \\ Q\hat{\phi}\Big|_{\widetilde{\theta}=0} & = \widetilde{\psi}^\mu(t) + \theta F^\mu(t) & =: \hat{\psi}^\mu(t,\theta) \\ DQ\hat{\phi}\Big|_{\widetilde{\theta}=0} & = F^\mu(t) & = D\hat{\psi}^\mu(t,\theta). \end{array} \tag{7.11}$$

Hence, we have defined two new $N = 1$ superfields; the bosonic field $\hat{X}^\mu(t,\theta)$ and the fermionic field $\hat{\psi}^\mu(t,\theta)$. The supersymmetry transformations of these superfields take the simple form

$$\begin{array}{llll} \delta\hat{X} & = \epsilon Q\hat{X} & = \epsilon Q\hat{\phi}\big| & = \epsilon\,\hat{\psi} \\ \delta\hat{\psi} & = \epsilon Q\hat{\psi} & = \epsilon QQ\hat{\phi}\big| & = \epsilon\,\dot{\hat{X}}. \end{array} \tag{7.12}$$

Using the equations (7.9) and (7.11), the dimensional reduction can be performed.

$$\begin{aligned} S &= \int dtd\theta^+ d\theta^- D_+\phi^\mu E_{\mu\nu} D_-\phi^\nu \\ &= -\int dtd\theta d\widetilde{\theta}\, \left[\tfrac{1}{\sqrt{2}}(D+Q)\phi^\mu E_{\mu\nu} \tfrac{1}{\sqrt{2}}(D-Q)\phi^\nu\right] \\ &= -\frac{1}{2}\int dtd\theta Q\big[(D+Q)\phi^\mu E_{\mu\nu}(D-Q)\phi^\nu\big]\Big|_{\widetilde{\theta}=0} \\ &= -\frac{1}{2}\int dtd\theta\big[-DQ\phi^\mu E_{\mu\nu}D\phi^\nu + DQ\phi^\mu E_{\mu\nu}Q\phi^\nu + Q^2\phi^\mu E_{\mu\nu}D\phi^\nu - Q^2\phi^\mu E_{\mu\nu}Q\phi^\nu \\ &\quad -D\phi^\mu E_{\mu\nu,\tau}Q\phi^\tau D\phi^\nu + D\phi^\mu E_{\mu\nu,\tau}Q\phi^\tau Q\phi^\nu - Q\phi^\mu E_{\mu\nu,\tau}Q\phi^\tau D\phi^\nu - Q\phi^\mu E_{\mu\nu,\tau}Q\phi^\tau Q\phi^\nu \\ &\quad +D\phi^\mu E_{\mu\nu}DQ\phi^\nu + D\phi^\mu E_{\mu\nu}Q^2\phi^\nu + Q\phi^\mu E_{\mu\nu}DQ\phi^\nu + Q\phi^\mu E_{\mu\nu}Q^2\phi^\nu\big]\Big|_{\widetilde{\theta}=0} \\ &= -\frac{1}{2}\int dtd\theta\big[-D\hat{\psi}^\mu E_{\mu\nu}D\hat{X}^\nu + D\hat{\psi}^\mu E_{\mu\nu}\hat{\psi}^\nu + i\dot{\hat{X}}^\mu E_{\mu\nu}D\hat{X}^\nu - i\dot{\hat{X}}^\mu E_{\mu\nu}\hat{\psi}^\nu \\ &\quad -D\hat{X}^\mu E_{\mu\nu,\tau}\hat{\psi}^\tau D\hat{X}^\nu + D\hat{X}^\mu E_{\mu\nu,\tau}\hat{\psi}^\tau\hat{\psi}^\nu - \hat{\psi}^\mu E_{\mu\nu,\tau}\hat{\psi}^\tau D\hat{X}^\nu + \hat{\psi}^\mu E_{\mu\nu,\tau}\hat{\psi}^\tau\hat{\psi}^\nu \\ &\quad +D\hat{X}^\mu E_{\mu\nu}D\hat{\psi}^\nu + D\hat{X}^\mu E_{\mu\nu}i\dot{\hat{X}}^\nu + \hat{\psi}^\mu E_{\mu\nu}D\hat{\psi}^\nu + \hat{\psi}^\mu E_{\mu\nu}i\dot{\hat{X}}^\nu\big] \\ &= \int dtd\theta\big[-iG_{\mu\nu}D\hat{X}^\mu\dot{\hat{X}}^\nu - G_{\mu\nu}\hat{\psi}^\mu\nabla\hat{\psi}^\nu + \big(D\hat{X}^\mu D\hat{X}^\nu\hat{\psi}^\tau + \tfrac{1}{3}\hat{\psi}^\mu\hat{\psi}^\nu\hat{\psi}^\tau\big)H_{\mu\nu\tau}\big], \end{aligned} \tag{7.13}$$

where we defined the connection ∇ as $G_{\mu\nu}\nabla\hat{\psi}^\nu := G_{\mu\nu}D\hat{\psi}^\nu + G_{\mu\nu,\tau}D\hat{X}^\nu\hat{\psi}^\tau$, and H is the torsion $H_{\mu\nu\tau} := \frac{1}{2}(B_{\mu\nu,\tau} + B_{\tau\mu,\nu} + B_{\nu\tau,\mu})$.

Hence, if the manifest $N = (1,1)$ sigma model in $D = 2$ (4.1) is dimensionally reduced to a $D = 1$ sigma model with manifest $N = 1$ supersymmetry, we get the action (7.13)

$$S = \int dtd\theta\Big[-iG_{\mu\nu}D\phi^\mu\dot{\phi}^\nu - G_{\mu\nu}\psi^\mu\nabla\psi^\nu + \big(D\phi^\mu D\phi^\nu\psi^\tau + \tfrac{1}{3}\psi^\mu\psi^\nu\psi^\tau\big)H_{\mu\nu\tau}\Big].$$

It is interesting to note, that this action is equivalent with the Hamiltonian formulation of the $N = (1,1)$ model in two dimensions. The geometry of this model has been interpreted in terms of generalized complex geometry in [35].

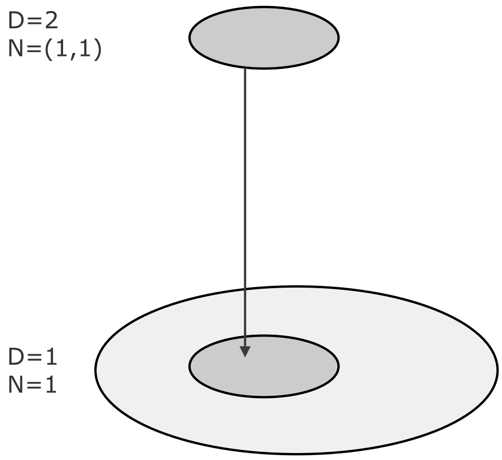
Comparing this action with the most general $N = 1$ sigma model in one dimension, given in equation (4.6) as

$$S = \frac{1}{2}\int dt\, d\theta \Big(-i\, g_{\mu\nu} D\phi^\mu \dot{\phi}^\nu + \frac{1}{3} h_{\mu\nu\tau} D\phi^\mu D\phi^\nu D\phi^\tau - h_{\mu\nu}\psi^\mu \nabla\psi^\nu$$
$$+\frac{1}{3} I_{\mu\nu\tau}\psi^\mu\psi^\nu\psi^\tau - 2i f_{\mu\nu}\dot{\phi}^\mu\psi^\nu + m_{\mu\nu\tau}\psi^\nu\psi^\tau D\phi^\mu + n_{\mu\nu\tau} D\phi^\mu D\phi^\nu \psi^\tau \Big),$$

one finds (using integration by parts) that all terms except $f_{\mu\nu}\dot{\phi}^\mu\psi^\nu$ are recovered. More specific, we recover the special case of the action (4.6) where all the undetermined tensors in (4.6) are constructed of the metric G and torsion H as

$$\begin{array}{llll} g_{\mu\nu} = G_{\mu\nu}, & h_{\mu\nu\tau} = G_{\mu\nu,\tau}, & h_{\mu\nu} = G_{\mu\nu}, & I_{\mu\nu\tau} = H_{\mu\nu\tau}, \\ f_{\mu\nu} = 0, & n_{\mu\nu\tau} = H_{\mu\nu\tau}, & h_{\mu\lambda}(A_\nu)^\lambda_\tau + m_{\nu\mu\tau} = G_{\mu\nu,\tau}. & \end{array} \tag{7.14}$$

Note, that this is not the most general form of a $N = 1$ model in one dimension. For example, $f_{\mu\nu} = 0$ and the term $G_{\mu\nu,\tau} D\phi^\mu D\phi^\nu D\phi^\tau$ vanishes due to the symmetry of the metric $G_{\mu\nu}$. The reduction can be illustrated graphically as



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7.2.2 Reduction via manifest $N = 2a$ model in one dimension

The $N = 2a$ models are described by unconstrained, real $N = 2$ superfields and can be obtained by dimensional reduction from the $N = (1, 1)$ model in two dimensions. The most general $N = 2a$ action is given by

$$S = \int dt d^2\theta \big(D_1\phi^\mu E_{\mu\nu} D_2\phi^\nu + l_{\mu\nu} D_1\phi^\mu D_1\phi^\nu + m_{\mu\nu} D_2\phi^\mu D_2\phi^\nu \big) \tag{7.15}$$

where $\phi = \phi(t, \theta^1, \theta^2)$ is a real $N = 2$ superfield with components

$$\phi\big| =: X, \quad D_1\phi\big| =: \lambda, \quad D_2\phi\big| =: \psi, \quad D_1 D_2\phi\big| =: F \tag{7.16}$$

and the supersymmetry derivatives fulfill the algebra $D_1^2 = D_2^2 = i\partial_t, \quad \{D_1, D_2\} = 0$. When the two-forms $l_{\mu\nu}$ and $m_{\mu\nu}$ vanish, this action is obtained by dimensional from the

$N = (1,1)$ model in two dimensions (4.1). The two couplings l, m correspond to non-Lorentz invariant terms in the two-dimensional action. By construction, we see that they are both anti-symmetric.

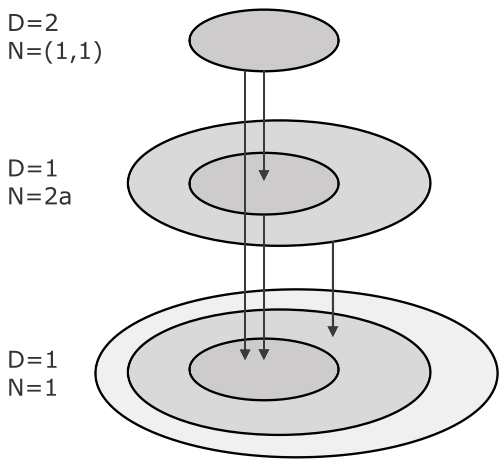
Reducing one of the supersymmetries of the manifest $N = 2a$ model (7.15), the reduction of the first term ($l = m = 0$) recovers the same $N = 1$ action as when performing the reduction directly from the $N = (1,1)$ model in two dimensions, as done in detail in the previous subsection. It is interesting to see which terms in the $N = 1$ action correspond to the non-Lorentz invariant terms l, m. The full $N = 2a$ model (7.15), with non-vanishing l and m, reduces to

$$\begin{aligned} S &= \int dt d\theta \big[-iG_{\mu\nu} D\phi^\mu \dot{\phi}^\nu - (G_{\mu\nu} + s_{\mu\nu})\psi^\mu \nabla \psi^\nu + \frac{1}{3} S_{\mu\nu\tau} D\phi^\mu D\phi^\nu D\phi^\tau \\ &\quad + (H_{\mu\nu\tau} - T_{\mu\nu\tau})(D\phi^\mu D\phi^\nu \psi^\tau + \frac{1}{3}\psi^\mu \psi^\nu \psi^\tau) - 2it_{\mu\nu}\dot{\phi}^\mu \psi^\nu \big] \end{aligned} \tag{7.17}$$

where $L_{\mu\nu\tau} := \frac{1}{2}(l_{\mu\nu,\tau} + l_{\nu\tau,\mu} + l_{\tau\mu,\nu})$, $M_{\mu\nu\tau} := \frac{1}{2}(m_{\mu\nu,\tau} + m_{\nu\tau,\mu} + m_{\tau\mu,\nu})$, $s_{\mu\nu} := l_{\mu\nu} - m_{\mu\nu}$, $t_{\mu\nu} := l_{\mu\nu} + m_{\mu\nu}$, $S_{\mu\nu\tau} := L_{\mu\nu\tau} - M_{\mu\nu\tau}$ and $T_{\mu\nu\tau} := L_{\mu\nu\tau} + M_{\mu\nu\tau}$. Is this the most general $N = 1$ supersymmetric sigma model in one dimension? All terms in the action (4.6) are indeed recovered, with

$$\begin{array}{llll} g_{\mu\nu} = G_{\mu\nu}, & h_{\mu\nu} = G_{\mu\nu} + s_{\mu\nu}, & h_{\mu\nu\tau} = G_{\mu\nu,\tau} + S_{\mu\nu\tau}, & f_{\mu\nu} = t_{\mu\nu}, \\ I_{\mu\nu\tau} = H_{\mu\nu\tau} - T_{\mu\nu\tau}, & n_{\mu\nu\tau} = H_{\mu\nu\tau} - T_{\mu\nu\tau}, & h_{\mu\lambda}(A_\nu)^\lambda_\tau + m_{\nu\mu\tau} = G_{\mu\nu,\tau} + s_{\mu\nu,\tau}. & \end{array} \tag{7.18}$$

This is a more general action than (7.13), obtained by dimensional reduction from the $N = (1,1)$ model. In (7.17), the term $S_{\mu\nu\tau} D\phi^\mu D\phi^\nu D\phi^\tau$ does not necessarily vanish due to the antisymmetry of $S_{\mu\nu\tau}$. Also, here $f_{\mu\nu}$ does not vanish, and anti-symmetric tensors $s_{\mu\nu}, T_{\mu\nu\tau}$ are added to the couplings. Still, in the most general action (4.6), the couplings are arbitrary and need not be closed. Here, $H_{\mu\nu\tau}, S_{\mu\nu\tau}$ and $T_{\mu\nu\tau}$ are constructed from derivatives and are by construction closed. Hence, the most general $N = 1$ model in one dimension cannot be constructed by dimensional reduction even from a manifest $N = 2a$ model in one dimension. The figure from previous subsection can now be enlarged to



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7.2.3 Reduction from $N = (2,0)$ model

Now turning to the reduction from the $N = (2,0)$ model. The most general renormalizable, Lorentz-invariant $N = (2,0)$ model in two dimensions is given by [10]

$$S = \int d^2x d^2\theta_+ \big[\underbrace{-\frac{i}{2}(K_\mu \partial_+ \Phi^\mu - K_{\bar\mu} \partial_+ \Phi^{\bar\mu})}_{\mathcal{L}_1} + \underbrace{(f_{ab}\Psi^a\Psi^b + f_{a\bar b}\Psi^a\Psi^{\bar b} + f_{\bar a\bar b}\Psi^{\bar a}\Psi^{\bar b})}_{\mathcal{L}_2} \big], \quad (7.19)$$

where Φ, Ψ are complex bosonic and fermionic superfields satisfying the chirality conditions $\bar{\mathcal{D}}_-\Phi^\mu = \bar{\mathcal{D}}_-\Psi^a = 0$. Reduced to a manifest $N = 2$ model in one dimension, one achieves the $N = 2b$ sigma model, constructed from complex chiral $N = 2$ superfields. These models can be constructed from the real $N = 2a$ model (7.15) by introducing complex supersymmetry derivatives $\mathcal{D} := D_1 + iD_2 = \frac{\partial}{\partial\theta} + i\bar\theta\partial_t$, $\theta := \theta^1 + i\theta^2$ fulfilling the algebra $\mathcal{D}^2 = 0, \{\mathcal{D}, \bar{\mathcal{D}}\} = 2i\partial_t$ and complex superfields Φ fulfilling the chirality condition $\bar{\mathcal{D}}\Phi = 0$. The most general $N = 2b$ action is then given by

$$S = \frac{1}{4}\int dt d^2\theta \big[iG_{\mu\bar\mu}\mathcal{D}\Phi^\mu\bar{\mathcal{D}}\bar\Phi^{\bar\mu} + \frac{1}{2}\big(B_{\mu\nu}\mathcal{D}\Phi^\mu\mathcal{D}\Phi^\nu + B_{\bar\mu\bar\nu}\bar{\mathcal{D}}\bar\Phi^{\bar\mu}\bar{\mathcal{D}}\bar\Phi^{\bar\nu}\big)\big]. \quad (7.20)$$

Reducing one of the supersymmetries of the $N = (2,0)$ model (7.19) yields the $N = (1,0)$ action [10]

$$S = -\int d^2x d\theta_+ \big[\underbrace{i(\partial_{\bar\mu}K_\mu D_-\phi^{\bar\mu}\partial_+\phi^\mu + \partial_\mu K_{\bar\mu} D_-\phi^\mu\partial_+\phi^{\bar\mu})}_{\mathcal{L}_1} + \underbrace{G_{ab}(\phi)\psi^a\nabla\psi^b}_{\mathcal{L}_2} \big], \quad (7.21)$$

where $G_{ab}\psi^a\nabla\psi^b = G_{ab}\psi^a D_-\psi^b + \psi^a A_{\mu ab} D\phi^\mu\psi^b$. In complex coordinates, the metric and the B-field can be expressed in terms of the vector potential K_μ as

$$G_{\mu\bar\mu} = \frac{1}{2}(\partial_\mu K_{\bar\mu} + \partial_{\bar\mu}K_\mu), \quad B_{\mu\bar\mu} = \frac{1}{2}(\partial_\mu K_{\bar\mu} - \partial_{\bar\mu}K_\mu). \quad (7.22)$$

Using this, the $N = (1,0)$ action (7.21) can be written in terms of real $N = (1,0)$ fields as

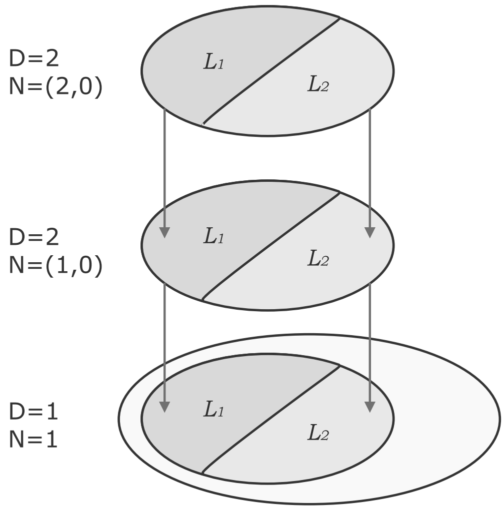
$$S = -\int d^2x d\theta_+ \big[i(G_{\mu\nu} + B_{\mu\nu})D\phi^\mu\partial_+\phi^\nu + G_{ab}\psi^a\nabla\psi^b \big]. \quad (7.23)$$

Further reducing this sigma model to one dimension yields the $N = 1$ model

$$S = \int dt d\theta \big[\underbrace{iG_{\mu\nu}D\phi^\mu\dot\phi^\nu + \frac{1}{3}H_{\mu\nu\tau}D\phi^\mu D\phi^\nu D\phi^\tau}_{\mathcal{L}_1} + \underbrace{G_{ab}\psi^a\nabla\psi^b}_{\mathcal{L}2} \big], \quad (7.24)$$

where the torsion $H_{\mu\nu\tau} := \frac{1}{2}(B_{\mu\nu,\tau} + B_{\nu\tau,\mu} + B_{\tau\mu,\nu})$ is closed. The bosonic part $\mathcal{L}_1$ of the $N = (2,0)$ model reduces to the bosonic part of the one-dimensional model. The bosonic part of the $N = (1,0)$ model admits $N = (2,0)$ supersymmetry if and only if the target manifold is hermitian. If the torsion vanishes, the manifold is Kähler [8]. The same conditions are valid for the bosonic part of the one-dimensional sigma model obtained by dimensional reduction from the $N = (2,0)$ model, as already mentioned in section 7.1. Including the fermionic part $\mathcal{L}_2$, the action admits extended supersymmetry if in addition $A_{\mu ab}$ is a holomorphic connection.

The above reduction scheme can be illustrated with the picture



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7.3 Geometry of $D = 1$ models compared to higher dimensions

As we have already seen, extended supersymmetry in one dimension impose weaker constraints on the geometry of the target space, than the same amount of supersymmetry does in two dimensions. This is an implication of the fact than in one dimension it is possible to construct an action with more couplings amongst the fields than in two dimensions, since in two dimensions they are ruled out by the Lorentz invariance [17]. In one dimension, there is no restriction relating the number of bosonic and fermionic fields [19]. Not only the actions, though, but also the supersymmetry transformations can be constructed in a more general way in one dimension than in higher dimensions, as seen in equation (7.2). When restricted to the bosonic sector of the theory, that is a one dimensional sigma model with only $N = 1$ bosonic superfields, but no fermionic superfields, most of the constraints vanish. The constraints we are left with tell us, that the action admits $N = 2$ supersymmetry $\delta\phi^\mu = \epsilon I^\mu_\nu D\phi^\nu$ if

- I is a complex structure preserving the metric,
- I fulfills a generalized Yano tensor condition, i.e. the Yano tensor condition for a connection with torsion, $\nabla^{(+)}_{(\mu} I^\tau_{\nu)} = 0$ and
- I fulfills the equation $\partial_{[\mu}(I^\tau_\nu h_{|\tau|\sigma\lambda]}) - 2I^\tau_{[\mu}\partial_{[\tau}h_{\nu\sigma\lambda]]} = 0$, for which there is no clear geometrical interpretation.

In two dimensions, the last two constraints are replaced by the condition, that I is covariantly constant with respect to a connection with torsion, $\nabla^{(+)}_{(\mu} I^\tau_{\nu)} = 0$.

Another difference between the one dimensional and the two dimensional sigma models is, that the existence of three supersymmetries implies four in $D = 2$, but not in $D = 1$. In two dimensions, $N = 3$ supersymmetry of the same chirality require two anti-commuting complex structures I_1, I_2, which can be used to generate a fourth supersymmetry. This is not the case in $D = 1$, where supersymmetry impose weaker constraints on the target space and there are no complex structures that can be used to generate $N = 4$ supersymmetry [12].

Moving to higher supersymmetries $N = 4$, the off-shell multiplets containing only four physical bosons and four fermions were studied in [34] and reviewed above. Such off-shell multiplets which do not contain any auxiliary fields exist only in one dimension [36]. In one dimension, it is possible to change between different supermultiplets by writing the auxiliary fields as time-derivatives of physical bosons and vice versa. E.g., if a bosonic auxiliary component of a $N = 4$ multiplet is transformed under a supersymmetry transformation as

$$\delta A \sim \text{parameter} \cdot \partial_t(\text{physical fermions}), \tag{7.25}$$

then one can replace A by a physical bosonic field u with $\partial_t u = A$ and the transformation properties [36]

$$\delta u \sim \text{parameter} \cdot (\text{physical fermions}). \tag{7.26}$$

Then the term quadratic in A turns into a kinetic term for the bosonic field u, and we have constructed a new supermultiplet with an additional physical boson u.

8 Summary and conclusions

In this master thesis, the geometric constraints arising on the target space when imposing supersymmetry on one-dimensional sigma models has been studied and compared to the situation in two dimensions. This has been done in several steps. First, the results in the area have been gathered and analyzed in section 7. The relevant results have then been developed in more detail in section 7.1. The sigma model in one dimension is given by

$$\begin{aligned} S &= \int dt\, d\theta \Big(-\frac{i}{2}\, g_{\mu\nu} D\phi^\mu \dot{\phi}^\nu + \frac{1}{3!} h_{\mu\nu\tau} D\phi^\mu D\phi^\nu D\phi^\tau - \frac{1}{2} h_{ab} \psi^a \nabla \psi^b \\ &\quad + \frac{1}{3!} I_{abc} \psi^a \psi^b \psi^c - i f_{\mu a} \dot{\phi}^\mu \psi^a + \frac{1}{2} m_{\mu ab} \psi^a \psi^b D\phi^\mu + \frac{1}{2} n_{\mu\nu a} D\phi^\mu D\phi^\nu \psi^a \Big). \end{aligned}$$

As we see, the action can take a more general form than in higher dimensions, where many of the terms present in one dimension are ruled out by the Lorentz invariance. Further, there are both bosonic and fermionic superfields present. In two dimensions, the $N = (1,1)$ sigma model cannot contain dynamical fermionic superfields, since they generate fields with high spin. The $N = (2,0)$ model on the other hand, may contain fermionic superfields since chirality condition can be used to avoid the higher spin fields. But not only can the sigma model be written in a more general form than higher dimensional models. Also the supersymmetry transformations take a more general form, as can be seen by comparing the $N = (2,2)$ supersymmetry transformation in two dimensions

$$\delta\phi^\mu = \epsilon D\phi^\nu J_\nu^{(\pm)\mu}$$

with the $N = 2$ supersymmetry transformations in one dimension

$$\begin{aligned} \delta\phi^\mu &= \epsilon I^\mu_\nu D\phi^\nu + \epsilon e^\mu_a \psi^a \\ \delta\psi^a &= \epsilon I^a_b \nabla\psi^b - (A_\mu)^a_b \delta\phi^\mu \psi^b + i\epsilon e^a_\mu \dot{\phi}^\mu + \epsilon E^a_{\mu\nu} D\phi^\mu D\phi^\nu \\ &\quad + \epsilon M^a_{bc} \psi^b \psi^c + \epsilon F^a_{b\mu} \psi^b D\phi^\mu. \end{aligned}$$

Naturally, since both the sigma model and the supersymmetry transformation take more general forms than in higher dimensions, the geometric constraints arising when requiring invariance of the action and closure of the algebra are more general and weaker than in higher dimensions. Closure of the algebra on ϕ and ψ result in fifteen different constraints on the target manifold, invariance of the action under the supersymmetry transformation in fourteen constraints. There is no clear geometrical interpretation of all these constraints; we do not get a Kähler manifold or such, as in the two dimensional case.

When restricted to the bosonic superfields, most of the constraints vanish. Only the first two terms in the above sigma model survives, and the supersymmetry transformation take the simple form

$$\delta\phi^\mu = \epsilon I^\mu_\nu D\phi^\nu$$

resembling the two-dimensional case. The algebra closes off-shell under the condition that I is a complex structure. The (bosonic sector of the) one dimensional sigma model is invariant under this $N = 2$ supersymmetry transformation provided that the metric is hermitian with respect to I and the two conditions

$$\begin{aligned} &\nabla^{(+)}_{(\mu} I^\nu_{\tau)} = 0, \\ &\partial_{[\mu}(I^\tau_\nu h_{|\tau|\sigma\lambda]}) - 2 I^\tau_{[\mu} \partial_{[\tau} h_{\nu\sigma\lambda]]} = 0. \end{aligned}$$

are fulfilled. The first of these constraints can be interpreted as a generalized Yano tensor condition with torsion, but the second has no clear geometrical interpretation. The bosonic part of the sigma model thus resembles the two-dimensional model in some aspects, but not quite. If the torsion h vanishes, the only conditions we are left with are

$$\begin{aligned} I^{\mu}_{\lambda} I^{\lambda}_{\nu} &= -\delta^{\mu}_{\nu} && \text{almost complex structure} \\ N(I)^{\mu}_{\nu\tau} &= 0 && \text{integrability} \\ I_{\mu\nu} &= -I_{\nu\mu} && \text{hermitian metric} \\ \nabla_{\mu} I_{\nu\tau} &= 0 && \text{covariantly constant} \end{aligned}$$

and so the target space is a Kähler manifold. The bosonic sector of the one-dimensional sigma model can be obtained by dimensional reduction from the $N = (2, 0)$ model in two dimensions. In that case, the torsion will be closed and I will fulfill the much stronger condition $\nabla^{(+)}_{\mu} I^{\nu}_{\tau} = 0$.

To see which parts of the one-dimensional sigma model correspond to two-dimensional models, dimensional reduction from models in $D = 2$ was performed. Both the $N = (1, 1)$ model and the $N = (2, 0)$ model were dimensionally reduced to one-dimensional sigma models. In addition, the manifest $N = 2a$ model in one dimension was reduced to a $N = 1$ model. It was found, that neither of the models can generate the most general sigma model. The two-dimensional models can recover only some of the terms in the one-dimensional model, as expected. The couplings that were recovered had additional constraints such as being created from the metric or being closed. The model most close to the most general sigma model in one dimension was of course obtained by reduction of a manifest $N = 2$ model in one dimension. But also in this case, the couplings obtained were not arbitrary and some had the constraint of being closed. The reduction schemes are given in detail in section 7.2.

Finally in section 7.3, the differences between the sigma models in one dimension and higher dimensions were explored in detail. As we have seen, the geometry constraints, the supersymmetry transformation and the sigma model in itself are more general in one dimension. Another difference is, that the existence of $N = 3$ supersymmetry implies $N = 4$ in two dimensions, but not in the one-dimensional case. Also, in one dimension, off-shell $N = 4$ multiplets with no auxiliary fields exist and it is possible to switch between different multiplets. Such multiplets do not exist in higher dimensions.

A Appendix

A.1 Field equations for $N = (1,1)$ susy sigma model in $D = 2$

The $N = (1,1)$ susy sigma model in $D = 2$ is given by $S = \int d^2\xi d^2\theta D_+\phi^\mu E_{\mu\nu} D_-\phi^\nu$, where $E_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}$. The equations of motion are derived from $\delta S = 0$, which is equivalent to the Euler-Lagrange equations $D_i\Big(\frac{\partial\mathcal{L}}{\partial(D_i\phi^\mu)}\Big) - \frac{\partial\mathcal{L}}{\partial\phi^\mu} = 0$, where $\mathcal{L}$ is the Lagrangian density $\mathcal{L} = D_+\phi^\mu E_{\mu\nu} D_-\phi^\nu$. First we calculate the parts in Euler-Lagrange.

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial(D_-\phi^\mu)} &= \frac{\partial(D_+\phi^\tau)}{\partial(D_-\phi^\mu)} E_{\tau\nu} D_-\phi^\nu - D_+\phi^\tau \frac{\partial E_{\tau\nu}}{\partial(D_-\phi^\mu)} D_-\phi^\nu - D_+\phi^\nu E_{\nu\mu} \frac{\partial(D_-\phi^\mu)}{\partial(D_-\phi^\mu)} = -D_+\phi^\nu E_{\nu\mu} \\
D_+\Big(\frac{\partial\mathcal{L}}{\partial(D_+\phi^\mu)}\Big) &= D_+(E_{\mu\nu} D_-\phi^\nu) = E_{\mu\nu,\tau} D_+\phi^\tau D_-\phi^\nu + E_{\mu\nu} D_+ D_-\phi^\nu \\
D_-\Big(\frac{\partial\mathcal{L}}{\partial(D_-\phi^\mu)}\Big) &= D_-(-D_+\phi^\nu E_{\nu\mu}) = -D_- D_+\phi^\nu E_{\nu\mu} + D_+\phi^\nu E_{\nu\mu,\tau} D_-\phi^\tau \\
\frac{\partial\mathcal{L}}{\partial\phi^\mu} &= D_+\phi^\tau E_{\tau\nu,\mu} D_-\phi^\nu
\end{aligned}
\tag{A.1}
$$

Now inserting these expressions into the Euler-Lagrange equations and using the symmetry and anti-symmetry of $G_{\mu\nu}$ and $B_{\mu\nu}$, respectively yields

$$
\begin{aligned}
0 &= G_{\mu\nu} D_+ D_-\phi^\nu - D_- D_+\phi^\nu G_{\nu\mu} + B_{\mu\nu} D_+ D_-\phi^\nu - D_- D_+\phi^\nu B_{\nu\mu} \\
&\quad + E_{\mu\nu,\tau} D_+\phi^\tau D_-\phi^\nu + D_+\phi^n u E_{\nu\mu,\tau} D_-\phi^\tau - D_+\phi^\nu E_{\tau\nu,\mu D_-\phi^\nu} \\
\Leftrightarrow\quad 0 &= 2G_{\mu\nu} D_+ D_-\phi^\nu + [E_{\mu\nu,\tau} + E_{\tau\mu,\nu} - E_{\tau\nu,\mu}] D_+\phi^\tau D_-\phi^\nu \\
\Leftrightarrow\quad 0 &= D_+ D_-\phi^\sigma + \Big(\underbrace{\frac{1}{2} G^{\sigma\mu}[G_{\mu\nu,\tau} + G_{\tau\mu,\nu} - G_{\tau\nu,\mu}]}_{\Gamma^{(0)\sigma}_{\tau\nu}} + \underbrace{\frac{1}{2} G^{\sigma\mu}[B_{\mu\nu,\tau} + B_{\tau\mu,\nu} - B_{\tau\nu,\mu}]}_{T^\sigma_{\tau\nu}}\Big) D_+\phi^\tau D_-\phi^\nu \\
\Leftrightarrow\quad 0 &= D_+ D_-\phi^\sigma + \Gamma^{(+)\sigma}_{\tau\nu} D_+\phi^\tau D_-\phi^\nu \\
\Leftrightarrow\quad 0 &= \nabla^{(+)}_+ D_-\phi^\mu.
\end{aligned}
\tag{A.2}
$$

A.2 Manifest invariance of $N=(1,1)$ sigma model under $N=(1,1)$ transformations

Because of the properties of the Grassmann coordinates, $\theta^+\theta^+ = \theta^-\theta^- = 0$, any product of $N = (1,1)$-superfields can be written as $\phi = a + b\theta^+ + c\theta^- + d\theta^+\theta^-$. The Berezin integral is so defined that $\int d^2\theta(a + b\theta^+ + c\theta^- + d\theta^+\theta^-) = d$, i.e. it picks out the $\theta^+\theta^-$-component.

Consider a supersymmetry transformation of a superfield,

$$
\begin{aligned}
\delta\phi^\mu &= i\epsilon^+ Q_+\phi^\mu + i\epsilon^- Q_-\phi^\mu \\
&= i\epsilon^+(\partial_+ - i\theta^+\partial_{++})\phi^\mu + i\epsilon^-(\partial_- - i\theta^-\partial_=)\phi^\mu \\
&= i\epsilon^+(\psi^\mu_+ + \theta^- F^\mu - i\theta^+\partial_{++}X^\mu - i\theta^+\theta^-\partial_{++}\psi^\mu_-) \\
&\quad + i\epsilon^-(\psi^\mu_- - \theta^+ F^\mu - i\theta^-\partial_= X^\mu - i\theta^-\theta^+\partial_=\psi^\mu_+) \\
&= \underbrace{i\epsilon\psi^\mu}_{\delta X^\mu} + \underbrace{\theta^+(-\epsilon^+\partial_{++}X^\mu + \epsilon^- F^\mu) + \theta^-(-\epsilon^-\partial_= X^\mu - \epsilon^+ F^\mu)}_{\theta\delta\psi^\mu} \\
&\quad + \theta^+\theta^- \underbrace{(\epsilon^+\partial_{++}\psi^\mu_- - \epsilon^-\partial_=\psi^\mu_+)}_{\delta F^\mu}.
\end{aligned}
\tag{A.3}
$$

Hence, the $\theta^+\theta^-$-component of $\delta\phi^\mu$ is proportional to a total space-time derivative of ψ^μ. Using partial integration together with boundary conditions $\psi^\mu(x_0) = \psi^\mu(x_1) = 0$, we find the invariance of the $N = (1,1)$ action under a $N = (1,1)$ supersymmetry transformation.

$$\begin{aligned} \delta S &= \int d^2x d^2\theta i\epsilon Q\mathcal{L} \\ &= \int d^2x(\epsilon^+\partial_{++}\psi^\mu_- - \epsilon^-\partial_=\psi^\mu_+) \\ &= 0. \end{aligned} \tag{A.4}$$

A.3 Integrability conditions for distributions

The distribution $T_pM^\pm$, where $T_pM = T_pM^+ \oplus T_pM^-$ is called *integrable* when

$$X, Y \in T_pM^\pm \quad \Rightarrow [X,Y] \in T_pM^\pm. \tag{A.5}$$

Let us introduce projection operators $P^\pm := \frac{1}{2}(1 \mp iJ)$, where J is an almost complex structure $J^2 = -1$. Then for every $X \in T_pM$, $P^\pm X \in T_pM^\pm$, and so the above condition implies $[P^\pm X, P^\pm Y] \in T_pM^\pm$ and so the integrability condition can be rewritten as

$$P^\mp[P^\pm X, P^\pm Y] = 0. \tag{A.6}$$

This expression can further be rewritten using the Nijenhuis tensor.

$$\begin{aligned} 0 &= P^\mp[P^\pm X, P^\pm Y] \\ &= \frac{1}{2}(1 \pm iJ)[\frac{1}{2}(1 \mp iJ)X, \frac{1}{2}(1 \mp iJ)Y] \\ &= \frac{1}{8}(1 \pm iJ)\left([X,Y] \mp i[X,JY] \mp i[JX,Y] - [JX,JY]\right) \\ &= \frac{1}{8}\left([X,Y] \mp i[X,JY] \mp i[JX,Y] - [JX,JY] \pm iJ[X,Y] + J[X,JY] + J[JX,Y] \mp iJ[JX,JY]\right) \\ &= \frac{1}{8}\Big(\underbrace{[X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]}_{=:N(X,Y)} \pm i(\underbrace{J[X,Y] - [JX,Y] - [X,JY] - J[JX,JY]}_{=JN(X,Y)})\Big) \\ &= \frac{1}{8}(1 \pm iJ)N(X,Y) \\ &= \frac{1}{4}P^\mp N(X,Y). \end{aligned} \tag{A.7}$$

$(P^+ + P^-)N(X,Y) = N(X,Y)$ finally implies

$$N(X,Y) = 0, \tag{A.8}$$

i.e. the integrability condition is equivalent with the vanishing of the Nijenhuis tensor. The Nijenhuis tensor is a $\binom{1}{2}$-tensor, and is in local coordinates written as

$$N^\mu_{\sigma\nu} = J^\mu_\alpha J^\alpha_{[\sigma,\nu]} + J^\beta_{[\sigma} J^\mu_{\nu],\beta}. \tag{A.9}$$

$$
\begin{aligned}
N(X,Y) &= [X,Y]+J[JX,Y]+J[X,JY]-[JX,JY] \\
&= (X^\nu\partial_\nu Y^\mu - Y^\nu + \partial_\nu X^\mu)\partial_\mu + J_\alpha^\beta dx^\alpha \otimes \partial_\beta\Big[J_\nu^\mu X^\nu \partial_\mu (Y^\sigma\partial_\sigma) - Y^\sigma\partial_\sigma(J_\nu^\mu X^\nu\partial_\mu)\Big] \\
&\quad + J_\alpha^\beta dx^\alpha \otimes \partial_\beta\Big[X^\tau\partial_\tau(J_\nu^\mu Y^\nu\partial_\mu) - J_\nu^\mu Y^\nu\partial_\mu(X^\tau\partial_\tau)\Big] - J_\nu^\mu X^\nu\partial_\mu(J_\alpha^\beta Y^\alpha\partial_\beta) \\
&\quad + J_\alpha^\beta Y^\alpha\partial_\beta(J_\nu^\mu X^\nu\partial_\mu) \\
&= \Big[X^\nu\partial_\nu Y\mu - Y^\nu\partial_\nu X^\mu + J_\alpha^\mu J_\nu^\beta X^\nu\partial_\beta Y^\alpha - J_\alpha^\mu Y^\sigma J_{\nu,\sigma}^\alpha X^\nu - J_\alpha^\mu Y^\sigma J_\nu^\alpha\partial_\sigma X^\nu \\
&\quad + J_\alpha^\mu X^\tau J_{\nu,\tau}^\alpha Y\nu + J_\alpha^\nu X^\tau J_\nu^\alpha\partial_\tau Y^\nu - J_\alpha^\mu J_\nu^\beta Y^\nu\partial_\beta X^\alpha - J_\nu^\beta X^\nu J_{\alpha,\beta}^\mu Y^\alpha \\
&\quad - J_\nu^\beta X^\nu J_\alpha^\mu\partial_\beta Y^\alpha + J_\alpha^\beta Y^\alpha J_{\mu,\beta}^\mu X^\nu + J_\alpha^\beta Y^\alpha J_\nu^\mu\partial_\beta X^\nu\Big]\partial_\mu \\
&= X^\nu Y^\sigma\underbrace{\Big[-J_\alpha^\mu J_{\nu,\sigma}^\alpha + J_\alpha^\mu J_{\sigma,\nu}^\alpha - J_\nu^\beta J_{\sigma,\beta}^\mu + J_\sigma^\beta J_{\nu,\beta}^\mu\Big]}_{=N_{\sigma\nu}^\mu}\partial_\mu .
\end{aligned}
$$

A.4 Dimensional reduction of manifest $N=(2,2)$ sigma model in $D=2$

The action $S=\int d^2xd^2\theta d^2\bar{\theta}K(\phi,\bar{\phi})$ is manifestly invariant under $N=(2,2)$ supersymmetry transformations if and only if K is a scalar function of $N=(2,2)$ superfields. The action is invariant under Kähler gauge transformations $\delta K=\Lambda(\phi)+\bar{\Lambda}(\bar{\phi})$ [7]. The lowest representation is carried by chiral superfields $\bar{\mathbb{D}}_\pm\phi=0$. The covariant derivatives and the SUSY generators are given by

$$
\begin{cases} \mathbb{Q}_\pm &= \partial_\pm - i\bar{\theta}^\pm\partial_{\pm\pm} \\ \bar{\mathbb{Q}}_\pm &= \bar{\partial}_\pm - i\theta^\pm\partial_{\pm\pm} \end{cases}
\begin{cases} \mathbb{D}_\pm &= \partial_\pm + i\bar{\theta}^\pm\partial_{\pm\pm} \\ \bar{\mathbb{D}}_\pm &= \bar{\partial}_\pm + i\theta^\pm\partial_{\pm\pm} \end{cases}
\qquad (A.10)
$$

fulfilling the algebra $\{\mathbb{D}_\pm,\bar{\mathbb{D}}_\pm\}=2i\partial_{\pm\pm}$. Using these $N=(2,2)$ supersymmetry generators, we now define $N=(1,1)$ generators as

$$
\begin{aligned}
D_\pm &= \tfrac{1}{\sqrt{2}}(\mathbb{D}_\pm + \bar{\mathbb{D}}_\pm) \\
Q_\pm &= \tfrac{1}{\sqrt{2}}(\mathbb{D}_\pm - \bar{\mathbb{D}}_\pm).
\end{aligned}
\qquad (A.11)
$$

Using the chirality condition we get the two relations

$$
\begin{aligned}
D_\pm\phi &= Q_\pm\phi \\
-D_\pm\bar{\phi} &= Q_\pm\bar{\phi}.
\end{aligned}
\qquad (A.12)
$$

Hence, the second supersymmetry is given by $\delta_2\phi=i\epsilon^\alpha Q_\alpha\phi=i\epsilon^\alpha D_\alpha\phi$ and $\delta_2\bar{\phi}=i\epsilon^\alpha Q_\alpha\phi=-i\epsilon^\alpha D_\alpha\phi$ or, in more compact notation

$$
\delta_2\begin{pmatrix}\phi\\ \bar{\phi}\end{pmatrix}=\epsilon^\alpha D_\alpha\begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix}\begin{pmatrix}\phi\\ \bar{\phi}\end{pmatrix}=\epsilon^\alpha D_\alpha\phi^\nu J_\nu^\mu. \qquad (A.13)
$$

J_ν^μ squares to minus one and it's Nijenhuis tensor vanishes. In other words, J_ν^μ is a complex structure.

Using $\mathbb{D}_+\mathbb{D}_-\bar{\mathbb{D}}_+\bar{\mathbb{D}}_+ = D_+D_-Q_+Q_-$, the manifest $N=(2,2)$ action can be reduced to a $N=(1,1)$ action:

$$\begin{aligned} S &= \int d^2x\, d^2\theta\, d^2\bar{\theta} K(\phi,\bar{\phi}) \\ &= \int d^2x \mathbb{D}_+\mathbb{D}_-\bar{\mathbb{D}}_+\bar{\mathbb{D}}_- K(\phi,\bar{\phi})\Big| \\ &= \int d^2x D_+D_-Q_+Q_- K(\phi,\bar{\phi})\Big| \\ &= -2\int d^2x D_+D_-\Big(\frac{\partial^2 K}{\partial\phi^\mu\partial\bar{\phi}^\nu}D_+\phi^\mu D_-\bar{\phi}^\nu - \frac{\partial^2 K}{\partial\phi^\mu\partial\bar{\phi}^\nu}D_-\phi^\mu D_+\bar{\phi}^\nu\Big)\Big| \\ &= -2\int d^2x\, d^2\theta \frac{\partial^2 K}{\partial\phi^\mu\partial\bar{\phi}^\nu}D^\alpha\phi^\mu D_\alpha\bar{\phi}^\nu\Big|, \end{aligned} \tag{A.14}$$

where partial integration was used and $D^\alpha D_\alpha = \varepsilon^{\alpha\beta}D_\beta D_\alpha = D_+D_- - D_-D_+$.

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