M.Sc. Thesis

Supersymmetric Quantum Mechanics and Integrability

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Abstract

This master's thesis investigates the relationship between supersymmetry and integrability in quantum mechanics. This is done by finding a suitable way to systematically add more supersymmetry to the system. Adding more supersymmetry will give constraints on the potential which will lead to an integrable system. A possible way to explore the integrability of supersymmetric quantum mechanics was introduced in a paper by Crombrugghe and Rittenberg in 1983, their method has been used as well as another approach based on expanding a $N = 1$ system by introducing complex structures. $N = 3$ or more supersymmetry is shown to give an integrable system.

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Contents

1 Populärvetenskaplig sammanfattning

Det finns två olika grupper av partiklar i naturen. En av dessa kallas för fermioner och är den typ av partiklar som bygger upp all materia omkring oss. Den andra typen av partiklar, som är ansvariga för att förmedla olika typer av krafter, kallas för bosoner. Ett exempel på bosoner är fotoner som förmedlar den elektromagnetiska kraften.

Det finns många olösta problem och oförklarade fenomen inom fysiken som skulle vara närmare en lösning om det fanns en relation, eller symmetri, mellan bosonerna och fermionerna. Den här möjliga symmetrin kallas för supersymmetri och är ett viktigt område inom teoretisk fysik.

Ett fysikaliskt system kallas för integrerbart om det går att lösa dess rörelseekvationer. Det är ofta ett svårt problem att avgöra om ett system är integrerbart eller inte, men genom att införa fler symmetrier i ett system blir systemet mer regelbundet och enklare att lösa. Det är här kopplingen mellan supersymmetri och integrabilitet går att se. Om det tillförs mer supersymmetri till ett system kommer systemet även få andra symmetrier vilket till slut kommer leda till att det blir integrerbart.

Genom att på ett systematiskt sätt tillföra supersymmetri till kvantmekaniska system har relationen mellan supersymmetri och integrabilitet utforskats. Slutsatsen är att 3 supersymmetrier (ofta kallat N=3) är tillräckligt för att göra ett system integrerbart. Detta har visats med två olika angreppssätt, dels genom att använda en metod som beskrivs i en artikel från 1983 av Crombrugghe och Rittenberg, och dels genom en ny metod som använder sig av så kallade komplexa strukturer.

2 Introduction

Supersymmetry, often abbreviated SUSY, is a suggested symmetry in nature which relates *bosonic* states to *fermonic* states. Supersymmetry was first introduced by Gel'fand and Likhtman [1], Ramond [2] and Neveu and Schwarz [3] in 1971 and plays an important part in most versions of string theory, but has since then also been combined with other areas of physics such as quantum field theory, where it has been suggested as a possible solution to the well known hierarchy problem. Another very interesting role of supersymmetry is that it increases the accuracy of the high energy unification of the electromagnetic, strong and weak interactions.

The main idea of supersymmetry is that there exist operators which can take a fermionic state and transform it into a bosonic state and vice versa. A fermionic state has half-integer spin, and describes one or several matter particles such as electrons and protons. A bosonic state has integer spin, and describes one or several force carrying particles such as photons, who carry the electromagnetic force, or Z-bosons, who is one of the particles who carry the weak force. An important property of these different kinds of particles is that they obey different *statistics*, fermions are forbidden to be in the same quantum state as each other by the Pauli exclusion principle while bosons have no such restrictions.

A consequence of supersymmetry is the, yet to be verified, existence of supersymmetric partner particles to the known particles. For example there should be a bosonic partner to the electron, usually called the selectron and so on. So far none of these, so called *superpartners*, have been found in nature. If there were superpartners to the known particles with the same mass as the ordinary particles, as one might expect from the theory, these would have been discovered by now. Instead the current understanding is that supersymmetry is a broken symmetry, where the superpartners are allowed to have larger mass than the ordinary particles and that this has made them escape detection. It is in the context of supersymmetry breaking where supersymmetric quantum mechanics (SQM) was first discussed by Witten [4] in 1981 as a kind of simplified setting for supersymmetry. Supersymmetric quantum mechanics will be the main area of this thesis starting with a general introduction in section 4.

The concept of *integrability* can be described in various more or less technical ways. Some of these will be presented in section 3 of this thesis. The intuitive picture of an integrable system is that the system is sufficiently simple so that it is solvable and non-chaotic. A system will become more constrained and typically simpler when more symmetries are included into the system.

The idea of supersymmetric quantum mechanics has been its own research area since its introduction in 1981. The starting point of this thesis was to review a a paper on SQM by Crombrugghe and Rittenberg from 1983 [5]. The two concepts of supersymmetry and integrability in quantum mechanics are naturally related because supersymmetry will impose conditions on the system which will lead to more symmetries, as well as restricting the system in such a way that it will eventually become integrable when a certain amount of symmetry is introduced. This feature of supersymmetric quantum mechanics is explored in the paper by Crombrugghe and Rittenberg (CR) and the results are presented in different sections of this thesis starting with the ansatz for supercharges in section 6 and continued in the form of an example with six supercharges in section 7.

The formalism for SQM developed in the CR paper proved hard to generalize which lead to the introduction of another more modern description of SQM based on the treatment on SQM in the book *Mirror symmetry* [6]. This new approach is explained in section 8 and it's generalization to more supersymmetry is introduced in section 9.

In this thesis we will impose different amounts of supersymmetry on quantum mechanical systems and see how this changes the integrability of the system. It can be seen that, when enough supersymmetry is introduced, the system will become integrable, as will be shown in section 10.

3 Integrability

A physical system is considered integrable in the Liouville sense, which is the most common definition, if its equations of motion can be solved by solving a finite number of algebraic equations and computing a finite number of integrals. A brief introduction to the most important concepts in integrability including the Liouville theorem will be given in this section. Most of the material presented here is based on [7]. The content in this section will be based on classical systems but everything that is said here will also be true for quantum mechanical systems. The difference being that quantum mechanical systems usually require different techniques for finding conserved quantities, and by that determine the integrability of the system. Unfortunately it is in general very hard to prove the opposite, that a system is not integrable, so the SQM cases that will be the subject of this thesis can only be divided in to those who are integrable and those who may or may not be integrable.

3.1 The Liouville theorem

The state of a classical system can be described by a point in it's phase space given by coordinates in terms of momentum p_i and position q_i. The equations of motion are then given by

$$\dot{q}_i = \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \tag{1}$$

where H is the Hamilitonian of the system. It is also given that for any function F_i

$$\dot{F}_i = \{F_i, H\} \tag{2}$$

where, in this section, $\{\cdot,\cdot\}$ denotes the Poisson bracket defined by

$$\{A, B\} = \sum_i \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} - \frac{\partial B}{\partial p_i}\frac{\partial A}{\partial q_i} \tag{3}$$

for the coordinates q_i and p_i we have that

$$\{q_i, q_j\} = 0, \quad \{p_i, p_j\} = 0, \quad \{p_i, q_j\} = \delta_{ij} \tag{4}$$

As a side remark we note that in the quantum mechanical systems which will be the subject of the rest of this thesis, the Poisson bracket for the coordinates p_i and q_i will be changed to the commutator $[\cdot, \cdot]$ for the operators p_i and x_i.

$$\{\cdot, \cdot\} \rightarrow \frac{i}{\hbar}[\cdot, \cdot], \quad \hbar = 1 \tag{5}$$

The Liouville theorem states that if we have a phase space with dimension $2n$ and we can find n functions in *involution* whose Poisson bracket with the Hamiltonian vanish, the system will be integrable. Since H will typically be among the F_is this means that a one dimensional system is always integrable and a two dimensional system only needs one additional conserved quantity to become integrable and so on.

$$\{F_i, F_j\} = 0, \qquad \{F_j, H\} = \dot{F}_j = 0 \tag{6}$$

This theorem is not very hard to prove and shows that the solutions to the equations of motion are easy to find if the functions F_i are known. A sketchy version of this proof is given in the appendix.

An example where this property is trivially seen is the harmonic oscillator

$$H = \sum_{i=1}^{n} \frac{1}{2}(p_i^2 + \omega_i^2 x_i^2) \tag{7}$$

where the conserved quantities are $F_i = \frac{1}{2}(p_i^2 + \omega_i^2 x_i^2)$.

3.2 Action angle variables and Lax pairs

There are several other ways to describe integrable systems, although they will not be used in this thesis, two of them are an important part of a general discussion on integrability.

Action-angle variables can be used to describe the phase space of a system in such a way that it is foliated into submanifolds, who are spanned by the *angle variables.* The other directions in the phase space will, as the name suggests, be spanned by so called *action variables.* In most $2n$ ($<\infty$) dimensional cases the submanifolds will be n dimensional tori which explains the term *angle variables.*

Lax pairs are a way to reformulate a Hamiltonian system in terms of two matrices L and M such that the equations of motions takes the form

$$\dot{L} = [M, L] \tag{8}$$

The main point of describing the system in this way, is that the constants of motion are relatively simple to find once the system has been recast into Lax pairs.

The Liouville theorem states that it is possible to make a transformation of an integrable the system into coordinates who are functions of the F_i and some other variables Ψ_i. These can be used to construct action angle variables which in turn can be used to create Lax pairs [7]. If we have the coordinates

$$\dot{I}_j = 0, \quad \dot{\theta}_j = \frac{\partial H}{\partial I_j} \tag{9}$$

and matrices E_i and H_i, $i = 1, ..., n$ that are representations of the lie algebra

$$[H_i, H_j] = [E_i, E_j] = 0, \quad [H_i, E_j] = 2\delta_{ij} E_i \tag{10}$$

We can then create the matrices L and M by

$$L = \sum_{j=1}^{n} I_j H_j + 2 I_j \theta_j E_j, \quad M = -\sum_{j=1}^{n} \frac{\partial H}{\partial I_j} E_j \tag{11}$$

Which will satisfy the relation $\dot{L} = [M, L]$. However, there is usually no need to express the system in terms of Lax pairs if the action angle variables are already known.

4 Supersymmetric Quantum Mechanics

Supersymmetric quantum mechanics was introduced as a simple example of a supersymmetric system in order to highlight the breaking of supersymmetry in a simple setting [4]. Since then it has been an active research area, often used as a testing ground for new ideas in supersymmetry but also as a research area in its own right. There is a number of introductory texts on SQM which give a nice introduction to the subject. Some of the material covered there will be reproduced here for completeness, and also to introduce some notation and terminology. First a very simple, but quite illustrative, example will be given in the form of the supersymmetric harmonic oscillator. This is a nice example where the introduction of supersymmetry is very straightforward. This system will lead to the SQM algebra which will be used in the rest of this thesis, sometimes in a slightly modified form. After the harmonic oscillator is presented there will be an example of how to calculate the conserved supercharges using the Noether procedure. This will be followed by a more general, but still simple example of $N = 1$ SQM.

4.1 The one dimensional SQM harmonic oscillator

This example can be found in any introductory text on supersymmeric quantum mechanics, see for instance [8]. Since it is simple but still quite illustrative it will be repeated here as a kind of crash course on basic SQM concepts.

The standard harmonic oscillator with the Hamiltonian (constant factors like Plancks constant $\hbar$, masses m and the frequency ω will be suppressed when possible)

$$H_B = \frac{1}{2}(p^2 + x^2) = \frac{1}{2}(a^\dagger a + aa^\dagger) = (a^\dagger a + \frac{1}{2}) \tag{12}$$

where we have defined the bosonic annihilation and creation operators

$$a = \frac{1}{2}(x + ip) \quad \text{and} \quad a^\dagger = \frac{1}{2}(x - ip) \tag{13}$$

with the property

$$[a, a^\dagger] = 1 \tag{14}$$

This Hamiltonian has a very straightforward fermonic analog created by introducing the new fermonic annihilation and creation operators ψ and $\psi^\dagger$ which satisfy

$$\{\psi, \psi^\dagger\} = 1 \tag{15}$$

Where $\{\cdot, \cdot\}$ denotes the anticommutator $\{a, b\} = ab + ba$. Using ψ and $\psi^\dagger$ we can construct the fermonic harmonic oscillator

$$H_F = \frac{1}{2}(\psi^\dagger\psi + \psi\psi^\dagger) = (\psi^\dagger\psi - \frac{1}{2}) \tag{16}$$

Note that we now have the opposite sign on the constant term due to the difference in commutation and anti commutation condition on the operators between the bosonic and fermionic case. We now have one Hamiltonian for bosonic states and one for fermionic states. The total Hamiltonian is given by the sum of these which gives us the supersymmetric Hamiltonian

$$H = H_B + H_F = (a^\dagger a + \psi^\dagger\psi) \tag{17}$$

using the a and ψ we can now construct two new operators Q and $Q^\dagger$ by

$$Q = a\psi^\dagger \quad \text{and} \quad Q^\dagger = a^\dagger\psi \tag{18}$$

commonly known as supercharges. One can check that

$$\begin{aligned}\{Q, Q^\dagger\} &= a\psi^\dagger a^\dagger\psi + a^\dagger\psi a\psi^\dagger = aa^\dagger\psi^\dagger\psi + a^\dagger a\psi\psi^\dagger = (1 + a^\dagger a)\psi^\dagger\psi + a^\dagger a\psi\psi^\dagger \\ &= a^\dagger a + \psi^\dagger\psi = H\end{aligned} \tag{19}$$

and in a similar way that

$$[Q, H] = [Q^\dagger, H] = 0 \tag{20}$$

together, the relations (19) and (20) is the so called *superalgebra* we want to have for the supercharges.

The supercharges Q and $Q^\dagger$ act on a state by exchanging one bosonic state for one fermionic state and vice versa. So that

$$\begin{aligned} Q \ \mid n_B, n_F\rangle &= \mid n_B - 1, n_F + 1\rangle, \\ Q^\dagger \mid n_B, n_F\rangle &= \mid n_B + 1, n_F - 1\rangle \end{aligned} \tag{21}$$

Where $| n_B, n_F\rangle$ denotes a state in Fock space where the number of bosons in the system is n_B and the number of fermions is n_F. It should also be noted that $(\psi^\dagger)^2 = 0$ which means that n_F can only take the values 1 or 0. An implication of this is that the degeneracy of the energy levels is two, except for the ground state which has zero energy.

The supercharges acts as operators which exchanges bosons for fermions and vice versa, while keeping the energy of the system constant. In other words we have a symmetry in the system with respect to changing bosons into fermions. This type of system which has one supercharge and its Hemitian conjugate is referred to as a $N = 1$ system.

4.2 SQM algebra

So far we have only looked at the supersymmetric harmonic oscillator, which is of course a simple and already very constrained system. What about other systems?

We can use the superalgebra from above to define what we mean by a SQM system, but there is also the possibility to generalize it a bit to include the possibility of having more than Q and $Q^\dagger$. This can be done by using a superalgebra

$$\begin{aligned}
&\{Q_i, Q_j^\dagger\} = 2H\delta_{ij} \\
&\{Q_i, Q_j\} = \{Q_i^\dagger, Q_j^\dagger\} = 0 \\
&[Q_i, H] = [Q_i^\dagger, H] = 0
\end{aligned} \tag{22}$$

Where we now allow for more than 2 supercharges. Imposing that the Hamiltonian commute with more supercharges will of course put more constraints on the potential. This is a central point for this thesis. In fact we can guess that if we keep adding more supercharges, the system will at some point be so restricted that it can have at most a quadratic potential. At this point the system will essentially be a standard SQM harmonic oscillator which is of course a very simple and solvable system.

4.3 A simple connection to integrability

The SQM algebra (22) above can be used immediately to find new symmetries in the system. This is a very straightforward way to see the connection between supersymmetry and the extra symmetries it leads to, which can potentially make the SQM systems integrable.

If we consider the Hamiltonian H, a general supercharge Q_i or $Q_i^\dagger$ together with some additional known symmetries of the system which we can call F_j. The given relations are

$$
\begin{aligned}
&[Q_i, H] = [Q_j^\dagger, H] = 0 \\
&[F_j, H] = 0
\end{aligned}
\tag{23}
$$

It is now easy to see that we are able to construct extra symmetries by

$$
\hat{F}_{ki} = [F_k, Q_i] \tag{24}
$$

These new symmetries will also commute with the Hamiltonian, which can be seen by the Jacobi identity, and are obviously consequences of supersymmetry. This simple way of finding symmetries makes it easy to realize that systems which already have a lot of symmetry will become very restricted by construction when supersymmetry is introduced.

4.4 Graded vector spaces

The supercharges and Hamiltonian in supersymmetric quantum mechanics form a closed algebra (22). Before we move on there is a comment that should be made on the properties of this algebra. In the typical case there would be a bracket operator that will give the relatationship between different elements, in this case however we have two different kinds of brackets depending on which elements we want to operate on. This type of space is called a *graded vector space*. In this case we have a $\mathbb{Z}_2$ graded vector space also known as a *super vector space*. We can think of this as having a decomposition of the vector space into two separate spaces.

$$
V = V_0 \oplus V_1 \tag{25}
$$

The elements in this space have a property called parity, denoted by $|\cdot|$, which in this case can be either 0 or 1 depending on whether it belongs to the even or odd part of the space which correspods to the bosonic and fermionic elements respectively.

$$|X] = \begin{cases} 0, & X \in V_0 \\ 1, & X \in V_1 \end{cases} \tag{26}$$

The bracket between two elements can now be defined as.

$$[X, Y] = XY - (-1)^{|X||Y|} YX \tag{27}$$

Throughout the rest of this thesis the symbol $\{\cdot\,,\cdot\}$ will always denote an *anti-commutator* which means that we have two elements with odd parity so that $|X||Y| = 1$ and $[\cdot\,,\cdot]$ will denote a standard commutator which means that we have at least one even element so that $|X||Y| = 0$.

5 A representation of $N = 1$ SQM

Now that we have seen a very simple example of SQM we can move on to a bit more general example. This is still very simple but also useful and will be referred to and expanded on later.

If we use the ansatz often given in introductory texts on SQM

$$Q = \sum_i \psi_i^\dagger (p_i - iW_i(x)) \quad , \quad Q^\dagger = \sum_j \psi_j (p_j + iW_j(x)) \tag{28}$$

$$\{\psi_i, \psi_j\} = \{\psi_i^\dagger, \psi_j^\dagger\} = 0 \quad , \quad \{\psi_i^\dagger, \psi_j\} = \delta_{ij} \tag{29}$$

$$[p_i, p_j] = [x_i, x_j] = 0 \quad , \quad [p_i, x_j] = -i\delta_{ij} \tag{30}$$

satisfying the algebra

$$\{Q, Q^\dagger\} = 2H \quad , \quad \{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0 \tag{31}$$

where ψ_i are odd variables representing the fermionic degrees of freedom in the system with $\psi_i^\dagger$ as its conjugate momentum, p_i is momentum and W_i are functions of position x.

5.1 Calculating the supercharges

While it's easy to verify that the ansatz (28) will give a reasonable Hamiltonian, by simply calculating the anti-commutator $\{Q, Q^\dagger\}$ it is possible to find the supercharges using a given Lagrangian. This will be done by using the so called Noether procedure, which is a standard method for calculating conserved quantities. The method uses the assumption that the variation of the action of the system vanishes. Consider simple system of a single variable x with a Lagrangian given by

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}(h'(x))^2 + \frac{i}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - h''(x)\bar{\psi}\psi \tag{32}$$

With odd variables ψ and $\bar{\psi}$. Together with the transformations

$$\begin{aligned} \delta x &= \epsilon\bar{\psi} - \bar{\epsilon}\psi \\ \delta\psi &= \epsilon(i\dot{x} + h'(x)) \\ \delta\bar{\psi} &= \bar{\epsilon}(-i\dot{x} + h'(x)) \end{aligned} \tag{33}$$

where ϵ is a fermonic variation parameter. If we let $\epsilon = \epsilon(t)$ it's possible to calculate the conserved supercharges using the Noether procedure.

Setting the variation of the action to zero gives the conserved currents.

$$\delta S = \delta \int L\, dt = 0 \tag{34}$$

Computing this variation and inserting the transformations in (33)

$$\begin{aligned} &\delta \int L\, dt = \int \Big(\dot{x}\delta\Big(\frac{dx}{dt}\Big) - h'(x)h''(x)\delta(x) + \frac{i}{2}(\delta(\bar{\psi})\dot{\psi} + \bar{\psi}\delta(\psi) - \delta(\dot{\bar{\psi}})\psi + \\ &- \dot{\bar{\psi}}\delta(\psi)) - h''(x)\delta(\bar{\psi})\psi - h''(x)\bar{\psi}\delta(\psi)\Big) dt \\ &= \int \Big(- \ddot{x}(\epsilon\bar{\psi} - \bar{\epsilon}\psi) - h'(x)h''(x)(\epsilon\psi\bar{\psi} - \bar{\epsilon}) + \frac{i}{2}\bar{\epsilon}\dot{\psi}(-i\dot{x} + h'(x)) + \\ &+ \frac{i}{2}\bar{\psi}\dot{\epsilon}(i\dot{x} + h'(x)) + \frac{i}{2}\bar{\psi}\epsilon(i\ddot{x} + \frac{d}{dt}h'(x)) - \frac{i}{2}\dot{\bar{\epsilon}}\psi(-i\dot{x} + h'(x)) + \\ &- \frac{i}{2}\bar{\epsilon}\psi(-i\ddot{x} + \frac{d}{dt}h'(x)) - \frac{i}{2}\dot{\bar{\psi}}\epsilon(i\dot{x} + h'(x)) + \\ &- \bar{\epsilon}\psi(-i\dot{x} + h'(x))h''(x) - \bar{\psi}\epsilon(i\dot{x} + h'(x))h''(x)\Big) dt \\ &= \int \Big(- \ddot{x}\epsilon\bar{\psi} + \ddot{x}\bar{\epsilon}\bar{\psi} - h'(x)h''(x)\epsilon\bar{\psi} + h'(x)h''(x)\bar{\epsilon}\psi - \frac{i}{2}\dot{\bar{\epsilon}}\psi(-i\dot{x} + h'(x)) + \\ &- \frac{i}{2}\bar{\epsilon}\psi(-i\ddot{x} + \frac{d}{dt}h'(x)) + \frac{i}{2}\bar{\psi}\dot{\epsilon}(i\dot{x} + h'(x)) + \frac{i}{2}\bar{\psi}\epsilon(-i\ddot{x} + \frac{d}{dt}h'(x)) + \\ &- \frac{i}{2}\dot{\bar{\epsilon}}\psi(-i\dot{x} + h'(x)) - \frac{i}{2}\bar{\epsilon}\psi(-i\ddot{x} + \frac{d}{dt}h''(x)) + \frac{i}{2}\bar{\psi}\dot{\epsilon}(i\dot{x} + h'(x)) + \\ &+ \frac{i}{2}\bar{\psi}\epsilon(-i\ddot{x} + \frac{d}{dt}h'(x)) - \bar{\epsilon}\psi(-i\dot{x} + h'(x))h''(x) - \epsilon\bar{\psi}(i\dot{x} + h'(x))h''(x)\Big) dt \\ &= \int -i\dot{\bar{\epsilon}}\psi(-i\dot{x} + h'(x)) - i\dot{\epsilon}\bar{\psi}(i\dot{x} + h'(x))\, dt = \int -i\dot{\epsilon}Q - i\dot{\bar{\epsilon}}\overline{Q}\, dt \end{aligned} \tag{35}$$

Noting that three ψ and $\bar{\psi}$ in any combination multiply to zero removes one of the terms resulting from differentiating $h''(x)\bar{\psi}\psi$ in the first step. Partial integration and inserting the transformations has been used in the second step. And again partially integrating some terms in the third step.

We can now identify the supercharges

$$\begin{aligned} Q &= \bar{\psi}(i\dot{x} + h'(x)) \\ \overline{Q} &= \psi(-i\dot{x} + h'(x)) \end{aligned} \tag{36}$$

We now see that this corresponds, up to a constant, to the ansatz for the supercharges given in (28). The difference being that we have now derived them from a Lagrangian.

5.2 Constraints on the potential

The hamiltonian for a system with the supercharges (28) is given by

$$\begin{aligned} &\{Q, Q^\dagger\} = \sum_{i,j} \{\psi_i^\dagger (p_i - iW_i), \psi_j(p_j + iW_j)\} \\ &= \sum_{i,j} \psi_i^\dagger \psi_j [(p_i - iW_i), (p_j + iW_j)] + (p_j + iW_j)(p_i - iW_i)\{\psi_i^\dagger, \psi_j\} \\ &= \sum_{i,j} i\psi_i^\dagger \psi_j ([p_i, W_j] - [W_i, p_j]) + \sum_i (p_i + iW_i)(p_i - iW_i) \\ &= \sum_{i,j} \frac{i}{2}([\psi_i^\dagger, \psi_j] + \delta_{ij})([p_i, W_j] + [p_j, W_i]) + \sum_i (p_i^2 + W_i^2 - i[p_i, W_j]) \\ &= \sum_i p_i^2 + W_i^2 + \sum_{i,j} [\psi_i^\dagger, \psi_j] \partial_i W_j = 2H \end{aligned} \tag{37}$$

The functions W_i have so far been assumed to be arbitrary functions of position. But we will now see that in order to satisfy the SQM algebra we will have to impose some constraints on these functions. the condition that $\{Q^\dagger, Q^\dagger\} = 0$ gives some conditions on W

$$\begin{aligned} &\{Q^\dagger, Q^\dagger\} = \sum_{i,j} \{\psi_i(p_i + iW), \psi_j(p_j + iW)\} \\ &= \sum_{i,j} \psi_i \psi_j [p_i + iW_i, p_j + iW_j] + (p_j + iW_j)(p_i + iW_i)\{\psi_i, \psi_j\} \\ &= \sum_{i,j} i\psi_i\psi_j([p_i, W_j] + [W_i, p_j]) = \sum_{i,j} \psi_i\psi_j(\partial_i W_j - \partial_j W_i) = 0 \end{aligned} \tag{38}$$

$$\Rightarrow \partial_i W_j - \partial_j W_i = 0 \tag{39}$$

which is solved by

$$W_i = \partial_i W \tag{40}$$

So we now have that all of the functions W_i have to be derivatives of some function W. While this certainly limits the possible choices for the W_is it is not possible to see that the system will be integrable. This means that we will have to include more supersymmetry, this will be the topic of the next section.

6 A supercharge recipe

We will now move away from the standard formulations of SQM for a while and take a look on another possible approach.

An article by Crombrugghe and Rittenberg (CR) from 1983 on SQM [5] gives a recipe for creating SQM supercharges in a straightforward way which will be described in this section. The method described in the article use an ansatz for supercharges based on the assumption that the supercharges contain only linear terms in fermionic degrees of freedom.

The CR paper creates a framework that is very structured and can be used to find additional symmetries in the system. Unfortunately the only conclusions drawn in the paper regarding extra symmetries and integrability are only presented in a quite narrow setting, with a certain amount of dimensions and certain amounts of supersymmetry. This will be explained briefly in subsection 6.4 below and in the discussion in section 11.

6.1 The setting

The ansatz made for the supercharges looks like

$$Q^{\alpha} = \frac{1}{\sqrt{2}} \sum_{i=1}^{r} \sum_{n=1}^{M} A_{in}^{\alpha} C_{in} \tag{41}$$

where the second sum is over different particles. To avoid cluttering the notations this will be suppressed here without loosing any vital properties of the charges. The ansatz then looks like

$$Q^{\alpha} = \frac{1}{\sqrt{2}} \sum_{i=1}^{r} A_i^{\alpha} C_i \tag{42}$$

The C_is represent odd degrees of freedom with properties that will be discussed below. The A_is are functions of even degrees of freedom, and it will turn out that they have to be chosen in a specific way for this ansatz to fulfill a SQM algebra and also give a reasonable Hamiltonian.

While this is a very nicely packaged form for the supercharges it has some potential drawbacks. The supercharges are real which means that we get a non standard form of the super algebra where the supercharges will anti commute among each other and the anticommutator of a supercharge with it self, not its conjugate, will produce the Hamiltonian.

$$\{Q^\alpha, Q^\beta\} = 2H\delta_{\alpha\beta} \quad , \quad [Q^\alpha, H] = 0 \tag{43}$$

It can be shown that a simple complex rotation of the supercharges will give back the standard SQM algebra, so this is not a problem but only slightly inconvenient. In the same way, the fermionic degrees of freedom also satisfy an algebra which differs from the one usually seen in the SQM literature. This means that the fermionic degrees of freedom will satisfy an algebra which is not as naturally related to that of x and p as they are in the standard formulation.

$$\{C_i, C_j\} = 2\delta_{ij}, \quad [p_i, x_j] = i\delta_{ij} \tag{44}$$

This can of course be cured in the same way as with the supercharges by simply making a complex rotation. But the standard analogy between bosonic and fermionic variables, where their algebra differs by exchanging commutation for anti-commutation is lost. This means that this ansatz is arguably less physically intuitive than the usual ansatz given in the previous section.

6.2 Constraints

We have now introduced the setting used for this representation of SQM. There will of course be some restrictions on the different components of the supercharges in (41). This will be done by making sure (43) is satisfied

$$\{Q^\alpha, Q^\beta\} = \frac{1}{2}\sum_{i,j=1}^{r}(A_i^\alpha C_i A_j^\beta C_j + A_j^\beta C_j A_i^\alpha C_i)$$

$$= \frac{1}{2}\sum_{i,j=1}^{r}(A_i^\alpha A_j^\beta \{C_i, C_j\} - [A_i^\alpha, A_j^\beta] C_j C_i)$$

$$= \frac{1}{2}\sum_{i,j=1}^{r}(A_i^\alpha A_j^\beta 2\delta_{ij} - [A_i^\alpha, A_j^\beta] C_j C_i)$$

$$= \sum_{i=1}^{r} A_i^\alpha A_i^\beta - \frac{1}{2}\sum_{i,j=1}^{r}[A_i^\alpha, A_j^\beta] C_j C_i$$

$$= \sum_{i=1}^{r} A_i^\alpha A_i^\beta - \frac{1}{4}\sum_{i,j=1}^{r}([A_i^\alpha, A_j^\beta](C_j C_i - C_j C_i + 2\delta_{ij})$$

$$= \sum_{i=1}^{r} A_i^\alpha A_i^\beta - \frac{1}{4}\sum_{i,j=1}^{r}[A_i^\alpha, A_j^\beta][C_j, C_i] - \frac{1}{2}\sum_{i=1}^{r}[A_i^\alpha, A_i^\beta]$$

$$= \frac{1}{2}\sum_{i=1}^{r}\{A_i^\alpha, A_i^\beta\} - \frac{1}{4}\sum_{i,j=1}^{r}[A_i^\alpha, A_j^\beta][C_j, C_i] = 2H\delta_{\alpha\beta} \tag{45}$$

(45) should be 0 for $\alpha \neq \beta$ which gives us the condition

$$\sum_{i=1}^{r}\{A_i^\alpha, A_i^\beta\} = 0 \quad , \quad \alpha \neq \beta \tag{46}$$

This can be fulfilled by using matrices O such that

$$A_i^\alpha = \sum_j O_{ij}^\alpha A_j^N \tag{47}$$

where N is the number of Qs and with orthogonal matrices O^α.

$$0 = \sum_i \{A_i^\alpha, A_i^\beta\} = \sum_{i,j,k}\{O_{ij}^\alpha A_j^N, O_{ik}^\beta A_k^N\} = \sum_{i,j,k} O_{ij}^\alpha O_{ik}^\beta \{A_j^N, A_k^N\} \tag{48}$$

setting $\beta = N$ gives us $O^\beta = I$ and we see that the O^αs are antisymmetric, this gives us

$$O^\alpha O^{\alpha T} = -1 \tag{49}$$

so the O^αs have to satisfy a *Clifford algebra*

$$\{O^\alpha, O^\beta\} = -2\delta^{\alpha\beta} \tag{50}$$

with this choice (46) will be automatically true, but any choice of As that satisfies (46) will work.

In [5] properties of clifford algebras are discussed at length. However, the only thing we need to know for this thesis is that there exist real valued matrix representations of the Clifford algebras and that it is relatively easy to find such representations. Therefore the discussion about these algebras and their representations will be kept to a minimum here.

The last term in (45) can only cancel if the terms containing the same C_i and C_j cancel

$$\begin{aligned} & [A_i^\alpha, A_j^\beta][C_j, C_i] + [A_j^\alpha, A_i^\beta][C_i, C_j] \\ \Rightarrow \quad & ([A_i^\alpha, A_j^\beta] - [A_j^\alpha, A_i^\beta])[C_j, C_i] = 0 \end{aligned} \tag{51}$$

so that

$$[A_i^\alpha, A_j^\beta] = [A_j^\alpha, A_i^\beta] \quad , \quad \alpha \neq \beta \tag{52}$$

Now that we have found the properties and restrictions of the bosonic functions A_i the ansatz (42) is quite convenient, since it lets us construct arbitrarily many supercharges as long as we can find matrices that satisfies (50). On the other hand, the constraints on A_i given by (52) can become a large system of differential equations depending on the x and p dependence of the A_i functions. This will become clear when four and more supercharges are used in later sections.

6.3 Correspondence with the standard formulation

Now we will find an example where this recipe is applied and see that it corresponds to the standard formulation in the previous section. If we use a very simple ansatz for the As:

$$A_1^N = A_1^1 = p_n, \quad A_2^N = A_2^1 = W_n \tag{53}$$

together with

$$O_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{54}$$

this will give us

$$A_1^1 = p_n \quad , \quad A_2^1 = W_n \quad , \quad A_1^2 = W_m \quad , \quad A_2^2 = -p_m \tag{55}$$

so that the supercharges for the system will be

$$\begin{aligned} Q_1 &= \sum_n C_{1n} p_n + C_{2n} W_n \\ Q_2 &= \sum_m C_{1m} W_m - C_{2m} p_m \end{aligned} \tag{56}$$

Which will give supercharges that already look somewhat similar to the supercharges (28) in the previous section. By applying constraints on the As given by (52) we see that the potential term has to satisfy a relation

$$[A_{1n}^1, A_{1m}^2] = [p_n, W_m] = [A_{1m}^1, A_{1n}^2] = [p_m, W_n] \tag{57}$$

which gives us a condition on W_n

$$\Rightarrow \partial_n W_m - \partial_m W_n = 0 \tag{58}$$

which completely agrees with the previous section. The solution to this is as before that

$$W_n = \partial_n W \tag{59}$$

The relation between the odd degrees of freedom in the two different cases is easy to see if we make a rotation of the supercharges

$$Q = \frac{1}{\sqrt{2}}(Q_1 - iQ_2) \tag{60}$$

and using the supercharges in (56) we get

$$\begin{aligned} Q &= \frac{1}{\sqrt{2}} \sum_i C_{1i} p_i + C_{2i} W_i - iC_{1i} W_i + iC_{2i} p_i \\ &= \frac{1}{\sqrt{2}} \sum_i (C_{1i} + iC_{2i}) p_i - iW_i (C_{1i} + iC_{2i}) \\ &= \frac{1}{\sqrt{2}} \sum_i (C_{1i} + iC_{2i})(p_i - iW_i) \end{aligned} \tag{61}$$

From this we can easily identify the odd variables from the previous section as

$$\psi_i^\dagger = \frac{1}{\sqrt{2}}(C_{1i} + iC_{2i}) \quad , \quad \psi_i = \frac{1}{\sqrt{2}}(C_{1i} - iC_{2i}) \tag{62}$$

We have seen that in this simple case it is easy to find the relation between the ansatz made in the article by Crombrugghe and Rittenberg and the more standard formulation found in most of the more recent papers on the subject.

The restrictions that we have found on the potential term is so far not very strong and it will in general not make the system integrable unless there are also some other constraints on the potential.

6.4 Extra symmetries

The CR paper presents some ways of finding extra symmetries in a SQM system. There are a number of examples where this is investigated but unfortunately they use a very specific setting i.e $\mathbb{R}^2$ and four supercharges and so on. These turned out to be hard to generalize, and the goal for this thesis is to find a relation between supersymmetry in quantum mechanics and integrability in a more general context.

One way to find extra symmetries which are a consequence of supersymmetry that is also presented in the paper by CR is based on the choice of bosonic variables A_i and the different clifford matrices O^α (note that $A_i^\alpha = \sum O_{ij}^\alpha A_j$. If we define f^l by

$$[A_j^\alpha, A_k^\beta] = i \sum_l f_{jk}^l B_l \tag{63}$$

where the B_l are some operators and the f^l are matrices. We get extra symmetries in the form

$$G^k = g_{ij}^k [C_i, C_j] \tag{64}$$

where g^k is some matrix that commutes with f^l. This symmetry is of course a consequence of supersymmetry but it will only be something bilinear in C_is and not a very interesting symmetry. In fact, considering the form of the Hamiltonian (45) it is not hard to guess that such symmetries exist. Some of these symmetries can be found by realizing that the clifford matrices O^α commute with f_l but in general this does not aid much in finding the relation between SQM and integrability.

7 Adding more symmetry using Crombrugghe and Rittenberg recipe

We have seen that we need to impose more supersymmetry on the systems to make them integrable. In order to do so we will use the CR recipe but in higher dimension. This will first be done with four supercharges, corresponding to a $N = 2$ system, and we will se a simple way to relate this to the previous results.

We will also use the ansatz to create a system with eight supercharges. The constraints on the potential will then be calculated when using six of these, which corresponds to a $N = 3$ system. This will constrain the system to having at most a quadratic potential which corresponds to a harmonic oscillator. Including two additional supercharges, creating a $N = 4$ system will force the system to be free.

7.1 Systems with four supercharges

One possible way to create a system with four supercharges inspired by the previous results is to add one more dimension. The natural ansatz for the bosonic A_is in the supercharges will then be

$$A_1 = p_x \ , \quad A_2 = p_y \ , \quad A_3 = \partial_x W \ , \quad A_4 = \partial_y W \tag{65}$$

which gives us an ansatz for a supercharge

$$Q_4 = p_{x_i} C_{1i} + p_{y_i} C_{2i} + \partial_{x_i} W C_{3i} + \partial_{y_i} W C_{4i} \tag{66}$$

We note for later use, that in a complex setting

$$\mathbb{R}^{2n} = \mathbb{C}^n, \quad z_i = x_i + iy_i \tag{67}$$

the supercharge may be written

$$Q_4 = p_{x_i} C_{1i} + p_{y_i} C_{2i} + \frac{1}{2} \operatorname{Re}(\partial_{z_i} W(z)) C_{3i} - \frac{1}{2} \operatorname{Im}(\partial_{z_i} W(z)) C_{4i} \tag{68}$$

under the assumption that W is a holomorphic function.

From (45) this ansatz gives us

$$
\begin{aligned}
2H &= p_{x_i}^2 + p_{y_i}^2 + \frac{1}{4}(\partial_{x_i}W)^2 + \frac{1}{4}(\partial_{y_i}W)^2 + \\
&- \frac{1}{4}([p_{x_i}, \partial_{x_j}W][C_{3j}, C_{1i}] + [p_{x_i}, \partial_{y_j}W][C_{4j}, C_{1i}] + \\
&+ [p_{y_i}, \partial_{x_j}W][C_{3j}, C_{2i}] + [p_{y_i}, \partial_{y_j}W][C_{4j}, C_{2i}]) \\
&= p_{x_i}^2 + p_{y_i}^2 + \frac{1}{4}(\partial_{x_i}W)^2 + \frac{1}{4}(\partial_{y_i}W)^2 + \\
&- \frac{1}{4}(\partial_{x_j}\partial_{x_i}W[C_{3j}, C_{1i}] + \partial_{x_i}\partial_{y_j}W[C_{4j}, C_{1i}] + \\
&+ \partial_{y_i}\partial_{x_j}W[C_{3j}, C_{2i}] + \partial_{y_i}\partial_{y_j}W[C_{4j}, C_{2i}])
\end{aligned}
\tag{69}
$$

The full set of supercharges are

$$
\begin{aligned}
Q_1 &= p_{y_i}C_{1i} - p_{x_i}C_{2i} + \partial_{y_i}WC_{3i} - \partial_{x_i}WC_{4i} \\
Q_2 &= \partial_{x_i}WC_{1i} - \partial_{y_i}WC_{2i} - p_{x_i}C_{3i} + p_{y_i}C_{4i} \\
Q_3 &= \partial_{y_i}WC_{1i} + \partial_{x_i}WC_{2i} - p_{y_i}C_{3i} - p_{x_i}C_{4i} \\
Q_4 &= p_{x_i}C_{1i} + p_{y_i}C_{2i} + \partial_{x_i}WC_{3i} + \partial_{y_i}WC_{4i}
\end{aligned}
\tag{70}
$$

Where the Clifford matrices in (47) are

$$
\begin{aligned}
O^1 &= i\sigma_2 \otimes \mathbb{1} \\
O^2 &= \sigma_3 \otimes i\sigma_2 \\
O^3 &= \sigma_1 \otimes 1\sigma_2
\end{aligned}
\tag{71}
$$

and the conditions on the functions $\partial_i W$ from (52) are

$$
\partial_{x_i}\partial_{x_j}W + \partial_{y_i}\partial_{y_j}W = 0 \tag{72}
$$

which tells us that W is the real part of a holomorphic function. This is a stronger condition than the one we had before, when W_i was only restricted to be the partial derivatives of some function. However it is not so restrictive that the system necessarily becomes integrable. We will see that this result gives a good hint on an alternative ansatz for supercharges later when we come back to $N = 2$ systems using complex variables where the concept of holomorphic functions become very natural.

7.2 Systems with six supercharges

We will now use an ansatz for a system with six supercharges and see how this adds more restrictions to the potential function. We need at least five different Clifford matrices in order to get six supercharges using the recipe. This means that we have to use eight by eight matrices. We will use an ansatz for the supercharges in 4 dimensions and make use of the recipe from CR. With this setup we can create eight supercharges which in the complex case corresponds to a $N = 4$ system. A priori we can have any functions A_i that can be functions of $x_i, i = 1, 2, 3, 4$ but they will be restricted when we impose that the Qs satisfy the supersymmetry algebra. The eight supercharges we get when using

$$\begin{aligned}
&O^1 = \sigma_3 \otimes i\sigma_2 \otimes \mathbb{1} \qquad && O^2 = \sigma_1 \otimes i\sigma_2 \otimes \mathbb{1} \\
&O^3 = i\sigma_2 \otimes \mathbb{1} \otimes \sigma_3 \qquad && O^4 = i\sigma_2 \otimes \mathbb{1} \otimes \sigma_1 \\
&O^5 = \mathbb{1} \otimes \sigma_3 \otimes i\sigma_2 \qquad && O^6 = \mathbb{1} \otimes \sigma_1 \otimes i\sigma_2 \\
&O^7 = i\sigma_2 \otimes i\sigma_2 \otimes i\sigma_2 &&
\end{aligned} \tag{73}$$

will be

$$\begin{aligned}
Q_1 &= p_3C_1 + p_4C_2 - p_1C_3 - p_2C_4 - A_7C_5 - A_8C_6 + A_5C_7 + A_6C_8 \\
Q_2 &= A_7C_1 + A_8C_2 - A_5C_3 - A_6C_4 + p_3C_5 + p_4C_6 - p_1C_7 - p_2C_8 \\
Q_3 &= A_5C_1 - A_6C_2 + A_7C_3 - A_8C_4 - p_1C_5 + p_2C_6 - p_3C_7 + p_4C_8 \\
Q_4 &= A_6C_1 + A_5C_2 + A_8C_3 + A_7C_4 - p_2C_5 - p_1C_6 - p_4C_7 - p_3C_8 \\
Q_5 &= p_2C_1 - p_1C_2 - p_4C_3 + p_3C_4 + A_6C_5 - A_5C_6 - A_8C_7 + A_7C_8 \\
Q_6 &= p_4C_1 - p_3C_2 + p_2C_3 - p_1C_4 + A_8C_5 - A_7C_6 + A_6C_7 - A_5C_8 \\
Q_7 &= A_8C_1 - A_7C_2 - A_6C_3 + A_5C_4 - p_4C_5 + p_3C_6 + p_2C_7 - p_1C_8 \\
Q_8 &= p_1C_1 + p_2C_2 + p_3C_3 + p_4C_4 + A_5C_5 + A_6C_6 + A_7C_7 + A_8C_8
\end{aligned} \tag{74}$$

7.2.1 Complex supercharges

Using these Qs to find the restrictions on the As is inconvenient because of the number of combinations of derivatives of As that can appear. This is somewhat simplified if we instead consider complex supercharges.

If we define new, complex supercharges as linear combinations of the real supercharges in (74) in the following way:

$$\begin{aligned}
\widehat{Q}_1 &= \frac{1}{\sqrt{2}}(Q_4 + iQ_1) \\
\widehat{Q}_2 &= \frac{1}{\sqrt{2}}(Q_2 + iQ_5) \\
\widehat{Q}_3 &= \frac{1}{\sqrt{2}}(Q_3 + iQ_6) \\
\widehat{Q}_4 &= \frac{1}{\sqrt{2}}(Q_7 + iQ_8)
\end{aligned} \tag{75}$$

the reason for this particular choice is not evident here but it is one of the choices which makes it possible to redefine the variables in the supercharges in a very convenient way.

We already have a hint from the standard form of the supercharges used in (28) that it is helpful to create bosonic functions in the form

$$b_i = (A_i(x) + ip_i) \tag{76}$$

and this combination of supercharges lets us define such variables in a straightforward way.

It is obvious that the new Qs in (75) will satisfy the SQM algebra (85) for complex supercharges if the real Qs in (74) satisfy the algebra (43) used for real supercharges due to

$$\widehat{Q}_a = Q_i + iQ_j \tag{77}$$

Which gives us that for $a \neq b$

$$\begin{aligned}
&\{\widehat{Q}_a, \widehat{Q}_b\} = \{Q_i + iQ_j, Q_k + iQ_l\} \\
&= \{Q_i, Q_k\} + i\{Q_i, Q_l\} + i\{Q_j, Q_k\} - \{Q_j, Q_l\} \\
&= H(\delta_{ik} + i\delta_{il} + i\delta_{jk} - \delta_{jl}) = 0
\end{aligned} \tag{78}$$

and for $a = b$ we get

$$\begin{aligned}
&\{\widehat{Q}_a, \widehat{Q}_a\} = \{Q_i + iQ_j, Q_i + iQ_j\} \\
&= \{Q_i, Q_i\} + i\{Q_i, Q_j\} + i\{Q_j, Q_i\} - \{Q_j, Q_j\} \\
&= H(\delta_{ii} + i\delta_{ij} + i\delta_{ji} - \delta_{jj}) = 0
\end{aligned} \tag{79}$$

And in the case with one of the charges conjugated

$$
\begin{aligned}
&\{\widehat{Q}_a, \widehat{Q}_a^\dagger\} = \{Q_i + iQ_j, Q_i - iQ_j\} \\
&= \{Q_i, Q_i\} - i\{Q_i, Q_j\} + i\{Q_j, Q_i\} - \{Q_j, Q_j\} \\
&= H(\delta_{ii} + i\delta_{ij} + i\delta_{ji} + \delta_{jj}) = 2H
\end{aligned} \tag{80}
$$

The new supercharges can be written

$$
\begin{aligned}
\widehat{Q}_1 &= \frac{1}{\sqrt{2}}(b_3^i\psi_1^i + b_4^i\psi_2^i + \bar{b}_1^i\bar{\psi}_3^i + \bar{b}_2^i\bar{\psi}_4^i) \\
\widehat{Q}_2 &= \frac{1}{\sqrt{2}}(b_2^i\psi_1^i + \bar{b}_1^i\bar{\psi}_2^i - b_4^i\psi_3^i - \bar{b}_3^i\bar{\psi}_4^i) \\
\widehat{Q}_3 &= \frac{1}{\sqrt{2}}(b_4^i\bar{\psi}_1^i - b_3^i\bar{\psi}_2^i + b_2^i\bar{\psi}_3^i - b_1^i\bar{\psi}_4^i) \\
\widehat{Q}_4 &= \frac{1}{\sqrt{2}}(b_1^i\psi_1^i - \bar{b}_2^i\bar{\psi}_2^i - \bar{b}_3^i\bar{\psi}_3^i + b_4^i\psi_4^i)
\end{aligned} \tag{81}
$$

where we have defined new variables inspired by the form of the supercharges in (28).

$$
\begin{aligned}
&\psi_1^i = C_1^i + iC_8^i \quad \psi_2^i = C_2^i + iC_7^i \quad \psi_3^i = C_3^i + iC_6^i \quad \psi_4^i = C_4^i + iC_5^i \\
&b_1^i = A_8^i + ip_1^i \qquad b_2^i = A_7^i + ip_2^i \quad b_3^i = A_6^i + ip_3^i \qquad b_4^i = A_5^i + ip_4^i
\end{aligned}
$$

These new supercharges will now have the standard algebra and the fermionic variables will follow the relation given in (29). So we have used the recipe to create eight supercharges, made a rotation of the charges and redefined the fermionic variables. The end result is that we have a $N = 4$ algebra with the standard properties created with the CR recipe.

Below we will give an example of how the restrictions on the bosonic functions A_i can be calculated. This will be done with six of the supercharges, which will correspond to a $N = 3$ system. The complete calculation is quite lengthy, and what is given here is just a small part, but hopefully enough to give a clear idea of the method.

$$
\begin{aligned}
&\{\widehat{Q}_1, \widehat{Q}_1\} = \frac{1}{2}\{b_3^i\psi_1^i + b_4^i\psi_2^i + \bar{b}_1^i\bar{\psi}_3^i + \bar{b}_2^i\bar{\psi}_4^i, b_3^j\psi_1^j + b_4^j\psi_2^j + \bar{b}_1^j\bar{\psi}_3^j + \bar{b}_2^j\bar{\psi}_4^j\} \\
&= \frac{1}{2}([b_3^i, b_3^j]\psi_1^i\psi_1^j + [b_3^i, b_4^j]\psi_1^i\psi_2^j + [b_3^i, \bar{b}_1^j]\psi_1^i\bar{\psi}_3^j + [b_3^i, \bar{b}_2^j]\psi_1^i\bar{\psi}_4^j + \\
&+ [b_4^i, b_3^j]\psi_2^i\psi_1^j + [b_4^i, b_4^j]\psi_2^i\psi_2^j + [b_4^i, \bar{b}_1^j]\psi_2^i\bar{\psi}_3^j + [b_4^i, \bar{b}_2^j]\psi_2^i\bar{\psi}_4^j + \\
&+ [\bar{b}_1^i, b_3^j]\bar{\psi}_3^i\psi_1^j + [\bar{b}_1^i, b_4^j]\bar{\psi}_3^i\psi_2^j + [\bar{b}_1^i, \bar{b}_1^j]\bar{\psi}_3^i\bar{\psi}_3^j + [\bar{b}_1^i, \bar{b}_2^j]\bar{\psi}_3^i\bar{\psi}_4^j + \\
&+ [\bar{b}_2^i, b_3^j]\bar{\psi}_4^i\psi_1^j + [\bar{b}_1^i, b_4^j]\bar{\psi}_4^i\psi_2^j + [\bar{b}_2^i, \bar{b}_1^j]\bar{\psi}_4^i\bar{\psi}_3^j + [\bar{b}_2^i, \bar{b}_2^j]\bar{\psi}_4^i\bar{\psi}_4^j) = 0
\end{aligned} \tag{82}
$$

for this to be true we have to impose

$$
\begin{array}{llll}
[b_2^i, b_4^j] = 0 & [b_3^i, \bar{b}_1^j] = 0 & [b_3^i, \bar{b}_2^j] = 0 & [b_4^i, \bar{b}_1^j] = 0 \\
[b_4^i, \bar{b}_2^j] = 0 & [\bar{b}_1^i, \bar{b}_2^j] = 0 & &
\end{array}
$$

Since we are interested in the constraints on the functions A_i and not on the b_is we plug in the definitions of the b_is

$$
\begin{array}{rr}
[A_6^i, p_4^j] + [p_3^i, A_5^j] = 0 & -[A_6^i, p_1^j] + [p_3^i, A_8^j] = 0 \\
-[A_6^i, p_2^j] + [p_3^i, A_7^j] = 0 & -[A_5^i, p_1^j] + [p_4^i, A_8^j] = 0 \\
-[A_5^i, p_2^j] + [p_4^i, A_7^j] = 0 & -[A_5^i, p_2^j] + [p_4^i, A_7^j] = 0 \\
-[A_8^i, p_2^j] - [p_1^i, A_7^j] = 0 &
\end{array}
$$

the conditions from $\{\widehat{Q}_2, \widehat{Q}_2\} = 0$ gives additional constraints

$$
\begin{array}{rr}
[A_7^i, p_4^j] + [p_2^i, A_5^j] = 0 & -[A_8^i, p_3^j] - [p_1^i, A_6^j] = 0 \\
-[A_5^i, p_3^j] + [p_4^i, A_6^j] = 0 &
\end{array}
$$

and in the same way $\{\widehat{Q}_3, \widehat{Q}_3\} = 0$ gives

$$
[A_5^i, p_1^j] + [p_4^i, A_8^j] = 0 \quad [A_6^i, p_2^j] + [p_3^i, A_7^j] = 0
$$

$\{\widehat{Q}_1, \widehat{Q}_2\} = 0$ gives

$$
\begin{array}{ll}
[A_8^i, p_2^j] - [p_1^i, A_7^j] = 0 & -[A_7^i, p_3^j] - [p_2^i, A_6^j] = 0 \\
[p_3^i, A_6^j] - [p_2^i, A_7^j] = 0 &
\end{array}
$$

using the above conditions the x_i dependence of the A's is reduced to

$$
A_8 = A_8(x_1), \quad A_7 = A_7(x_2), \quad A_6 = A_6(x_3), \quad A_5 = A_5(x_4)
$$

and lastly the conditions

$$
\begin{array}{l}
\{\widehat{Q}_1, \widehat{Q}_3\} = 0 \Rightarrow [A_8^i, p_1^j] + [A_7^i, p_2^j] = 0 \\
\{\widehat{Q}_2, \widehat{Q}_3\} = 0 \Rightarrow [A_6^i, p_3^j] + [A_8^i, p_1^j] = 0 \\
\{\widehat{Q}_1, \widehat{Q}_2^\dagger\} = 0 \Rightarrow [A_8^i, p_1^j] + [p_4^i, A_5^j] = 0
\end{array} \tag{83}
$$

which gives the final form of the A's

$$A_8^i = ax_1^i, \quad A_7^i = -ax_2^i, \quad A_6^i = -ax_3^i, \quad A_5^i = ax_4^i$$

so that the b's become

$$b_1^i = ax_1^i + ip_1^i, \quad b_2^i = -ax_2^i + ip_2^i, \quad b_3^i = -ax_3^i + ip_3^i, \quad b_4^i = ax_4^i + ip_4^i$$

since all of the b's will now commute, the hamiltonian for this system becomes

$$2H = \frac{1}{2}\sum_{n=1}^{4}(b_n^i \bar{b}_n^i + [b_n^i, \bar{b}_n^i]\bar{\psi}_n^i \psi_n^i) \tag{84}$$

where $[b_n^i, \bar{b}_n^i] = d_n$ are constants, making this system a 4 dimensional SQM harmonic oscillator. The differing signs on the a's can be solved by choosing a different ansatz, using other $\widehat{Q}$'s. This also means that the $N = 4$ system where all of the $\widehat{Q}$'s are used, will become a free system.

The calculations above clearly illustrates why there is a need to find another formulation of SQM which is easier to work with. Apart from being tedious to work with, this method also make it very hard to see directly how the increasing amount of symmetry affects the constraints on the potential.

The comments made on this result in the CR paper is essentially that there is no reason to believe that something new and exciting will happen is we generalize the system e.g by adding dimensions. This agrees with the result we get in the later sections of this thesis when $N = 3$ is explored without any assumptions of that kind.

The result we have from this calculation is useful even if it is done in a restricted setting. We know that the SQM harmonic oscillator is an integrable system, and this is what we are after. It is also shown in [5] that using only five supercharges will actually constrain the potential to being at most quadratic when working in $\mathbb{R}^4$. In this thesis we have chosen not to look closer at this result since using an odd number of supercharges is very hard to generalize into the standard setting of complex supercharges.

8 $N = 2$ using complex variables

We have seen that using an ansatz for real supercharges gives us an opportunity to generalize the form of the charges such that it is relatively easy to find as many supercharges as wee need. On the other hand, the constraints on the potential become tedious to calculate, and the algebra of the supercharges and the fermionic variables become different from the standard perhaps more physical form. Instead we will take a look at a $N = 2$ system using a complex variables in order to see if it can be generalized to create more supercharges in a methodical way. This will help us find a systematic description of what happens to the conditions on the potential when imposing more supersymmetry on the system. Most of the material in this section is based on chapter 10 in the book *Mirror symmetry* [6].

We want to use the standard SQM algebra

$$\{Q_\alpha, Q^\dagger_\beta\} = 2H\delta_{\alpha\beta} \quad , \quad \{Q_\alpha, Q_\beta\} = \{Q^\dagger_\alpha, Q^\dagger_\beta\} = 0 \tag{85}$$

and define

$$z^i = x^i + iy^i \tag{86}$$

and also complex odd variables

$$\begin{aligned} \psi^i = \psi^{x^i} + i\psi^{y^i} \quad &, \quad \psi^{\dagger i} = \psi^{\dagger x^i} + i\psi^{\dagger y^i} \\ \psi^{\bar{\imath}} = \psi^{x^i} - i\psi^{y^i} \quad &, \quad \psi^{\dagger \bar{\imath}} = \psi^{\dagger x^i} - i\psi^{\dagger y^i} \end{aligned} \tag{87}$$

so that

$$\{\psi^i, \psi^{\dagger\bar{\imath}}\} = \{\psi^{\dagger i}, \psi^{\bar{\imath}}\} = 1 \tag{88}$$

We want to find a set two supercharges and their conjugates who satisfy (85). This can of course be done in a similar way as before by finding a Lagrangian and a set of supersymmetry transformations and then using the Noether procedure. Since this method has already been presented earlier in this thesis and since finding supercharges is not the aim here we will simply give a set of Qs. Using the results later in this chapter it can easily be seen that this is completely in line with the supercharges found using the Noether procedure in the $N = 1$ case shown in a previous section. We will also present the relation between the supercharges given here and the ones found using the recipe of Crombrugghe and Rittenberg.

We have the supercharges, with implicit sums over i.

$$\begin{aligned} Q_+ = \psi^i p_i - \frac{i}{2}\psi^{\bar{\imath}}\partial_{\bar{\imath}}\overline{W} \quad &, \quad Q_- = \psi^{\dagger i} p_i + \frac{i}{2}\psi^{\dagger\bar{\imath}}\partial_{\bar{\imath}}\overline{W} \\ Q^\dagger_+ = \psi^{\dagger\bar{\imath}} p_{\bar{\imath}} + \frac{i}{2}\psi^{\dagger i}\partial_i W \quad &, \quad Q^\dagger_- = \psi^{\bar{\imath}} p_{\bar{\imath}} - \frac{i}{2}\psi^i\partial_i W \end{aligned} \tag{89}$$

We want to find the conditions on the so far arbitrary function W. This is of course done by imposing that the Qs obey the SQM algebra. Most of the terms here will vanish immediately due to the commutation and anti-commutation properties of the fermionic and bosonic variables in the supercharges.

$$\begin{aligned} \{Q_+, Q_+\} &= -\frac{i}{2}[p_i, \partial_{\bar{j}}\overline{W}]\psi^i\psi^{\bar{j}} - \frac{i}{2}[\partial_{\bar{i}}\overline{W}, p_j]\psi^{\bar{i}}\psi^j \\ &= -i[p_i, \partial_{\bar{j}}\overline{W}]\psi^i\psi^{\bar{j}} = 0 \end{aligned} \tag{90}$$

the above expression gives the condition on $\overline{W}$

$$\partial_i\partial_{\bar{j}}\overline{W} = 0 \tag{91}$$

which means that $\operatorname{Re} W$ and $\operatorname{Im} W$ must be harmonic functions, which is the condition we have on the real part of a holomorphic function. As expected this result is exactly the she as when we used the real supercharges in (70).

We also calculate the conditions on W due to the conjugate supercharges.

$$\begin{aligned} \{Q_+^\dagger, Q_+^\dagger\} &= \frac{i}{2}[p_{\bar{i}}, \partial_j W]\psi^{\dagger\bar{i}}\psi^{\dagger j} + \frac{i}{2}[\partial_i W, p_{\bar{j}}]\psi^{\dagger i}\psi^{\dagger\bar{j}} \\ &= i[p_{\bar{i}}, \partial_j W]\psi^{\dagger\bar{i}}\psi^{\dagger j} = 0 \end{aligned} \tag{92}$$

which gives the condition

$$\partial_{\bar{i}}\partial_j W = 0 \tag{93}$$

which gives exactly the same condition as (91).

We have two more combinations to check.

$$\{Q_+, Q_-^\dagger\} = -\frac{i}{2}[p_i, \partial_j W]\psi^i\psi^j - \frac{i}{2}[\partial_{\bar{i}}\overline{W}, p_{\bar{j}}]\psi^{\bar{i}}\psi^{\bar{j}} = 0 \tag{94}$$

the last expression is zero due to the symmetry in $i \leftrightarrow j$ in the commutators of even variables and antisymmetry in $i \leftrightarrow j$ in the products of ψ, so it does not give any further constraints on W.

$$\begin{aligned} \{Q_+, Q_-\} &= \frac{i}{2}\{\psi^i, \psi^{\dagger\bar{j}}\}\partial_{\bar{j}}\overline{W}p_i + \frac{i}{2}[p_i, \partial_{\bar{j}}\overline{W}]\psi^i\psi^{\dagger\bar{j}} + \\ &- \frac{i}{2}\{\psi^{\bar{i}}, \psi^{\dagger j}\}p_j\partial_{\bar{i}}\overline{W} - \frac{i}{2}[\partial_{\bar{i}}\overline{W}, p_j]\psi^{\bar{i}}\psi^{\dagger j} \\ &= i(\delta_{ij}\partial_{\bar{j}}\overline{W}p_i - \delta_{ij}p_j\partial_{\bar{i}}\overline{W}) + \frac{i}{2}[p_i, \partial_{\bar{j}}\overline{W}]\psi^i\psi^{\dagger\bar{j}} - \frac{i}{2}[\partial_{\bar{i}}\overline{W}, p_j]\psi^{\bar{i}}\psi^{\dagger j} \\ &= i[\partial_{\bar{i}}\overline{W}, p_i] + \frac{i}{2}[p_i, \partial_{\bar{j}}\overline{W}]\psi^i\psi^{\dagger\bar{j}} - \frac{i}{2}[\partial_{\bar{i}}\overline{W}, p_j]\psi^{\bar{i}}\psi^{\dagger j} = 0 \end{aligned} \tag{95}$$

all three terms are zero due to condition (91) on W. The same condition on W arise from the other combinations of Q.

The hamiltonian becomes

$$
\begin{aligned}
2H &= \{Q_+, Q_+^\dagger\} \\
&= \{\psi^i, \psi^{\dagger\bar{\jmath}}\}p_{\bar{\jmath}}p_i + \frac{i}{2}[p_i, \partial_j W]\psi^i\psi^{\dagger j} - \frac{i}{2}[\partial_{\bar{\imath}}\overline{W}, p_{\bar{\jmath}}]\psi^i\psi^{\dagger\bar{\jmath}} + \\
&+ \frac{i}{4}\{\psi^{\bar{\imath}}, \psi^{\dagger j}\}\partial_{\bar{\imath}}\overline{W}\partial_j W \\
&= p_i p_{\bar{\imath}} + \frac{1}{4}\partial_{\bar{\imath}}\overline{W}\partial_i W + \frac{i}{2}([p_i, \partial_j W]\psi^i\psi^{\dagger j} + [p_{\bar{\jmath}}, \partial_{\bar{\imath}}\overline{W}]\psi^{\bar{\imath}}\psi^{\dagger\bar{\jmath}}) \\
&= p_i p_{\bar{\imath}} + \frac{1}{4}\partial_{\bar{\imath}}\overline{W}\partial_i W + \frac{1}{2}(\psi^i\psi^{\dagger j}\partial_i\partial_j W + \psi^{\bar{\imath}}\psi^{\dagger\bar{\jmath}}\partial_{\bar{\jmath}}\partial_{\bar{\imath}}\overline{W})
\end{aligned}
\tag{96}
$$

8.1 Relation to real supercharges

By using a rotation it is easy to find the relation between the supercharges with in the $N = 2$ complex setting above and the real supercharges from (70). We can see that

$$
\begin{aligned}
Q_+^\dagger &= \frac{1}{\sqrt{2}}(Q_1 - iQ_2) \\
&= \frac{1}{\sqrt{2}}(p_x(-C_2 - iC_1) + ip_y(-iC_1 - C_2) + \\
&+ \operatorname{Re}\partial_z W(z)(-C_4 - iC_3) + i\operatorname{Im}\partial_z W(z)(-iC_3 - C_4)) \\
&= \frac{1}{\sqrt{2}}(p_z(-C_2 - iC_1) + \partial_z W(z)(-iC_3 - C_4))
\end{aligned}
\tag{97}
$$

which means that we can identify the relation between the odd variables in the two different formulations.

$$
\begin{aligned}
\psi^{\dagger\bar{\imath}} &= -\frac{1}{\sqrt{2}}(C_2 + iC_1) \\
\psi^{\dagger i} &= \sqrt{2}(-C_3 + iC_4)
\end{aligned}
\tag{98}
$$

The rest of the supercharges and odd variables can be related in the same way using the other two supercharges.

9 Creating a more general ansatz

We have now seen that by using complex variables we can find a representation of $N = 2$ SQM. If we could find a way to rotate the supercharges in (89) in such a way that it is easier to recognize the $N = 1$ (28) supercharges among the four supercharges in $N = 2$ this would give a strong hint on how to expand the system from $N = 1$ to $N = 2$ and then to $N = 3$ in systematic way which would be very helpful to us. Finding such a general ansatz for supercharges would make it much easier to see what happens to the constraints on the potential when systematically adding more supersymmetry to the system.

It turns out that it is quite straightforward to find this relation between the $N = 1$ and $N = 2$ case, this will be shown in this section.

9.1 Compact form of the supercharges

By inserting the definition of the variables in the supercharges from (89) and making a linear combination of the supercharges, we can calculate new supercharges which will satisfy the same algebra. We start of by expanding Q_+ and $Q_-^\dagger$ in terms of their real variables x and y. The complex function W is split into its real and imaginary part $W = W_1 + iW_2$ in the expression below.

$$\begin{aligned}
Q_+ &= \psi^{x^i} p_{x^i} + \psi^{y^i} p_{y^i} + \frac{1}{2}(\psi^{x^i}(\partial_{y^i} W_1 - \partial_{x^i} W_2) - \psi^{y^i}(\partial_{x^i} W_1 + \partial_{y^i} W_2) + \\
&+ i(-\psi^{x^i} p_{y^i} + \psi^{y^i} p_{x^i} - \frac{1}{2}(\psi^{x^i}(\partial_{x^i} W_1 + \partial_{y^i} W_2) + \psi^{y^i}(\partial_{y^i} W_1 - \partial_{x^i} W_2))) \\
&= \psi^{x^i}(p_{x^i} + \partial_{y^i} W_1 - i p_{y^i} - i\partial_{x^i} W_1) + \psi^{y^i}(p_{y^i} - \partial_{x^i} W_1 + i p_{x^i} - i\partial_{y^i} W_1)
\end{aligned} \tag{99}$$

$$\begin{aligned}
Q_-^\dagger &= \psi^{x^i} p_{x^i} + \psi^{y^i} p_{y^i} - \frac{1}{2}(\psi^{x^i}(\partial_{y^i} W_1 - \partial_{x^i} W_2) - \psi^{y^i}(\partial_{x^i} W_1 + \partial_{y^i} W_2) + \\
&- i(-\psi^{x^i} p_{y^i} + \psi^{y^i} p_{x^i} + \frac{1}{2}(\psi^{x^i}(\partial_{x^i} W_1 + \partial_{y^i} W_2) + \psi^{y^i}(\partial_{y^i} W_1 - \partial_{x^i} W_2))) \\
&= \psi^{x^i}(p_{x^i} - \partial_{y^i} W_1 + i p_{y^i} - i\partial_{x^i} W_1) + \psi^{y^i}(p_{y^i} + \partial_{x^i} W_1 - i p_{x^i} - i\partial_{y^i} W_1)
\end{aligned} \tag{100}$$

The aim was to find the supercharges used in the $N = 1$ system (28), this is done by using a rotation.

$$\frac{1}{2}(Q_-^\dagger + Q_+) = (\psi^{x^i}(p_{x^i} - i\partial_{x^i}W_1) + \psi^{y^i}(p_{y^i} - i\partial_{y^i}W_1)$$
$$= \sum_i \psi^i(p_i - i\partial_i W_1) \tag{101}$$

We can now see that the $N = 2$ case is the same system as $N = 1$ with additional supercharges added. The other two supercharges are given by the rotation

$$\frac{1}{2}(Q_-^\dagger - Q_+) = (\psi^{x^i}(ip_{y^i} - \partial_{y^i}W_1) + \psi^{y^i}(-ip_{x^i} + \partial_{x^i}W_1)$$
$$= \sum_{ij} i\psi^i J_{ij}(p_j + i\partial_j W_1) \tag{102}$$

where

$$J_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{103}$$

The form of the supercharge in (102) gives a hint of how we can generalize to include more supercharges into the system. The choice of the matrix J corresponds to a specific choice of rotation we made. If we instead use (102) as an ansatz and determine what conditions we have on J we can find more matrices I, K and so on and use this to create systems with more supersymmetry.

9.2 Complex structure

We will now use an anstatz inspired by (102). Note that the i in (102) has been absorbed, this will of course only contribute an overall factor, and as such it will not change the constraints on J.

$$Q^1 = \sum_j (p_j - iW_j)\psi_j \tag{104}$$

together with

$$Q^2 = \sum_j (p_j + iW_j)J_{jk}\psi_k \tag{105}$$

The conditions we have to impose on J is determined by imposing the standard SQM algebra (85). First let's check $\{Q^{1\dagger}, Q^2\}$

$$
\begin{aligned}
\{Q^{1\dagger}, Q^2\} &= \sum_k \sum_{ij} \{(p_k + iW_k)\psi_k^\dagger, (p_i + iW_i)J_{ij}\psi_j\} \\
&= \sum_k \sum_{ij} [(p_k + iW_k), (p_i + iW_i)] J_{ij} \psi_k^\dagger \psi_j + \\
&\quad + \{\psi_k^\dagger, \psi_j\}(p_i + iW_i)(p_k + iW_k) J_{ij} + \\
&= \sum_k \sum_{ij} [(p_k + iW_k), (p_i + iW_i)] J_{ij} \psi_k^\dagger \psi_j \\
&\quad + (p_i + iW_i)(p_j + iW_j) J_{ij} = 0
\end{aligned}
\tag{106}
$$

The first term vanishes due to $W_i = \partial_i W$. For the last term to vanish we have to impose $J_{ij} = -J_{ji}$

$$
\begin{aligned}
&(p_i + iW_i)(p_j + iW_j)J_{ij} + (p_j + iW_j)(p_i + iW_i)J_{ji} \\
&= \left[\text{setting } J_{ij} = -J_{ji}\right] \\
&= ((p_i - iW_i)(p_j - iW_j) - (p_j - iW_j)(p_i - iW_i))J_{ij} \\
&= i([p_j, W_i] - [p_i, W_j])J_{ij} \\
&= (\partial_j \partial_i W - \partial_i \partial_j W)J_{ij} = 0
\end{aligned}
\tag{107}
$$

We also get a condition on J_{ij} from $\{Q^1, Q^{1\dagger}\} = \{Q^2, Q^{2\dagger}\} = 2H$

$$
\begin{aligned}
\{Q^{1\dagger}, Q^1\} &= \sum_{ij} [(p_j + iW_j), (p_i - iW_i)]\psi_j^\dagger \psi_i + (p_i - iW_i)(p_i + iW_i) \\
&= \{Q^{2\dagger}, Q^2\} = \sum_{kl} \sum_{ij} [(p_k - iW_k), (p_i + iW_i)] J_{ij} J_{kl} \psi_l^\dagger \psi_j + \\
&\quad + \{\psi_l^\dagger, \psi_j\}(p_i + iW_i)(p_k - iW_k) J_{ij} J_{kl} \\
&= \sum_{kl} \sum_{ij} [(p_k - iW_k), (p_i + iW_i)] J_{ij} J_{kl} \psi_l^\dagger \psi_j + \\
&\quad + (p_i + iW_i)(p_k - iW_k) J_{ij} J_{kj}
\end{aligned}
\tag{108}
$$

so that $\sum_j J_{ij} J_{kj} = JJ^t = I$ in order to get the same Hamiltonian. This in turn means that $J^2 = J(J^t)^t = J(-J)^t = -\mathbb{1}$.

These properties of J is exactly what we have for a complex structure. Note that there is usually a metric multiplying the complex structure, but we assume that we have a flat space so the metric will be an identity matrix.

9.3 Constraints on the potential

In order for the first term in (108) to vanish we have to impose

$$\sum_{ijk}[(p_k - iW_k),(p_i + iW_i)]J_{ij}\psi_k\psi_j = 2\sum_{ijk}\partial_k\partial_i W J_{ij}\psi_k\psi_j = 0 \tag{109}$$

$$\begin{aligned} &\Rightarrow \sum_i \partial_k\partial_i W J_{ij}\psi_k\psi_j + \partial_j\partial_i W J_{ik}\psi_j\psi_k \\ &= \sum_i(\partial_k\partial_i W J_{ij} - \partial_j\partial_i W J_{ik})\psi_k\psi_j = 0 \end{aligned} \tag{110}$$

$$\Rightarrow \sum_i \partial_i J_{i[j}\partial_{k]}W = 0 \tag{111}$$

where $W_i = \partial_i W$ has been used. The constraint on W is that it has to be a holomorphic function with respect to the complex structure J. This is a very useful result which will give us a natural way to describe the constraints on W when we add more supercharges.

10 $N = 3$ using complex structure

In the last section we found a way to express supercharges by using a supercharge of the same form as in a $N = 1$ system together with a supercharge with a complex structure. The constraints on that system turned out to be that the function W had to be holomorphic with respect to the complex structure. In this section it will be shown that adding more supercharges by using another complex structure will create the same constraint for the additional complex structure. There will also be more constraints on W depending on the combination of the two complex structures used in the new system. It is also shown that the choice of the second complex structure cannot be completely arbitrary but that they have to satisfy a specific commutation relation.

10.1 Additional constraints

We will use the same supercharges and algebra as in the previous section but with one additional charge added by using an additional complex structure I.

$$Q^3 = \sum_j (p_j + iW_j)I_{jk}\psi_k \tag{112}$$

Adding this extra Q gives constraints on the complex structures I and J from

$$
\begin{aligned}
\{Q^2, Q^{3\dagger}\} = & \sum_{ij}\sum_{kl}[(p_i + iW_i), (p_k - iW_k)]J_{ij}I_{kl}\psi_j\psi_l^\dagger + \\
& + \{\psi_j, \psi_l^\dagger\}(p_k - iW_k)(p_i + iW_i)J_{ij}I_{kl}
\end{aligned} \tag{113}
$$

First term has to be zero for all j and l:

$$
\begin{aligned}
& \sum_{ij}\sum_{kl}[(p_i + iW_i), (p_k - iW_k)]J_{ij}I_{kl}\psi_j\psi_l^\dagger \\
& = 2\sum_{ij}\sum_{kl}\partial_k\partial_i W\, J_{ij}I_{kl}\psi_j\psi_l^\dagger = 0
\end{aligned} \tag{114}
$$

$$
\begin{aligned}
& \Rightarrow \sum_{i}\sum_{k}\partial_k\partial_i W\, J_{ij}I_{kl} = \sum_{i}\sum_{k}\partial_j\partial_i W\, J_{ik}I_{kl} \\
& = \sum_{i}\sum_{k}\partial_j\partial_l W\, J_{ik}I_{ki} = 0 \quad \forall j, l
\end{aligned} \tag{115}
$$

where (111) has been used. This term is zero if the product $\sum_k J_{ak}I_{kb}$ is antisymmetric in $a \leftrightarrow b$.

We can now see the relation we have to impose in the complex structures.

$$
JI = -(JI)^t = -I^t J^t = -IJ \tag{116}
$$

So we know that J and I have to anticommute.

And we can also see that the product JI is a complex structure

$$
(JI)(JI) = -JIIJ = JJ = -\mathbb{1} \tag{117}
$$

Apart from this constraint on the complex structure we also get additional constraints on W. First of all W now has to be holomorphic with respect to both J and$I$ which is quite hard to satisfy and greatly limits the choice of W. We also have to make the second term of (113) vanish. This will also put a quite strong constraint on W.

$$
\begin{aligned}
&\sum_{ijk}(p_k - iW_k)(p_i + iW_i)J_{ij}I_{kj} \\
&= \sum_{ijk}(iW_i p_k - iW_k p_i)J_{ij}I_{kj} \\
&= \sum_{ijk} iW_i p_k J_{ij}I_{kj} - \sum_{ijk} iW_i p_k J_{kj}I_{ij} \\
&= \sum_{ijk} iW_i p_k J_{ij}I_{kj} + \sum_{ijk} iW_i p_k J_{ij}I_{kj} \\
&= \sum_{ijk} iW_i J_{ij}I_{kj}(p_k + p_i) = 0
\end{aligned}
\tag{118}
$$

so that we have to impose

$$
\sum_{ijk} \partial_i W J_{ij} I_{kj} = 0 \tag{119}
$$

The condition that the potential has to be holomorphic with respect to both I and J has been explored in a paper from 1983 by Alvarez-Gaumé and Freedman [9] where it is concluded that the only solution is to have a quadratic potential, which will make the system integrable.

10.2 Beyond $N = 3$

The natural way to continue would be to include yet another supercharge like the one in (112) but with a third complex structure K. The constraints on the potential resulting from this would of course be yet another holomorphicity condition on W and it also follows that W has to satisfy more constraints similar to (119). Since the system is already integrable this will not be explored further here. A remark on the third complex structure K, which follows directly from the reasoning on J and I above is that we may choose

$$
K = IJ \tag{120}
$$

which leads us directly to the following properties of I,J and K

$$
\begin{aligned}
&I^2 = J^2 = K^2 = IJK = -1 \\
&IJ = -JI = K, \quad KI = -IK = J, \quad JK = -KJ = I
\end{aligned}
\tag{121}
$$

Which is the relation satisfied by quaternions. Among other things this means that we have now exhausted the number of supercharges we can construct without using linear combinations of the four we already have. This is because any new complex structure satisfying (121) can always be written in terms of a linear combination of I, J, K and $\mathbb{1}$.

11 Discussion

This thesis has investigated the constraints that arise on the potential of a SQM hamiltonian as more supersymmetry is introduced into the system. The purpose of this was to find at which point the system becomes integrable due to the amount of symmetry imposed on the system. This has been done by finding an ansatz for supercharges and systematically imposing more supersymmetry on a general quantum mechanical system.

Supersymmetry in quantum mechanics was explored in a paper from 1983 by Crombrugghe and Rittenberg. The goal of this paper was not expressly to find out under which circumstances and amounts of supersymmetry quantum mechanical systems become integrable. The parts of the paper which touches the extra symmetries that arise due to supersymmetry, and weather or not this makes systems integrable are there mainly because it's natural relation to SQM, and it is never claimed in the paper that this relation is fully explored and entirely conclusive.

In the CR paper an ansatz is used using real valued supercharges and a slightly non standard SQM algebra. This is in itself not a very big issue but it turns out that it makes it hard to generalize and analyze in the general case, without making assumptions on how many dimensions one has in relation to the amount of supersymmetry. This problem is due to that the model is based on finding real matrix representations of elements in a Clifford algebra and using these to construct the supercharges. In the general case this means that one has to use technical properties of Clifford algebra representations, which obscures the possible physical interpretations. Even under circumstances when the ansatz is easy to use, the paper only goes so far as to conclude that systems with five supercharges in four dimensions become integrable, and that there is no reason to believe that anything new will happen when going to higher dimension. While this seems reasonable, a more conclusive and less complicated reasoning would be desirable. CR also presents a ways of finding some conserved quantities that arise due to the addition of supersymmetry to a system. This is of course relevant to the integrability of the system but most of the additional symmetries that are presented are hard to generalize to a general case. The additional odd symmetry discussed in 6.4 is also not very useful and there are no conclusions drawn from this regarding the integrability of the SQM systems.

The ansatz used in the CR paper is very concise and gives a compact form on the conditions one has to impose on the bosonic part of the supercharges, e.g the potential. The ansatz also benefits from being easy to use for constructing supercharges in low number of dimensions when the clifford matrices are easy to find. On the other hand, the fact that the algebra used for these supercharges is not the standard SQM algebra, also means that the fermionic variables in the supercharges do not satisfy their usual algebra. This makes the supercharges hard to interpret physically and also hard to relate to other known formulations of SQM.

Instead of using the formulation of SQM from CR it is possible to find a general way of writing supercharges using an ansatz inspired by the simplest case of $N = 1$ SQM. This option has been explored and has been compared and related to the result from using the CR ansatz. This new ansatz makes it very straightforward to introduce more supercharges into the system and has made it easy to express the kind of conditions on the potential that arise from adding more supersymmetry.

It can be seen from this new formulation that, in $N = 2$ systems, the restrictions on the system means that the potential has to correspond to the real part of some holomorphic function. This is in agreement with the result found in most introductory texts on SQM, but with this ansatz the result is even easier to find than when using other formulations. There is no conclusion drawn from this regarding the integrability of the system. The reason for this is that there are no known results that points to that this condition on the potential is a strong enough constraint to make the system integrable. In $N = 3$ systems the potential is restricted to being the real part of a function which is holomorphic with respect to two complex structures, along with a constraint given by a combination of the coplex structures. These constraints that arise from $N = 3$ are very hard to satisfy and leads to that the system becomes integrable. This corresponds to the finding of CR that the system is integrable when using six (actually already at five according to CR) supercharges. In this case the potential has to be the real part of a function which is holomorphic with respect to two complex structures which, according to Alvarez-Gaumé and Freedman [9], means that it can be at most quadratic, making the system integrable.

Proof of Liouville theorem

We will now give a somewhat sketchy proof of the Liouvile theorem. The purpose of this is to give an idea of how the conditions on the system and its conserved quantities makes the system solvable rather than to give a completely rigorous proof. The full version can be found in [7].

Consider a system with a phase space M of dimension $2n$ equipped with a one form $\alpha = \sum_i p_i dq_i$ and a symplectic two-form

$$\omega = d\alpha = \sum_j dp_j \wedge dq_j \tag{122}$$

If we define the vector field $X_A = \omega^{ij}\partial_j A \partial_i$ it can be shown that (see for instance [7] or [10])

$$\{A, B\} = X_A(B) = \omega(X_A, X_B) \tag{123}$$

We want to be able to do a transformation so that the conserved quantities F_i are among the coordinates of the system:

$$\omega = \sum_j dp_j \wedge dq_j = \sum_j dF_j \wedge d\psi_j \tag{124}$$

If we can find such a transformation the solutions of the equations of motion will become trivial

$$\begin{aligned} \dot{F}_j &= \{H, F_j\} = 0, \\ \dot{\psi}_j &= \{H, \psi_j\} = \frac{\partial H}{\partial F_j} = \Omega_j \end{aligned} \tag{125}$$

The Ω_js will be time independent since they only depend on F_j. So the solutions will be given by

$$\begin{aligned} F(t)_j &= F(0)_j, \\ \psi(t)_j &= \psi(0)_j + \Omega_j t \end{aligned} \tag{126}$$

We now introduce a new function S such that on M_f given by $F_j(p, q) = f_i$

$$S(F, q) = \int_{m_0}^{m} \alpha = \int_{q_0}^{q} \sum_i^n p_i(f, q) dq_i \tag{127}$$

If such a function exists, the integral in (127) cannot depend on the path from m_0 to m which means that we want

$$d\alpha|_{M_f} = \omega|_{M_f} = 0 \tag{128}$$

The vector space X_{F_i} is tangent to M_f because $X_{F_i}(F_j) = \{F_i, F_j\} = 0$. We can now see that $\omega|_{M_f} = \omega(F_i, F_j) = \{F_i, F_j\} = 0$. This makes it possible to write

$$p_i = \frac{\partial S}{\partial q_i} \tag{129}$$

and if we also define

$$\psi_i = \frac{\partial S}{\partial F_i} \tag{130}$$

we see that

$$dS = \sum_i \psi_i dF_i + p_i dq_i \tag{131}$$

and by using $d^2S = 0$ we finally get

$$\omega = \sum_j dp_j \wedge dq_j = \sum_j dF_j \wedge d\psi_j \tag{132}$$

□

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