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Applications of Supergeometry

by

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Abstract

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The master thesis considers certain mathematical aspects of Supergeometry which are widely used in the context of supersymmetric field theories whithin the path integral approach. In this work we discuss three topics. First we state all the basic mathematical structure involving the introduction of non-commutative variables and how to construct derivation and integration. Later on we introduce the concept of $\mathbb{R}^{m|n}$ and supermanifolds, working out some simple examples to understand the presented definitions. At last we use our calculations and the developed concepts and properties to present a method for the evaluation of a family of integrals. This technique is called Localization and reduces integration to the evaluation over a set of fixed points. In this work we consider the localization for a finite dimesional setup when the set of points is discrete.

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Contents

To my Family

Chapter 0

Introduction

Supergeometry studies the introduction of non commutative variables as coordinates and the properties of the "spaces" they describe. The discovery of Bose-Fermi supersymmetry has made unavoidable the knowledge of Supergeometry for a theoretical physicist [1]. It is known that this mathematical structure is applied in diverse fields of physics, as Supersymmetric Quantum Field Theory, Supersymmetric Gauge theory or Superstring theory [5]. Supergeometry is a subject that can be easily found in literature, treated with sufficient depth and often within its applications in any of the physics fields mentioned before. Sometimes an elemental and basic approach is difficult to see. The purpouse of the present work is to present some of the mathematical background behind Supergeometry in a clear and straightforward way, with the introduction and proof of some basic concepts and theorems in subjects such as Grassmaniann Variables and Supermanifolds. It is as well purpouse of this work to achieve enough formality to make the reader understand the behavior of mathematical structures in supergeometry.

We will center our attention on three topics. In the beginning we will state all the basic mathematical structure involving the introduction of grassmaniann variables and how to construct derivation and integration. Later on we are going to go further into the concept of $\mathbb{R}^{m|n}$ and supermanifolds, working out some simple examples to understand the presented definitions. At last we will use the developed concepts and properties coming from them, to present a method for the evaluation of a family of integrals that reduces to the evaluation over a set of points, making the calculation easier. This technique is called localization and it's widely used in the context of supersymmetric field theories whithin the path integral approach. In this work we consider the localization for a finite dimesional setup when the set of points is discrete. One reason for this is that the mathematical structure needed for showing how localization appears is nicely simplified and the main objective of this work is to help the reader get in touch with the behaviour of supergeometric spaces and understand how they bring new properties that help making calculations.

This master thesis is in nature mathematical, but it intends to present mathematical structures used in supersymmetric physics, not just by stating results and definitions but by giving proofs with enough formality that make the nature of the results understandable and show a glimpse of how supergeometry and the addition of grassmaniann variables brings the gift of new properties that help physicists calculations.

Chapter 1

Grassmannian Variables and $\mathbb{R}^{m|n}$

The introduction of non commutative (Grassmaniann) variables is well known as strategy in the construction of theories involving fermionic fields [5]. They will come to hand when it's necessary to have a classical analogue of anticommuting operators in order to develop a quantum theory through quantization. Different approaches have been taken to generalize concepts we are already familiar with but in the case of Grassmaniann variables [1] [2]. In this chapter our main goal is to go through the necessary mathematical background involving the introduction of Grassmannian variables. We will introduce the concepts of grassmannian variables and calculus defined over them, as well as its generalization to a space involving grassmannian variables and variables in $\mathbb{R}^m$. We will explain the concept of differentiation and integration, putting some attention on the change of variables in an integral over spaces including non commutative variables.

1.1 Grassmannian Variables

We are going to suppose that the reader is already familiar with real numbers as commutative variables $\{X^1, X^2, ..., X^m\}$ and differential calculus defined over $\mathbb{R}^m$. Later on we are going to center our attention on non commutative variables θ_i that fulfill the following relations

$$\theta_i\theta_j = -\theta_j\theta_i \qquad \theta_j X^A = X^A\theta_j \tag{1.1}$$

where the X^A represent the usual real numbers as commutative variables. It's easy to see that those relations imply that $\theta_i^2 = 0$. The θ variables satisfying 1.1 are called grassmannian variables.

1.1.1 Grassmannian differentiation

In order to further develop the concept of integration, we need to define differentiation involving Grassmaniann variables. In doing so, we would like it to respect the basic properties of differentiation in $\mathbb{R}^m$. We would need first to set how a function of this variables would be defined. Let $F(\theta_1, \theta_2, ..., \theta_n)$ be a function of grassmaniann variables. As the variables satisfy 1.1 the θ_i variables should appear at most to the power of one. Noted that, we can write a general function in the following way

$$F = f^0 + f^i\theta_i + f^{ij}\theta_i\theta_j + ... + f^{12...n}\theta_1\theta_2...\theta_n \tag{1.2}$$

where the f are real coefficients and the Einstein summation convention is used.

We would say that a function is even when it only has even number of the θ_i variables in each factor in its expansion 1.2. In the same way we would say it's odd when the number of θ_i is odd in each of the factors. Not every function can be considered as even or odd, so this is not a complete classification. Functions that are not even nor odd can exist, with even and odd factors mixed, in such case we can always express the function as a sum of even and odd functions.

Now, we can safely define a differentiation over this functions by setting it to satisfy some basic properties. This properties come from the natural generalization of the properties found in differentiation on $\mathbb{R}^m$.

Definition 1. The right derivative $\frac{\overrightarrow{d}}{d\theta_i}$ is defined to satisfy the following properties.

$$\frac{\overrightarrow{d}\,\theta_i}{d\theta_j} = \delta_{ij} \tag{1.3}$$

$$\frac{\overrightarrow{d}\,(f \cdot g)}{d\theta_i} = (-1)^{|f|} f \frac{\overrightarrow{d}\,g}{d\theta_i} + \frac{\overrightarrow{d}\,f}{d\theta_i} g \tag{1.4}$$

$$\frac{\overrightarrow{d}\,(af + bg)}{d\theta_i} = a\frac{\overrightarrow{d}\,(f)}{d\theta_i} + b\frac{\overrightarrow{d}\,(g)}{d\theta_i} \tag{1.5}$$

Where f and g are functions, a and b are real numbers and $|f|$ is 1 if f is an odd function and 0 if it's even function of the θ_i variables.

In the case when the function is not even or odd, we can safely compute the derivative for it's odd and even parts separately by the properties stated above.

Applying this properties to a general function, we can see that the right derivative would be calculated explicitly as

$$\frac{\overrightarrow{d}}{d\theta_i}F = f^i + \delta_{ij}f^{jk}\theta_k - \delta_{ij}f^{kj}\theta_k + \\ + \delta_{ij}f^{jkl}\theta_k\theta_l - \delta_{ij}f^{kjl}\theta_k\theta_l + \delta_{ij}f^{klj}\theta_k\theta_l + ... \tag{1.6}$$

where δ_{ij} is the kronecker delta and again Einstein summation convention is used.

Definition 2. The left derivative $\frac{\overleftarrow{d}}{d\theta_i}$ can be defined as

$$\frac{\overrightarrow{d}}{d\theta_j}\theta_i = \theta_i\frac{\overleftarrow{d}}{d\theta_j}$$

Choosing between the left or right derivative is just a matter of convention. Any conclusion made by choosing the left or right derivative should be equivalent. We have to choose one convention for the porpouse of this work, so in the following we will use $\frac{d}{d\theta_i}$ for the right derivative and $\frac{\partial}{\partial\theta_i}$ for the partial derivative induced by it (treating the other θ variables as constant).

1.1.2 Grassmannian Integration

We can also define an integration over the functions of Grassmaniann variables. This Integration will be defined by some basic properties we will require it to satisfy.

Definition 3. The Grassmmannian Integration (or Berezin integration) is defined by the following rules

$$\int (af(\theta_i) + bg(\theta_i))\, d\theta_i = a\int f(\theta_i)d\theta_i + b\int g(\theta i)d\theta_i \tag{1.7}$$

$$\int d\theta_i = 0 \tag{1.8}$$

$$\int \theta_i d\theta_j = \delta_{ij} \tag{1.9}$$

$$\int F_1(\theta_1)F_2(\theta_2)...F_n(\theta_n)d\theta_1 d\theta_2...d\theta_n = \\ = \int F_1(\theta_1)d\theta_1 \int F_2(\theta_2)d\theta_2... \int F_n(\theta_n)d\theta_n \tag{1.10}$$

As we can see, this definition resembles some kind of differentiation, but we will see later how it helps to achieve a natural generalization for the concept of integration in a more general space. Using this properties on a general function 1.2 we find that

$$\begin{aligned}
&\int F d\theta_1 d\theta_2 ... d\theta_n = \\
&= \int (f^0 + f^i\theta_i + f^{ij}\theta_i\theta_j + ... + f^{12...n}\theta_1\theta_2...\theta_n) d\theta_1 d\theta_2 ... d\theta_n = \\
&= f^{12...n}
\end{aligned} \tag{1.11}$$

As only the factor involving all the θ_i will survive the integration.

1.1.2.1 Change of variables in grasmmannian integration

Now we can study how this definition of integration behaves under a change of variables. If we want to perform a change of the variables $\{\theta_1, \theta_2, ..., \theta_n\}$ the integration measure $d\theta_1 d\theta_2 ... d\theta_n$ has to transform accordingly. We are familiar with the commutative variables case, on which if we perform the change of variables

$$\tilde{X}^A = f^A(X^1, ..., X^m)$$

we have the following relation between integrals in the old and new variables

$$\int G(\tilde{X}^1(X^1, ..., X^m)...\tilde{X}^m(X^1, ..., X^m)) \left|\frac{\partial \tilde{X}^A}{\partial X^B}\right| dX^1...dX^m = \int G(\tilde{X}^1, ..., \tilde{X}^m) d\tilde{X}^1...d\tilde{X}^m \tag{1.12}$$

Where $\frac{\partial \tilde{X}^A}{\partial X^B}$ is the jacobian matrix,

$$\begin{pmatrix}
\frac{\partial \tilde{X}^1}{\partial X^1} & \frac{\partial \tilde{X}^1}{\partial X^2} & \cdots & \frac{\partial \tilde{X}^1}{\partial X^m} \\
\frac{\partial \tilde{X}^2}{\partial X^1} & \frac{\partial \tilde{X}^2}{\partial X^2} & \cdots & \frac{\partial \tilde{X}^2}{\partial X^m} \\
\vdots & \vdots & \cdots & \vdots \\
\frac{\partial \tilde{X}^m}{\partial X^1} & \frac{\partial \tilde{X}^m}{\partial X^2} & \cdots & \frac{\partial \tilde{X}^m}{\partial X^m}
\end{pmatrix}$$

In a very simple example, we have that in $\mathbb{R}$ and under the change of variable $\tilde{X} = cX$ the integral measure transforms $d\tilde{X} = cdX$.

In the case of a one-dimensional Grassmaniann space, under the change of variable $\tilde{\theta} = c\theta$, we have by using the properties of Grassmannian Integration

$$\int \theta d\theta = \int \tilde{\theta} d\tilde{\theta} = \int \theta c d\tilde{\theta} = 1 \tag{1.13}$$

From which we can see that $d\tilde{\theta}_i = \frac{1}{c} d\theta_i$.

From this simple example we can see that if we use Grassmannian variables, the relation between integration measures works in a different way. If we make a change of coordinates $\tilde{\theta}_i = g_i(\theta_1, \theta_2, ..., \theta_n)$, we have to impose the new variables to satisfy 1.1, so they should become noncommutative variables again, in other words, the function g_i must have only odd numbers of the old variables in each factor (i.e g_i has to be an odd function of θ_i). That means that the functions g_i should have the following expantion on the θ variables:

$$g_i(\theta_j) = g_i^k \theta_k + g_i^{klm} \theta_k \theta_l \theta_m + ... \tag{1.14}$$

Moreover, for the change to become a proper one we must demand this function to be invertible. If we denote by $j_i(\tilde{\theta}_j)$ the inverse function of $g_i(\theta_j)$, it should satisfy

$$g_i(j_j(\tilde{\theta}_k)) = \tilde{\theta}_i \tag{1.15}$$

And as we have seen in 1.2, it can be expanded as (noting that it should be odd as well),

$$j_i(\tilde{\theta}_j) = j_i^k \tilde{\theta}_k + j_i^{klm} \tilde{\theta}_k \tilde{\theta}_l \tilde{\theta}_m + ...$$

then if we use both expansions in 1.15 we have that the relation that the inverse function has to satisfy is

$$\begin{aligned} g_i(j_j(\tilde{\theta}_k)) = {} & g_i^k \left[j_k^r \tilde{\theta}_r + j_k^{rsm} \tilde{\theta}_r \tilde{\theta}_s \tilde{\theta}_m + ... \right] \\ & + g_i^{klp} \left[j_k^r \tilde{\theta}_r + j_k^{rsm} \tilde{\theta}_r \tilde{\theta}_s \tilde{\theta}_m + ... \right] \left[j_l^r \tilde{\theta}_r + j_l^{rsm} \tilde{\theta}_r \tilde{\theta}_s \tilde{\theta}_m + ... \right] \left[j_p^r \tilde{\theta}_r + j_p^{rsm} \tilde{\theta}_r \tilde{\theta}_s \tilde{\theta}_m + ... \right] + ... \\ & = \tilde{\theta}_i \end{aligned} \tag{1.16}$$

It's easy to see from this latter equation that the only term which includes only one $\tilde{\theta}$ in each factor is $g_i^k j_k^r \tilde{\theta}_r$. If we want it to satisfy 1.15 we should ask that

$$g_i^k j_k^r = \delta_{ir} \tag{1.17}$$

This condition is equivalent to ask that the matrix $j^i_j = \left[g^i_j\right]^{-1}$. From this we can see that the invertibility of the function g_i depends only on the fact that g^i_j should be invertible as a matrix.

With a change of variables of this form we can work out how the integration is affected. Using the previous definition 3 of grassmannian integration we have that

$$\int \theta_1 ... \theta_n d\theta_1 ... d\theta_n = \int \tilde{\theta}_1 ... \tilde{\theta}_n d\tilde{\theta}_1 ... d\tilde{\theta}_n = 1 \tag{1.18}$$

but we can use the properties of grassmannian variables to work out this relation. Expanding each of the new variables in the old ones we have that

$$\begin{aligned}
&\tilde{\theta}_1 ... \tilde{\theta}_n = \\
&= \left(g^k_1 \theta_k + g^{klm}_1 \theta_k \theta_l \theta_m + ...\right) ... \left(g^k_n \theta_k + g^{klm}_n \theta_k \theta_l \theta_m + ...\right) \\
&= h^{1...n} \theta_1 ... \theta_n
\end{aligned} \tag{1.19}$$

Since any other term involved is of higher degree than one in at least one of the θ_i. At the same time using the developed concept of grasmmannian differentiation we can find $h^{12...n}$ by

$$h^{12...n} = \frac{\partial(\tilde{\theta}_1 ... \tilde{\theta}_n)}{\partial(\theta_1 ... \theta_n)} \tag{1.20}$$

after repeated use of 1.6, and arranging the factors properly, we find the following relation

$$\frac{\partial(\tilde{\theta}_1 ... \tilde{\theta}_n)}{\partial(\theta_1 ... \theta_n)} = \sum_{\sigma \in S_n} \prod_{i=1}^{n} \frac{\partial \tilde{\theta}_{\sigma(i)}}{\partial \theta_i} Sgn(\sigma) = det \left| \frac{\partial \tilde{\theta}_i}{\partial \theta_j} \right| \tag{1.21}$$

Now, using 1.19 into 1.18

$$\begin{aligned}
&\int \tilde{\theta}_1 ... \tilde{\theta}_n d\tilde{\theta}_1 ... d\tilde{\theta}_n = \\
&= \int \left(g^k_1 \theta_k + g^{klm}_1 \theta_k \theta_l \theta_m + ...\right) ... \left(g^k_n \theta_k + g^{klm}_n \theta_k \theta_l \theta_m + ...\right) d\tilde{\theta}_1 ... d\tilde{\theta}_n \\
&= \int h^{1...n} \theta_1 ... \theta_n d\tilde{\theta}_1 ... d\tilde{\theta}_n \\
&= \int \theta_1 ... \theta_n d\theta_1 ... d\theta_n
\end{aligned} \tag{1.22}$$

As this equation holds, we can see that $\int h^{1...n}\theta_1...\theta_n d\tilde{\theta}_1...d\tilde{\theta}_n = \int \theta_1...\theta_n d\theta_1...d\theta_n$, which shows that the relation between the integration measures is held by $h^{1...n}$. But we already found a simple expression for $h^{1...n}$ in 1.21, so using that we can conclude

$$\int \theta_1...\theta_n d\theta_1...d\theta_n = \int \theta_1...\theta_n det\left|\frac{\partial\tilde{\theta}_i}{\partial\theta_j}\right| d\tilde{\theta}_1...d\tilde{\theta}_n \tag{1.23}$$

Moreover, from that equation we can finally obtain the relation between the integration measures

$$d\theta_1...d\theta_n \frac{1}{det\left|\frac{\partial\tilde{\theta}_i}{\partial\theta_j}\right|} = d\tilde{\theta}_1...d\tilde{\theta}_n \tag{1.24}$$

This tells us that in the case of integration in Grassmannian variables, if we make a change of variables $\tilde{\theta}_i = g_i(\theta_1, ..., \theta_n)$ the change in the integration measure will be given by the following equation

$$\int F(\tilde{\theta}_i(\theta_j)) \frac{1}{det\left|\frac{\partial\tilde{\theta}_i}{\partial\theta_j}\right|} d\theta_1...d\theta_n = \int F(\tilde{\theta}_i) d\tilde{\theta}_1...d\tilde{\theta}_n \tag{1.25}$$

Which differs from the case in $\mathbb{R}^m$ where we had 1.12, as we have here the inverse of what would be the jacobian matrix in this case.

1.2 Mixed commutative and Grassmannian variables, $\mathbb{R}^{m|n}$

We can now think of a more general space, a space that mixes both commutative and Grassmannian variables. It is more abstract space than $\mathbb{R}^m$ as we can't picture it graphicly, but we can define it as the space described by coordinates $\{X^1, ..., X^m, \theta_1, ..., \theta_n\}$ where X^A are commutative coordinates and the θ_i are Grasmmannian coordinates. In that sense, it will be easier to extrapolate old concepts and ideas from $\mathbb{R}^m$ as we will see it as the addition of new Grassmaniann coordinates in a real space. In the following we will call this space $\mathbb{R}^{m|n}$.

1.2.1 Details on $\mathbb{R}^{m|n}$

As we stated before, we can picture $\mathbb{R}^{m|n}$ as a space with the first m coordinates being commutative and the next n being Grassmaniann. The first attempt the reader can make to understand this is to think of commutative coordinates as real numbers. This can be true for some but not for every coordinate system. A change of coordinates in $\mathbb{R}^{m|n}$ can mix both, Grassmaniann and real coordinates, with the only restriction of keeping the commutation relations 1.1. For example in $\mathbb{R}^{2|2}$ we can make a change of variables $\tilde{X^A} = \theta_1\theta_2 + X^A$ and $\tilde{\theta}_i = \theta_i$. In that case the new $\tilde{X}^A$ coordinates will not be real numbers but a mixture of both, with the only condition that they commute with each other and with the $\tilde{\theta}_i$ variables. Taking that into account, we can now define properly $\mathbb{R}^{m|n}$

Definition 4. We will call $\mathbb{R}^{m|n}$ to be the space satisfying that in any coordinate system $\{X^1, ..., X^m, \theta_1, ..., \theta_n\}$ that describes it, the coordinates have the following properties

$$X^A X^B = X^B X^A, \quad X^A\theta_i = \theta_i X^A \quad \text{and} \quad \theta_i\theta_j = -\theta_j\theta_i \tag{1.26}$$

Where the X^A are not necessarily real numbers. We can always express the commutative coordinates X^A as $X^A = r^A + f^A(\theta_i)$ where r^A is a number in $\mathbb{R}$ and f^A a function of θ_i coordinates with coefficients in $\mathbb{R}$.

Definition 5. The real projector $\pi : \mathbb{R}^{m|n} \to \mathbb{R}^m$ is a function that acts in $\mathbb{R}^{m|n}$ in the following way:

$$\left(X^1, ..., X^m, \theta_1, ..., \theta_n\right) \xrightarrow{\pi} \left(r^1, ..., r^m\right) \tag{1.27}$$

This projector will come to hand when we will develop the concept of a supermanifold. We can now go safely onto the definition of differentiation and integration in $\mathbb{R}^{m|n}$

1.2.2 Differentiation in $\mathbb{R}^{m|n}$

Now that we are familiar with the concept of differentiation with respect to grassmann variables, we can define how to do it in $\mathbb{R}^{m|n}$. First of all we must define how a function looks in this new space. In $\mathbb{R}^{m|n}$ a function has the following form.

$$F = f^i(X^1, ..., X^m)\theta_i + f^{ij}(X^1, ..., X^m)\theta_i\theta_j + ... + f^{12...n}(X^1, ..., X^m)\theta_1\theta_2...\theta_n \tag{1.28}$$

We will also use the concept of even and odd function for the case where the function has only even or odd number of the θ_i variables in each factor.

Now we need to establish a relationship between both kinds of variables, so we state that the following relation holds.

$$\frac{\partial X^i}{\partial \theta_j} = 0$$

And of course we would like to have a differentiation defined for this functions. We will just mix the concepts of differentiation in $\mathbb{R}^m$ and Grassmaniann differentiation by using both in the following way

Definition 6. The derivatives of the function F will be given by this rules

$$\frac{\partial}{\partial\theta_i}F = f^i(X^1,...,X^m)+\delta_{ij}f^{jk}(X^1,...,X^m)\theta_k - \delta_{ij}f^{kj}(X^1,...,X^m)\theta_k+ \\ + \delta_{ij}f^{jkl}(X^1,...,X^m)\theta_k\theta_l - \delta_{ij}f^{kjl}(X^i,...,X^m)\theta_k\theta_l+ \\ + \delta_{ij}f^{klj}(X^1,...,X^m)\theta_k\theta_l + ... \tag{1.29}$$

$$\frac{\partial}{\partial X^A}F = \frac{\partial}{\partial X^A}f^i(X^1,...,X^m)\theta_i + \frac{\partial}{\partial X^A}f^{ij}(X^1,...,X^m)\theta_i\theta_j + ... \\ ... + \frac{\partial}{\partial X^A}f^{12...n}(X^1,...,X^m)\theta_1\theta_2...\theta_n \tag{1.30}$$

1.2.2.1 Differentiable functions

Such as in the real variables case, we can now define a concept of differentiability for functions in $\mathbb{R}^{m|n}$.

Definition 7. We will call a function differentiable when the real projection of the coefficients $f^{12...i}$ in 1.28, $\pi(f^{12...i}) \in \mathbb{R}$, are differentiable in the usual definition for $\mathbb{R}^m$.

Note that the nature of grassmaniann variables makes that differentiability depends only on the real projection of the commutative variables, as we don't have a notion of continuity or even numerical value for Grassmaniann variables.

1.2.3 Integration in $\mathbb{R}^{m|n}$

We can work out now how to perform integration in $\mathbb{R}^{m|n}$. We already went through integration in the pure Grassmaniann case, and as we are taking commutative variables to be independent of Grassmaniann variables, for a function in $\mathbb{R}^{m|n}$ we have

$$\begin{aligned}
&\int F d\theta_1 d\theta_2 ... d\theta_n = \\
&\quad = \int (f^i\theta_i + f^{ij}\theta_i\theta_j + ... + f^{1...n}\theta_1...\theta_n) d\theta_1...d\theta_n dX^1...dX^m = \\
&\quad = \int f^{12...n} dX^1...dX^m
\end{aligned} \tag{1.31}$$

where now all the coefficients $f^{12...i}$ are functions of the X^A variables.

1.2.3.1 change of variables in $\mathbb{R}^{m|n}$

If we make a change of variables in $\mathbb{R}^{m|n}$, the change in the integration measure is not trivial. In order to prove the right relationship between the change of coordinates, let us study first some particular cases.

In order to make the change of variables a proper one, we should consider the parity of the coordinate change. The new coordinates should satisfy equations 1.1, so the change functions should be even for the new even variables and odd for the new odd ones. If we consider that, a proper change of variables will be in the form

$$\begin{aligned}
\tilde{X}^A &= f^A(X^B, \theta_i) \\
\tilde{\theta}_i &= g_i(X^A, \theta_k)
\end{aligned} \tag{1.32}$$

where the f^A are even functions (even number of θ_i in each factor) and the g^i are odd (odd number of θ_i in each factor) functions of θ_i. As we saw before, in order to make this change become a proper one we should also ask this functions to be invertible. We already showed in 1.17 how for the θ variables the invertibility depends only on the first coefficient of the expantion of the function g_i. For the X variables the invertibility is asked for the coefficient independent of the θ variables in f^A, as the relation

$$f^A|_{f^{-1}} = f_{0A}|_{f^{-1}} + f^{ij}_{1A}|_{f^{-1}}\theta_i\theta_j + ... = X^A \tag{1.33}$$

where $f^{-1} = \left[\left(f^1\right)^{-1}, \left(f^2\right)^{-1}, ..., \left(f^m\right)^{-1}\right]$, should be satisfied. From this last equation we can see that, as the only term independent of the θ variables will come from $f_{0A}|_{f^{-1}}$, in order to fullfil the condition for the inverse function we only need to satisfy

$$f_{0A}|_{f^{-1}_{0A}} = X^A \tag{1.34}$$

where $f_{0A}^{-1} = \left[\left(f_{0A}^{1}\right)^{-1}, \left(f_{0A}^{2}\right)^{-1}, ..., \left(f_{0A}^{m}\right)^{-1} \right]$. This relation is equivalent to the function f_{0A} being invertible.

Having discused the properties of the change of variables, let us consider first the special case where the transformations have the form

$$\begin{aligned} \tilde{X}^A &= f^A(X^B) \\ \tilde{\theta}_i &= g_i(\theta_k) \end{aligned} \tag{1.35}$$

As we have no dependence between the odd and even variables, we can just see the change of variables as two separate changes in the spaces $U_1 = gen(X^i)$ and $U_2 = gen(\theta_i)$. In order to make the whole change of variables, we can break it into two steps, each one making the change separately in every space. We already know the transformation rules for each space so it is now simple to conclude that the change of coordinates will be reflected in the integral in the following way

$$\begin{aligned} \int F(X^A, \theta_j) det \left| \frac{\partial X^A}{\partial X^B} \right| det \left| \frac{\partial \tilde{\theta}_i}{\partial \theta_j} \right|^{-1} dX^1 ... dX^m d\theta_1 ... d\theta_n = \\ = \int F(\tilde{X}^A, \tilde{\theta}_j) d\tilde{X}^1 ... d\tilde{X}^m d\tilde{\theta}_1 ... d\tilde{\theta}_n \end{aligned} \tag{1.36}$$

The same conclusion can be developed for the two cases where the transformations have the form

$$\begin{aligned} \tilde{X}^A &= f^A(X^B) \\ \tilde{\theta}_i &= g_i(X^A, \theta_k) \end{aligned} \tag{1.37}$$

Or

$$\begin{aligned} \tilde{X}^A &= f^A(X^B, \tilde{\theta}_i) \\ \tilde{\theta}_i &= g_i(\theta_k) \end{aligned} \tag{1.38}$$

As in those cases we can interpret the transformation as two different ones, one first on U_1 (or U_2) and then another change leaving constant the odd (or even) variables respectively.

The Jacobian matrix of a transformation can be expressed as the multiplication of two jacobian matrices of transformations such that their composition results in the original one. That is, if we have two transformations t_1 and t_2 such that $T = t_1 \circ t_2$ then we have that,

$$Jac(T) = Jac(t_1)Jac(t_2)$$

For a coordinate transformation in $\mathbb{R}^{m|n}$ we have that the jacobian matrix has the form:

$$\begin{pmatrix} \frac{\partial \tilde{X}}{\partial X} & \frac{\partial \tilde{X}}{\partial \theta} \\ \frac{\partial \tilde{\theta}}{\partial X} & \frac{\partial \tilde{\theta}}{\partial \theta} \end{pmatrix} \tag{1.39}$$

Where $\frac{\partial \tilde{X}}{\partial X}, \frac{\partial \tilde{\theta}}{\partial \theta}$ are even block matrices and $\frac{\partial \tilde{X}}{\partial \theta}, \frac{\partial \tilde{\theta}}{\partial X}$ are odd block matrices of the respective partial derivatives.

Moreover, we can see that we can decompose the jacobian matrix as follows

$$\begin{pmatrix} \frac{\partial \tilde{X}}{\partial X} & \frac{\partial \tilde{X}}{\partial \theta} \\ \frac{\partial \tilde{\theta}}{\partial X} & \frac{\partial \tilde{\theta}}{\partial \theta} \end{pmatrix} = \begin{pmatrix} I & 0 \\ \frac{\partial \tilde{\theta}}{\partial X}\frac{\partial \tilde{X}}{\partial X}^{-1} & \frac{\partial \tilde{\theta}}{\partial \theta} \end{pmatrix} \begin{pmatrix} \frac{\partial \tilde{X}}{\partial X} & 0 \\ 0 & \frac{\partial \tilde{\theta}}{\partial \theta}^{-1} \end{pmatrix} \begin{pmatrix} I & \frac{\partial \tilde{X}}{\partial X}^{-1}\frac{\partial \tilde{X}}{\partial \theta} \\ 0 & \frac{\partial \tilde{\theta}}{\partial \theta} - \frac{\partial \tilde{\theta}}{\partial X}(\frac{\partial \tilde{X}}{\partial X})^{-1}\frac{\partial \tilde{X}}{\partial \theta} \end{pmatrix} \tag{1.40}$$

The existence of the inverse matrices $\frac{\partial \tilde{X}}{\partial X}^{-1}$ and $\frac{\partial \tilde{X}}{\partial X}^{-1}$ comes from the fact that the coordinate change should be invertible and the jacobian of the inverse transformation is just the inverse jacobian matrix of the original one. Therefore the invertibility of the jacobian matrices is ensured by the invertibility of the change of coordinates. Noting that this decomposition respects that each of the factors is respectively odd or even, we can say they represent the jacobian matrix of three proper changes of variables. By the definition of the jacobian matrix we can conclude that those changes represent the following cases.

$$\begin{pmatrix} I & 0 \\ \frac{\partial\tilde{\theta}}{\partial X}\frac{\partial\tilde{X}}{\partial X}^{-1} & \frac{\partial\tilde{\theta}}{\partial\theta} \end{pmatrix} \qquad \text{A transformation of the class 1.37}$$

$$\begin{pmatrix} \frac{\partial\tilde{X}}{\partial X} & 0 \\ 0 & \frac{\partial\tilde{\theta}}{\partial\theta}^{-1} \end{pmatrix} \qquad \text{A transformation of the class 1.35}$$

$$\begin{pmatrix} I & \frac{\partial\tilde{X}}{\partial X}^{-1}\frac{\partial\tilde{X}}{\partial\theta} \\ 0 & \frac{\partial\tilde{\theta}}{\partial\theta} - \frac{\partial\tilde{\theta}}{\partial X}\left(\frac{\partial\tilde{X}}{\partial X}\right)^{-1}\frac{\partial\tilde{X}}{\partial\theta} \end{pmatrix} \qquad \text{A transformation of the class 1.38} \tag{1.41}$$

We already know how the integration measure changes with those kinds of transformations. Splitting the main transformation as three separate ones and following what we found in 1.36, it's clear that our new transformation factor would be

$$det\left|I\right| det\left|\frac{\partial\tilde{\theta}}{\partial\theta}\right|^{-1} det\left|\frac{\partial\tilde{X}}{\partial X}\right| det\left|\frac{\partial\tilde{\theta}}{\partial\theta}^{-1}\right|^{-1} det\left|I\right| det\left|\frac{\partial\tilde{\theta}}{\partial\theta} - \frac{\partial\tilde{\theta}}{\partial X}\left(\frac{\partial\tilde{X}}{\partial X}\right)^{-1}\frac{\partial\tilde{X}}{\partial\theta}\right|^{-1} =$$
$$= det\left|\frac{\partial\tilde{X}}{\partial X}\right| det\left|\frac{\partial\tilde{\theta}}{\partial\theta} - \frac{\partial\tilde{\theta}}{\partial X}\left(\frac{\partial\tilde{X}}{\partial X}\right)^{-1}\frac{\partial\tilde{X}}{\partial\theta}\right|^{-1} \tag{1.42}$$

So in a general integration on $\mathbb{R}^{m|n}$ we have that under a proper change of variables the following relation is satisfied

$$\int F(X^A,\theta_i) det\left|\frac{\partial\tilde{X}}{\partial X}\right| det\left|\frac{\partial\tilde{\theta}}{\partial\theta} - \frac{\partial\tilde{\theta}}{\partial X}\left(\frac{\partial\tilde{X}}{\partial X}\right)^{-1}\frac{\partial\tilde{X}}{\partial\theta}\right|^{-1} dX^1...dX^m d\theta_1...d\theta_n =$$
$$= \int F(\tilde{X}^A,\tilde{\theta}_j) d\tilde{X}^1...d\tilde{X}^m d\tilde{\theta}_1...d\tilde{\theta}_n \tag{1.43}$$

This relation motivates us to state a definition for the generalization to $\mathbb{R}^{m|n}$ of the determinant of a matrix in $\mathbb{R}^m$, called Berezinian after the Russian mathematician Felix Berezin.

Definition 8. A Matrix with entries in $\mathbb{R}^{m|n}$ is written as

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1.44}$$

Where A and D are invertible matrices with even functions as entries and B and C have odd functions as entries in $\mathbb{R}^{m|n}$. The Berezinian of the matrix M ($Ber(M)$) is given by

$$Ber(M) = det\,|A|\,det\,\left|D - CA^{-1}B\right|^{-1} \tag{1.45}$$

The Berezinian has the following properties

1. If M_1 and M_2 are even matrices, then $Ber(M_1M_2) = Ber(M_1)Ber(M_2)$
2. $Ber(M) = e^{(Tr[log(A)]-Tr[log(D)])}$. Where $Tr(A)$ and $Tr(D)$ are the traces of the matrices A and D.

The relation between the integration measures after a change of variables T in $\mathbb{R}^{m|n}$ is related closely to the Berezinian. Rewriting equation 1.43 we see that this relation takes the form

$$Ber[Jac(T)]dX^1...dX^m d\theta_1...d\theta_n = d\tilde{X}^1...d\tilde{X}^m d\tilde{\theta}_1...d\tilde{\theta}_n \tag{1.46}$$

The Berezinian as a generalization of the determinant helps us to state the change of the integral under a change of variables in $\mathbb{R}^{m|n}$ in a way analogue to $\mathbb{R}^m$. We also see that it's properties are also analogue to the properties of the determinant.

With the introduction of $\mathbb{R}^{m|n}$ and the structure of calculus over it with the knowledge of how changes of variables work, we have the necessary tools to go further onto the nature of $\mathbb{R}^{m|n}$. The relations found in this chapter will be used in the following chapter, where we will present the basic concepts around supermanifolds.

Chapter 2

Supermanifolds

The nowadays essential knowledge of supergeometry in theoretical physics, makes very useful a quite simple and clear introductory definition of what is a supermanifold. Supermanifolds are "spaces" that arise after the consideration of using Grassmaniann variables as coordinates. In this chapter we will go through the definition of a Supermanifold and our goal is to present an useful yet simple definition for it and the concept of integration over them. In doing so, we will work out some simple examples in order to succesfuly become familiar with this kind of spaces.

2.1 What is a Supermanifold?

There are several definitions of a Supermanifold found in the literature [1] [6] [5], each of which follow a different formal approach. In the following we will present a definition that tries to be more clear and simple as it takes advantage from the common definition of a Manifold. It will show a Supermanifold as a Manifold with coordinates not in $\mathbb{R}^n$ but in $\mathbb{R}^{m|n}$.

2.1.1 The topology of $\mathbb{R}^{m|n}$

Before we define more formally what is a supermanifold, we need to give a topology to $\mathbb{R}^{m|n}$. We will use the definition made by De Witt [1], but in a clearer way, using the real projector defined in 5. This topology will take profit of the fact that $\mathbb{R}^{m|n}$ has some coordinate system in which its commutative coordinates are just real numbers.

Definition 9. Let O_A be a subset of $\mathbb{R}^{m|n}$, we will call this subset open if $\pi(O_A) \in \mathbb{R}^m$, the image under the real projector, is an open set under the usual topology in $\mathbb{R}^m$

2.1.2 the definition of Supermanifold

Definition 10. A supermanifold with dimension $m|n$ is a space M with a collection of pairs (U_A, ϕ_A) where the U_A are subsets of M and the ϕ_A are the coordinate functions $\phi_A : U_A \to O_A$ where O_A is an open set of $\mathbb{R}^{m|n}$. This collection of pairs satisfy the following properties

- $\bigcup_A U_A = M$
- $\phi_A \circ \phi_B^{-1} : \mathbb{R}^{m|n} \to \mathbb{R}^{m|n}$, that we will call the transition function, should be differentiable (in the sense of 7)

2.1.3 Diffeomorphisms in Supermanifolds

As in manifolds theory, we can find functions $\phi : M \to \tilde{M}$ where M with (U_A, ϕ_A) and $\tilde{M}$ with $(\tilde{U}_B, \tilde{\phi}_B)$ are Supermanifolds. We will say that such function is differentiable when it's translation onto $\mathbb{R}^{m|n}$ is differentiable, i.e when we have that for certain A and B, $\phi(U_A) \cap \tilde{U}_B \neq \emptyset$ (ϕ takes points from U_A to $\tilde{U}_B$), then the translated function in $\mathbb{R}^{m|n}$ ($\tilde{\phi_B} \circ \phi \circ \phi_A^{-1}$, which goes from $\phi_A(U_A)$ to $\tilde{\phi}_B(\tilde{U}_B)$) is differentiable in the sense of 7. If we also require that ϕ is a bijection and it has a differentiable inverse, we will call ϕ a diffeomorphism. Diffeomorphisms are important for the fact that they not just only translate points between supermanifolds, but some of its properties. For example, if two supermanifolds share a diffeomorphism, they have the same dimension. They are important as well when we want to understand when supermanifolds share the same structure, as topological properties are also invariant under diffeomorphisms.

2.2 Integration on Supermanifolds

Integration over a supermanifold is performed in the local coordinates, which are in $\mathbb{R}^{m|n}$. In order to define a proper integration, we should make it to be invariant under the change between coordinates. That means that the integral should be defined in a way that it does not change if we perform it in two different coordinate systems. To achieve that, the integration measure should have a density function that transforms in a way that makes the integral to remain unchanged. Then the integration of a function F is written as

$$\int F\mu(\theta, X) d\theta dX \tag{2.1}$$

Suppose we have the change of coordinates

$$\tilde{X}^A = [\phi_A \circ \phi_B^{-1}]^A(X^1, ..., X^m, \theta_1, ..., \theta_n)$$
$$\tilde{\theta}_i = [\phi_A \circ \phi_B^{-1}]_i(X^1, ..., X^m, \theta_1, ..., \theta_n) \tag{2.2}$$

Where $[\phi_A \circ \phi_B^{-1}]^A$ and $[\phi_A \circ \phi_B^{-1}]_i$ denote the respective components of the transition function. We already proved that the integration measure changes as 1.46. The integration density function μ should transform under this change in a way that keeps the integral invariant. If we require that μ changes in the following way

$$\frac{1}{Ber[Jac(\phi_A \circ \phi_B^{-1})]}\mu(X^A, \theta_i) = \tilde{\mu}(\tilde{X}^A, \tilde{\theta}_i) \tag{2.3}$$

We have that the integration is invariant under the mentioned change, as we have

$$\int F(\tilde{X}^A(X^B), \tilde{\theta}_k(\theta_l))\frac{1}{Ber[Jac(\phi_A \circ \phi_B^{-1})]}\mu(\tilde{X}^A(X^B), \tilde{\theta}_k(\theta_l))Ber[Jac(\phi_A \circ \phi_B^{-1})]d\theta dX =$$
$$= \int F(\tilde{X}^A(X^B), \tilde{\theta}_k(\theta_l))\ \mu(\tilde{X}^A(X^B), \tilde{\theta}_k(\theta_l))d\theta dX = \int F(\tilde{X}^A, \tilde{\theta}_j)\tilde{\mu}(\tilde{X}^A, \tilde{\theta}_i)d\tilde{\theta}d\tilde{X} \tag{2.4}$$

With this density function chosen we have achieved that under the change of coordinates the integral remains the same, which allows us to consider it a proper and consistent definition of integration in the Supermanifold. With this definition in mind we can work out some simple examples of Supermanifolds.

2.3 Basic examples of Supermanifolds

In order to further understand the concept of Supermanifold, and integration defined on it, we are going to present two important and simple examples of supermanifolds. They follow from a generalization of the known Tangent and Cotangent Bundle of an usual Manifold.

2.3.1 Tangent Bundle with reversed parity $T[1]M$

For the construction of supermanifolds, we will just specify the transition functions between the coordinates. The coordinates are in $\mathbb{R}^{m|n}$ but the way we glue them together will result in two different Supermanifolds. The first one is called the Tangent bundle with reversed parity, as it comes from the generalization of tangent bundle in a usual Manifold. We will see the grassmann coordinates as a tangent space over the commutative coordinates, and because of that they will transform as vectors in the usual sense for manifolds. The even coordinates transform such as in an usual Manifold, so we have that the transition functions between coordinates are of the form $\tilde{X^A} = f^A(X^B)$. We add the odd coordinates in a way that their transition function behaves as the transition function for tangent vector in a Manifold, so we have that $\tilde{\theta}^A = \frac{\partial f^A}{\partial X^B}\theta^B$

2.3.1.1 Functions on $T[1]M$

We already know how a function in $\mathbb{R}^{m|n}$ looks like. For a supermanifold we will ask for a function to be an invariant object under change of coordinates, and to look like a function in each of the local coordinate systems. Therefore, if G is a function on $T[1]M$, equating its expression in two different coordinates, it follows that

$$\begin{aligned} G &= g_0 + g_{AB}\theta^A\theta^B + g_{ABC}\theta^A\theta^B\theta^C + ... + g_{1...m}\theta^1...\theta^m \\ &= \tilde{g}_0 + \tilde{g}_{AB}\frac{\partial f^A}{\partial X^C}\theta^C\frac{\partial f^B}{\partial X^D}\theta^D + \tilde{g}_{ABC}\frac{\partial f^A}{\partial X^C}\theta^C\frac{\partial f^B}{\partial X^D}\theta^D\frac{\partial f^C}{\partial X^E}\theta^E + ... + \tilde{g}_{1...m}\frac{\partial f^1}{\partial X^D}\theta^D...\frac{\partial f^m}{\partial X^E}\theta^E \end{aligned} \tag{2.5}$$

From that last relation we can calculate the relation between $g_{1...m}$ and $\tilde{g}_{1...m}$ by looking for the factor with all the θ variables in the second line of the latter equation. The result of this calculation is

$$g_{1...m} = det\left|\frac{\partial f^A}{\partial X^B}\right|\tilde{g}_{1...m} \tag{2.6}$$

This shows us that this coefficient transforms as a top-form in the manifold described by the X coordinates. As we saw in 1.31 this coefficient is the one that will come to hand when we perform the integration in the $\mathbb{R}^{m|n}$ coordinates. This shows that when we integrate a function in $T[1]M$ we are performing an integration of a top-form in the manifold described by the commutative coordinates and the transition functions f^A.

Moreover we can see that the other coefficients transform as forms in the M manifold, as for example by looking the factors proportional to $\theta^A\theta^B$we can see the coefficient g_{AB} is transformed

$$g_{AB} = \left[\frac{\partial f^C}{\partial X^B}\frac{\partial f^D}{\partial X^B} - \frac{\partial f^C}{\partial X^B}\frac{\partial f^D}{\partial X^A}\right]\tilde{g}_{CD}$$

which is exactly how a two-form transforms in M.

2.3.1.2 integration on $T[1]M$

In order to define integration over a supermanifold we must introduce the concept of density function, as we saw before. To present how integration works for $T[1]M$ we have to calculate the jacobian matrix for the coordinate change functions, and then introduce a density function as we stated in the preceding section. Recalling 1.39 we have that in this case the jacobian matrix is

$$J = \begin{pmatrix} \frac{\partial \tilde{X}}{\partial X} & \frac{\partial \tilde{X}}{\partial \theta} \\ \frac{\partial \tilde{\theta}}{\partial X} & \frac{\partial \tilde{\theta}}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial X} & 0 \\ \frac{\partial \tilde{\theta}}{\partial X} & \frac{\partial f}{\partial X} \end{pmatrix} \tag{2.7}$$

Which makes the berezinian $Ber[J] = det\left|\frac{\partial f}{\partial X}\right|(det\left|\frac{\partial f}{\partial X}\right|)^{-1} = 1$. So we can see that integration is naturally defined as we need no density function in order to define it. This is logical as we showed before how integration of functions in $T[1]M$ is analogue to the integration of a top-form over the manifold described by the X coordinates, which is canonically defined without any problem.

2.3.2 Cotangent Bundle with reversed parity $T^*[1]M$

The second supermanifold we want to construct, comes from a generalization of the cotangent bundle. We will add the grassmann coordinates as if they where cotangent vectors in an usual manifold. We will consider again that the even coordinates transform with a generic transformation f. The odd coordinates will transform as forms in the cotangent space. As we know from differential geometry it means that the θ variables will transform in the following way

$$\tilde{\theta}_A = \frac{\partial X^B}{\partial f^A}\theta_B$$

2.3.2.1 Functions on $T^*[1]M$

Again we want a function in $T^*[1]M$ to be an invariant object under changes of coordinates. It should also look like what we know is a function in $\mathbb{R}^{m|n}$ in local coordinates. By asking this invariance we have that if V is a function in $T^*[1]M$

$$\begin{aligned} V &= v^0 + v^{AB}\theta_A\theta_B + v^{ABC}\theta_A\theta_B\theta_C + ... + f^{1...m}\theta_1...\theta_m \\ &= \tilde{v}^0 + \tilde{v}^{AB}\frac{\partial X^C}{\partial f^A}\theta_C\frac{\partial X^D}{\partial f^B}\theta_D + \tilde{v}^{ABC}\frac{\partial X^C}{\partial f^A}\theta_C\frac{\partial X^D}{\partial f^B}\theta_D\frac{\partial X^E}{\partial f^C}\theta_E + ... + \tilde{v}^{1...m}\frac{\partial X^D}{\partial f^1}\theta_D...\frac{\partial X^E}{\partial f^m}\theta_E \end{aligned} \tag{2.8}$$

We can also calculate the relation between $v^{1...m}$ and $\tilde{v}^{1...m}$ by looking again for the factor with all the θ variables. The result of this calculation in this case is

$$v^{1...m} = det\left|\frac{\partial X^A}{\partial f^B}\right|\tilde{v}^{1...m} \tag{2.9}$$

We can see that this transformation rule is analogue to how an antisymmetric top-multivector field will transform. With multivector field we refer to an object that transforms as a vector field in each of its indexes. We can see that all the coefficients of this function will transform as antisymmetric multivector fields, for example, v^{AB} will transforms as

$$v^{AB} = \left[\frac{\partial X^A}{\partial f^C}\frac{\partial X^B}{\partial f^D} - \frac{\partial X^A}{\partial f^D}\frac{\partial X^B}{\partial f^C}\right]\tilde{v}^{CD}$$

For this case is easy to see that this is the rule under a antisymmetric two-multivector field will transform, as we have that if v is now an antisymmetric two-multivector field and f a change of coordinates such that $\tilde{X^C} = f^C(X^A)$

$$\begin{aligned} v^{AB} = v^A \otimes v^B - v^B \otimes v^A &= \frac{\partial X^A}{\partial f^C}\frac{\partial X^B}{\partial f^D}\tilde{v}^C \otimes \tilde{v}^D - \frac{\partial X^A}{\partial f^D}\frac{\partial X^B}{\partial f^C}\tilde{v}^D \otimes \tilde{v}^C \\ &= \left[\frac{\partial X^A}{\partial f^C}\frac{\partial X^B}{\partial f^D} - \frac{\partial X^A}{\partial f^D}\frac{\partial X^B}{\partial f^C}\right]\tilde{v}^{CD} \end{aligned} \tag{2.10}$$

where $\otimes$ is the tensorial product between two spaces with coordinates X^A.

2.3.2.2 integration in $T^*[1]M$

As we saw before, to define integration on a supermanifold we should calculate the jacobian matrix of the coordinate change functions. The jacobian matrix for this manifold is calculated as follows

$$J^* = \begin{pmatrix} \frac{\partial\tilde{X}}{\partial X} & \frac{\partial\tilde{X}}{\partial\theta} \\ \frac{\partial\tilde{\theta}}{\partial X} & \frac{\partial\tilde{\theta}}{\partial\theta} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial X} & 0 \\ \frac{\partial\tilde{\theta}}{\partial X} & \frac{\partial X}{\partial f} \end{pmatrix} \tag{2.11}$$

We have that the berezinian is $Ber[J] = det\left|\frac{\partial f}{\partial X}\right|(det\left|\frac{\partial X}{\partial f}\right|)^{-1} = det\left|\frac{\partial f}{\partial X}\right|^2$, as $\frac{\partial X}{\partial f}$ is the inverse matrix of $\frac{\partial f}{\partial X}$. Unlike the precedent example, the berezinian here is not constant so we have that the integration measure is not invariant under changes of coordinates. As we saw before, in order to define a proper integration we have to define a density function that transforms in the following way

$$\frac{1}{det\left|\frac{\partial f}{\partial X}\right|^2}\mu(X^A,\theta^i) = \tilde{\mu}(\tilde{X}^A,\tilde{\theta}^i)$$

Now that we have presented some basic notions of supergeometry and how integration over a supermanifold is closely related to integration over $\mathbb{R}^{m|n}$, we will concentrate in some important properties brought by this new spaces and grassmaniann calculus. In the next chapter we will show how we can evaluate integrals, otherwise difficult to do so, by just centering our attention in discrete points thanks to the nature of $\mathbb{R}^{m|n}$ and transformations that will leave invariant the berezinian.

Chapter 3

Supersymmetries and Localization

We can study how all the mathematical structures in $\mathbb{R}^{m|n}$ behave under the change of variables. We will be interested in transformations that make the berezinian invariant and the new properties induced in Integration. This particular transformations will help us to find new properties in mathematical objects such as functions or integrals over this space. In this chapter we will go further in exploring this transformations, first by finding the conditions we have to impose in order to achieve invariance of the berezinian and then by investigating new properties this transformations will help to show in a special family of integrals. We will see that in some cases we can reduce integration to a simple evaluation over a set of points. A similar property is showed for a simple specific case in [2], but we will expand this case to a more general one in the last part of this chapter.

3.1 Supersymmetries

Lets suppose we make the following change of variables.

$$
\begin{aligned}
\delta X^A &= \epsilon f^A(X^B, \theta_j) = \epsilon(f^j_{0A}\theta_j + f^{jkl}_{1A}\theta_j\theta_k\theta_l) \\
\delta\theta_i &= \epsilon g_i(X^A, \theta_j) = \epsilon(g_{0i} + g^{jk}_{1i}\theta_j\theta_k)
\end{aligned}
\tag{3.1}
$$

Where ϵ is a grassmannian variable satisfying 1.1.

This change of variables has a particular property that will become very useful. This change of variables is such that $\delta^2 = 0$. As we proved in chapter 1, the integration measure changes with the Berezinian which is defined as

$$Ber\begin{pmatrix} \frac{\partial \tilde{X}}{\partial X} & \frac{\partial \tilde{X}}{\partial \theta} \\ \frac{\partial \tilde{\theta}}{\partial X} & \frac{\partial \tilde{\theta}}{\partial \theta} \end{pmatrix} = det\left|\frac{\partial \tilde{X}}{\partial X}\right| det\left|\frac{\partial \tilde{\theta}}{\partial \theta} - \frac{\partial \tilde{\theta}}{\partial X}\left(\frac{\partial \tilde{X}}{\partial X}\right)^{-1}\frac{\partial \tilde{X}}{\partial \theta}\right|^{-1} \tag{3.2}$$

For the particular kind of transformations we are going to think about from now on, we have that this berezinian simplifies much. The elements of the jacobian matrix will be

$$\begin{aligned} \frac{\partial \tilde{X}^A}{\partial X^B} &= \delta_{AB} + \epsilon\frac{\partial f^A}{\partial X^B} & \qquad \frac{\partial \tilde{\theta}_i}{\partial X^B} &= \epsilon\frac{\partial g_i}{\partial X^B} \\ \frac{\partial \tilde{X}^A}{\partial \theta_j} &= \epsilon\frac{\partial f^A}{\partial \theta_j} & \qquad \frac{\partial \tilde{\theta}_i}{\partial \theta_j} &= \delta_{ij} + \epsilon\frac{\partial g_i}{\partial \theta_j} \end{aligned} \tag{3.3}$$

As we stated before, the berezinian simplifies in this case. The matrix $\frac{\partial \tilde{\theta}}{\partial X}\frac{\partial \tilde{X}}{\partial X}^{-1}\frac{\partial \tilde{X}}{\partial \theta}$ is equal to 0 because it has factor of $\epsilon^2 = 0$, so the calculation of the Berezinian reduces to

$$Ber\begin{pmatrix} \frac{\partial \tilde{X}}{\partial X} & \frac{\partial \tilde{X}}{\partial \theta} \\ \frac{\partial \tilde{\theta}}{\partial X} & \frac{\partial \tilde{\theta}}{\partial \theta} \end{pmatrix} = det\left|\frac{\partial \tilde{X}}{\partial X}\right| det\left|\frac{\partial \tilde{\theta}}{\partial \theta}\right|^{-1} \tag{3.4}$$

We also can calculate $det\left|\frac{\partial \tilde{X}}{\partial X}\right|$ by noting that $\frac{\partial \tilde{X}^A}{\partial X^B}\cdot\frac{\partial \tilde{X}^C}{\partial X^D} = 0$ if $A \neq B$ and $C \neq D$. Then it's straightforward to derive that

$$det\left|\frac{\partial \tilde{X}}{\partial X}\right| = \sum_{\sigma \in S_n} Sgn(\sigma)\prod_{A=1}^{m}\frac{\partial \tilde{X}^A}{\partial X^{\sigma(A)}} = \prod_{A=1}^{m}\frac{\partial \tilde{X}^A}{\partial X^A} \tag{3.5}$$

Using now what we showed in 3.3 and noting again the fact that $\epsilon^2 = 0$ we have

$$\prod_{A=1}^{m}\frac{\partial \tilde{X}^A}{\partial X^A} = \prod_{i=A}^{m}(1 + \epsilon\frac{\partial f_A}{\partial X^A}) = 1 + \sum_A \epsilon\frac{\partial f_A}{\partial X^i} \tag{3.6}$$

As any other terms involved are of order ≥ 2 in ϵ. So we have that

$$det\left|\frac{\partial \tilde{X}}{\partial X}\right| = 1 + \sum_A \epsilon\frac{\partial f^A}{\partial X^A} \tag{3.7}$$

In the exact same way we can calculate $det\left|\frac{\partial \tilde{\theta}}{\partial \theta}\right|$ obtaining that

$$det\left|\frac{\partial \tilde{\theta}}{\partial \theta}\right| = 1 + \sum_i \epsilon\frac{\partial g_i}{\partial \theta_i} \tag{3.8}$$

If we want to perform a transformation that leaves the integration measure invariant, we must require the Berezinian to be equal to 1. As we can see from 3.4 this is the same as requiring

$$
\begin{aligned}
1+\sum_A \epsilon \frac{\partial f^A}{\partial X^A} &= 1+\sum_i \epsilon \frac{\partial g_i}{\partial \theta_i} \\
\sum_A \epsilon \frac{\partial f^A}{\partial X^A} &= \sum_i \epsilon \frac{\partial g_i}{\partial \theta_i}
\end{aligned} \tag{3.9}
$$

which we can express as the vanishing of some kind of divergence of the transformation seen as a vector field.

$$
\sum_A \frac{\partial f^A}{\partial X^A} - \frac{\partial g_i}{\partial \theta_i} = 0 \tag{3.10}
$$

Changes of variables that satisfy that and make the berezinian 1, will be called supersymmetries, as the represent some sort of symmetry in the $\mathbb{R}^{m|n}$ space.

3.1.1 Action invariance

In the following we will be interested in the explicit calculation of integrals in the form

$$
\int F e^{-S} d\theta dX \tag{3.11}
$$

Where S is an even function and F a function in $\mathbb{R}^{m|n}$. More precisely we want to develop the cases where S is invariant under the proposed change of variables of 3.1. In the study of this kind of integrals we will present some important properties that show up when we choose S to be the variation under the change of variables of some other function U (i.e $S = \delta(U)$), in other words, S being invariant under such change, as we already know that $\delta^2 = 0$.

If we also choose F such that $\delta F = 0$, the invariance of the functions S and F will make the integral $\int F e^{-S} d\theta dX$ vanish in almost every point, letting the only contributions come from the points where $g_{0k} = 0$ (for some k) in 3.1. In order to see this we will perform the following change of variables, i.e by choosing $\epsilon = [-\theta_k(\frac{1}{g_{0k}} - \frac{g_{1k}^{lm}}{g_{0k}^2}\theta_l\theta_m)]$ in 3.1

$$\delta X^A = [-\theta_k(\frac{1}{g_{0k}} - \frac{g_{1k}^{lm}}{g_{0k}^2}\theta_l\theta_m)](f_{0i}^j\theta_j + f_{1i}^{jrs}\theta_j\theta_r\theta_s)$$
$$\delta\theta_i = [-\theta_k(\frac{1}{g_{0k}} - \frac{g_{1k}^{lm}}{g_{0k}^2}\theta_l\theta_m)](g_{0i} + g_{1i}^{js}\theta_j\theta_s) \qquad (3.12)$$

For this particular change we have that $\delta\theta_k = -\theta_k$ which makes $\tilde{\theta}_k = 0$. That means that after the change of variables, we will have no explicit dependence on the $\tilde{\theta}_k$ variable. The absence of $\tilde{\theta}_k$ after the change of variables will make the integral in 3.11 vanish after the θ-space integration. That comes from the fact that the integration is only different from 0 when we have factors which contain every θ variable and because F and S are invariant and the berezinian is one for 3.12 we will not have new $\tilde{\theta}_k$ that would bring $\frac{0}{0}$ factors. But we have to be careful on when that change of variables is a proper one. We can see that in the points where $g_{0k} = 0$ the change has a singularity, which makes the change not doable in such points. By the arguments given above the integral will vanish everywhere where $g_{0k} \neq 0$, therefore only the points where it is 0 will contribute to the integral $\int Fe^{-S}d\theta dX$.

3.2 Localization principle

We are interested in the calculation of integrals of the form 3.11. For that we are going to use a supersymmetry property that allows the integral to become an evaluation of some function over a set of points by centering our attention in the points where the coefficient g_{0k} in 3.1 is 0 for some k. We already saw that these points play an important role in the calculation of integrals involving the invariant function S. This set of points can be continous (i.e in two dimensions the function $X^2 - Y^2$ vanishes over the circle centered at (0,0) with radius 1) but in order to grasp the nature of the technique we want to develop, we will center on the case where this set of points is discrete. We will assume then that the function g_{0k} has only one point where it vanishes. In the following we will study some relations that are going to be useful in the calculation of the mentioned integrals. In order to develop the concept of localization we will require that $\mathbb{R}^{m|n}$ has grassmaniann dimension $n = 2q$ for $q \in \mathbb{N}$, this will not only allow us to achieve a more powerful result but is also the condition for spaces on which Supersymmetric quantum field theory is formulated. We will also require more from the change of variables 3.1; we will ask to the functions g_{1k}^{ij} to have a minimum at X^0

3.2.1 Some important properties

If we have a function $F(X,\theta)$ we can express the variation under the change of coordinates as

$$\delta F = \frac{\partial F}{\partial X^A}\delta X^A + \frac{\partial F}{\partial \theta_i}\delta\theta_i \tag{3.13}$$

recalling again 3.1 we have that

$$\delta F = \epsilon\frac{\partial F}{\partial X^A}f^A + \epsilon\frac{\partial F}{\partial \theta_i}g_i \tag{3.14}$$

By using this expression we have that the integration of δF vanishes in all space :

$$\begin{aligned}\int \delta F d\theta dX = \epsilon\int (\frac{\partial F}{\partial X^A}f^A + \frac{\partial F}{\partial \theta_i}g_i)d\theta dX &= -\epsilon\int (F\frac{\partial f^A}{\partial X^A} - F\frac{\partial g_i}{\partial \theta_i})d\theta dX\\ &= -\epsilon\int F(\frac{\partial f^A}{\partial X^A} - \frac{\partial g_i}{\partial \theta_i})d\theta dX = 0\end{aligned} \tag{3.15}$$

Where we performed integration by parts assuming that the function F behaves properly, making the boundary terms vanish. As well we used in the last relation what we found in 3.10. We just showed that if we integrate a function δF over all the space, the integral is equal to zero.

$$\int \delta F d\theta dX = 0 \tag{3.16}$$

Now let us consider a more general type of integral. Let G and U be functions in $\mathbb{R}^{m|n}$, with G such that $\delta G = 0$. Using 3.16 We have that the following relation holds

$$\int \delta(G\cdot U\cdot e^{-\delta U\cdot t})d\theta dX = \int \delta G\cdot U\cdot e^{-\delta U\cdot t}d\theta dX + \int G\cdot\delta U\cdot e^{-\delta U\cdot t}d\theta dX = 0 \tag{3.17}$$

As $\delta e^{\delta U\cdot t} = t\delta^2 U e^{\delta U\cdot t} = 0$.

But remembering that we chose $\delta G = 0$, the latter expression simplifies to

$$\int G\cdot\delta U\cdot e^{-\delta U\cdot t}d\theta dX = 0 \tag{3.18}$$

This expression can be identified as the t-derivative of another integral,

$$\frac{\partial}{\partial t}\int G\cdot e^{-\delta U\cdot t}d\theta dX=\int G\cdot\delta U\cdot e^{-\delta U\cdot t}d\theta dX=0 \tag{3.19}$$

This found relation allows us to conclude that the integral $\int G\cdot e^{\delta U\cdot t}d\theta dX$ does not depend on the variable t. We can choose any value for t and the integral will have the same value, therefore we must only center our attention in the terms which have not explicit dependence on t in order to calculate the integral.

We can also show that the integral has no specific dependence on the choosing of the function U. To do so, we pick a variation of U, $U'=U+\lambda\Delta U$. By the same argument developed before we have that

$$\int\delta\left(G\cdot\Delta Ut\cdot e^{-\delta(U+\lambda\Delta U\cdot t)}\right)d\theta dX=\int G\cdot\delta(\Delta U)t\cdot e^{-\delta(U+\lambda\Delta U)\cdot t}d\theta dX=0 \tag{3.20}$$

Which is the derivative with respect to λ of the integral. We see then that it has no dependence on λ and we can choose it, for example, equal to 0 or 1 and the integral remains unchanged. That shows that we can choose arbitrarily the function U without affecting the integral.

3.2.2 Evaluation of the Integral

In this section we will pursue the evaluation of an integral in the form of 3.11. In doing so, we will make use of the properties showed in the latter section. We will see that the integral is reduced to a discrete sum of terms which depend on the point X^0 where the coefficients in the supersymmetry transformation behave like having a fixed point. We will show first a simple yet significant case, where the main structure of the general case we we will approach later becomes more reachable.

3.2.2.1 The simple case of $\mathbb{R}^{2|2}$

As we stated before, we want to evaluate an integral of the form

$$\int Fe^{-S}d\theta dX \tag{3.21}$$

We want to use the properties that we recently showed that arise from a supersymmetry transformation and conditions over F and S. If we want to apply the results from last section we will have to choose a general action of the form

$$S = \frac{1}{\epsilon}\delta U$$

for some function U in $\mathbb{R}^{m|n}$. Because of what we have seen in preceding sections, we can imagine already that the integral will be reduced to an evaluation of some function in the points where g_{0k} vanishes. Let us first show a simple example for $\mathbb{R}^{2|2}$. Let us choose as supersymmetry transformation the following

$$\delta\theta_1 = \epsilon\left[\frac{1}{2}(X^1)^2 + \frac{1}{2}(X^2)^2 + \theta_1\theta_2\right] \qquad \delta\theta_2 = \epsilon\left[\frac{1}{2}\frac{((X^1)^2+(X^2)^2)^2}{X^1X^2} + \frac{((X^1)^2+(X^2)^2)}{X^1X^2}\theta_1\theta_2\right]$$
$$\delta X^1 = \epsilon\left[X^1\theta_1\right] \qquad \delta X^2 = \epsilon\left[-X^2\theta_1\right] \tag{3.22}$$

Where we can see that for $X^0 = (0,0)$, $\frac{1}{2}(X^1)^2 + \frac{1}{2}(X^2)^2$ vanishes. If we choose the function U in the following way

$$U = \left(1 - [(X^1)^2 + (X^2)^2]\right)\theta_1 + X^1X^2\theta_2 \tag{3.23}$$

We can compute the action S by calculating the variation of the function U through the supersymmetry transformation stated above 3.22

$$\begin{aligned}
S = \frac{1}{\epsilon}\delta U &= \frac{1}{\epsilon}\frac{\partial U}{\partial X^A}\delta X^A + \frac{1}{\epsilon}\frac{\partial U}{\partial\theta_i}\delta\theta_i \\
&= \left(1 - [(X^1)^2 + (X^2)^2]\right)\left(\frac{1}{2}(X^1)^2 + \frac{1}{2}(X^2)^2 + \theta_1\theta_2\right) + \\
&\quad + X^1X^2\left(\frac{1}{2}\frac{((X^1)^2+(X^2)^2)^2}{X^1X^2} + \frac{((X^1)^2+(X^2)^2)}{X^1X^2}\theta_1\theta_2\right) \\
&= \left[\frac{1}{2}(X^1)^2 + \frac{1}{2}(X^2)^2 + \theta_1\theta_2\right]
\end{aligned} \tag{3.24}$$

Let us calculate now the following integral

$$I = \int Fe^{-\left[\frac{1}{2}(X^1)^2 + \frac{1}{2}(X^2)^2 + \theta_1\theta_2\right]\cdot t}d\theta_1 d\theta_2 dX^1 dX^2 \tag{3.25}$$

We showed before that this integral is independent of t if we require that $\delta F = 0$. In order to calculate it we should look for factors which don't depend on this variable, as any other term should be depreciable for the fact that t can take any value. If we use the expansion of the exponential $e^{\theta_1\theta_2 \cdot t} = (1 - \theta_1\theta_2 t)$ we have that

$$\begin{aligned} I &= \int Fe^{-\left[\frac{1}{2}(X^1)^2+\frac{1}{2}(X^2)^2\right]\cdot t}\left(1 - t\theta_1\theta_2\right) d\theta_1 d\theta_2 dX^1 dX^2 \\ &= \int Fe^{-\left[\frac{1}{2}(X^1)^2+\frac{1}{2}(X^2)^2\right]\cdot t} d\theta_1 d\theta_2 dX^1 dX^2 - \int Fe^{-\left[\frac{1}{2}(X^1)^2+\frac{1}{2}(X^2)^2\right]\cdot t}\left(t\theta_1\theta_2\right) d\theta_1 d\theta_2 dX^1 dX^2 \end{aligned} \tag{3.26}$$

This latter expression could lead us to say that the integral vanishes as it seems to depend everywhere on t. This is not right as we have to take care of the dependence of t showed by the remaining exponential term. An easy way to do so is making the change of variables $\tilde{X}^A = \sqrt{t} X^A$ which makes $dX^1 dX^2 = \frac{1}{t} d\tilde{X}^1 d\tilde{X}^2$. Performing this change of variables we have

$$\begin{aligned} I &= \int Fe^{-\left[\frac{1}{2}(X^1)^2+\frac{1}{2}(X^2)^2\right]\cdot t} d\theta_1 d\theta_2 dX^1 dX^2 - \int Fe^{-\left[\frac{1}{2}(X^1)^2+\frac{1}{2}(X^2)^2\right]\cdot t}\left(t\theta_1\theta_2\right) d\theta_1 d\theta_2 dX^1 dX^2 \\ &= \int \tilde{F} e^{-\left[\frac{1}{2}(\tilde{X}^1)^2+\frac{1}{2}(\tilde{X}^2)^2\right]} d\theta_1 d\theta_2 \frac{1}{t} d\tilde{X}^1 d\tilde{X}^2 - \int \tilde{F} e^{-\left[\frac{1}{2}(\tilde{X}^1)^2+\frac{1}{2}(\tilde{X}^2)^2\right]}\left(\theta_1\theta_2\right) d\theta_1 d\theta_2 d\tilde{X}^1 d\tilde{X}^2 \end{aligned} \tag{3.27}$$

Where $\tilde{F}$ is the function F expressed in the $\tilde{X}$ variables. Now $\tilde{F}$ has dependence on t, so we must take care. If we make an expansion of F around the point X_{min} we have that

$$F = F|_{X_{min}} + F_1|_{X_{min}} X^1 + F_2|_{X_{min}} X^2 + \ldots \tag{3.28}$$

And now we can find that after the change of variables we will have $\tilde{F}$ expressed as

$$\tilde{F} = F|_{X_{min}} + \frac{1}{\sqrt{t}} F_1|_{X_{min}} \tilde{X}^1 + \frac{1}{\sqrt{t}} F_2|_{X_{min}} \tilde{X}^2 + \ldots \tag{3.29}$$

We are looking for the terms independent of t in 3.27. It's easy to see that using the latter expression for $\tilde{F}$, the first term in 3.27is always depending on t, so it will vanish because of the independence of the integral. For the second term we will only take the factor independent of t in $\tilde{F}$. Doing so we can see that the integral I can be expressed as

$$I = \int \tilde{F} e^{-\left[\frac{1}{2}(\tilde{X}^1)^2 + \frac{1}{2}(\tilde{X}^2)^2\right]} d\theta_1 d\theta_2 \frac{1}{t} d\tilde{X}^1 d\tilde{X}^2 - \int \tilde{F} e^{-\left[\frac{1}{2}(\tilde{X}^1)^2 + \frac{1}{2}(\tilde{X}^2)^2\right]} (\theta_1\theta_2)\, d\theta_1 d\theta_2 d\tilde{X}^1 d\tilde{X}^2$$
$$= -\int F|_{X_{min}} e^{-\left[\frac{1}{2}(\tilde{X}^1)^2 + \frac{1}{2}(\tilde{X}^2)^2\right]} (\theta_1\theta_2)\, d\theta_1 d\theta_2 d\tilde{X}^1 d\tilde{X}^2 \tag{3.30}$$

If we now perform the integral over the θ_i variables, and if we remember that by the rules of Grassmann integration only the terms that show $\theta_1\theta_2$ will survive, we have that

$$I = -\int f^0|_{X_{min}} e^{-\left[\frac{1}{2}(\tilde{X}^1)^2 + \frac{1}{2}(\tilde{X}^2)^2\right]} d\tilde{X}^1 d\tilde{X}^2 \tag{3.31}$$

Where we have used that we can expand F as $F = f^0 + f^1\theta_1 + f^2\theta_2 + f^{12}\theta_1\theta_2$.

The latter integral belongs to the category of Gaussian integrals, which we know how to calculate. Doing so we get that we can calculate very easily the integral I. We have then that

$$I = -2\pi f^0|_{X_{min}} \tag{3.32}$$

3.2.2.2 The general case

Now we can think of calculating this family of integrals for a more general case of dimension. We will show that for this case we will also find that the integral reduces to an evaluation over the point X_{min}. As we stated before, we are going to work in the case where $n = 2q$ for $\mathbb{R}^{m|n}$, and we will ask some conditions as well over the function S. We have that the function S would be expressed as

$$S = \frac{1}{\epsilon}\frac{\partial U}{\partial X^A}\delta X^A + \frac{1}{\epsilon}\frac{\partial U}{\partial \theta_i}\delta\theta_i \tag{3.33}$$

With this condition satisfied, we already showed that we can choose U without any restrictions, so for our porpouse we will require the following

$$\frac{\partial^2 U}{\partial X^B \partial X^A}|_{X_{min}} = \frac{\partial U}{\partial X^A}|_{X_{min}} = 0 \tag{3.34}$$

$$\text{for } i \neq k \quad \frac{\partial U}{\partial \theta_i}|_{X_{min}} = 0 \tag{3.35}$$

This will bring as consequence that $\frac{\partial S}{\partial X^A}|_{X_{min}} = 0$ as we have

$$\frac{\partial S}{\partial X^B}_{|X_{min}} = \frac{1}{\epsilon}\frac{\partial^2 U}{\partial X^B \partial X^A}_{|X_{min}} \delta X^A{}_{|X_{min}} + \frac{1}{\epsilon}\frac{\partial U}{\partial X^A}_{|X_{min}} \frac{\partial \delta X^A}{\partial X^B}_{|X_{min}} + \\ + \frac{1}{\epsilon}\frac{\partial^2 U}{\partial X^B \partial \theta_i}_{|X_{min}} \delta\theta_i{}_{|X_{min}} + \frac{1}{\epsilon}\frac{\partial U}{\partial \theta_i}_{|X_{min}} \frac{\partial \delta\theta_i}{\partial X^B}_{|X_{min}} \tag{3.36}$$

The first three terms are zero after imposing the conditions 3.34. The last term $\frac{\partial U}{\partial \theta_i}_{|X_{min}} \frac{\partial \delta\theta_i}{\partial X^B}_{|X_{min}}$ vanishes because of condition 3.35 when $i \neq k$, and when $i = k$ it does because $\frac{\partial \delta\theta_k}{\partial X^B}$ vanishes in X_{min}.

We can expand this action on the X^A variables, centering the expansion in the point X_{min} that satisfies that $g_{0k} = 0$, in the following way

$$S = S|_{X_{min}} + \frac{1}{2}\sum_{AB} S_{AB}(X^A - X^A_{min})(X^B - X^B_{min}) + ... \tag{3.37}$$

Where $S_{AB} = \frac{\partial^2 S}{\partial X^A \partial X^B}|_{X_{min}}$.

As we can use 3.19 in this case, we can add time dependence to an integral of the form 3.21 and we can now express such integral as

$$\int F e^{-tS} d\theta dX = \int F e^{-tS|_{X_{min}} - \frac{t}{2}\sum_{AB} S_{AB}(X^A - X^A_{min})(X^B - X^B_{min})} d\theta dX \tag{3.38}$$

Where the fact that we keep only the terms to second order of the X^A variables, comes from the property that the integral has only contributions in X_{min}. This comes, as we showed, from the invariance of F and S as well from the fact that the transformation is a supersymmetry satisfying 3.10. Because of this porperty we can safely center our attention in a small neighborhood around that point, making the expansion up to second order a good approximation. We can expand the function F in the X^A variables, centering the expansion in X_{min}, in the following way

$$F = F|_{X_{min}} + \sum_A F_A(X^A - X^A_{min}) + \frac{1}{2}\sum_{AB} F_{AB}(X^A - X^A_{min})(X^B - X^B_{min}) + ... \tag{3.39}$$

Where $F_A = \frac{\partial F}{\partial X^A}|_{X_{min}}$ and $F_{AB} = \frac{\partial^2 F}{\partial X^A \partial X^B}|_{X_{min}}$.

If now we make the change of variables $\sqrt{t}X^A = \tilde{X}^A$, we have that

$$\int F e^{-tS|_{X_{min}} - \frac{t}{2}\sum_{AB} S_{AB}(X^A - X^A_{min})(X^B - X^B_{min})} d\theta dX =$$
$$= \int \frac{\tilde{F}}{t^{\frac{m}{2}}} e^{-tS|_{X_{min}}} e^{-\frac{1}{2}\sum_{AB} S_{AB}(\tilde{X}^A - \tilde{X}^A_{min})(\tilde{X}^B - \tilde{X}^B_{min})} d\theta d\tilde{X} \tag{3.40}$$

Where $\tilde{F}$ is the function F expressed in the new variables. if we look into the expansion of F we have that

$$\tilde{F} = F|_{X_{min}} + \frac{1}{\sqrt{t}} \sum_A F_A(\tilde{X}^A - \tilde{X}^A_{min}) + \frac{1}{t}\frac{1}{2} \sum_{AB} F_{AB}(\tilde{X}^A - \tilde{X}^A_{min})(\tilde{X}^B - \tilde{X}^B_{min}) + ... \tag{3.41}$$

We would like to find the terms independent of t in 3.40. For that we have to find the factor proportional to $t^{\frac{m}{2}}$ in $\tilde{F}e^{-tS|_{X_{min}}}$. Let us first expand the exponential in a taylor series

$$e^{-tS|_{X_{min}}} = 1 + \sum_i \frac{(-1)^i t^i S|^i_{X_{min}}}{i!} \tag{3.42}$$

As we showed before, S is written as

$$S = S^0 + S^{ij}\theta_i\theta_j + S^{ijkl}\theta_i\theta_j\theta_k\theta_l + ... \tag{3.43}$$

But if we recall 3.35 and the fact that $g_{0k}|_{X_0} = 0$ we see that $S^0|_{X_0} = \frac{\partial U}{\partial \theta_i} g_{0i} = 0$ will come as a consequence.

Now we have to work out the terms depending on $t^{\frac{m}{2}}$. We see that t factors can come as well from F as from the expansion of the exponential. The terms coming from F have negative powers, and those that come from the exponential positive. Using 3.42 and 3.41, we find that the only terms with that dependence on t are

$$\sum_{p=0}^{\frac{n}{2}-\frac{m}{2}} \left[\frac{t^{\frac{m}{2}+\frac{p}{2}}}{[\frac{m}{2}+\frac{p}{2}]!} (-1)^{\frac{m}{2}+\frac{p}{2}} \left(S^{ij}\theta_i\theta_j + S^{ijkl}\theta_i\theta_j\theta_k\theta_l + ... \right) |^{\frac{m}{2}+\frac{p}{2}}_{X_{min}} \right] \cdot$$
$$\cdot \left[\frac{1}{p! t^{\frac{p}{2}}} \sum_{A_1...A_p} F_{A_1...A_p}|_{X_{min}} (\tilde{X}^{A_1} - \tilde{X}^{A_1}_{min})(\tilde{X}^{A_2} - \tilde{X}^{A_2}_{min})...(\tilde{X}^{A_p} - \tilde{X}^{A_p}_{min}) \right] \tag{3.44}$$

This is nor a beautiful or simple result. the fact that $\left(S^{ij}\theta_i\theta_j + S^{ijkl}\theta_i\theta_j\theta_k\theta_l + ...\right)|^{\frac{r}{2}}_{X_{min}} = 0$ for $r > n$ tells us that for the case when $m > n$ the integral will not show any term independent of t and therefore will vanish. If $m < n$, we need to perform the integration over the θ_i variables and in order to do so, we must find which term depends on all the θ_i at the same time in 3.44. This task depends on the numbers n, m and on the number p. This fact makes difficult to achieve a generalization over arbitrarily dimensions of $\mathbb{R}^{m|n}$. We will instead state a solution for the case when $m = n$ by noticing that in that case the above factor simplifies greatly. We can note as well, that some of the most important examples in supersymmetric physics are builded using that specific case, i.e the first idea of superspace comes from the observation that a 4-dimensional Poincaré group generalized to superspace can be seen as $\mathbb{R}^{4|4}$ [5]. In the case $m = n$, we have that 3.44 can be written as

$$\left[(-1)^{\frac{n}{2}}\frac{t^{\frac{n}{2}}}{[\frac{n}{2}]!}\left(S^{ij}\theta_i\theta_j + S^{ijkl}\theta_i\theta_j\theta_k\theta_l + ...\right)|^{\frac{n}{2}}_{X_{min}}\right]\cdot[F|_{X_{min}}] \tag{3.45}$$

but as n is the number of θ_i variables available, we can clearly state that

$$(S^{ij}\theta_i\theta_j + S^{ijkl}\theta_i\theta_j\theta_k\theta_l + ...)|^{\frac{n}{2}}_{X_{min}} = S^{12}...S^{(n-1)n}|_{X_{min}}\theta_1...\theta_n \tag{3.46}$$

As any other term will include repeated θ_i variables, and therefore will vanish. The factor that we are looking for (In this case the factor depending on $t^{\frac{n}{2}}$) will be just the following simple expression

$$(-1)^{\frac{n}{2}}\frac{t^{\frac{n}{2}}}{[\frac{n}{2}]!}F|_{X_{min}}S^{12}...S^{(n-1)n}|_{X_{min}}\theta_1...\theta_n = (-1)^{\frac{n}{2}}\frac{t^{\frac{n}{2}}}{[\frac{n}{2}]!}f^0|_{X_{min}}S^{12}...S^{(n-1)n}|_{X_{min}}\theta_1...\theta_n \tag{3.47}$$

Where f^0 is the term independent of the θ_i in F, as all the terms including this variables in F will vanish by 1.1. This term is at the same time the one including all the θ_i variables and will allow us to do the integral over this variables in an easy way.

We can see that the only term that doesn't have a explicit dependence on t in the integral 3.40 is

$$I = \frac{(-1)^{\frac{n}{2}}}{[\frac{n}{2}]!}\int f^0|_{X_{min}}S^{12}...S^{(n-1)n}|_{X_{min}}\theta_1...\theta_n e^{-\frac{1}{2}\sum_{AB}S_{AB}(\tilde{X}^A-\tilde{X}^A_{min})(\tilde{X}^B-\tilde{X}^B_{min})}d\theta dX \tag{3.48}$$

If we now perform the integration on the θ_i space, recalling the fact that the only contribution will come from the term with all the θ_i variables in the same factor, we have

$$I = \frac{(-1)^{\frac{n}{2}}}{[\frac{n}{2}]!} \int f^0|_{X_{min}} S^{12}...S^{(n-1)n}|_{X_{min}} e^{-\frac{1}{2}\sum_{AB} S^0_{AB}(\tilde{X}^A-\tilde{X}^A_{min})(\tilde{X}^B-\tilde{X}^B_{min})} dX \tag{3.49}$$

where $S^0_{AB} = \frac{\partial S^0}{\partial X^A \partial X^B}|_{X_{min}}$

Notice that we have taken in the exponential the terms independent of θ_i in S_{AB}, i.e S^0_{AB}. What we have now is a Gaussian integral in the X^A variables, which can be calculated explicitly. By doing so we obtain that the integral reduces to

$$I = \frac{(-1)^{\frac{n}{2}}}{[\frac{n}{2}]!} f^0|_{X_{min}} S^{12}...S^{(n-1)n} \sqrt{\frac{(2\pi)^n}{detS^0_{AB}}} \tag{3.50}$$

Taking S^0_{AB} as a matrix in the AB indexes.

This result shows a dependence on S and therefore on U, which contradicts what we have proved before. This tells us that there must be a simplification of this result that shows no dependence on S. Because of that the S depending factor

$$\frac{S^{12}...S^{(n-1)n}|_{X_{min}}}{\sqrt{detS^0_{AB}}} \tag{3.51}$$

has to be simplified. To do so we will express it in terms of the function U, as S is closely related to it. The function U is expanded as

$$U = U^0 + U^i\theta_i + U^{ij}\theta_i\theta_j + U^{ijk}\theta_i\theta_j\theta_k + U^{ijkl}\theta_i\theta_j\theta_k\theta_l + ... \tag{3.52}$$

recalling 3.1 and 3.33, as well as the conditions already imposed on U 3.34 3.35, we can express everything in terms of U. Doing so we arrive to

$$S^{ij}|_{X_{min}} = [U^k g^{ij}_{1k}]|_{X_{min}} \tag{3.53}$$

$$S^0_{AB}|_{X_{min}} = [U^k \frac{\partial^2 g_{0k}}{\partial X^A \partial X^B}]|_{X_{min}} \tag{3.54}$$

In both cases with k fixed following our notation ($g_{0k} = 0$ in X_{min}). Using this, is now simple to express 3.51 in terms of the function U.

$$\frac{S^{12}...S^{(n-1)n}|_{X_{min}}}{\sqrt{detS^0_{AB}}} =$$

$$= \frac{U^{k\frac{n}{2}}|_{X_{min}} g^{12}_{1k}...g^{(n-1)n}_{1k}|_{X_{min}}}{U^{k\frac{n}{2}}|_{X_{min}}\sqrt{\sum_{\sigma\in S_n} Sgn(\sigma)\prod_A \frac{\partial^2 g_{0k}}{\partial X^A \partial X^{\sigma(A)}}|_{X_{min}}}}$$

$$= \frac{g^{12}_{1k}...g^{(n-1)n}_{1k}|_{X_{min}}}{\sqrt{\sum_{\sigma\in S_n} Sgn(\sigma)\prod_A \frac{\partial^2 g_{0k}}{\partial X^A \partial X^{\sigma(A)}}|_{X_{min}}}}$$

$$= \frac{g^{12}_{1k}...g^{(n-1)n}_{1k}|_{X_{min}}}{\sqrt{det\left|\frac{\partial^2 g_{0k}}{\partial X^A \partial X^B}|_{X_{min}}\right|}} \tag{3.55}$$

We can see how the dependence on U disappears as we wanted, and in perfect agreement with our proofs. The final result for our integral doesn't depend on S, but in the supersymmetry transformation for the θ_k variable (with fixed k), the transformation for which g_{0k} vanishes at X_0. Finally this result can be stated properly as

$$I = \frac{(-1)^{\frac{n}{2}}}{[\frac{n}{2}]!} f^0|_{X_{min}}\sqrt{(2\pi)^n}\frac{g^{12}_{1k}...g^{(n-1)n}_{1k}|_{X_{min}}}{\sqrt{det\left|\frac{\partial^2 g_{0k}}{\partial X^A \partial X^B}|_{X_{min}}\right|}} \tag{3.56}$$

It is important to remember that we have taken conditions over F and U as well on the supersymmetry transformation of the space. We proved that the integral is invariant under changes of U so the conditions imposed for this function in 3.34 and 3.35 are just to make simpler the calculation of the integral. We can forget about this from now on.

Let us now state everything in a clear and compact way

Theorem 1. Given the functions F, S and U with the supersymmetry transformation in $\mathbb{R}^{n|n}(n = 2q$ for $q \in \mathbb{N})$

$$\delta X^A = \epsilon f^A(X^B, \theta_j) = \epsilon(f^j_{0A}\theta_j + f^{jkl}_{1A}\theta_j\theta_k\theta_l)$$
$$\delta\theta_i = \epsilon g_i(X^A, \theta_j) = \epsilon(g_{0i} + g^{lm}_{1i}\theta_l\theta_m)$$

With $g_{0k}|_{X_{min}} = 0$ for some point X_{min} and g_{1k}^{lm} with a minimum in such point,

we have that if

$$F = f^0 + f^i\theta_i + f^{ij}\theta_i\theta_j + ...$$

is such that $\delta F = 0$ and $S = \frac{1}{\epsilon}\delta U$ for some function U, then the calculation of the integral

$$\int Fe^{-S}d\theta dX$$

is reduced to

$$\int Fe^{-S}d\theta dX = \frac{(-1)^{\frac{n}{2}}}{[\frac{n}{2}]!}f^0|_{X_{min}}\sqrt{(2\pi)^n}\frac{g_{1k}^{12}...g_{1k}^{(n-1)n}|_{X_{min}}}{\sqrt{det\left|\frac{\partial^2 g_{0k}}{\partial X^A \partial X^B}|_{X_{min}}\right|}}$$

As we can see, this case is reduced to the result we found in 3.32 for $\mathbb{R}^{2|2}$ as in that case all the conditions are satisfied.

The only strong condition remains always in F that should be such that $\delta F = 0$. We can always build the supersymmetry, given the function F, under which it's invariant and then using some function U find S fulfilling the other conditions.

Chapter 4

Summary and conclusions

In this thesis we have presented some basic aspects of supergeometry and how we can use inherent properties of supergeometric spaces such as $\mathbb{R}^{m|n}$ to develop methods that simplify greatly calculations otherwise difficult to make.

Through the study of grassmanian variables introduction, we have constructed a basic notion of calculus for spaces which have both, commutative and grassmaniann coordinates. We have defined formally the space $\mathbb{R}^{m|n}$ and showed carefully how change of variables affects integration on this space and defined the berezinian, which acts as a generalization of the determinant.

After getting familiar with $\mathbb{R}^{m|n}$ we presented a definition of supermanifold showing it as a manifold with coordinates in $\mathbb{R}^{m|n}$, by giving to $\mathbb{R}^{m|n}$ a topology inherited from $\mathbb{R}^{m}$. Later on, and using all the theory developed for integration in $\mathbb{R}^{m|n}$, we explored how to define integration on a supermanifold, taking care of making integrals invariant under changes of coordinates. To further support the understanding of the concept of supermanifold, we presented two basic examples: The tangent bundle with reversed parity $T[1]M$ and the cotangent bundle with reversed parity $T^*[1]M$. These supermanifolds come from the addition of grasmaniann variables transforming as a vector in the tangent space (for $T[1]M$) and a form in the cotangent space (for $T^*[1]M$). We described how functions and integration look like in both spaces.

Finally we showed how the mathematical structure surrounding $\mathbb{R}^{m|n}$ brings new properties that help us in the calculation of a wide family of integrals. In doing so, we explored transformations that left invariant the berezinan (of the jacobian matrix). We called these transformations supersymmetries. Later we saw how integration of functions related to these supersymmetries simplified significantly, as they vanished everywhere but in a set of points acting like fixed points of the supersymmetry. We showed that choosing

S as the variation of another function U under the supersymmetry and choosing G invariant, the integral $\int Fe^{-St}dXd\theta$ was independent of the parameter t and the function U.

This property is the main tool we used in showing the localization principle, which reduces the integral $\int Fe^{-St}dXd\theta$ to an evaluation over a set of points. We began the demonstration of the localization principle by calculating a simple example for $\mathbb{R}^{2|2}$, in which the integral was calculated as the evaluation of a function in a single point. After that we went through a more general case, but still making some restrictions to make the mathematical procedure remain simple. We centered in the case where the supersymmetry has a single fixed point and where $m = n$ with $n = 2q$. We also imposed some conditions on the function U. In this particular case and by using all the developed concepts of calculus in $\mathbb{R}^{m|n}$, we showed that the integration became discrete and its result was just an evaluation over the fixed point of a function related to both F and the supersymmetry transformation.

This master thesis went through basic yet important aspects of supergeometry, by showing calculations and proofs that will allow the reader to further understand supersymmetric spaces properties that help the simplification of calculations needed by physicists in supersymmetric theories. The way the information was presented will hopefully help the reader to familiarize with grassmaniann coordinates, $\mathbb{R}^{m|n}$ and supermanifolds as tools needed for understanding formulations of physical theories such as supersymmetric quantum field theory.

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