Thesis for the Degree of Master of Science in Physics

Auxiliary Fields in Supersymmetric Sigma Models and the relation to Complex Geometry

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Abstract

In this master thesis we introduce the basics of supersymmetry in terms of superspace and superfields. We then show that a two-dimensional sigma model admits N=(2,2) supersymmetry iff the target space geometry is bi-hermitian. In order for the supersymmetry algebra to close off-shell, auxiliary fields are needed in the model. These are introduced via a formulation of the model in terms of N=(2,2) semi-chiral superfields. We then introduce generalized complex geometry, and define generalized Kähler geometry (GKG). Since bi-hermitian geometry and (GKG) are equivalent, we then try to realize GKG directly from the sigma model. In doing so, we must redefine the auxiliary fields into fields transforming in T^*, and we discuss different ways to make such redefintions.

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Contents

1 Introduction

In this master thesis we will look at the connections between supersymmetric sigma models and complex geometry. We will see that when we impose enough supersymmetry on the sigma model, certain restrictions on the geometry in the target space will arise. We will concentrate on a two-dimensional sigma model, since that is the dimension relevant for string theory. With one supersymmetry, no restrictions will arise. It is when we make the model N=(2,2) supersymmetric that interesting things will happen. First we will show that if we have a N =(2,2) sigma model with chiral superfields, the geometry of the target space has to be Kähler. More generally, without any restrictions on the N=(2,2) superfields, we will show that the geometry has to be bi-hermitian. This will be referred to as the 'Gates, Hull and Roček theorem' (GHR-theorem). Bi-hermitian means that there are two complex structures, and the metric is hermitian with the respect to both of them, and the complex structures are covariantly constant.

We will also look at a generalization of complex geometry, called Generalized Complex Geometry (GCG). In ordinary complex geometry we have complex structures, mapping the tangent space to itself. In GCG we have a Generalized Complex Structure, mapping the sum of the tangent space and the cotangent space to itself. One remarkable fact is that there is a subclass of GCG's, called Generalized Kähler Geometry (GKG), which is equivalent to the bi-hermitian geometry mentioned above. Therefore it should be possible to realize GKG in the target space directly from the sigma model. In doing so, we will need fields living in the cotangent space. These fields have to be auxiliary, since we want the physical properties of the sigma model to be the same. These auxiliary fields are also needed when the complex structures do not coummute, in order for the algebra to close off-shell. The auxiliary fields can be defined in different ways, and we get different GCS.

When we realize GKG in the target space from a sigma model, we will closely follow the paper [1]. In addition to what is done in that paper, we here define the auxiliary fields in the model differently, and we will therefore obtain a different set GCS's, but with the auxiliary fields integrated out, the model looks the same as in [1].

The thesis will be organized as follows: section 2 will consist of some background material in order to introduce supersymmetry, including a very short introduction to Lie Groups, the Poincaré Group, and representations thereof. In section 3 we will introduce the supersymmetry algebra, and the Super-Poincaré group, and start looking for representations of this larger group, inspired by the way of proceeding in the Poincaré case. We will introduce superfields and superspace, a generalization of Minkowski space. We will also introduce the sigma model. In section 4 we will look at the parts needed of complex geometry in order to see the connection to supersymmetry. In section 5 we will make the sigma model supersymmetric, and in section 6 there will the first example on how the restrictions of the geometry in target space arise from the N=(2,2) supersymmetric sigma model. Section 7 treats the GRH-theorem, and in section 8 we look at the basics of GCG. In section 9 and 10 we talk about how to identify geometrical objects from supersymmetry transformations, and how to treat auxiliary fields. In sections 11 and 12 we will realize GKG from the sigma model, and in section 13 there will be a discussion about the results. In the appendix we will look at a special case of the general model treated in section 12, and see simplifications that can be made. In the appendix we will also discuss how the results obtained in section 12 can be understood from a geometrical point of view.

2 Background material

In this thesis we will discuss the connection between supersymmetric sigma models and geometry. Here we present some background material which is needed in order to develop supersymmetry. Of course, much more background material could be included. Also, the treatment of the material I have included is far from rigorous or exhaustive. This section is based on the reference [2].

2.1 Lie groups

Loosely speaking, a Lie Group is a group where the group elements depend smoothly on a set of parameters, ϵ^a, a=1,2 ... They play a central role in physics. For example, if you want to describe rotations, the angle of rotation will be the parameter that the group elements will depend smoothly on. First, we define what a representation of a group is:

A representation of a group is an assignment of each element of the group g, to a linear operator that acts on elements of a linear vector space,

$$g \to D(g) \tag{2.1}$$

such that the identity of the group is mapped to the identity operator, and the group multiplication law is preserved, that is

$$D(e) = 1, \tag{2.2}$$

$$D(g_1)D(g_2) = D(g_1 g_2). \tag{2.3}$$

In a given representation, since the group elements depends smoothly on the parameters, we can expand a group element around the identity, with infinitesimal parameters:

$$D(\epsilon) \approx 1 + i\epsilon_a X^a. \tag{2.4}$$

The same index upstairs and downstairs implies summation over that index. That convention will be used throughout this master thesis. The X^a are called the generators of the group. It can be shown that one can always write a general group element as

$$D(\epsilon) = e^{i\epsilon_a X^a}. \tag{2.5}$$

Since it's a group, we should have

$$e^{i\alpha_a X^a} e^{i\beta_a X^a} = e^{i\delta_a X^a} \tag{2.6}$$

for some δ_a as a function of α_a and β_a. Since the operators do not necessarily commute, in general we have $e^A e^B \neq e^{A+B}$. Therefore δ_a is not simply $\delta_a = \alpha_a + \beta_a$. Instead, one has to use the Baker-Hausdorff formula

$$e^A e^B = e^{(A+B+\frac{1}{2}[A,B]+\ldots)}. \tag{2.7}$$

By using Baker-Hausdorff on the left hand side of (2.6), and then expanding the exponential, it can be shown that a necessary and sufficient condition for it to be true for some $\delta_a = \delta_a(\alpha_a, \beta_a)$ is that the generators fulfill the following identity:

$$\left[X^a, X^b\right] = f_c^{ab} X^c, \tag{2.8}$$

where $[A, B]$ is the commutator between A and B and f_c^{ab} are constants called the structure constants. The above relation between the generators is called the Lie Algebra of the group. The explicit form of the generators depend on the representation used, but the structure constants do not.

2.2 The Poincaré group

An example of a Lie group is the Poincaré group. In four space-time dimensions it contains 10 generators. 4 for translations in space-time, 3 for rotations in space and 3 for boosts. It is sometimes denoted ISO(1,3). Here the SO(1,3)-part is for the rotations in space-time, and the I includes the translations. Its Lie Algebra is given by

$$[P_\mu, P_\nu] = 0 \tag{2.9}$$
$$[P_\mu, J_{\rho\sigma}] = i(\eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho) \tag{2.10}$$
$$[J_{\mu\nu}, J_{\rho\sigma}] = i(\eta_{\nu\rho} J_{\mu\sigma} - \eta_{\mu\rho} J_{\nu\sigma} - \eta_{\nu\sigma} J_{\mu\rho} + \eta_{\mu\sigma} J_{\nu\rho}). \tag{2.11}$$

The translations are generated by P_μ. The rotations and boosts are generated by $J_{\mu\nu}$, which by itself forms the Lorentz group, denoted SO(1,3). $J_{\mu\nu}$ is antisymmetric, so it has six independent components, as it should.

We want to find how the coordinates in Minkowski space transforms under the Poincaré group. We also need to find how functions over Minkowski space transform. To see how a coordinate transforms, we will define Minkowski space as the quotient space (Poincaré group/Lorentz group). The reason to take this complicated approach is that it is easy to generalize to the supersymmetric case. A point x in Minkowski space is then given by

$$h(x) = e^{ix^\mu P_\mu}. \tag{2.12}$$

The action of the Poincaré group on the coordinates of this quotient space is given by left multiplication:

$$h(gx) = h(x') \equiv gh(x)(modSO(1,3)). \tag{2.13}$$

In this thesis, we will never have to "mod out" any Lorentz transformations, but this is the proper way to define it. If $g = e^{(i\eta^\nu P_\nu)}$ is a translation, we can calculate how a translation changes the coordinates:

$$h(x') = e^{(i\eta^\nu P_\nu)} e^{(ix^\mu P_\mu)} = e^{i(\eta^\mu + x^\mu) P_\mu} = h(\eta + x). \tag{2.14}$$

This follows since two different generators of translations commute. Therefore, not too surprising,

$$x' = x + \eta \Rightarrow \delta x = \eta. \tag{2.15}$$

Later on, when we've introduced superspace and superfields, we will generalize this procedure, and then we will have non-vanishing commutators between generators of translations, and therefore a more nontrivial result.

2.3 Field representations of the Poincaré group

A field over Minkowski space is just a function from Minkowski space into some other vector space, $x^\mu \to f(x^\mu)$. When we make a coordinate change, $x^\mu \to x'^\mu$, the fields will change into new fields of the new coordinates, $f(x^\mu) \to f'(x'^\mu)$, in some way. There are different types of fields. For example, a scalar field is defined by the following transformation law:

$$f(x^\mu) = f'(x'^\mu). \tag{2.16}$$

Under an infinitesimal coordinate transformation $x^\mu \to x^\mu + \delta x^\mu$, the scalar field transforms as

$$\delta f(x) \equiv f'(x) - f(x) = f'(x' - \delta x) - f(x) = -\delta x^\mu \partial_\mu f(x), \tag{2.17}$$

where we made a Taylor expansion in the last line. For an infinitesimal transformation, we also have

$$\delta f(x) = i\eta^\mu P_\mu f(x), \tag{2.18}$$

where P_μ is the representation of the generator of translations acting on scalar fields. If we plug in (2.15) into (2.18), and compare to (2.17), we can see that the generators of translation are represented by

$$P_\mu = i\partial_\mu \tag{2.19}$$

when acting on scalar fields.

2.4 Spinors

Another representation of the Lorentz group is carried by so called spinors. In this thesis we will use two component spinors denoted Ψ^α. α will take the two values {+,-}. The following matrices are very useful in spinor space:

$$C_{\alpha\beta} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad C^{\alpha\beta} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{2.20}$$

We can rise and lower spinor indices with them as follows:

$$\Psi_\alpha = \Psi^\beta C_{\beta\alpha} \quad \Psi^\alpha = C^{\alpha\beta}\Psi_\beta. \tag{2.21}$$

One can show, that if Ψ^α transforms with N, Ψ_α will transform with the inverse N^{-1}, so the product $\Psi^\alpha\Psi_\alpha$ will be invariant. In this sense, the $C_{\alpha\beta}$ acts like a metric in spinor space.

3 Supersymmetry

In nature there are two types of particles, bosons and fermions. The main difference between them is that they obey different statistics. The reason for this is that there can only be one fermion per quantum state, whereas for bosons there are no such restrictions. It is possible to extend the Poincaré group to include fermionic generators Q_α. These generators transform bosons into fermions and fermions into bosons. They transform as spinors under the Lorentz group, and therefore carry a spinor index. This new group will be called the Super-Poincaré group. The main references for the material on supersymmetry are [3] and [4].

3.1 Super-Poincaré algebra

The algebra for this new group will look slightly different in different dimensions. In this master thesis we will consider supersymmetry in two dimensions. In addition to (2.9), the Super-Poincaré algebra in two dimensions reads

$$\begin{aligned} [P_\mu, Q_\alpha] &= 0 \\ [J_{\mu\nu}, Q_\alpha] &= ([\rho_\mu, \rho_\nu])_\alpha^{\ \beta} Q_\beta \\ \{Q_\alpha, Q_\beta\} &= \rho^\mu_{\alpha\beta} P_\mu . \end{aligned} \tag{3.1}$$

The last line is the anticommutator between the supersymmetry generators, and ρ_μ satisfies the Clifford algebra $\{\rho^\mu, \rho^\nu\} = 2\eta^{\mu\nu}$. We see that the Q's are invariant under translation, transforms as a spinor under Lorentz-transformations, and that the anticommutator between two Q's is a translation.

Now we need to find a representation of the Super-Poincaré group. For this, we introduce superspace. Superspace is parametrized by the normal commuting variables x^μ, and in addition to those, we have anticommuting variables θ^α. The anticommuting variables are called Grassman variables, and we group them into a two-component spinor

$$\begin{pmatrix} \theta^+ \\ \theta^- \end{pmatrix} . \tag{3.2}$$

Functions on superspace are called superfields. Since the Grassman variables anticommute, $\theta_i\theta_j + \theta_j\theta_i = 0$, it follows that $\theta_i^2 = 0$. Therefore we can expand any superfield in the anticommuting variables, and the expansion will terminate eventually. For example, a function of one single anticommuting variable has the general form

$$f(\theta) = a + b\theta \quad a, b \in R.$$

When we work with Grassman variables, we have to know how to differentiate them. Later we will construct supersymmetric actions of superspace, so we need to know how to integrate them, too.

3.2 Grassman Calculus

Differentiation is defined by

$$\frac{\partial \theta^\alpha}{\partial \theta^\beta} \equiv \partial_\beta \theta^\alpha = \delta^\alpha_\beta. \tag{3.3}$$

We want the integral of a superfield over superspace to resemble the integral $\int_{-\infty}^{\infty}$ of ordinary functions over ordinary coordinates. So we demand the integral to be linear and invariant under translations, that is

$$\int d\theta \sum C_i f_i(\theta) = \sum C_i \int d\theta f_i(\theta) \quad \text{and} \quad \int d\theta f(\theta+\epsilon) = \int d\theta f(\theta). \tag{3.4}$$

Since $f(\theta) = a + b\theta$, these two conditions give us

$$\int d\theta(a + b\theta + b\epsilon) = (a{+}b\epsilon)\int d\theta{+}b\int d\theta\theta = \int d\theta(a + b\theta) = a\int d\theta{+}b\int d\theta\theta. \tag{3.5}$$

Comparing the second line with the last line we see that this can only hold if $b\epsilon \int d\theta = 0$. $b\int d\theta\theta$ on the other hand is arbitrary. This leads to the definition

$$\int d\theta = 0 \quad \text{and} \quad \int d\theta\theta = 1. \tag{3.6}$$

If we have several variables $\theta^1, \theta^2 \dots \theta^n$, we get

$$\int d\theta^j \theta^i = \delta^{ij} \quad \text{and} \quad \int d\theta^i = 0 \tag{3.7}$$

Finally, from these definitions we can see that we have a connection between differentiation and integration:

$$\int d\theta = \frac{\partial}{\partial \theta}. \tag{3.8}$$

This last line will be useful when we start to construct supersymmetric Lagrangians.

3.3 Representations of Supersymmetry

Now we need to find a representation of the supersymmetry translation generators, Q_α, acting on superfields. We will follow the same procedure as we did in the Minkowski space case. We define Superspace as the quotient (Super-Poincaré group/Lorentz group). An element in this space is written

$$h(x, \theta) = e^{i(x^\mu P_\mu + \theta^\alpha Q_\alpha)}. \tag{3.9}$$

For a supersymmetry translation $g = e^{i\epsilon^\alpha Q_\alpha}$ in superspace, we can calculate how the coordinates changes:

$$\begin{aligned} h(x', \theta') = gh(x) &= e^{i\epsilon^\alpha Q_\alpha} e^{i(x^\mu P_\mu + \theta^\beta Q_\beta)} \\ &= e^{i((x^\mu - \frac{i}{2}\epsilon^\alpha \rho^\mu_{\alpha\beta}\theta^\beta)P_\mu + (\epsilon^\alpha + \theta^\alpha)Q_\alpha)} = h(x - \frac{i}{2}\epsilon\rho\theta, \epsilon + \theta), \end{aligned} \tag{3.10}$$

where we used the Baker-Hausdorff formula and the supersymmetry algebra, equations (2.7) and (3.1). From this we see that the coordinates changes as

$$\begin{aligned} \delta\theta^{\alpha} &= \epsilon^{\alpha} \\ \delta x^{\mu} &= -\frac{i}{2}\epsilon^{\alpha}\rho^{\mu}_{\alpha\beta}\theta^{\beta}. \end{aligned} \tag{3.11}$$

A scalar superfield changes as

$$\delta\phi(x,\theta) = \phi'(x'-\delta x, \theta'-\delta\theta) - \phi(x,\theta) = -\delta x^{\mu}\partial_{\mu}\phi - \delta\theta^{\alpha}\partial_{\alpha}\phi. \tag{3.12}$$

In terms of the supersymmetry generator this change is written as

$$\delta\phi = i\epsilon^{\alpha}Q_{\alpha}\phi. \tag{3.13}$$

Plugging in (3.11) in (3.12) and comparing to (3.13), we find that the generators are represented by

$$Q_{\alpha} = i\frac{\partial}{\partial\theta^{\alpha}} + \frac{1}{2}\rho^{\mu}_{\alpha\beta}\theta^{\beta}\partial_{\mu} \tag{3.14}$$

when acting on scalar superfields.

In addition to the supersymmetry generators, we want to have derivatives on superfields. We want these derivatives to be supertranslation-invariant, so they should anticommute with the generators. They are called covariant derivatives, and are found to be

$$D_{\alpha} = \frac{\partial}{\partial\theta^{\alpha}} + \frac{i}{2}\rho^{\mu}_{\alpha\beta}\theta^{\beta}\partial_{\mu} \tag{3.15}$$

The coviariant derivatives obey the same algebra as the supersymmetry generators.

As mentioned above, a superfield can be expanded in the Grassman variables. The coefficients in the expansion are called the component fields, and will be fields over the commuting variables. For example, if we have a function of two Grassman variables, $\phi(x^{\mu}, \theta^{\alpha})$, $\alpha = \{+, -\}$, we will get

$$\phi(x,\theta) = A(x) + \theta^{\alpha}\Psi_{\alpha}(x) + \theta^{\alpha}\theta_{\alpha}F(x).$$

With the use of our covariant derivatives, we can define the component fields by

$$\begin{aligned} A(x) &\equiv \phi(x,\theta)| \\ \Psi_{\pm} &\equiv D_{\pm}\phi(x,\theta)| \\ F(x) &\equiv D_{+}D_{-}\phi(x,\theta)|, \end{aligned} \tag{3.16}$$

where the bar in the end of each line means putting $\theta = 0$ in the expression. This way of writing will be very convenient when start writing actions over superspace. The reason is this relation, which can be seen from (3.8) and (3.15):

$$\int d\theta_{\alpha} = D_{\alpha}|. \tag{3.17}$$

3.4 Sigma Models

A sigma model is a set of maps ϕ^i from a manifold Σ, with coordinates x, to a target manifold T. The map is given by extremizing the action

$$S = \int_\Sigma g_{ij}(\phi)\partial^\mu \phi(x)^i \partial_\mu \phi(x)^j dx. \tag{3.18}$$

where g_{ij} is the metric of the target space.

The equations of motion for a free string in String theory is derived by extremizing the area that the string sweeps out in space-time. The area is written as an integral over the world sheet of a string, and in a certain gauge the action looks like a two-dimensional Sigma model. So Σ will be the world sheet of a string, and T will be space-time.

We can extend Σ to a superspace, to make our sigma model world sheet supersymmetric. When we do that, we will find connections between the amount of supersymmetry on the sigma model, and the geometry on the target manifold. It will turn out that the geometry will be different kinds of complex geometry. We will therefore now have a brief look of what complex geometry is.

4 Complex Geometry

We will here give a brief introduction to complex manifolds. We assume that the basic aspects of differential geometry are known, and refer to [5] for a more complete treatment.

4.1 Complex Manifolds

If M is an even-dimensional differentiable manifold, one can locally introduce complex coordinates. If $\{x^\mu, y^\nu\}$ are the original coordinates,

$$\begin{aligned} z &= x + iy \\ \bar{z} &= x - iy \end{aligned} \tag{4.1}$$

can be taken as the complex coordinates. The target space and dual space will be spanned by the following bases

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} \\ \frac{\partial}{\partial \bar{z}} &= \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \\ dz &= dx + idy \\ d\bar{z} &= dx - idy. \end{aligned} \tag{4.2}$$

The basis vectors of the tangent space are called holomorphic and anti-holomorphic vectors. If the transition functions between different patches with complex coordinates are holomorphic, M is called a complex manifold. On a complex manifold one can globally define a tensor J that maps the target space to itself. It acts on the basis vectors as

$$\begin{aligned} J\frac{\partial}{\partial z} &= i\frac{\partial}{\partial z} \\ J\frac{\partial}{\partial \bar{z}} &= -i\frac{\partial}{\partial \bar{z}} \end{aligned} \tag{4.3}$$

It follows that $J^2 = -1$. The tensor J can be used to construct a projection operator:

$$P_\pm = \frac{1}{2}(1 \mp iJ) \tag{4.4}$$

It will project out the holomorphic and anti-holomorphic part of a vector, and the tangent space is split into one holomorphic and one anti-holomorphic part.

Next we will look at the case when we have a manifold M with a tensor J which fulfills $J^2 = -1$. What does it take for M to be a complex manifold? The reason we do this, is that in the sigma models further on, this is the case we will find.

M is called an *almost* complex manifold if it has a tensor that fulfills $J^2 = -1$. Like above we can define projection operators, and the tangent space will be split into on holomorphic and one anti-holomorphic part, but not necessarily with the basis above. If the Lie bracket of two holomorphic vectors is again holomorphic, J is called integrable. This will be the case if

$$P_- [P_+X, P_+Y] = 0 \quad \forall X, Y \in T_pM^+ \tag{4.5}$$

where T_pM^+ denotes the holomorphic part of the tangent space. The above equation is equivalent to the vanishing of the so called Nijenhuis tensor, which in coordinate form is defined as (The bracket [,] around indices means anti-symmetrization with respect to those indices.)

$$N^\rho_{\mu\nu} = J^\rho_\lambda \partial_{[\nu} J^\lambda_{\mu]} + \partial_\lambda J^\rho_{[\nu} J^\lambda_{\mu]} \tag{4.6}$$

This can be seen by plugging in (4.4) in (4.5) and expanding. If the Nijenhuis tensor vanishes M is a complex manifold. That is, one can choose complex coordinates, and the transition functions between different such patches will be holomorphic.

4.2 Kähler Manifolds

In this section we will introduce some definitions that in the end will allow us to define what we mean by a Kähler manifold. It will be those manifolds that will arise later.

If M is a complex manifold, and the metric tensor obeys

$$J^i_k g_{ij} J^j_l = g_{kl} \tag{4.7}$$

then g is called a hermitian metric, and M is called an hermitian manifold.

From the definition of a hermitian metric, it follows that on a hermitian manifold, J is antisymmetric in the following way:

$$\begin{aligned} J^i_k g_{ij} J^j_l &= g_{kl} = g_{lk} \Rightarrow \\ J^i_k g_{ij} J^j_l J^l_m &= J^l_m g_{lk} \Rightarrow \\ -J^i_k g_{ij} \delta^j_m &= -J^i_k g_{im} = J^l_m g_{lk} \end{aligned} \tag{4.8}$$

Therefore, on an hermitian manifold, we can define a two-form, the **Kähler form**:

$$\Omega = J^k_i g_{kj} dx^i \wedge dx^j \tag{4.9}$$

In complex coordinates, the Kähler form is

$$\Omega = 2i g_{\mu\bar{\nu}} dz^\mu \wedge d\bar{z}^{\bar{\nu}}. \tag{4.10}$$

We will use bars over the anti-holomorphic indices. If the Kähler form is closed, $d\Omega = 0$, then the hermitian manifold is called a **Kähler manifold**. For a Kähler manifold, the metric will obey

$$g_{\mu\bar{\nu},\lambda} = g_{\lambda\bar{\nu},\mu} \qquad g_{\mu\bar{\nu},\bar{\lambda}} = g_{\mu\bar{\lambda},\bar{\nu}} \tag{4.11}$$

The comma means differentiation with respect to the variable after the comma. This can be shown from the definition $d\Omega = 0$. If

$$g_{\mu\bar{\nu}} = \partial_\mu \partial_{\bar{\nu}} K \tag{4.12}$$

for some function K, (4.11) will be satisfied. It can be shown that the converse is also true, any Kähler metric can be expressed as (4.12), locally. K is called the *Kähler potential*.

Finally it can be shown that a hermitian manifold is a Kähler manifold if and only if

$$\nabla_\mu J = 0 \tag{4.13}$$

where ∇_μ is the Levi-Civita connection associated with the metric g.

5 Supersymmetric Sigma Models

In this section we will make the sigma model supersymmetric. The model will then contain both bosons and fermions, and they are transformed into each other under a supersymmetry transformation.

5.1 Supersymmetry on the world sheet

We can extend the string world sheet to a superspace by introducing anti-commuting variables, and letting the fields living on the world sheet become superfields:

$$\phi(\sigma,\tau) \to \phi(\sigma,\tau,\theta^+,\theta^-). \tag{5.1}$$

The supersymmetric generalization of the sigma model given above, equation (3.18), is

$$\begin{aligned} S &= \int d^2x d^2\theta g_{ij}(\phi) D^\alpha \phi^i(x,\theta) D_\alpha \phi^j(x,\theta) \\ &= 2i \int d^2x d^2\theta g_{ij}(\phi) D_+ \phi^i(x,\theta) D_- \phi^j(x,\theta). \end{aligned} \tag{5.2}$$

We can expand the superfield living on the world sheet in the anti-commuting variables:

$$\phi(x,\theta) = A(x) + \theta^\alpha \Psi_\alpha(x) + \theta_\alpha \theta^\alpha F(x). \tag{5.3}$$

We can plug this expansion into (5.2) and then integrate out the Grassman variables. The equations of motion derived from the resulting expression will show that F(x) is an auxiliary field, its field equation will give $F=0$. It can also be shown that A(x) describes a boson, and $\Psi(x)$ a fermion.

Now we want to find out what happens to the components of a superfield under a superfield transformation. With the supersymmetry generators given by

$$Q_\alpha = i\frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}\rho^\mu_{\alpha\beta}\theta^\beta\partial_\mu \tag{5.4}$$

we find

$$\begin{gathered} \delta(\epsilon)\phi \equiv i\epsilon^\alpha Q_\alpha \phi \equiv \delta A(x) + \theta^\alpha \delta\Psi_\alpha + \theta^\alpha\theta_\alpha \delta F(x) = \\ i\epsilon^\alpha \left(i\frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}\rho^\mu_{\alpha\beta}\theta^\beta\partial_\mu \right)\left(A(x) + \theta^\alpha\Psi_\alpha(x) + \theta^\alpha\theta_\alpha F(x)\right). \end{gathered} \tag{5.5}$$

We identify what is in front of each power of θ to find that the components transforms as

$$\delta A(x) = -\epsilon^\alpha \Psi_\alpha, \tag{5.6}$$

$$\delta\Psi_\alpha = \frac{i}{2}\epsilon^\alpha \rho^\mu_{\alpha\beta}\partial_\mu A(x) - \epsilon_\alpha F(x), \tag{5.7}$$

$$\delta F(x) = \epsilon^\alpha \rho^\mu_{\alpha\beta}\partial_\mu \Psi^\beta(x). \tag{5.8}$$

We see that the bosonic field A(x) is transformed into a fermionic field, and vice versa. When performing the integral, only the $\theta^\alpha\theta_\alpha$ component of a superfield will be non-vanishing, because of (3.7). Since $\delta F(x)$ is a total derivative under a supersymmetry transformation, we see that an action written in terms of superfields will be invariant under a supersymmetry. This can also be seen by exploiting the relation

$$\int d\theta = D| \tag{5.9}$$

and observing that $D| = -iQ|$ (c.f. equations (3.14) and (3.15)). Namely, if we have an arbitrary function of superfields, $L(\phi)$, under a supersymmetry it transforms as

$$\delta S = \int d^2x d^2\theta \delta L(\phi) = \int d^2x D^2 \epsilon^\alpha Q_\alpha L(\phi)| = -\int d^2x D^2 i\epsilon^\alpha D_\alpha L(\phi)| \doteq 0 \tag{5.10}$$

where the last equality means '0 up to total derivatives' and follows from the algebra of the covariant derivatives, especially from the fact that $D_\pm^2 \propto \rho^\mu \partial_\mu$.

5.2 The algebra in light-cone coordiantes

We now make a specific choice of the matrices ρ^μ:

$$\begin{aligned} \left(\rho^0\right)_\beta^{\ \alpha} &= \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \\ \left(\rho^1\right)_\beta^{\ \alpha} &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \end{aligned} \tag{5.11}$$

Plugging this matrices into the anticommutator between the Q's in (3.1) gives us

$$\begin{aligned} \{Q_\pm, Q_\pm\} &= 2Q_\pm^2 = \left(\rho^\mu\right)_\pm^{\ \gamma} C_{\gamma\pm} P_\mu = \partial_0 \pm \partial_1 \\ \{Q_+, Q_-\} &= \left(\rho^\mu\right)_+^{\ \gamma} C_{\gamma -} P_\mu = 0 \end{aligned} \tag{5.12}$$

where $C_{\alpha\beta}$ is the spinor space metric, equation (2.20).

If we make a change of coordinates to so called light-cone coordinates,

$$\begin{aligned} \xi_{+\!\!+} &= \sigma + \tau \\ \xi_{=} &= \sigma - \tau \end{aligned} \tag{5.13}$$

we get a very nice look of the algebra for the N=(1,1) and supersymmetry generators:

$$Q_+^2 = i\partial_{+\!\!+}, \quad Q_-^2 = i\partial_{=}, \quad \{Q_+, Q_-\} = 0 \tag{5.14}$$

where $\partial_{\substack{+\!\!+\\=}} = \frac{1}{2}\left(\partial_0 \pm \partial_1\right)$. These coordinates will be used in what follows.

6 The first example of the connection between supersymmetry and complex geometry

In the last section, we constructed a N=(1,1) supersymmetric sigma model. Now we want to impose more supersymmetry on the sigma model, and make it N=(2,2) supersymmetric. We will do this by writing a manifestly N=(2,2) supersymmetric action, which upon reduction to its N=(1,1) form again looks like (5.2). However, we will see that now the metric is not arbitrary anymore, it must be Kähler.

6.1 A Sigma Model with chiral fields

When we want to write a manifestly N=(2,2) supersymmetric action, we will need four Grassman variables, and four covariant derivatives. From two independent N=(1,1) subalgebras, we can construct complex N=(2,2) covariant derivatives:

$$\begin{aligned} \mathbb{D}_\alpha &= \frac{1}{2}(D^1_\alpha + iD^2_\alpha) \\ \bar{\mathbb{D}}_\alpha &= \frac{1}{2}(D^1_\alpha - iD^2_\alpha). \end{aligned} \tag{6.1}$$

In the N=(1,1) sigma model, equation (5.2), we only want N=(1,1) *scalar* superfields to be present. One kind of superfields which will provide us with this are *chiral superfields*. A chiral superfield is defined by

$$\bar{\mathbb{D}}_\pm \phi = 0. \tag{6.2}$$

The complex conjugate of a chiral superfield is called an anti-chiral superfield, and obeys the complex conjugate of the above. We will see that when we reduce a manifestly N=(2,2) action with chiral superfields, only the scalar part will survive.

6.2 The reduction to its N=(1,1) form

The most general manifestly N=(2,2) supersymmetric action of chiral and anti-chiral fields is written as

$$S = \int d^2\xi d^2\theta d^2\bar{\theta} K(\phi^a, \bar{\phi}^{\bar{a}}). \tag{6.3}$$

where $K(\phi^a, \bar{\phi}^{\bar{a}})$ is a real valued function, and the index $\{a\}$ label different fields. We want to reduce this to a manifest N=(1,1) action, and compare to (5.2). Going backwards, using (6.1), we can define N=(1,1) covariant derivatives and supersymmetry generators:

$$D_\alpha = \mathbb{D}_\alpha + \bar{\mathbb{D}}_\alpha \tag{6.4}$$

$$Q_\alpha = i\left(\mathbb{D}_\alpha - \bar{\mathbb{D}}_\alpha\right). \tag{6.5}$$

Let ϕ be our N=(2,2) chiral field. We can expand ϕ in the second set of Grassman variables, and denote the lowest component in the expansion by X, that is

$$\phi^a| = X^a. \tag{6.6}$$

X^a is a N=(1,1) scalar superfield. As usual, when integrating the Grassman variables, we will use the connection between integration and differentiation, $\int d\theta = D|$. In this case, we can also use equation (6.1) to show that $\int d^2\theta d^2\bar{\theta} = \mathbb{D}^2\bar{\mathbb{D}}^2| = -\frac{i}{2}D^2Q_+Q_-|$. Therefore we need to know how Q_α acts on X^a. Using (6.4), (6.5) and (6.2) we find that

$$\begin{aligned} Q_\pm X^a &= iD_\pm X^a \\ Q_\pm \bar{X}^{\bar{a}} &= -iD_\pm \bar{X}^{\bar{a}}. \end{aligned} \tag{6.7}$$

Now we have everything we need to reduce (6.3) to its N=(1,1) form:

$$\begin{aligned} S &= \int d^2\xi d^2\theta d^2\bar{\theta} K(\phi^a, \bar{\phi}^{\bar{a}}) = \int d^2\xi \mathbb{D}^2\bar{\mathbb{D}}^2 K(\phi^a, \bar{\phi}^{\bar{a}})| = -\frac{i}{2}\int d^2\xi D^2 Q_+Q_- K(\phi^a, \bar{\phi}^{\bar{a}})| \\ &= -\frac{i}{2}\int d^2\xi D^2 \left(K_{a\bar{b}} D_+X^a D_-\bar{X}^{\bar{b}} + K_{\bar{a}b} D_+\bar{X}^{\bar{a}} D_- X^b\right), \end{aligned} \tag{6.8}$$

where $K_a \equiv \frac{\partial K}{\partial X^a}$, and so on. As expected, only the scalar field in the ϕ-expansion survived. Let us group X^a and $\bar{X}^{\bar{a}}$ into a vector X^A, with the collective index $\{A\} = \{a, \bar{a}\}$. Then we can write the result as

$$S = \int d^2\xi d^2\theta d^2\bar{\theta} K(\phi^a, \bar{\phi}^{\bar{a}}) = -\frac{i}{2}\int d^2\xi D^2 \left(g_{AB} D_+X^A D_- X^B\right). \tag{6.9}$$

Where the matrix g_{AB} is

$$g_{AB} = \begin{pmatrix} 0 & K_{a\bar{b}} \\ K_{\bar{a}b} & 0 \end{pmatrix}. \tag{6.10}$$

Comparing to (5.2) and keeping (4.12) in mind, we see that the target space of the N=(2,2) sigma model has to be Kähler. The Lagrangian in the original sigma model serves as the Kähler potential.

7 The Gates, Hull and Roček theorem

In the last section, we started with a manifestly N=(2,2) supersymmetric sigma model. Reduceding it to N=(1,1)-form, we found that the target space geometry has to be Kähler. In this section, we will take the opposite path. We will start with a manifestly N=(1,1) model. Then we will investigate what must be fulfilled if we demand that the model has a second supersymmetry. The action we will use is

$$S = \int d^2\xi d^2\theta D_+\phi^\mu(x,\theta)E_{\mu\nu}(\phi)D_-\phi^\nu(x,\theta). \tag{7.1}$$

where $E_{\mu\nu} = g_{\mu\nu} + B_{\mu\nu}$. Now we have included a antisymmetric background field $B_{\mu\nu} = -B_{\nu\mu}$. This model is manifestly N=(1,1) supersymmetric, since it is written in terms of N=(1,1) superfields. The second supersymmetry has to act on the fields as

$$\delta^2(\epsilon)\phi^\mu = \epsilon^+ D_+\phi^\nu J^\mu_{(+)\nu} + \epsilon^- D_-\phi^\nu J^\mu_{(-)\nu}, \tag{7.2}$$

where $J^\mu_{(\pm)\nu}$ are two tensor fields in the target space. This ansatz is unique for dimensional reasons.

We now demand two things. Firstly, the second supersymmetry must obey the supersymmetry algebra, (5.14). That is, the commutator of two supersymmetry transformations must close to a translation:

$$[\delta_2(\epsilon_1), \delta_2(\epsilon_2)] = \epsilon_1^\alpha\epsilon_2^\beta\{Q_\alpha, Q_\beta\} = 2i\epsilon_1^+\epsilon_2^+\partial_{++} + 2i\epsilon_1^-\epsilon_2^-\partial_{=}. \tag{7.3}$$

Secondly, the action (7.1) should be invariant under (7.2).

Performing the calculations, we find the following conditions: (For details, we refer to [6].)

- The action is invariant under the second supersymmetry provided that:

$$J^\mu_{(\pm)\rho}g_{\mu\nu} = -g_{\mu\rho}J^\mu_{(\pm)\nu} \tag{7.4}$$

$$\nabla^\pm_\rho J^\mu_{(\pm)\nu} = J^\mu_{(\pm)\nu,\rho} + \Gamma^{\pm\mu}_{\rho\sigma}J^\sigma_{(\pm)\nu} - \Gamma^{\pm\sigma}_{\rho\nu}J^\mu_{(\pm)\sigma} = 0, \tag{7.5}$$

  where $\Gamma^{\pm\mu}_{\rho\nu}$ is defined by

$$\Gamma^{\pm\mu}_{\rho\nu} = \Gamma^\mu_{\rho\nu} \pm g^{\mu\sigma}H_{\sigma\rho\nu}. \tag{7.6}$$

  $\Gamma^\mu_{\rho\nu}$ are the Christoffel symbols formed by the metric $g_{\mu\nu}(\phi)$, and $H_{\sigma\rho\nu}$ is the torsion, determined by the field $B_{\mu\nu}(\phi)$:

$$H_{\sigma\rho\nu} = \frac{1}{2}\left(B_{\mu\rho,\sigma} + B_{\rho\sigma,\mu} + B_{\sigma\mu,\rho}\right). \tag{7.7}$$

- The supersymmetry algebra is fulfilled provided that the tensors $J_{(\pm)}$ obey

$$J^\mu_{(\pm)\nu}J^\rho_{(\pm)\mu} = -\delta^\rho_\nu \tag{7.8}$$

$$N^\rho_{\mu\nu} = J^\rho_{(\pm)\lambda}\partial_{[\nu}J^\lambda_{(\pm)\mu]} + \partial_\lambda J^\rho_{(\pm)[\nu}J^\lambda_{(\pm)\mu]} = 0. \tag{7.9}$$

This will be refered to as the 'Gates, Hull and Roček theorem'(GHR-theorem).

What do all these conditions mean? Keeping the section about Complex Geometry in mind, equation (7.8) means that the two tensors $J_\pm$ are complex structures. Equation (7.9) shows that these two complex structures are integrable, so the target manifold is a Complex Manifold. Further, comparing (7.4) and (4.8), we see that the metric has to be hermitian with the respect to both complex structures. Finally, we see from (7.5) that $J_\pm$ has to be constant with respect to a torsionful connection. The geometry with the objects $(g, J_\pm, H)$ fulfilling these conditions is called a *bi-hermitian* geometry

However, there is one more complication. Only if the two complex structures commute, closure of the algebra is achieved off-shell. If the two complex structures do not commute, we have to use the field equations derived from the action to get the algebra to close. That is unwanted, since then this result only works for this particular model, and we want it to be as general as possible. In order to get off-shell closure of the algebra, one way out is to introduce auxiliary fields. We will see examples below of how this can be done.

(This general result agrees with the special case treated in the last section. There we found that what corresponds to E_{ij} is symmetric, so $B_{ij} \equiv 0$. The extra term in the Christoffel symbols, eq. (7.6), therefore vanishes. Then (4.13) is fulfilled. Also, $J_+ = J_-$, so we are not forced to use the field equations in order to get the algebra to close.)

A recent generalization of complex geometry, called generalized complex geometry, naturally contains the bi-hermitian geometry in its framework. In generalized complex geometry, one considers maps from $T \oplus T^*$ to itself. If we introduce auxiliary fields that transform in T^*, we can hope to realize generalized complex geometry in the target space directly from the sigma model. We will now introduce the basic aspects of generalized complex geometry.

8 Generalized Complex Geometry

In complex geometry, we have a (1,1)-tensor which squares to -1. This tensor is usually denoted by J, and is called an almost complex structure. If this tensor obeys certain integrability conditions, the manifold is a complex manifold. Generalized complex geometry was introduced in [7] and developed in [8]. In Generalized complex geometry, one considers maps from the sum of the tangent space and its dual space, $T \oplus T^*$, to itself. We will see below that generalized complex geometry contains ordinary complex geometry as a special case. This summary of the basics aspects of this new geometry follows very closely the paper [10].

8.1 Basics of Generalized Complex Geometry

Let us write an element in $T \oplus T^*$ as $X + \xi$, where $X \in T$ and $\xi \in T^*$. On $T \oplus T^*$, we have a natural pairing of the elements:

$$\langle X + \xi, Y + \eta \rangle = i_X \eta + i_Y \xi. \tag{8.1}$$

where $i_X \eta$ is the interior product. Given a coordinate basis (∂_μ, dx^μ), the interior product between X and η is simply $X^\mu \eta_\mu$. This pairing defines a metric $\mathcal{I}$ on $T \oplus T^*$, which in the given coordinate basis is:

$$\mathcal{I} = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix}. \tag{8.2}$$

A generalized almost complex structure $\mathcal{J}$ is a map

$$\mathcal{J} : T \oplus T^* \to T \oplus T^* \tag{8.3}$$

which obeys $\mathcal{J}^2 = -1$, and $\mathcal{I}$ has to be hermitian with respect to $\mathcal{J}$, that is:

$$\mathcal{J}^t \mathcal{I} \mathcal{J} = \mathcal{I}. \tag{8.4}$$

Similarly to ordinary complex geometry, $\mathcal{J}$ has eigenvalues $\pm i$, and $T \oplus T^*$ splits into two pieces, determined by the eigenvalue of the elements. The generalized almost complex structure can be used to define projection operators onto these two pieces: $\Pi_\pm = \frac{1}{2}(1 \mp i\mathcal{J})$. As before, $\mathcal{J}$ is called integrable if the bracket between two elements in the +i part of $T \oplus T^*$ is still in the +i part. The difference is that it is not the Lie Bracket this time, it is something called the Courant Bracket[1], defined by

$$[X + \xi, Y + \eta]_C = [X, Y] + \mathcal{L}_X \eta + \mathcal{L}_Y \xi - \frac{1}{2} d\left(i_X \eta - i_Y \xi\right). \tag{8.5}$$

The first bracket is the normal Lie Bracket, the $\mathcal{L}$'s denote the Lie derivative, and d is the exterior derivative. The generalized almost complex structure is integrable if

$$\Pi_\mp \left[\Pi_\pm (X + \xi), \Pi_\pm (Y + \eta)\right]_C = 0. \tag{8.6}$$

[1] The Courant Bracket does not in general fulfil the Jacobi identity. However, if there is a subspace $L \subset T \oplus T^*$ which is closed under the Courant Bracket and isotropic with the respect to the natural pairing, the Courant Bracket does satisfy the Jacobi identity. This is a reason to demand hermicity of the natural pairing, equation (8.4).

We see that if we only consider the tangent space, everything above, apart from the condition of hermicity of $\mathcal{I}$, reduces to the description of ordinary complex geometry. Finally, in the coordinate basis (∂_μ, dx^μ), the generalized almost complex structure $\mathcal{J}$ can be written as

$$\mathcal{J} = \begin{pmatrix} J & P \\ L & K \end{pmatrix}, \tag{8.7}$$

where $J : T \to T$, $P : T^* \to T$, $L : T \to T^*$, $K : T^* \to T^*$ are the tensor fields J^μ_ν, $P^{\mu\nu}$, $L_{\mu\nu}$ and K^ν_μ. In terms of these tensors, the condition $\mathcal{J}^2 = -1$ can be rewritten as

$$\begin{aligned} J^2 + PL &= -1 \\ JP + PK &= 0 \\ LP + K^2 &= -1 \\ KL + LJ &= 0 \end{aligned} \tag{8.8}$$

and the condition about hermicity of the metric with the respect to the generalized complex structure, $\mathcal{J}^t I \mathcal{J} = I$, can be written as

$$\begin{aligned} J &= -K^t \\ P &= -P^t \\ L &= -L^t. \end{aligned} \tag{8.9}$$

8.2 Examples of Generalized Complex Geometry

The ordinary complex geometry is included in the generalized complex geometry. This is because if we have a complex structure, we can form a generalized complex structure by:

$$\mathcal{J}_J = \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix}. \tag{8.10}$$

This object fulfills (8.3), (8.4) and (8.6) as long as J is a complex structure.

Another example is a symplectic structure. A symplectic structure is a two-form ω which fulfills $d\omega = 0$. A Kähler manifold has both a complex structure and a symplectic structure. In this case

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}. \tag{8.11}$$

is a generalized complex structure.

8.3 Generalized Kähler Geometry

As mentioned above, if we have a Kähler manifold, we can form two generalized complex structures, $\mathcal{J}_\omega$ and $\mathcal{J}_J$. Using equations (4.7) and (4.10), it is not hard to show that these two commute. Also, the product of the two will be

$$-\mathcal{J}_J \mathcal{J}_\omega = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \equiv \mathcal{G}, \tag{8.12}$$

which is positive definite.

Inspired by these results, Gualteri [8] made the following definition:

A generalized complex manifold is said to be generalized Kähler if there are two commuting generalized complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$, such that $\mathcal{G} \equiv -\mathcal{J}_1\mathcal{J}_2$ is a positive definite metric on $T \oplus T^*$.

It is shown in [8] that there is a map between the bi-hermitian geometry discussed above, and generalized Kähler geometry. Namely, the geometrical objects on a bi-hermitian manifold, $(J_\pm, g, B)$, defines a generalized Kähler geometry with generalized complex structures

$$\mathcal{J}_{1,2} = -\frac{1}{2}\begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}\begin{pmatrix} J_+ \pm J_- & -\left(\omega_+^{-1} \mp \omega_-^{-1}\right) \\ \omega_+ \mp \omega_- & -\left(J_+^t \pm J_-^t\right) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}, \quad (8.13)$$

where $\omega_\pm = gJ_\pm$.

Remark:The defintion of generalized Kähler geometry is inspired by the Kähler geometry, in which the metric is always Riemannian. Therefore the generalized Kähler metric is defined to be positive definite. In bi-hermitian geometry the metric is not necessarily positive definite, it can have different signatures. The definition of generalized Kähler geometry should therefore be extended to metrics with arbitrary signature.

We now want to start looking for sigma models, which realizes generalize Complex Geometry in the target space. Doing so, we must in some way introduce fields which lie in T^*.

9 How to read off the complex structures

In the next sections we will have explicit expressions for the second supersymmetry that the fields in the action undergo. From those we will read off geometrical objects, which will turn out to be complex structures and generalized complex structures. But which are the guiding principles when reading off the geometrical objects? We will for example come across the case where we have the set of fields $\{X, \Psi_\alpha\}$. Here X are bosonic and serves as coordinates, and Ψ_α are fermionic fields, which transforms in the tangent space. The second supersymmetry transformation will then always have the form

$$\delta \begin{pmatrix} X \\ \Psi_\alpha \end{pmatrix} = \epsilon \mathcal{M} \mathcal{D} \begin{pmatrix} X \\ \Psi_\alpha \end{pmatrix}. \tag{9.1}$$

ϵ is the transformations parameter. The matrix $\mathcal{D}$ is a matrix of covariant derivatives, which is needed in order to

- Make sure that bosonic fields are transformed into fermionic fields, and vice versa.
- Make sure that the Lorentz transformations properties are the same on both sides of the equation.

Since the covariant derivatives are odd objects, for example $D_+\Psi_-$ will be bosonic. The insertion of the derivatives makes all the objects on the right hand side transform in the tangent space, and $\mathcal{M}$ is a tensor

$$\mathcal{M} : T \to T \tag{9.2}$$

In the previous section, when we treated chiral fields, we saw an example of this structure, equation (6.7). In the notation introduced here, equation (6.7) would be written as

$$\delta^{\pm} \begin{pmatrix} X^a \\ \bar{X}^{\bar{a}} \end{pmatrix} = \epsilon^{\pm} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} D_\pm & 0 \\ 0 & D_\pm \end{pmatrix} \begin{pmatrix} X^a \\ \bar{X}^{\bar{a}} \end{pmatrix}. \tag{9.3}$$

If we instead want to realize generalized complex geometry, we will need fields transforming in the co-tangent space. If $\{S_\alpha\}$ are such fields, the second supersymmetry transformation will be of the form

$$\delta \begin{pmatrix} X \\ S_\alpha \end{pmatrix} = \epsilon \mathcal{M} \mathcal{D} \begin{pmatrix} X \\ S_\alpha \end{pmatrix}, \tag{9.4}$$

where now $\mathcal{M}$ is a map

$$\mathcal{M} : T \oplus T^* \to T \oplus T^*. \tag{9.5}$$

In the examples we will encounter, we will see that this $\mathcal{M}$ turns out to be a generalized complex structure.

10 Auxiliary fields

As seen in the GHR-theorem, we will in general need auxiliary fields in the action in order to get the supersymmetry algebra to close off-shell. In order to realize generalized complex geometry in the target space directly from the sigma model, we would like these auxiliary fields to transform in the co-tangent space.

10.1 Ambiguity in choosing the auxiliary fields

Below, we will introduce so called semi-chiral N=(2,2) superfields, introduced in [11]. When reduced to a N=(1,1) action, the semi-chiral fields will leave us with both bosonic and fermionic N=(1,1) fields. The bosonic ones will be our coordinates in the target space. The fermionic fields will be transforming in the tangent space under a change of coordinates, and their field equations will turn out to be algebraic. This means that they are auxiliary, using their field equations we can solve for them in terms of the bosonic fields.

But instead of using their field equations to eliminate them, we will redefine them into fields transforming in the co-tangent space. We will then use their supersymmetry transformations in order to identify transformations $T \oplus T^* \to T \oplus T^*$. However, there is an ambiguity in the redefinitions. We will investigate two natural redefinitions, and both will lead to objects which fulfill everything that is needed in order to obtain the status as generalized complex structures, with one difference: one of them will not fulfill equation (8.4), that is $\mathcal{J}^t I \mathcal{J} = I$, whereas the other one will.

The reason we are interested in these investigations is that if we integrate out the auxiliary fields, we know from the GHR-theorem that the second supersymmetry transformation will provide us with two complex structures. We also know that we can form generalized complex structures with the aid of the Gualteri map, equation (8.13). However, that map is pretty complicated, and we would like to find a shorter way to introduce generalized complex geometry directly from the sigma model, with the help of auxiliary fields.

10.2 Going from N=(1,1) to N=(2,2) with auxiliary fields

We will here take a little detour before beginning the big investigations. In the next sections, we will start with a manifestly N=(2,2) supersymmetric sigma model. We will then reduce it to its N=(1,1) form in order say something about the geometry in the target space, and especially hope to identify generalized complex geometry. Another approach would have been to do as with did arriving to the GHR-theorem. We could start with a manifestly N=(1,1) supersymmetric sigma model, and included fields transforming in the co-tangent space. These fields must be auxiliary, since we do not want any new physical degrees of freedom. We could then make an ansatz of the most general second supersymmetry transformation that these fields can undergo, and see if we find any restrictions to the geometry in the target space. However, this approach is difficult, for at least two different reasons.

Firstly there is not a unique way to introduce auxiliary fields in the action. For example, both

$$S = \int d^2\xi d^2\theta \left(S_{+i} E^{ij} S_{-j} + S_{(+i} D_{-)} \phi^i \right) \tag{10.1}$$

and

$$S = \int d^2\xi d^2\theta \left(D_+ \phi^i E_{ij} D_- \phi^j + S_{+i} A^{ij} S_{-j} \right) \tag{10.2}$$

gives (7.1) back when $S_\pm$ are integrated out. In the first action, E^{ij} is the inverse of E_{ij}, and in the second action, A^{ij} is an arbitrary invertible matrix.

Secondly, if we choose one of the possible extensions of (7.1), and we want to see what it takes for it to have a second supersymmetry, we have to make an ansatz for the second supersymmetry. When we have the auxiliary spinor fields at hand, the most general form of the second supersymmetry reads ([14]):

$$\begin{aligned}
\delta^{(\pm)} \phi^\mu &= \epsilon^\pm \left(D_\pm \phi^\nu J_\nu^{(\pm)\mu} + S_{\pm\mu} P^{(\pm)\mu\nu} \right) \\
\delta^{(\pm)} S_{\pm\mu} &= \epsilon^\pm \Big(i\partial_{\substack{+ \\ =}} \phi^\nu L_{\mu\nu}^{(\pm)} - D_\pm S_{\pm\nu} K_\mu^{(\pm)\nu} + S_{\pm\nu} S_{\pm\sigma} N_\mu^{(\pm)\nu\sigma} \\
&\quad + D_\pm \phi^\nu D_\pm \phi^\rho M_{\mu\nu\rho}^{(\pm)} + D_\pm \phi^\nu S_{\pm\sigma} Q_{\mu\nu}^{(\pm)\sigma} \Big) \\
\delta^{(\pm)} S_{\mp\mu} &= \epsilon^\pm \Big(D_\pm S_{\mp\nu} R_\mu^{(\pm)\nu} + D_\mp S_{\pm\nu} Z_\mu^{(\pm)\nu} + D_\pm D_\mp \phi^\nu T_{\mu\nu}^{(\pm)} \\
&\quad + S_{\pm\rho} D_\mp \phi^\nu U_{\mu\nu}^{(\pm)\rho} + D_\pm \phi^\nu S_{\mp\rho} V_{\mu\nu}^{(\pm)\rho} \\
&\quad + D_\pm \phi^\nu D_\mp \phi^\rho X_{\mu\nu\rho}^{(\pm)} + S_{\pm\nu} S_{\mp\rho} Y_\mu^{(\pm\nu\rho)} \Big).
\end{aligned} \tag{10.3}$$

The $(\pm)$-superscript on the objects on the right hand side label different tensors, and are *not* spinor indices. This ansatz is more general than the supersymmetry transformations we discussed in the last section. For example $S_\alpha S_\beta$ is bosonic, and ϕ can be transformed into that object. These transformations are derived in the following way: First you identify the dimension of the different ingredients, namely the fields, the transformation parameters and the covariant derivatives. On the right hand side we take the different possible combinations matching the dimension of the left hand side. We also have to match the transformation properties under the rest of the Super Poincaré group, that is matching the spinor indices.

These transformations should be compared to (7.2). The introduction of auxiliary fields complicate things considerably. If one tries the same program as before, namely demand closure of the algebra and invariance of the action under this second supersymmetry, one will end up with more than a hundred differential and algebraic equations that must be fulfilled, see for example [10]. The question of what is the geometry of the target space in this very general case has not been finally resolved yet.

So, instead of starting with a N=(1,1) model with auxiliary fields, and demanding it to have a second supersymmetry, we will start with a manifestly N=(2,2) supersymmetric model. We will then reduce it to its N=(1,1) form, and in this way try to identify the geometry, similarly as was done in the first example.

11 Generalized Kähler Geometry from a topological model

In this section we will start from a manifestly N=(2,2) supersymmetric action, and reduce it to its N=(1,1) form. After the reduction, will try to identify the geometry. The way of proceeding will be similar to the case when we found the ordinary Kähler geometry in the target manifold. We will use so called semi-chiral fields, which will be defined below. With these fields, we will have auxiliary fields left after the reduction.

11.1 Topological model

We will start with a topological model, that is, a model with no dynamics. The reason we do this is for the calculation to be as simple as possible, but still we get an interesting result. We will use the same notation as above, with the N=(2,2) spinor derivatives $\mathbb{D}_\alpha$ and $\bar{\mathbb{D}}_\alpha$. From them we form the N=(1,1) derivatives and the non-manifest supersymmetry generators as

$$\begin{aligned} D_\alpha &= \mathbb{D}_\alpha + \bar{\mathbb{D}}_\alpha \\ Q_\alpha &= i\left(\mathbb{D}_\alpha - \bar{\mathbb{D}}_\alpha\right). \end{aligned} \tag{11.1}$$

We now define complex left and right semi-chiral fields as fields that obey

$$\begin{aligned} \bar{\mathbb{D}}_+ \mathbb{X}_L &= 0 \\ \mathbb{D}_- \mathbb{X}_R &= 0. \end{aligned} \tag{11.2}$$

These semi-chiral fields will after the reduction leave us with (auxiliary) spinor N=(1,1) fields, together with the scalar fields. From the semi-chiral fields we define the N=(1,1) superfields X_L, X_R, Ψ_{L-} and Ψ_{R+} as

$$\begin{aligned} \mathbb{X}_L| &= X_L \\ \mathbb{X}_R| &= X_R \\ Q_- \mathbb{X}_L| &= \Psi_{L-} \\ Q_+ \mathbb{X}_R| &= \Psi_{R+} \end{aligned} \tag{11.3}$$

where the bar means independent of the second pair of $\theta' s$.

In our topological model we will use left semi-chiral fields only, so the action is

$$S = \int d^2\xi d^2\theta d^2\bar{\theta} K(\mathbb{X}_L, \bar{\mathbb{X}}_L). \tag{11.4}$$

When reducing this action to its N=(1,1) form, we need to know how the Q's act on the fields. This we find from (11.1) and (11.2). For example (from now on we suppress the label of the semi-chiral fields, since we are only considering the left going ones)

$$\begin{aligned} & iD_+ - Q_+ = 2i\bar{\mathbb{D}}_+ \Longrightarrow (iD_+ - Q_+)\mathbb{X} = 2i\bar{\mathbb{D}}_+\mathbb{X} = 0 \\ & \Longrightarrow Q_+\mathbb{X}| = Q_+ X = iD_+ X. \end{aligned} \tag{11.5}$$

Performing the same exercise with the other fields we find

$$\begin{aligned}
Q_+X &= iD_+X \\
Q_-X &= \Psi_- \\
Q_+\Psi_- &= iD_+\Psi_- \\
Q_-\Psi_- &= -i\partial_= X \\
Q_+\bar{X} &= -iD_+\bar{X} \\
Q_-\bar{X} &= \bar{\Psi}_- \\
Q_+\bar{\Psi}_- &= -iD_+\bar{\Psi}_- \\
Q_-\bar{\Psi}_- &= -i\partial_= \bar{X}.
\end{aligned} \tag{11.6}$$

Integrating out the second set of $\theta' s$, we find the action to be

$$\begin{aligned}
S = &\int d^2\xi d^2\theta d^2\bar{\theta} K(\mathbb{X},\bar{\mathbb{X}}) = \int d^2\xi \mathbb{D}^2\bar{\mathbb{D}}^2 K(\mathbb{X},\bar{\mathbb{X}})| = -\frac{i}{2}\int d^2\xi D^2 Q_+Q_- K(\mathbb{X},\bar{\mathbb{X}})| \\
&= -\frac{i}{2}\int d^2\xi D^2 Q_+\left(K_a\Psi^a_- + K_{\bar{a}}\bar{\Psi}^{\bar{a}}_-\right)| \\
&= -\frac{i}{2}\int d^2\xi D^2 \Big(iK_{ab}D_+X^b\Psi^a_- - iK_{a\bar{b}}D_+\bar{X}^{\bar{b}}\Psi^a_- \\
&+ iK_aD_+\Psi^a_- + iK_{\bar{a}b}D_+X^b\bar{\Psi}^{\bar{a}}_- - K_{\bar{a}\bar{b}}D_+\bar{X}^{\bar{b}}\bar{\Psi}^{\bar{a}}_- - iK_{\bar{a}}D_+\bar{\Psi}^{\bar{a}}_-\Big) \\
&= -\frac{i}{2}\int d^2\xi D^2 \left(2iK_{a\bar{b}}D_+X^a\bar{\Psi}^{\bar{b}}_- - 2iK_{\bar{a}b}D_+\bar{X}^{\bar{a}}\Psi^b_-\right),
\end{aligned} \tag{11.7}$$

where we in the last step has performed partial integration on the K_i-terms, and as usual assumed that the fields vanishes on the boundary. By defining the index $\{A\} = \{a, \bar{a}\}$, and the matrix

$$\omega_{AB} = \begin{pmatrix} 0 & 2iK_{a\bar{b}} \\ -2iK_{\bar{a}b} & 0 \end{pmatrix} \tag{11.8}$$

we can write the result in a more compact way as

$$S = -\frac{i}{2}\int d^2\xi D^2 \left(D_+X^A\omega_{AB}\Psi^B_-\right). \tag{11.9}$$

ω_{AB} is a symplectic form. As a final touch, we define

$$S_{A-} = \omega_{AB}\Psi^B_- \tag{11.10}$$

and we get

$$S = -\frac{i}{2}\int d^2\xi D^2 \left(D_+X^A S_{A-}\right) \tag{11.11}$$

S_{A-} is a field that lies in T^*, so by looking at the supersymmetry transformations of X^A and S_{A-}, we hope to identify a generalized complex geometry in the target manifold.

11.2 Identifying the geometry

Now we have a model with fields in $T \oplus T^*$, $D_+ X^A$ and S_{A-}. What we will do now is to calculate how X^A and S_{A-} transforms under the second supersymmetry, which is generated by (11.1). In these transformations, we will identify different tensors, and we will show that these tensors are the ones you need in order to form a generalized complex structure. In this subsection, the calculations will follow the ones made in [1]. In the next subsection, we will modify the way of obtaining the supersymmetry transformations of the auxiliary fields, and this modified method will give us a simpler calculation.

Equation (11.6) can be written in a more compact way with our new index A:

$$Q_+ X^A = J^A_B D_+ X^B \qquad Q_+ \Psi^A_- = J^A_B D_+ \Psi^B_- \tag{11.12}$$

$$Q_- X^A = \Psi^A_- \qquad Q_- \Psi^A_- = -i\partial_= X^A \tag{11.13}$$

where

$$J^A_B = \begin{pmatrix} i\delta^a_b & 0 \\ 0 & -i\delta^a_b \end{pmatrix} \tag{11.14}$$

is a complex structure. With equations (11.12), (11.13) and (11.10), we can calculate how X^A and S_{A-} transforms under (11.1):

$$\begin{aligned}
\delta^{(+)} X^A &= \epsilon^+ J^A_B D_+ X^B \\
\delta^{(-)} X^A &= \epsilon^- Q_- X^A = \epsilon^- \Psi^A_- = \epsilon^- \omega^{AB} S_{B-} \\
\delta^{(+)} S_{A-} &= \delta^{(+)} \left(\omega_{AB} \Psi^B_-\right) = \delta^{(+)} \left(\omega_{AB}\right) \Psi^B_- + \omega_{AB} \delta^{(+)} \left(\Psi^B_-\right) \\
&= \partial_E \left(\omega_{AB}\right) \delta^+ \left(X^E\right) \Psi^B_- + \epsilon^+ \omega_{AB} J^B_C D_+ \Psi^C_- \\
&= \partial_E \left(\omega_{AB}\right) \delta^+ \left(X^E\right) \Psi^B_- + \epsilon^+ D_+ \left(\omega_{AB} J^B_C \Psi^C_-\right) - \epsilon^+ \partial_E \left(\omega_{AB}\right) D_+ \left(X^E\right) J^B_C \Psi^C_- \\
&= -\epsilon^+ D_+ S_{B-} J^B_A - \epsilon^+ \partial_E \left(\omega_{AB}\right) D_+ \left(X^E\right) J^B_C \omega^{CD} S_{D-} + \partial_E \left(\omega_{AB}\right) \delta^+ \left(X^E\right) \omega^{BD} S_{D-} \\
&= -\epsilon^+ D_+ S_{B-} J^B_A + \epsilon^+ \partial_E \left(\omega^{CD}\right) D_+ \left(X^E\right) J^B_C \omega_{AB} S_{D-} - \partial_E \left(\omega^{BD}\right) \delta^+ \left(X^E\right) \omega_{AB} S_{D-} \\
&= -\epsilon^+ D_+ S_{B-} J^B_A + \omega_{AB} \left(\epsilon^+ J^B_C D_+ \left(X^E\right) - \delta^B_C \delta^{(+)} \left(X^E\right)\right) \partial_E \left(\omega^{CD}\right) S_{D-} \\
\delta^{(-)} S_{D-} &= \delta^{(-)} \left(\omega_{AB} \Psi^B_-\right) \\
&= \partial_E \left(\omega_{AB}\right) \delta^{(-)} \left(X^E\right) \Psi^B_- + \omega_{AB} \delta^{(-)} \left(\Psi^B_-\right) \\
&= -i\epsilon^- \omega_{AB} \partial_= X^B + \partial_E \left(\omega_{AB}\right) \delta^{(-)} \left(X^E\right) \omega^{BD} S_{D-}.
\end{aligned} \tag{11.15}$$

This is a special case of the general case, equation (10.3). Typically, when trying to plug in the general ansatz in a N=(1,1) model, and see what it takes for it to admit N=(2,2) supersymmetry, the higher index tensors in (10.3) turn out to be derivatives of the two-index tensors. Therefore, we will here identify the two-index tensors, and hope that they gives us the generalized complex structures. In the (+)-transformations we have the tensors

$$\begin{aligned} J &: T \to T \\ J^t &: T^* \to T^* \end{aligned} \tag{11.16}$$

and we identify the map $\mathcal{J}_J : T \oplus T^* \to T \oplus T^*$

$$\mathcal{J}_J = \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix}. \tag{11.17}$$

From the (-)-transformation we have the two maps

$$\begin{aligned} \omega &: T \to T^* \\ \omega^{-1} &: T^* \to T \end{aligned} \tag{11.18}$$

which gives a second map $\mathcal{J}_\omega : T \oplus T^* \to T \oplus T^*$

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{pmatrix}. \tag{11.19}$$

From the section about generalized complex geometry, we know that these two objects are indeed generalized complex structures. Actually, a manifold with these two general complex strucures is generalized Kähler. So we have seen that in our topological model, the target manifold has to be generalized Kähler.

11.3 Another way to obtain the result

Instead of making the redefintion (11.10), we can keep the two fields, D_+X^A and Ψ_-^A, living in one tangent space each. We can then formally take two copies of the tangent space, and identify the maps

$$\mathcal{M}_\pm : T_1 \oplus T_2 \to T_1 \oplus T_2$$

from the non-manifest supersymmetry transformation that the fields undergo. In the notation introduced in equation (9.1), we find the non-manifest transformation from equations (11.12) and (11.13) to be

$$\begin{aligned} \delta^+ \begin{pmatrix} X \\ \Psi_- \end{pmatrix} &= \epsilon^+ \mathcal{M}_{(+)} \mathcal{D}_{(+)} \begin{pmatrix} X \\ \Psi_- \end{pmatrix} \\ \delta^- \begin{pmatrix} X \\ \Psi_- \end{pmatrix} &= \epsilon^- \mathcal{M}_{(-)} \mathcal{D}_{(-)} \begin{pmatrix} X \\ \Psi_- \end{pmatrix} \end{aligned} \tag{11.20}$$

where the matrices $\mathcal{D}_{(\pm)}$ and $\mathcal{M}_{(\pm)}$ are given by

$$\mathcal{D}_{(+)} = \begin{pmatrix} D_+ & 0 \\ 0 & D_+ \end{pmatrix} \qquad \mathcal{M}_{(+)} = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \tag{11.21}$$

$$\mathcal{D}_{(-)} = \begin{pmatrix} D_-^2 & 0 \\ 0 & I \end{pmatrix} \qquad \mathcal{M}_{(-)} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{11.22}$$

I denotes the identity matrix. What we seek for is a map from $T \oplus T^*$ to itself. The action (11.9) provides us with a map from T to T^*, namely

ω_{AB}, given by eq. (11.8). So we can map our fields (D_+X, Ψ_-) living in $T_1 \oplus T_2$, to fields $(D_+X, \omega\Psi_-)$ living in $T \oplus T^*$ by the map $\hat{\omega} : T_1 \oplus T_2 \to T \oplus T^*$ given in matrix form by

$$\hat{\omega} = \begin{pmatrix} I & 0 \\ 0 & \omega \end{pmatrix}. \tag{11.23}$$

Combining the maps $\mathcal{M}_{(\pm)}$ and $\hat{\omega}$, we get two maps $\mathcal{J}_\pm : T \oplus T^* \to T \oplus T^*$ given by

$$\begin{aligned}
\mathcal{J}_+ = \hat{\omega}\mathcal{M}_{(+)}\hat{\omega}^{-1} &= \begin{pmatrix} I & 0 \\ 0 & \omega \end{pmatrix}\begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & \omega^{-1} \end{pmatrix} = \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix} \\
\mathcal{J}_- = \hat{\omega}\mathcal{M}_{(-)}\hat{\omega}^{-1} &= \begin{pmatrix} I & 0 \\ 0 & \omega \end{pmatrix}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & \omega^{-1} \end{pmatrix} = \begin{pmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{pmatrix},
\end{aligned} \tag{11.24}$$

where we used the property $\omega J = -J^t\omega$, which is true because of the special form of the matrices ω, equation (11.8). We see that this result agrees with the one in the previous section, and the calculations are simpler. In the sections to come, we will use this method when treating the auxiliary fields.

12 Generalized Kähler Geometry from the sigma model

In this section we will treat the full N=(2,2) action with both left and right semi-chiral fields. After the reduction to its N=(1,1) form, we will have auxiliary fields left. In order to find generalized complex geometry in the target space, we will redefine the auxiliary fields into fields that transform in T^*. This can be done in different ways, and we will find different generalized complex structures. No matter how we redefine the auxiliary fields, when we integrate them out, we will end up with the good old N=(1,1) supersymmetric action (7.1). We will again follow [1], and in addition to what is done in that paper, we will try a new redefintion of the auxiliary fields.

When we have both left and right semi-chiral fields, the action reads

$$S = \int d^2\xi d^2\theta d^2\bar{\theta} K(\mathbb{X}_L, \bar{\mathbb{X}}_L, \mathbb{X}_R, \bar{\mathbb{X}}_R). \tag{12.1}$$

When reducing this action to its N=(1,1) form, the calculations will proceed as in the previous section.

When integrating out the second pair of θ's, we need to know how $Q_\pm$ acts on the different fields. This we find from equations (11.1), (11.2) and (11.3). Using these definitions we find

$$\begin{aligned}
Q_+ X_L^A &= J^A_B D_+ X_L^B \\
Q_+ X_R^{A'} &= \Psi_{R+}^{A'} \\
Q_+ \Psi_{L-}^A &= J^A_B D^+ \Psi_{L-}^B \\
Q_+ \Psi_{R+}^{A'} &= -\partial_{++} X_R \\
Q_- X_L^A &= \Psi_{L-}^A \\
Q_- X_R^{A'} &= J^{A'}_{B'} D_- X_R^{B'} \\
Q_- \Psi_{R+}^{A'} &= J^{A'}_{B'} D_+ \Psi_{R+}^{B'} \\
Q_- \Psi_{R+}^{A'} &= J^{A'}_{B'} D_- \Psi_{R+}^{B'},
\end{aligned} \tag{12.2}$$

where J^A_B and $J^{A'}_{B'}$ are complex structures of the form

$$J^A_B = \begin{pmatrix} i\delta^a_b & 0 \\ 0 & -i\delta^{\bar{a}}_{\bar{b}} \end{pmatrix}. \tag{12.3}$$

We find the N=(1,1) form of the action to be

$$
\begin{aligned}
S = &\int d^2\xi d^2\theta d^2\bar{\theta} K(\mathbb{X}_L, \bar{\mathbb{X}}_L, \mathbb{X}_R, \bar{\mathbb{X}}_R) = -\frac{i}{2}\int d^2\xi D^2 Q_+ Q_- K(\mathbb{X}_L, \bar{\mathbb{X}}_L, \mathbb{X}_R, \bar{\mathbb{X}}_R)| = \\
&= -\frac{i}{2}\int d^2\xi D^2 \Big(D_+ X_L^A m_{AB'} D_- X_R^{B'} + \Psi_{R+}^{A'} n_{A'B} \Psi_{L-}^B \\
&+ \Psi_{L-}^A (2i\omega_{AB}^L D_+ X_L^B + i p_{AB'} D_+ X_R^{B'}) \\
&- \Psi_{R+}^{A'} (2i\omega_{A'B'}^R D_- X_R^{B'} + i q_{A'B} D_- X_L^B) \Big),
\end{aligned}
\tag{12.4}
$$

where the matrices are given by

$$
m_{AB'} = \begin{pmatrix} -K_{ab'} & K_{a\bar{b}'} \\ K_{\bar{a}b'} & -K_{\bar{a}\bar{b}'} \end{pmatrix} \qquad n_{A'B} = \begin{pmatrix} K_{a'b} & K_{a'\bar{b}} \\ K_{\bar{a}'b} & K_{\bar{a}'\bar{b}} \end{pmatrix} \tag{12.5}
$$

$$
p_{AB'} = \begin{pmatrix} K_{ab'} & K_{a\bar{b}'} \\ -K_{\bar{a}b'} & -K_{\bar{a}\bar{b}'} \end{pmatrix} \qquad q_{A'B} = \begin{pmatrix} K_{a'b} & K_{a'\bar{b}} \\ -K_{\bar{a}'b} & -K_{\bar{a}'\bar{b}} \end{pmatrix} \tag{12.6}
$$

$$
\omega_{AB}^L = \begin{pmatrix} 0 & K_{a\bar{b}} \\ -K_{\bar{a}b} & 0 \end{pmatrix} \qquad \omega_{A'B'}^R = \begin{pmatrix} 0 & K_{a'\bar{b}'} \\ -K_{\bar{a}'b'} & 0 \end{pmatrix}. \tag{12.7}
$$

12.1 Two different ways of obtaining Generalized Complex Structures

We will now discuss two different ways of obtaining generalized complex structures in the target space. The first approach will be to redefine the auxiliary fields into fields transforming in T^*, with the redefinition inspired by the field equations that one finds for the auxiliary fields. Then we will look at the second supersymmetry transformation that the fields undergo, and from that read off maps from $T \oplus T^*$ to itself. We will verify that these maps fulfil everything needed in order to be generalized complex structures, except that the natural paring is not hermitian with respect to them. This first approach is the one pursued in [1].

The second approach will be similar to the first one, but this time the redefinition of the fields will be inspired by the one we did for the topological model, equation (11.10). We will see that this redefinition gives us generalized complex structures which preserve the natural pairing, and the expressions will be much simpler.

12.2 Generalized Complex Geometry in the target space, version 1

The field equations obtained by varying the action are

$$
\begin{aligned}
n_{A'B}\Psi_{L-}^B - \Big(2i\omega_{A'B'}^R D_- X_R^{B'} + i q_{A'B} D_- X_L^B\Big) = 0 \\
\Psi_{R+}^{A'} n_{A'B} - \Big(2i\omega_{AB}^L D_+ X_L^B + i p_{AB'} D_+ X_R^{B'}\Big) = 0.
\end{aligned}
\tag{12.8}
$$

These equations are algebraic, and very easy to solve. We find solutions to be

$$
\begin{aligned}
\Psi^A_{L-} &= u^{AA'}\left(2i\omega^R_{A'B'}D_-X^{B'}_R + iq_{A'B}D_-X^B_L\right)\\
\Psi^{A'}_{R+} &= u^{AA'}\left(2i\omega^L_{AB}D_+X^B_L + ip_{AB'}D_+X^{B'}_R\right),
\end{aligned}
\tag{12.9}
$$

where $u^{AA'}$ is the inverse of $n_{A'A}$.

Inspired by these equations, we define the fields $S_\pm$ by

$$
\begin{aligned}
u^{AA'}S_{A'-} &\equiv \Psi^A_{L-} - u^{AA'}\left(2i\omega^R_{A'B'}D_+X^{B'}_R + iq_{A'B}D_-X^B_L\right)\\
u^{AA'}S_{A+} &\equiv \Psi^{A'}_{R+} - u^{AA'}\left(2i\omega^L_{AB}D_+X^B_L + ip_{AB'}D_+X^{B'}_R\right).
\end{aligned}
\tag{12.10}
$$

If we insert this in the action, equation (12.4), we find

$$
\begin{aligned}
S = -\frac{i}{2}\int d^2\xi D^2\Big(&-2D_+X^A_L\omega^L_{AB}u^{BB'}q_{B'C}D_-X^C_L\\
&D_+X^A_L\left(m_{AA'} - 4\omega^L_{AB}u^{BB'}\omega^R_{B'A'}\right)D_-X^{A'}_R\\
&+D_+X^{A'}_R p_{AA'}u^{AB'}q_{B'B}D_-X^B_L + 2D_+X^{A'}_R p_{AA'}u^{AB'}\omega^R_{B'C'}D_-X^{C'}_R + S_{A+}u^{AB'}S_{B'-}\Big)\\
&\equiv -\frac{i}{2}\int d^2xD^2\left(D_+\phi^t\mathbb{E}D_-\phi + S^t_+\mathbb{U}S_-\right).
\end{aligned}
\tag{12.11}
$$

In the last step switched to matrix notation, and introduced the matrices

$$\phi = \begin{pmatrix} X^A_L \\ X^{A'}_R \end{pmatrix} \tag{12.12}$$

$$S_+ = \begin{pmatrix} S_{A+} \\ 0 \end{pmatrix} \tag{12.13}$$

$$S_- = \begin{pmatrix} 0 \\ S_{A'-} \end{pmatrix} \tag{12.14}$$

$$\mathbb{E} = \begin{pmatrix} -2\omega^L uq & m - 4\omega^L u\omega^R \\ p^t uq & 2p^t u\omega^L \end{pmatrix} \qquad \mathbb{U} = \begin{pmatrix} 0 & u^{AB'} \\ 0 & 0 \end{pmatrix} \tag{12.15}$$

$\mathbb{E}$ is the sum of the metric and the B-field, and we see that the action has the form (10.2). If the fields $S_\pm$ are integrated out, we will get back the original sigma model (7.1).

12.2.1 The Generalized Complex Structures

We will here calculate how the fields $\{X_L, X_R, S_-, S_+\}$ transforms under the second supersymmetry, which is generated by $Q_\pm$. If we group the original fields $\{X_L, X_R, \Psi_{L-}, \Psi_{R+}\}$ into a column vector

$$\begin{pmatrix} X_L \\ X_R \\ \Psi_{L-} \\ \Psi_{R+} \end{pmatrix} \tag{12.16}$$

we find from equation (12.2) that the second supersymmetry acts on the fields as

$$\begin{aligned}
\delta^{+}\begin{pmatrix} X_L \\ X_R \\ \Psi_{L-} \\ \Psi_{R+} \end{pmatrix} &= \epsilon^{+}\mathcal{M}_{(+)}\mathcal{D}_{(+)}\begin{pmatrix} X_L \\ X_R \\ \Psi_{L-} \\ \Psi_{R+} \end{pmatrix} \\
\delta^{-}\begin{pmatrix} X_L \\ X_R \\ \Psi_{L-} \\ \Psi_{R+} \end{pmatrix} &= \epsilon^{-}\mathcal{M}_{(-)}\mathcal{D}_{(-)}\begin{pmatrix} X_L \\ X_R \\ \Psi_{L-} \\ \Psi_{R+} \end{pmatrix}
\end{aligned} \tag{12.17}$$

where the matrices $\mathcal{D}_{(\pm)}$ and $\mathcal{M}_{(\pm)}$ are given by

$$\mathcal{D}_{(+)} = \begin{pmatrix} D_+ & 0 & 0 & 0 \\ 0 & D_+^2 & 0 & 0 \\ 0 & 0 & D_+ & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \quad \mathcal{M}_{(+)} = \begin{pmatrix} J & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & J & 0 \\ 0 & -I & 0 & 0 \end{pmatrix} \tag{12.18}$$

$$\mathcal{D}_{(-)} = \begin{pmatrix} D_-^2 & 0 & 0 & 0 \\ 0 & D_- & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & D_- \end{pmatrix} \quad \mathcal{M}_{(-)} = \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & J & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & J \end{pmatrix}. \tag{12.19}$$

Here $\mathcal{M}_{(\pm)}$ are maps $\mathcal{M}_{(\pm)} : T_1 \oplus T_2 \to T_1 \oplus T_2$. With the redefinition (12.10), we map the fields $\Psi_\pm$ to the cotangent space. The map is given by

$$C = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ -2i\omega^L & -ip & 0 & n^t \\ -iq & -2i\omega^R & n & 0 \end{pmatrix} \tag{12.20}$$

The inverse of C is given by

$$C^{-1} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ iuq & 2iu\omega^R & 0 & u \\ 2iu^t\omega^L & iu^tp & u^t & 0 \end{pmatrix}. \tag{12.21}$$

With this transformation, we get the two maps $\mathcal{J}_\pm : T \oplus T^* \to T \oplus T^*$ by:

$$\begin{aligned}
\mathcal{J}_+ &= CM_{(+)}C^{-1} \\
&= \begin{pmatrix} J & 0 & 0 & 0 \\ 2iu^t\omega^L & iu^tp & u^t & 0 \\ 2(pu^t\omega^L - i\omega^L J) & (pu^tp - n^t) & -ipu^t & 0 \\ (inJuq - iqJ + 4\omega^R u^t\omega^L) & 2(\omega^R u^t p + nJu\omega^R) & -2i\omega^R u^t & nJu \end{pmatrix}. \\
\mathcal{J}_- &= CM_{(-)}C^{-1} \\
&= \begin{pmatrix} iuq & 2iu\omega^R & 0 & u \\ 0 & J & 0 & 0 \\ 2(\omega^L uq + in^tJu^t\omega^L) & (-ipJ + 4\omega^L u\omega^R + in^tJu^tp) & n^tJu^t & -2i\omega^L u \\ quq - n & 2(quq\omega^R - i\omega^R J) & 0 & -iqu \end{pmatrix}
\end{aligned} \tag{12.22}$$

These are the two maps found in [1], obtained with the new method. They fulfil $\mathcal{J}^2 = -1$, and they also commute, $[\mathcal{J}_+, \mathcal{J}_-] = 0$ (see [1]). In [1] it is also shown that their product gives a metric on $T \oplus T^*$. It is easy to see that the top right corners of the two generalized complex does not fulfil equation (8.9). Therefore, the natural pairing, equation (8.2), is not hermitian with respect to these objects, that is

$$\mathcal{J}_\pm^t \mathcal{I} \mathcal{J}_\pm \neq \mathcal{I}. \tag{12.23}$$

Actually, some simplifications can be made in equation (12.22). We have the following connections between the matrices in (12.5):

$$\begin{aligned} p &= -iJn^t \\ q &= -iJn. \end{aligned} \tag{12.24}$$

So we find

$$\begin{aligned} pu^t\omega^L - i\omega^L J = -iJn^t u^t \omega^L - i\omega^L J = i\omega^L J - i\omega^L J = 0 \\ pu^t p - n^t = (-iJn^t)u^t(-iJn^t) - n^t = n^t - n^t = 0 \\ quq - n = (-iJn)u(-iJn) - n = n - n = 0 \\ qu\omega^R - i\omega^R J = (-iJn)u\omega^R + iJ\omega^R = -iJ\omega^R + iJ\omega^R = 0. \end{aligned} \tag{12.25}$$

This inserted in (12.22) gives us

$$\begin{aligned} \mathcal{J}_+ &= \begin{pmatrix} J & 0 & 0 & 0 \\ 2iu^t\omega^L & iu^t p & u^t & 0 \\ 0 & 0 & -ipu^t & 0 \\ (inJuq - iqJ + 4\omega^R u^t \omega^L) & 2(\omega^R u^t p + nJu\omega^R) & -2i\omega^R u^t & nJu \end{pmatrix} \\ \mathcal{J}_- &= \begin{pmatrix} iuq & 2iu\omega^R & 0 & u \\ 0 & J & 0 & 0 \\ 2(\omega^L uq + in^t Ju^t \omega^L) & (-ipJ + 4\omega^L u\omega^R + in^t Ju^t p) & n^t Ju^t & -2i\omega^L u \\ 0 & 0 & 0 & -iqu \end{pmatrix}. \end{aligned} \tag{12.26}$$

12.3 Generalized Complex Geometry in the target space, version 2

In this section we will make the same redefinition of the auxiliary fields as we did for the topological model. After this is done, we will identify two Generalized Complex Structures, and this time they will not only be less complicated, but also will the natural pairing $\mathcal{I}$ be hermitian.

We start by making the redefinitions of the auxiliary fields Ψ into fields that transform in the co-tangent space:

$$\begin{aligned} S_{A-} &\equiv \omega^L_{AB}\Psi^B_{L-} \\ S_{A'+} &\equiv \omega^R_{A'B'}\Psi^{B'}_{R+} \end{aligned} \tag{12.27}$$

which in matrix form looks like

$$\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & \omega^L & 0 \\ 0 & 0 & 0 & \omega^R \end{pmatrix} \tag{12.28}$$

If we carry along $\mathcal{M}_{(\pm)}$ with these redefinitions, we get two transformations $T \oplus T^*$ to itself:

$$\mathcal{J}_+ = \hat{\omega}\mathcal{M}_{(+)}\hat{\omega}^{-1} = \begin{pmatrix} J & 0 & 0 & 0 \\ 0 & 0 & 0 & (\omega^R)^{-1} \\ 0 & 0 & -J^t & 0 \\ 0 & -\omega^R & 0 & 0 \end{pmatrix} \tag{12.29}$$

and

$$\mathcal{J}_- = \hat{\omega}\mathcal{M}_{(-)}\hat{\omega}^{-1} = \begin{pmatrix} 0 & 0 & (\omega^L)^{-1} & 0 \\ 0 & J & 0 & 0 \\ -\omega_L & 0 & 0 & 0 \\ 0 & 0 & 0 & -J^t \end{pmatrix}. \tag{12.30}$$

That these objects fulfil conditions $\mathcal{J}_\pm^2 = -1$ and $\mathcal{J}_\pm^t \mathcal{I} \mathcal{J}_\pm = \mathcal{I}$ is easy to check. They are also integrable, which can be checked using equation (8.6) (Equation (8.6) is written in coordinate form in [10], and with those coordinate relations integrability follows quite easily. Integrability for the structures obtained in the previous section, equation (12.26), we have not checked, since they are much more complicated) They are therefore generalized complex structures.

They also commute, and their product is

$$\mathcal{G} = -\mathcal{J}_+\mathcal{J}_- = \begin{pmatrix} 0 & 0 & -J(\omega^L)^{-1} & 0 \\ 0 & 0 & 0 & (\omega^R)^{-1}J^t \\ -J^t\omega^L & 0 & 0 & 0 \\ 0 & \omega^R J & 0 & 0 \end{pmatrix}. \tag{12.31}$$

From equation (12.7), we see that

$$\begin{aligned} -J^t\omega^L &= i\begin{pmatrix} 0 & K_{a\bar{b}} \\ K_{\bar{a}b} & 0 \end{pmatrix} \equiv g^L \\ \omega^R J &= i\begin{pmatrix} 0 & K_{a'\bar{b}'} \\ K_{\bar{a}'b'} & 0 \end{pmatrix} \equiv g^R, \end{aligned} \tag{12.32}$$

where we in the last steps denoted the matrices by $g^{L/R}$, because of the obvious similarities with a Kähler metric. With these names , the product between the two generalized complex structures can be written as

$$\mathcal{G} = \begin{pmatrix} 0 & 0 & (g^L)^{-1} & 0 \\ 0 & 0 & 0 & (g^R)^{-1} \\ g^L & 0 & 0 & 0 \\ 0 & g^R & 0 & 0 \end{pmatrix}. \tag{12.33}$$

This again looks like the generalized Kähler metric one forms from the data provided from a Kähler manifold. However, the metric in the target space of the sigma model is identified to be the symmetric part of equation (12.15), which involves $g^{L/R}$ in a complicated manner.

To make sure that everything is consistent, we must check that we get the original action, equation (7.1), when the auxiliary fields are integrated out.

With the redefinitions (12.27), our action (12.4) now reads

$$\begin{aligned} S = -\frac{i}{2}\int d^2\xi D^2\Big(& D_+X_L^A m_{AB'}D_-X_R^{B'} + \omega_R^{A'B'}S_{B'+}n_{A'A}\omega_L^{AB}S_{B-} \\ & \omega_L^{AB}S_{B-}(2i\omega^L_{AC}D_+X_L^C + ip_{AB'}D_+X_R^{B'}) \\ & -\omega_R^{A'B'}S_{B'+}(2i\omega^R_{A'C'}D_-X_R^{C'} + iq_{A'B}D_-X_L^B)\Big), \end{aligned} \tag{12.34}$$

where $\omega_L^{AB}, \omega_R^{A'B'}$ are the inverses of $\omega^L_{AB}, \omega^R_{A'B'}$.

The field equations for $S_\pm$ are

$$\begin{aligned} \omega_R^{A'B'}S_{B'+}n_{A'A}\omega_L^{AB} - (2i\omega^L_{AC}D_+X_L^C + ip_{AB'}D_+X_R^{B'})\omega_L^{AB} &= 0 \\ \omega_R^{A'B'}n_{A'A}\omega_L^{AB}S_{B-} - \omega_R^{A'B'}(2i\omega^R_{A'C'}D_-X_R^{C'} + iq_{A'B}D_-X_L^B) &= 0 \end{aligned} \tag{12.35}$$

with the solutions

$$\begin{aligned} S_{B'+} &= \omega^R_{B'A'}(2i\omega^L_{AC}D_+X_L^C + ip_{AC'}D_+X_R^{C'})u^{AA'} \\ S_{B-} &= \omega^L_{BA}u^{AA'}(2i\omega^R_{A'C'}D_-X_R^{C'} + iq_{A'C}D_-X_L^C). \end{aligned} \tag{12.36}$$

Plugging this into (12.34), we get

$$\begin{aligned} S = -\frac{i}{2}\int d^2\xi D^2\Big(& D_+X_L^A m_{AA'}D_-X_R^{A'} \\ & +(2i\omega^L_{AC}D_+X_L^C + ip_{AC'}D_+X_R^{C'})u^{AA'}(2i\omega^R_{A'B'}D_-X_R^{B'} + iq_{A'B}D_-X_L^B) \\ & -(2i\omega^L_{AC}D_+X_L^C + ip_{AC'}D_+X_R^{C'})u^{AA'}(2i\omega^R_{A'B'}D_-X_R^{B'} + iq_{A'B}D_-X_L^B) \\ & -(2i\omega^L_{AC}D_+X_L^C + ip_{AC'}D_+X_R^{C'})u^{AA'}(2i\omega^R_{A'B'}D_-X_R^{B'} + iq_{A'B}D_-X_L^B)\Big) \\ = -\frac{i}{2}\int d^2\xi D^2\Big(& -2D_+X_L^A\omega^L_{AB}u^{BB'}q_{B'C}D_-X_L^C \\ & D_+X_L^A\left(m_{AA'} - 4\omega^L_{AB}u^{BB'}\omega^R_{B'A'}\right)D_-X_R^{A'} \\ & +D_+X_R^{A'}p_{AA'}u^{AB'}q_{B'B}D_-X_L^B + 2D_+X_R^{A'}p_{AA'}u^{AB'}\omega^R_{B'C'}D_-X_R^{C'}\Big). \end{aligned} \tag{12.37}$$

Everything works out as it should, and we end up with the same action as before, equation (12.11) (except for the auxiliary fields).

12.4 Possible modifications

We saw above that we can get generalized complex structures which preserves the natural pairing from the sigma model with semi-chiral fields. But, the auxiliary fields occurs in the action (12.34) in a complicated way, and their field equations are non-trivial. It would be preferred to introduce auxiliary fields in a manner similar to what is done in (12.11). Then we can read off the complex structures in the top-left corner of the generalized complex structures, since the field equations for the auxiliary fields are $S_\pm = 0$. But with the redefinitions made in equation (12.10), the generalized complex structures do not preserve the natural pairing. It is therefore natural to seek for yet another redefinition of the auxiliary fields, such that the generalized complex structures preserve the natural pairing. After the redefinition we would still like to have the generalized complex structures in the form

$$\mathcal{J}_\pm = \begin{pmatrix} J_\pm & * \\ * & * \end{pmatrix}. \tag{12.38}$$

Achieving this may also take us one step closer to understand the Gualteri-map directly from the sigma model, because of the structure of the map, equation (8.13).

With the objects we have at hand, the most general redefinition of the auxiliary fields which preserves the top left corner is

$$\begin{aligned} \tilde{S}_{A'+} &= \Omega^{B'}_{A'(+)} S_{B'+} \\ \tilde{S}_{A-} &= \Omega^{B}_{A(-)} S_{B-}. \end{aligned} \tag{12.39}$$

The plan is to calculate how the generalized complex structures obtained in equation (12.22) changes under the above field transformations, and then choose $\Omega_\pm$ such that the transformed generalized complex structures preserve the natural pairing. This seems to be hard to do. For example, $\mathcal{J}_+$ would transform into

$$\tilde{\mathcal{J}} = \begin{pmatrix} J & 0 & 0 & 0 \\ 2iu^t\omega^L & iu^tp & u^t\Omega^{-1}_{(+)} & 0 \\ 0 & 0 & -i\Omega_{(+)}pu^t\Omega^{-1}_{(+)} & 0 \\ \Omega_{(-)}(inJuq - iqJ + 4\omega^R u^t \omega^L) & 2\Omega_{(-)}(\omega^R u^t p + nJu\omega^R) & -2i\Omega_{(-)}\omega^R u^t & \Omega_{(-)}nJu\Omega^{-1}_{(-)} \end{pmatrix}. \tag{12.40}$$

It is impossible to choose a non-trivial Ω_+ such that equation (8.9) is fulfilled.

13 Discussion

In this thesis we have investigated the connection between supersymmetric sigma models and the geometry in the target space. We have seen that if a sigma model has N=(2,2) supersymmetry, the target space geometry has to be bi-hermitian. In order for the algebra to close off-shell in general, one has to introduce auxiliary fields. Via the Gualteri map, equation (8.13), a bi-hermitian geometry is equivalent to a subset of Generalized Complex Geometry (GCG), namely Generalized Kähler Geometry (GKG). In GCG one considers the sum of the tangent space and the cotangent space. Since we know that bi-hermitian geometry and GKG are equivalent, we have tried to realize GKG straight from the sigma model with the help of the auxiliary fields, which we have constructed to transform in the co-tangent space.

We have mainly focused on a manifestly N=(2,2) supersymmetric action with semi-chiral fields. After the reduction, the semi-chiral fields leave us with auxiliary spinor fields which transforms in the tangent space. In order to realize GCG, we need fields transforming in the co-tangent space. We have therefore redefined the auxiliary fields into fields transforming in T^*. However, there is an ambiguity in how to redefine the auxiliary fields, and different redefinitions obtain different things. With the natural redefinition inspired by the field equations for the auxiliary spinors, equation (12.10), the Generalized Complex Structures (GCS) do not leave the natural pairing invariant, they do not fulfil equation (8.4). If we instead make the redefinition as in equation (12.27), we end up with GCS that do fulfill (8.4), but the price we paid was to end up with a messy action. No matter how we redefine the auxiliary fields, upon integrating them out we always end up with the same N=(1,1) sigma model.

What we ultimately have searched for is a set of auxiliary fields such that the Gualteri map is understood directly from the sigma model. That is, from the sigma model with semi-chiral fields we can integrate out the auxiliary fields. After this is done, we can read off the metric, the B-field and complex structures. From this data we can form GCS from the Gualteri map, equation (8.13). These structures will be given in terms of derivatives of the Lagrangian we used in the N=(2,2) action. In the other approach we would keep the auxiliary fields. We would define them in such a way that we can, from their supersymmetry transformations, read of the same GCS as we get from the Gualteri map. By looking at equation (8.13), it seems like a first step would be to at least define the auxiliary fields such that we have the complex structures in the top left corner. However, in the last section we saw that if we require the GCS to at the same time preserve the natural pairing, this seems to be hard to do.

In the appendix, we will choose a particularly simple Lagrangian, and see if we can get a manageable look of the GCS obtained from the Gualteri map. Even with this simple Lagrangian, what we get from the Gualteri map is a mess. This gives an indication that it might be hard to understand the Gaulteri map directly from the sigma model written in Lagrangian form. However, such a direct realization of the Gualteri map has been achieved in the Hamiltonian formulation of the sigma model, where less of the supersymmetry is manifest (see [12] and [13]). Therefore, pursuing a realization of the Gualteri map from the Lagranian formulation of the sigma model seems worthwhile.

14 Appendix 1 - Looking for solutions

As seen from equation (12.15), the expression for the metric in the general N=(2,2) sigma model with semi-chiral fields is pretty complicated. Here we will choose a simple form of the potential $K(\mathbb{X}_L, \bar{\mathbb{X}}_L, \mathbb{X}_R, \bar{\mathbb{X}}_R)$, and see what simplifications can be made.

We will choose the Lagranian K to be

$$\begin{aligned} K &= \mathbb{X}_L^A \delta_{AA'} \mathbb{X}_R^{A'} + K^L(\mathbb{X}_L, \bar{\mathbb{X}}_L) + K^R(\mathbb{X}_R, \bar{\mathbb{X}}_R) \\ &\equiv K^{LR}(\mathbb{X}_L, \bar{\mathbb{X}}_L, \mathbb{X}_R, \bar{\mathbb{X}}_R) + K^L(\mathbb{X}_L, \bar{\mathbb{X}}_L) + K^R(\mathbb{X}_R, \bar{\mathbb{X}}_R). \end{aligned} \tag{14.1}$$

This is a model with the fields $\{X_L, X_R\}$ coupled together with a simple coupling K^{LR}.

Since this K is a special case of the general K in equation (12.1), we can re-use the results from that section. With the auxiliary fields integrated out we found the N=(1,1) form of the action to be

$$\begin{aligned} S = -\frac{i}{2} \int d^2\xi D^2 \Big(& -2D_+ X_L^A \omega^L_{AB} u^{BB'} q_{B'C} D_- X_L^C \\ & D_+ X_L^A \left(m_{AA'} - 4\omega^L_{AB} u^{BB'} \omega^R_{B'A'} \right) D_- X_R^{A'} \\ & + D_+ X_R^{A'} p_{AA'} u^{AB'} q_{B'B} D_- X_L^B + 2D_+ X_R^{A'} p_{AA'} u^{AB'} \omega^R_{B'C'} D_- X_R^{C'} \Big) \\ \equiv -\frac{i}{2} \int & d^2x D^2 \left(D_+ \phi^t \mathbb{E} D_- \phi \right), \end{aligned} \tag{14.2}$$

where the matrices in the first line are given by (12.5), and the sum of the metric and B-field $\mathbb{E}$ is given by equation (12.15). With the special form of K, (14.1), these matrices read

$$m_{AA'} = -\begin{pmatrix} \delta_{aa'} & 0 \\ 0 & \delta_{\bar{a}'\bar{a}} \end{pmatrix} \qquad p_{AA'} = \begin{pmatrix} \delta_{aa'} & 0 \\ 0 & -\delta_{\bar{a}\bar{a}'} \end{pmatrix} \tag{14.3}$$

$$n_{A'A} = \begin{pmatrix} \delta_{a'a} & 0 \\ 0 & \delta_{\bar{a}\bar{a}'} \end{pmatrix} \qquad q_{A'A} = \begin{pmatrix} \delta_{a'a} & 0 \\ 0 & -\delta_{\bar{a}'\bar{a}} \end{pmatrix} \tag{14.4}$$

$$\omega^L_{AB} = \begin{pmatrix} 0 & K^L_{a\bar{b}} \\ -K^L_{\bar{a}b} & 0 \end{pmatrix} \qquad \omega^R_{A'B'} = \begin{pmatrix} 0 & K^R_{a'\bar{b}'} \\ -K^R_{\bar{a}'b'} & 0 \end{pmatrix}. \tag{14.5}$$

This inserted in $\mathbb{E}$ gives

$$\mathbb{E} = \begin{pmatrix} -2\omega^L uq & m - 4\omega^L u\omega^R \\ p^t uq & 2p^t u\omega^R \end{pmatrix} = \begin{pmatrix} -2iJ\omega^L & -I - 4\omega^L\omega^R \\ I & -2iJ\omega^R \end{pmatrix}. \tag{14.6}$$

14.1 The complex structures

The non-manifest supersymmetry transformations of $\{X_L, X_R\}$ are as usual found from equations (11.1) and (11.2), and we find them to be

$$
\begin{aligned}
\delta^+ X_L^A &\equiv \epsilon^+ Q_+ X_L^A = \epsilon^+ J^A_B D_+ X_L^B \\
\delta^+ X_R^{A'} &\equiv \epsilon^+ Q_+ X_R^{A'} = \epsilon^+ \Psi^{A'}_{R+} = \epsilon^+ \left(2iu^{AA'} \omega^L_{AB} D_+ X_L^B iu^{AA'} p_{AB'} D_+ X_R^{B'}\right) \\
\delta^- X_L^A &\equiv \epsilon^- Q_- X_L^A = \epsilon^- \Psi^A_{L-} = \epsilon^- \left(2iu^{AA'} \omega^R_{A'B'} D_- X_R^{B'} + iu^{AA'} q_{A'B} D_- X_l^B\right) \\
\delta^- X_R^{A'} &\equiv \epsilon^- Q_- X_R^{A'} = \epsilon^- J^{A'}_{B'} D_- X_R^{B'},
\end{aligned}
\tag{14.7}
$$

where the last lines in the two middle equations follows from the field equations for the spinors, equation (12.9). We know from the GHR-theorem that we can read off two complex structures from these transformations, one from the (+)-transformation and one from the (-)-transformation:

$$
\begin{aligned}
J_+ &= \begin{pmatrix} J & 0 \\ 2iu^t \omega^L & iu^t p \end{pmatrix} \\
J_- &= \begin{pmatrix} iuq & 2iu\omega^R \\ 0 & J' \end{pmatrix}.
\end{aligned}
\tag{14.8}
$$

Plugging in the matrices (14.3) we find the complex strucutres

$$
\begin{aligned}
J_+ &= \begin{pmatrix} J & 0 \\ 2i\omega^L & J' \end{pmatrix} \\
J_- &= \begin{pmatrix} J & 2i\omega^R \\ 0 & J' \end{pmatrix}.
\end{aligned}
\tag{14.9}
$$

These two tensors fulfills $J_\pm^2 = -1$, and $[J_+, J_-] \neq 0$, as expected.

14.2 The metric

The matrix $\mathbb{E}$ is the sum of the B-field and the metric, and we extract the metric by taking its symmetric part:

$$
\begin{aligned}
g = \frac{\mathbb{E} + \mathbb{E}^t}{2} &= \frac{1}{2} \begin{pmatrix} -2iJ\omega^L & -I - 4\omega^L\omega^R \\ I & -2iJ'\omega^R \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -2i(\omega^L)^t J^t & I \\ -I - 4(\omega^R)^t(\omega^L)^t & -2i(\omega^R)^t J'^t \end{pmatrix} \\
&= -2 \begin{pmatrix} iJ\omega^L & \omega^L\omega^R \\ \omega^R\omega^L & iJ'\omega^R \end{pmatrix}.
\end{aligned}
\tag{14.10}
$$

This follows from the facts that the ω's are antisymmetric, J is symmetric and $\omega J = -J\omega$. This metric should be hermitian with respect to both complex structures, which in its simplified form is given by equation (14.9). A not too long calculation yields that

$$
J_\pm^t g J_\pm = g \tag{14.11}
$$

so the metric is indeed bi-hermitian. We extract the B-field from the antisymmetric part of $\mathbb{E}$:

$$B = \frac{\mathbb{E} - \mathbb{E}^t}{2} = \begin{pmatrix} 0 & -(I + 2\omega^L\omega^R) \\ I + 2\omega^R\omega^L & 0 \end{pmatrix}. \tag{14.12}$$

Since we are dealing with a bi-hermitian metric, we can form two Kähler forms:

$$\begin{aligned} \Omega_+ = J_+^t g &= \begin{pmatrix} -2i\omega^L + 4i\omega^L\omega^R\omega^L & -2J\omega^L\omega^R \\ 2J'\omega^R\omega^L & 2i\omega^R \end{pmatrix} \\ \Omega_- = J_-^t g &= \begin{pmatrix} -2i\omega^L & 2J\omega^L\omega^R \\ -2J'\omega^R\omega^L & -4i\omega^R\omega^L\omega^R + 2i\omega^R \end{pmatrix}. \end{aligned} \tag{14.13}$$

14.3 The Generalized Complex Structures

Now we have all the objects we need in order to form generalized complex structures. From the section about generalized complex geometry, we know that a manifold with the objects $\{J_\pm, g, B\}$ always defines a generalized Kähler geometry, with generalized complex structures

$$\mathcal{J}_\pm = -\frac{1}{2}\begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}\begin{pmatrix} J_+ \pm J_- & -\left(\Omega_+^{-1} \mp \Omega_-^{-1}\right) \\ \Omega_+ \mp \Omega_- & -\left(J_+^t \pm J_-^t\right) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}. \tag{14.14}$$

In this matrix, we want to take the inverse of the Ω's. By looking at equation (14.13), we see that this is complicated. I have tried, and so far I have only reached the result that the resulting generalized complex structures will be very complicated.

We have arrived at this point by choosing the mixed part of the Lagrangian as simple as possible, and keeping the left and right parts general. Still, the generalized complex structures formed by the Gualteri-map are very complicated.

15 Appendix 2 - A geometrical approach

In the previous sections, we have used a manifestly N=(2,2) supersymmetric model with semi-chiral fields, and showed that the geometry of the target space is generalized Kähler, or equivalently, bi-hermitian. We have also showed that the use of chiral fields instead of semi-chiral fields gives rise to ordinary Kähler geometry, which is a special case of generalized Kähler, in the target space. There is a third kind of fields that one can define, called twisted chiral fields:

$$\bar{\mathbb{D}}_+\chi = \mathbb{D}_-\chi = \mathbb{D}_+\bar{\chi} = \bar{\mathbb{D}}_-\bar{\chi} = 0. \tag{15.1}$$

These fields gives rise to bi-hermitian geometry in the target space, too. (The twisted chiral fields can be re-defined into chiral fields, only when both fields are present in a model one has to distinguish between the two).

One can ask oneself how many different fields is it possible to define, that will give rise to bi-hermitian geometry? The answer is: no more than these three. This can be proven in the following way:

First we start with a manifestly N=(2,2) supersymmetric sigma model, with a Lagrangian a general function of chiral, twisted chiral and semi-chiral fields: $K = K(\phi, \chi, \mathbb{X}_L, \mathbb{X}_R)$. We reduce it down to its N=(1,1) form, and read off the geometrical objects $\{J_\pm, g, B\}$. We did this in the last section, with a special case of the function K. In the general case, everything will be much more complicated, but at the end of the day (or week) you end up with expressions for the geometrical objects, in these coordinates (the fields in the action serves as coordinates in the target manifold).

Then we take a second, geometrical, approach. We start with a bi-hermitian manifold, and show that we can find coordinates in which $\{J_\pm, g, B\}$ looks exactly as the ones arising from the sigma model. So for any bi-hermitian manifold, chiral, twisted chiral and semi-chiral fields can be used as coordinates.

As a bonus, it is also shown that there exists a generalized Kähler potential, namely a function in which the geometry is encoded. This is the case in ordinary Kähler geometry, where the metric is given as second derivatives of a function, the Kähler potential, see equation (4.12). In the case of generalized Kähler geometry, the metric will still be given by second derivatives of a function, but this time in a nonlinear manner.

Below I will sketch how the proof is made. The proof involves some geometrical technicalities, which I will not treat with any rigor. The whole proof is found in [9].

15.1 The split of the tangent space

We can decompose the tangent space into

$$T = ker[J_+, J_-] \oplus coker[J_+, J_-]. \tag{15.2}$$

It was shown already in [6] that for the kernel of the commutator between the two complex structures, chiral and twisted chiral fields could be chosen as coordinates. The problem was the co-kernel. Were the semi-chiral fields enough, or do we need more? Now we will show that they are enough.

15.2 The $coker[J_+, J_-]$

We will assume that $ker[J_+, J_-] = 0$, for simplicity. Important for the whole proof are so called *Poisson structures*. A Poisson structure is an antisymmetric bi-vector $\pi^{\mu\nu}$ that satisfies the differential equation

$$\pi^{\mu\nu}\partial_\nu\pi^{\rho\sigma} + \pi^{\rho\nu}\partial_\nu\pi^{\sigma\mu} + \pi^{\sigma\nu}\partial_\nu\pi^{\nu\rho} = 0. \tag{15.3}$$

If π is invertible, then π^{-1} is a symplectic structure. This is the property that we will use.

We find in [9] that given a bi-hermitian manifold with objects $(J_\pm, g, B)$, the following object is a Poisson structure:

$$\sigma = [J_+, J_-]g^{-1}. \tag{15.4}$$

Since we assume $[J_\pm, J_\pm] \neq 0$, σ is invertible, and $\sigma^{-1} \equiv \Omega$ is a symplectic structure. It can be shown that it is possible to choose so called Darboux coordinates for Ω. This can be done with respect to either one of the complex structures. Let $\{q, \bar{q}, p, \bar{p}\}$ be Darboux coordinates in which J_+ looks like

$$J_+ = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \equiv \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}. \tag{15.5}$$

In these coordinates, Ω looks like

$$\Omega = dq^a \wedge dp^a + d\bar{q}^{\bar{a}} \wedge d\bar{p}^{\bar{a}} \tag{15.6}$$

If we instead choose Darboux coordinates $\{Q, \bar{Q}, P, \bar{P}\}$ where J_- looks like

$$J_- = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \equiv \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}, \tag{15.7}$$

then Ω looks like

$$\Omega = dQ^{a'} \wedge dP^{a'} + d\bar{Q}^{\bar{a}'} \wedge d\bar{P}^{\bar{a}'} \tag{15.8}$$

We now have two sets of coordinates, $\{q, \bar{q}, p, \bar{p}\}$ and $\{Q, \bar{Q}, P, \bar{P}\}$, in which the symplectic structure Ω looks the same. From classical mechanics, we know that a coordinate transformation $\{p, q\} \to \{P, Q\}$ which preserves Ω is called a canonical transformation. In such a coordinate transformation, we have a generating function K, which always is a function of half the old coordinates, an half of the new ones.

Let K depend on the old coordinates $\{q\}$ and on the new coordinates $\{P\}$. The canonical transformation is then generated by the function $K(q, P)$, and you find the new coordinates from the equations

$$\begin{aligned} p &= \frac{\partial K}{\partial q} \\ Q &= \frac{\partial K}{\partial P}. \end{aligned} \tag{15.9}$$

Now we will choose to be in the coordinates $\{q, P\}$. Since J_+ is a (1,1)-tensor, it will be given by

$$J_+ = \left(\frac{\partial(q,p)}{\partial(q,P)}\right)^{-1} \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \left(\frac{\partial(q,p)}{\partial(q,P)}\right) \tag{15.10}$$

in the new coordinates. We find the transformation matrix to be

$$\left(\frac{\partial(q,p)}{\partial(q,P)}\right) = \begin{pmatrix} 1 & 0 \\ \frac{\partial p}{\partial q} & \frac{\partial p}{\partial P} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{\partial^2 K}{\partial q^2} & \frac{\partial^2 p}{\partial P \partial q} \end{pmatrix}, \tag{15.11}$$

where the last step follows from (15.9).

We introduce the notation

$$K_{LR} \equiv \begin{pmatrix} K_{aa'} & K_{a\bar{b}'} \\ K_{\bar{a}b'} & K_{\bar{a}\bar{b}'} \end{pmatrix}. \tag{15.12}$$

The subscript L means differentiation with respect to $\{q^a, \bar{q}^{\bar{a}}\}$, and the subscript R means differentiation with respect to $\{P^{a'}, \bar{P}^{\bar{a}'}\}$. The reason we choose these subscripts is that we will identify the coordinates $\{q, P\}$ with the fields $\{X_L, X_R\}$.

If we now compute J_+ in the $\{q, P\}$ coordinates, we find

$$\begin{aligned} J_+ = &\begin{pmatrix} 1 & 0 \\ -(K_{LR})^{-1}K_{LL} & (K_{LR})^{-1} \end{pmatrix} \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} 1 & 0 \\ K_{LL} & K_{LR} \end{pmatrix} = \\ &\begin{pmatrix} J & 0 \\ (K_{LR})^{-1}C_{LL} & (K_{LR})^{-1}JK_{LR} \end{pmatrix} \end{aligned} \tag{15.13}$$

where $C_{LL} = JK_{LL} - K_{LL}J$. Let us stop here for a while, and compare to what we got from the sigma model. In the previous section, we treated the sigma model with semi-chiral fields, and we ended up with the following J_+ and J_-:

$$\begin{aligned} J_+ &= \begin{pmatrix} J & 0 \\ 2iu^t\omega^L & u^tJp \end{pmatrix} \\ J_- &= \begin{pmatrix} iuq & 2iu\omega^R \\ 0 & J' \end{pmatrix}, \end{aligned} \tag{15.14}$$

where $\{u = n^{-1}, p, \omega^L\}$ are given by equations (12.5). In the new notation, these matrices can be written as

$$\begin{aligned} n &= K_{RL} \\ q &= JK_{RL} \\ 2i\omega^L &= C_{LL}. \end{aligned} \tag{15.15}$$

If we now compare what we got from the sigma model, (15.14) with what we got from the geometric construction, equation (15.13), we see that they are the same.

Similarly, we can compute J_-, which we now have in $\{Q, P\}$ coordinates, in the $\{q, P\}$ coordinates. We find

$$J_- = \left(\frac{\partial(Q,P)}{\partial(q,P)}\right)^{-1} \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \left(\frac{\partial(Q,P)}{\partial(q,P)}\right). \tag{15.16}$$

The transformation matrix is now

$$\left(\frac{\partial(Q,P)}{\partial(q,P)}\right) = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial P} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 K}{\partial q \partial P} & \frac{\partial^2 p}{\partial P \partial P} \\ 0 & 1 \end{pmatrix}, \tag{15.17}$$

where the last step again follows from (15.9). So J_- is given by

$$\begin{aligned} J_- = &\begin{pmatrix} (K_{RL})^{-1} & -K_{RL})^{-1} K_{RR} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} K_{RL} & K_{RR} \\ 0 & 1 \end{pmatrix} = \\ &\begin{pmatrix} (K_{RL})^{-1} J K_{RL} & (K_{RL})^{-1} C_{RR} \\ 0 & J \end{pmatrix}, \end{aligned} \tag{15.18}$$

where $C_{RR} = JK_{RR} - K_{RR}J$. If we make the notational modification as above, we see from equation (15.14) that this is the same expression for J_- as we got from the sigma model in the previous section.

Finally, we compute Ω in $\{q, P\}$-coordinates. In the coordinates $\{q, p\}$ it looks like

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{15.19}$$

Since Ω is a (0,2)-tensor, it transforms as

$$\left(\frac{\partial(Q,P)}{\partial(q,P)}\right)^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left(\frac{\partial(Q,P)}{\partial(q,P)}\right) \tag{15.20}$$

when we make the change of coordinates $\{q, p\} \to \{q, P\}$. Therefore we find Ω in the mixed coordinates to be:

$$\begin{aligned} \Omega = &\begin{pmatrix} 1 & (K_{LL})^t \\ 0 & (K_{LR})^t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ K_{LL} & K_{LR} \end{pmatrix} = \\ &\begin{pmatrix} 0 & K_{LR} \\ -K_{RL} & 0 \end{pmatrix}. \end{aligned} \tag{15.21}$$

In the calculation we have used $(K_{LR})^t = K_{RL}$.

From (15.4) we find the metric to be

$$g = \Omega[J_+, J_-]. \tag{15.22}$$

Since Ω, J_+ and J_- all are given in terms of second derivatives of K in these coordinates, the metric g will also be given in terms of second derivatives of K. Therefore, it is natural to refer to K as a generalized Kähler potential.

To compute g is a bit cumbersome, and equally cumbersome is it to compute the metric which one obtains from the sigma model with semi-chiral fields. Luckily, the final result of the two calculations again agrees.

Here we illustrate the result with K such that $K_{LR} = K_{RL} = I$ (as in the calculation we did with the sigma model in the previous section). We then find that

$$[J_+, J_-] = \begin{pmatrix} J & 0 \\ 2iC_{LL} & J \end{pmatrix} \begin{pmatrix} J & 2iC_{RR} \\ 0 & J \end{pmatrix} - \begin{pmatrix} J & 2iC_{RR} \\ 0 & J \end{pmatrix} \begin{pmatrix} J & 0 \\ 2iC_{LL} & J \end{pmatrix} = \begin{pmatrix} 4C_{RR}C_{LL} & 4iJC_{RR} \\ 4iC_{LL}J & -4C_{LL}C_{RR} \end{pmatrix}. \tag{15.23}$$

Since $C_{RR} = \omega^R$ and $C_{LL} = \omega^L$, we find the metric to be

$$g = \Omega[J_+, J_-] = -4 \begin{pmatrix} iJ\omega^L & \omega^L\omega^R \\ \omega^R\omega^L & iJ\omega^R \end{pmatrix}, \tag{15.24}$$

which up to an overall factor agrees with the metric we found from the sigma model with semi-chiral fields, equation (14.10).

15.3 The general case

In the last section we treated the case $ker[J_+, J_-] = 0$, and showed that semi-chiral fields can be used as coordinates for the co-kernel. In the general case, both the kernel and the co-kernel of $[J_+, J_-]$ will be non-empty. In this case, more tools from Poisson geometry are needed, and it is shown in [9] that in the general case, it is possible to find coordinates where all the geometrical objects looks the same as they do when we have a sigma model with chiral, twisted chiral and semi-chiral fields. Also in the general case there exists a function K in which builds up the metric, a generalized Kähler potential.

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