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Crosscap Number: Handcuff Graphs and Unknotting Number

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CROSSCAP NUMBER: HANDCUFF GRAPHS AND UNKNOTTING NUMBER

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Many types of invariants are used in the study of knots. Some are based on polynomials, some are purely algebraic, and some have their origins in geometry. One of the best known geometric knot invariants is the genus of a knot. A closely related but lesser-known invariant, crosscap number, was first introduced by Bradd Evans Clark in 1978. This thesis primarily concerns crosscap number two knots. Starting with a list of knots found to have crosscap number two by Burton and Ozlen using a linear programing approach, we verify, though are unable to expand, this list using a computer search. Surfaces that realize the minimal crosscap number of these knots are moreover found to arise from low-complexity handcuff diagrams. We also find a knot for which a single crossing change simultaneously lowers the unknotting number and raises the crosscap number. The proof utilizes signature to bound unknotting number from below. This result is a non-orientable analogue of a result for genus given in a paper by Scharlemann and Thompson. The result is further expanded to two infinite families of knots, one non-hyperbolic and one hyperbolic, which have the same property.

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Contents

Chapter 1

Introduction

The field of knot theory, a subbranch of low-dimensional topology, stands out among topics of current research for the ease with which much of its vocabulary can be described, even to those who have little background in the area. This is largely due to its geometric nature.

Intuitively speaking, a mathematical knot is what results when a piece of stretchy string is tangled in some way and the ends of the string are then glued together. Unlike a fixed subset of Euclidean space, a knot can be moved about and the string can be stretched or contracted, as long as the string is not cut or allowed to pass through itself. More formally, a knot K is an equivalence class of embeddings of the standard circle S^1 in compactified three-space S^3, where equivalence is up to ambient isotopy. We focus on smooth knots, whose embeddings and isotopies are C^∞. Knots are often studied using diagrams, projections of the knot onto $\mathbb{R}^2$ with information added at each crossing point to indicate which strand goes over the other.

Because the knot itself is a more general object than any one of its diagrams, knot theory focuses on characteristics of the knot itself which are invariant under ambient isotopy. This thesis addresses a particular knot invariant which arises from the study of surfaces in S^3.

For this we will need the following classical theorem:

Theorem 1.1 (Classification Theorem for Closed Surfaces)**.** *Every closed, connected surface is homeomorphic to either:*

1. *The connected sum of g tori for $g \geq 0$ (where the 0-holed torus is understood to be the sphere), or*

2. *The connected sum of k real projective planes.*

"Crosscap" is another term for a real projective plane.

A spanning surface of a knot K is a surface in S^3 with K as its only boundary component. Gluing a disk along this component gives a closed surface which can be classified using the theorem above. Thus the numbers g and k can be used to characterize the surface. We have the following definitions:

Definition 1.2. *Given an orientable spanning surface S of a knot K, the genus of S, denoted $g(S)$, is the genus of the surface obtained by gluing a disk along K.*

Definition 1.3. *Given a knot K, the genus of K is defined by*

$$g(K) = \min\{g(S) : S \textit{ is an orientable spanning surface for } K\}$$

Applying the same gluing technique to non-orientable spanning surfaces gives the corresponding definition:

Definition 1.4. *The crosscap number of a knot K is defined as:*
$cc(K) = \min\{k : k$ *is the crosscap number of a non-orientable spanning surface of K with a disk glued along K.* $\}$

Because both genus and crosscap number of a knot are defined as the minimum over all spanning surfaces, they are not dependent on a particular knot projection and thus are knot invariants. For this same reason, however, both genus and crosscap number can be challenging to compute.

The genus of a knot has been well studied since 1935, when Herbert Seifert proved that every knot has an orientable spanning surface by providing an algorithm to construct such a surface from any diagram of the knot [28]. This algorithm is dependent on the diagram that is used and does not guarantee a minimal genus surface unless applied to diagrams of one of several classes of knots. For example, when the algorithm is applied to a reduced alternating diagram of an alternating knot it gives a minimal genus surface as shown in [9] and elsewhere.

This thesis concerns the comparatively new and related topic of crosscap number. Chapter 2 discusses basic properties of crosscap number, including some prior results on how to compute it, known bounds, and behavior under connected sum. In chapter 3 we present explicit crosscap minimizing surfaces for some knots for which the crosscap number was known to be two. We show that these surfaces for crosscap number two knots arise from low complexity handcuff graphs. Chapter 4 is an adaptation of a result relating genus and unknotting number to crosscap number. An initial example showing that the crosscap number can be raised while lowering the unknotting number is expanded to two infinite families of knots with this characteristic. The last chapter contains some thoughts for future work.

Chapter 2

About Crosscap Number

The systematic study of crosscap number began in 1978 with a paper by Bradd Evens Clark titled “Crosscaps and Knots” [5]. In addition to giving the definition above, Clark made some preliminary observations. He showed that at least one of the checkerboard surfaces of a knot projection must be nonorientable, thus demonstrating that every knot has a crosscap number. To complete the definition he defined the crosscap number of the unknot to be zero.

Clark observed the following result of the definition of crosscap number:

Proposition 2.1. *[5] Let K be a knot. Then $cc(K) = 1 - \chi(S)$ where S is a nonorientable spanning surface with maximal Euler characteristic.*

Any orientable surface with boundary can be made nonorientable by adding a twisted band along one edge of the surface (this has the effect of a Riedemeister I move on the knot). The new, nonorientable surface has Euler characteristic one lower than the original surface. Thus if S is a minimal genus spanning surface for K and S' is the nonorientable surface formed by adding a twisted band, then $\chi(S') = \chi(S) - 1 = 1 - 2g(K) - 1 = -2g(K)$. Thus $cc(S') = 1 + 2g(K)$ and we have the upper bound:

$$cc(K) \leq 2g(K) + 1.$$

Clark was uncertain if this bound was best possible, but a concrete example exhibiting equality was provided by Murakami and Yasuhara in 1995 [21].

The following are also due to Clark:

Proposition 2.2. *[5, Prop 2.2] For a knot K, $cc(K) = 1$ if and only if K is a $(2,n)$ cable.*

Proof. Suppose $cc(K) = 1$. Then there is a Möbius band that spans the knot. The knot is then the $(2,n)$ cable of the centerline of the band. On the other hand, if K is a $(2,n)$ cable of some knot K', then it is possible to construct a spanning surface with crosscap number 1 by reversing the process just mentioned. Assuming the knot is non-trivial, we have that $cc(K) = 1$. □

Corollary 2.3. *[5, Cor 2.3] There exist knots with arbitrarily large genus that have crosscap number one.*

Proof. In particular, $(2,n)$ torus knots are $(2,n)$ cables of the unknot and thus have crosscap number one. Yet $g(T(2,n)) = \frac{1}{2}(n-1)$. □

2.1 Computing crosscap number

There is no known algorithm for computing the crosscap number of a generic knot. However, various people have found computations for specific classes of knots. For two bridge knots, Bessho found in 1994 [3] that the crosscap number is the minimal length of all expansions of odd type of all continued fractions corresponding to the knot. An effective algorithm for calculating this number was presented by Hirasawa and Teragaito in 2005 [11].

In 2004 Teragaito [30] showed that the torus knot $T(p, q)$ has crosscap number $N(p, q)$ if pq is even and $N(pq-1, p^2)$ if pq is odd, where $N(s, t)$ is the minimal genus of a closed, connected, non-orientable surface contained in the lens space $L(s, t)$.

In 2007 Ichihara and Mizushima [13] proved that the crosscap number of a pretzel knot $P(p_1, p_2, ..., p_N)$ is $N-1$ if one of the p_i is even, and N if all the p_i are odd. That is, the checkerboard surface obtained from the standard projection of a pretzel knot is crosscap minimizing so long as that surface is nonorientable. If it is orientable, a crosscap minimizing surface can be obtained by adding a twisted band to the surface.

In 2012 Colin Adams and Thomas Kindred [2] found an algorithm to determine the crosscap number of alternating knots. Their algorithm is constructive and follows the general pattern of Seifert's algorithm for finding an orientable spanning surface.

Here is a summary of the algorithm:

1. Start with any regular, reduced projection of the knot. Unlike Seifert's algorithm, the projection does not need to be oriented.

2. At each crossing of the projection, erase the projection and connect the stands in one of the two possible, non-crossing ways:

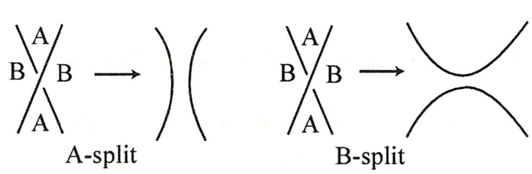


Figure 2.1: Two ways to rejoin a knot. This is figure 10 in [2]

Once all the crossings have been cut and reconnected, you have a set of disjoint circles in the projection plane.

3. Fill in each circle with a disk as in Seifert's algorithm, moving any circle contained in another circle to slightly above the projection plane to avoid intersection.

4. Attach twisted bands at each place where a crossing was removed in such a way that the boundary of the new surface is the original knot. Note that unlike Seifert's algorithm, this may involve attaching both ends of the same band to a single disk.

5. A crosscap minimizing surface is guaranteed to be among the surfaces so constructed, so the final step of the algorithm is to compare the surfaces and determine which has the minimal crosscap number.

The surfaces constructed in this way are called *layered surfaces.* Note that due to the choices involved in step 2), a knot with n crossings produces 2^n layered surfaces. However, the Euler characteristic of a surface built from c circles spanning an alternating knot with n crossings is $c - n$ [2], so the final step can be simplified by strategically choosing the connections in step 2) to maximize the number of circles and ignoring the other surfaces. Adams and Kindred applied their algorithm to find the crosscap number of all alternating knots through nine crossings and all two-component links through eight crossings. For the definition of the crosscap number of a two-component link, see Definition 2.12 below.

2.2 Bounds

Instead of computing crosscap number directly some researchers have found upper and or lower bounds for this invariant. In some cases the correspondence of known bounds has made it possible to determine the crosscap number of specific knots.

In a paper primarily about the crosscap number of connected sums, Murakami and Yasuhara [21] found the following bounds:

Proposition 2.4. *[21, Prop 1.3] For any knot K, $cc(K) \leq \lfloor \frac{c(K)}{2} \rfloor$ where $c(K)$ is the the crossing number of K.*

They provided examples showing this bound is sharp. Another inequality from the same paper:

Proposition 2.5. *[21, Prop 1.6] Let $e_2(K)$ denote the minimum number of generators of $H_1(\Sigma, \mathbb{Z})$ where Σ is the double branched cover of S^3 over K. Then $e_2(K) \leq cc(K)$.*

Building on the results of Adams and Kindred and using Proposition 2.4 above, Kalfagianni and Lee [15] found the following bounds for alternating knots:

Theorem 2.6. *[15, Theorem 1.2] For K an alternating, non-torus knot, let $J_K(t) = \alpha_K t^n + \beta_K t^{n-1} + \cdots + \beta'_K t^{s+1} + \alpha'_K t^s$ be the Jones polynomial of K. Let $T_K = |\beta_K| + |\beta'_K|$ and let $s_K = s - n$ be the degree span of $J_K(t)$. Then:*

$$\left\lceil \frac{T_K}{3} \right\rceil + 1 \leq cc(K) \leq \min\{T_K + 1, \left\lfloor \frac{s_K}{2} \right\rfloor\}$$

One of the upper bounds is given by combining Proposition 2.4 with a result of Kauffman [16] which says alternating knots have crossing number equal to s_K. The proof of the lower bound and the other upper bound utilizes the result of Dasbach and Lin [8], which says that if the minimal number of twist regions for an alternating knot is at least two, then the minimal number of twist regions in an alternating diagram of the knot is T_K. This is combined with the Adams-Kindred algorithm and an argument using normal surface theory to give the bounds above. They found examples to demonstrate that each of these bounds are sharp. Specifically, the knot

10_3 has crosscap number 3 and $\left\lceil \frac{T_{10_3}}{3} \right\rceil + 1 = 3$. Both the lower bound and the remaining upper bound are reached by 10_{123}. Kalfagianni and Lee applied their lower bound to the 1778 alternating knots with crossing number 10, 11, or 12, improving the known lower bound for 1472 of these knots. For 283 of these knots their lower bound met the known upper bound, thus determining the crosscap number.

Attempting to find an approach to crosscap number that works for all knots, Burton and Ozlen used integer programming and normal surface theory to create three algorithms in 2012 [4]. The first two of these will either determine the crosscap number or reduce the question to two possible values. These algorithms are, in their own estimation, however, computationally impractical. The third algorithm does not compute the crosscap number directly but it does provide an upper bound. Running this last algorithm on all knots with 12 crossings or less they were able to improve the known upper bound for several hundred knots. In addition, they found 27 knots for which their upper bound matched the previously known lower bound, thus establishing the crosscap number of these knots. In each case the crosscap number was two.

Figure 2.2 is Table 1 from [4]. It shows the 27 knots whose crosscap number Burton and Ozlen found. In their notation $C(K)$ is used for $cc(K)$.

2.3 Connected Sum

One question which has been studied is the behavior of crosscap number under connected sum. Given two knots K_1 and K_2, their *connected sum* is a knot formed by removing an arc from each of K_1, K_2 and attaching the knots along the break. The connected sum is denoted $K_1 \# K_2$. See Figure 2.3 below.

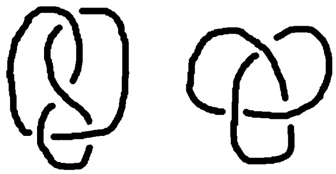
This operation is well defined for oriented knots [24, p. 46]. Using a cut and

KnotInfo name	Dowker-Thistlethwaite name	Genus	Previous bounds on $C(K)$	New value of $C(K)$
8_{20}	$8n_1$	2	$[2,4]$	2
10_{125}	$10n_{15}$	3	$[2,4]$	2
10_{126}	$10n_{17}$	3	$[2,4]$	2
10_{139}	$10n_{27}$	4	$[2,3]$	2
10_{140}	$10n_{29}$	2	$[2,4]$	2
10_{142}	$10n_{30}$	3	$[2,4]$	2
10_{145}	$10n_{14}$	2	$[2,4]$	2
10_{161}	$10n_{31}$	3	$[2,5]$	2
$11n_{102}$	$11n_{102}$	2	$[2,4]$	2
$11n_{104}$	$11n_{104}$	4	$[2,4]$	2
$11n_{135}$	$11n_{135}$	3	$[2,5]$	2
$12n_{0121}$	$12n_{0121}$	2	$[2,4]$	2
$12n_{0233}$	$12n_{0233}$	4	$[2,4]$	2
$12n_{0235}$	$12n_{0235}$	4	$[2,4]$	2
$12n_{0242}$	$12n_{0242}$	5	$[2,3]$	2
$12n_{0404}$	$12n_{0404}$	2	$[2,4]$	2
$12n_{0474}$	$12n_{0474}$	4	$[2,4]$	2
$12n_{0475}$	$12n_{0475}$	3	$[2,4]$	2
$12n_{0522}$	$12n_{0522}$	3	$[2,4]$	2
$12n_{0575}$	$12n_{0575}$	4	$[2,4]$	2
$12n_{0581}$	$12n_{0581}$	3	$[2,4]$	2
$12n_{0582}$	$12n_{0582}$	2	$[2,4]$	2
$12n_{0591}$	$12n_{0591}$	4	$[2,5]$	2
$12n_{0721}$	$12n_{0721}$	4	$[2,4]$	2
$12n_{0725}$	$12n_{0725}$	5	$[2,3]$	2
$12n_{0749}$	$12n_{0749}$	3	$[2,5]$	2
$12n_{0851}$	$12n_{0851}$	3	$[2,5]$	2

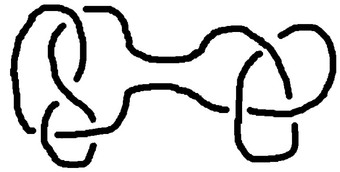
Figure 2.2: Knots with crosscap number two found by Burton and Ozlen

paste argument it is possible to see that $g(K_1\#K_2) = g(K_1)+g(K_2)$. Take a minimal genus, orientable spanning surface for each of K_1, K_2. When performing the connected sum, these two surfaces combine to form an orientable spanning surface for $K_1\#K_2$. Thus $g(K_1\#K_2) \leq g(K_1) + g(K_2)$. Taking a minimal genus surface Σ for $K_1\#K_2$, it is possible to find an arc in Σ such that cutting along that arc gives orientable spanning surfaces for K_1 and K_2. Thus $g(K_1\#K_2) \geq g(K_1) + g(K_2)$. This yields the desired equality. More complete details on why such an arc exists can be found in [24, Theorem 5A14].

Clark applied this argument to nonorientable surfaces. Taking minimal crosscap



(a) K_1 and K_2



(b) $K_1 \# K_2$

Figure 2.3: Connected Sum

number surfaces for K_1 and K_2, one can create a non-oreintable spanning surface for $K_1 \# K_2$, so $cc(K_1 \# K_2) \leq cc(K_1) + cc(K_2)$. However, a minimal crosscap number surface for $K_1 \# K_2$ may divide into an orientable and a nonorientable surface. Without loss of generality suppose K_1 is the knot that is left with the orientable surface. For this to happen we must have $cc(K_1) = 2g(K_1) + 1$. Let S_1, S_2 be crosscap minimizing surfaces for K_1 and K_2. Furthermore let S_1' be a minimal genus surface for K_1. Then by our assumption S_1 can be obtained by adding a twisted band to S_1'. Thus $\chi(S_1) = \chi(S_1') - 1$. Note that $S_1' \# S_2$ is non-orientable. We have $\chi(S_1' \# S_2) = \chi(S_1') + \chi(S_2) = \chi(S_1) + \chi(S_2) + 1$, so $cc(S_1' \# S_2) = 1 - \chi(S_1' \# S_2) = -\chi(S_1) - \chi(S_2)$. But $cc(S_1 \# S_2) = 1 - \chi(S_1 \# S_2) = 1 - \chi(S_1) - \chi(S_2)$. We have $cc(S_1' \# S_2) = cc(S_1 \# S_2) - 1$. Thus:

Theorem 2.7. *[5, Theorem 2.8] see also [21, Proposition 4.1] Let K_1, and K_2 be knots. Then*

$$cc(K_1) + cc(K_2) - 1 \leq cc(K_1 \# K_2) \leq cc(K_1) + cc(K_2).$$

Moreover, we have:

Theorem 2.8. *[21, Prop 4.3] $cc(K_1 \# K_2) = cc(K_1) + cc(K_2)$ if and only if $cc(K_i) < 2g(K_i)$ for at least one of $i = 1, 2$.*

As a corollary, Murakami and Yasuhara pointed out that $cc(K_1\#K_2\#\cdots\#K_n) = cc(K_1)+cc(K_2)+\cdots+cc(K_n)$ if $cc(K_i) = 2$ for $i = 1, \ldots, n$. They provided a concrete example where $cc(K_1\#K_2) = cc(K_1)+cc(K_2)-1$ by proving that $cc(7_4) = 3 < 2g(7_4)$, so $cc(7_4\#7_4) = 5$.

2.4 Slope

Another interesting area of study concerns the slope of a spanning surface. Slope is defined using the longitude and meridian of a torus.

Definition 2.9. *Let T be a torus. Their are two essential closed curves on T that generate $\pi_1(T)$. One is called the meridian, and the other is called the longitude of T.*

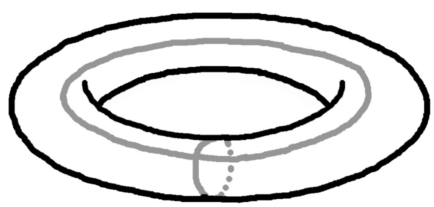


Figure 2.4: The longitude and meridian of a torus

Definition 2.10. *For a spanning surface S of a knot K, let N be a regular neighborhood of K and let $L = \partial N \cap S$. Then L wraps once longitudinally around N and m times meridionally. The number m is called the* slope *of S.*

If S is an orientable surface then the slope of S is 0. This follows from considering L as an element in the homology of the knot exterior. For non-orientable surfaces, the question of what slopes can be achieved by minimal crosscap number surfaces is richer. In 2002 Ichihara, Ohtouge, and Teragaito proved:

Theorem 2.11. *[12, Theorem 2] If K has crosscap number two, then a crosscap minimizing surface for K has slope a multiple of four, there are at most two such slopes for any such K, and those slopes are either 4 or 8 apart. If they are 8 apart, then K is the figure eight knot and the two slopes are -4 and 4.*

They also describe an infinite family of crosscap number two knots that have crosscap minimizing surfaces with distinct slopes. The family contains only two hyperbolic knots, the pretzel knot $P(-2, 37)$ and its mirror image. Ramirez-Losada and Valdez-Sanchez later proved that this family gives all such knots [22].

2.5 Miscellaneous results

A number of other results relating to crosscap number are mentioned here.

The following definition for the crosscap number of a 2-component link was given by Zhang in 2006:

Definition 2.12. *[32] For L a 2-component link:*

$$cc(L) = \min\{\beta_1(S) | S \textit{ is a connected nonorientable surface bounding } L\}$$

where β_1 denotes the first Betti number.

Using the techniques of Murakami and Yasuhara [21], Zhang found bounds for the split union of two links similar to those found by Murakami and Yasuhara for the connected sum of knots. There are also parallel results for the relationship between crosscap number and crossing number, and between crosscap number and the minimum number of generators of the second homology of double branched cover of the link.

In [18] Yoko Mizuma and Yukihiro Tsutsumi investigated the relationship between crosscap number and essential tangle decompositions. Their main result:

Theorem 2.13. *[18, Theorem 1.2] Let K be a knot with two disjoint and non-parallel essential Conway spheres S_1 and S_2. Let B_1, B_2 be the two disjoint 3-balls bounded by S_1, S_2 respectively. Let C be the $S_2 \times I$ between S_1 and S_2. Suppose none of $B_i \cap K$ consists of two parallel strings and that at least one of the four strings of $C \cap K$ is not parallel to any of the other three in C. Then $cc(K) \geq 4$ and $g(K) \leq 2$.*

Concretely, they showed that the Kinoshita-Terasaka knot and the Conway knot both have crosscap number 4. They also found:

Proposition 2.14. *[18, Prop 1.5] Let K be a knot in S^3 which admits an essential 2-string tangle decomposition, and let K^τ be a mutant of K. Suppose $cc(K) \leq cc(K^\tau)$. Then, if $cc(K)$ is odd, $cc(K^\tau) = cc(K)$. If $cc(K)$ is even, $|cc(K^\tau) - cc(K)| \leq 1$.*

This proposition is interesting mainly because it is a rare example of the crosscap number behaving "better" than genus. That is, there is no such relationship between $g(K)$ and $g(K^\tau)$.

The following proposition, due to Teragaito, follows from an investigation of building manifolds which contain Klein bottles in their complement using Dehn surgery.

Proposition 2.15. *[29, Cor 1.4] A genus one, crosscap number two knot is a doubled knot.*

Chapter 3

Knots with cc(K) = 2

This chapter concerns the discovery of crosscap minimizing surfaces for the crosscap number two knots found by Burton and Ozlen and the search for more surfaces of the same type.

Many of the knots from table 2.2 are three strand pretzel knots with one p_i even, so the checkerboard surface arising from their pretzel projection is crosscap minimizing. The table below shows these knots with the corresponding pretzel name on the right.

8_{20}	P(3,-3,2)	12_{n233}	P(2,3,-7)	12_{n522}	P(3,4,-5)
10_{125}	P(5,-3,2)	12_{n235}	P(2,-3,-7)	12_{n581}	P(3,3,-6)
10_{126}	P(-5,3,2)	12_{n242}	P(2,-3,-7)	12_{n582}	P(-3,3,6)
10_{140}	P(4,3,-3)	12_{n474}	P(3,-4,5)	12_{n721}	P(2,-5,5)
10_{142}	P(-4,3,3)	12_{n475}	P(-3,4,5)	12_{n725}	P(-2,5,5)

The remaining knots from Burton and Ozlen's table are: $10_{139}, 10_{145}, 10_{161}, 11_{n102}, 11_{n104}, 11_{n135}, 12_{n121}, 12_{n404}, 12_{n575}, 12_{n591}, 12_{n749}$, and 12_{n851}. These knots are non-alternating, so the surfaces arising from the Adams-Kindred algorithm are not guaranteed to be crosscap minimizing.

Theorem 3.1. *Crosscap minimizing surfaces for the remaining knots in Burton and Ozlen's table can be built from the handcuff diagram* 2_1.

Before proving Theorem 3.1 it will be necessary to give some definitions and establish notation.

3.1 Handcuff notation

Definition 3.2. *A handcuff graph is a graph consisting of two vertices and three edges, with one edge connecting the two vertices and the other two edges forming loops, one at each vertex.*

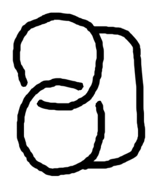


Figure 3.1: A handcuff graph

Hiromasa Moriuchi has compiled a table of prime handcuff diagrams up to seven crossings [19]. In this context, "prime" means these graphs cannot be decomposed into simpler but non-trivial graphs by cutting along one, two, or three edges. The above example is the handcuff graph 2_1 from Moriuchi's table. The full table is included in the appendix in Figure A.2 for reference.

Definition 3.3. *A surface arising from a handcuff graph is formed by replacing each vertex with a disk, and each edge* e *with* $e \times I$*. The subspace* $e \times I$ *may be given any number of half twists before it is connected to the appropriate disk.*

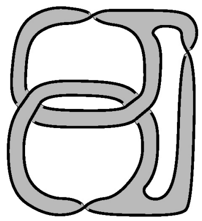
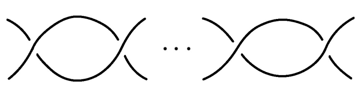


Figure 3.2: A surface arising from the graph 2_1

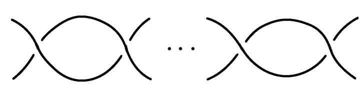
Note that for the boundary of such a surface to be a knot, the edges which form loops must be given an odd number of half twists.

Definition 3.4. *Let $hgX_Y(i,j,k)$ denote the knot that forms the boundary of the surface arising from the handcuff graph X_Y in Moriuchi's table with i half twists given to the loop higher up or farther to the left, j half twists to the other loop, and k half-twists to the thickened edge connecting the vertices.*

The numbers i, j and k correspond to the number of crossings created by twisting the thickened edges. For consistency, these crossings will be counted as positive or negative according to the following convention:



(a) positive twists



(b) negative twists

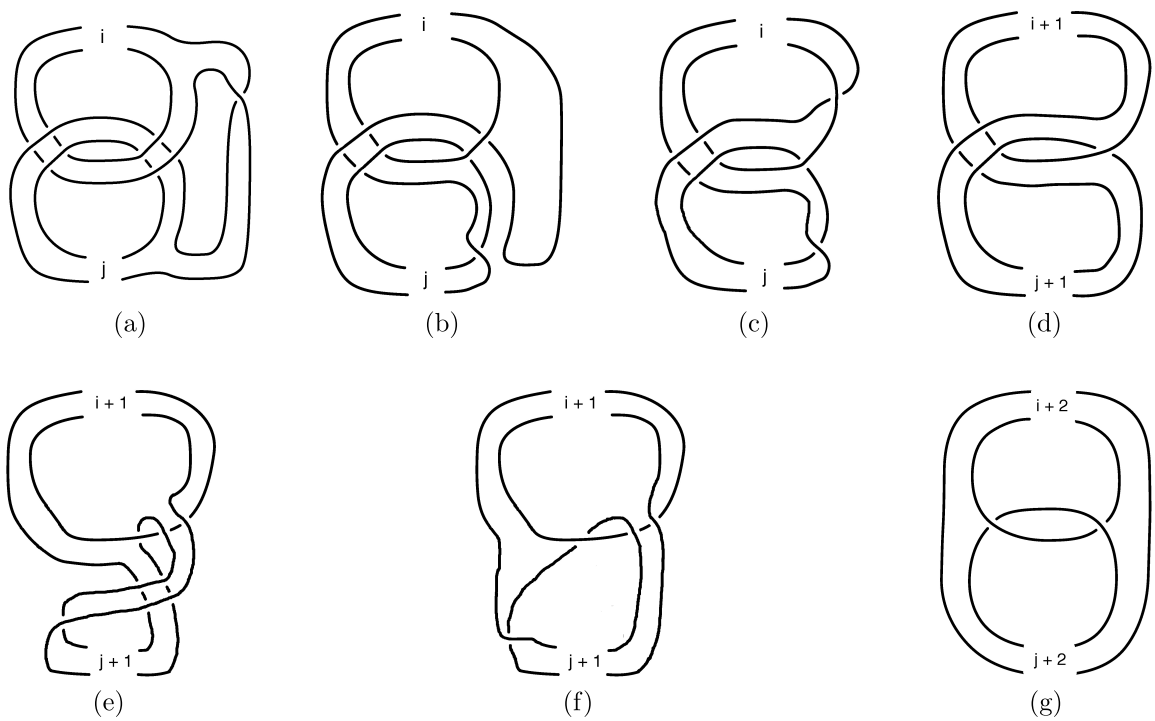
Figure 3.3: How to determine positive and negative twists

The surface arising from a handcuff graph contains exactly two Mobius bands, so knots that bound such a surface have crosscap number at most 2. It is possible that such a knot has crosscap number one, but no examples were found in our search.

The following observation about a special subset of knots arising from the handcuff graph 2_1 will be used in the following chapter:

Proposition 3.5. $hg2_1(i,j,1) = P(-(i+2), -(j+2), 2)$.

Proof.



Rotating this last image 90° to the left gives the pretzel knot $P(-(i+2), 2, -(j+2))$, and by a well known property of pretzel knots $P(-(i+2), 2, -(j+2)) = P(-(i+2), -(j+2), 2)$. $\square$

3.2 Finding crosscap number two knots

The software program SnapPy [7] can take a diagram of a knot and compare the knot complement with the manifolds in its census. This census includes the complements of all knots with 14 or fewer crossings. If the program finds an isomorphism between the knot complements, then the knots themselves are identical. This enables the efficient identification of knots that form the boundary of surfaces arising from handcuff diagrams.

Proof of Theorem 3.1. The following table lists the twelve knots in question with their standard name on the left and the handcuff notation on the right. These identifications were made using SnapPy.

10_{139}	$hg2_1(1,1,-1)$	12_{n121}	$hg2_1(-1,-3,-1)$
10_{145}	$hg2_1(-1,-1,-1)$	12_{n404}	$hg2_1(-1,-1,-3)$
10_{161}	$hg2_1(1,-1,-1)$	12_{n575}	$hg2_1(1,1,-3)$
11_{n102}	$hg2_1(1,1,-2)$	12_{n591}	$hg2_1(-1,3,-1)$
11_{n104}	$hg2_1(-1,-1,-2)$	12_{n749}	$hg2_1(1,-3,-1)$
11_{n135}	$hg2_1(1,-1,-2)$	12_{n851}	$hg2_1(1,-1,-3)$

□

A search for more knots with crosscap number two was conducted using the handcuff graphs in Moriuchi's table, and allowing all combinations of i, j and k with i and j odd, $-19 \leq i, j \leq 21$, and $-20 \leq k \leq 21$. These numbers were selected for computational ease. Starting with one half twist on each band, -10 to 10 full twists were added. To simplify the input of the diagrams, twists were added using a process called Dehn filling. To add twists between two strands of a knot using Dehn filling, first carve out a small torus around both strands. Then fill in the torus in such a way that the meridian of the new solid torus follows a loop that wraps once meridianally and n times longitudinally around the hole left by the torus that was carved out. This changes the knot exterior to the exterior of a knot with n full twists added between the strands. Using this process, we can input a diagram with loops added at i, j and k and let the computer do the fillings and identify the resulting knot exterior. The exact code used is in Appendix A, Figure A.1.

This search found surfaces for a number of alternating knots, some pretzel knots, the knots mentioned above from Butron and Ozlen's table, and a number of thirteen and fourteen crossing knots. Thus there were no new knots with crossing number less than or equal to twelve that were discovered to have crosscap number two through this process. A complete list of the knots with their handcuff surfaces are also included in the appendix.

Chapter 4

Raising crosscap number while lowering unknotting number

The purpose of this chapter is to explore the non-orientable parallel of a result for orientable surfaces. The original theorem is from a paper by Martin Scharlemann and Abigail Thompson from 1988 [26]. Before stating their result, it will be necessary to define a knot invariant known as unknotting number.

4.1 Unknotting number

Most of this section is adapted from [25].

Definition 4.1. *Given a diagram of a knot, consider any point at which one strand passes over the other. A diagram of a (usually different) knot can be formed by changing which strand goes over and which goes under. This operation is known as crossing change.*

A crossing change can be accomplished by Dehn filling, using one full twist to turn a negative crossing into a positive one or vice-versa. In this case the small torus that is carved out is called a crossing circle and the disk that the crossing circle bounds is called a crossing disk.

Proposition 4.2. *[25, Lemma 1.4] Any knot can be transformed into the unknot by a sequence of crossing changes. Moreover, there is such a sequence that has length less than or equal to half the number of crossings of the knot.*

Proof. Consider a knot diagram. Chose a point on the knot that is not a crossing and chose an orientation around the knot. Travel around the knot, starting at the chosen point and following the orientation. Each time you encounter a crossing, check if you are on the over-strand or under-strand and if you have encountered this crossing before. If it is the first time you have encountered the crossing and you are on the over-strand, do nothing. If it is the first time you have encountered the crossing and you are on the under-strand, perform a crossing change at this crossing. The second time you encounter the crossing you will be on the under-strand. Once the knot has been completely traversed, the resulting knot is the unknot.

Note that in this process the number of crossing changes performed was less than or equal to the number of crossings in the projection. We can start with a diagram with a minimal number of crossings and perform this process twice, starting at the same base point each time but traveling in opposite directions. The second time we perform the process we will change exactly the crossings we left alone the first time, and leave alone exactly the crossings that we changed before. The crossing number of the knot is equal to the sum of the crossings changed the first time and the crossings changed the second time, so one of these two numbers must be less than or equal to half the crossing number. □

Definition 4.3. *The minimal number of crossing changes necessary to transform a knot K into the unknot is called the* unknotting number *of K. These crossing changes* *need not occur in the same knot diagram. That is, the knot may be isotoped between crossing changes. Unknotting number is traditionally denoted* $u(K)$*.*

The above proposition can be written:

$$u(K) \leq \frac{1}{2}c(K).$$

In 1985 Scharlemann proved that the unknotting number of the connected sum of two non-trivial knots is at least two [27]. It is conjectured, but unfortunately not known, that unknotting number is additive under connected sum. As recently as 2013 attempts have been made to expand Scharlemann's proof to this context [23].

We do however have the following:

Proposition 4.4. *[25, Cor 1.6] Given knots* $K_1, K_2, \ldots, K_n$*,*

$$u(K_1 \# K_2 \# \cdots \# K_n) \leq u(K_1) + u(K_2) + \cdots + u(K_n).$$

Proof. If $K = K_1 \# K_2$, then a set of unkotting crossing changes for K_1 and a set of unknotting crossing changes for K_2 combine to make a set of unknotting crossing changes for K, which may or may knot be minimal. If K_1 or K_2 is also a connect sum of simpler knots, the same argument applies. □

The following well known result for torus knots will be used in a later section:

Proposition 4.5. *[25, Theorem 3.3] The unknotting number of the torus knot* $T(p,q)$ *is:*

$$u(T(p,q)) = \frac{(p-1)(q-1)}{2}.$$

4.2 The Scharlemann-Thompson result

When performing a crossing change, let K_L denote the new knot created by the crossing change.

Theorem 4.6. *[26, Cor 1.10] There exists a knot K with a crossing change L such that $u(K) > u(K_L)$, but $g(K) < g(K_L)$.*

Their argument requires the following definition:

Definition 4.7. *[26, Def 1.7] A knot K is* totally knotted *if, for every minimal genus Seifert surface S of K with regular neighborhood $n(S)$, the boundary of $n(S)$ is incompressible in $S^3 \backslash n(S)$.*

Scharlemann and Thompson show that for any totally knotted knot K with Λ a set of crossing disks for crossing changes that unknot K, then $\min\{g(S)|S$ is a Seifert surface for K disjoint from $\Lambda\}$ is larger than the genus of K.

From there, they prove that:

Theorem 4.8. *[26, Theorem 1.9] Given a totally knotted knot K and Λ a set of crossing disks that unknot K with Λ of minimal size, there is a subset $P \subset \Lambda$ such that $g(K_P) > g(K)$.*

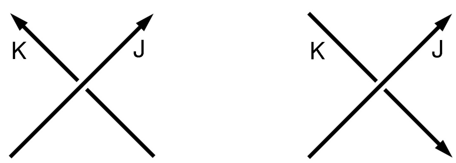
Theorem 4.6 follows as a corollary.

A specific example of a knot and a crossing change that lowers the unkotting number and raises the genus, due to Chuck Livingston, is presented in the appendix to [26]. The authors also state: “Boileau and Murakami have shown us others.”

4.3 Signature

This section is largely adapted from [24]. For the following proof we will need to use a knot invariant called signature. To define this invariant, we first begin with an orientable spanning surface S for K. Let $x_1, \ldots, x_n$ be a basis for $H_1(S)$ consisting of simple loops. Thicken the surface by taking $S \times [0,1]$, and let x_i^+ be a copy of x_i in $S \times \{1\}$ for each $i = 1, \ldots, n$. We also need the definition:

Definition 4.9. *Given two oriented knots in a projection plane, K and J, the linking number of K and J, $\ell(K,J)$ is found by labeling each crossing where J goes over K by 1 or -1 according to the following:*



(a) +1 (b) -1

Figure 4.1:

and then summing over these values.

Definition 4.10. *Given S, x_i, and x_i^+ for $i = 1, \ldots, n$ as above, the* Seifert matrix V is the $n \times n$ matrix with entries $v_{ij} = \ell(x_i, x_j^+)$.

Note that the Seifert matrix V is dependent on S, the orientation of S, and the choice of basis for $H_1(S)$.

Let $A = V + V^T$. Choosing a different orientation for S changes V to V^T, so A is unaffected by the choice of orientation. Moreover, a change of basis for $H_1(S)$ only changes A up to congruence. A is a symmetric matrix, so by the spectral theorem A is congruent to a diagonal matrix. By Sylvester's law of inertia, congruent diagonal matrices have the same number of positive entries and the same number of negative entires.

Definition 4.11. *Let n^+ be the number of positive entries in a diagonal matrix D, and n^- be the number of negative entires. Then the* signature *of D is $\sigma(D) = n^+ - n^-$.*

Note that $\sigma(D)$ is the number of positive eigenvalues of A minus the number of negative eigenvalues. According to Wikipedia, signature is $n^- - n^+$, but here we use the convention of Rolfsen.

Rolfsen shows [24, p. 218 and following] that $\sigma(D)$ is dependent only on K, not on the choice of S, enabling the following definition:

Definition 4.12. *The* signature *of a knot K is $\sigma(K) = \sigma(D)$ for a diagonal matrix congruent to A arising from any Seifert surface of K.*

The following propositions will be useful in section 4.4.

Proposition 4.13. *Signature is additive under connect sum.*

Proof. Let S be a Seifert surface for $K_1 \# K_2$, and let S_i be the Seifert surface for K_i obtained by splitting S as in section 2.3. Then a basis for $H_1(S)$ can be obtained by taking the union of the bases for $H_1(S_1)$ and $H_1(S_2)$, so a Seifert matrix for $K_1 \# K_2$ is:

$$V = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}$$

where V_i is the Seifert matrix for K_i corresponding to S_i. Thus $V + V^T$ can be diagonalized by diagonalizing $V_1 + V_1^T$ and $V_2 + V_2^T$ separately. □

Proposition 4.14. *[6, Theorem 6.8.2] Let K_+ and K_- be two knots that have identical projections except at one crossing (so K_- is obtained from K_+ by a crossing* *change and vice-versa). Then:*

$$\sigma(K_+) - \sigma(K_-) \in \{-2, 0, 2\}.$$

This proposition can be proved by constructing surfaces for K_+, K_-, and the knot with the relevant crossing split, and using the determinant of the relative matrices.

We then have the following connection between signature and unknotting number.

Proposition 4.15. *[6, cor 6.8.3] Let K be a knot. Then:*

$$u(K) \geq \frac{1}{2}|\sigma(K)|.$$

Proof. Let O denote the unknot. $u(O) = \sigma(O) = 0$, and by Proposition 4.14 a crossing change can alter the signature by at most two. Thus $|\sigma(K)| \leq 2u(K)$ and the result follows. □

The following is a special case of Theorem 1.17 in [14] and so is included here without proof.

Proposition 4.16. *The pretzel knot $P(m, n, t)$ with t even has signature:*

$$\sigma(P(m, n, t)) = -(sign(m)(|m| - 1) + sign(n)(|n| - 1)) + 1 + sign(n \cdot m).$$

4.4 The Examples

Proposition 4.17. *There exists a knot such that changing a crossing produces a knot with a lower unknotting number and a higher crosscap number.*

Proof. From SnapPy we have that $8_{19} = hg2_1(1, 1, 1)$, and since 8_{19} is not a $(2, n)$-cable of another knot we know that $cc(8_{19}) = 2$. Changing the indicated crossing produces a seven crossing knot as shown by Figure 4.2.

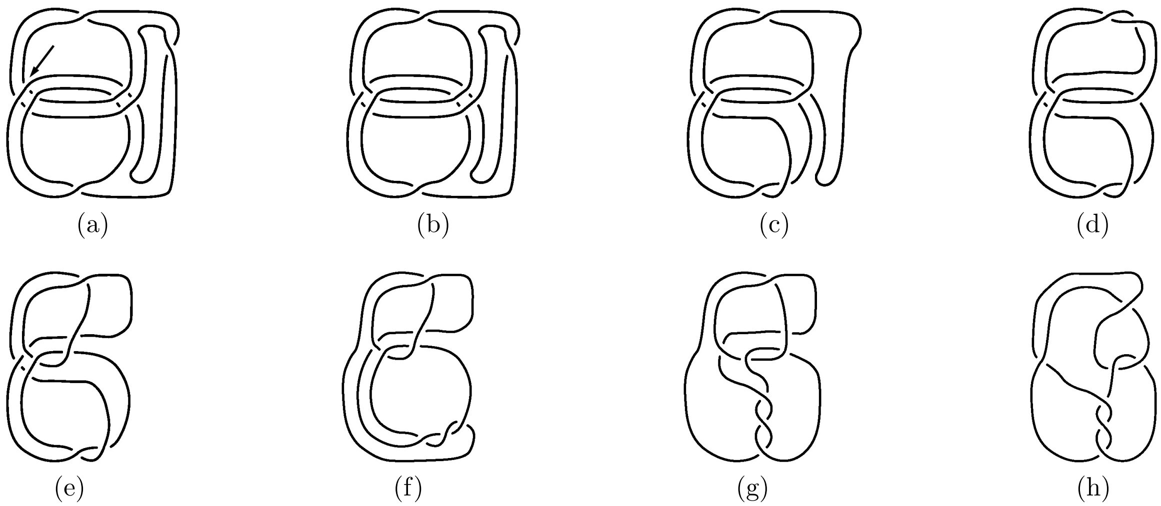


Figure 4.2: Transforming 8_{19} into 7_5 with a crossing change

This resulting knot is 7_5. It is known that $u(8_{19}) = 3$ while $u(7_5) = 2$. A lower bound for these unknotting numbers can be found using proposition 4.15, which meets the upper bound found by explicitly finding crossing changes that unknot 8_{19} and 7_5. Thus the unknotting number has gone down as a result of the crossing change between Figure 4.2 (a) and (b). As the final diagram of 7_5 is alternating, running the Adams-Kindred algorithm verifies that $cc(7_5) = 3$. Thus the crosscap number has gone up. □

By building on this example it is possible to say more.

Theorem 4.18. *There exists an infinite family of knots such that changing a crossing produces a knot with a lower unknotting number and a higher crosscap number.*

Proof. Let $\#_n K$ denote the connected sum of n copies of K.

Claim:

$$cc(8_{19}\#(\#_n 3_1)) = 2 + n \text{ and } cc(7_5\#(\#_n 3_1)) = 3 + n \text{ for all } n \geq 1.$$

Proof. (of claim) We have: $cc(3_1) = 1 < 2g(3_1) = 2$, and $cc(8_{19}) = 2 < 2g(8_{19}) = 6$. Thus by Theorem 2.8 $cc(8_{19}\#3_1) = 2 + 1 = 3$. Moreover $cc(7_5) = 3 < 2g(7_5) = 4$ so $cc(7_5\#3_1) = 3 + 1 = 4$. Thus the claim holds for $n = 1$.

Suppose the claim is true for n. Since genus is additive under connected sum, $g(8_{19}\#(\#_n 3_1)) = 3 + n$. We have:

$$cc(8_{19}\#(\#_n 3_1)) = 2 + n < 2(3 + n) = 2g(8_{19}\#(\#_n 3_1)),$$

so by Theorem 2.8 $cc(8_{19}\#(\#_{n+1} 3_1)) = 2 + (n + 1)$.

Similarly if $cc(7_5\#(\#_n 3_1)) = 3+n$, then $cc(7_5\#(\#_n 3_1)) < 2g(7_5\#(\#_n 3_1))$, and so

$$cc(7_5\#(\#_{n+1} 3_1)) = 3 + (n + 1).$$

We have established the claim. □

Thus we have an infinite family such that changing the crossing noted above gives a knot with higher crosscap number. Using Propositions 4.15 and 4.13 above we have:

$$u(8_{19}\#(\#_n 3_1)) \geq \frac{1}{2}|\sigma(8_{19}\#(\#_n 3_1))| = \frac{1}{2}|\sigma(8_{19}) + n\sigma(3_1)| = \frac{1}{2}| - 6 - 2n| = 3 + n.$$

Meanwhile from Proposition 4.4:

$$u(7_5\#(\#_n 3_1)) \leq u(7_5) + nu(3_1) = 2 + n.$$

Thus

$$u(8_{19}\#(\#_n 3_1)) \geq 3+n > 2+n \geq u(7_5\#(\#_n 3_1)).$$

So for this family of knots, the indicated crossing change also lowers the unknotting number. $\square$

The transformation of 8_{19} into 7_5 via a single crossing change gives another infinite family of knots for which the unknotting number can be lowered while the crosscap number rises. The final step of the proof utilizes a classic result of Thurston's, as formulated in a paper by Colin Adams.

Theorem 4.19. *[1, p. 125]: Given a link L with hyperbolic complement, if one does* $\frac{p_i}{q_i}$ *Dehn surgery on some subset of the components of the link, then the resulting manifold is hyperbolic for all but a finite set of* (p_i, q_i)*.*

Theorem 4.20. *There exists an infinite family of hyperbolic knots such that changing a crossing produces a knot with a lower unknotting number and a higher crosscap number.*

Proof. A similar series of pictures to those in figure 4.2 transforms $hg2_1(1, n, 1)$ into the following:

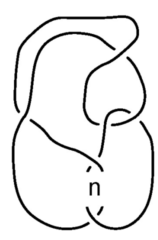


Figure 4.3: The knot K_n

For the remainder of the proof let this knot be denoted K_n. By Proposition 3.5 $hg2_1(1, n, 1)$ is the pretzel knot $P(-3, -(n+2), 2)$, and thus has crosscap number 2 by

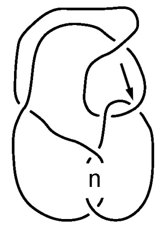


Figure 4.4: K_n with indicated crossing

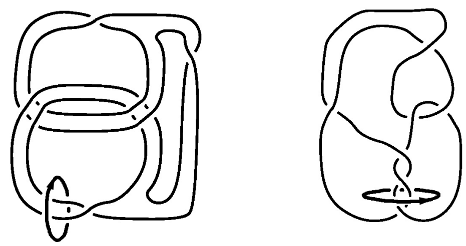


Figure 4.5: 8_{19} and K_n with crossing links

[13]. The knot K_n, on the other hand, is alternating and thus by the Adams-Kindred algorithm we have $cc(K_n) = 3$.

Then using Propositions 3.5 and 4.16:

$$\sigma(hg2_1(1, n, 1)) = \sigma(P(-3, -(n + 2), 2)) = -(-2 - (n + 1)) + 1 + 1 = n + 5.$$

Thus $u(hg2_1(1, n, 1)) \geq \frac{1}{2}|\sigma(hg2_1(1, n, 1))| = \dfrac{n+5}{2}$. On the other hand, changing the crossing indicated in figure 4.4 transforms K_n into the torus knot $T(2, n + 2)$.

So using Proposition 4.5:

$$u(K_n) \leq 1 + u(T(2, n + 2)) = 1 + \frac{n+1}{2} = \frac{n+3}{2}.$$

As this construction works for any n, we have another infinity family, built by applying $\frac{1}{n-1}$ surgery to the links in figure 4.5.

The complements of the links in figure 4.5 are hyperbolic, so by Theorem 4.19 the knots obtained by $\frac{1}{n}$ surgery are also hyperbolic for all but a finite number of n. $\square$

Chapter 5

Speculations

In the previous chapter we found a examples of knots and crossings such that changing that particular crossing lowers the unkotting number while raising the crosscap number. There is more than one way this question could be expanded. For example, is there a knot or set of knots such that every sequence of crossing changes that transforms it into the unknot contains at least one crossing change which raises the crosscap number? Is there a knot such that every crossing change which lowers the unknotting number also raises the crosscap number? This second question is an adaptation of the question concerning genus which is asked by Sharlemann and Thompson at the conclusion of [26].

It seems likely that the $(2, n)$-cabled knots provide the example asked for in the first question. By Proposition 2.2 such knots have crosscap number 1. Thus the only way there could be a $(2, n)$-cabled knot with an unknotting sequence such that no crossing change raises the crosscap number is if that unknotting sequence gives a sequence of knots, all of which are $(2, n)$-cables for various n until the sequence terminates with the unknot. This seems to imply that each crossing change in this sequence merely untwists the Möbius band which forms the crosscap minimizing surface and that the resulting $(2,1)$-cabled knot could be unknotted without raising the crosscap number. This last condition seems unlikely. The $(2,1)$-cabled knots, moreover, probably provide a positive answer to the second question.

The simplest example, the $(2,1)$-cable of the trefoil, is shown below:

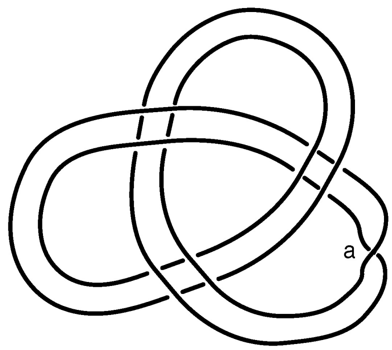


Figure 5.1: (2,1)-cable of the trefoil

This knot is 13_{n4587}, which has an unknown unknotting number. Because it is a satellite knot, the main result of [27] implies $u(13_{n4587}) \geq 2$, and from the diagram we can see $u(13_{n4587}) \leq 4u(3_1) = 4$. Changing any crossing in this diagram except the one marked a gives either 10_{152} or 12_{n426}. Both of these knots have crosscap number 3, so in this instance the crosscap number has gone up. But $u(10_{152}) = 4$, so the unknotting number cannot have gone down by changing to this knot. The unknotting number of 12_{n426} is unknown, but it is less than 3. Thus the unknotting number may or may not have gone down.

Changing the crossing a gives another crosscap number 1 knot, 13_{n4639}. From here we can obtain either 10_{154} or 12_{n830} by a crossing change. Both of these knots have crosscap number 3 and unknotting number 3. Thus this simplest example has not provided a counter example to the suggestion that (2,n)-cabled knots must have their crosscap number go up as you unknot them. Yet, without a more universal technique it is impossible to say there is not another diagram of 13_{n4587} with a crossing change that provides another crosscap number one knot.

The search in chapter 3 could of course be expanded by looking at non-prime handcuff graphs, and at the low crossing theta graphs also provided by Moriuchi [20]. Prompted by the results in chapter 3, however, it would also be interesting to look for an upper bound on the crossing number of $hgX_Y(m,n,t)$ in terms of n,m,t and the crossing number of X_Y. If found, this bound would enable an enumeration of crosscap number two knots up to a given number of crossings. It seems likely that such a bound exists provided m,n, and t are large enough, but this statement would not be enough for enumeration and it may be hard to prove. There is a classical conjecture that the crossing number of satellite knots is bounded below by some function of the crossing number of the companion knot, but the bounds that have been found require such large numbers that they are unsatisfactory [10] [17].

Appendix A

Handcuff computations

Due to the fact that $\frac{1}{n}$ Dehn surgery adds $2n$ crossings to a knot, a slightly different notation is adopted in this appendix from the one in chapter 3. In the following tables the numbers in parentheses indicate how many full twists have been given to the diagram with one twist in each loop, and either one or no twists in the connecting band. Thus $X_Y E(i,j,k) = hgX_Y(1+2i, 1+2j, 2k)$ and $X_Y O(i,j,k) = hgX_Y(1+2i, 1+2j, 1+2k)$.

After loading the diagram, the code below was run with the appropriate label given for each diagram. For some values of i, j and k SnapPy was unable to complete the computation and gave the error message "SnapPeaFatalError." Multiple iterations were found to eliminate a large number of these errors, so the code was designed to collect these values of i, j and k and run them through the computation again. The figure shows one such iteration after the initial code.

The tables below show the standard knot name on the left followed by the handcuff notation on the right. Often a given knot has more than one handcuff notation. Much of this redundancy is due to symmetries of the underlying handcuff graph.

Figure A.1: SnapPy code for 2_1E

```
MM = M.copy()
Excep=[]
for i in range(-10,11):
     for j in range(-10,11):
         for k in range(-10,11):
             M = MM.copy()
             M.dehn_fill((1,i),1)
             M.dehn_fill((1,j),2)
             M.dehn_fill((1,k),3)
             N = '2_1E(' + str(i) + ',' + str(j) + ',' + str(k) + ')'
             try:
                 ID = M.identify()
                 print N, ID
             except SnapPeaFatalError:
                 ID = 'SnapPeaFatalError'
                 Excep.append([i,j,k])
                 print N, ID
             except ValueError:
                 ID = 'ValueError'
                 Excep.append([i,j,k])
                 print N, ID
print Excep
Excep2=[]
for w in Excep:
    M = MM.copy()
    M.dehn_fill((1,w[0]),1)
    M.dehn_fill((1,w[1]),2)
    M.dehn_fill((1,w[2]),3)
    N = '2_1E(' + str(w[0]) + ',' + str(w[1]) + ',' + str(w[2]) + ')'
    try:
        ID = M.identify()
        print N, ID
    except SnapPeaFatalError:
        ID = 'SnapPeaFatalError'
        Excep2.append(w)
        print N, ID
    except ValueError:
        ID = 'ValueError'
        Excep2.append(w)
        print N, ID
print Excep2
```

Table A.1: Alternating Knots

Knot Name	Handcuff Notation			
4_1	$2_1E(1,1,0)$	$2_1O(-2,-2,0)$	$4_1E(1,3,0)$	$4_1E(3,1,0)$
5_2	$2_1E(1,-1,0)$	$2_1E(-1,1,0)$	$2_1O(-2,0,0)$	$2_1O(0,-2,0)$
	$4_1E(1,1,0)$			
6_1	$4_1E(2,2,0)$	$6_5E(2,4,0)$	$6_5E(4,2,0)$	
6_2	$2_1E(2,1,0)$	$2_1E(1,2,0)$	$2_1O(-2,-3,0)$	$2_1O(-3,-2,0)$
	$4_1E(1,4,0)$	$4_1E(4,1,0)$		
7_2	$4_1E(0,2,0)$	$4_1E(2,0,0)$	$6_5E(2,2,0)$	
7_3	$2_1E(1,-2,0)$	$2_1E(-2,1,0)$	$2_1O(-2,1,0)$	$2_1O(1,-2,0)$
	$4_1E(0,1,0)$	$4_1E(1,0,0)$		
8_1	$6_5E(3,3,0)$			
8_2	$2_1E(1,3,0)$	$2_1E(3,1,0)$	$2_1O(-4,-2,0)$	$2_1O(-2,-4,0)$
	$4_1E(1,5,0)$	$4_1E(5,1,0)$		
8_4	$4_1E(3,2,0)$	$4_1E(2,3,0)$	$6_5E(5,2,0)$	$6_5E(2,5,0)$
8_5	$2_1E(2,2,0)$	$2_1O(-3,-3,0)$		
9_2	$6_5E(1,3,0)$	$6_5E(3,1,0)$		
9_3	$2_1E(1,-3,0)$	$2_1E(-3,1,0)$	$2_1O(-2,2,0)$	$2_1O(2,-2,0)$
	$4_1E(-1,1,0)$	$4_1E(1,-1,0)$		
9_4	$4_1E(2,-1,0)$	$4_1E(-1,2,0)$	$6_5E(2,1,0)$	$6_5E(1,2,0)$
10_2	$2_1E(1,4,0)$	$2_1E(4,1,0)$	$2_1O(-5,-2,0)$	$2_1O(-2,-5,0)$
	$4_1E(6,1,0)$	$4_1E(1,6,0)$		
10_4	$6_5E(3,4,0)$	$6_5E(4,3,0)$		
10_8	$4_1E(4,2,0)$	$4_1E(2,4,0)$	$6_5E(2,6,0)$	$6_5E(6,2,0)$

Table A.2: Alternating Knots Continued

10_{46}	$2_1E(3,2,0)$	$2_1E(2,3,0)$	$2_1O(-3,-4,0)$	$2_1O(-4,-3,0)$
10_{61}	$4_1E(3,3,0)$			
11_{547}	$4_1E(2,-2,0)$	$4_1E(-2,2,0)$	$6_5E(2,0,0)$	$6_5E(0,2,0)$
11_{550}	$6_5E(0,3,0)$	$6_5E(3,0,0)$		
11_{551}	$2_1E(1,-4,0)$	$2_1E(-4,1,0)$	$2_1O(-2,3,0)$	$2_1O(3,-2,0)$
	$4_1E(1,-2,0)$	$4_1E(-2,1,0)$		
12_{a722}	$2_1E(5,1,0)$	$2_1E(1,5,0)$	$2_1O(-2,-6,0)$	$2_1O(-6,-2,0)$
	$4_1E(1,7,0)$	$4_1E(7,1,0)$		
12_{a838}	$2_1E(2,4,0)$	$2_1E(4,2,0)$	$2_1O(-3,-5,0)$	$2_1O(-5,-3,0)$
12_{a1157}	$4_1E(2,5,0)$	$4_1E(5,2,0)$	$6_5E(2,7,0)$	$6_5E(7,2,0)$
12_{a1214}	$2_1E(3,3,0)$	$2_1O(-4,-4,0)$		
12_{a1242}	$4_1E(3,4,0)$	$4_1E(4,3,0)$		
12_{a1278}	$6_5E(3,5,0)$	$6_5E(5,3,0)$		
12_{a1286}	$6_5E(4,4,0)$			
13_{a4834}	$6_5E(-1,3,0)$	$6_5E(3,-1,0)$		
13_{a4874}	$2_1E(1,-5,0)$	$2_1E(-5,1,0)$	$2_1O(-2,4,0)$	$2_1O(4,-2,0)$
	$4_1E(1,-3,0)$			
13_{a4866}	$4_1E(2,-3,0)$	$4_1E(-3,2,0)$	$6_5E(-1,2,0)$	$6_5E(2,-1,0)$
14_{a12197}	$2_1E(1,6,0)$	$2_1E(6,1,0)$	$2_1O(-2,-7,0)$	$2_1O(-7,-2,0)$
	$4_1E(1,8,0)$	$4_1E(8,1,0)$		
14_{a13328}	$2_1E(2,5,0)$	$2_1E(5,2,0)$	$2_1O(-3,-6,0)$	$2_1O(-6,-3,0)$
14_{a17701}	$4_1E(6,2,0)$	$4_1E(2,6,0)$	$6_5E(2,8,0)$	$6_5E(8,2,0)$
14_{a18246}	$2_1E(3,4,0)$	$2_1E(4,3,0)$	$2_1O(-5,-4,0)$	$2_1O(-4,-5,0)$
14_{a18510}	$4_1E(5,3,0)$	$4_1E(3,5,0)$		
14_{a19420}	$6_5E(6,3,0)$	$6_5E(3,6,0)$		
14_{a19484}	$4_1E(4,4,0)$			
14_{a19524}	$6_5E(4,5,0)$	$6_5E(5,4,0)$		

Table A.3: Non-alternating Pretzel Knots

Knot Name	Handcuff Notation			
8_{20}	$2_1O(-3,0,0)$	$2_1O(0,-3,0)$		
10_{125}	$2_1E(-1,3,0)$	$2_1E(3,-1,0)$	$2_1O(-4,0,0)$	$2_1O(0,-4,0)$
10_{126}	$2_1E(-2,2,0)$	$2_1E(2,-2,0)$	$2_1O(-3,1,0)$	$2_1O(1,-3,0)$
10_{140}	$4_1E(0,3,0)$			
10_{142}	$4_1E(0,0,0)$			
12_{n233}	$2_1E(-3,2,0)$	$2_1E(2,-3,0)$	$2_1O(-3,2,0)$	$2_1O(2,-3,0)$
12_{n235}	$2_1E(4,-1,0)$	$2_1E(-1,4,0)$	$2_1O(-5,0,0)$	$2_1O(0,-5,0)$
12_{n242}	$2_1E(-1,-2,1)$	$2_1E(-2,-1,1)$	$2_1O(1,0,-1)$	$2_1O(0,1,-1)$
	$2_1O(0,2,0)$	$2_1O(2,0,0)$		
12_{n474}	$4_1E(-1,0,0)$	$4_1E(0,-1,0)$		
12_{n475}	$4_1E(0,4,0)$			
12_{522}	$4_1E(3,-1,0)$	$4_1E(-1,3,0)$		
12_{n581}	$6_5E(1,1,0)$			
12_{n582}	$6_5E(1,4,0)$	$6_5E(4,1,0)$		
12_{n721}	$2_1E(-2,3,0)$	$2_1E(3,-2,0)$	$2_1O(-4,1,0)$	$2_1O(1,-4,0)$
12_{n725}	$2_1E(-2,-2,0)$	$2_1O(1,1,0)$		

Table A.4: Non-pretzel knots found to have crosscap number two by Burton and Ozlen

Knot Name	Handcuff Notation			
10_{139}	$2_1E(-1,-1,1)$	$2_1O(0,0,-1)$		
10_{145}	$2_1E(0,0,1)$	$2_1O(-1,-1,-1)$		
10_{161}	$2_1O(-1,0,-1)$	$2_1O(0,-1,-1)$		
11_{n102}	$2_1E(0,0,-1)$	$2_1O(-1,-1,1)$		
11_{n104}	$2_1E(-1,-1,-1)$	$2_1O(0,0,1)$		
11_{n135}	$2_1E(-1,0,-1)$	$2_1E(0,-1,-1)$	$2_1O(-1,0,1)$	$2_1O(0,-1,1)$
12_{n121}	$2_1E(0,1,1)$	$2_1E(1,0,1)$	$2_1O(-1,-2,-1)$	$2_1O(-2,-1,-1)$
12_{n404}	$2_1E(0,0,2)$	$2_1O(-1,-1,-2)$		
12_{n575}	$2_1E(-1,-1,2)$	$2_1O(0,0,-2)$		
12_{n591}	$2_1E(0,-2,1)$	$2_1E(-2,0,1)$	$2_1O(-1,1,-1)$	$2_1O(1,-1,-1)$
12_{n749}	$2_1E(1,-1,1)$	$2_1E(-1,1,1)$	$2_1O(0,-2,-1)$	$2_1O(-2,0,-1)$
12_{n851}	$2_1E(0,-1,2)$	$2_1E(-1,0,2)$	$2_1O(0,-1,-2)$	$2_1O(-1,0,-2)$

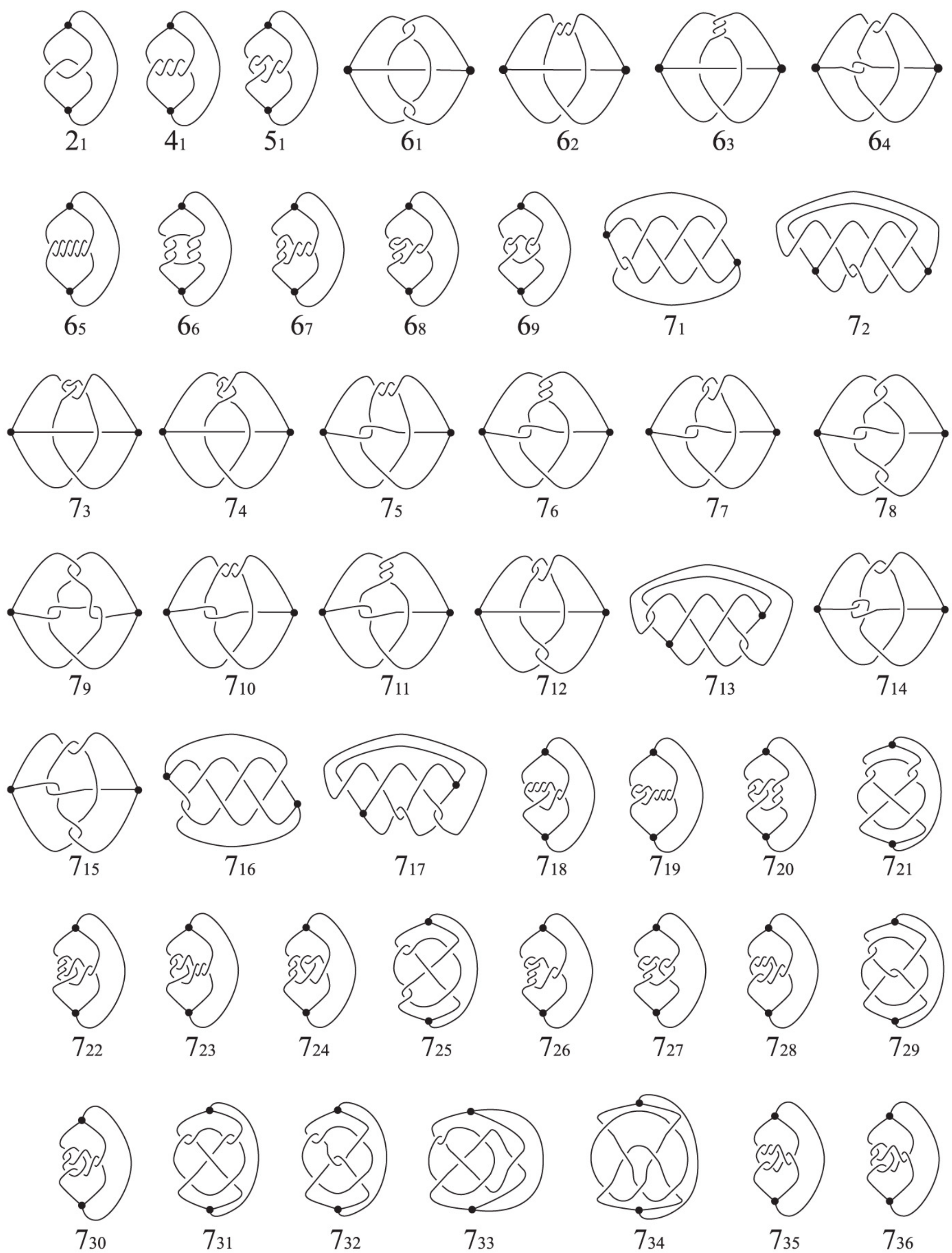
Table A.5: Non-alternating 13-crossing knots

Knot Name	Handcuff Notation			
13_{n469}	$2_1E(1,0,-1)$	$2_1E(0,1,-1)$	$2_1O(-2,-1,1)$	$2_1O(-1,-2,1)$
13_{n1153}	$2_1E(-1,-2,-1)$	$2_1E(-2,-1,-1)$	$2_1O(0,1,1)$	$2_1O(1,0,1)$
13_{n2872}	$2_1E(1,-1,-1)$	$2_1E(-1,1,-1)$	$2_1O(0,-2,1)$	$2_1O(-2,0,1)$
13_{n2969}	$2_1E(0,0,-2)$	$2_1O(-1,-1,2)$		
13_{n3061}	$2_1E(-1,-1,-2)$	$2_1O(0,0,2)$		
13_{n3082}	$2_1E(-2,0,-1)$	$2_1E(0,-2,-1)$	$2_1O(-1,1,1)$	$2_1O(1,-1,1)$
13_{n4738}	$2_1E(-1,0,-2)$	$2_1E(0,-1,-2)$	$2_1O(-1,0,2)$	$2_1O(0,-1,2)$
13_{n5016}	$4_1O(-1,-1,1)$			

Table A.6: Non-alternating 14-crossing knots

Knot Name	Handcuff Notation			
14_{n3611}	$2_1E(0,1,2)$	$2_1E(1,0,2)$	$2_1O(-2,-1,-2)$	$2_1O(-1,-2,-2)$
14_{n6004}	$2_1E(2,-4,0)$	$2_1E(-4,2,0)$	$2_1O(-3,3,0)$	$2_1O(3,-3,0)$
14_{n6006}	$2_1E(-1,5,0)$	$2_1E(5,-1,0)$	$2_1O(-6,0,0)$	$2_1O(0,-6,0)$
14_{n6022}	$2_1E(-1,-4,0)$	$2_1E(-4,-1,0)$	$2_1O(0,3,0)$	$2_1O(3,0,0)$
14_{n6023}	$2_1E(-1,-2,-2)$	$2_1E(-2,-1,2)$	$2_1O(1,0,-2)$	$2_1O(0,1,-2)$
14_{n12204}	$4_1E(0,-2,0)$	$4_1E(-2,0,0)$		
14_{n12205}	$4_1E(0,5,0)$			
14_{n12939}	$4_1E(-2,3,0)$	$4_1E(3,-2,0)$		
14_{n14254}	$2_1E(2,0,1)$	$2_1E(0,2,1)$	$2_1O(-3,-1,-1)$	$2_1O(-1,-3,-1)$
14_{n15069}	$2_1E(0,0,3)$	$2_1O(-1,-1,-3)$		
14_{n15961}	$6_5E(0,1,0)$	$6_5E(1,0,0)$		
14_{n15962}	$6_5E(5,1,0)$	$6_5E(1,5,0)$		
14_{n16364}	$2_1E(-3,0,1)$	$2_1E(0,-3,1)$	$2_1O(-1,2,-1)$	$2_1O(2,-1,-1)$
14_{n16886}	$6_5E(0,4,0)$	$6_5E(4,0,0)$		
14_{n18095}	$2_1E(-1,-1,3)$	$2_1O(0,0,-3)$		
14_{n18935}	$5_1E(0,1,0)$			
14_{n21316}	$2_1E(3,-3,0)$	$2_1E(-3,3,0)$	$2_1O(2,-4,0)$	$2_1O(-4,2,0)$
14_{n21318}	$2_1E(-2,4,0)$	$2_1E(4,-2,0)$	$2_1O(-5,1,0)$	$2_1O(1,-5,0)$
14_{n21324}	$2_1E(-2,-3,0)$	$2_1E(-3,-2,0)$	$2_1O(1,2,0)$	$2_1O(2,1,0)$
14_{n21882}	$2_1E(2,-1,1)$	$2_1E(-1,2,1)$	$2_1O(0,-3,-1)$	$2_1O(-3,0,-1)$
14_{n24552}	$4_1E(-1,-1,0)$			
14_{n24553}	$4_1E(-1,4,0)$	$4_1E(4,-1,0)$		
14_{n24834}	$2_1E(0,-2,2)$	$2_1E(-2,0,2)$	$2_1O(1,-1,-2)$	$2_1O(-1,1,-2)$
14_{n26238}	$2_1E(0,-1,3)$	$2_1E(-1,0,3)$	$2_1O(-1,0,-3)$	$2_1O(0,-1,-3)$
14_{n27136}	$2_1E(1,-1,2)$	$2_1E(-1,1,2)$	$2_1O(-2,0,-2)$	$2_1O(0,-2,-2)$

Figure A.2: Table of prime handcuff graphs from [19]



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