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Formalizing Categorical and Algebraic Constructions in Operator Theory

William Benjamin Grilliette
University of Nebraska-Lincoln, evilpii@juno.com

FORMALIZING CATEGORICAL AND ALGEBRAIC CONSTRUCTIONS IN OPERATOR THEORY

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William Benjamin Grilliette

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FORMALIZING CATEGORICAL AND ALGEBRAIC CONSTRUCTIONS IN OPERATOR THEORY

William Benjamin Grilliette, Ph. D.

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Adviser: David R. Pitts

In this work, I offer an alternative presentation theory for C*-algebras with applicability to various other normed structures. Specifically, the set of generators is equipped with a nonnegative-valued function which ensures existence of a C*-algebra for the presentation. This modification allows clear definitions of a "relation" for generators of a C*-algebra and utilization of classical algebraic tools, such as Tietze transformations.

To those who laid the foundation, built upon it, and will build it higher.

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Chapter 1

Background

1.1 Introduction

This dissertation consists of one primary theme, a new view of presentation theory for C*-algebras. While there already exists a presentation theory for C*-algebras, featured in [4] and [27], a key feature of the perspective given in this work is its ability to use combinatorial algebra techniques, such as Tietze transformations and other formal manipulations. In particular, Theorems 3.9.7 and 4.7.2 yield Tietze transformation results analogous to the well-known group theory result of [38].

Also, the methods by which this presentation theory is built enable new types of relations to be imposed within the C*-algebras. Specifically, the continuous and analytic functional calculi allow relations such as "$\sin(x) = 0$", "$x \geq 0$", and "$\|x\| \leq \lambda$" to be imposed, among innumerable others not viable in pure algebraic settings. Table 1.1 gives a listing of the main examples presented within this work, a presentation for each, and where each is located.

The notion of applying combinatorial algebra to C*-algebras is not completely new, considered previously in [19]. However, this treatment uses only *-algebraic

Table 1.1: Main Examples in Considered in this Work

C*-algebra	Sample Presentation	Example
$\mathbb{C}^{2n+1}, n \in \mathbb{W} := \mathbb{N} \cup \{0\}$	$\left\langle (x, n\pi) \,\middle\vert\, \begin{array}{c} \sin(x) = 0, \\ x^*x = xx^* \end{array} \right\rangle_{\mathbf{1C^*}}$	3.6.1
$\mathbb{C}^{2n+2}, n \in \mathbb{W}$	$\left\langle \left(x, \frac{\pi}{2} + n\pi\right) \,\middle\vert\, \begin{array}{c} \cos(x) = 0, \\ x^*x = xx^* \end{array} \right\rangle_{\mathbf{1C^*}}$	3.6.2
$C[0,1]$	$\langle (x,1) \,\vert x = x^* \rangle_{\mathbf{1C^*}}$	3.5.2
$C(\mathbb{T})$	$\langle (x,1) \,\vert x^*x = xx^* = \mathbb{1} \rangle_{\mathbf{1C^*}}$	3.5.3
$C([-2,-1] \cup [1,2])$	$\langle (x,2) \,\vert x = x^*, x^*x \geq \mathbb{1} \rangle_{\mathbf{1C^*}}$	3.11.6
$C\left(A_{1,2}\right)$	$\langle (x,2) \,\vert x^*x = xx^*, x^*x \geq \mathbb{1} \rangle_{\mathbf{1C^*}}$	3.11.7
$C\left(\prod_{\lambda \in \Lambda} \overline{\mathbb{D}}\right), \Lambda \neq \emptyset$	$\left\langle \Lambda, \mathbf{1}_{[0,\infty)} \,\middle\vert\, \begin{array}{c} xy = yx, \\ xy^* = y^*x \end{array} \;\forall x, y \in \Lambda \right\rangle_{\mathbf{1C^*}}$	3.15.7
$\mathcal{T}$	$\langle (x,1) \,\vert x^*x = \mathbb{1} \rangle_{\mathbf{1C^*}}$	3.5.4
$\begin{bmatrix} C[0,1] & C_0(0,1] \\ C_0(0,1] & C[0,1] \end{bmatrix}$	$\left\langle (x,2) \,\middle\vert x = x^2 \right\rangle_{\mathbf{1C^*}}$	3.12.7
$\begin{bmatrix} C[0,1] & C_0(0,1) \\ C_0(0,1) & C[0,1] \end{bmatrix}$	$\left\langle (p,1), (q,1) \,\middle\vert\, \begin{array}{c} p = p^* = p^2, \\ q = q^* = q^2 \end{array} \right\rangle_{\mathbf{1C^*}}$	3.10.2
$C[0,1] *_{\mathbb{C}} C(\mathbb{T})$	$\langle (x,2) \,\vert x^*x \geq \mathbb{1}, xx^* \geq \mathbb{1} \rangle_{\mathbf{1C^*}}$	3.11.8
$C[0,1] *_{\mathbb{C}} \mathcal{T}$	$\langle (x,2) \,\vert x^*x \geq \mathbb{1} \rangle_{\mathbf{1C^*}}$	3.11.9
$C_0\left([-1,0) \cup (0,1]\right)$	$\langle (x,1) \,\vert x = x^* \rangle_{\mathbf{C^*}}$	4.6.2
$C_0(0,1]$	$\langle (x,1) \,\vert x \geq 0 \rangle_{\mathbf{C^*}}$	4.6.5
$C_0\left(\left(\prod_{\lambda \in \Lambda} \overline{\mathbb{D}}\right) \setminus \left\{\vec{0}\right\}\right), \Lambda \neq \emptyset$	$\left\langle \Lambda, \mathbf{1}_{[0,\infty)} \,\middle\vert\, \begin{array}{c} xy = yx, \\ xy^* = y^*x \end{array} \;\forall x, y \in \Lambda \right\rangle_{\mathbf{C^*}}$	4.6.7
$\begin{bmatrix} C[0,1] & C_0(0,1] \\ C_0(0,1] & C_0(0,1] \end{bmatrix}$	$\left\langle (x,2) \,\middle\vert x = x^2 \right\rangle_{\mathbf{C^*}}$	4.6.8
$\begin{bmatrix} C[0,1] & C_0(0,1) \\ C_0(0,1) & C_0[0,1) \end{bmatrix}$	$\left\langle (p,1), (q,1) \,\middle\vert\, \begin{array}{c} p = p^* = p^2, \\ q = q^* = q^2 \end{array} \right\rangle_{\mathbf{C^*}}$	4.6.9

relations, which limits the manipulations which can be performed. In Remark 2.4.1.13 of [19], it is conjectured that a Tietze transformation theorem for C*-algebras would require so many assumptions as to be practically worthless. Using only *-algebraic relations, this may well be true, but the perspective of the current work attains such theorems with few initial assumptions by means of relations made from the functional calculus as described above. A more detailed comparison is given in Section 4.3, showing that the current work extends that of [19].

Further, since there does already exist a presentation theory for C*-algebras, several sections of this work are made specifically for recapturing these well-known ideas in the context of the new perspective. Notions such as abelianization (Sections 3.4 and 4.5), unitization (Section 4.4), free products (Sections 3.10 and 4.8), separability (Sections 3.14 and 4.9), and projectivity (Sections 3.15 and 4.10) are all considered and given very natural characterizations, directly reflecting their classical interpretations. Also, many of the examples of the existing presentation theory are shown to coincide appropriately with the new perspective of the current work, as seen in Table 1.1.

Lastly, the methods used in this work for C*-algebras have potential to be used in other normed algebraic settings. In particular, the foundational work in Chapter 2 and the constructions within Sections 3.1 and 4.1 are very general and can be translated into other settings with relative ease. For this reason, Chapter 2 holds more general results than are needed for the main thrust of constructing C*-algebras. However, it is of note that in the general setting of Chapter 2, many well-known constructions of normed objects reappear without any notion of linearity or distance.

To outline the contents of this dissertation, the remainder of this chapter explains the fundamental failure of applying classical combinatorial methods to normed structures, as well as some previous work which has developed as a result. Chapter 2 describes the foundational structure for the presentation theory, a *crutched set*, which encodes the norm data. Chapter 3 builds the presentation theory for unital C*-algebras from the ground work of Chapter 2 along with the classical notions explained in Section 1.2. These constructions and characterizations here are very algebraic and categorical in flavor, so some readers may want to read Appendices A and B for relevant background. Chapter 4 repeats this process for general C*-algebras, noting differences where necessary.

1.2 Classical Situation: Sets & Free Algebras

This section considers the algebraic notion of a free algebra, derived as a reflection along a forgetful functor. This construction is classical and well-known, but it is very central to the contents of Chapter 2. Here, these ideas will be treated summarily, stating results without proof. Full treatments of these notions can be found in most resources on the subject, such as [6] and [25].

Fix a commutative ring R with unit 1. Let $R\mathbf{1Alg}$ denote the category whose objects are unital R-algebras and whose arrows are unital R-algebra homomorphisms under composition. Explicitly, $\mathrm{Ob}(R\mathbf{1Alg})$ is the class of all unital R-algebras, and for $A, B \in \mathrm{Ob}(R\mathbf{1Alg})$, $R\mathbf{1Alg}(A, B)$ is the set of all unital R-algebra homomorphisms from A to B.

Let **Set** denote the category whose objects are sets and whose arrows are functions under composition. Explicitly, $\mathrm{Ob}(\mathbf{Set})$ is the class of all sets, and for $S, T \in \mathrm{Ob}(\mathbf{Set})$, $\mathbf{Set}(S, T)$ is all functions from S to T.

As every $A \in \mathrm{Ob}(R\mathbf{1Alg})$ is a set, there is a natural "forgetful" map to $\mathrm{Ob}(\mathbf{Set})$ where one regards A as a mere set, ignoring all its R-algebra structure. Similarly, given $A, B \in \mathrm{Ob}(R\mathbf{1Alg})$ and $\phi \in R\mathbf{1Alg}(A, B)$, ϕ is firstly a function from A to B, meaning $\phi \in \mathbf{Set}(A, B)$. One can quickly check that these two associations define a functor $F_{R\mathbf{1Alg}} : R\mathbf{1Alg} \to \mathbf{Set}$, where one ignores all the algebraic data from $R\mathbf{1Alg}$. This is a prime example of a "forgetful" functor.

Now, fix $S \in \mathrm{Ob}(\mathbf{Set})$, thought of as a set of generators. The objective is to build a reflection of S along $F_{R\mathbf{1Alg}}$. Specifically, a *reflection* of S along $F_{R\mathbf{1Alg}}$ is a $C \in \mathrm{Ob}(R\mathbf{1Alg})$ equipped with $\eta \in \mathbf{Set}\left(S, F_{R\mathbf{1Alg}}C\right)$ such that for any $B \in \mathrm{Ob}(R\mathbf{1Alg})$ and $\phi \in \mathbf{Set}\left(S, F_{R\mathbf{1Alg}}B\right)$, there is a unique $\hat{\phi} \in R\mathbf{1Alg}(C, B)$ such that $F_{R\mathbf{1Alg}}\hat{\phi} \circ \eta = \phi$. In short, any function from the generation set S into a unital R-algebra determines a unique extension to the reflection object.

To construct this universal object, let M_S be the set of all finite sequences of elements from S, thought of as non-commuting monomials. Specifically, one requires that the empty list u be included in M_S. Under concatenation of lists, M_S is naturally a monoid with unit u, free monoid on S.

Next, let A_S be the set of all functions from M_S to R whose support is finite, thought of as non-commuting polynomials with coefficients from R. Under point-wise addition and scalar multiplication, A_S is naturally an R-module, the free module on M_S. Further, each function can be written uniquely as an R-linear sum of functions with singleton support and value 1, δ_l for each $l \in M_S$.

Vector multiplication is determined by the usual polynomial formula. Explicitly, given $p = \sum_{j=1}^{n} \lambda_j \delta_{l_j}$ and $r = \sum_{k=1}^{q} \mu_k \delta_{m_k}$,

$$pr := \sum_{j=1}^{n} \sum_{k=1}^{q} \lambda_j \mu_k \delta_{l_j m_k},$$

where $l_j m_k$ is the product in M_S.

Under these operations, it is a standard exercise to show A_S to be an R-algebra with unit δ_u. Specifically, A_S resides in $\mathrm{Ob}(R\mathbf{1Alg})$. As such, one can consider $F_{R\mathbf{1Alg}}A_S$, this algebra without its structure. There is a canonical map $\eta_S : S \to A_S$ by $\eta_S(s) := \delta_s$ for the singleton listing of s alone. Similarly, it is a standard exercise to show the unital R-algebra A_S equipped with η_S is a reflection of S along $F_{R\mathbf{1Alg}}$, the free unital R-algebra on S.

Further, since S was arbitrary, Proposition A.5.1 states that there is a unique functor $L : \mathbf{Set} \to R\mathbf{1Alg}$ such that $LS = A_S$, and $L \dashv F_{R\mathbf{1Alg}}$ by Theorem A.5.2.

In most purely algebraic settings, a "free object" can be regarded as reflection along a forgetful functor to **Set** from a particular category of interest. The above construction is mimicked in each setting, yielding such objects as free modules, free groups, and the like. The universal mapping property of the reflection in each case is typically called the "free mapping property" as it has no restriction on where a generator can be mapped in the target object.

1.3 Failure of Freeness for Normed Settings

This section considers the failure of the classical notion of a free object in many normed algebraic contexts. This fact is well-known in the literature and has motivated many new developments and constructions in the field with hopes of remedying the issue, including this present work. As such, this issue will be considered in detail.

Fix $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $\mathbb{F}\mathbf{NVec}_1$ denote the category whose objects are normed $\mathbb{F}$-vector spaces and whose arrows are $\mathbb{F}$-linear transformations which are contractive. Explicitly, $\mathrm{Ob}\,(\mathbb{F}\mathbf{NVec}_1)$ is the class of normed $\mathbb{F}$-vector spaces, and for $A, B \in \mathrm{Ob}\,(\mathbb{F}\mathbf{NVec}_1)$, $\mathbb{F}\mathbf{NVec}_1(A, B)$ is the set of all contractive $\mathbb{F}$-linear transformations from A to B.

Let $\mathscr{C}$ be a subcategory of $\mathbb{F}\mathbf{NVec}_1$. That is, $\mathrm{Ob}(\mathscr{C})$ is a subclass of $\mathrm{Ob}(\mathbb{F}\mathbf{NVec}_1)$.

Further, for all $A, B \in \mathrm{Ob}(\mathscr{C})$, $\mathscr{C}(A,B) \subseteq \mathbb{F}\mathbf{NVec}_1(A,B)$ such that identity arrows are present and compositions remain within $\mathscr{C}$.

As every $A \in \mathrm{Ob}(\mathscr{C})$ is a set, there is a natural forgetful map to $\mathrm{Ob}(\mathbf{Set})$ where one regards A as a mere set, ignoring all its algebraic and topological structure. Similarly, given $A, B \in \mathrm{Ob}(\mathscr{C})$ and $\phi \in \mathscr{C}(A,B)$, ϕ is firstly a function from A to B, meaning $\phi \in \mathbf{Set}(A,B)$. One can quickly check that these two associations define a functor $F_{\mathscr{C}} : \mathscr{C} \to \mathbf{Set}$, where one ignores all the algebraic and topological data from $\mathscr{C}$.

As in Section 1.2, fix S from $\mathrm{Ob}(\mathbf{Set})$. One would like to find a reflection of S along $F_{\mathscr{C}}$, but unfortunately, this is quite rare. To show this, let $\mathbb{0} := \{0\}$, the zero space.

Proposition 1.3.1. *Let $S \neq \emptyset$. If there is $V \in \mathrm{Ob}(\mathscr{C})$ such that $V \ncong_{\mathbb{F}\mathbf{NVec}_1} \mathbb{0}$, then S has no reflection along $F_{\mathscr{C}}$.*

Proof. Assume for purposes of contradiction that S had a reflection (R, η) along $F_{\mathscr{C}}$. As $V \ncong_{\mathbb{F}\mathbf{NVec}_1} \mathbb{0}$, there is $v \in V$ with $\|v\|_V \neq 0$. For $n \in \mathbb{N}$, define $\phi_n \in \mathbf{Set}\left(S, F_{\mathscr{C}} V\right)$ by $\phi_n(s) := nv$, a constant function. Then, there must exist $\hat{\phi}_n \in \mathscr{C}(R,V)$ such that $F_{\mathscr{C}}\hat{\phi}_n \circ \eta = \phi_n$ for all $n \in \mathbb{N}$.

Define $r_s := \eta(s) \in R$. As each $\hat{\phi}_n$ is an arrow in $\mathbb{F}\mathbf{NVec}_1$, for all $n \in \mathbb{N}$ and $s \in S$,

$$\|r_s\|_R \geq \left\|\hat{\phi}_n\left(r_s\right)\right\|_V = \left\|\left(F_{\mathscr{C}}\hat{\phi}_n \circ \eta\right)(s)\right\|_V = \|\phi_n(s)\|_V = n\|v\|_V.$$

As $\|v\|_V \neq 0$, the right-hand side increases without bound. Hence, $\|r_s\|_R$ cannot have a finite value for any $s \in S$, which cannot occur in R. As such, this R is complete fiction.

□

This proposition has said something quite poignant. Unless one restricts to a trivial class of normed vector spaces, e.g. just isomorphic copies of $\mathbb{O}$, or considers an empty set of generators, there is no normed $\mathbb{F}$-vector space with the free mapping property, regardless of all other restrictions of object class or sets of contractive homomorphisms.

Since the free mapping property is a cornerstone to many constructions in pure algebra, particularly presentation theory, this is a most discouraging fact. In particular, what the above proposition states is that the category of C*-algebras and *-homomorphisms cannot have nontrivial free objects, nor can its unital counterpart. However, this statement encompasses many other settings, such as operator algebras and Banach algebras with any class of contractive homomorphisms.

1.4 Previous Work on This Problem

This section considers existing work in the literature dealing with the critical issue addressed within Section 1.3. Since there is no free object in any nontrivial subcategory of normed spaces by Proposition 1.3.1, some sacrifice must be made to remedy the situation, and each of the following references takes an approach to that end. Each work accepts that no free object exists within the category of C*-algebras and *-homomorphisms, but focuses mainly on creating universal objects in this category which are subject to certain relations.

However, in the definition of "relation" itself, there has been much debate. As found in standard references [13] and [25], when one considers an algebraic context with a free object, such as a group or ring, a *relation* is simply an element of this free object. Yet, without a free object, how does one then define "relation"?

For a fixed set S, one response to this question can be found in [27]: "any conditions that can possibly hold for a map $j : S \to \mathcal{B}$ into a C*-algebra, with one proviso." Explicitly, the map j is termed a *representation* of relations R if all conditions with R hold within $\mathcal{B}$ for $j(S)$. For the proviso, it is required that conditions R respect inclusions in the following way. If $\phi : \mathcal{B} \to \mathcal{C}$ is an isometric *-homomorphism, then $j : S \to \mathcal{B}$ is a representation of R in $\mathcal{B}$ if and only if $\phi \circ j$ is a representation of R in $\mathcal{C}$. However, while there are examples of the types of conditions are shown, like norm bounds and *-polynomials, the actual criteria for such a condition are left nebulous.

All authors agree that *-polynomials, which are easily forced via quotient methods as in the algebraic case, should be considered as "relations". Most also include norm bounds as "relations" since one can restrict to certain types of "admissable" representations to build a C*-algebra with a universal mapping property, as shown in [4].

However, the analytic functional calculus also is most agreeable with *-homomorphism, as is the continuous functional calculus, when applicable. Specifically, one would desire "relations" such as "$\sin(x) = 0$" and "$0 \leq x \leq \mathbb{1}$". Several explicit examples are shown in [4] and [27].

However, within a C*-algebra, some conditions can force certain norm conditions as well. For example, if one considers the *-algebraic conditions $x = x^*$ and $x = x^2$, the defining notion of a projection, an operator x satisfying these must have norm at most 1. As such, some authors do not necessitate a norm bound when other conditions impose one. Specifically, universal C*-algebras of graphs and other combinatorial objects have such conditions, as shown in [18], and [36]. Further, there are sets of conditions which do not enforce norm bounds, such as $x = x^2$ and others cited in [27].

While all of these are examples of what a "relation" should be, a clear definition remains elusive. Hence, one returns to the base question of how to replace the free object in the picture of universal algebra for C*-algebras and *-homomorphisms. Within [21], a "free C*-algebra" is defined to be the *-monoid C*-algebra of the free *-monoid on a given set, the universal C*-algebra on a set of contractions. Similarly [14] stated that this algebra is "the closest one gets to free C*-algebras", though in [27], it is noted that this algebra is clearly not free, corroborated by Proposition 1.3.1.

One potential replacement is suggested in [22]. For a set S, one forms the free $\mathbb{C}$-algebra B_S on elements of S and their formal adjoints, much like in Section 1.2. From here, one considers the class of functions $f : S \to \mathcal{B}(\mathcal{H}_f)$, each inducing a *-representation of B_S. Also, one considers the functions $n : S \to [0, \infty)$, each inducing a semi-norm on B_S. Together, these are used to create a locally convex *-algebra, thought of as a non-commutative version of $C(\mathbb{C})$. However, though it does have a connection to a certain kind of freeness, this is not a C*-algebra. It is more closely related to the pro-C*-algebras developed in [35], created by changing categories to topological *-algebras over $\mathbb{C}$ and continuous *-homomorphisms.

There are more categorical approaches as well. In [29] and [28], a "C*-relation" is defined by considering a full subcategory of representations. Explicitly, for a fixed set S, the "null C*-relation" is the category defined as follows: the objects are functions $j : S \to \mathcal{B}$ while the arrows between $(j, \mathcal{B})$ and $(k, \mathcal{C})$ are *-homomorphisms $\phi : \mathcal{B} \to \mathcal{C}$ such that $\phi \circ j = k$, making the appropriate triangle commute. A general "C*-relation" is then a subcategory of this structure, subject to certain axioms. Then, the universal C*-algebra of this relation would be the initial object in this subcategory, an object with precisely one outgoing arrow for every other object. In category theory, constructions like this "null C*-relation" are rather standard to realize a particular universal object as an initial object, such as a *comma category*. However, this point of view obscures the classical picture established in Section 1.2, as well as the intuitive notion of a "relation" described above.

In a different direction, [31] considers a functor, unital C*-algebras and unital *-homomorphisms to groups and group homomorphisms by taking the unitary group. Here, the functor is shown to have a left adjoint, namely the group C*-algebra functor, and a few of its functorial properties are considered.

In [33] and [34], several forgetful functors are considered to try and understand the categorical nature of the algebraic theory of C*-algebras and *-homomorphisms. Specifically, [34] considers the forgetful functor to Banach *-algebras, recognizing the enveloping C*-algebra as its left adjoint. Similarly, [33] considers forgetful functors to the unit ball, the self-adjoint part of the ball, and the positive part of the ball. In each case, a left adjoint exists, recreating the C*-algebraic structure. Further, they each explore the operations to build the equational theory. However, both recognize that the "free C*-algebra", again the universal C*-algebra of a set of contractions, is difficult to understand so this equational theory is very vague and unclear.

However, the spirit of [31], [33], and [34] reflects that of Section 1.2. With a similar spirit, the author presents another alternative in the following chapter, very functorial and algebraic in flavor.

1.5 The Thesis due to Gerbracht

When a substantial portion of this present work had been completed, the thesis [19] due to Eberhard Hans-Alexander Gerbracht came to the author's attention. As the author was initially unaware of [19], several ideas overlap between the present work and that of Gerbracht. In particular, both consider modifying the construction of Section 1.2 with a category of sets equipped with a nonnegative-valued function, a presentation theory for C*-algebras, and Tietze transformations for this theory.

While there is overlap, [19] adheres strictly to an algebraic view of C*-algebras.

The only relations considered throughout the work are *-polynomials in the generators, implemented by norming or topologizing the *-algebra over $\mathbb{C}$ subject to those relations. The Tietze transformations, while proven with the nonnegative-valued function in play, are only defined and used when the *-polynomials imply a bound on the norms of all the generators. Further, Remark 2.4.1.13 in [19] states that an analog of Tietze's transformation theorem for C*-algebras would require so many assumptions as to be practically worthless.

However, in the present work, the "relations" defined in Section 3.5 are created as limits of *-polynomials, allowing for the use of the functional calculi to impose conditions used often in functional analysis. Respectively, Tietze transformations of Section 3.9 allow for the addition and removal of such elements. Further, Theorem 3.9.7 and Corollary 3.9.8 establish analogs of the Tietze transformation theorem seen in group theory without major assumptions. Throughout, the nonnegative function is retained and shown to have great influence on the resultant C*-algebra, creating a "bifurcation" theory presented in Section 3.16.

Also, Section 2.2 does a detailed categorical analysis of the underlying category used in the modified construction, making several claims made in [19] more precise. Moreover, the constructions done in this analysis are used to streamline the respective C*-algebraic constructions, taking advantage of the adjoint relationship at the core of both works.

Lastly, this present work introduces a more general version of this auxiliary category with applications to normed algebraic structures equipped with bounded linear maps. Section 2.3 also does a detailed categorical analysis of this structure for comparison to the original "contractive" version, as well as future use with functional analysis.

Chapter 2

Crutched Sets

In this chapter, another object is defined and explored, creating a working environment for a forgetful functor.

In Section 1.2, the forgetful functor from unital R-algebras and their homomorphisms to sets and their functions has a left adjoint. That is, one can forget the algebraic structure and then reconstruct it, up to quotient. However, as observed in Section 1.3, the forgetful functor from normed $\mathbb{F}$-algebras and contractive homomorphisms to sets and functions does not have a left adjoint. One cannot reconstruct the normed algebra structure simply from sets alone in the same manner.

Since a normed algebra cannot be reconstructed if all its structure is stripped away, something more must be retained. Specifically, as Proposition 1.3.1 shows, the norm is the component causing the issue. Hence, the central notion taken here is that of a forgetful functor which strips away all data save two components: the underlying set and the norm. This object was previously introduced in [19], which also recognized this norm issue.

The upcoming forgetful functor should be compared to the well-known “unit ball” functor. Explicitly, the functor goes from Banach spaces and contractive maps to sets and functions by associating a Banach space with its closed unit ball and a contraction with its restriction to the unit ball. As shown in [1], every set S has a reflection along this functor, namely $\ell^1(S)$. Similarly in [33], the unit ball functor from unital C*-algebras and *-homomorphisms to sets also has a left adjoint, specifically the universal C*-algebra of a set of contractions.

2.1 Definitions & Basic Results

As explained in the introduction to this chapter, the objective is to construct a category so that the forgetful functor from C*-algebras and *-homomorphisms will have a left adjoint. Explicitly, the objects will be a set with a "sizing" function.

Definition. A *crutched set* is a pair (S, f), where S is a set and f a function from S to $[0, \infty)$. The function f is called the *crutch function*. For $s \in S$, s is said to be *crutched* by the value $f(s)$, and $f(s)$ is the *crutch value* of s.

In Section 3.1, the nomenclature "crutched" becomes more clear, where this nonnegative-valued function supports, much like a crutch, the algebraic construction of Section 1.2 to the construction of a C*-algebra. Arguably, one could call this property "normed", but the author chooses not to use this term as there is no linearity assumed on f. Indeed, f is simply *any* set mapping from S to $[0, \infty)$. In Chapter 3, this level of generality is shown to be quite powerful and useful. This object was also considered in [19].

Example 2.1.1. Given any normed vector space V, define $f_V : V \to [0, \infty)$ by $f_V(v) := \|v\|_V$, the norm function. Then, (V, f_V) is a crutched set. This is the most key example of a crutched set as it will be half of the forgetful functor in Section 3.1.

Example 2.1.2. Let $(a_n)_{n\in\mathbb{N}} \subset [0,\infty)$. Define $f_{\vec{a}} : \mathbb{N} \to [0,\infty)$ by $f_{\vec{a}}(n) := a_n$. Then, $(\mathbb{N}, f_{\vec{a}})$ is a crutched set. Truthfully, the sequence is actually a function already, but $f_{\vec{a}}$ puts it into the notation of the definition.

In many cases in Chapter 3, it will be advantageous to regard a crutched set as a collection of pairs, an element of S and a nonnegative real value, rather than a set and a function. As such, it will be a common practice to write a crutched set as set of pairs, like the previous example, when the set is countable. Since a function is fundamentally a set of pairs, this second notation essentially regards the underlying set as an index for the crutch values.

Example 2.1.3. Let $S := \{s,t\}$ and $f : S \to [0,\infty)$ be a crutch function. Let $\lambda := f(x)$ and $\mu := f(y)$. Then, (S,f) can be also written as

$$\{(s,\lambda),(t,\mu)\}.$$

The arrows between two crutched sets should preserve the structure, specifically the crutch function. To that end, the following definitions are made purposefully analogous to the notion of linear continuity for normed structures.

Definition. Given two crutched sets (S,f) and (T,g), a function $\phi : S \to T$ is *bounded* if there is $M \geq 0$ such that for all $s \in S$, $g(\phi(s)) \leq Mf(s)$. This will be denoted $\phi : (S,f) \to (T,g)$. Let

$$\text{crh}(\phi) := \inf\left\{M \in [0,\infty) : g(\phi(s)) \leq Mf(s) \forall s \in S\right\},$$

the *crutch bound* of ϕ. If $\text{crh}(\phi) \leq 1$, ϕ is *constrictive.*

Similarly, use of existing terminology like "norm" or "contraction" is avoided, as there is no concept of linearity or distance in this setting. However, as these notions are analogous, familiar results follow immediately from definition.

First, the relationship between the crutch bound of a bounded function and the crutch functions of its domain and codomain directly mirrors the relationship between the norm of a bounded linear map and the norms of its domain and codomain.

Proposition 2.1.4. *Let (S, f) and (T, g) be crutched sets and $\phi : (S, f) \to (T, g)$ be bounded. Then, for all $s \in S$,*

$$g(\phi(s)) \leq \operatorname{crh}(\phi) f(s).$$

Proof. For $n \in \mathbb{N}$, there is $M_n \in \{M \in [0, \infty) : g\left(\phi(s)\right) \leq M f(s) \forall s \in S\}$ such that $\operatorname{crh}(\phi) \leq M_n \leq \operatorname{crh}(\phi) + \dfrac{1}{n}$. For each $s \in S$,

$$g(\phi(s)) \leq M_n f(s) \leq \left(\operatorname{crh}(\phi) + \frac{1}{n}\right) f(s).$$

Letting $n \to \infty$, $g(\phi(s)) \leq \operatorname{crh}(\phi) f(s)$.

□

Observe that as a result, if $\phi : (S, f) \to (T, g)$ is constrictive, $g(\phi(s)) \leq f(s)$ for all $s \in S$. This is taken as definition for the maps considered in [19].

The above proposition immediately yields the following result regarding compositions of bounded functions, reflecting its counterpart for bounded linear maps.

Corollary 2.1.5. *Let (S, f), (T, g), and (U, h) be crutched sets and $\phi : (S, f) \to (T, g)$ and $\psi : (T, g) \to (U, h)$ be bounded. Then, $\psi \circ \phi : S \to U$ is bounded and*

$$\operatorname{crh}(\psi \circ \phi) \leq \operatorname{crh}(\psi) \operatorname{crh}(\phi).$$

Proof. For each $s \in S$,

$$h((\psi \circ \phi)(s)) \leq \operatorname{crh}(\psi) g(\phi(s)) \leq \operatorname{crh}(\psi) \operatorname{crh}(\phi) f(s).$$

□

Corollary 2.1.6. *Let (S,f), (T,g), and (U,h) be crutched sets and $\phi : (S,f) \to (T,g)$ and $\psi : (T,g) \to (U,h)$ be constrictive. Then, $\psi \circ \phi : (S,f) \to (U,h)$ is constrictive.*

Proof. As ψ and ϕ are constrictive,

$$\operatorname{crh}(\psi \circ \phi) \leq \operatorname{crh}(\psi) \operatorname{crh}(\phi) \leq 1.$$

□

Also, the computation of the crutch bound can be reformulated from an infimum to a supremum in a familiar way.

Proposition 2.1.7. *Let (S,f) and (T,g) be crutched sets and $\phi : (S,f) \to (T,g)$ be bounded. Then,*

$$\operatorname{crh}(\phi) = \sup\left(\left\{\frac{g(\phi(s))}{f(s)} : s \notin f^{-1}(0)\right\} \cup \{0\}\right).$$

Proof. Let $L := \sup\left(\left\{\frac{g(\phi(s))}{f(s)} : s \notin f^{-1}(0)\right\} \cup \{0\}\right)$. For all $s \notin f^{-1}(0)$,

$$0 \leq g(\phi(s)) \leq \operatorname{crh}(\phi) f(s)$$

so

$$0 \leq \frac{g(\phi(s))}{f(s)} \leq \operatorname{crh}(\phi).$$

Thus, $L \leq \operatorname{crh}(\phi)$.

For $s \notin f^{-1}(0)$, $\dfrac{g(\phi(s))}{f(s)} \leq L$ so $g(\phi(s)) \leq Lf(s)$. For $s \in f^{-1}(0)$,

$$0 \leq g(\phi(s)) \leq \operatorname{crh}(\phi) f(s) = 0.$$

Then, $g(\phi(s)) = 0 = Lf(s)$. Therefore, $\operatorname{crh}(\phi) \leq L$.

□

From this result, alternate criteria for boundedness can be devised.

Proposition 2.1.8. *Let (S, f) and (T, g) be crutched sets. A function $\phi : S \to T$ is bounded if and only if*

$$\sup\left(\left\{\frac{g(\phi(s))}{f(s)} : s \notin f^{-1}(0)\right\} \cup \{0\}\right) < \infty$$

and $g(\phi(s)) = 0$ for all $s \in f^{-1}(0)$.

Proof. ($\Rightarrow$) This direction is the content of Proposition 2.1.7.

($\neg \Rightarrow \neg$) Assuming that ϕ is not bounded, then for each $M \geq 0$, there is $s_M \in S$ such that $g\left(\phi\left(s_M\right)\right) > Mf\left(s_M\right)$. If some $s_M \in f^{-1}(0)$, then

$$g\left(\phi\left(s_M\right)\right) > Mf\left(s_M\right) = 0.$$

If $s_M \notin f^{-1}(0)$ for all $M \geq 0$, then

$$\frac{g\left(\phi\left(s_M\right)\right)}{f\left(s_M\right)} > M$$

for every $M \geq 0$. Hence,

$$\sup\left(\left\{\frac{g(\phi(s))}{f(s)} : s \notin f^{-1}(0)\right\} \cup \{0\}\right) = \infty.$$

□

Now, observe that the criterion on $f^{-1}(0)$ is necessary. Without linearity in ϕ, $f^{-1}(0)$ does not necessarily get mapped into $g^{-1}(0)$.

Example 2.1.9. Let V and W be normed vector spaces and $\phi : V \to W$ be a bounded linear function. Let f_V and f_W be crutch functions on V and W, respectively, defined as in Example 2.1.1. By Propositions 2.1.7 and 2.1.8, ϕ is a bounded function from (V, f_V) to (W, f_W) and $\text{crh}(\phi) = \|\phi\|_{\mathcal{B}(V,W)}$. As in Example 2.1.1, this is the key example as it will be the other half of the forgetful functor in Section 3.1.

Example 2.1.10. Given a crutched set (S, f), let $id_{(S,f)} : S \to S$ by $id_{(S,f)}(s) := s$, the identity function. Then, as $f \circ id_{(S,f)} = f$, $id_{(S,f)}$ is constrictive with

$$\text{crh}\left(id_{(S,f)}\right) = \begin{cases} 1, & S \neq f^{-1}(0), \\ 0, & \text{otherwise.} \end{cases}$$

Example 2.1.11. Let $S := T := \mathbb{N}$. Define $f : S \to [0, \infty)$ by $f(n) := n$ and $g : T \to [0, \infty)$ by $g(n) := \dfrac{1}{n}$. Further, let $\phi : S \to T$ by $\phi(n) := n$. Then, for each $n \in S$,

$$\frac{g(\phi(n))}{f(n)} = \frac{\frac{1}{n}}{n} = \frac{1}{n^2} \leq 1,$$

meaning ϕ is bounded and $\text{crh}(\phi) = 1$ by Propositions 2.1.7 and 2.1.8. In particular, ϕ is constrictive.

However, let $\psi : T \to S$ by $\psi(n) := n$, the inverse set map of ϕ. For $n \in T$,

$$\frac{f(\psi(n))}{g(n)} = \frac{n}{\frac{1}{n}} = n^2.$$

Thus, ψ is unbounded by Proposition 2.1.8.

2.2 Category of Crutched Sets & Constrictive Maps

This section is dedicated a detailed study of crutched sets and constrictive functions between them. This combination of objects and maps was considered previously in [19]. For notation, the symbol $\mathbf{CSet}_1$ will be used to denote the following data:

- $\mathrm{Ob}\left(\mathbf{CSet}_1\right) :=$ the class of all crutched sets;
- For $(S, f), (T, g) \in \mathrm{Ob}(\mathbf{CSet}_1)$, define

$$\mathbf{CSet}_1((S, f), (T, g)) := \{\phi \in \mathbf{Set}(S, T) : \phi \text{ constrictive from } (S, f) \text{ to } (T, g)\}.$$

Equipping this structure with function composition, which is well-known to be associative with identity maps as the units of this operation, $\mathbf{CSet}_1$ is a category from Corollary 2.1.6 and Example 2.1.10.

Proposition 2.2.1. $\mathbf{CSet}_1$ *is a category.*

With this new structure defined, one considers some of its basic properties and constructions. Many of these will be very familiar to those with experience with **Set**. However, this is done for three particular reasons.

First, it is good practice to understand a new class of mathematical objects when they are defined. There may well be interesting uses of these objects that become clear when their structure is observed.

Second, the designed use of $\mathbf{CSet}_1$ is to replace **Set** in the construction of Section 1.2. If $\mathbf{CSet}_1$ and **Set** were equivalent as categories, then one could just rearrange the construction and make **Set** the target instead. Hence, one should show that the two are genuinely not the same structure.

Last, and probably most interesting, the basic constructions immediately resemble their counterparts in normed structures. This means that with simply sets and positive functions, subject to "non-increasing" maps, the traditional notions of normed structures are partially recovered. This seems to indicate the dependency of these notions on the positive function, not algebraic structure or notions of linearity or distance.

For completeness, several standard categorical notions will be applied to describe $\mathbf{CSet}_1$. However, many of these are not directly applicable to the overall purpose of constructing C*-algebras. Three particular results are the most related to the construction of Section 3.1. The first is the "disjoint union" coproduct in Proposition 2.2.9, which gives a standard decomposition not only of a crutched set, but of the resulting C*-algebras in Corollaries 3.2.5 and 4.2.5. The second is the distinction between $\mathbf{CSet}_1$ and **Set**, which is a direct corollary to Proposition 2.2.11. The third is Proposition 2.2.13, which generalizes the failure result of Proposition 1.3.1 to subcategories of $\mathbf{CSet}_1$.

To begin, consider the primary properties of constrictive mappings, listed below:

- an *isomorphism* is an invertible map;

- a *section* is a left-invertible map, its dual notion a *retraction*;

- a *monomorphism* is a left-cancelable map, its dual notion an *epimorphism*.

This proposition gives necessary and sufficient criteria for each of these notions in $\mathbf{CSet}_1$, adding precision the statements made in Remark 1.1.9 from [19].

Proposition 2.2.2. *Let (S, f) and (T, g) be crutched sets and $\phi : (S, f) \to (T, g)$ be constrictive.*

1. *ϕ is a monomorphism in $\mathbf{CSet}_1$ iff ϕ is one-to-one;*
2. *ϕ is an epimorphism in $\mathbf{CSet}_1$ iff ϕ is onto;*
3. *ϕ is a section in $\mathbf{CSet}_1$ iff ϕ is one-to-one, $g \circ \phi = f$, and for all $t \notin \phi(S)$, there is $s_t \in S$ such that $f(s_t) \leq g(t)$;*
4. *ϕ is a retraction in $\mathbf{CSet}_1$ iff for all $t \in T$, $\phi^{-1}(t) \cap f^{-1}(g(t)) \neq \emptyset$;*
5. *ϕ is an isomorphism in $\mathbf{CSet}_1$ iff ϕ is one-to-one, onto, and $g \circ \phi = f$.*

Proof. 1. ($\Rightarrow$) Assume that ϕ is a monomorphism in $\mathbf{CSet}_1$. Let $s, \hat{s} \in S$ such that $\phi(s) = \phi(\hat{s})$. Let $U := \{0\}$ and $h(0) := \max\{f(s), f(\hat{s})\}$. Define $\alpha(0) := s$ and $\beta(0) := \hat{s}$. Note that α and β are both constrictive, and

$$(\phi \circ \alpha)(0) = \phi(s) = \phi(\hat{s}) = (\phi \circ \beta)(0).$$

Hence, $\phi \circ \alpha = \phi \circ \beta$, meaning $\alpha = \beta$. Therefore, $s = \alpha(0) = \beta(0) = \hat{s}$.

($\Leftarrow$) Assume ϕ is one-to-one. For any crutched set (U, h), let $\alpha, \beta : (U, h) \to (S, f)$ be constrictive such that $\phi \circ \alpha = \phi \circ \beta$. For all $u \in U$, $\phi(\alpha(u)) = \phi(\beta(u))$. Since ϕ is one-to-one, $\alpha(u) = \beta(u)$, meaning $\alpha = \beta$.

2. ($\Rightarrow$) Assume ϕ is an epimorphism in $\mathbf{CSet}_1$. Let $U := \{0,1\}$ and $h(u) := 0$. Define $\alpha, \beta : T \to U$ by $\alpha(t) := 0$ and

$$\beta(t) := \begin{cases} 0, & t \in \operatorname{ran}(\phi), \\ 1, & t \notin \operatorname{ran}(\phi). \end{cases}$$

Note that α and β are both constrictive, and for all $s \in S$,

$$(\alpha \circ \phi)(s) = 0 = (\beta \circ \phi)(s).$$

Thus, $\alpha \circ \phi = \beta \circ \phi$ so $\alpha = \beta$. Therefore, for all $t \in T$, $\beta(t) = \alpha(t) = 0$, meaning $T = \operatorname{ran}(\phi)$.

($\Leftarrow$) Assume ϕ is onto. For any crutched set (U, h), let $\alpha, \beta : (T, g) \to (U, h)$ be constrictive such that $\alpha \circ \phi = \beta \circ \phi$. For all $t \in T$, there is some $s \in S$ such that $t = \phi(s)$. Thus, $\alpha(t) = (\alpha \circ \phi)(s) = (\beta \circ \phi)(s) = \beta(t)$ so $\alpha = \beta$.

3. ($\Rightarrow$) Assume that ϕ is a section in $\mathbf{CSet}_1$. Then, there is a constrictive $\psi : (T, g) \to (S, f)$ such that $\psi \circ \phi = id_{(S,f)}$. From basic function results, ϕ must be one-to-one. For all $s \in S$,

$$f(s) = \left(f \circ id_{(S,f)}\right)(s) = (f \circ \psi \circ \phi)(s) \leq (g \circ \phi)(s) \leq f(s)$$

so $f(s) = (g \circ \phi)(s)$, meaning $f = g \circ \phi$. Lastly, let $s_t := \psi(t)$ for each $t \in T$. Then, $f(s_t) \leq g(t)$.

($\Leftarrow$) Assuming the result, define $\psi : T \to S$ by

$$\psi(t) := \begin{cases} s, & t = \phi(s), \\ s_t, & t \notin \phi(S). \end{cases}$$

As ϕ is one-to-one, this is a well-defined function, and $\psi \circ \phi = id_{(S,f)}$ by design. To prove ψ constrictive, observe that for $s \in S$,

$$f(\psi(\phi(s))) = f(s) = g(\phi(s))$$

and for $t \notin \phi(S)$,

$$f(\psi(t)) = f(s_t) \leq g(t).$$

4. ($\Rightarrow$) Assume that ϕ is a retraction in $\mathbf{CSet}_1$. Then, there is a constrictive $\psi : (T, g) \to (S, f)$ such that $\phi \circ \psi = id_{(T,g)}$. For $t \in T$, let $s_t := \psi(t)$. Observe that $\phi(s_t) = t$ and

$$g(t) = \left(g \circ id_{(T,g)}\right)(t) = (g \circ \phi \circ \psi)(t) \leq f(\psi(t)) = f(s_t) \leq g(t).$$

Thus, $g(t) = f(s_t)$ so $s_t \in \phi^{-1}(t) \cap f^{-1}(g(t))$.

($\Leftarrow$) Assuming the result, let $s_t \in \phi^{-1}(t) \cap f^{-1}(g(t))$ and define $\psi : T \to S$ by $\psi(t) := s_t$. Then, $\phi \circ \psi = id_{(T,g)}$ by design. To prove ψ constrictive, observe that for all $t \in T$,

$$f(\psi(t)) = f(s_t) = g(t).$$

5. ($\Rightarrow$) Assume that ϕ is an isomorphism in $\mathbf{CSet}_1$. Then, ϕ is both a section and a retraction, in particular also an epimorphism. Hence, ϕ is one-to-one, onto, and $f = g \circ \phi$.

($\Leftarrow$) Assuming the result, ϕ is an epimorphism as it is onto. Further, $T \backslash \phi(S) = \emptyset$ by this fact, meaning ϕ is further a section. Hence, ϕ is an isomorphism.

$\square$

It is of some note that each of the conditions in Item 5 are necessary. In particular, the condition $f = g \circ \phi$ is reminiscent of isometry in normed spaces. However, these crutch functions are not linear, nor do they reflect any idea of metric distance. Hence, this condition alone does not imply even monomorphism, let alone isomorphism.

Example 2.2.3. Let $S := \{0\}$, $f(0) := 1$, and $g(0) := 0$. Define $\phi : S \to S$ by $\phi(0) := 0$, a constrictive map. However, while ϕ is both monic and epic, it is not a section or retraction. This example concretely demonstrates the statement made in Remark 1.1.9 of [19] about monic and epic constrictions, which are not sections or retractions.

Example 2.2.4. Let $S := \mathbb{N}$, $f(n) := 1$, $T := \{0\}$, and $g(0) := 1$. Define $\phi : S \to T$ by $\phi(n) := 0$, a constrictive map. Then, ϕ is a retraction and $g \circ \phi = f$, but it is not a monomorphism.

Similarly, define $\varphi : T \to S$ by $\varphi(0) := 1$, another constrictive map. Then, φ is section, but it is not an epimorphism.

Next, consider the standard universal constructions in $\mathbf{CSet}_1$. First, an equalizer of two parallel morphisms is a universal way to compare maps. Explicitly, for two maps $\alpha, \beta : (S, f) \to (T, g)$, an *equalizer* of α and β is a crutched set (K, k) equipped with a constrictive map $\iota : (K, k) \to (S, f)$ satisfying

- $\alpha \circ \iota = \beta \circ \iota$,
- for a crutched set (U, h) and a constrictive map $\phi : (U, h) \to (S, f)$ such that $\alpha \circ \phi = \beta \circ \phi$, there is a unique $\hat{\phi} : (U, h) \to (K, k)$ such that $\iota \circ \hat{\phi} = \phi$.

As it happens, this notion characterizes the substructures in this category.

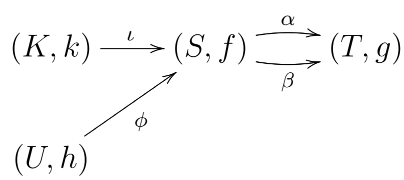
Proposition 2.2.5. *Let (S, f) and (T, g) be crutched sets and $\alpha, \beta : (S, f) \to (T, g)$ constrictive maps. Let*

$$K := \{s \in S : \alpha(s) = \beta(s)\},$$

$k := f|_K$, and $\iota : K \to S$ by $\iota(s) := s$. Then, (K, k) equipped with ι is an equalizer of α and β.

Further, given any $L \subseteq S$, define $l := f|_L$. Then, (L, l) can be realized as an equalizer of two parallel arrows from (S, f).

Proof. From definition, ι is constrictive and $\alpha \circ \iota = \beta \circ \iota$. To check the universal property, let (U, h) be a crutched set and $\phi : (U, h) \to (S, f)$ be constrictive such that $\alpha \circ \phi = \beta \circ \phi$.



Then, for all $u \in U$, $(\alpha \circ \phi)(u) = (\beta \circ \phi)(u)$. Hence, $\phi(u) \in K$ so define $\hat{\phi} := \phi|^K$, restricting its codomain. Since the crutch function is likewise restricted, $\hat{\phi}$ is constrictive. Also, $\phi = \iota \circ \hat{\phi}$ by expansion of codomain.

Assume that there was $\varphi : (U, h) \to (K, k)$ such that $\phi = \iota \circ \varphi$. Then, $\iota \circ \varphi = \iota \circ \hat{\phi}$ and as ι is one-to-one, $\varphi = \hat{\phi}$.

For (L, l), let $T := \{0, 1\}$ and $g(t) := 0$. Define $\alpha, \beta : S \to T$ by $\alpha(s) := 0$ and

$$\beta(s) := \begin{cases} 0, & s \in L, \\ 1, & s \notin L. \end{cases}$$

Then, (L, l) will be an equalizer of α and β.

$\square$

The dual notion is a *coequalizer*, similar to a quotient. The crutch function in this case sharply reflects the quotient norm in normed algebraic structures.

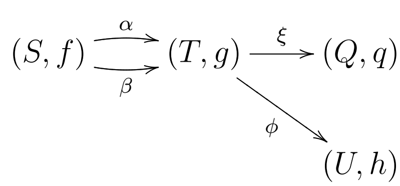
Proposition 2.2.6. *Let (S, f) and (T, g) be crutched sets and $\alpha, \beta : (S, f) \to (T, g)$ constrictive maps. Let*

$$P := \left\{(\alpha(s), \beta(s)) \in T^2 : s \in S\right\}$$

and $\sim_P$ be the equivalence relation on T generated by P. Define $Q := T/\sim_P$, $q([t]) := \inf\{g(\tau) : \tau \sim_P t\}$, and $\xi : T \to Q$ by $\xi(t) := [t]$. Then, (Q, q) equipped with ξ is a coequalizer of α and β.

Further, given any equivalence relation $\sim$ on T, define $r : T/\sim \to [0, \infty)$ by $r([t]) := \inf\{g(\tau) : \tau \sim t\}$. Then, $(T/\sim, r)$ can be realized as a coequalizer of two parallel arrows to (T, g).

Proof. From definition, ξ is constrictive and $\xi \circ \alpha = \xi \circ \beta$. To check the universal property, let (U, h) be a crutched set and $\phi : (T, g) \to (U, h)$ be constrictive such that $\phi \circ \alpha = \phi \circ \beta$.



Consider $\sim_\phi := \left\{(t, \tau) \in T^2 : \phi(t) = \phi(\tau)\right\}$. Note that $\sim_\phi$ is an equivalence relation. Further, for all $s \in S$, $(\phi \circ \alpha)(s) = (\phi \circ \beta)(s)$. Hence, $P \subseteq \sim_\phi$ so $\sim_P \subseteq \sim_\phi$. Thus, if $t \sim_P \tau$, $t \sim_\phi \tau$, or rather, $\phi(t) = \phi(\tau)$. Hence, define $\hat{\phi} : Q \to U$ by $\hat{\phi}([t]) := \phi(t)$. By the above argument, this is well-defined. For all $t \in T$,

$$\left(\hat{\phi} \circ \xi\right)(t) = \hat{\phi}([t]) = \phi(t),$$

meaning $\hat{\phi} \circ \xi = \phi$. Now, for all $\tau \in [t]$,

$$\left(h \circ \hat{\phi}\right)([t]) = (h \circ \phi)(\tau) \leq g(\tau).$$

Hence, $\left(h \circ \hat{\phi}\right)([t]) \leq q([t])$, meaning $\hat{\phi}$ is constrictive.

Assume there was some other constrictive $\varphi : (Q, q) \to (U, h)$ such that $\phi = \varphi \circ \xi$. Then, for all $t \in T$,

$$\varphi([t]) = (\varphi \circ \xi)(t) = \phi(t) = \hat{\phi}([t]).$$

Thus, $\varphi = \hat{\phi}$.

For $(T/\sim, r)$, let $S :=\sim$ and $f(t, \tau) := \max\{g(t), g(\tau)\}$. Define $\alpha, \beta : S \to T$ by $\alpha(t, \tau) := t$ and $\beta(\tau) := \tau$. Then, $(T/\sim, r)$ will be an coequalizer of α and β.

□

While equalizers and coequalizers compare parallel maps, the next pair of constructions are more designed to combine sets of objects into a new structure. For an index set I and crutched sets $(S_i, f_i)_{i \in I}$, a *product* of $(S_i, f_i)_{i \in I}$ is a crutched set (P, f) equipped with constrictive maps $\pi_i : (P, f) \to (S_i, f_i)$ for each $i \in I$ satisfying for any other crutched set (U, h) and constrictive maps $\phi_i : (U, h) \to (S_i, f_i)$ for $i \in I$, there is a unique $\hat{\phi} : (U, h) \to (P, f)$ such that $\pi_i \circ \hat{\phi} = \phi_i$. Specifically, the product in $\mathbf{CSet}_1$ should be compared to the ℓ^∞-sum of normed spaces.

Proposition 2.2.7. *For an index set I, let (S_i, f_i) be crutched sets for $i \in I$. Define*

$$P := \left\{ \vec{s} \in \mathbf{Set}\left(I, \bigcup_{i \in I} S_i\right) : \vec{s}(i) \in S_i \forall i \in I, \sup\left\{f_i\left(\vec{s}(i)\right) : i \in I\right\} < \infty \right\},$$

$f : P \to [0, \infty)$ by $f\left(\vec{s}\right) := \sup\left\{f_i\left(\vec{s}(i)\right) : i \in I\right\}$, and $\pi_i : P \to S_i$ by $\pi_i\left(\vec{s}\right) := \vec{s}(i)$. Then, (P, f) equipped with $(\pi_i)_{i \in I}$ is a product of $((S_i, f_i))_{i \in I}$.

Proof. From definition, π_i is constrictive for each $i \in I$. To check the universal property, let (U, h) be a crutched set and $\phi_i : (U, h) \to (S_i, f_i)$ be constrictive for all $i \in I$.

$$\begin{array}{ccc} (P, f) & \xrightarrow{\pi_i} & (S_i, f_i) \\ & \nearrow_{\phi_i} & \\ (U, h) & & \end{array}$$

For each $u \in U$, observe that $(f_i \circ \phi_i)(u) \leq h(u)$. Hence,

$$\sup \{(f_i \circ \phi_i)(u) : i \in I\} < \infty$$

so define $\phi : U \to P$ by $\phi(u)(i) := \phi_i(u)$. Then, $\pi_i \circ \phi = \phi_i$ for each $i \in I$. Also,

$$(f \circ \phi)(u) = \sup \{(f_i \circ \phi_i)(u) : i \in I\} \leq h(u),$$

making ϕ constrictive.

Assume there was some other constrictive $\varphi : (U, h) \to (P, f)$ such that $\pi_i \circ \varphi = \phi_i$. Then, for each $i \in I$ and $u \in U$,

$$(\pi_i \circ \varphi)(u) = \phi_i(u)$$

Hence, $\varphi(u)(i) = \phi_i(u) = \phi(u)(i)$, meaning $\varphi(u) = \phi(u)$. Therefore, $\varphi = \phi$.

□

Notice that there are times when a product of nontrivial objects is trivial.

Example 2.2.8. Define $S := \{0\}$ and $f_n(0) := n$ for all $n \in \mathbb{N}$. Then,

$$\prod_{n \in \mathbb{N}}^{\mathbf{CSet_1}} (S, f_n) \cong_{\mathbf{CSet_1}} \left(\emptyset, \mathbf{0}_{[0,\infty)}\right),$$

the empty set and the empty function into $[0, \infty)$. Further, the canonical projections from the product to S are empty functions, which are hardly onto mappings.

The dual notion is a *coproduct*. In $\mathbf{CSet}_1$, the coproduct is more closely related to the disjoint union in **Set**. This is of interest as it gives a canonical way of writing any crutched set in terms of singletons. This will be of great use throughout the main results of this work, starting with Corollaries 3.2.5 and 4.2.5.

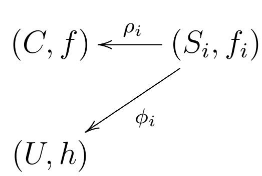
Proposition 2.2.9. *For an index set I, let (S_i, f_i) be crutched sets for $i \in I$. Define*

$$C := \left\{ (i, s) \in I \times \left(\bigcup_{i \in I} S_i \right) : s \in S_i \right\},$$

$f : C \to [0, \infty)$ by $f(i, s) := f_i(s)$, and $\rho_i : S_i \to C$ by $\rho_i(s) := (i, s)$. Then, (C, f) equipped with $(\rho_i)_{i \in I}$ is a coproduct of $((S_i, f_i))_{i \in I}$.

Further, for any crutched set (T, g), then (T, g) is a coproduct of $(\{(t, g(t))\})_{t \in T}$ when equipped with the standard inclusion maps.

Proof. From definition, ρ_i is constrictive for each $i \in I$. To check the universal property, let (U, h) be a crutched set and $\phi_i : (S_i, f_i) \to (U, h)$ be constrictive for all $i \in I$.



Define $\phi : C \to U$ by $\phi(i, s) := \phi_i(s)$. Then, for each $i \in I$ and $s \in S_i$,

$$(\phi \circ \rho_i)(s) = \phi(i, s) = \phi_i(s)$$

meaning $\phi \circ \rho_i = \phi_i$. Also,

$$(h \circ \phi)(i, s) = (h \circ \phi_i)(s) \leq f_i(s) = f(i, s)$$

so ϕ is constrictive.

Assume that there was some other constrictive $\varphi : (C, f) \to (U, h)$ such that $\varphi \circ \rho_i = \phi_i$. Then, for each $i \in I$ and $s \in S_i$,

$$\varphi(i, s) = (\varphi \circ \rho_i)(s) = \phi_i(s) = \phi(i, s).$$

Therefore, $\varphi = \phi$.

Given a crutched set (T, g), note that

$$\left\{(t, \tau) \in T \times \left(\bigcup_{t \in T} \{t\}\right) : \tau \in \{t\}\right\} = \{(t, t) : t \in T\} \cong_{\mathbf{Set}} T$$

and

$$f(t, t) = g|_{\{t\}}(t) = g(t).$$

Thus, the second result follows.

□

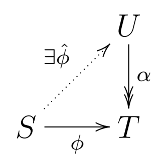
As $\mathbf{CSet}_1$ has all products and equalizers, all the other standard limit processes can be performed. Dually, colimit processes follow from the existence of all coproducts and coequalizers. Summarily, this may be stated as follows.

Corollary 2.2.10. *The category* $\mathbf{CSet}_1$ *is categorically complete and cocomplete.*

Further, an empty product yields a terminal object, $\{(0, 0)\}$, and the empty coproduct an initial object, $\left(\emptyset, \mathbf{0}_{[0,\infty)}\right)$.

However, **Set** also shares these completion properties. This is not unexpected as $\mathbf{CSet}_1$ adds relatively little structure to **Set**. Indeed, this is actually desired so as to remain close to the classical construction of Section 1.2.

Yet, $\mathbf{CSet}_1$ is not equivalent to **Set** as categories. To see this, recall that every object in **Set** is projective with respect to all epimorphisms in **Set**. Explicitly, what this means is that given sets S, T, U and functions $\phi : S \to T$ and $\alpha : U \to T$ such that α is onto, one can lift ϕ along α to U. This is shown in the commutative diagram below.



For any set S, this is clear as any pre-images in U will do. However, the idea of constriction almost completely forbids this behavior in $\mathbf{CSet}_1$.

Proposition 2.2.11. *Let (S, f) be a crutched set.*

1. *(S, f) is projective relative to all epimorphisms in $\mathbf{CSet}_1$ iff $S = \emptyset$.*

2. *(S, f) is injective relative to all monomorphisms in $\mathbf{CSet}_1$ iff $S \neq \emptyset$ and $f = 0$.*

Proof. 1. ($\Leftarrow$) Assume that $S = \emptyset$. Then, $f = \mathbf{0}_{[0,\infty)}$. As $(\emptyset, \mathbf{0}_{[0,\infty)})$ is initial, it is trivially projective relative any class of maps in $\mathbf{CSet}_1$.

($\neg \Leftarrow \neg$) For purposes of contradiction, assume that $S \neq \emptyset$ and (S, f) is projective relative to all epimorphisms. For each $n \in \mathbb{N}$, define $g, h_n : S \to [0, \infty)$ by $g(s) := 0$ and $h_n(s) := n$. Also, let $\phi, \alpha_n : S \to S$ by $\phi(s) := \alpha_n(s) := s$. Then, consider the following diagram in $\mathbf{CSet}_1$ for each n.

$$\begin{array}{ccc} & & (S, h_n) \\ & & \big\downarrow{\scriptstyle \alpha_n} \\ (S, f) & \xrightarrow[\phi]{} & (S, g) \end{array}$$

Since α_n is onto and (S, f) projective to epimorphisms, there must be a constrictive $\phi_n : (S, f) \to (S, h_n)$ such that $\phi = \alpha_n \circ \phi_n$. Then, for each $s \in S$ and $n \in \mathbb{N}$,

$$s = \phi(s) = (\alpha \circ \phi_n)(s) = \phi_n(s)$$

and

$$n = (h_n \circ \phi_n)(s) \leq f(s).$$

Thus, f cannot have a finite value, contradicting that (S, f) was a crutched set.

2. ($\Rightarrow$) Assume that (S, f) is injective relative to all monomorphisms. Let $\mathbf{0}_S : \emptyset \to S$ and $\mathbf{0}_{\{0\}} : \emptyset \to \{0\}$ be the empty functions into S and $\{0\}$, respectively. Consider the following diagram in $\mathbf{CSet}_1$.

$$\begin{array}{ccc} (S, f) & & \\ {\scriptstyle \mathbf{0}_S}\big\uparrow & & \\ (\emptyset, \mathbf{0}_{[0,\infty)}) & \xrightarrow[\mathbf{0}_{\{0\}}]{} & \{(0,0)\}. \end{array}$$

As (S, f) is injective relative to $\mathbf{0}_{\{0\}}$, there must be a constrictive map from $\{(0, 0)\}$ to (S, f). Hence, there is a function from a nonempty set into S, forcing $S \neq \emptyset$.

Define $h : S \to [0, \infty)$ by $h(s) := 0$. Also, let $\phi, \alpha : S \to S$ by $\phi(s) := \alpha(s) := s$.

Then, consider the following diagram in $\mathbf{CSet}_1$.

$$\begin{array}{ccc} (S,f) & & \\ {\scriptstyle\phi}\big\uparrow & & \\ (S,f) & \overset{}{\underset{\alpha}{\rightarrowtail}} & (S,h). \end{array}$$

Then, there is a constriction $\hat{\phi} : (S,h) \to (S,f)$ such that $\phi = \hat{\phi} \circ \alpha$. Then, for each $s \in S$,

$$s = \phi(s) = \left(\hat{\phi} \circ \alpha\right)(s) = \hat{\phi}(s)$$

and

$$0 \leq f(s) = \left(f \circ \hat{\phi}\right)(s) \leq h(s) = 0.$$

($\Leftarrow$) Assume that $f = 0$ and $S \neq \emptyset$. Let (T,g) and (U,h) be crutched sets and $\alpha : (T,g) \to (U,h)$ be a monomorphism. Define $\hat{U} := \operatorname{ran}(\alpha)$ and observe that $\alpha|^{\hat{U}}$ is bijective. Given any $\phi : T \to S$, choose any $s_0 \in S$ and define $\hat{\phi} : U \to S$ by

$$\hat{\phi}(u) := \begin{cases} \phi(s), & u = \alpha(s), \\ s_0, & u \notin \hat{U}. \end{cases}$$

As α is one-to-one, this is a well-defined function. Clearly, $\phi = \hat{\phi} \circ \alpha$, and since $f = 0$, $\hat{\phi}$ is trivially constrictive.

□

There is precisely one isomorphism class of a projective object relative to all epimorphisms in $\mathbf{CSet}_1$, but $\mathbf{Set}$ has a proper class of such isomorphism classes. Hence, the distinction follows.

Corollary 2.2.12. $\mathbf{CSet}_1$ *and* $\mathbf{Set}$ *are not equivalent as categories.*

To close this section on $\mathbf{CSet}_1$, this category can be used to extend the failure result of Proposition 1.3.1. As before, let $\mathscr{C}$ be a subcategory of $\mathbf{CSet}_1$. There is a natural forgetful map from $\mathrm{Ob}(\mathscr{C})$ to $\mathrm{Ob}(\mathbf{Set})$ where one strips away the crutch function. Similarly, given $(S, f), (T, g) \in \mathrm{Ob}(\mathscr{C})$ and $\phi \in \mathscr{C}((S, f), (T, g))$, $\phi \in \mathbf{Set}(S, T)$ by definition of $\mathbf{CSet}_1$. One can quickly check that these two associations define a functor $F_{\mathscr{C}} : \mathscr{C} \to \mathbf{Set}$, where one ignores all the numeric properties from $\mathscr{C}$.

Proposition 2.2.13. *Let $S \neq \emptyset$. Assume that for each $n \in \mathbb{N}$, there is an object $(S_n, f_n) \in \mathrm{Ob}(\mathscr{C})$ with an element $s_n \in f_n^{-1}([n, \infty))$. Then, S has no reflection along $F_{\mathscr{C}}$.*

Proof. For purposes of contradiction, assume that (R, f) equipped with $\eta : S \to F_{\mathscr{C}} R$ is a reflection of S along $F_{\mathscr{C}}$. For each $n \in \mathbb{N}$, define $\phi_n \in \mathbf{Set}\left(S, F_{\mathscr{C}} S_n\right)$ by $\phi_n(s) := s_n$, a constant function. Then, there is a unique $\hat{\phi}_n \in \mathscr{C}\left((R, f), (S_n, f_n)\right)$ such that $F_{\mathscr{C}} \hat{\phi}_n \circ \eta = \phi_n$ for all $n \in \mathbb{N}$.

For each $s \in S$, let $r_s := \eta(s) \in R$ and observe that for each $n \in \mathbb{N}$,

$$f\left(r_s\right) \geq f_n\left(\hat{\phi}_n\left(r_s\right)\right) = f_n\left(\left(F_{\mathscr{C}} \hat{\phi}_n \circ \eta\right)(s)\right) = f_n\left(\phi_n(s)\right) = f_n\left(s_n\right) \geq n.$$

Hence, $f\left(r_s\right)$ cannot have finite value for any $s \in S$, which cannot occur in (R, f). As such, this reflection is fiction.

□

In the case of Proposition 1.3.1, all the S_n were the same nontrivial normed $\mathbb{F}$-vector space and the s_n multiples of a nonzero vector. Thus, the above proposition genuinely resolves to Proposition 1.3.1 when $\mathscr{C}$ is a nontrivial subcategory of $\mathbb{F}\mathbf{NVec}_1$.

However, this generalization allows the elements of increasing size to come from different objects in $\mathscr{C}$, which seems to sour any possibility of classical free objects in most categories of interest. For example, the any subcategory of $\mathbf{CSet}_1$ containing the singleton crutched sets $\{(0, n)\}$ for $n \in \mathbb{N}$ cannot have a reflection along the forgetful functor for any nonempty set S.

2.3 Category of Crutched Sets & Bounded Maps

This section is dedicated a study of crutched sets and bounded functions between them. This section is tangential to the remainder of this work, but has potential application to other categories of normed algebraic objects. For completeness, a similar analysis of its categorical structure will be done in comparison $\mathbf{CSet}_1$. Of particular note, Proposition 2.3.12 is an analogue of the failure result in Proposition 1.3.1, destroying most avenues for classical free objects in normed algebraic categories.

For notation, the symbol $\mathbf{CSet}_\infty$ will be used to denote the following data:

- $\mathrm{Ob}\left(\mathbf{CSet}_\infty\right) :=$ the class of all crutched sets;
- For each $(S, f), (T, g)$ in $\mathrm{Ob}(\mathbf{CSet}_\infty)$, define

$$\mathbf{CSet}_\infty((S, f), (T, g)) := \{\phi \in \mathbf{Set}(S, T) : \phi \text{ bounded from } (S, f) \text{ to } (T, g)\}.$$

Equipping this structure with function composition, which is well-known to be associative with identity maps as the units of this operation, $\mathbf{CSet}_\infty$ is a category from Corollary 2.1.5 and Example 2.1.10.

Proposition 2.3.1. $\mathbf{CSet}_\infty$ *is a category.*

At first glance, $\mathbf{CSet}_\infty$ is very similar to $\mathbf{CSet}_1$, and most of its constructions are identical. However, there are some notable distinctions between the two, reminiscent of the differences between considering Banach spaces with bounded linear maps and contractive linear maps.

To begin, consider the primary properties of bounded mappings.

Proposition 2.3.2. *Let* (S, f) *and* (T, g) *be crutched sets and* $\phi : (S, f) \to (T, g)$ *be bounded. Define* $K := T \setminus \phi(S)$, $h := g|_K$, *and*

$$\lambda := \inf \left\{ \frac{g(\phi(s))}{f(s)} : s \notin f^{-1}(0) \right\}.$$

1. ϕ *is a monomorphism in* $\mathbf{CSet}_\infty$ *iff* ϕ *is one-to-one;*

2. ϕ *is an epimorphism in* $\mathbf{CSet}_\infty$ *iff* ϕ *is onto;*

3. ϕ *is a section in* $\mathbf{CSet}_\infty$ *iff* ϕ *is one-to-one,* $\lambda > 0$, *and there is a bounded function* $\alpha : (K, h) \to (S, f)$*;*

4. ϕ *is a retraction in* $\mathbf{CSet}_\infty$ *iff there are* $(s_t)_{t \in T} \subseteq S$ *such that* $\phi(s_t) = t$ *for all* $t \in T$, $f(s_t) = 0$ *for all* $t \in g^{-1}(0)$ *and*

$$\sup \left(\left\{ \frac{f(s_t)}{g(t)} : t \notin g^{-1}(0) \right\} \cup \{0\} \right) < \infty;$$

5. ϕ *is an isomorphism in* $\mathbf{CSet}_\infty$ *iff* ϕ *is one-to-one, onto, and* $\lambda > 0$.

Proof. 1. Use the same proof as Proposition 2.2.2, considering bounded α and β.

2. Use the same proof as Proposition 2.2.2, considering bounded α and β.

3. ($\Rightarrow$) Assume that ϕ is a section in $\mathbf{CSet}_\infty$. Then, there is a bounded $\psi : (T, g) \to (S, f)$ such that $\psi \circ \phi = id_{(S,f)}$. From basic function results, ϕ must be one-to-one. Letting $\alpha := \psi|_K$, $\alpha : (K, h) \to (S, f)$ is bounded as ψ was. If

there is $s \notin f^{-1}(0)$, observe that

$$0 < f(s) = \left(f \circ id_{(S,f)}\right)(s) = (f \circ \psi \circ \phi)(s) \leq \text{crh}(\psi)(g \circ \phi)(s)$$

so $\text{crh}(\psi) \neq 0$ and

$$\frac{1}{\text{crh}(\psi)} \leq \frac{(g \circ \phi)(s)}{f(s)}.$$

Hence, $\lambda \geq \dfrac{1}{\text{crh}(\psi)} > 0$.

If $S = f^{-1}(0)$, then $\lambda = \infty$ by convention.

($\Leftarrow$) Assuming the conclusion, define $\psi : T \to S$ by

$$\psi(t) := \begin{cases} s, & t = \phi(s), \\ \alpha(t), & t \in K. \end{cases}$$

As ϕ is one-to-one, this is a well-defined function, and $\psi \circ \phi = id_{(S,f)}$ by design. To prove ψ bounded, note that for all $t \in K$,

$$f(\alpha(t)) \leq \text{crh}(\alpha)h(t) = \text{crh}(\alpha)g(t)$$

since α is bounded. If $t = \phi(s)$ for some $s \in S$, consider when $g(t) = 0$. If $f(s) \neq 0$, then $\lambda = 0$, contradicting the assumption. Thus, $f(s) = 0 \leq \text{crh}(\alpha)g(t)$.

If $g(t) \neq 0$, $f(s) \neq 0$ by Proposition 2.1.8, meaning $\lambda \neq \infty$. Hence,

$$f(s) = \frac{f(s)}{(g \circ \phi)(s)} \cdot (g \circ \phi)(s) \leq \frac{1}{\lambda} g(t).$$

Therefore, for all $t \in T$,

$$f(\psi(t)) \leq \max\left\{\frac{1}{\lambda}, \operatorname{crh}(\alpha)\right\} g(t),$$

meaning ψ is bounded.

4. ($\Rightarrow$) Assume that ϕ is a retraction in $\mathbf{CSet}_\infty$. Then, there is a bounded $\psi : (T, g) \to (S, f)$ such that $\phi \circ \psi = id_{(T,g)}$. For $t \in T$, let $s_t := \psi(t)$. Observe that $\phi(s_t) = t$. Also, by Proposition 2.1.8, $f(s_t) = 0$ for all $t \in g^{-1}(0)$ and

$$\sup\left(\left\{\frac{f(s_t)}{g(t)} : t \notin g^{-1}(0)\right\} \cup \{0\}\right) < \infty.$$

   ($\Leftarrow$) Assuming the result, define $\psi : T \to S$ by $\psi(t) := s_t$. Then, $\phi \circ \psi = id_{(T,g)}$ by design. Further, by Proposition 2.1.8, ψ is bounded.

5. ($\Rightarrow$) Assume that ϕ is an isomorphism in $\mathbf{CSet}_\infty$. Then, ϕ is both a section and a retraction, in particular also an epimorphism. Hence, ϕ is one-to-one, onto, and $\lambda > 0$.

   ($\Leftarrow$) Assuming the result, ϕ is an epimorphism as it is onto. Further, $T \backslash \phi(S) = \emptyset$ by this fact, meaning ϕ is further a section. Hence, ϕ is an isomorphism.

□

Much like Proposition 2.2.2, each of the criteria in Item 5 are necessary. In particular, the infimum criterion is identical to the notion of "bounded below" for bounded linear maps, but like its "isometric" counterpart in Proposition 2.2.2, this fact alone does not imply monomorphism, let alone isomorphism. Examples 2.2.3 and 2.2.4 also demonstrate the necessity of the criteria in Item 5, but Example 2.1.11 demonstrates this bounded below idea in a less trivial way.

Next, equalizers for parallel arrows in $\mathbf{CSet}_\infty$ are computed precisely the same way they are in $\mathbf{CSet}_1$.

Proposition 2.3.3. *Let (S, f) and (T, g) be crutched sets and $\alpha, \beta : (S, f) \to (T, g)$ bounded maps. Let*

$$K := \{s \in S : \alpha(s) = \beta(s)\},$$

$k := f|_K$, and $\iota : K \to S$ by $\iota(s) := s$. Then, (K, k) equipped with ι is an equalizer of α and β.

Since the notions of equalizer in $\mathbf{CSet}_1$ and $\mathbf{CSet}_\infty$ determine the same object up to isomorphism in $\mathbf{CSet}_1$, the following definition seems very natural.

Definition. Given a crutched set (S, f), a *crutched subset* of (S, f) is a pair (K, k), where $K \subseteq S$ and $k = f|_K$.

Similarly, coequalizers for parallel arrows in $\mathbf{CSet}_\infty$ are also share the same structure as their $\mathbf{CSet}_1$ counterparts.

Proposition 2.3.4. *Let (S, f) and (T, g) be crutched sets and $\alpha, \beta : (S, f) \to (T, g)$ bounded maps. Let*

$$P := \big\{(\alpha(s), \beta(s)) \in T^2 : s \in S\big\}$$

and $\sim_P$ be the equivalence relation on T generated by P. Define $Q := T/\sim_P$, $q([t]) := \inf\{g(\tau) : \tau \sim_P t\}$, and $\xi : T \to Q$ by $\xi(t) := [t]$. Then, (Q, q) equipped with ξ is a coequalizer of α and β.

Again, as the notions of coequalizer correspond between the two categories in question, the following definition appears sensical.

Definition. Given a crutched set (S, f) and an equivalence relation $\sim$ on S, the *crutched quotient set* of (S, f) by $\sim$ is (Q, q), where $Q := S/\sim$ and $q([t]) := \inf\{f(\tau) : \tau \sim t\}$.

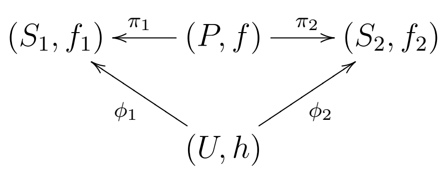
Turning attention toward products, $\mathbf{CSet}_\infty$ begins to show more differences from $\mathbf{CSet}_1$. Computation of binary product in $\mathbf{CSet}_\infty$ is identical to its $\mathbf{CSet}_1$ counterpart.

Proposition 2.3.5. *Let (S_1, f_1) and (S_2, f_2) be crutched sets. Define*

$$P := S_1 \times S_2,$$

$f : P \to [0, \infty)$ by $f(s_1, s_1) := \max\{f_1(s_1), f_2(s_2)\}$, and $\pi_i : P \to S_i$ by $\pi_i(s_1, s_2) := s_i$ for $i = 1, 2$. Then, (P, f) equipped with $(\pi_i)_{i=1,2}$ is a product of $((S_i, f_i))_{i=1,2}$.

Proof. From definition, π_i is constrictive for each $i = 1, 2$. To check the universal property, let (U, h) be a crutched set and $\phi_i : (U, h) \to (S_i, f_i)$ be bounded for each $i = 1, 2$.



For each $u \in U$ and $i = 1, 2$, observe that

$$(f_i \circ \phi_i)(u) \leq \operatorname{crh}(\phi_i) h(u) \leq \max\{\operatorname{crh}(\phi_1), \operatorname{crh}(\phi_2)\} h(u)$$

so

$$\max\{(f_1 \circ \phi_1)(u), (f_2 \circ \phi_2)(u)\} \leq \max\{\operatorname{crh}(\phi_1), \operatorname{crh}(\phi_2)\} h(u).$$

Hence, define $\phi : U \to P$ by $\phi(u) := (\phi_1(u), \phi_2(u))$. By the above,

$$f(\phi(u)) \leq \max\{\operatorname{crh}(\phi_1), \operatorname{crh}(\phi_2)\} h(u),$$

meaning ϕ is bounded. Also, $\pi_i \circ \phi = \phi_i$ for each $i = 1, 2$.

Assume there was some other bounded $\varphi : (U, h) \to (P, f)$ such that $\pi_i \circ \varphi = \phi_i$. Then, for each $i = 1, 2$ and $u \in U$,

$$(\pi_i \circ \varphi)(u) = \phi_i(u)$$

Hence, $\varphi(u)(i) = \phi_i(u) = \phi(u)(i)$, meaning $\varphi(u) = \phi(u)$. Therefore, $\varphi = \phi$.

□

As $\mathbf{CSet}_\infty$ has binary products and has a terminal object, namely $\{(0, 0)\}$, it immediately has any finitary product by iteration of the binary product. However, $\mathbf{CSet}_\infty$ does not have arbitrary product objects. This is similar to the case of Banach spaces with bounded linear maps.

Example 2.3.6. For $n \in \mathbb{N}$, let $S_n := [0, \infty)$ and $f_n : S_n \to [0, \infty)$ by $f_n(\lambda) := \lambda$. Assume for purposes of contradiction that $((S_n, f_n))_{n \in \mathbb{N}}$ has a product (P, f) in $\mathbf{CSet}_\infty$. For $n \in \mathbb{N}$, define $\phi_n : S_1 \to S_n$ by $\phi_n(\lambda) := \lambda$, each a constrictive map with $\operatorname{crh}(\phi_n) = 1$. Then, there is a unique bounded function $\phi : (S_1, f_1) \to (P, f)$ such that $\phi_n = \pi_n \circ \phi$ for each $n \in \mathbb{N}$. By Proposition 2.1.5,

$$1 = \operatorname{crh}(\phi_n) \leq \operatorname{crh}(\pi_n) \operatorname{crh}(\phi).$$

Thus, $\operatorname{crh}(\pi_n) \neq 0$.

Let $T := \{0\}$ and $g : T \to [0, \infty)$ by $g(0) := 1$. For $n \in \mathbb{N}$, define $\psi_n : T \to S_n$ by

$\psi_n(0) := n\,\mathrm{crh}\,(\pi_n)$, each a bounded map with $\mathrm{crh}\,(\psi_n) = n\,\mathrm{crh}\,(\pi_n)$. Then, there is a unique bounded function $\psi : (T, g) \to (P, f)$ such that $\psi_n = \pi_n \circ \psi$ for all $n \in \mathbb{N}$. In this case,

$$n\,\mathrm{crh}\,(\pi_n) = \mathrm{crh}\,(\psi_n) \leq \mathrm{crh}\,(\pi_n)\,\mathrm{crh}(\psi).$$

Hence, $n \leq \mathrm{crh}(\psi)$ for all $n \in \mathbb{N}$, contradicting that ψ was bounded. Thus, $((S_n, f_n))_{n\in\mathbb{N}}$ cannot have a product in $\mathbf{CSet}_\infty$.

Therefore, as $\mathbf{CSet}_1$ and $\mathbf{Set}$ both have arbitrary products, $\mathbf{CSet}_\infty$ must be distinct from both.

Corollary 2.3.7. $\mathbf{CSet}_\infty$ *is not equivalent to* **Set** *or* $\mathbf{CSet}_1$ *as categories.*

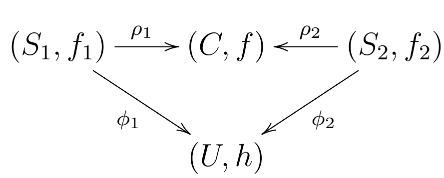
Similarly, $\mathbf{CSet}_\infty$ also has binary coproducts, computed just as in $\mathbf{CSet}_1$.

Proposition 2.3.8. *Let* (S_1, f_1) *and* (S_2, f_2) *be crutched sets. Define*

$$C := \{(i, s) \in \{1, 2\} \times (S_1 \cup S_2) : s \in S_i\},$$

$f : C \to [0, \infty)$ *by* $f(i, s) := f_i(s)$*, and* $\rho_i : S_i \to C$ *by* $\rho_i(s) := (i, s)$ *for* $i = 1, 2$*. Then,* (C, f) *equipped with* $(\rho_i)_{i=1,2}$ *is a coproduct of* $((S_i, f_i))_{i=1,2}$*.*

Proof. From definition, ρ_i is constrictive for each $i = 1, 2$. To check the universal property, let (U, h) be a crutched set and $\phi_i : (S_i, f_i) \to (U, h)$ be bounded.



For each $i = 1, 2$ and $s \in S_i$, observe that

$$h(\phi_i(s)) \leq \operatorname{crh}(\phi_i) f_i(s) \leq \max\{\operatorname{crh}(\phi_1), \operatorname{crh}(\phi_2)\} f_i(s)$$

so

$$h(\phi_i(s)) \leq \max\{\operatorname{crh}(\phi_1), \operatorname{crh}(\phi_2)\} f(i, s).$$

Hence, define $\phi : C \to U$ by $\phi(i, s) := \phi_i(s)$. By the above,

$$h(\phi(i, s)) \leq \max\{\operatorname{crh}(\phi_1), \operatorname{crh}(\phi_2)\} f(i, s),$$

meaning ϕ is bounded. Also, $\phi \circ \rho_i = \phi_i$ for each $i = 1, 2$.

Assume that there was some other bounded $\varphi : (C, f) \to (U, h)$ such that $\varphi \circ \rho_i = \phi_i$. Then, for each $i = 1, 2$ and $s \in S_i$,

$$\varphi(i, s) = (\varphi \circ \rho_i)(s) = \phi_i(s) = \phi(i, s).$$

Therefore, $\varphi = \phi$.

□

As $\mathbf{CSet}_\infty$ has binary coproducts and has an initial object, namely $\left(\emptyset, \mathbf{0}_{[0,\infty)}\right)$, it immediately has any finitary coproduct by iteration of the binary coproduct. However, just as with products, $\mathbf{CSet}_\infty$ does not have arbitrary coproduct objects.

Example 2.3.9. For $n \in \mathbb{N}$, define $S_n := \{0\}$ and $f_n : S_n \to [0, \infty)$ by $f_n(0) := 1$. Assume for purposes of contradiction that $((S_n, f_n))_{n \in \mathbb{N}}$ has a coproduct (C, f) in $\mathbf{CSet}_\infty$. For $n \in \mathbb{N}$, define $\phi_n : S_n \to S_1$ by $\phi_n(0) := 0$, each constrictive with $\operatorname{crh}(\phi_n) = 1$. Then, there is a unique bounded function $\phi : (C, f) \to (S_1, f_1)$ such that $\phi_n = \phi \circ \rho_n$ for each $n \in \mathbb{N}$. By Proposition 2.1.5,

$$1 = \operatorname{crh}(\phi_n) \leq \operatorname{crh}(\phi) \operatorname{crh}(\rho_n).$$

Thus, $\operatorname{crh}(\rho_n) \neq 0$.

Let $T := \mathbb{N}$ and $g : T \to [0, \infty)$ by $g(n) := n \operatorname{crh}(\rho_n)$. Define $\psi_n : S_n \to T$ by $\psi_n(0) := n$, each a bounded map with $\operatorname{crh}(\psi_n) = n \operatorname{crh}(\rho_n)$. Then, there is a unique bounded function $\psi : (C, f) \to (T, g)$ such that $\psi_n = \psi \circ \rho_n$ for all $n \in \mathbb{N}$. In this case,

$$n \operatorname{crh}(\rho_n) = \operatorname{crh}(\psi_n) \leq \operatorname{crh}(\psi) \operatorname{crh}(\rho_n).$$

Hence, $n \leq \operatorname{crh}(\psi)$ for all $n \in \mathbb{N}$, contradicting that ψ was bounded. Thus, $((S_n, f_n))_{n \in \mathbb{N}}$ cannot have a coproduct in $\mathbf{CSet}_\infty$.

Still, as $\mathbf{CSet}_\infty$ has all finitary products and equalizers, all finitary limit processes may be performed. Likewise, finitary colimit processes follow from finitary coproducts and coequalizers. In summary, these facts can be stated in the following way.

Corollary 2.3.10. *The category* $\mathbf{CSet}_\infty$ *is finitely complete and finitely cocomplete.*

To close the comparison between $\mathbf{CSet}_\infty$ and $\mathbf{CSet}_1$, the standard projective and injective objects can be completely characterized.

Proposition 2.3.11. *Let* (S, f) *be a crutched set.*

1. (S, f) *is projective relative to all epimorphisms in* $\mathbf{CSet}_\infty$ *iff* $\operatorname{card}(S) < \aleph_0$ *and* $f(s) \neq 0$ *for all* $s \in S$.

2. (S, f) *is injective relative to all monomorphisms in* $\mathbf{CSet}_\infty$ *iff* $S \neq \emptyset$ *and* $f = 0$.

Proof. 1. ($\Leftarrow$) If $\operatorname{card}(S) = 0$, then $f = \mathbf{0}_{[0,\infty)}$. As $(\emptyset, \mathbf{0}_{[0,\infty)})$ is initial, it is trivially projective relative any class of maps in $\mathbf{CSet}_\infty$.

Assume $0 < \text{card}(S) < \aleph_0$. Let (T, g) and (U, h) be crutched sets and $\alpha : (U, h) \to (T, g)$ be an epimorphism. Given a bounded function $\phi : (S, f) \to (T, g)$, consider the diagram below in $\mathbf{CSet}_\infty$.

$$\begin{array}{ccc} & & (U,h) \\ & & \downarrow{\scriptstyle \alpha} \\ (S,f) & \xrightarrow[\phi]{} & (T,g) \end{array}$$

Since $S \neq \emptyset$, $T \neq \emptyset$. Consequently, neither is U as α is onto. For each $s \in S$, choose $u_s \in \alpha^{-1}(\phi(s))$. Define $\hat{\phi} : S \to U$ by $\phi(s) := u_s$. Note that $\phi = \alpha \circ \hat{\phi}$. Further, as $f(s) \neq 0$ for all $s \in S$ and S finite,

$$\sup\left(\left\{\frac{g(\phi(s))}{f(s)} : s \notin f^{-1}(0)\right\} \cup \{0\}\right) < \infty$$

so ϕ is bounded by Proposition 2.1.8.

$(\neg \Leftarrow \neg)$ For purposes of contradiction, assume first that there is $s_0 \in S$ such that $f(s_0) = 0$ and that (S, f) is projective relative to all epimorphisms. Define $g, h : S \to [0, \infty)$ by $g(s) := 0$ and $h(s) := 1$. Also, let $\phi, \alpha : S \to S$ by $\phi(s) := \alpha(s) := s$. Then, consider the following diagram in $\mathbf{CSet}_\infty$.

$$\begin{array}{ccc} & & (S,h) \\ & & \downarrow{\scriptstyle \alpha} \\ (S,f) & \xrightarrow[\phi]{} & (S,g) \end{array}$$

Since α is onto and (S, f) projective to epimorphisms, there must be a bounded

$\hat{\phi} : (S, f) \to (S, h)$ such that $\phi = \alpha \circ \hat{\phi}$. Then, for each $s \in S$,

$$s = \phi(s) = \left(\alpha \circ \hat{\phi}\right)(s) = \hat{\phi}(s)$$

so

$$1 = \left(h \circ \hat{\phi}\right)(s_0) \leq \operatorname{crh}\left(\hat{\phi}\right) f\left(s_0\right) = 0,$$

which is nonsense.

Assume instead that $\operatorname{card}(S) \geq \aleph_0$, that f is strictly positive, and that (S, f) is projective relative to all epimorphisms. Let $(s_n)_{n \in \mathbb{N}}$ be a sequence of distinct elements in S. Define $T := \mathbb{N}$ and $g, h : T \to [0, \infty)$ by $g(n) := nf\left(s_n\right)$ and $h(n) := 0$. Consider $\alpha : T \to T$ by $\alpha(n) := n$, and $\phi : S \to T$ by

$$\phi(s) := \begin{cases} n, & s = s_n, \\ 1, & s \neq s_n. \end{cases}$$

Observe that ϕ and α are both bounded. Consider the following diagram in $\mathbf{CSet}_\infty$.

$$\begin{array}{ccc} & & (T, g) \\ & & \downarrow{\scriptstyle \alpha} \\ (S, f) & \xrightarrow[\phi]{} & (T, h) \end{array}$$

By assumption, there is a bounded function $\hat{\phi} : (S, f) \to (T, g)$ such that $\phi = \alpha \circ \hat{\phi}$. Then, for each $n \in \mathbb{N}$,

$$n = \phi\left(s_n\right) = \left(\alpha \circ \hat{\phi}\right)\left(s_n\right) = \hat{\phi}(s_n)$$

and

$$nf(s_n) = g(n) = g\left(\hat{\phi}(s_n)\right) \leq \operatorname{crh}\left(\hat{\phi}\right) f(s_n).$$

Therefore, $n \leq \operatorname{crh}\left(\hat{\phi}\right)$ for all $n \in \mathbb{N}$, contradicting that $\hat{\phi}$ was bounded.

2. Use the same proof as Proposition 2.2.11, considering bounded maps.

□

Notice that the inclusion of more maps between objects increased the number of projective objects, from one unique object in $\mathbf{CSet}_1$ to a countable family of isomorphism classes in $\mathbf{CSet}_\infty$. However, the number of injective objects remained unchanged.

To conclude discussion of $\mathbf{CSet}_\infty$, this category can also extend the failure results of Propositions 1.3.1 and 2.2.13. As before, let $\mathscr{C}$ be a subcategory of $\mathbf{CSet}_\infty$. There is a natural forgetful map from $\mathrm{Ob}(\mathscr{C})$ to $\mathrm{Ob}(\mathbf{Set})$ where one strips away the crutch function. Similarly, given $(S, f), (T, g) \in \mathrm{Ob}(\mathscr{C})$ and $\phi \in \mathscr{C}((S, f), (T, g))$, $\phi \in \mathbf{Set}(S, T)$ by definition of $\mathbf{CSet}_\infty$. One can quickly check that these two associations define a functor $F_\mathscr{C} : \mathscr{C} \to \mathbf{Set}$, where one ignores all the numeric properties from $\mathscr{C}$.

Proposition 2.3.12. *Let $S \in \mathrm{Ob}(\mathbf{Set})$ be an infinite set. Assume that there is an object $(T, g) \in \mathrm{Ob}(\mathscr{C})$ with elements $t_n \in g^{-1}([n, \infty))$ for all $n \in \mathbb{N}$. Then, S has no reflection along $F_\mathscr{C}$.*

Proof. For purposes of contradiction, assume that (R, f) equipped with $\eta : S \to F_\mathscr{C} R$ is a reflection of S along $F_\mathscr{C}$. Define $\phi : S \to T$ by $\phi(s) := t_1$. Then, there is a unique bounded $\hat{\phi} : (R, f) \to (T, g)$ such that $F_\mathscr{C}\hat{\phi} \circ \eta = \phi$. For each $s \in S$, define $r_s := \eta(s)$ and observe that

$$1 \leq g(t_1) = g(\phi(s)) = g\left(\left(F_\mathscr{C}\hat{\phi} \circ \eta\right)(s)\right) = g\left(\hat{\phi}(r_s)\right) \leq \operatorname{crh}\left(\hat{\phi}\right) f(r_s).$$

Thus, $f(r_s) \neq 0$.

Let $(s_j)_{j=1}^{\infty} \subseteq S$ be distinct. For each $j \in \mathbb{N}$, choose $n_j \in \mathbb{N}$ such that $n_j \geq j \cdot f\left(r_{s_j}\right)$. Define $\psi : S \to T$ by

$$\psi(s) := \begin{cases} t_{n_j}, & s = s_j, \\ t_1, & s \neq s_j. \end{cases}$$

Then, there is a unique $\hat{\psi} : (R, f) \to (T, g)$ such that $F_{\mathscr{C}}\hat{\psi} \circ \eta = \psi$. By Proposition 2.1.8,

$$\begin{aligned} \operatorname{crh}\left(\hat{\psi}\right) &\geq \frac{g\left(\hat{\psi}\left(r_{s_j}\right)\right)}{f\left(r_{s_j}\right)} \\ &= \frac{g\left(\left(F_{\mathscr{C}}\hat{\psi} \circ \eta\right)(s_j)\right)}{f\left(r_{s_j}\right)} \\ &= \frac{g\left(\psi\left(s_j\right)\right)}{f\left(r_{s_j}\right)} \\ &= \frac{g\left(t_{n_j}\right)}{f\left(r_{s_j}\right)} \\ &\geq \frac{n_j}{f\left(r_{s_j}\right)} \\ &\geq \frac{j \cdot f\left(r_{s_j}\right)}{f\left(r_{s_j}\right)} \\ &= j \end{aligned}$$

for all $j \in \mathbb{N}$. Then, $\hat{\psi}$ is unbounded, a contradiction.

□

The above proposition does not have quite the impact that Propositions 1.3.1 and 2.2.13 had due to the loss of the constrictive property. To illustrate this, consider the following examples.

Example 2.3.13. Consider the entire category $\mathbf{CSet}_{\infty}$. Given a finite set S, let $R := S$,

$\eta := id_S$, and $f : R \to [0, \infty)$ by $f(r) := 1$ for all $r \in R$. Given a crutched set (T, g) and $\phi \in \mathbf{Set}(S, T)$, define $\hat{\phi} : R \to T$ by $\hat{\phi} := \phi$. Observe that $\hat{\phi}$ is trivially bounded by Proposition 2.1.8, and $F_{\mathbf{CSet}_\infty} \hat{\phi} \circ \eta = \phi$ in an apparent way. Further, if $\varphi : (R, f) \to (T, g)$ such that $F_{\mathbf{CSet}_\infty} \varphi \circ \eta = \phi$, observe that for all $s \in S$,

$$\phi(s) = (F_{\mathbf{CSet}_\infty} \varphi \circ \eta)(s) = \varphi(s).$$

Thus, $\varphi = \hat{\phi}$. Hence, (R, f) equipped with η is a reflection of S along $F_{\mathbf{CSet}_\infty}$.

Example 2.3.14. Consider the category of $\mathbb{F}$-Banach spaces with bounded linear maps, $\mathbb{F}\mathbf{Ban}_\infty$. Given a finite set S, let $R := \ell^1(S)$ with its usual norm and $\eta : S \to R$ by $\eta(s) := \delta_s$, the point mass at s. Given another $\mathbb{F}$-Banach space X and a set map $\phi : S \to X$, define $\hat{\phi} : R \to X$ by

$$\hat{\phi}(x) := \sum_{s \in S} x_s \phi(s),$$

where $x = \sum_{s \in S} x_s \delta_s$ is the decomposition of x with respect to the linear basis $(\delta_s)_{s \in S}$. A quick check shows that $\hat{\phi}$ is an $\mathbb{F}$-linear transformation, and since $\ell^1(S)$ is finite-dimensional, $\hat{\phi}$ is automatically continuous. Further,

$$\left(F_{\mathbb{F}\mathbf{Ban}_\infty} \hat{\phi} \circ \eta\right)(s) = \hat{\phi}(\delta_s) = \phi(s)$$

so $F_{\mathbb{F}\mathbf{Ban}_\infty} \hat{\phi} \circ \eta = \phi$.

If $\varphi : R \to X$ such that $F_{\mathbb{F}\mathbf{Ban}_\infty} \varphi \circ \eta = \phi$, observe that for all $s \in S$,

$$\phi(s) = (F_{\mathbb{F}\mathbf{Ban}_\infty} \varphi \circ \eta)(s) = \varphi(\delta_s).$$

Hence, $\varphi(\delta_s) = \hat{\phi}(\delta_s)$ so by linearity, $\varphi = \hat{\phi}$. Therefore, R equipped with η is a reflection along $F_{\mathbb{F}\mathbf{Ban}_\infty}$.

What Proposition 2.3.12 has done is forbidden classical free objects generated by countable or larger sets in nontrivial categories of normed structures with bounded maps, i.e., copies of the zero space $\mathbb{0}$. Classical free objects may still exist for finite generation sets as shown in the above two examples, but this would require more particular attention to the type of structure.

Chapter 3

A Presentation Theory for 1C*

This chapter presents a treatment of a familiar construction from a revised perspective, that of an adjoint functor pair, in Sections 3.1 and 3.2. From this, a presentation theory for unital C*-algebras is developed in Section 3.3 and following, heavily grounded in both the classical notions of Section 1.2 and standard references like [13] and [25].

The most notable result of this chapter is the Tietze transformation theorem, Theorem 3.9.7. This guarantees that presentations for the same unital C*-algebra can be formally manipulated to be identical. This Tietze calculus is then used to characterize several unital C*-algebras, particularly the C*-algebras generated by a single type of invertible element in Section 3.11. Another more well-known example, the unital C*-algebra of a single idempotent, is formally computed in Example 3.12.7. Examples 3.6.1, 3.6.2, and 3.8.2 also demonstrate relations constructed via the functional calculus, which may be of future interest.

Also, while unital C*-algebras are the focus of this chapter, it is observed at key sections that the notions and methods described here can be adapted to other structures of interest, such as Banach algebras.

3.1 The Modified Construction for $\mathbf{1C}^*$

With an understanding of $\mathbf{CSet}_1$, attention returns to modifying the construction of Section 1.2. This construction will be familiar to anyone who has studied universal C*-algebras. In particular, this method should be thought of as a generalization of the constructions done in [4] and [21] with the viewpoint of [33]. The use of the crutch function is analogous to the "$\mathcal{X}$-norms" in [22], but the universal object created here will be an actual C*-algebra, as opposed to a general pro-C*-algebra.

The modified construction shown in the present work is not entirely new, previously done for general C*-algebras and LMC*-algebras within Section 1.3 of [19]. However, this presentation of the material explicitly carried the universal maps of both free *-semigroup and free *-algebra constructions throughout each result. Section 3 of [29] also does this construction for general C*-algebras. The present work aims to codify the construction for C*-algebras as a left adjoint functor, gaining all the abstract results of that characterization.

In this presentation, work will be done first with *unital* C*-algebras, rather than the more general non-unital. This is done for two specific reasons.

First, the inclusion of a unit completes the analytic and continuous functional calculi in the following sense. Nonzero constant *-polynomials can now be considered, allowing the entire spectrum of an operator to be separated through the Stone-Weierstrass theorem. This yields a finer picture of the algebra considered.

Second, the non-unital case can be recovered from the unital case. This will be done in Section 4.2 using the unitization functor of Section B.5.

To begin, let $\mathbf{1C}^*$ denote the category of unital C*-algebras and unital *-homomorphisms. Explicitly, $\mathrm{Ob}\,(\mathbf{1C}^*)$ is the class of all unital C*-algebras, and for $\mathcal{A}, \mathcal{B} \in \mathrm{Ob}\,(\mathbf{1C}^*)$, $\mathbf{1C}^*(\mathcal{A}, \mathcal{B})$ is the set of all unital *-homomorphisms from $\mathcal{A}$ to $\mathcal{B}$.

Note that the zero algebra, $\mathbb{0}$, will be considered as a *unital* C*-algebra for the purposes of this work. Specifically, it will be thought of as the unique unital C*-algebra where $0 = \mathbb{1}$, or equivalently, $C(\emptyset)$, continuous functions on the empty topological space.

As in Example 2.1.1, every $\mathcal{A} \in \mathrm{Ob}\left(\mathbf{1C^*}\right)$ is a set with a nonnegative function $f_{\mathcal{A}} : \mathcal{A} \to [0, \infty)$ by $f_{\mathcal{A}}(a) := \|a\|_{\mathcal{A}}$. Thus, there is a natural forgetful map to $\mathrm{Ob}\left(\mathbf{CSet}_1\right)$, where one regards $\mathcal{A}$ as a crutched set $(\mathcal{A}, f_{\mathcal{A}})$, ignoring all structure except the norm function. Similarly, given $\mathcal{A}, \mathcal{B} \in \mathrm{Ob}\left(\mathbf{1C^*}\right)$ and $\phi \in \mathbf{1C^*}(\mathcal{A}, \mathcal{B})$, ϕ is firstly a function from $\mathcal{A}$ to $\mathcal{B}$, but it is a standard fact that $\|\phi(a)\|_{\mathcal{B}} \leq \|a\|_{\mathcal{A}}$ for all $a \in \mathcal{A}$. Hence, $\phi \in \mathbf{CSet}_1\left((\mathcal{A}, f_{\mathcal{A}}), (\mathcal{B}, f_{\mathcal{B}})\right)$ as in Example 2.1.9. One can quickly check that these two associations define a functor $F^{\mathbf{CSet}_1}_{\mathbf{1C^*}} : \mathbf{1C^*} \to \mathbf{CSet}_1$, where one ignores all data from $\mathbf{1C^*}$ save the set and norm.

Now, fix (S, f) from $\mathrm{Ob}\left(\mathbf{CSet}_1\right)$, thought of as a set of generators normed by their values under f. The objective is to build a reflection of (S, f) along $F^{\mathbf{CSet}_1}_{\mathbf{1C^*}}$. First, the norm structure of a C*-algebra will force any element crutched by 0 to be the zero element, so these elements are removed. Let $S_f := S \setminus f^{-1}(0)$.

Next, the adjoint structure will be encoded. Let $S_{f,*} := S_f \uplus S_f := \{0, 1\} \times S_f$, the disjoint union of S_f with itself. The original set S_f is identified with $\{0\} \times S_f$ while elements of $\{1\} \times S_f$ are denoted s^*, formal adjoints of elements in S_f. As such, it is standard to consider $S_{f,*} := S_f \cup \{s^* : s \in S_f\}$.

To encode the multiplicative structure, let $M_{S,f}$ be the set of all finite sequences of elements from $S_{f,*}$, thought of as non-commuting monomials. Specifically, one requires that the empty list u be included in $M_{S,f}$. Under concatenation of lists, $M_{S,f}$ is naturally a monoid with unit u. However, it also has a natural involution by reversing order and swapping presence/absence of the *. Hence, $M_{S,f}$ is a *-monoid, the free *-monoid on S_f.

For additive structure, let $A_{S,f}$ be the set of all functions from $M_{S,f}$ to $\mathbb{C}$ whose support is finite, thought of as non-commuting polynomials with coefficients from $\mathbb{C}$. Under point-wise addition and scalar multiplication, $A_{S,f}$ is naturally a $\mathbb{C}$-vector space. Further, each function can be written uniquely as a $\mathbb{C}$-linear sum of functions with singleton support and value 1, denoted δ_l for each $l \in M_{S,f}$.

Vector multiplication is determined by the usual polynomial formula. Explicitly, given $p = \sum_{j=1}^{n} \lambda_j \delta_{l_j}$ and $r = \sum_{k=1}^{q} \mu_k \delta_{m_k}$,

$$pr := \sum_{j=1}^{n} \sum_{k=1}^{q} \lambda_j \mu_k \delta_{l_j m_k},$$

where $l_j m_k$ is the product in $M_{S,f}$. Similarly, the adjoint operation is determined in an equally natural way. Explicitly, given $p = \sum_{j=1}^{n} \lambda_j \delta_{l_j}$,

$$p^* = \sum_{j=1}^{n} \overline{\lambda}_j \delta_{l_j^*},$$

where l_j^* is the adjoint in $M_{S,f}$.

Under these operations, it is a standard exercise to show $A_{S,f}$ to be an involutive $\mathbb{C}$-algebra with unit δ_u, the free unital *-algebra over $\mathbb{C}$ on S_f. The non-unital version of this *-algebra and its properties were detailed in Sections 1.3.3-4 of [19].

To continue the construction, one must norm $A_{S,f}$, which is where the numeric value of the crutch function arises. First, a faithful representation of $A_{S,f}$ is constructed. This proof was given previously in Lemma 3.7 of [29]. It is included here with more detail for completeness.

Lemma 3.1.1. *There exist a Hilbert space $\mathcal{H}$ and a unital *-homomorphism $\pi_0 : A_{S,f} \to \mathcal{B}(\mathcal{H})$, which is one-to-one and satisfies $\|\pi_0(\delta_s)\|_{\mathcal{B}(\mathcal{H})} = f(s)$ for all $s \in S_f$.*

Proof. Let $X := S_f \uplus S_f$, again the disjoint union of S_f with itself, but to distinguish it from $S_{f,*}$, X will be written as

$$X = \{x_s : s \in S_f\} \cup \{y_s : s \in S_f\}.$$

Let G denote the free group on X and $\mathcal{H} := \ell^2(G)$.

For $g \in G$, define $U_g : \mathcal{H} \to \mathcal{H}$ by left translation, explicitly given by

$$U_g(\vec{v})(h) := \vec{v}\left(g^{-1}h\right)$$

for all $h \in G$. This operator is well-known to be unitary with several important properties. In particular, for $g, h \in G$, $U_g^* = U_g^{-1} = U_{g^{-1}}$ and $U_g U_h = U_{gh}$. Letting $\vec{e}_g$ denote the point-mass at g in $\mathcal{H}$, observe that for $g, h \in G$,

$$U_g(\vec{e}_h) = \vec{e}_{gh}.$$

From this, the unitaries are $\mathbb{C}$-linearly independent. Explicitly, suppose that for some distinct $(g_j)_{j=1}^n \subset G$ and $(\lambda_j)_{j=1}^n \subset \mathbb{C}$,

$$\sum_{j=1}^{n} \lambda_j U_{g_j} = 0.$$

Then, evaluating at $\vec{e}_{\mathbb{1}_G}$, the point-mass for the identity in G,

$$\sum_{j=1}^{n} \lambda_j \vec{e}_{g_j} = 0.$$

However, the point-masses are an orthonormal basis for $\mathcal{H}$, meaning $\lambda_j = 0$ for all $j = 1, \ldots, n$.

To build the representation, observe that for $s \in S_f$ and $g \in G$,

$$\left(U_{x_s} - U_{y_s}\right)\left(\vec{e}_g\right) = U_{x_s}\left(\vec{e}_g\right) - U_{y_s}\left(\vec{e}_g\right) = \vec{e}_{x_s g} - \vec{e}_{y_s g}.$$

Since $x_s \neq y_s$ in G, $x_s g \neq y_s g$ for all $g \in G$, ensuring that $\vec{e}_{x_s g} \neq \vec{e}_{y_s g}$. Thus, $U_{x_s} - U_{y_s} \neq 0$. For all $s \in S_f$, define

$$T_s := \frac{f(s)}{\|U_{x_s} - U_{y_s}\|_{\mathcal{B}(\mathcal{H})}}\left(U_{x_s} - U_{y_s}\right)$$

and

$$T_{s^*} := T_s^* = \frac{f(s)}{\|U_{x_s} - U_{y_s}\|_{\mathcal{B}(\mathcal{H})}}\left(U_{x_s^{-1}} - U_{y_s^{-1}}\right).$$

Define $\pi_{00} : M_{S,f} \to \mathcal{B}(\mathcal{H})$ for $l = t_1 \cdots t_n \in M_{S,f}$ by

$$\pi_{00}(l) := T_{t_1} \cdots T_{t_n}, \quad \pi_{00}(u) := \mathbb{1}_{\mathcal{B}(\mathcal{H})},$$

encoding multiplicativity and the adjoint. Then, defining $\pi_0 : A_{S,f} \to \mathcal{B}(\mathcal{H})$ by $\pi_0(\delta_l) := \pi_{00}(l)$ and extending by $\mathbb{C}$-linearity, one can check that π_0 is a unital *-homomorphism. By design, $\|\pi_0\left(\delta_s\right)\|_{\mathcal{B}(\mathcal{H})} = f(s)$.

To show π_0 one-to-one, consider first a monomial $m := \prod_{j=1}^{n} s_j^{\epsilon_j}$, where ϵ_j detects the presence or absence of *. Then,

$$\begin{aligned}
\pi_0\left(\delta_m\right) &= \prod_{j=1}^{n} T_{s_j}^{\epsilon_j} \\
&= \left(\prod_{j=1}^{n} \frac{f\left(s_j\right)}{\left\|U_{x_{s_j}} - U_{y_{s_j}}\right\|_{\mathcal{B}(\mathcal{H})}}\right)\left(U_{x_{s_1}^{\epsilon_1} \cdots x_{s_n}^{\epsilon_n}} - U_{x_{s_1}^{\epsilon_1} \cdots x_{s_{n-1}}^{\epsilon_{n-1}} y_{s_n}^{\epsilon_n}} + \cdots\right),
\end{aligned}$$

where ϵ_j equivalently stands for the presence or absence of -1. Observe that the alternating product $x_1^{\epsilon_1} y_2^{\epsilon_2} x_3^{\epsilon_3} \cdots$ has no collapsing in G. Further, none of the other products involved have either this form or length, after reduction in G. Hence, $U_{x_1^{\epsilon_1} y_2^{\epsilon_2} x_3^{\epsilon_3} \cdots}$ is $\mathbb{C}$-linearly independent from any other term in the above sum.

Consider a general $a = \sum_{k=1}^{p} \lambda_k \delta_{m_k} \in A_{S,f}$, where $m_k \neq m_q$ for $k \neq q$. Then, each $m_k = \prod_{j=1}^{n_k} s_{j,k}^{\epsilon_{j,k}}$ has a term of form $U_{x_{1,k}^{\epsilon_{1,k}} y_{2,k}^{\epsilon_{2,k}} x_{3,k}^{\epsilon_{3,k}} \cdots}$. If $U_{x_{1,k}^{\epsilon_{1,k}} y_{2,k}^{\epsilon_{2,k}} x_{3,k}^{\epsilon_{3,k}} \cdots} = U_{x_{1,q}^{\epsilon_{1,q}} y_{2,q}^{\epsilon_{2,q}} x_{3,q}^{\epsilon_{3,q}} \cdots}$ for some k, q, then $x_{1,k}^{\epsilon_{1,k}} y_{2,k}^{\epsilon_{2,k}} x_{3,k}^{\epsilon_{3,k}} \cdots = x_{1,q}^{\epsilon_{1,q}} y_{2,q}^{\epsilon_{2,q}} x_{3,q}^{\epsilon_{3,q}} \cdots$, which forces $n_k = n_q$, $x_{j,k} = x_{j,q}$, $y_{j,k} = y_{j,q}$, and $\epsilon_{j,k} = \epsilon_{j,q}$ for all $j = 1, \ldots, n_k$. Hence, $m_k = m_q$, meaning these alternating products are unique to each monomial term.

Therefore, if $\pi_0(a) = 0$, $\lambda_k = 0$ for all $k = 1, \ldots, p$. Then, $a = 0$, and π_0 is, thereby, one-to-one.

□

With this representation, $A_{S,f}$ can be normed. Theorem 1.3.6.1 from [19] is the analogous version of this result.

Lemma 3.1.2. *For each $a \in A_{S,f}$, define*

$$\mathscr{S}_a := \left\{ \|\pi(a)\|_{\mathcal{B}} : \begin{array}{c} \mathcal{B} \text{ a unital } C^*\text{-algebra}, \\ \pi : A_{S,f} \to \mathcal{B} \text{ a unital } {}^*\text{-homomorphism}, \\ \|\pi(\delta_s)\|_{\mathcal{B}} \leq f(s) \forall s \in S_f \end{array} \right\}.$$

and $\rho_{S,f} : A_{S,f} \to [0, \infty)$ by $\rho_{S,f}(a) := \sup \mathscr{S}_a$. Then, $\rho_{S,f}$ is a sub-multiplicative norm on $A_{S,f}$ satisfying the C^*-property.

Proof. Fix $a \in A_{S,f}$. First, $\mathscr{S}_a$ is nonempty since $\|\pi_0(a)\|_{\mathcal{B}(\mathcal{H})} \in \mathscr{S}_a$, where π_0 is the representation of Lemma 3.1.1.

Next, this supremum is shown to be finite. Write a as

$$a = \sum_{j=1}^{n} \lambda_j \left(\prod_{k=1}^{m_j} \delta_{t_{j,k}} \right),$$

where each $t_{j,k}$ is a singleton list. For each $t_{j,k}$, if $t_{j,k} \in S_f$, let $s_{j,k} := t_{j,k}$. Otherwise, let $s_{j,k} := t^*_{j,k}$. Given any $\pi : A_{S,f} \to \mathcal{B}$ such that $\|\pi(\delta_s)\|_{\mathcal{B}} \leq f(s)$ for all $s \in S_f$, observe that

$$\begin{aligned} \|\pi(a)\|_{\mathcal{B}} &\leq \sum_{j=1}^{n} |\lambda_j| \left(\prod_{k=1}^{m_j} \left\|\pi\left(\delta_{t_{j,k}}\right)\right\|_{\mathcal{B}} \right) \\ &= \sum_{j=1}^{n} |\lambda_j| \left(\prod_{k=1}^{m_j} \left\|\pi\left(\delta_{s_{j,k}}\right)\right\|_{\mathcal{B}} \right) \\ &\leq \sum_{j=1}^{n} |\lambda_j| \left(\prod_{k=1}^{m_j} f\left(s_{j,k}\right) \right), \end{aligned}$$

which is independent of $\mathcal{B}$ and π. Thus, $\rho_{S,f}(a) < \infty$.

Also, $0 \leq \|\pi_0(a)\|_{\mathcal{B}(\mathcal{H})} \leq \rho_{S,f}(a)$. Since π_0 is one-to-one, $\|\pi_0(a)\|_{\mathcal{B}(\mathcal{H})} = 0$ if and only if $a = 0$. Therefore, $\rho_{S,f}(a) = 0$ if and only if $a = 0$.

Now, for any $a, b \in A_{S,f}$ and $\lambda \in \mathbb{C}$, the following conditions hold since $\|\cdot\|_{\mathcal{B}}$ is a C*-norm and π a *-homomorphism.

$$\|\pi(a+b)\|_{\mathcal{B}} \leq \|\pi(a)\|_{\mathcal{B}} + \|\pi(b)\|_{\mathcal{B}} \leq \rho_{S,f}(a) + \rho_{S,f}(b),$$

$$\|\pi(ab)\|_{\mathcal{B}} \leq \|\pi(a)\|_{\mathcal{B}} \|\pi(b)\|_{\mathcal{B}} \leq \rho_{S,f}(a)\rho_{S,f}(b),$$

$$\|\pi(a^*a)\|_{\mathcal{B}} = \|\pi(a)\|^2_{\mathcal{B}},$$

$$\|\pi(\lambda a)\|_{\mathcal{B}} = |\lambda| \|\pi(a)\|_{\mathcal{B}}.$$

Thus, by taking suprema, $\rho_{S,f}$ is a norm on $A_{S,f}$ satisfying the C*-property.

□

Thus, $A_{S,f}$ is a unital *-algebra over $\mathbb{C}$ with a C*-norm. Therefore, the completion, denoted $\mathcal{A}_{S,f}$, is a unital C*-algebra, residing in $\mathrm{Ob}\left(\mathbf{1C}^*\right)$. As such, one can consider $F_{\mathbf{1C}^*}^{\mathbf{CSet}_1}\mathcal{A}_{S,f}$, this algebra with only its norm. There is a canonical association $\eta_{S,f} : S \to \mathcal{A}_{S,f}$ by

$$\eta_{S,f}(s) := \begin{cases} \delta_s, & s \in S_f, \\ 0, & s \notin S_f. \end{cases}$$

The unital C*-algebra $\mathcal{A}_{S,f}$ equipped with $\eta_{S,f}$ is a candidate for the reflection of (S,f) along $F_{\mathbf{1C}^*}^{\mathbf{CSet}_1}$. Theorem 1.3.7.1 in [19] gives the analogous result.

Lemma 3.1.3. *The function $\eta_{S,f}$ is constrictive from (S,f) to $F_{\mathbf{1C}^*}^{\mathbf{CSet}_1}\mathcal{A}_{S,f}$.*

Proof. For each $s \in S_f$,

$$\|\eta_{S,f}(s)\|_{\mathcal{A}_{S,f}} = \|\delta_s\|_{\mathcal{A}_{S,f}} = \rho_{S,f}\left(\delta_s\right) = f(s).$$

For $s \notin S_f$,

$$\|\eta_{S,f}(s)\|_{\mathcal{A}_{S,f}} = \|0\|_{\mathcal{A}_{S,f}} = 0 = f(s).$$

□

Theorem 3.1.4. *The unital C*-algebra $\mathcal{A}_{S,f}$ equipped with $\eta_{S,f}$ is a reflection of (S,f) along $F_{\mathbf{1C}^*}^{\mathbf{CSet}_1}$.*

Proof. Let $\mathcal{B} \in \mathrm{Ob}\left(\mathbf{1C}^*\right)$ and $\phi \in \mathbf{CSet}_1\left((S,f), F_{\mathbf{1C}^*}^{\mathbf{CSet}_1}\mathcal{B}\right)$. For each $s \in S_f$, let $b_s := \phi(s)$ and $b_{s^*} := b_s^*$. Define $\hat{\phi} : M_{S,f} \to \mathcal{B}$ for $l = t_1 \cdots t_n \in M_{S,f}$ by

$$\hat{\phi}(l) := b_{t_1} \cdots b_{t_n}, \quad \hat{\phi}(u) = \mathbb{1}_{\mathcal{B}},$$

encoding multiplicativity and the adjoint. Then, defining $\tilde{\phi} : A_{S,f} \to \mathcal{B}$ by $\tilde{\phi}(\delta_l) := \hat{\phi}(l)$ and extending by $\mathbb{C}$-linearity, one can check that $\tilde{\phi}$ is a unital *-homomorphism.

For all $a \in A_{S,f}$,

$$\left\|\tilde{\phi}(a)\right\|_{\mathcal{B}} \leq \rho_{S,f}(a) = \|a\|_{\mathcal{A}_{S,f}},$$

so $\tilde{\phi}$ is contractive and, therefore, continuous. Extend $\tilde{\phi}$ by continuity to $\varphi \in \mathbf{1C}^*\left(\mathcal{A}_{S,f}, \mathcal{B}\right)$. Observe that for each $s \in S_f$,

$$\left(F_{\mathbf{1C}^*}^{\mathbf{CSet}_1}\varphi \circ \eta_{S,f}\right)(s) = F_{\mathbf{1C}^*}^{\mathbf{CSet}_1}\varphi\left(\delta_s\right) = \varphi\left(\delta_s\right) = \tilde{\phi}\left(\delta_s\right) = \hat{\phi}(s) = \phi(s).$$

For $s \notin S_f$,

$$0 \leq \|\phi(s)\|_{\mathcal{B}} \leq f(s) = 0$$

so $\|\phi(s)\|_{\mathcal{B}} = 0$, meaning $\phi(s) = 0$. Therefore,

$$\left(F_{\mathbf{1C}^*}^{\mathbf{CSet}_1}\varphi \circ \eta_{S,f}\right)(s) = F_{\mathbf{1C}^*}^{\mathbf{CSet}_1}\varphi\left(0\right) = \varphi\left(0\right) = 0 = \phi(s).$$

Thus, $F_{\mathbf{1C}^*}^{\mathbf{CSet}_1}\varphi \circ \eta_{S,f} = \phi$.

Assume there was some other $\psi \in \mathbf{1C}^*\left(\mathcal{A}_{S,f}, \mathcal{B}\right)$ such that $F_{\mathbf{1C}^*}^{\mathbf{CSet}_1}\psi \circ \eta_{S,f} = \phi$. Then, for each $s \in S_f$,

$$\phi(s) = \left(F_{\mathbf{1C}^*}^{\mathbf{CSet}_1}\psi \circ \eta_{S,f}\right)(s) = F_{\mathbf{1C}^*}^{\mathbf{CSet}_1}\psi\left(\delta_s\right) = \psi\left(\delta_s\right)$$

Hence, $\psi = \varphi$ by $\mathbb{C}$-linearity, multiplicativity, and continuity.

□

Further, since (S, f) was arbitrary, Proposition A.5.1 states that there is a unique functor $1\mathrm{C}^*\mathrm{Alg} : \mathbf{CSet}_1 \to \mathbf{1C}^*$ such that $1\mathrm{C}^*\mathrm{Alg}(S, f) = \mathcal{A}_{S,f}$, and $1\mathrm{C}^*\mathrm{Alg} \dashv F_{\mathbf{1C}^*}^{\mathbf{CSet}_1}$ by Theorem A.5.2.

Also, observe that the target category used need not be $\mathbf{1C}^*$. In particular, this construction can easily be adapted for the category of general C*-algebras and

*-homomorphisms, which is done in Section 4.1. Indeed, this can be adapted for the category of normed algebras and contractive homomorphisms, and subcategories thereof. Investigation of other settings may prove fruitful for further study.

3.2 Properties of the Functor $1\mathrm{C}^*\mathrm{Alg}$

With the adjoint pair $1\mathrm{C}^*\mathrm{Alg} \dashv F_{\mathbf{1C^*}}^{\mathbf{CSet}_1}$ exhibited in Section 3.1, attention turns to its immediate properties. This study first recovers and generalizes results from previous work, particularly the isomorphism of Proposition 3.2.4. Combining the isomorphism with the functorial characterization yields a canonical free product decomposition result in Corollary 3.2.5. Lastly, Proposition 3.2.6 determines a projectivity criterion that will be used extensively in conjunction with the free product for the main results of this work.

To begin, the universal property of the adjoint pair can be restated three different ways. The first is a direct translation of the definition of the left adjoint. It is also formally similar to the classical free mapping property.

Theorem 3.2.1 (Explicit Universal Property of $1\mathrm{C}^*\mathrm{Alg} \dashv F_{\mathbf{1C^*}}^{\mathbf{CSet}_1}$)**.** *Let (S, f) be a crutched set and $\mathcal{B}$ be a unital C*-algebra. Then for any constrictive map $\phi : (S, f) \to F_{\mathbf{1C^*}}^{\mathbf{CSet}_1}\mathcal{B}$, there is a unique unital *-homomorphism $\hat{\phi} : 1\mathrm{C}^*\mathrm{Alg}(S, f) \to \mathcal{B}$ such that $\hat{\phi}\left(\eta_{S,f}\left(s\right)\right) = \phi(s)$ for all $s \in S$.*

The condition that $\phi : (S, f) \to F_{\mathbf{1C^*}}^{\mathbf{CSet}_1}\mathcal{B}$ be constrictive is precisely that for all $s \in S$, $\|\phi(s)\|_{\mathcal{B}} \leq f(s)$. Theorem 1.3.7.1 in [19] gives the analogous result.

However, in many applications, a crutch function may not be readily available or gleaned from context, but this is not a horrible impediment. For any particular unital C*-algebra $\mathcal{B}$, one can "steal" its norm to fabricate a crutch function. This second form of the universal property is visually even closer to the free mapping property.

Corollary 3.2.2 (Norm-Stealing Form)**.** *Let S be a set and $\mathcal{B}$ be a unital C*-algebra. For any function $\phi : S \to \mathcal{B}$, define $f_\phi : S \to [0, \infty)$ by $f_\phi(s) := \|\phi(s)\|_{\mathcal{B}}$. Then, there is a unique unital *-homomorphism $\hat{\phi} : 1\mathrm{C}^*\mathrm{Alg}\,(S, f_\phi) \to \mathcal{B}$ such that $\hat{\phi}\left(\eta_{S,f_\phi}(s)\right) = \phi(s)$ for all $s \in S$.*

The third form of the universal property is termed the *scaled-free mapping property*, because when an element $s \in S$ can be mapped anywhere within a unital C*-algebra $\mathcal{B}$, $\eta_{S,f}(s)$ is sent to a nonnegative scalar multiple of this location.

Corollary 3.2.3 (Scaled-Free Mapping Property Form)**.** *Let (S, f) be a crutched set and $\mathcal{B}$ be a unital C*-algebra. Then, for any function $\phi : S \to \mathcal{B}$, there is a unique unital *-homomorphism $\hat{\phi} : 1\mathrm{C}^*\mathrm{Alg}(S, f) \to \mathcal{B}$ such that for all $s \in S$,*

$$\|\phi(s)\|_{\mathcal{B}} \cdot \hat{\phi}\left(\eta_{S,f}(s)\right) = f(s) \cdot \phi(s).$$

Proof. Let $\mathcal{A} := 1\mathrm{C}^*\mathrm{Alg}(S, f)$ and define $\varphi : S \to \mathcal{B}$ by

$$\varphi(s) := \begin{cases} \dfrac{f(s)}{\|\phi(s)\|_{\mathcal{B}}}\phi(s), & \|\phi(s)\|_{\mathcal{B}} \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that for all $s \in S$, $\|\varphi(s)\|_{\mathcal{B}} \leq f(s)$, making φ constrictive from (S, f) to $F^{\mathbf{CSet_1}}_{\mathbf{1C^*}}\mathcal{B}$. By Theorem 3.2.1, there is a unique unital *-homomorphism $\hat{\phi} : \mathcal{A} \to \mathcal{B}$ such that for all $s \in S$,

$$\hat{\phi}\left(\eta_{S,f}(s)\right) = \varphi(s) = \begin{cases} \dfrac{f(s)}{\|\phi(s)\|_{\mathcal{B}}}\phi(s), & \|\phi(s)\|_{\mathcal{B}} \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

For all $s \in S$ satisfying $\|\phi(s)\|_{\mathcal{B}} \neq 0$, a multiplication yields the desired equality. In the case that $\|\phi(s)\|_{\mathcal{B}} = 0$, then $\phi(s) = 0$ so

$$\|\phi(s)\|_{\mathcal{B}} \cdot \hat{\phi}\left(\eta_{S,f}(s)\right) = 0 = f(s) \cdot \phi(s).$$

□

For this reason, the unital C*-algebra $1\mathrm{C}^*\mathrm{Alg}(S, f)$ is termed the *scaled-free unital C*-algebra on* (S, f).

If one follows the construction in Section 3.1 with the constant function $\mathbf{1}_S : S \to \{1\}$, it is precisely the construction of the universal C*-algebra on a set S of contractions, studied in depth within [21] as a *-monoid algebra. In fact, this is precisely the only type of scaled-free unital C*-algebra, up to *-isomorphism. This is due primarily to the linearity of *-homomorphism. Also, this result generalizes Conclusion 4.1.2.9 in [19], which considers only strictly positive f.

Proposition 3.2.4 (Uniqueness of $1\mathrm{C}^*\mathrm{Alg}(S, f)$)**.** *Given a crutched set* (S, f), *let* $\mathbf{1}_{S_f} : S_f \to \{1\}$ *be the constant function. Then,* $1\mathrm{C}^*\mathrm{Alg}(S, f) \cong_{\mathbf{1C^*}} 1\mathrm{C}^*\mathrm{Alg}\left(S_f, \mathbf{1}_{S_f}\right)$.

Proof. Let $\mathcal{A} := 1\mathrm{C}^*\mathrm{Alg}(S, f)$ and $\mathcal{B} := 1\mathrm{C}^*\mathrm{Alg}\left(S_f, \mathbf{1}_{S_f}\right)$. Define $\phi : S \to \mathcal{B}$ by

$$\phi(s) := \begin{cases} f(s)\eta_{S_f,\mathbf{1}_{S_f}}(s), & s \in S_f, \\ 0, & s \notin S_f. \end{cases}$$

Observe that for all $s \in S_f$, $\|\phi(s)\|_{\mathcal{B}} = f(s)$ and for all $s \notin S_f$,

$$\|\phi(s)\|_{\mathcal{B}} = \|0\|_{\mathcal{B}} = 0 = f(s).$$

By Theorem 3.2.1, there is a unique unital *-homomorphism $\hat{\phi} : \mathcal{A} \to \mathcal{B}$ such that

$\hat{\phi}\left(\eta_{S,f}(s)\right) = \phi(s)$ for all $s \in S$.

Similarly, define $\varphi : S_f \to \mathcal{A}$ by $\varphi(t) := \frac{1}{f(t)}\eta_{S,f}(t)$. Then, for all $t \in S_f$,

$$\|\varphi(t)\|_{\mathcal{A}} = \frac{1}{f(t)}f(t) = 1.$$

By Theorem 3.2.1, there is a unique unital *-homomorphism $\hat{\varphi} : \mathcal{B} \to \mathcal{A}$ such that $\hat{\varphi}\left(\eta_{S_f,\mathbf{1}_{S_f}}(t)\right) = \varphi(t)$ for all $t \in S_f$.

Note that for each $t \in S_f$,

$$\left(\hat{\phi} \circ \hat{\varphi}\right)\left(\eta_{S_f,\mathbf{1}_{S_f}}(t)\right) = \frac{1}{f(t)}\hat{\phi}\left(\eta_{S,f}(t)\right) = \frac{1}{f(t)}f(t)\eta_{S_f,\mathbf{1}_{S_f}}(t) = \eta_{S_f,\mathbf{1}_{S_f}}(t).$$

By Theorem 3.2.1, $\hat{\phi} \circ \hat{\varphi} = id_{\mathcal{B}}$.

Symmetrically, for each $s \in S_f$,

$$\left(\hat{\varphi} \circ \hat{\phi}\right)\left(\eta_{S,f}(s)\right) = f(s)\hat{\varphi}\left(\eta_{S_f,\mathbf{1}_{S_f}}(s)\right) = f(s)\frac{1}{f(s)}\eta_{S,f}(s) = \eta_{S,f}(s).$$

For each $s \notin S_f$,

$$\left(\hat{\varphi} \circ \hat{\phi}\right)\left(\eta_{S,f}(s)\right) = \left(\hat{\varphi} \circ \hat{\phi}\right)(0) = 0 = \eta_{S,f}(s).$$

By Theorem 3.2.1, $\hat{\varphi} \circ \hat{\phi} = id_{\mathcal{A}}$.

□

In [21], this algebra is called the "free C*-algebra" on S. However, the statement is qualified that the algebra is "free" precisely in the sense of Theorem 3.2.1. Also, the scaled-free mapping property of Corollary 3.2.3 substantiates the statement that this algebra is "the closest one gets to free C*-algebras" in [14].

Similarly, [33] and [34] create this same algebra by considering another forgetful functor. Explicitly, to every $\mathcal{A}$ in $\mathrm{Ob}\,(\mathbf{1C}^*)$, one associates the set

$$U_{\mathcal{A}} := \{a \in \mathcal{A} : \|a\|_{\mathcal{A}} \leq 1\}\,,$$

the unit ball. Also, for a map $\phi \in \mathbf{1C}^*(\mathcal{A}, \mathcal{B})$, it is a standard fact that $\phi\,(U_{\mathcal{A}}) \subseteq U_{\mathcal{B}}$. Thus, one can quickly show that these associations define a functor $U : \mathbf{1C}^* \to \mathbf{Set}$, where all structure is lost except the unit ball and the appropriately restricted maps. Both papers show that this functor has a left adjoint, creating the algebra of contractions.

However, with the functor U, the norm has been hardcoded by the choice of $U_{\mathcal{A}}$. Specifically, let $L : \mathbf{Set} \to \mathbf{1C}^*$ be the left adjoint to U. Then, the universal property of $L(S)$ is that given a unital C*-algebra $\mathcal{B}$ and a function $\phi : S \to U(\mathcal{B})$, there is a unique unital *-homomorphism $\hat{\phi} : L(S) \to \mathcal{B}$ such that $\hat{\phi}\,(\delta_s) = \phi(s)$. Hence, any element of S *must* be sent to an element of norm at most 1.

What the construction in Section 3.1 has done is allowed the norms of generators to vary, encoding the numeric data in the crutch function rather than the choice of a subset. Indeed, the f in Theorem 3.2.1 is fixed prior to construction, but has no restriction otherwise. In particular, it need not be constant or bounded.

Also, the forgetful functor $F^{\mathbf{CSet}_1}_{\mathbf{1C}^*}$ only removes structure, not altering the underlying set in any way. This aspect seems to give a more natural "forgetful" feel like the classical situation of Section 1.2.

Proposition 3.2.4 states that the properties of the unit ball functor are recovered via this more general construction. Arguably, one can choose to scale all generators to norm 1, but in some cases, it may be preferable to let individual generators have different crutched values. This issue of allowing norms of generators to vary is a driving concept behind the remainder of this chapter.

As stated before, Section 1.3 of [19] forms the non-unital algebra of contractions in a similar way to Section 3.1 of the present work. Section 4.1.2 of [19] holds a comparable analysis of the structure of this object. However, while the initial formulation in Section 1.1 of [19] mentions the forgetful functor and the adjoint situation, the categorical properties are not exploited in the work.

Specifically, $1\mathrm{C}^*\mathrm{Alg}$ has been shown to be a left adjoint functor in Theorem 3.1.4, which ensures that it preserves all categorical colimits by Proposition A.5.4. A fundamental type of colimit is the coproduct. As summarized in [40], $\mathbf{1C^*}$ has all coproducts, namely the free product amalgamated along the identity. As such, for an index set I and unital C*-algebras $(\mathcal{A}_i)_{i\in I}$, their free product amalgamated along their identities will be denoted $\coprod_{i\in I}^{\mathbf{1C^*}} \mathcal{A}_i$.

In regard to notation, the unital free product is usually denoted by "$*_{\mathbb{C}}$", indicating the merger of the identities. The "$\coprod$" notation will be used interchangeably with the "$*_{\mathbb{C}}$" notation, but preference will be given to the "$\coprod$" with arbitrary index sets.

Recall that Proposition 2.2.9 described the "disjoint union" crutched set, which gave a canonical decomposition of a crutched set into singleton crutched sets. Combining this characterization with Proposition 3.2.4, the following canonical form is taken.

Corollary 3.2.5. *Given a crutched set (S, f),*

$$1\mathrm{C}^*\mathrm{Alg}(S,f) \cong_{\mathbf{1C^*}} \coprod_{s\in S_f}^{\mathbf{1C^*}} 1\mathrm{C}^*\mathrm{Alg}\left(\{(s,f(s))\}\right) \cong_{\mathbf{1C^*}} \coprod_{s\in S_f}^{\mathbf{1C^*}} 1\mathrm{C}^*\mathrm{Alg}\left(\{(s,1)\}\right).$$

In the case $\mathrm{card}(S) = 2$ and $f(s) > 0$ for all $s \in S$, this result can be stated in the traditional notation as

$$1\mathrm{C}^*\mathrm{Alg}(S,f) \cong_{\mathbf{1C^*}} 1\mathrm{C}^*\mathrm{Alg}\left(\{(s_1,1)\}\right) *_{\mathbb{C}} 1\mathrm{C}^*\mathrm{Alg}\left(\{(s_2,1)\}\right).$$

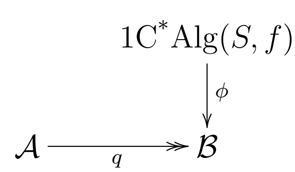
Decompositions and characterizations such as this will be used extensively in the remainder of this chapter, particularly Sections 3.9 and 3.10.

Lastly, most pure algebraic contexts have the free object become automatically projective with respect to all surjections due to its universal property, which allows arbitrary mapping of its generators. However, the universal property for $1\mathrm{C}^*\mathrm{Alg}(S, f)$ has a restriction on where its generators can be sent, namely by constrictive mapping.

Thus, there is some care which needs to be taken here with regard to the crutch function. Nevertheless, the scaled-free unital C*-algebra is projective relative to all surjections, like its algebraic counterpart. The author would like to thank and acknowledge Dr. Terry Loring for the functional calculus method used here, rather than proximinality, providing a more constructive and simple proof.

Proposition 3.2.6. *Given a crutched set* (S, f)*,* $1\mathrm{C}^*\mathrm{Alg}(S, f)$ *is projective with respect to all surjections in* $\mathbf{1C}^*$*.*

Proof. Consider the following diagram in $\mathbf{1C}^*$



where $q : \mathcal{A} \to \mathcal{B}$ is surjective. For each $s \in S$, let $b_s := (\phi \circ \eta_{S,f})(s)$. If $b_s = 0$, let $a_s := 0$.

If $b_s \neq 0$, choose $\hat{a}_s \in q^{-1}(b_s)$. Define $g_s : \left[0, \|\hat{a}_s\|_{\mathcal{A}}^2\right] \to \mathbb{R}$ by

$$g_s(\mu) := \begin{cases} 1, & 0 \leq \mu \leq \|b_s\|_{\mathcal{B}}^2, \\ \dfrac{\|b_s\|_{\mathcal{B}}}{\sqrt{\mu}}, & \mu > \|b_s\|_{\mathcal{B}}^2, \end{cases}$$

a continuous function. Also, notice that for $\mu \in \left[0, \|\hat{a}_s\|_{\mathcal{A}}^2\right]$,

$$\left(id_{\left[0,\|\hat{a}_s\|_{\mathcal{A}}^2\right]} \cdot g_s^2\right)(\mu) = \begin{cases} \mu, & 0 \leq \mu \leq \|b_s\|_{\mathcal{B}}^2, \\ \|b_s\|_{\mathcal{B}}^2, & \mu > \|b_s\|_{\mathcal{B}}^2. \end{cases}$$

Let $a_s := \hat{a}_s g_s\left(\hat{a}_s^* \hat{a}_s\right)$, created by applying the continuous functional calculus to the positive element $\hat{a}_s^* \hat{a}_s$. By the continuous functional calculus, the following equalities hold:

$$\begin{aligned} \|a_s\|_{\mathcal{A}}^2 &= \|g_s\left(\hat{a}_s^* \hat{a}_s\right) \hat{a}_s^* \hat{a}_s g_s\left(\hat{a}_s^* \hat{a}_s\right)\|_{\mathcal{A}} = \left\|\hat{a}_s^* \hat{a}_s g_s^2\left(\hat{a}_s^* \hat{a}_s\right)\right\|_{\mathcal{A}} \\ &= \left\|\left(id_{\left[0,\|\hat{a}_s\|_{\mathcal{A}}^2\right]} \cdot g_s^2\right)\left(\hat{a}_s^* \hat{a}_s\right)\right\|_{\mathcal{A}} = \sup\left\{\mu g_s^2(\mu) : \mu \in \sigma_{\mathcal{A}}\left(\hat{a}_s^* \hat{a}_s\right)\right\} \\ &= \|b_s\|_{\mathcal{B}}^2 \end{aligned}$$

and

$$q\left(a_s\right) = b_s g_s\left(b_s^* b_s\right) = b_s \mathbb{1}_{\mathcal{B}} = b_s.$$

Define $\varphi : (S, f) \to F_{\mathbf{1C^*}}^{\mathbf{CSet}_1} \mathcal{A}$ by $\varphi(s) := a_s$, which is constrictive by design. By Theorem 3.2.1, there is a unique unital *-homomorphism $\hat{\varphi} : 1\mathrm{C}^*\mathrm{Alg}(S, f) \to \mathcal{A}$ such that $\hat{\varphi}\left(\eta_{S,f}(s)\right) = \varphi(s) = a_s$ for all $s \in S$. Observe that for each $s \in S$,

$$\left(q \circ \hat{\varphi}\right)\left(\eta_{S,f}(s)\right) = q\left(a_s\right) = b_s = \phi\left(\eta_{S,f}(s)\right).$$

Therefore, by Theorem 3.2.1, $q \circ \hat{\varphi} = \phi$.

□

3.3 Definitions & Conventions for $1C^*$

In most pure algebraic categories with a free object, a *relation* is precisely an element of this free object. The primary reason for this definition is that in these categories, every object has a free object of appropriate size which maps onto it. Hence, by that category's first isomorphism theorem, the target object is isomorphic to a quotient of a free object. That is, the kernel of the map encodes the algebraic data of the target object not already present in the free object.

In the next example, the C*-algebras $1\mathrm{C}^*\mathrm{Alg}(S,f)$ perform this very task.

Example 3.3.1. Given a unital C*-algebra $\mathcal{B}$, let $S := \mathcal{B}$, the underlying set of $\mathcal{B}$, and $f : S \to [0,\infty)$ by $f(s) := \|s\|_{\mathcal{B}}$. Define $\phi : S \to \mathcal{B}$ by $\phi(s) := s$, the identity map. Trivially, ϕ is a constriction from (S,f) to $F^{\mathbf{CSet_1}}_{\mathbf{1C^*}}\mathcal{B}$. By Theorem 3.2.1, there is a unique unital *-homomorphism $\hat{\phi} : 1\mathrm{C}^*\mathrm{Alg}(S,f) \to \mathcal{B}$ such that $\hat{\phi}\,(\eta_{S,f}(s)) = \phi(s)$ for all $s \in S$. Then, for all $b \in \mathcal{B}$, $b = \hat{\phi}\,(\eta_{S,f}(b))$. Hence, $\hat{\phi}$ is surjective.

Thus, in parallel to the pure algebraic cases, the following definitions are made.

Definition. For a crutched set (S,f), a *C*-relation* on (S,f) is an element of $1\mathrm{C}^*\mathrm{Alg}(S,f)$. An element of $\eta_{S,f}(S)$ itself is a *generator*.

Definition. For a crutched set (S,f) and C*-relations $R \subseteq 1\mathrm{C}^*\mathrm{Alg}(S,f)$ on (S,f), let J_R be the two-sided, norm-closed ideal generated by R in $1\mathrm{C}^*\mathrm{Alg}(S,f)$. Then, the *unital C*-algebra presented on* (S,f) *subject to* R is

$$\langle S, f | R \rangle_{\mathbf{1C^*}} := 1\mathrm{C}^*\mathrm{Alg}(S,f)/J_R,$$

the quotient C*-algebra of $1\mathrm{C}^*\mathrm{Alg}(S,f)$ by J_R.

Note that $\langle S, f | R \rangle_{\mathbf{1C^*}}$ is a unital C*-algebra. Also, by Example 3.3.1, every unital

C*-algebra has a presentation in this sense. In parallel to the algebraic notion of presentation, the following definitions describe how a particular unital C*-algebra was formed.

Definition. Let $\mathcal{A}$ be a unital C*-algebra.

1. $\mathcal{A}$ is *finitely generated* in $\mathbf{1C}^*$ if there is a crutched set (S, f) and C*-relations R on (S, f) such that $\mathrm{card}(S) < \aleph_0$ and $\mathcal{A} \cong_{\mathbf{1C}^*} \langle S, f|R\rangle_{\mathbf{1C}^*}$.

2. $\mathcal{A}$ is *finitely related* in $\mathbf{1C}^*$ if there is a crutched set (S, f) and C*-relations R on (S, f) such that $\mathrm{card}(R) < \aleph_0$ and $\mathcal{A} \cong_{\mathbf{1C}^*} \langle S, f|R\rangle_{\mathbf{1C}^*}$.

3. $\mathcal{A}$ is *finitely presented* in $\mathbf{1C}^*$ if there is a crutched set (S, f) and C*-relations R on (S, f) such that $\mathrm{card}(S), \mathrm{card}(R) < \aleph_0$ and $\mathcal{A} \cong_{\mathbf{1C}^*} \langle S, f|R\rangle_{\mathbf{1C}^*}$.

Analogously, one also defines *countably generated*, *countably related*, and *countably presented* by easing the strict inequality on the cardinalities to allow equality. Most of the examples presented in this work will be finitely presented, and many of those will be singly generated.

Definitions of "relation" and "universal C*-algebra" are made in Definition 2.2.1 of [19] by embedding the complex *-algebra over a set S into the scaled-free C*-algebra on (S, f). Specifically, [19] defines a *relation* to be an element of the complex *-algebra, *not* the C*-algebra. This choice prevents use of norm limits in relations, restricting attention only to *-algebraic combinations of the generators.

However, allowing relations to arise from the C*-algebra itself allows different kinds of conditions to be implemented. This view of *C*-relation* will be utilized and demonstrated throughout the remainder of this present work.

Also, one should check to see how the presentations defined in [19] correspond to those considered in this present work. As [19] handles non-unital C*-algebras, this argument will be set aside until Section 4.3, where a respective non-unital presentation theory is developed.

Notice also that if one can perform a scaled-free construction analogous to Section 3.1 in another category of normed algebraic objects, analogs of these definitions can be made and utilized.

As is convention for presentation theories, one blurs the distinction between $s \in S$ and $[\eta_{S,f}(s)] \in \langle S, f|R\rangle_{\mathbf{1C^*}}$, considering the latter a singleton monomial in the algebra. Even though this convention does neglect the quotienting that is happening, it eases notation and helps intuition. However, one should be very wary of where generators are located and what quotient processes have occurred.

Another convention for presentation theories is to write relations equationally. Specifically, $r = 0$ in $\langle S, f|R\rangle_{\mathbf{1C^*}}$ for all $r \in R$. Many times, a presentation can be written more easily or more intuitively by replacing some or all of these equational statements with an equivalent one. For most examples, this is very useful and instructive. However, for most general proofs, regarding R as a set of elements is far more useful than as a set of equations.

Further, if $R = \emptyset$, $J_R = 0$ so $\langle S, f|\emptyset\rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} 1\mathrm{C}^*\mathrm{Alg}(S, f)$. Thus, for simplicity and consistency of notation, the scaled-free unital C*-algebra on (S, f) will be denoted $\langle S, f|\emptyset\rangle_{\mathbf{1C^*}}$ rather than $1\mathrm{C}^*\mathrm{Alg}(S, f)$.

For a finite set $S = \{s_1, \ldots, s_n\}$, the general notation above can be simplified a bit. Recall Example 2.1.3, where a crutched set was written as

$$(S, f) = \{(s_1, \lambda_1), \ldots, (s_n, \lambda_n)\},$$

directly associating $s_j \in S$ with $\lambda_j := f(s_j)$. Similarly, if S is finite, notation and intuition may be aided by writing the presentation in the following way.

$$\langle (s_1, \lambda_1), \ldots, (s_n, \lambda_n) | R\rangle_{\mathbf{1C^*}} := \langle S, f | R\rangle_{\mathbf{1C^*}}$$

As a presentation is built out of universal constructions, specifically the adjoint functor 1C*Alg and the C*-quotient, it satisfies a universal property. Theorem 2.2.5 of [19] is the analogous result.

Theorem 3.3.2 (Universal Property of a Presentation)**.** *Let R be C*-relations on (S, f) and $\mathcal{B}$ a unital C*-algebra. Let $\phi : (S, f) \to F^{\mathbf{CSet_1}}_{\mathbf{1C^*}}\mathcal{B}$ be a constriction and $\hat{\phi} : \langle S, f|\emptyset\rangle_{\mathbf{1C^*}} \to \mathcal{B}$ the unital *-homomorphism guaranteed by Theorem 3.2.1. If $R \subseteq \ker\left(\hat{\phi}\right)$, then there is a unique unital *-homomorphism $\tilde{\phi} : \langle S, f|R\rangle_{\mathbf{1C^*}} \to \mathcal{B}$ such that $\tilde{\phi}(s) = \phi(s)$.*

Proof. Let $q : \langle S, f|\emptyset\rangle_{\mathbf{1C^*}} \to \langle S, f|R\rangle_{\mathbf{1C^*}}$ be the quotient map. Given $\phi : (S, f) \to F^{\mathbf{CSet_1}}_{\mathbf{1C^*}}\mathcal{B}$ is a constriction, then Theorem 3.2.1 provides a unique unital *-homomorphism $\hat{\phi} : \langle S, f|\emptyset\rangle_{\mathbf{1C^*}} \to \mathcal{B}$ such that $\phi(s) = \hat{\phi}(s)$ for all $s \in S$. By hypothesis, $R \subseteq \ker\left(\hat{\phi}\right)$ so if J_R is the two-sided norm-closed ideal generated by R, $J_R \subseteq \ker\left(\hat{\phi}\right)$. Thus, there is a unique unital *-homomorphism $\tilde{\phi} : \langle S, f|R\rangle_{\mathbf{1C^*}} \to \mathcal{B}$ such that $\tilde{\phi} \circ q = \hat{\phi}$. In particular, for all $s \in S$, $\tilde{\phi}(s) = \hat{\phi}(s) = \phi(s)$.

Assume there was another $\psi : \langle S, f|R\rangle_{\mathbf{1C^*}} \to \mathcal{B}$ such that $\psi(s) = \phi(s)$ for all $s \in S$. Then, $(\psi \circ q)(s) = \phi(s)$ for all $s \in S$. Hence, by Theorem 3.2.1, $\psi \circ q = \hat{\phi} = \tilde{\phi} \circ q$. By the universal property of the quotient, $\psi = \tilde{\phi}$.

□

While the statement of the above theorem is verbose and buried in notation, the intuition behind it is natural. Given a constrictive mapping ϕ of the generators (S, f) to a unital C*-algebra $\mathcal{B}$ where all C*-relations R "evaluate" to 0, there is a unique unital *-homomorphism $\tilde{\phi}$ from $\langle S, f|R\rangle_{\mathbf{1C^*}}$ to $\mathcal{B}$ with $\tilde{\phi}(s) = \phi(s)$ for each $s \in S$. As such, many would call $\langle S, f|R\rangle_{\mathbf{1C^*}}$ the "universal unital C*-algebra of (S, f) subject to R".

3.4 Construction: Abelianization for $\mathbf{1C^*}$

Before computing examples, one particular construction is characterized first to make these calculations easier to manage. A well-known construction is the abelianization, a canonical way of making an algebra commutative. In Section B.4, the abelianization is realized as a left adjoint functor to a natural forgetful functor. For notation, let $\mathbf{C1C^*}$ denote the category of commutative unital C*-algebras with unital *-homomorphisms and $\mathrm{Ab}_1 : \mathbf{1C^*} \to \mathbf{C1C^*}$ the abelianization functor.

For the remainder of this section, fix a crutched set (S, f) and C*-relations R on (S, f). Define $\mathcal{F} := \langle S, f|\emptyset\rangle_{\mathbf{1C^*}}$, $\mathcal{A} := \langle S, f|R\rangle_{\mathbf{1C^*}}$, and $q_R : \mathcal{F} \to \mathcal{A}$ the quotient map. Composing a presentation for $\mathrm{Ab}_1(\mathcal{A})$ is straight-forward and natural, merely forcing the generators and their adjoints to commute. The analogous result was proven in Proposition 3.2.2 of [19].

Proposition 3.4.1. *Given a crutched set (S, f) and C*-relations R on (S, f),*

$$\mathrm{Ab}_1\left(\langle S, f|R\rangle_{\mathbf{1C^*}}\right) \cong_{\mathbf{C1C^*}} \langle S, f\,|R \cup \{st - ts, st^* - t^*s : s, t \in S\}\rangle_{\mathbf{1C^*}}\,.$$

Proof. Let $\hat{R} := R \cup \{st - ts, st^* - t^*s \,:\, s, t \in S\}$, $\hat{\mathcal{A}} := \left\langle S, f \,\middle|\, \hat{R} \right\rangle_{\mathbf{1C^*}}$, and $q_{\hat{R}} : \mathcal{F} \to \hat{\mathcal{A}}$ be the quotient map. Observe that $\hat{\mathcal{A}}$ is commutative. Briefly, the generators commute with one another and their adjoints, meaning all unital polynomials in the generators commute. Since these polynomials are dense in $\hat{\mathcal{A}}$, $\hat{\mathcal{A}}$ is commutative. Further, note that $R \subseteq \ker(q_{\hat{R}})$. By Theorem 3.3.2, there is a unique unital *-

homomorphism $\rho : \mathcal{A} \to \hat{\mathcal{A}}$ such that $\rho(s) = q(s) = s$ for all $s \in S$.

The unital C*-algebra $\hat{\mathcal{A}}$ equipped with ρ is a candidate for the abelianization of $\mathcal{A}$. To show the universal property, let $\mathcal{B}$ be a commutative, unital C*-algebra and $\phi : \mathcal{A} \to \mathcal{B}$ a unital *-homomorphism. Since $\mathcal{B}$ is commutative, $\hat{R} \subseteq \ker(\phi \circ q)$. By Theorem 3.3.2, there is a unique unital *-homomorphism $\hat{\phi} : \hat{\mathcal{A}} \to \mathcal{B}$ such that $\hat{\phi}(s) = (\phi \circ q)(s) = (\phi \circ \rho)(s) = \phi(s)$ for all $s \in S$.

□

However, this characterization can be made far more concrete using the Gelfand duality. Proposition 4.2.1.7 in [19] has a similar description, but it is restricted to finitely many generators and *-algebraic relations. The following construction and result have neither of these two restrictions. Further, the proof demonstrated here will use existing facts about the Gelfand duality and the scaled-free unital C*-algebra, yielding a shorter proof.

For notation, let **Comp** stand for the category of compact Hausdorff spaces with continuous functions, $C : \textbf{Comp} \to \textbf{C1C}^*$ the functor associating a space X with continuous functions on X, and $\Delta : \textbf{C1C}^* \to \textbf{Comp}$ the functor associating a C*-algebra $\mathcal{A}$ with its maximal ideal space. For $\lambda \geq 0$, let $D_\lambda := \{\mu \in \mathbb{C} : |\mu| \leq \lambda\}$, the closed disc of radius λ.

To devise this concrete description, the characters on $\mathcal{F}$ are determined. Using this, each C*-relation can be associated to a continuous function via the Gelfand theory. Then, in direct parallel to the classical geometric results, the zero set of these functions yields the maximal ideal space of $\text{Ab}_1(\mathcal{A})$.

To begin, let

$$X := \prod_{s \in S}^{\textbf{Comp}} D_{f(s)}$$

be equipped with the projection maps $\pi_s : X \to D_{f(s)}$. Given any $\vec{x} \in X$, define

$\phi_{\vec{x}} : S \to \mathbb{C}$ by $\phi_{\vec{x}}(s) := \pi_s(\vec{x})$, formalizing the tuple $\vec{x}$ as a function. By Theorem 3.2.1, there exists a unique unital *-homomorphism $\hat{\phi}_{\vec{x}} : \mathcal{F} \to \mathbb{C}$ such that $\hat{\phi}_{\vec{x}}(s) = \phi_{\vec{x}}(s) = \pi_s(\vec{x})$ for all $s \in S$. Furthermore, all characters from $\mathcal{F}$ take this form.

Lemma 3.4.2. *Given any unital *-homomorphism $\psi : \mathcal{F} \to \mathbb{C}$, there is a unique $\vec{x} \in X$ such that $\psi = \hat{\phi}_{\vec{x}}$.*

Proof. Let $\vec{x} := (\psi(s))_{s\in S}$, the tuple defined by the images of the generators under ψ. Since $|\psi(s)| \leq \|s\|_{\mathcal{F}} = f(s)$, $\vec{x} \in X$. Observe that for all $s \in S$,

$$\hat{\phi}_{\vec{x}}(s) = \pi_s(\vec{x}) = \psi(s).$$

By Theorem 3.2.1, $\hat{\phi}_{\vec{x}} = \psi$.

If there was $\vec{y} \in X$ such that $\hat{\phi}_{\vec{y}} = \psi = \hat{\phi}_{\vec{x}}$, then for all $s \in S$,

$$\pi_s(\vec{x}) = \hat{\phi}_{\vec{x}}(s) = \hat{\phi}_{\vec{y}}(s) = \pi_s(\vec{y}).$$

Hence, $\vec{x} = \vec{y}$.

□

Next, each C*-relation is associated to a continuous function on X. For $r \in \mathcal{F}$, define $g_r : X \to \mathbb{C}$ by $g_r(\vec{x}) := \hat{\phi}_{\vec{x}}(r)$. If $\mathcal{F}$ were commutative, this would be the Gelfand transform of r. However, the continuity of g_r can be readily proven from first principles.

Lemma 3.4.3. *For each $r \in \mathcal{F}$, $g_r \in C(X)$.*

Proof. Recall from the construction in Section 3.1 that $\mathcal{F}$ is a norm-completion of the unital *-polynomials in S. For any unital *-polynomial p and $\vec{x} \in X$, observe that

$$g_p(\vec{x}) = \hat{\phi}_{\vec{x}}(p) = p(\vec{x}),$$

just evaluation of the polynomial function p at $\vec{x}$. Thus, each $g_p \in C(X)$ for each unital *-polynomial p. For any $r \in \mathcal{F}$, there is a sequence $(p_n)_{n\in\mathbb{N}}$ of unital *-polynomials in S such that $p_n \to r$ in norm. Observe that for any $\vec{x} \in X$,

$$\begin{aligned} |g_{p_n}(\vec{x}) - g_r(\vec{x})| &= |\phi_{\vec{x}}(p_n) - \phi_{\vec{x}}(r)| \\ &= |\phi_{\vec{x}}(p_n - r)| \\ &\leq \|p_n - r\|_{\mathcal{F}} \end{aligned}$$

so $g_{p_n} \to g_r$ uniformly. Hence, $g_r \in C(X)$.

□

Finally, let $X_R := \bigcap_{r\in R} g_r^{-1}(0)$, the common zero set of $(g_r)_{r\in R}$ and a closed subspace of X. This becomes the maximal ideal space of $\mathrm{Ab}_1(\mathcal{A})$.

Theorem 3.4.4. *For a crutched set (S, f) and C*-relations R on (S, f),*

$$\mathrm{Ab}_1(\langle S, f|R\rangle_{\mathbf{1C^*}}) \cong_{\mathbf{C1C^*}} C\left(\bigcap_{r\in R} g_r^{-1}(0)\right).$$

Proof. By Proposition 3.4.1, $\hat{\mathcal{A}}$ is an abelianization of $\mathcal{A}$. This presentation will be used to create the desired isomorphism.

For $s \in S$, let $\rho_s := \pi_s|_{X_R}$, the restriction of the projection map to X_R. Notice that $\|\rho_s\|_{C(X_R)} \leq f(s)$ for all $s \in S$. Define $\psi : S \to C(X_R)$ by $\psi(s) := \rho_s$. By Theorem 3.2.1, there exists a unique unital *-homomorphism $\hat{\psi} : \mathcal{F} \to C(X_R)$ such that $\hat{\psi}(s) = \psi(s) = \rho_s$.

For $\vec{x} \in X_R$, let $\epsilon_{\vec{x}} : C(X_R) \to \mathbb{C}$ by $\epsilon_{\vec{x}}(a) := a(\vec{x})$, the evaluation map, a well-known unital *-homomorphism. For all $s \in S$ and $\vec{x} \in X_R$,

$$\left(\epsilon_{\vec{x}} \circ \hat{\psi}\right)(s) = \epsilon_{\vec{x}}(\rho_s) = \rho_s(\vec{x}) = \pi_s(\vec{x}) = \hat{\phi}_{\vec{x}}(s).$$

By Theorem 3.2.1, $\epsilon_{\vec{x}} \circ \hat{\psi} = \hat{\phi}_{\vec{x}}$ for all $\vec{x} \in X_R$. For all $r \in R$ and $\vec{x} \in X_R$,

$$\left(\epsilon_{\vec{x}} \circ \hat{\psi}\right)(r) = \hat{\phi}_{\vec{x}}(r) = g_r(\vec{x}) = 0.$$

Since $C(X_R)$ is commutative, $\hat{R} \subseteq \ker\left(\hat{\psi}\right)$. By Theorem 3.3.2, there is a unique unital *-homomorphism $\tilde{\psi} : \hat{\mathcal{A}} \to C(X_R)$ such that $\tilde{\psi}(s) = \psi(s) = \rho_s$ for all $s \in S$. By the Stone-Weierstrass Theorem, $(\rho_s)_{s\in S}$ generate $C(X_R)$ so $\tilde{\psi}$ is onto.

Applying Δ, $\Delta\left(\tilde{\psi}\right) : \Delta(C(X_R)) \to \Delta\left(\hat{A}\right)$ is one-to-one. To show this map onto, let $\gamma : \hat{\mathcal{A}} \to \mathbb{C}$ be a character. Then, $\gamma \circ q_{\hat{R}}$ is a character on $\mathcal{F}$ so by Lemma 3.4.2, there is a unique $\vec{x} \in X$ such that $\gamma \circ q_{\hat{R}} = \hat{\phi}_{\vec{x}}$. For all $r \in R$,

$$g_r(\vec{x}) = \hat{\phi}_{\vec{x}}(r) = (\gamma \circ q_{\hat{R}})(r) = 0$$

so $\vec{x} \in X_R$. Therefore,

$$\epsilon_{\vec{x}} \circ \tilde{\psi} \circ q_{\hat{R}} = \epsilon_{\vec{x}} \circ \hat{\psi} = \hat{\phi}_{\vec{x}} = \gamma \circ q_{\hat{R}}$$

so by Theorem 3.3.2, $\gamma = \epsilon_{\vec{x}} \circ \tilde{\psi} = \Delta\left(\tilde{\psi}\right)(\epsilon_{\vec{x}})$. Thus, $\Delta\left(\tilde{\psi}\right)$ is onto, implying that $\hat{\mathcal{A}} \cong_{\mathbf{C1C^*}} C(X_R)$.

$\square$

Consequently, if $\mathcal{A}$ was commutative originally, $\mathcal{A}$ is completely described by this theorem. If $\mathcal{A}$ was not commutative, this result above gives spectral containments for the generators.

Corollary 3.4.5. *For a crutched set* (S, f) *and C*-relations* R *on* (S, f)*,*

$$\pi_s\left(\bigcap_{r\in R} g_r^{-1}(0)\right) \subseteq \sigma_{\mathcal{A}}(s) \subseteq D_{f(s)},$$

for all $s \in S$*.*

3.5 Well-Known Examples

Examples of many C*-relations are already in existence and readily accessible. In particular, *-polynomials in the generators (S, f) are C*-relations. In fact, many important types of operators are immediately characterized in this way.

As a first example, an element x is *normal* if $x^*x = xx^*$.

Example 3.5.1 (A normal element, [4])*.* For $\lambda \geq 0$, consider

$$\langle (x, \lambda) | x^*x = xx^* \rangle_{\mathbf{1C^*}}.$$

Note that x is normal and generates this algebra, so it is commutative. In this case, $r = x^*x - xx^*$ so

$$g_r(\mu) = \overline{\mu}\mu - \mu\overline{\mu} = 0.$$

Hence, $g_r^{-1}(0) = D_\lambda$. By Theorem 3.4.4,

$$\langle (x, \lambda) | x^*x = xx^* \rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} C\left(D_\lambda\right) \cong_{\mathbf{1C^*}} \begin{cases} \mathbb{C}, & \lambda = 0, \\ C\left(\overline{\mathbb{D}}\right), & \lambda > 0, \end{cases}$$

since $D_\lambda \cong_{\mathbf{Top}} D_1 = \overline{\mathbb{D}}$ for all $\lambda > 0$.

An element x is *self-adjoint* if $x^* = x$.

Example 3.5.2 (A self-adjoint element, [4]). For $\lambda \geq 0$, consider

$$\langle (x, \lambda) | x^* = x \rangle_{\mathbf{1C^*}}.$$

Then, $xx^* = x^2 = x^*x$ so x is normal and generates this algebra. Thus, it is commutative. In this case, $r = x - x^*$ so

$$g_r(\mu) = \mu - \overline{\mu} = 2\imath\Im(\mu)$$

Hence, $g_r^{-1}(0) = [-\lambda, \lambda]$. By Theorem 3.4.4,

$$\langle (x, \lambda) | x^* = x \rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} C[-\lambda, \lambda] \cong_{\mathbf{1C^*}} \begin{cases} \mathbb{C}, & \lambda = 0, \\ C[0,1], & \lambda > 0, \end{cases}$$

since $[-\lambda, \lambda] \cong_{\mathbf{Top}} [0, 1]$ for all $\lambda > 0$.

An operator x is *unitary* if $x^*x = xx^* = \mathbb{1}$.

Example 3.5.3 (A unitary element, [4]). For $\lambda \geq 0$, consider

$$\langle (x, \lambda) | x^*x = xx^* = \mathbb{1} \rangle_{\mathbf{1C^*}}.$$

Then, x is normal and generates this algebra, so it is commutative. In this case, $r_1 = x^*x - \mathbb{1}$ and $r_2 = xx^* - \mathbb{1}$ so

$$g_{r_1}(\mu) = g_{r_2}(\mu) = |\mu|^2 - 1.$$

Hence, $g_{r_j}^{-1}(0) = \mathbb{T} \cap D_\lambda$ for $j = 1, 2$. By Theorem 3.4.4,

$$\langle (x,\lambda) | x^*x = xx^* = \mathbb{1} \rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} C\left(\mathbb{T} \cap D_\lambda\right) \cong_{\mathbf{1C^*}} \begin{cases} \mathbb{0}, & 0 \leq \lambda < 1, \\ C(\mathbb{T}), & \lambda \geq 1. \end{cases}$$

An operator x is *isometric* if $x^*x = \mathbb{1}$. Symmetrically, x is *coisometric* if $xx^* = \mathbb{1}$. These two notions are dual weakenings of the criterion for being unitary and are well-studied in the Toeplitz algebra.

Example 3.5.4 (An isometry, the Toeplitz algebra, [4]). For $\lambda \geq 0$, consider

$$\mathcal{A} := \langle (x,\lambda) | x^*x = \mathbb{1} \rangle_{\mathbf{1C^*}}.$$

If $0 \leq \lambda < 1$, then $\|\mathbb{1}\|_{\mathcal{A}} = \|x^*x\|_{\mathcal{A}} \leq \lambda^2 < 1$. Hence, $0 = \mathbb{1}$ so $\mathcal{A} \cong_{\mathbf{1C^*}} \mathbb{0}$.

Consider when $\lambda \geq 1$. In this case, $r = x^*x - \mathbb{1}$ so

$$g_r(\mu) = |\mu|^2 - 1.$$

Hence, $g_r^{-1}(0) = \mathbb{T}$. By Corollary 3.4.5, $\sigma_{\mathcal{A}}(x) \supseteq \mathbb{T}$, $0 \neq \mathbb{1}$, and $\|x\|_{\mathcal{A}} = 1$. Let $T \in \mathcal{B}\left(\ell^2\right)$ be the unilateral shift and $\mathcal{T} := C^*(T) \subset \mathcal{B}\left(\ell^2\right)$, the Toeplitz algebra. Recall that $T^*T = \mathbb{1}$ and $\|T\|_{\mathcal{B}(\ell^2)} = 1$. Then, there is a unique unital *-homomorphism $\phi : \mathcal{A} \to \mathcal{T}$ such that $\phi(x) = T$ by Theorem 3.3.2. In particular, this shows that x cannot be normal since this would force T to be unitary, which is not so.

Let $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be the universal *-representation of $\mathcal{A}$. Then, $\pi(x)$ is a proper isometry on $\mathcal{H}$ and $\mathcal{A} \cong_{\mathbf{1C^*}} C^*(\pi(x))$. By Coburn's Theorem in [8], there is a unique *-homomorphism $\varphi : \mathcal{T} \to C^*(\pi(x))$ such that $\varphi(T) = \pi(x)$. In particular,

$$\varphi(\mathbb{1}) = \varphi(T^*T) = \varphi(T)^*\varphi(T) = \pi(x)^*\pi(x) = \pi(x^*x) = \pi(\mathbb{1}) = \mathbb{1}$$

so this map is also unital. Let $\psi := \left(\pi|^{C^*(\pi(x))}\right)^{-1} \circ \varphi$.

Therefore,

$$(\psi \circ \phi)(x) = \psi(T) = \left(\pi|^{C^*(\pi(x))}\right)^{-1}(\pi(x)) = x$$

By Theorem 3.3.2, $\psi \circ \phi = id_{\mathcal{A}}$. Similarly,

$$(\phi \circ \psi)(T) = \phi\left(\left(\pi|^{C^*(\pi(x))}\right)^{-1}(\pi(x))\right) = \phi(x) = T.$$

By Coburn's Theorem, $\phi \circ \psi = id_{\mathcal{T}}$. Hence,

$$\langle (x,\lambda)|x^*x = \mathbb{1}\rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} \begin{cases} \mathbb{0}, & 0 \leq \lambda < 1, \\ \mathcal{T}, & \lambda \geq 1. \end{cases}$$

Dually, consider

$$\langle (y,\lambda)|yy^* = \mathbb{1}\rangle_{\mathbf{1C^*}}.$$

Then, $x := y^*$ is an isometry, and by the same arguments as above,

$$\langle (y,\lambda)|yy^* = \mathbb{1}\rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} \begin{cases} \mathbb{0}, & 0 \leq \lambda < 1, \\ \mathcal{T}, & \lambda \geq 1. \end{cases}$$

An operator x is *idempotent* if $x^2 = x$. This operator is a *projection* if $x^2 = x^* = x$.

Example 3.5.5 (A projection). For $\lambda \geq 0$, consider

$$\langle (x,\lambda)|x^2 = x^* = x\rangle_{\mathbf{1C^*}}.$$

Then, x is normal and generates this algebra, so it is commutative. In this case,

$r_1 = x^2 - x$ and $r_2 = x - x^*$ so

$$g_{r_1}(\mu) = \mu^2 - \mu$$

and

$$g_{r_2}(\mu) = 2\imath\Im(\mu).$$

Hence, $g_{r_1}^{-1}(0) = \{0,1\} \cap D_\lambda$ and $g_{r_2}^{-1}(0) = [-\lambda, \lambda]$. By Theorem 3.4.4,

$$\left\langle (x,\lambda) | x^2 = x^* = x \right\rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} C\left(\{0,1\} \cap [-\lambda,\lambda]\right) \cong_{\mathbf{1C^*}} \begin{cases} \mathbb{C}, & 0 \le \lambda < 1, \\ \mathbb{C} \oplus \mathbb{C}, & \lambda \ge 1. \end{cases}$$

Observe that for each example above, there were two cases depending on the crutch value λ. This "bifurcating" behavior is of particular note and becomes far more interesting as the examples become increasingly more complex. It has been observed previously in papers such as [15], [16], and [17], usually when the parameter approaches 0. This notion of bifurcation is the key point of Section 3.16.

3.6 Example: An Analytic Relation, Sine

In all the preceding examples, the C*-relations used have only been *-polynomials. However, this need not be the case. Specifically, one can use the analytic and continuous functional calculi to impose other relations. This section demonstrates a C*-relation built from the analytic functional calculus.

Specifically, recall that the function $\sin : \mathbb{C} \to \mathbb{C}$ is given by the uniformly convergent power series

$$\sin(\lambda) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \lambda^{2n+1}.$$

Thus, for any unital C*-algebra $\mathcal{B}$ and $x \in \mathcal{B}$, $\sin(x) \in C^*(x)$ by the analytic functional calculus. Hence, one can consider the algebra

$$\langle (x, \lambda) | \sin(x) = 0 \rangle_{\mathbf{1C^*}}$$

for $\lambda \in [0, \infty)$.

Example 3.6.1 (Sine and Normality)*.* For simplicity and tractability, consider the algebra

$$\mathcal{A}_\lambda := \langle (x, \lambda) | x^* x = x x^*, \sin(x) = 0 \rangle_{\mathbf{1C^*}},$$

which is a quotient of the previous one. Then, x is normal and generates this algebra, so it is commutative. In this case, $r_1 = x^* x - x x^*$ and $r_2 = \sin(x)$ so

$$g_{r_1}(\mu) = 0$$

and

$$g_{r_2}(\mu) = \sin(\mu).$$

Hence, $g_{r_2}^{-1}(0) = \{\pi n : n \in \mathbb{Z}\} \cap D_\lambda$. By Theorem 3.4.4,

$$\mathcal{A}_\lambda \cong_{\mathbf{1C^*}} C\left(\{\pi n : n \in \mathbb{Z}\} \cap D_\lambda\right) \cong_{\mathbf{1C^*}} \mathbb{C}^{2n+1},$$

for each $\pi n \leq \lambda < \pi(n+1)$ and $n \in \mathbb{W}$. Thus, there are precisely $\aleph_0$ distinct isomorphism classes as λ varies.

Example 3.6.2 (Cosine and Normality)*.* Similarly, consider the algebra

$$\mathcal{C}_\lambda := \langle (x, \lambda) | x^* x = x x^*, \cos(x) = 0 \rangle_{\mathbf{1C^*}}.$$

Following the same arguments as above,

$$\mathcal{C}_\lambda \cong_{\mathbf{1C^*}} \begin{cases} \mathbb{O}, & 0 \le \lambda < \dfrac{\pi}{2}, \\ \mathbb{C}^{2n+2}, & \dfrac{\pi}{2} + \pi n \le \lambda < \dfrac{\pi}{2} + \pi(n+1), n \in \mathbb{W}. \end{cases}$$

As above, there are precisely $\aleph_0$ distinct isomorphism classes as λ varies. Also, $\mathcal{C}_\lambda \not\cong_{\mathbf{1C^*}} \mathcal{A}_\mu$ for all $\lambda, \mu \in [0, \infty)$.

In the examples of Section 3.5, each presentation only had two distinct isomorphism classes as the crutched value λ varied: one of $\mathbb{O}$ or $\mathbb{C}$, and a more interesting case. Here, there are far more, caused by the functional calculus in play.

Sine and cosine each have countably many zeroes, and as λ increases, more and more are included into the spectrum of x. Thus, the crutched value λ can have a great deal of influence on the algebra, demonstrating more dramatically the "bifurcating" behavior noted in Section 3.5.

3.7 Example: A Continuous Relation, Positivity

While the previous section used the analytic functional calculus to create a C*-relation, this section shall use the continuous functional calculus to do the same.

In particular, recall that an operator x is *positive* if $x = x^*$ and $\sigma(x) \subset [0, \infty)$, written $x \ge 0$. This definition can be characterized using a single C*-relation in the following way.

Let $p : \mathbb{R} \to \mathbb{R}$ by

$$p(\mu) := \begin{cases} 0, & \mu < 0, \\ \mu, & \mu \ge 0, \end{cases}$$

a continuous function. For any operator x, let $\Re(x) := \dfrac{1}{2}(x + x^*)$, the *real part* of x.

Since $\Re(x)$ is self-adjoint, $\sigma(\Re(x)) \subset \mathbb{R}$. Hence, by the continuous functional calculus, $p(\Re(x)) \in C^*(1, \Re(x)) \subseteq C^*(1, x)$.

Since $p(\Re(x))$ can be realized as a limit of $\mathbb{C}$-polynomials in $\Re(x)$, $p(\Re(x))$ is normal. Further,

$$\sigma(p(\Re(x))) = p(\sigma(\Re(x))) \subset [0, \infty)$$

by the continuous functional calculus. Thus, these two facts together show $p(\Re(x))$ is self-adjoint, and therefore, positive, regardless of x.

Proposition 3.7.1. *For a C*-algebra $\mathcal{A}$ and $x \in \mathcal{A}$, $x \geq 0$ if and only if $x = p(\Re(x))$.*

Proof. ($\Leftarrow$) As shown above, $p(\Re(x)) \geq 0$ so by assumption, $x = p(\Re(x)) \geq 0$.

($\Rightarrow$) Given that $x \geq 0$, $x = x^*$ so $\Re(x) = \dfrac{1}{2}(x + x) = x$. Then, note that

$$\sigma(x - p(x)) = (id_{\mathbb{R}} - p)\,(\sigma(x)) \subseteq \{0\}$$

so $x = p(x) = p(\Re(x))$.

$\square$

Thus, x is positive if and only if $x - p(\Re(x)) = 0$, and $x - p(\Re(x))$ is a C*-relation on (x, λ). However, this C*-relation is a bit bulky and obscuring so it will be written in a presentation by the more conventional "$x \geq 0$".

Example 3.7.2 (A positive element, [4]). For $\lambda \geq 0$, consider

$$\langle (x, \lambda) | x \geq 0 \rangle_{\mathbf{1C^*}}.$$

Then, x is normal and generates this algebra, so it is commutative. In this case, $r = p\left(\Re(x)\right) - x$ so

$$g_r(\mu) = p\left(\Re(\mu)\right) - \mu.$$

Hence, $g_r^{-1}(0) = [0, \lambda]$. By Theorem 3.4.4,

$$\langle (x, \lambda) | x \geq 0 \rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} C[0, \lambda] \cong_{\mathbf{1C^*}} \begin{cases} \mathbb{C}, & \lambda = 0, \\ C[0,1], & \lambda > 0, \end{cases}$$

since $[0, \lambda] \cong_{\mathbf{Top}} [0, 1]$ for all $\lambda > 0$.

This particular C*-relation now enables manipulation of the order structure in a presentation. Recall that given two self-adjoint operators, $x \geq y$ if $x - y \geq 0$. That is, $x - y$ is a positive operator in the sense above.

3.8 Norm Bounds as C*-relations

As cited in [4] and [27], norm bounds on C*-relations are desired as a type of "relation". This can be accomplished in the context of C*-relations using the order manipulation devised in Section 3.7.

Proposition 3.8.1 (Norm bounds)**.** *Let $\mathcal{A}$ be a C*-algebra, $a \in \mathcal{A}$, and $\lambda \in [0, \infty)$. Then, $\|a\| \leq \lambda$ if and only if $(a^*a)^2 \leq \lambda^2 a^*a$.*

Proof. ($\Rightarrow$) Observe that

$$\|a^*a\| = \|a\|^2 \leq \lambda^2.$$

As the spectral radius is bounded by the norm, $\sigma(a^*a) \subseteq \left[0, \lambda^2\right]$ so by the continuous functional calculus, $\sigma\left(\lambda^2 a^*a - (a^*a)^2\right) \subseteq \left[0, \dfrac{\lambda^4}{4}\right]$. Therefore, $(a^*a)^2 \leq \lambda^2 a^*a$.

($\Leftarrow$) Observe that $\sigma\left(\lambda^2 a^*a - (a^*a)^2\right) \subset [0, \infty)$ so by the continuous functional calculus, $\sigma\left(a^*a\right) \subseteq \left[0, \lambda^2\right]$. As the spectral radius of a normal element equals its norm,

$$\|a\|^2 = \|a^*a\| = r\left(a^*a\right) \leq \lambda^2$$

so $\|a\| \leq \lambda$.

$\square$

Combining Proposition 3.8.1 with Proposition 3.7.1, $\|a\| \leq \lambda$ if and only if $p\left(\Re\left(\lambda^2 a^* a - (a^* a)^2\right)\right) = \lambda^2 a^* a - (a^* a)^2$. Since $\lambda^2 a^* a - (a^* a)^2$ is already self-adjoint, this C*-relation reduces to $p\left(\lambda^2 a^* a - (a^* a)^2\right) = \lambda^2 a^* a - (a^* a)^2$. Much like in Example 3.7.2, the C*-relation for this norm condition is bulky. As such, it will be abbreviated in a presentation by the more conventional "$\|a\| \leq \lambda$".

Example 3.8.2. For $\lambda, \mu \in [0, \infty)$, consider the algebra

$$\mathcal{A}_{\lambda,\mu} := \langle (x, \lambda) | x^* x = x x^*, \| \exp(x) \| \leq \mu \rangle_{\mathbf{1C^*}}.$$

Then, x is normal and generates this algebra, so it is commutative. In this case, $r_1 = x^* x - x x^*$ and

$$r_2 = p\left(\mu^2 \exp(x)^* \exp(x) - (\exp(x)^* \exp(x))^2\right) - \mu^2 \exp(x)^* \exp(x) + (\exp(x)^* \exp(x))^2$$

so

$$g_{r_1}(\nu) = 0$$

and

$$g_{r_2}(\nu) = p\left(\mu^2 |\exp(\nu)|^2 - |\exp(\nu)|^4\right) - \mu^2 |\exp(\nu)|^2 + |\exp(\nu)|^4$$

Note that $g_{r_2}(\nu) = 0$ whenever when $\mu \geq |\exp(\nu)|$. Hence, $g_{r_2}^{-1}(0) = \exp^{-1}(D_\mu) \cap D_\lambda$. By Theorem 3.4.4, $\mathcal{A}_{\lambda,\mu} \cong_{\mathbf{1C^*}} C\left(D_\lambda \cap \exp^{-1}(D_\mu)\right)$.

Interpreting the spectrum,

$$\begin{aligned}\sigma_{\mathcal{A}_{\lambda,\mu}}(x) &= D_\lambda \cap \{\nu \in \mathbb{C} : |\exp(\nu)| \le \mu\} \\ &= D_\lambda \cap \{\nu \in \mathbb{C} : \exp(\Re(\nu)) \le \mu\}.\end{aligned}$$

If $\mu = 0$, then $\sigma_{\mathcal{A}_{\lambda,\mu}}(x) = \emptyset$ so $\mathcal{A}_{\lambda,\mu} \cong_{\mathbf{1C^*}} \mathbb{O}$. Otherwise,

$$\sigma_{\mathcal{A}_{\lambda,\mu}}(x) = D_\lambda \cap \{\nu \in \mathbb{C} : \Re(\nu) \le \ln(\mu)\},$$

the intersection of a disc and a half-plane. Thus, there are only the following situations.

1. If $\ln(\mu) < -\lambda$, the intersection is empty.
2. If $\ln(\mu) = -\lambda$, the half-plane is tangent to the disc, meaning the intersection is a singleton.
3. If $\lambda = 0, \ln(\mu) \ge 0$, the half-plane includes a degenerate disc, meaning the intersection is a singleton again.
4. If $\ln(\mu) \ge \lambda > 0$, the half-plane envelopes the disc, meaning the intersection is the disc.
5. In all other cases, the intersection is a full section of the disc, which is homeomorphic to a disc.

In summary,

$$\mathcal{A}_{\lambda,\mu} \cong_{\mathbf{1C^*}} \begin{cases} \mathbb{O}, & \mu = 0 \text{ or } \ln(\mu) < -\lambda, \\ \mathbb{C}, & \ln(\mu) = -\lambda \text{ or } \lambda = 0 \le \ln(\mu), \\ C\left(\overline{\mathbb{D}}\right), & \text{otherwise.} \end{cases}$$

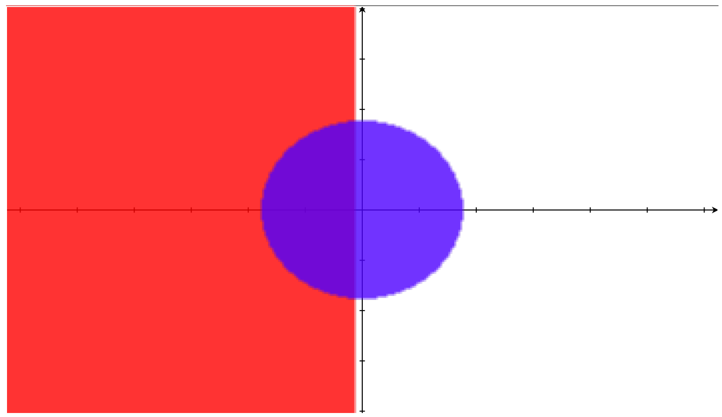


Figure 3.1: Intersection of a Disc and a Half-Plane

Norm conditions such as the last example are of particular interest to the study of “stable relations”, detailed in [27].

3.9 Tietze Transformations for 1C*

As noted in many of the previous sections, several different presentations can yield isomorphic unital C*-algebras, just as in pure algebra. In [38], a definitive and well-known criterion was developed for when two group presentations result in isomorphic groups.

In this section, the analog is proven for the presentation theory constructed in Section 3.3. This will be done systematically, describing each type of technique that will be used in the main result.

Section 2.4.1 of [19] considers an analogous calculus for its version of presentation theory. However, the relations used in [19] are restricted to *-polynomials within the free complex *-algebra, not the scaled-free C*-algebra. In Remark 2.4.1.13 of [19], it is conjectured that a Tietze transformation theorem for C*-algebras would require so many assumptions as to be practically worthless. However, Theorem 3.9.7 attains this result via C*-relations, as defined in Section 3.3, with few initial assumptions.

3.9.1 C*-relations

The classical result from [38] utilized several formal manipulations known as "Tietze transformations", two invertible operations. The first of these operations is the addition or removal of a "redundant" relation, a condition that is automatically implied by the others in play. This section rigorously considers this in the case for the presentation theory for $\mathbf{1C^*}$ developed in Section 3.3.

To be clear, the notion of redundancy is as follows. Let (S, f) be a crutched set and R a set of C*-relations on (S, f). Define $\mathcal{F} := \langle S, f|\emptyset\rangle_{\mathbf{1C^*}}$ and $\mathcal{A} := \langle S, f|R\rangle_{\mathbf{1C^*}}$. The set of C*-relations $Q \subseteq \mathcal{F}$ are *redundant* for $\mathcal{A}$ if $Q \subseteq J_R$, where J_R is the norm-closed, two-sided ideal generated by R in $\mathcal{F}$.

In short, as heuristically stated above, the C*-relations in Q are already forced by R. Indeed, observe that $R \cup Q \subseteq J_R$ so $J_{R\cup Q} \subseteq J_R$, where $J_{R\cup Q}$ is the norm-closed, two-sided ideal generated by $R \cup Q$ in $\mathcal{F}$. Similarly, $J_R \subseteq J_{R\cup Q}$. This implies the following chain of equalities.

$$\mathcal{A} = \langle S, f|R\rangle_{\mathbf{1C^*}} =: \mathcal{F}/J_R = \mathcal{F}/J_{R\cup Q} := \langle S, f|R \cup Q\rangle_{\mathbf{1C^*}}$$

Requiring that the C*-relations in Q be satisfied adds no new structure to $\mathcal{A}$.

Corollary 2.4.1.7 and Proposition 2.4.1.11 of [19] give the analogous isomorphism.

For concreteness, consider the following example of removing a C*-relation.

Example 3.9.1 (Removing a redundant C*-relation). Consider the unital C*-algebra below.

$$\left\langle (x, 1)|x^2 = x, x^3 = x\right\rangle_{\mathbf{1C^*}}$$

In this case, $S = \{x\}$, $f : S \to [0, \infty)$ by $f(x) = 1$, and $R = \left\{x^2 - x\right\}$. Letting $Q := \left\{x^3 - x\right\}$, then $R \cup Q = \left\{x^3 - x, x^2 - x\right\}$. Observe that

$$x^3 - x = x^3 - x^2 + x^2 - x = \left(x^2 - x\right) x + \left(x^2 - x\right) = \left(x^2 - x\right)(x + \mathbb{1}) \in J_R.$$

Thus, by the above,

$$\left\langle (x, 1) | x^2 = x, x^3 = x \right\rangle_{\mathbf{1C^*}} = \left\langle (x, 1) | x^2 = x \right\rangle_{\mathbf{1C^*}}.$$

Similarly, one can add redundant C*-relations without issue.

Example 3.9.2 (Adding a redundant C*-relation). Consider the unital C*-algebra below.

$$\left\langle (x, 1) | x = x^2 \right\rangle_{\mathbf{1C^*}}$$

In this case, $S = \{x\}$, $f : S \to [0, \infty)$ by $f(x) = 1$, and $R = \left\{x^2 - x\right\}$. Letting $Q := \left\{x^5 - x\right\}$, then $R \cup Q = \left\{x^2 - x, x^5 - x\right\}$, Observe that

$$\begin{aligned} x^5 - x &= x^5 - x^4 + x^4 - x^3 + x^3 - x^2 + x^2 - x \\ &= x^3\left(x^2 - x\right) + x^2\left(x^2 - x\right) + x\left(x^2 - x\right) + \left(x^2 - x\right) \\ &= \left(x^3 + x^2 + x + \mathbb{1}\right)\left(x^2 - x\right) \in J_R \end{aligned}$$

Thus, by the above,

$$\left\langle (x, 1) | x = x^2, x = x^5 \right\rangle_{\mathbf{1C^*}} = \left\langle (x, 1) | x = x^2 \right\rangle_{\mathbf{1C^*}}$$

Removing redundant C*-relations is a natural operation to create a simpler presentation. However, in practice, it is often useful to add redundant C*-relations as this may allow the next type of Tietze transformation, removal of redundant generators.

3.9.2 Generators

The second type of Tietze transformation involves the addition or removal of a "redundant" generator, one that can be recovered in terms of the others. This is similar in flavor to the reduction of a generating set for a vector space to a linear basis, removing all but those which are absolutely necessary to recover the original structure. This section rigorously considers this in the case for the presentation theory of Section 3.3.

To begin, let (S_0, f_0) be a crutched set and R_0 a set of C*-relations on (S_0, f_0). Define $\mathcal{F}_0 := \langle S_0, f_0 | \emptyset \rangle_{\mathbf{1C^*}}$, $\mathcal{A}_0 := \langle S_0, f_0 | R_0 \rangle_{\mathbf{1C^*}}$, and $q_0 : \mathcal{F}_0 \to \mathcal{A}_0$ the quotient map.

Let $G \subseteq \mathcal{F}_0$ and associate a new symbol s_g and a nonnegative value $\lambda_g \in \left[\|q_0(g)\|_{\mathcal{A}_0}, \infty\right)$ for each $g \in G$. Define $S_1 := \{s_g : g \in G\}$ and $f_1 : S_1 \to [0, \infty)$ by $f_1(s_g) := \lambda_g$, creating a new crutched set (S_1, f_1). Let

$$(S, f) := (S_0, f_0) \coprod^{\mathbf{CSet_1}} (S_1, f_1),$$

the disjoint union described in Proposition 2.2.9 and $\rho_j : (S_j, f_j) \to (S, f)$ the canonical inclusions for $j = 0, 1$. Define $\mathcal{F} := \langle S, f | \emptyset \rangle_{\mathbf{1C^*}}$. Theorem 3.1.4 and Proposition A.5.1 state that

$$\mathcal{F} \cong_{\mathbf{1C^*}} \mathcal{F}_0 \coprod^{\mathbf{1C^*}} \mathcal{F}_1 \cong_{\mathbf{1C^*}} \mathcal{F}_0 *_{\mathbb{C}} \mathcal{F}_1$$

with connecting maps $\hat{\rho}_j := 1\mathrm{C}^*\mathrm{Alg}(\rho_j)$. Let $R := \hat{\rho}_0(R_0) \cup \{s_g - \hat{\rho}_0(g) : g \in G\}$, taking the original C*-relations R_0 along with requirements that s_g be associated to g.

Define $\mathcal{A} := \langle S, f | R \rangle_{\mathbf{1C^*}}$ and $q : \mathcal{F} \to \mathcal{A}$ the quotient map. The objective is to show $\mathcal{A}$ isomorphic as a unital C*-algebra to $\mathcal{A}_0$.

$$\begin{array}{ccc} \mathcal{F}_0 & \xrightarrow{\hat{\rho}_0} & \mathcal{F} \\ {\scriptstyle q_0}\downarrow & & \downarrow{\scriptstyle q} \\ \mathcal{A}_0 & & \mathcal{A} \end{array}$$

First, observe that for each $r \in R_0$, $(q \circ \hat{\rho}_0)(r) = 0$ so by the universal property of the quotient, there is a unique unital *-homomorphism $\psi : \mathcal{A}_0 \to \mathcal{A}$ such that $\psi \circ q_0 = q \circ \hat{\rho}_0$, a candidate for the isomorphism.

To construct its inverse, define $\varphi : (S, f) \to F^{\mathbf{CSet}_1}_{\mathbf{1C^*}}\mathcal{A}_0$ by $\varphi(s) := q_0(s)$ and $\varphi(s_g) := q_0(g)$ for all $s \in S_0$ and $g \in G$, a constrictive function. By Theorem 3.2.1, there is a unique $\hat{\varphi} : \mathcal{F} \to \mathcal{A}_0$ such that $\hat{\varphi}(s) = q_0(s)$ and $\hat{\varphi}(s_g) = q_0(g)$ for all $s \in S_0$ and $g \in G$. Then, observe that for each $s \in S_0$, $(\hat{\varphi} \circ \hat{\rho}_0)(s) = q_0(s)$ so by Theorem 3.2.1, $\hat{\varphi} \circ \hat{\rho}_0 = q_0$. Then, for all $r \in R_0$ and $g \in G$,

$$\hat{\varphi}(\hat{\rho}_0(r)) = q_0(r) = 0$$

and

$$\hat{\varphi}(s_g - \hat{\rho}_0(g)) = q_0(g) - q_0(g) = 0.$$

By the universal property of the quotient, there is a unique unital *-homomorphism $\phi : \mathcal{A} \to \mathcal{A}_0$ such that $\hat{\varphi} = \phi \circ q$.

For all $s \in S_0$, observe that

$$(\phi \circ \psi)(q_0(s)) = (\phi \circ q \circ \hat{\rho}_0)(s) = (\hat{\varphi} \circ \hat{\rho}_0)(s) = q_0(s).$$

Thus, by Theorem 3.2.1, $\phi \circ \psi \circ q_0 = q_0$ so $\phi \circ \psi = id_{\mathcal{A}_0}$ by Theorem 3.3.2.

Similarly, for all $s \in S_0$ and $g \in G$,

$$(\psi \circ \phi)(q(s)) = (\psi \circ \hat{\varphi})(s) = (\psi \circ q_0)(s) = (q \circ \hat{\rho}_0)(s) = q(s)$$

and

$$\begin{aligned} (\psi \circ \phi)(q(s_g)) &= (\psi \circ \hat{\varphi})(s_g) \\ &= (\psi \circ q_0)(g) \\ &= (q \circ \hat{\rho}_0)(g) \\ &= q(s_g) - q(s_g - \hat{\rho}_0(g)) \\ &= q(s_g) \end{aligned}$$

Thus, by Theorem 3.2.1, $\psi \circ \phi \circ q = q$ so $\psi \circ \phi = id_{\mathcal{A}}$ by Theorem 3.3.2.

In summary,

$$\langle S, f|R\rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} \langle S_0, f_0|R_0\rangle_{\mathbf{1C^*}}.$$

Corollary 2.4.1.8 and Proposition 2.4.1.11 of [19] give the analogous isomorphism.

For concreteness, consider the following example of adding an unnecessary generator.

Example 3.9.3 (Adding an unnecessary generator)*.* Consider the unital C*-algebra below.

$$\left\langle (x,1)|x = x^2\right\rangle_{\mathbf{1C^*}}$$

In this case, $S_0 = \{x\}$, $f_0 : S_0 \to [0,\infty)$ by $f_0(x) = 1$, and $R_0 = \left\{x - x^2\right\}$. Let $S_1 := \{y\}$, a new symbol, and $g := xx^*x \in \mathcal{F}_0 := \langle (x,1)|\emptyset\rangle_{\mathbf{1C^*}}$. Observe that

$$\|xx^*x\|_{\mathcal{A}_0} \leq \|x\|_{\mathcal{A}_0} \|x^*x\|_{\mathcal{A}_0} = \|x\|^3_{\mathcal{A}_0} \leq 1.$$

Define $f_1 : S_1 \to [0, \infty)$ by $f_1(y) := 1$. In this case,

$$(S, f) := (S_0, f_0) \coprod^{\mathbf{CSet}_1} (S_1, f_1) \cong_{\mathbf{CSet}_1} \{(x, 1), (y, 1)\}.$$

Then, the above result states that

$$\left\langle (x, 1), (y, 1) \left| x = x^2, y = xx^*x \right. \right\rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} \left\langle (x, 1) | x = x^2 \right\rangle_{\mathbf{1C^*}}.$$

Similarly, one can remove an unnecessary generator.

Example 3.9.4 (Removing an unnecessary generator). Consider the unital C*-algebra below.

$$\left\langle (x, 1), (y, 1) | x = x^2, y = x^*x \right\rangle_{\mathbf{1C^*}}$$

In this case, $S = \{x, y\}$, $f : S \to [0, \infty)$ by $f(x) = 1$ and $f(y) = 1$, and $R = \left\{x - x^2, y - x^*x\right\}$. Let $S_0 := \{x\}$, $f_0 : S_0 \to [0, \infty)$ by $f_0(x) := 1$, and $R_0 := \left\{x - x^2\right\}$.

Letting $S_1 := \{y\}$ and $f_1 : S_1 \to [0, \infty)$ by $f_1(y) := 1$, note that

$$(S_0, f_0) \coprod^{\mathbf{CSet}_1} (S_1, f_1) \cong_{\mathbf{CSet}_1} (S, f)$$

and $R = R_0 \cup \{y - x^*x\}$. Then, the above result states that

$$\left\langle (x, 1), (y, 1) | x = x^2, y = x^*x \right\rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} \left\langle (x, 1) | x = x^2 \right\rangle_{\mathbf{1C^*}}.$$

However, there is some care to be taken in removing generators as done above. Specifically, consider the same example when the crutched value on y is $\dfrac{1}{4}$.

Example 3.9.5. Let

$$\mathcal{C} := \left\langle (x,1), \left(y, \frac{1}{4}\right) \middle| x = x^2, y = x^*x \right\rangle_{\mathbf{1C^*}}.$$

In this case, observe that

$$\|x\|_{\mathcal{C}}^2 = \|x^*x\|_{\mathcal{C}} = \|y\|_{\mathcal{C}} \leq \frac{1}{4}$$

so $\|x\|_{\mathcal{C}} \leq \frac{1}{2}$. However, if $\|x\|_{\mathcal{C}} \neq 0$,

$$\|x\|_{\mathcal{C}} = \left\|x^2\right\|_{\mathcal{C}} \leq \|x\|_{\mathcal{C}}^2$$

so $1 \leq \|x\|_{\mathcal{C}} \leq \frac{1}{2}$, which is nonsense. Hence, $x = 0$. Observe that $y = x^*x = 0$ so

$$\left\langle (x,1), \left(y, \frac{1}{4}\right) \middle| x = x^2, y = x^*x \right\rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} \mathbb{C}.$$

However, for $\mathcal{A} := \left\langle (x,1) | x = x^2 \right\rangle_{\mathbf{1C^*}}$, there is a unique unital *-homomorphism $\phi : \mathcal{A} \to \mathbb{C} \oplus \mathbb{C}$ by $\phi(x) = 1 \oplus 0$. Hence, $\|x\|_{\mathcal{A}} \geq \|1 \oplus 0\|_{\mathbb{C}\oplus\mathbb{C}} = 1$. Further, $\phi(\mathbb{1}) = 1 \oplus 1$ so $x \notin \overline{\text{span}}\{\mathbb{1}\}$. Therefore, $\mathcal{A} \ncong_{\mathbf{1C^*}} \mathbb{C}$.

Unlike the previous example where the extra generator could be removed without trouble, observe that the crutched value of y is strictly beneath the bound determined by x. Explicitly,

$$\|y\|_{\mathcal{C}} \leq \frac{1}{4}, \quad \|x^*x\|_{\mathcal{C}} = \|x\|_{\mathcal{C}}^2 \leq 1.$$

This discrepancy caused more reduction to occur within the quotient creating $\mathcal{C}$. Thus, one should be aware of the crutched values and their effect on the resulting quotient structure.

Like the addition and removal of redundant C*-relations, removal of an unnecessary generator is a natural choice, but the addition of an unnecessary generator is not as obvious. Addition of generators in the above way is used in tandem with adding redundant C*-relations to rearrange the presentation into something more familiar. Detailed, nontrivial examples of this will be done in Section 3.11.

3.9.3 Tietze Theorem for $\mathbf{1C}^*$

With an understanding of the different Tietze transformations, the main theorem can now be proven. This proof is based on the treatment given in Section III.5 of [3] for group presentations.

For this discussion, only a pair of unital C*-algebras will be considered. For $j = 1, 2$, fix a crutched set (S_j, f_j) and a set of C*-relations R_j on (S_j, f_j). Define $\mathcal{F}_j := \langle S_j, f_j | \, \emptyset \rangle_{\mathbf{1C^*}}$, $\mathcal{A}_j := \langle S_j, f_j | \, R_j \rangle_{\mathbf{1C^*}}$, and $q_j : \mathcal{F}_j \to \mathcal{A}_j$ the quotient map.

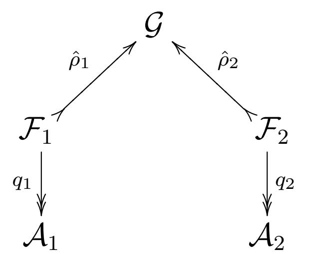
To prove the theorem, one considers $\mathcal{A}_1$ and $\mathcal{A}_2$ as quotients of a single, unified algebra. To build this structure, define

$$(T, g) := (S_1, f_2) \coprod^{\mathbf{CSet}_1} (S_2, f_2)$$

to be the disjoint union described in Proposition 2.2.9, $\rho_j : (S_j, f_j) \to (T, g)$ the canonical inclusions for $j = 1, 2$, and $\mathcal{G} := \langle T, g | \emptyset \rangle_{\mathbf{1C^*}}$. Theorem 3.1.4 and Proposition A.5.1 state that

$$\mathcal{G} \cong_{\mathbf{1C^*}} \mathcal{F}_1 \coprod^{\mathbf{1C^*}} \mathcal{F}_2 \cong_{\mathbf{1C^*}} \mathcal{F}_1 *_{\mathbb{C}} \mathcal{F}_2$$

with connecting maps $\hat{\rho}_j := 1\mathrm{C}^*\mathrm{Alg}\,(\rho_j)$ for $j = 1, 2$.



The following lemma is the key step in the main result, allowing $\mathcal{A}_1$ and $\mathcal{A}_2$ to be realized as quotients of $\mathcal{G}$. Further, the explicit C*-relations on (T, g) are determined.

Lemma 3.9.6. *Given the notation above, assume* $\Theta_j : \mathcal{G} \to \mathcal{F}_j$ *is a unital *-homomorphism satisfying that* $\Theta_j \circ \hat{\rho}_j = id_{\mathcal{F}_j}$*. Then,* $\ker(q_j \circ \Theta_j)$ *is the norm-closed, two-sided ideal* $\mathcal{J}_j$ *generated by* $\hat{\rho}_j(R_j) \cup \{s - (\hat{\rho}_j \circ \Theta_j)(s) : s \in S_{3-j}\}$ *in* $\mathcal{G}$*.*

Proof. For $r \in R_j$ and $s \in S_{3-j}$,

$$(q_j \circ \Theta_j)(\hat{\rho}_j(r)) = q_j\left(id_{\mathcal{F}_j}(r)\right) = q_j(r) = 0$$

and

$$\begin{aligned}
(q_j \circ \Theta_j)(s - (\hat{\rho}_j \circ \Theta_j)(s)) &= (q_j \circ \Theta_j)(s) - (q_j \circ \Theta_j \circ \hat{\rho}_j \circ \Theta_j)(s) \\
&= (q_j \circ \Theta_j)(s) - \left(q_j \circ id_{\mathcal{F}_j} \circ \Theta_j\right)(s) \\
&= (q_j \circ \Theta_j)(s) - (q_j \circ \Theta_j)(s) \\
&= 0.
\end{aligned}$$

Hence, $\hat{\rho}_j(R_j) \cup \{s - (\hat{\rho}_j \circ \Theta_j)(s) : s \in S_{3-j}\} \subseteq \ker(q_j \circ \Theta_j)$ so $\mathcal{J}_j \subseteq \ker(q_j \circ \Theta_j)$.

Let $\gamma : \mathcal{G} \to \mathcal{G}/\mathcal{J}_j$ be the quotient map. For all $s \in S_j$ and $t \in S_{3-j}$,

$$\begin{aligned}(\gamma \circ \hat{\rho}_j \circ \Theta_j)(s) &= (\gamma \circ \hat{\rho}_j \circ \Theta_j \circ \hat{\rho}_j)(s) \\ &= \left(\gamma \circ \hat{\rho}_j \circ id_{\mathcal{F}_j}\right)(s) \\ &= (\gamma \circ \hat{\rho}_j)(s) \\ &= \gamma(s)\end{aligned}$$

and

$$\begin{aligned}(\gamma \circ \hat{\rho}_j \circ \Theta_j)(t) &= (\gamma \circ \hat{\rho}_j \circ \Theta_j)(t) + \gamma(t - (\hat{\rho}_j \circ \Theta_j)(t)) \\ &= (\gamma \circ \hat{\rho}_j \circ \Theta_j)(t) + \gamma(t) - (\gamma \circ \hat{\rho}_j \circ \Theta_j)(t) \\ &= \gamma(t).\end{aligned}$$

By Theorem 3.2.1, $\gamma = \gamma \circ \hat{\rho}_j \circ \Theta_j$.

For $b \in \ker(q_j \circ \Theta_j)$, then $\Theta_j(b) \in \ker(q_j) = J_{R_j}$, the norm-closed, two-sided ideal generated by R_j in $\mathcal{F}_j$. Thus, $(\hat{\rho}_j \circ \Theta_j)(b) \in \mathcal{J}_j$. Also,

$$\gamma(b - (\hat{\rho}_j \circ \Theta_j)(b)) = \gamma(b) - (\gamma \circ \hat{\rho}_j \circ \Theta_j)(b) = \gamma(b) - \gamma(b) = 0$$

so $b - (\hat{\rho}_j \circ \Theta_j)(b) \in \ker(\gamma) = \mathcal{J}_j$. Therefore, $b \in \mathcal{J}_j$.

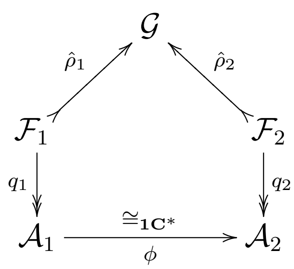
□

Now, the main result can be proven.

Theorem 3.9.7 (Tietze Theorem for $\mathbf{1C^*}$)**.** *$\mathcal{A}_1 \cong_{\mathbf{1C^*}} \mathcal{A}_2$ if and only if there is a sequence of four Tietze transformations changing the presentation of $\mathcal{A}_1$ into the presentation for $\mathcal{A}_2$.*

Proof. ($\Leftarrow$) If there is a sequence of four Tietze transformations changing the presentation of $\mathcal{A}_1$ to the presentation of $\mathcal{A}_2$, observe that each Tietze transformation is an isomorphism. As such, composing all these isomorphisms together yields a single isomorphism from $\mathcal{A}_1$ to $\mathcal{A}_2$.

($\Rightarrow$) Assuming that $\mathcal{A}_1 \cong_{\mathbf{1C^*}} \mathcal{A}_2$, let $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a unital *-isomorphism. First, maps Θ_j satisfying the conditions of Lemma 3.9.6 are created. The purpose of these maps is to relate generators in S_1 in terms of generators in S_2, and vice versa.



By Proposition 3.2.6, there is a unital *-homomorphism $\psi_2 : \mathcal{F}_2 \to \mathcal{F}_1$ such that $\phi \circ q_1 \circ \psi_2 = q_2$. Using the coproduct characterization of $\mathcal{G}$, there is a unique unital *-homomorphism $\Theta_1 : \mathcal{G} \to \mathcal{F}_1$ such that $\Theta_1 \circ \hat{\rho}_1 = id_{\mathcal{F}_1}$ and $\Theta_1 \circ \hat{\rho}_2 = \psi_2$.

Similarly, there is a unital *-homomorphism $\psi_1 : \mathcal{F}_1 \to \mathcal{F}_2$ such that $\phi^{-1} \circ q_2 \circ \psi_1 = q_1$. Likewise, there is a unique unital *-homomorphism $\Theta_2 : \mathcal{G} \to \mathcal{F}_2$ such that $\Theta_2 \circ \hat{\rho}_1 = \psi_1$ and $\Theta_2 \circ \hat{\rho}_2 = id_{\mathcal{F}_2}$.

Further, observe that

$$\phi \circ q_1 \circ \Theta_1 \circ \hat{\rho}_1 = \phi \circ q_1 \circ id_{\mathcal{F}_1} = \phi \circ q_1 = q_2 \circ \psi_1 = q_2 \circ \Theta_2 \circ \hat{\rho}_1$$

and

$$\phi \circ q_1 \circ \Theta_1 \circ \hat{\rho}_2 = \phi \circ q_1 \circ \psi_2 = q_2 = q_2 \circ id_{\mathcal{F}_2} = q_2 \circ \Theta_2 \circ \hat{\rho}_2$$

so by universal property of the coproduct, $\phi \circ q_1 \circ \Theta_1 = q_2 \circ \Theta_2$.

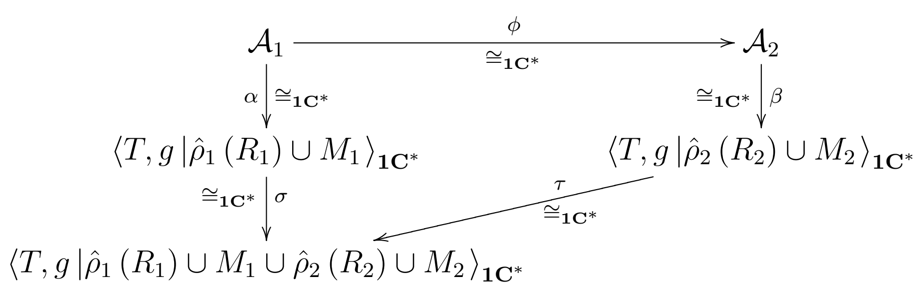
Next, ϕ is to be decomposed into a composition of Tietze isomorphisms. To this end, let $M_j := \{s - (\hat{\rho}_j \circ \Theta_j)(s) : s \in S_{3-j}\} \subset \mathcal{G}$ for $j = 1, 2$. By Lemma 3.9.6,

$\ker(q \circ \Theta_j)$ is the norm-closed, two-sided ideal generated by $\hat{\rho}_j(R_j) \cup M_j$. Observe that as ϕ is an isomorphism,

$$\ker(q_2 \circ \Theta_2) = \ker(\phi \circ q_1 \circ \Theta_1) = \ker(q_1 \circ \Theta_1).$$

Thus, the ideal generated by $\hat{\rho}_1(R_1) \cup M_1$ is the same as the ideal generated by $\hat{\rho}_2(R_2) \cup M_2$.

Therefore, there are C*-relation-adding and generator-adding Tietze isomorphisms $\alpha, \beta, \sigma, \tau$ below.



Fix $s \in S_1$. In $\langle T, g \,|\hat{\rho}_1(R_1) \cup M_1 \cup \hat{\rho}_2(R_2) \cup M_2 \rangle_{\mathbf{1C^*}}$, $(\sigma \circ \alpha \circ q_1)(s)$ is the generator $[s]$, and $(\tau \circ \beta \circ \phi \circ q_1)(s) = (\tau \circ \beta \circ q_2)(\psi_1(s))$ is $[\hat{\rho}_2(\psi_1(s))]$. Also,

$$[s] = [s - (\hat{\rho}_2 \circ \Theta_2)(s)] + [(\hat{\rho}_2 \circ \Theta_2)(s)] = [(\hat{\rho}_2 \circ \Theta_2)(s)] = [\hat{\rho}_2(\psi_1(s))]$$

in $\langle T, g \,|\hat{\rho}_1(R_1) \cup M_1 \cup \hat{\rho}_2(R_2) \cup M_2 \rangle_{\mathbf{1C^*}}$. Thus, $(\tau \circ \beta \circ \phi \circ q_1)(s) = (\sigma \circ \alpha \circ q_1)(s)$. As $s \in S_1$ was arbitrary, Theorem 3.2.1 states that $\tau \circ \beta \circ \phi \circ q_1 = \sigma \circ \alpha \circ q_1$. By the universal property of the quotient, $\tau \circ \beta \circ \phi = \sigma \circ \alpha$. As τ and β are invertible, $\phi = \beta^{-1} \circ \tau^{-1} \circ \sigma \circ \alpha$. Hence, ϕ is a sequence of isomorphisms given by Tietze transformations.

□

Notice that the main thrust of the theorem is the existence of the maps Θ_j, guaranteed by the projectivity in Proposition 3.2.6 and the coproduct decomposition of $\mathcal{G}$. Analogous theorems for other normed algebraic objects may well require similar results.

Now, a Tietze transformation is *elementary* if only one generator or C*-relation is changed. As such, any Tietze transformation where finitely many changes are made can be realized by a finite sequence of elementary Tietze transformations. Thus, the following corollary is the direct analog of the result from [38].

Corollary 3.9.8. *Given unital C^*-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are finitely presented in $\mathbf{1C^*}$, $\mathcal{A}_1 \cong_{\mathbf{1C^*}} \mathcal{A}_2$ if and only if there is a finite sequence of elementary Tietze transformations changing the presentation of $\mathcal{A}_1$ into the presentation for $\mathcal{A}_2$.*

3.9.4 An Example of Computing Tietze Transformations

With the main results proven, consider the following example of their application.

Example 3.9.9. From Examples 3.5.2 and 3.7.2, let

$$\mathcal{A} := \langle (x, 1) \,|\, x = x^* \rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} C[0, 1]$$

and

$$\mathcal{B} := \langle (y, 1) \,|\, y \geq 0 \rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} C[0, 1].$$

By Corollary 3.9.8, there is a finite sequence of elementary Tietze transformations that take the first presentation to the second.

To compute these transformations, recall from Example 3.5.2 that x is self-adjoint and $\sigma_{\mathcal{A}}(x) = [-1, 1]$. In $\mathcal{A}$, define $y := \frac{1}{2}x + \frac{1}{2}\mathbb{1}$. By the continuous functional calculus, $\sigma_{\mathcal{A}}(y) = [0, 1]$, meaning $y \geq 0$ and $\|y\|_{\mathcal{A}} \leq 1$. Thus, the following Tietze transformations can be performed on the presentation for $\mathcal{A}$, adding the generator y and C*-relations on it.

$$\begin{aligned}\mathcal{A} &\cong_{\mathbf{1C^*}} \left\langle (x,1),(y,1) \,\middle|\, \begin{array}{c} x = x^*, \\ y = \frac{1}{2}x + \frac{1}{2}\mathbb{1} \end{array} \right\rangle_{\mathbf{1C^*}} \\ &\cong_{\mathbf{1C^*}} \left\langle (x,1),(y,1) \,\middle|\, \begin{array}{c} x = x^*, \\ y = \frac{1}{2}x + \frac{1}{2}\mathbb{1}, y \geq 0 \end{array} \right\rangle_{\mathbf{1C^*}}\end{aligned}$$

Rearranging the C*-relation $y = \frac{1}{2}x + \frac{1}{2}\mathbb{1}$ obtains $x = 2y - \mathbb{1}$. This C*-relation can be added as follows.

$$\mathcal{A} \cong_{\mathbf{1C^*}} \left\langle (x,1),(y,1) \,\middle|\, \begin{array}{c} x = x^*, \\ y = \frac{1}{2}x + \frac{1}{2}\mathbb{1}, y \geq 0 \\ x = 2y - \mathbb{1} \end{array} \right\rangle_{\mathbf{1C^*}}$$

Notice that if $x = 2y - \mathbb{1}$, then

$$\begin{aligned}\frac{1}{2}x + \frac{1}{2} &= \frac{1}{2}(2y - \mathbb{1}) + \frac{1}{2} \\ &= y - \frac{1}{2}\mathbb{1} + \frac{1}{2} \\ &= y\end{aligned}$$

and

$$\begin{aligned}x^* &= (2y - \mathbb{1})^* \\ &= 2y^* - \mathbb{1} \\ &= 2y - \mathbb{1} \\ &= x.\end{aligned}$$

Hence, those C*-relations may be removed as follows.

$$\mathcal{A} \cong_{\mathbf{1C^*}} \left\langle (x,1),(y,1) \,\middle|\, \begin{array}{c} x = x^*, \\ y \geq 0 \\ x = 2y - \mathbb{1} \end{array} \right\rangle_{\mathbf{1C^*}}$$

$$\cong_{\mathbf{1C^*}} \left\langle (x,1),(y,1) \,\middle|\, \begin{array}{c} y \geq 0 \\ x = 2y - \mathbb{1} \end{array} \right\rangle_{\mathbf{1C^*}}$$

Lastly, by the continuous functional calculus, $\sigma_{\mathcal{A}}(2y - \mathbb{1}) = [-1, 1]$ so $\|2y - \mathbb{1}\|_{\mathcal{A}} = 1$. Therefore, the generator x is unnecessary and may be removed, yielding the final presentation of $\mathcal{B}$. In summary, the sequence of transformations performed is as follows.

$$\langle (x,1) \,|x = x^* \rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} \left\langle (x,1),(y,1) \,\middle|\, \begin{array}{c} x = x^*, \\ y = \frac{1}{2}x + \frac{1}{2}\mathbb{1} \end{array} \right\rangle_{\mathbf{1C^*}}$$

$$\cong_{\mathbf{1C^*}} \left\langle (x,1),(y,1) \,\middle|\, \begin{array}{c} x = x^*, \\ y = \frac{1}{2}x + \frac{1}{2}\mathbb{1}, y \geq 0 \end{array} \right\rangle_{\mathbf{1C^*}}$$

$$\cong_{\mathbf{1C^*}} \left\langle (x,1),(y,1) \,\middle|\, \begin{array}{c} x = x^*, \\ y = \frac{1}{2}x + \frac{1}{2}\mathbb{1}, y \geq 0 \\ x = 2y - \mathbb{1} \end{array} \right\rangle_{\mathbf{1C^*}}$$

$$\cong_{\mathbf{1C^*}} \left\langle (x,1),(y,1) \,\middle|\, \begin{array}{c} x = x^*, y \geq 0 \\ x = 2y - \mathbb{1} \end{array} \right\rangle_{\mathbf{1C^*}}$$

$$\cong_{\mathbf{1C^*}} \left\langle (x,1),(y,1) \,\middle|\, \begin{array}{c} y \geq 0 \\ x = 2y - \mathbb{1} \end{array} \right\rangle_{\mathbf{1C^*}}$$

$$\cong_{\mathbf{1C^*}} \langle (y,1) \,| y \geq 0 \rangle_{\mathbf{1C^*}}$$

3.9.5 Manipulation of the Crutch Function

As illustrated in the examples thus far, the crutch function itself plays a key role not only in the construction of a C*-algebra, but also in its resulting structure. While manipulation of the crutch function was not integral to the main result in Theorem 3.9.7, such a transformation can be useful to understand an algebra or reducing the possible number of cases to consider. To demonstrate these manipulations, let (S, f) be a crutched set, R a set of C*-relations on (S, f), $\mathcal{F} := \langle S, f|\emptyset\rangle_{\mathbf{1C^*}}$, $\mathcal{A} := \langle S, f|R\rangle_{\mathbf{1C^*}}$, and $q_{\mathcal{A}} : \mathcal{F} \to \mathcal{A}$ the quotient map.

First, any generator with crutch value 0 becomes 0 in the C*-algebra. Specifically, recall the association of generators $\eta_{S,f} : S \to \mathcal{F}$ from Section 3.1. For all $s \in f^{-1}(0)$, $\eta_{S,f}(s) = 0$. Hence, the entirety of $f^{-1}(0)$ is associated to 0 in $\mathcal{A}$ via $q_{\mathcal{A}}$, which allows a C*-relation-adding Tietze transformation.

$$\mathcal{A} \cong_{\mathbf{1C^*}} \left\langle S, f \,\middle|\, R \cup f^{-1}(0) \right\rangle_{\mathbf{1C^*}}$$

Use of the previously discussed transformations can reduce the C*-relations by replacing elements of $f^{-1}(0)$ with 0, as well as reduce the generation set.

All of the preceding examples have shown this with their "0-case", when all generators were crutched by 0. In particular, reworking Example 3.5.1 yields the following sequence of Tietze transformations.

Example 3.9.10.

$$\begin{aligned}\left\langle (x,0)\,|x^*x = xx^* \right\rangle_{\mathbf{1C^*}} &\cong_{\mathbf{1C^*}} \left\langle (x,0)\,|x^*x = xx^*, x = 0 \right\rangle_{\mathbf{1C^*}} \\ &\cong_{\mathbf{1C^*}} \left\langle (x,0)\,|x = 0 \right\rangle_{\mathbf{1C^*}} \\ &\cong_{\mathbf{1C^*}} \left\langle \emptyset, \mathbf{0}_{[0,\infty)}\,|\emptyset \right\rangle_{\mathbf{1C^*}} \\ &\cong_{\mathbf{1C^*}} \mathbb{C}\end{aligned}$$

Second, the "scaling isomorphism" developed in Proposition 3.2.4 extends to an analogous scaling isomorphism for a nonempty set of C*-relations. To elaborate, let $g : S \to [0,\infty)$ be a second crutch function on S with $S \setminus g^{-1}(0) = T := S \setminus f^{-1}(0)$. Then, Proposition 3.2.4 states that

$$\mathcal{F} \cong_{\mathbf{1C^*}} \langle T, \mathbf{1}_T|\emptyset\rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} \langle S, g|\emptyset\rangle_{\mathbf{1C^*}}.$$

Let $\Phi : \mathcal{F} \to \langle S, g|\emptyset\rangle_{\mathbf{1C^*}}$ be the connecting isomorphism, given on the generating set by

$$\Phi(s) = \begin{cases} \dfrac{f(s)}{g(s)} s, & s \in T, \\ s, & s \notin T. \end{cases}$$

If J_R is the ideal generated by R in $\mathcal{F}$, $\Phi(J_R)$ will be an ideal in $\langle S, g|\emptyset\rangle_{\mathbf{1C^*}}$ as Φ is an isomorphism. Then, $\Phi(J_R) \supseteq J_{\Phi(R)}$, the ideal generated by $\Phi(R)$ in $\langle S, g|\emptyset\rangle_{\mathbf{1C^*}}$. Symmetrically, $\Phi^{-1}\left(J_{\Phi(R)}\right) \supseteq J_R$ as Φ^{-1} is an isomorphism. Therefore, $J_{\Phi(R)} = \Phi(J_R)$, yielding the isomorphism below.

$$\langle S, f|R\rangle_{\mathbf{1C^*}} := \mathcal{F}/J_R \cong_{\mathbf{1C^*}} \langle S, g|\emptyset\rangle_{\mathbf{1C^*}} / J_{\Phi(R)} =: \langle S, g|\Phi(R)\rangle_{\mathbf{1C^*}}$$

Notice that the C*-relations may be *altered* in this process. What occurs is that the scale factor becomes intertwined with the original C*-relations defining $\mathcal{A}$, which could possibly complicate and mask the structure.

Example 3.9.11. A rework of Example 3.5.5 yields the following sequence of isomorphisms for $\lambda \geq 1$.

$$\begin{aligned}
\mathbb{C} \oplus \mathbb{C} &\cong_{\mathbf{1C^*}} \left\langle (x, \lambda) \,\middle|\, x = x^2 = x^* \right\rangle_{\mathbf{1C^*}} \\
&\cong_{\mathbf{1C^*}} \left\langle (x, 1) \,\middle|\, \lambda x = \lambda^2 x^2 = \lambda x^* \right\rangle_{\mathbf{1C^*}} \\
&\cong_{\mathbf{1C^*}} \left\langle (x, 1) \,\middle|\, \begin{array}{c} \lambda x = \lambda^2 x^2 = \lambda x^*, \\ x = \lambda x^2 \end{array} \right\rangle_{\mathbf{1C^*}} \\
&\cong_{\mathbf{1C^*}} \left\langle (x, 1) \,\middle|\, \begin{array}{c} \lambda x = \lambda^2 x^2 = \lambda x^*, \\ x = \lambda x^2 = x^* \end{array} \right\rangle_{\mathbf{1C^*}} \\
&\cong_{\mathbf{1C^*}} \left\langle (x, 1) \,\middle|\, \begin{array}{c} \lambda^2 x^2 = \lambda x^*, \\ x = \lambda x^2 = x^* \end{array} \right\rangle_{\mathbf{1C^*}} \\
&\cong_{\mathbf{1C^*}} \left\langle (x, 1) \,\middle|\, x = \lambda x^2 = x^* \right\rangle_{\mathbf{1C^*}}
\end{aligned}$$

Notice that in the last "normalized" presentation, x is no longer a projection, but rather λx is. While the generator has been scaled into the unit ball, the condition has been blurred by the introduction of λ into the C*-relation.

However, there are cases where this move is very advantageous. Reworking Example 3.5.1 yields the following isomorphisms for $\lambda > 0$.

Example 3.9.12.

$$\begin{aligned}
\langle (x,\lambda) \,|\, x^*x = xx^* \rangle_{\mathbf{1C^*}} &\cong_{\mathbf{1C^*}} \left\langle (x,1) \,\middle|\, \lambda^2 x^*x = \lambda^2 xx^* \right\rangle_{\mathbf{1C^*}} \\
&\cong_{\mathbf{1C^*}} \left\langle (x,1) \,\middle|\, \begin{array}{c} \lambda^2 x^*x = \lambda^2 xx^*, \\ x^*x = xx^* \end{array} \right\rangle_{\mathbf{1C^*}} \\
&\cong_{\mathbf{1C^*}} \langle (x,1) \,|\, x^*x = xx^* \rangle_{\mathbf{1C^*}} \\
&\cong_{\mathbf{1C^*}} C\left(\overline{\mathbb{D}}\right)
\end{aligned}$$

Lastly, several examples have shown situations where $\|s\|_{\mathcal{A}} < f(s)$. When $\|s\|_{\mathcal{A}}$ can be computed, it can be used to completely replace $f(s)$. To explain, recall the norm-stealing result of Corollary 3.2.2. Letting $h : S \to [0,\infty)$ be defined by $h(s) := \|s\|_{\mathcal{A}}$ and $\mathcal{H} := \langle S, h | \emptyset \rangle_{\mathbf{1C^*}}$, consideration of the map $\psi : S \to \mathcal{A}$ by $\psi(s) := s$ obtains a unique unital *-homomorphism $\hat{\psi} : \mathcal{H} \to \mathcal{A}$ such that $\hat{\psi}(s) = s$.

Similarly, note that $h(s) \leq f(s)$ for all $s \in S$ so the map $\phi : (S,f) \to F_{\mathbf{1C^*}}^{\mathbf{CSet}_1}\mathcal{H}$ by $\phi(s) := s$ creates a unique unital *-homomorphism $\hat{\phi} : \mathcal{F} \to \mathcal{H}$ such that $\hat{\phi}(s) = s$ by Theorem 3.2.1. Let $\hat{R} := \hat{\phi}(R)$, $\mathcal{B} := \left\langle S, h \,\middle|\, \hat{R} \right\rangle_{\mathbf{1C^*}}$, and $q_{\mathcal{B}} : \mathcal{H} \to \mathcal{B}$ be the quotient map. Diagrammatically, this situation is shown below.

$$\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\hat{\phi}} & \mathcal{H} \\
{\scriptstyle q_{\mathcal{A}}}\downarrow & \swarrow {\scriptstyle \hat{\psi}} & \downarrow{\scriptstyle q_{\mathcal{B}}} \\
\mathcal{A} & & \mathcal{B}
\end{array}$$

Observe that for all $s \in S$,

$$\left(\hat{\psi} \circ \hat{\phi}\right)(s) = \hat{\psi}(s) = s = q_{\mathcal{A}}(s).$$

so by Theorem 3.2.1, $\hat{\psi} \circ \hat{\phi} = q_{\mathcal{A}}$. For each $r \in R$,

$$\hat{\psi}\left(\hat{\phi}(r)\right) = \left(\hat{\psi} \circ \hat{\phi}\right)(r) = q_{\mathcal{A}}(r) = 0.$$

Theorem 3.3.2 states that there is a unique unital *-homomorphism $\beta : \mathcal{B} \to \mathcal{A}$ such that $\beta(s) = s$. Similarly, for all $r \in R$,

$$\left(q_{\mathcal{B}} \circ \hat{\phi}\right)(r) = q_{\mathcal{B}}\left(\hat{\phi}(r)\right) = 0.$$

Again, Theorem 3.3.2 produces a unique unital *-homomorphism $\alpha : \mathcal{A} \to \mathcal{B}$ such that $\alpha(s) = s$. Therefore, for all $s \in S$,

$$(\alpha \circ \beta)(s) = \alpha(s) = s$$

and

$$(\beta \circ \alpha)(s) = \beta(s) = s$$

so by Theorem 3.3.2, $\alpha \circ \beta = id_{\mathcal{B}}$ and $\beta \circ \alpha = id_{\mathcal{A}}$. In short,

$$\langle S, f | R \rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} \left\langle S, h \middle| \hat{R} \right\rangle_{\mathbf{1C^*}}.$$

In this case, the C*-relations appear to be changed, but practically, this is not the case. All that has really been done is the restatement of the same conditions in $\mathcal{H}$.

Example 3.9.13. Recall Example 3.5.3. In this case, $(S, f) = \{(x, \lambda)\}$ and $R = \{x^*x - \mathbb{1}, xx^* - \mathbb{1}\}$. For $\lambda \geq 1$, notice that

$$\|x\|_{\mathcal{A}}^2 = \|x^*x\|_{\mathcal{A}} = \|\mathbb{1}\|_{\mathcal{A}} = 1.$$

Note that $\hat{\phi}(R) = \{x^*x - \mathbb{1}, xx^* - \mathbb{1}\}$. Thus, the above isomorphism states

$$\begin{aligned}\langle (x, \lambda) \, | x^*x = xx^* = \mathbb{1} \rangle_{\mathbf{1C^*}} \quad &\cong_{\mathbf{1C^*}} \quad \langle (x, 1) \, | x^*x = xx^* = \mathbb{1} \rangle_{\mathbf{1C^*}} \\ &\cong_{\mathbf{1C^*}} \quad C(\mathbb{T}).\end{aligned}$$

By Theorem 3.9.7, each of these three manipulations can be done by means of the generator/C*-relation Tietze transformations. However, they are included here as another way of manipulating a presentation's crutch function directly, without adding generators or necessarily adding C*-relations.

3.10 Construction: Unital Free Products

A common heuristic for the free product, for groups or other structures, has been to gather all generators and relations from the factors, adding nothing more. Intuitively, this is precisely the correct notion, and its validity is demonstrated in Corollary 3.2.5 for the scaled-free unital C*-algebra. In this section, the result will be extended rigorously to general presentations, allowing them to be split or merged via the free product. This provides more formal manipulations for presentations, much like Tietze transformations of Section 3.9.

To be precise, recall that the free product being considered here includes amalgamation of the identities of the factors. This is done to be sure that the result will be once again a unital C*-algebra, residing in $\mathbf{1C^*}$. In [40], the free product with amalgamation of identities is shown to be the coproduct in $\mathbf{1C^*}$, satisfying the appropriate mapping property.

However, [40] demonstrates this object's existence by means of representations on a Hilbert space, rather than constructing via C*-algebraic results. With the presentation theory devised in Section 3.5, the free product can be shown to exist without direct reference to a Hilbert space representation. Corollary 3.3.3 of [19] gave the analogous result for its presentation theory.

As in Section 3.2, the unital free product is usually denoted by "$*_{\mathbb{C}}$", indicating the merger of the identities. The category theoretic "$\coprod$" notation will be used interchangeably with the "$*_{\mathbb{C}}$" notation, but preference will be given to the "$\coprod$" with arbitrary index sets.

To begin, let Γ be an index set, $(S_\gamma, f_\gamma)_{\gamma\in\Gamma}$ be crutched sets, and R_γ C*-relations on (S_γ, f_γ) for each γ. Define $\mathcal{F}_\gamma := \langle S_\gamma, f_\gamma | \emptyset \rangle_{\mathbf{1C^*}}$, $\mathcal{A}_\gamma := \langle S_\gamma, f_\gamma | R_\gamma \rangle_{\mathbf{1C^*}}$, and $q_\gamma : \mathcal{F}_\gamma \to \mathcal{A}_\gamma$ the quotient map for each γ. The $\mathcal{A}_\gamma$ will be the unital C*-algebras to merge.

Let

$$(S, f) := \coprod_{\gamma\in\Gamma}^{\mathbf{CSet_1}} (S_\gamma, f_\gamma),$$

the disjoint union described in Proposition 2.2.9, $\rho_\gamma : (S_\gamma, f_\gamma) \to (S, f)$ the canonical inclusions for each γ, and $\mathcal{F} := \langle S, f | \emptyset \rangle_{\mathbf{1C^*}}$. Theorem 3.1.4 and Proposition A.5.1 state that

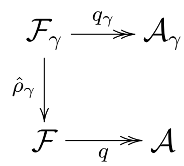
$$\mathcal{F} \cong_{\mathbf{1C^*}} \coprod_{\gamma\in\Gamma}^{\mathbf{1C^*}} \mathcal{F}_\gamma$$

with connecting maps $\hat{\rho}_\gamma := 1\mathrm{C}^*\mathrm{Alg}(\rho_\gamma)$. Let

$$R := \bigcup_{\gamma\in\Gamma} \hat{\rho}_\gamma (R_\gamma),$$

grouping all the C*-relations of the $\mathcal{A}_\gamma$ into one subset within the larger algebra $\mathcal{F}$.

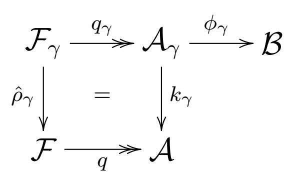
Define $\mathcal{A} := \langle S, f | R \rangle_{\mathbf{1C^*}}$, a candidate for the free product of the $\mathcal{A}_\gamma$, and $q : \mathcal{F} \to \mathcal{A}$ the quotient map. To create the connecting maps, fix $\gamma \in \Gamma$ and consider the following diagram in $\mathbf{1C}^*$.



Given $r \in R_\gamma$, observe that $\hat{\rho}_\gamma(r) \in R$ so $(q \circ \hat{\rho})(r) = 0$ by design. Thus, by the universal property of the quotient, there is a unique unital *-homomorphism $k_\gamma : \mathcal{A}_\gamma \to \mathcal{A}$ such that $k_\gamma \circ q_\gamma = q \circ \hat{\rho}_\gamma$.

Theorem 3.10.1. *The unital C*-algebra $\mathcal{A}$ equipped with unital *-homomorphisms $k_\gamma : \mathcal{A}_\gamma \to \mathcal{A}$ is a coproduct of $(\mathcal{A}_\gamma)_{\gamma \in \Gamma}$ in $\mathbf{1C}^*$.*

Proof. Let $\mathcal{B}$ be a unital C*-algebra and $\phi_\gamma : \mathcal{A}_\gamma \to \mathcal{B}$ be unital *-homomorphisms for each $\gamma \in \Gamma$. This situation is shown in the diagram below for each $\gamma \in \Gamma$.



As $\phi_\gamma \circ q_\gamma : \mathcal{F}_\gamma \to \mathcal{B}$ are unital *-homomorphisms, there is a unique unital *-homomorphism $\psi : \mathcal{F} \to \mathcal{B}$ such that $\psi \circ \hat{\rho}_\gamma = \phi_\gamma \circ q_\gamma$ for all $\gamma \in \Gamma$ by the coproduct characterization of $\mathcal{F}$. For $\gamma \in \Gamma$ and $r \in R_\gamma$, observe that

$$(\psi \circ \hat{\rho}_\gamma)(r) = (\phi_\gamma \circ q_\gamma)(r) = \phi_\gamma(0) = 0.$$

Thus, there is a unique unital *-homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ such that $\psi = \phi \circ q$.

For a fixed $\gamma \in \Gamma$, observe that

$$\phi \circ k_\gamma \circ q_\gamma = \phi \circ q \circ \hat{\rho}_\gamma = \psi \circ \hat{\rho}_\gamma = \phi_\gamma \circ q_\gamma$$

so $\phi \circ k_\gamma = \phi_\gamma$ by Theorem 3.3.2.

Assume that there was another unital *-homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ such that $\varphi \circ k_\gamma = \phi_\gamma$ for all $\gamma \in \Gamma$. Then,

$$\psi \circ \hat{\rho}_\gamma = \phi_\gamma \circ q_\gamma = \varphi \circ k_\gamma \circ q_\gamma = \varphi \circ q \circ \hat{\rho}_\gamma$$

so by the universal property of the coproduct $\mathcal{F}$, $\varphi \circ q = \psi = \phi \circ q$. Therefore, by the universal property of the quotient, $\varphi = \phi$.

□

In summary,

$$\langle S, f|R\rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} \coprod_{\gamma\in\Gamma}^{\mathbf{1C^*}} \langle S_\gamma, f_\gamma|R_\gamma\rangle_{\mathbf{1C^*}}.$$

In the case $\Gamma = \{1, 2\}$, this result can be stated in the traditional notation as

$$\langle S, f|R\rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} \langle S_1, f_1|R_1\rangle_{\mathbf{1C^*}} *_{\mathbb{C}} \langle S_2, f_2|R_2\rangle_{\mathbf{1C^*}}.$$

As a concrete example, consider the free product in $\mathbf{1C^*}$ of $\mathbb{C} \oplus \mathbb{C}$ with itself.

Example 3.10.2 (Pedersen's unital C*-algebra of two projections). From [32], one may consider the following unital C*-algebra

$$\mathcal{A} := \left\langle (p, 1), (q, 1) | p = p^* = p^2, q = q^* = q^2 \right\rangle_{\mathbf{1C^*}}.$$

Observe that in this case, $S = \{p, q\}$, $f : S \to [0, \infty)$ by $f(p) = f(q) = 1$, and $R = \left\{p - p^*, p - p^2, q - q^*, q - q^2\right\}$.

Let $S_1 := \{p\}$, $f_1 : S_1 \to [0, \infty)$ by $f_1(p) := 1$, and $R_1 := \left\{p - p^*, p - p^2\right\}$. Likewise, let $S_2 := \{q\}$, $f_2 : S_2 \to [0, \infty)$ by $f_2(q) := 1$, and $R_2 := \left\{q - q^*, q - q^2\right\}$. By Example 3.5.5, the unital C*-algebras $\langle S_j, f_j|R_j\rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} \mathbb{C} \oplus \mathbb{C}$ for $j = 1, 2$.

Letting $\rho_j : (S_j, f_j) \to (S, f)$ be the inclusions for $j = 1, 2$, observe that $R = \bigcup_{j=1}^{2} 1\mathrm{C}^{*}\mathrm{Alg}\,(\rho_j)\,(R_j)$. Thus, the above result states that

$$\begin{aligned}(\mathbb{C} \oplus \mathbb{C}) *_{\mathbb{C}} (\mathbb{C} \oplus \mathbb{C}) \quad &\cong_{\mathbf{1C^*}} \quad \langle S_1, f_1 | R_1 \rangle_{\mathbf{1C^*}} *_{\mathbb{C}} \langle S_2, f_2 | R_2 \rangle_{\mathbf{1C^*}} \\ &\cong_{\mathbf{1C^*}} \quad \langle S, f | R \rangle_{\mathbf{1C^*}}.\end{aligned}$$

Using the result of [32],

$$\mathcal{A} \cong_{\mathbf{1C^*}} \begin{bmatrix} C(X) & C_0(X \setminus \{0,1\}) \\ C_0(X \setminus \{0,1\}) & C(X) \end{bmatrix},$$

where $X := \sigma_{\mathcal{A}}(pqp)$. Notice that $0 \leq pqp$ and $\|pqp\|_{\mathcal{A}} \leq 1$. Hence, $\sigma_{\mathcal{A}}(pqp) \subseteq [0,1]$. For $\alpha \in [0,1]$, let

$$p_\alpha := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$q_\alpha := \begin{bmatrix} \alpha & \sqrt{\alpha - \alpha^2} \\ \sqrt{\alpha - \alpha^2} & 1 - \alpha \end{bmatrix}$$

in M_2. A quick arithmetic check shows that both p_α and q_α are projections, the same ones used in [32]. By Theorem 3.3.2, there is a unique $\phi_\alpha : \mathcal{A} \to M_2$ such that $\phi_\alpha(p) = p_\alpha$ and $\phi_\alpha(q) = q_\alpha$. Observe that

$$\phi_\alpha(pqp) = p_\alpha q_\alpha p_\alpha = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}$$

so $\sigma_{\mathcal{A}}(pqp) \supseteq \sigma_{M_2}(p_\alpha q_\alpha p_\alpha) = \{0, \alpha\}$. Therefore, $\sigma_{\mathcal{A}}(pqp) = [0,1]$.

In summary,

$$(\mathbb{C} \oplus \mathbb{C}) *_{\mathbb{C}} (\mathbb{C} \oplus \mathbb{C}) \cong_{\mathbf{1C^*}} \begin{bmatrix} C[0,1] & C_0(0,1) \\ C_0(0,1) & C[0,1] \end{bmatrix}$$

A common practice with presentation theories is to ignore the inclusion maps $\hat{\rho}_\gamma$, regarding each $\mathcal{F}_\gamma$ as a subalgebra of $\mathcal{F}$. The reason for this is seen in the above example, where the $\hat{\rho}_\gamma$ serve to partition the generators and C*-relations. Separated in this way, the smaller algebras can be more easily computed, leaving the free product construction to handle the other interactions.

3.11 Examples: Types of Invertibility

Example 3.5.3 considered a unitary, a specific type of invertible element. In this section, one considers unital C*-algebras generated by other types of invertible elements.

In particular, this section describes a minimal set of C*-relations imposing invertibility on an element. Following this, illustrative examples demonstrate the behavior of increasingly general types of invertible elements. Finally, a one-sided invertible is considered and its structure related to the previous examples.

3.11.1 C*-relations for Invertibility

A natural construct to consider for invertibility would be

$$\langle (x, \lambda), (y, \mu) | xy = yx = \mathbb{1} \rangle_{\mathbf{1C^*}}$$

for $\lambda, \mu \in [0, \infty)$, an analog of the ring of Laurent polynomials. However, while this C*-algebra does describe a pair of inverses, its definition requires two generators. In considering the C*-algebra generated by an invertible element x, how does its inverse y come into play?

In truth, consideration of y is unnecessary. The invertibility of x can be characterized entirely in terms of x itself by means of C*-relations built via the continuous functional calculus. Specifically, the positive part of x must be bounded away from 0.

Historically, the *positive part* of an operator x is determined by the polar decomposition in a faithful representation on a Hilbert space. This decomposition, however, gives a pleasant formula for the positive part, $(x^*x)^{\frac{1}{2}}$. Symmetrically, the polar decomposition of x^* yields the formula $(xx^*)^{\frac{1}{2}}$. In the decomposition, $x = u\,(x^*x)^{\frac{1}{2}} = (xx^*)^{\frac{1}{2}}\,v$ for appropriate partial isometries u and v. Both of $(x^*x)^{\frac{1}{2}}$ and $(xx^*)^{\frac{1}{2}}$ can arguably be called the "positive part" of x.

As it happens, these two C*-relations can completely characterize not only invertibility, but one-sided invertibility.

Proposition 3.11.1. *Let $\mathcal{A}$ be a unital C*-algebra, $x \in \mathcal{A}$, and $\mu \in (0, \infty)$. Then, there is $y \in \mathcal{A}$ satisfying $\|y\| \le \mu$ and $yx = \mathbb{1}$ if and only if $\mathbb{1} \le \mu^2 x^* x$. In this case, one can arrange that $y \in C^*(\mathbb{1}, x)$.*

Proof. ($\Rightarrow$) Assume that there is $y \in \mathcal{A}$ such that $\|y\| \le \mu$ and $yx = \mathbb{1}$. Then,

$$\mathbb{1} = \mathbb{1}^2 = \mathbb{1}^*\mathbb{1} = (yx)^*(yx) = x^*y^*yx \le \|y\|^2 x^*x \le \mu^2 x^*x.$$

($\Leftarrow$) Assume that $\mathbb{1} \le \mu^2 x^* x$. Define $q := (x^*x)^{\frac{1}{2}}$. By assumption, $\sigma_{\mathcal{A}}\left(\mu q - \mathbb{1}\right) \subset$

$[0,\infty)$ so by the continuous functional calculus,

$$\begin{aligned} \sigma_{\mathcal{A}}(\mu q) &\subset [1,\infty), \\ \sigma_{\mathcal{A}}(q) &\subset \left[\frac{1}{\mu},\infty\right), \\ \sigma_{\mathcal{A}}\left(q^{-1}\right) &\subset (0,\mu]. \end{aligned}$$

Define $u := xq^{-1}$. Observe that

$$u^*u = q^{-1}x^*xq^{-1} = (x^*x)^{\frac{-1}{2}}(x^*x)(x^*x)^{\frac{-1}{2}} = (x^*x)^{-1}(x^*x) = \mathbb{1},$$

meaning u is an isometry. Letting $y := q^{-1}u^* \in C^*(\mathbb{1},x)$,

$$yx = q^{-1}u^*uq = q^{-1}q = \mathbb{1}$$

and

$$\|y\| \leq \left\|q^{-1}\right\| \|u^*\| \leq \mu.$$

□

When considering operators on a Hilbert space, this criterion is usually termed "bounded below", as one would rewrite the condition as $\frac{1}{\mu}\mathbb{1} \leq (x^*x)^{\frac{1}{2}}$. For right-invertibility, one considers x^* in the above proposition to yield the following.

Corollary 3.11.2. *Let $\mathcal{A}$ be a unital C*-algebra, $x \in \mathcal{A}$ and $\mu \in (0,\infty)$. Then, there is $y \in \mathcal{A}$ satisfying $\|y\| \leq \mu$ and $xy = \mathbb{1}$ if and only if $\mathbb{1} \leq \mu^2 xx^*$. In this case, one can arrange that $y \in C^*(\mathbb{1},x)$.*

Together, these two facts give C*-relations for an invertible element.

Proposition 3.11.3. *Let $\mathcal{A}$ be a unital C*-algebra, $x \in \mathcal{A}$ and $\mu \in (0, \infty)$. Then, there is $y \in \mathcal{A}$ satisfying $\|y\| \leq \mu$ and $xy = yx = \mathbb{1}$ if and only if $\mathbb{1} \leq \mu^2 x^* x$ and $\mathbb{1} \leq \mu^2 x x^*$. In this case, $y \in C^*(\mathbb{1}, x)$.*

Proof. ($\Rightarrow$) Assuming that there is $y \in \mathcal{A}$ such that $xy = yx = \mathbb{1}$ with $\|y\| \leq \mu$, then y is both a left- and right-inverse to x with the appropriate norm bound. Hence, $\mathbb{1} \leq \mu^2 x^* x$ and $\mathbb{1} \leq \mu^2 x x^*$.

($\Leftarrow$) Assuming the result, then there are $y_1, y_2 \in C^*(\mathbb{1}, x)$ such that $y_1 x = x y_2 = \mathbb{1}$, $\|y_1\| \leq \mu$, and $\|y_2\| \leq \mu$. However, observe that

$$y_1 = y_1 \mathbb{1} = y_1 x y_2 = \mathbb{1} y_2 = y_2.$$

□

Now, observe that both of the C*-relations determined in this proposition are tied to the value μ, which serves as a bound on the norm of x^{-1}. This proposition has actually characterized the condition "x has an inverse of norm at most μ".

Unfortunately, there are no C*-relations in terms of x alone that characterize the condition "x has an inverse". Explicitly, the norm of x^{-1} cannot be allowed to grow without bound as shown below.

Example 3.11.4 (Necessity of bounds on inverses)*.* For $\lambda \in (0, \infty)$, let

$$\mathcal{F} := \langle (x, \lambda) | \emptyset \rangle_{\mathbf{1C^*}}.$$

For all $\mu \in (1, \infty)$, observe that $\lambda \oplus \dfrac{\lambda}{\mu} \in \mathbb{C} \oplus \mathbb{C}$ has inverse $\dfrac{1}{\lambda} \oplus \dfrac{\mu}{\lambda} \in \mathbb{C} \oplus \mathbb{C}$, and $\left\| \lambda \oplus \dfrac{\lambda}{\mu} \right\|_{\mathbb{C} \oplus \mathbb{C}} = \lambda$. By Theorem 3.2.1, there is a unique unital *-homomorphism $\phi_\mu : \mathcal{F} \to \mathbb{C} \oplus \mathbb{C}$ by $\phi_\mu(x) := \lambda \oplus \dfrac{\lambda}{\mu}$.

Assume there is a set of C*-relations R on $\{(x,\lambda)\}$ such that for any unital *-homomorphism $\phi : \mathcal{F} \to \mathcal{A}$ where $\phi(x)$ is invertible, $R \subseteq \ker(\phi)$. Define $\mathcal{C} := \langle (x,\lambda)|R\rangle_{\mathbf{1C^*}}$. Then, for each $\mu \in (1,\infty)$, $R \subseteq \ker(\phi_\mu)$ so by Theorem 3.3.2, there is a unique unital *-homomorphism $\hat{\phi}_\mu : \mathcal{C} \to \mathbb{C} \oplus \mathbb{C}$ such that $\hat{\phi}_\mu(x) = \lambda \oplus \dfrac{\lambda}{\mu}$. Thus,

$$\sigma_{\mathcal{C}}(x) \supseteq \sigma_{\mathbb{C}\oplus\mathbb{C}}\left(\phi_\mu(x)\right) = \sigma_{\mathbb{C}\oplus\mathbb{C}}\left(\lambda \oplus \frac{\lambda}{\mu}\right) = \left\{\lambda, \frac{\lambda}{\mu}\right\}$$

for all $\mu \in (1,\infty)$, meaning $\sigma_{\mathcal{C}}(x) \supseteq (0,\lambda]$. As $\sigma_{\mathcal{C}}(x)$ is well-known to be closed, $0 \in \sigma_{\mathcal{C}}(x)$. Therefore, x is not invertible in $\mathcal{C}$.

Notice what has happened in the example above is that x was not bounded below. Indeed, if there was $y \in \mathcal{C}$ such that $xy = yx = \mathbb{1}$, then the following occurs for each $\mu \in (1,\infty)$.

$$\phi_\mu(xy) = \phi_\mu(yx) = \phi_\mu(\mathbb{1})$$
$$\phi_\mu(x)\phi_\mu(y) = \phi_\mu(y)\phi_\mu(x) = 1 \oplus 1$$
$$\left(\lambda \oplus \frac{\lambda}{\mu}\right)\phi_\mu(y) = \phi_\mu(y)\left(\lambda \oplus \frac{\lambda}{\mu}\right) = 1 \oplus 1$$

Thus, $\phi_\mu(y) = \dfrac{1}{\lambda} \oplus \dfrac{\mu}{\lambda}$ so $\|y\|_{\mathcal{C}} \geq \left\|\dfrac{1}{\lambda} \oplus \dfrac{\mu}{\lambda}\right\|_{\mathbb{C}\oplus\mathbb{C}} = \dfrac{\mu}{\lambda}$. However, the right-hand side grows without bound, meaning $\|y\|_{\mathcal{C}}$ cannot have a finite value.

3.11.2 Commutative Cases

In all commutative cases, one does not require both the C*-relations of Proposition 3.11.3 since the commutativity makes them the same element. With this fact in hand, unital C*-algebras generated by a single element which is both normal and invertible can be characterized.

Example 3.11.5 (Positive invertible). For $\lambda, \mu \in [0, \infty)$, consider the unital C*-algebra

$$\mathcal{P} := \left\langle (x, \lambda) \,\middle|\, x \geq 0, \mu^2 x^* x \geq \mathbb{1}, \mu^2 x x^* \geq \mathbb{1} \right\rangle_{\mathbf{1C^*}}.$$

One can directly use Theorem 3.4.4 to concretely realize this algebra, but the presentation will be reduced to a simpler form first. This is done for two main reasons. Primarily, the reduced presentation will be used again in Subsection 3.11.3 for the main result of this section. Second, the reduced presentation will have only one C*-relation to consider, which eases the spectral computation in Theorem 3.4.4.

First, there is a trivial case to consider. If $\lambda\mu < 1$,

$$\|\mathbb{1}\|_{\mathcal{P}} \leq \mu^2 \|x^* x\|_{\mathcal{P}} \leq \mu^2 \lambda^2 < 1.$$

so as in Example 3.5.3, $\mathbb{1} = 0$ and $\mathcal{P} \cong_{\mathbf{1C^*}} \mathbb{O}$.

Assume $\lambda\mu \geq 1$. In $\mathcal{P}$, note that $x = p(\Re(x))$ implies that $\sigma_{\mathcal{P}}(x) \subset [0, \infty)$ and $x = x^*$ so the continuous functional calculus states that

$$\mu \left(x^* x\right)^{\frac{1}{2}} = \mu \left(x^2\right)^{\frac{1}{2}} = \mu x.$$

Hence,

$$\mu x - \mathbb{1} = \mu \left(x^* x\right)^{\frac{1}{2}} - \mathbb{1} = p\left(\Re\left(\mu \left(x^* x\right)^{\frac{1}{2}} - \mathbb{1}\right)\right) = p(\Re(\mu x - \mathbb{1}))$$

so a C*-relation-adding Tietze transformation yields

$$\mathcal{P} \cong_{\mathbf{1C^*}} \left\langle (x, \lambda) \,\middle|\, \begin{array}{c} x \geq 0, \mu^2 x^* x \geq \mathbb{1}, \mu^2 x x^* \geq \mathbb{1}, \\ \mu x \geq \mathbb{1} \end{array} \right\rangle_{\mathbf{1C^*}}.$$

Assuming only that $p(\Re(\mu x - \mathbb{1})) = \mu x - \mathbb{1}$, then $x = \frac{1}{\mu}\mathbb{1} + \frac{1}{\mu}p(\Re(\mu x - \mathbb{1}))$, a self-adjoint element. By the continuous functional calculus,

$$\begin{aligned}
\sigma_{\mathcal{P}}(\mu x - \mathbb{1}) &\subset [0, \infty) \\
\sigma_{\mathcal{P}}(\mu x) &\subset [1, \infty) \\
\sigma_{\mathcal{P}}(x) &\subset \left[\frac{1}{\mu}, \infty\right) \\
\sigma_{\mathcal{P}}\left(x^2\right) &\subset \left[\frac{1}{\mu^2}, \infty\right) \\
\sigma_{\mathcal{P}}\left(\mu^2 x^2\right) &\subset [1, \infty) \\
\sigma_{\mathcal{P}}\left(\mu^2 x^2 - \mathbb{1}\right) &\subset [0, \infty)
\end{aligned}$$

so

$$p(\Re(x)) = p(x) = x,$$

$$\begin{aligned}
p\left(\Re\left(\mu^2 x^* x - \mathbb{1}\right)\right) &= p\left(\Re\left(\mu^2 x^2 - \mathbb{1}\right)\right) \\
&= \mu^2 x^2 - \mathbb{1} \\
&= \mu^2 x^* x - \mathbb{1},
\end{aligned}$$

and likewise

$$p\left(\Re\left(\mu^2 x x^* - \mathbb{1}\right)\right) = \mu^2 x x^* - \mathbb{1}.$$

Thus, C*-relation-removing Tietze transformations yield

$$\begin{aligned}
\mathcal{P} &\cong_{\mathbf{1C^*}} \left\langle (x, \lambda) \;\middle|\; \begin{array}{c} \mu^2 x^* x \geq \mathbb{1}, \mu^2 x x^* \geq \mathbb{1}, \\ \mu x \geq \mathbb{1} \end{array} \right\rangle_{\mathbf{1C^*}} \\
&\cong_{\mathbf{1C^*}} \left\langle (x, \lambda) \;\middle|\; \mu^2 x x^* \geq \mathbb{1}, \mu x \geq \mathbb{1} \right\rangle_{\mathbf{1C^*}} \\
&\cong_{\mathbf{1C^*}} \left\langle (x, \lambda) \;\middle|\; \mu x \geq \mathbb{1} \right\rangle_{\mathbf{1C^*}}.
\end{aligned}$$

Considering this reduced presentation, note that x is normal. In this case,

$$r := p(\Re(\mu x - \mathbb{1})) - \mu x + \mathbb{1}$$

so

$$g_r(\nu) = p(\Re(\mu\nu - 1)) - \mu\nu + 1.$$

Note that $g_r(\nu) = 0$ whenever $\mu\nu - 1 \geq 0$. Then, $g_r^{-1}(0) = \left[\frac{1}{\mu}, \lambda\right]$. By Theorem 3.4.4,

$$\mathcal{P} \cong_{\mathbf{1C^*}} C\left[\frac{1}{\mu}, \lambda\right] \cong_{\mathbf{1C^*}} \begin{cases} \mathbb{C}, & \lambda\mu = 1, \\ C[0,1], & \lambda\mu > 1, \end{cases}$$

as $\left[\frac{1}{\mu}, \lambda\right] \cong_{\mathbf{Top}} [0,1]$ when $\lambda\mu > 1$.

In summary,

$$\mathcal{P} \cong_{\mathbf{1C^*}} \begin{cases} \mathbb{0}, & \lambda\mu < 1, \\ \mathbb{C}, & \lambda\mu = 1, \\ C[0,1], & \lambda\mu > 1. \end{cases}$$

Example 3.11.6 (Self-adjoint invertible). For $\lambda, \mu \in [0, \infty)$, consider the unital C*-algebra

$$\mathcal{S} := \left\langle (x, \lambda) \,\middle|\, x = x^*, \mu^2 x^* x \geq \mathbb{1}, \mu^2 x x^* \geq \mathbb{1} \right\rangle_{\mathbf{1C^*}}.$$

As in Example 3.11.5, if $\lambda\mu < 1$, $\mathbb{1} = 0$ and $\mathcal{S} \cong_{\mathbf{1C^*}} \mathbb{0}$.

Assume $\lambda\mu \geq 1$. The following C*-relation-removing Tietze transformation results as in Example 3.11.5.

$$\mathcal{S} \cong_{\mathbf{1C^*}} \left\langle (x, \lambda) \,\middle|\, x = x^*, \mu^2 x^* x \geq \mathbb{1} \right\rangle_{\mathbf{1C^*}}$$

Considering this reduced presentation, note that x is normal. In this case, $r_1 :=$

$x - x^*$ and $r_2 := p\left(\mu^2 x^* x - \mathbb{1}\right) - \mu^2 x^* x + \mathbb{1}$ so

$$g_{r_1}(\nu) = 2\imath\Im(\nu)$$

and

$$g_{r_2}(\nu) = p\left(\mu^2|\nu|^2 - 1\right) - \mu^2|\nu|^2 + 1.$$

Note that $g_{r_2}(\nu) = 0$ whenever $\mu^2|\nu|^2 - 1 \geq 0$. Let

$$A_{t_1,t_2} := \{\nu \in \mathbb{C} : t_1 \leq |\nu| \leq t_2\}$$

denote a closed annulus with inner radius t_1 and outer radius t_2. Then, $g_{r_2}^{-1}(0) = A_{\frac{1}{\mu},\lambda}$ and $g_{r_1}^{-1}(0) = [-\lambda, \lambda]$. By Theorem 3.4.4,

$$\mathcal{S} \cong_{\mathbf{1C^*}} C\left(\left[-\lambda, \frac{-1}{\mu}\right] \cup \left[\frac{1}{\mu}, \lambda\right]\right) \cong_{\mathbf{1C^*}} \begin{cases} \mathbb{C} \oplus \mathbb{C}, & \lambda\mu = 1, \\ C\left([-2,-1] \cup [1,2]\right), & \lambda\mu > 1, \end{cases}$$

as $\left[-\lambda, \frac{-1}{\mu}\right] \cup \left[\frac{1}{\mu}, \lambda\right] \cong_{\mathbf{Top}} [-2,-1] \cup [1,2]$ when $\lambda\mu > 1$.

In summary,

$$\mathcal{S}_{\lambda,\mu} \cong_{\mathbf{1C^*}} \begin{cases} \mathbb{0}, & \lambda\mu < 1, \\ \mathbb{C} \oplus \mathbb{C}, & \lambda\mu = 1, \\ C\left([-2,-1] \cup [1,2]\right), & \lambda\mu > 1. \end{cases}$$

Example 3.11.7 (Normal invertible). For $\lambda, \mu \in [0, \infty)$, consider the unital C*-algebra

$$\mathcal{N} := \left\langle (x, \lambda) \,\middle|\, x^* x = x x^*, \mu^2 x^* x \geq \mathbb{1}, \mu^2 x x^* \geq \mathbb{1} \right\rangle_{\mathbf{1C^*}}.$$

As in Example 3.11.5, if $\lambda\mu < 1$, $\mathbb{1} = 0$ and $\mathcal{N} \cong_{\mathbf{1C^*}} \mathbb{0}$.

Assume $\lambda\mu \geq 1$. Assuming only that $p\left(\Re\left(\mu^2 x^* x - \mathbb{1}\right)\right) = \mu^2 x^* x - \mathbb{1}$ and $x^*x = xx^*$, then

$$\begin{aligned} p\left(\Re\left(\mu^2 xx^* - \mathbb{1}\right)\right) &= p\left(\Re\left(\mu^2 x^* x - \mathbb{1}\right)\right) \\ &= \mu^2 x^* x - \mathbb{1} \\ &= \mu^2 xx^* - \mathbb{1}. \end{aligned}$$

Thus, a C*-relation-removing Tietze transformation yields

$$\mathcal{N} \quad \cong_{\mathbf{1C^*}} \quad \left\langle (x,\lambda) \,\middle|\, x^*x = xx^*, \mu^2 x^* x \geq \mathbb{1} \right\rangle_{\mathbf{1C^*}}.$$

Considering this reduced presentation, note that x is normal. In this case, $r_1 := x^*x - xx^*$ and $r_2 := p\left(\mu^2 x^* x - \mathbb{1}\right) - \mu^2 x^* x + \mathbb{1}$ so

$$g_{r_1}(\nu) = 0$$

and

$$g_{r_2}(\nu) = p\left(\mu^2 |\nu|^2 - 1\right) - \mu^2 |\nu|^2 + 1.$$

Then, $g_{r_2}^{-1}(0) = A_{\frac{1}{\mu},\lambda}$. By Theorem 3.4.4,

$$\mathcal{N} \cong_{\mathbf{1C^*}} C\left(A_{\frac{1}{\mu},\lambda}\right) \cong_{\mathbf{1C^*}} \begin{cases} C(\mathbb{T}), & \lambda\mu = 1, \\ C\left(A_{1,2}\right), & \lambda\mu > 1, \end{cases}$$

as $A_{\lambda,\lambda} \cong_{\mathbf{Top}} \mathbb{T}$ and $A_{\frac{1}{\mu},\lambda} \cong_{\mathbf{Top}} A_{1,2}$ when $\lambda\mu > 1$.

$$\mathcal{N}_{\lambda,\mu} \cong_{\mathbf{1C^*}} \begin{cases} \mathbb{0}, & \lambda\mu < 1, \\ C(\mathbb{T}), & \lambda\mu = 1, \\ C\left(A_{1,2}\right), & \lambda\mu > 1. \end{cases}$$

Recall that **Comp** denotes the category of compact Hausdorff spaces with continuous functions. Here, note that in a natural way,

$$\left[-\lambda, \frac{-1}{\mu}\right] \cup \left[\frac{1}{\mu}, \lambda\right] \cong_{\mathbf{Comp}} \left[\frac{1}{\mu}, \lambda\right] \prod^{\mathbf{Comp}} \{-1, 1\}$$

and

$$A_{\frac{1}{\mu},\lambda} \cong_{\mathbf{Comp}} \left[\frac{1}{\mu}, \lambda\right] \prod^{\mathbf{Comp}} \mathbb{T}$$

via the polar decomposition in $\mathbb{C}$. Recall also that the coproduct in $\mathbf{C1C^*}$ is the generalized tensor product so by the Gelfand duality,

$$\begin{aligned} C\left(\left[-\lambda, \frac{-1}{\mu}\right] \cup \left[\frac{1}{\mu}, \lambda\right]\right) &\cong_{\mathbf{1C^*}} C\left[\frac{1}{\mu}, \lambda\right] \otimes C(\{-1, 1\}) \\ &\cong_{\mathbf{1C^*}} C\left[\frac{1}{\mu}, \lambda\right] \coprod^{\mathbf{C1C^*}} C(\{-1, 1\}) \end{aligned}$$

and

$$C\left(A_{\frac{1}{\mu},\lambda}\right) \cong_{\mathbf{1C^*}} C\left[\frac{1}{\mu}, \lambda\right] \otimes C(\mathbb{T}) \cong_{\mathbf{1C^*}} C\left[\frac{1}{\mu}, \lambda\right] \coprod^{\mathbf{C1C^*}} C(\mathbb{T}).$$

This demonstrates the polar decomposition, splitting the invertible x into its positive and unitary parts. For the positive case, the unitary part is merely the identity. For the self-adjoint case, the unitary part has real spectrum, $\{-1, 1\}$. For the general normal case, the unitary part has full spectrum, $\mathbb{T}$.

However, it is the relationship to the coproduct in $\mathbf{C1C^*}$ that is of interest. This leads directly to the next case, a general invertible element.

3.11.3 C*-algebra of a General Invertible

With an understanding of normal invertible elements, attention now turns to the general case, where normality is not assumed. This will be done using not only the

Tietze transformations of Section 3.9, but also the unital free product of Section 3.10.

Example 3.11.8 (General invertible). For $\lambda, \mu \in [0, \infty)$, let

$$\mathcal{I} := \left\langle (x, \lambda) \,\middle|\, \mu^2 x^* x \geq \mathbb{1}, \mu^2 x x^* \geq \mathbb{1} \right\rangle_{\mathbf{1C^*}}.$$

The goal is to realize this algebra either explicitly or as a combination of familiar algebras, and this will be done by means of the Tietze transformations of Section 3.9.

First, as in Example 3.11.5, if $\lambda\mu < 1$, $\mathbb{1} = 0$ and $\mathcal{I} \cong_{\mathbf{1C^*}} \mathbb{O}$.

Assume now that $\lambda\mu \geq 1$. As the C*-relations were concocted via the polar decomposition, this decomposition will be used to split the algebra into two free-factors. Let $q := (x^* x)^{\frac{1}{2}}$, a self-adjoint element, and observe from the continuous functional calculus,

$$\begin{aligned} \sigma_{\mathcal{I}}\left(\mu^2 q^2 - \mathbb{1}\right) &\subset [0, \infty) \\ \sigma_{\mathcal{I}}\left(\mu^2 q^2\right) &\subset [1, \infty) \\ \sigma_{\mathcal{I}}\left(\mu q\right) &\subset [1, \infty) \\ \sigma_{\mathcal{I}}\left(\mu q - \mathbb{1}\right) &\subset [0, \infty) \end{aligned}$$

so $\mu q \geq \mathbb{1}$.

Similarly,

$$\begin{aligned} \sigma_{\mathcal{I}}\left(p(\mu q - \mathbb{1})\right) &\subset [0, \infty) \\ \sigma_{\mathcal{I}}\left(p(\mu q - \mathbb{1}) + \mathbb{1}\right) &\subset [1, \infty) \end{aligned}$$

so $p(\mu q - \mathbb{1}) + \mathbb{1}$ is invertible. Define $u := \mu x \left(p\left(\mu \left(x^* x\right)^{\frac{1}{2}} - \mathbb{1}\right) + \mathbb{1}\right)^{-1}$.

Here, q and u represent the positive and unitary parts of x. Also, $\|q\|_{\mathcal{I}} \leq \lambda$ and

$\|u\|_{\mathcal{I}} \leq \lambda\mu$. Using generator-adding Tietze transformations,

$$\begin{aligned}
\mathcal{I} &\cong_{\mathbf{1C^*}} \left\langle \begin{array}{c} (x,\lambda), \\ (q,\lambda) \end{array} \middle| \ \mu^2 x^* x \geq \mathbb{1}, \mu^2 x x^* \geq \mathbb{1}, q = (x^*x)^{\frac{1}{2}} \right\rangle_{\mathbf{1C^*}} \\
&\cong_{\mathbf{1C^*}} \left\langle \begin{array}{c} (x,\lambda), \\ (q,\lambda), (u, \lambda\mu) \end{array} \middle| \begin{array}{c} \mu^2 x^* x \geq \mathbb{1}, \mu^2 x x^* \geq \mathbb{1}, q = (x^*x)^{\frac{1}{2}}, \\ u = \mu x \left(p \left(\mu \left(x^* x \right)^{\frac{1}{2}} - \mathbb{1} \right) + \mathbb{1} \right)^{-1} \end{array} \right\rangle_{\mathbf{1C^*}}
\end{aligned}$$

In $\mathcal{I}$, observe that

$$\begin{aligned}
u^*u &= \mu \left(p \left(\mu \left(x^* x \right)^{\frac{1}{2}} - \mathbb{1} \right) + \mathbb{1} \right)^{-1} x^* \cdot \mu x \left(p \left(\mu \left(x^* x \right)^{\frac{1}{2}} - \mathbb{1} \right) + \mathbb{1} \right)^{-1} \\
&= \mu^2 \left(p \left(\mu q - \mathbb{1} \right) + \mathbb{1} \right)^{-1} \left(x^* x \right) \left(p \left(\mu q - \mathbb{1} \right) + \mathbb{1} \right)^{-1} \\
&= \mu^2 \left(\mu q - \mathbb{1} + \mathbb{1} \right)^{-1} q^2 \left(\mu q - \mathbb{1} + \mathbb{1} \right)^{-1} \\
&= \mu^2 \left(\mu q \right)^{-1} q^2 \left(\mu q \right)^{-1} \\
&= \mu^2 \left(\mu q \right)^{-2} q^2 \\
&= \mu^2 \mu^{-2} q^{-2} q^2 \\
&= \mathbb{1},
\end{aligned}$$

$$\begin{aligned}
uq &= \mu x \left(p \left(\mu \left(x^* x \right)^{\frac{1}{2}} - \mathbb{1} \right) + \mathbb{1} \right)^{-1} q \\
&= \mu x \left(p \left(\mu q - \mathbb{1} \right) + \mathbb{1} \right)^{-1} q \\
&= \mu x \left(\mu q - \mathbb{1} + \mathbb{1} \right)^{-1} q \\
&= \mu x \left(\mu q \right)^{-1} q \\
&= \mu \mu^{-1} x q^{-1} q \\
&= x,
\end{aligned}$$

and

$$
\begin{aligned}
uu^* &= \mu x\left(p\left(\mu\left(x^*x\right)^{\frac{1}{2}}-\mathbb{1}\right)+\mathbb{1}\right)^{-1}\cdot\mu\left(p\left(\mu\left(x^*x\right)^{\frac{1}{2}}-\mathbb{1}\right)+\mathbb{1}\right)^{-1}x^* \\
&= \mu^2 x\left(p\left(\mu\left(x^*x\right)^{\frac{1}{2}}-\mathbb{1}\right)+\mathbb{1}\right)^{-2}x^* \\
&= \mu^2 x\left(p\left(\mu q-\mathbb{1}\right)+\mathbb{1}\right)^{-2}x^* \\
&= \mu^2 x\left(\mu q-\mathbb{1}+\mathbb{1}\right)^{-2}x^* \\
&= \mu^2 x\left(\mu q\right)^{-2}x^* \\
&= \mu^2\mu^{-2}xq^{-2}x^* \\
&= xq^{-2}x^* \\
&= x\left(x^*x\right)^{-1}x^* \\
&= xx^{-1}\left(x^*\right)^{-1}x^* \\
&= \mathbb{1}
\end{aligned}
$$

so C*-relation-adding Tietze transformations yield

$$
\begin{aligned}
\mathcal{I} &\cong_{\mathbf{1C^*}} \left\langle \begin{array}{c}(x,\lambda),\\ (q,\lambda),(u,\lambda\mu)\end{array} \middle| \begin{array}{c}\mu^2x^*x\geq\mathbb{1},\mu^2xx^*\geq\mathbb{1},q=\left(x^*x\right)^{\frac{1}{2}},\\ u=\mu x\left(p\left(\mu\left(x^*x\right)^{\frac{1}{2}}-\mathbb{1}\right)+\mathbb{1}\right)^{-1},\\ \mathbb{1}\leq\mu q\end{array}\right\rangle_{\mathbf{1C^*}} \\
&\cong_{\mathbf{1C^*}} \left\langle \begin{array}{c}(x,\lambda),\\ (q,\lambda),(u,\lambda\mu)\end{array} \middle| \begin{array}{c}\mu^2x^*x\geq\mathbb{1},\mu^2xx^*\geq\mathbb{1},q=\left(x^*x\right)^{\frac{1}{2}},\\ u=\mu x\left(p\left(\mu\left(x^*x\right)^{\frac{1}{2}}-\mathbb{1}\right)+\mathbb{1}\right)^{-1},\\ \mathbb{1}\leq\mu q, uu^*=\mathbb{1}\end{array}\right\rangle_{\mathbf{1C^*}} \\
&\cong_{\mathbf{1C^*}} \left\langle \begin{array}{c}(x,\lambda),\\ (q,\lambda),(u,\lambda\mu)\end{array} \middle| \begin{array}{c}\mu^2x^*x\geq\mathbb{1},\mu^2xx^*\geq\mathbb{1},q=\left(x^*x\right)^{\frac{1}{2}},\\ u=\mu x\left(p\left(\mu\left(x^*x\right)^{\frac{1}{2}}-\mathbb{1}\right)+\mathbb{1}\right)^{-1},\\ \mathbb{1}\leq\mu q, u^*u=uu^*=\mathbb{1}\end{array}\right\rangle_{\mathbf{1C^*}}
\end{aligned}
$$

$$\cong_{\mathbf{1C^*}} \left\langle \begin{array}{c} (x,\lambda), \\ (q,\lambda),(u,\lambda\mu) \end{array} \middle| \begin{array}{c} \mu^2 x^* x \geq \mathbb{1}, \mu^2 x x^* \geq \mathbb{1}, q = (x^* x)^{\frac{1}{2}}, \\ u = \mu x \left(p \left(\mu (x^* x)^{\frac{1}{2}} - \mathbb{1} \right) + \mathbb{1} \right)^{-1}, \\ \mathbb{1} \leq \mu q, u^* u = u u^* = \mathbb{1}, x = uq \end{array} \right\rangle_{\mathbf{1C^*}}.$$

Assuming only $x = uq$, $u^*u = uu^* = \mathbb{1}$, and $p(\mu q - \mathbb{1}) = \mu q - \mathbb{1}$,

$$\begin{aligned} (x^*x)^{\frac{1}{2}} &= ((uq)^* uq)^{\frac{1}{2}} \\ &= (qu^*uq)^{\frac{1}{2}} \\ &= \left(q^2\right)^{\frac{1}{2}} \\ &= q, \end{aligned}$$

$$\begin{aligned} p\left(\mu^2 x^* x - \mathbb{1}\right) &= p(\mu^2 q^2 - \mathbb{1}) \\ &= \mu^2 q^2 - \mathbb{1} \\ &= \mu^2 x^* x - \mathbb{1}, \end{aligned}$$

and

$$\begin{aligned} \mu x \left(p \left(\mu (x^* x)^{\frac{1}{2}} - \mathbb{1} \right) + \mathbb{1} \right)^{-1} &= \mu x \left(p \left(\mu ((uq)^* uq)^{\frac{1}{2}} - \mathbb{1} \right) + \mathbb{1} \right)^{-1} \\ &= \mu uq \left(p \left(\mu (qu^* uq)^{\frac{1}{2}} - \mathbb{1} \right) + \mathbb{1} \right)^{-1} \\ &= \mu uq \left(p \left(\mu \left(q^2\right)^{\frac{1}{2}} - \mathbb{1} \right) + \mathbb{1} \right)^{-1} \\ &= \mu uq \left(p \left(\mu q - \mathbb{1} \right) + \mathbb{1} \right)^{-1} \\ &= \mu uq \left(\mu q - \mathbb{1} + \mathbb{1} \right)^{-1} \\ &= \mu uq \left(\mu q \right)^{-1} \\ &= \mu uq \mu^{-1} q^{-1} \\ &= u. \end{aligned}$$

Since u is unitary, note that

$$\mathbb{1} = uu^* \leq \mu^2 uq^2 u^*$$

so

$$\begin{aligned} p\left(\mu^2 xx^* - \mathbb{1}\right) &= p\left(\mu^2 (uq)(uq)^* - \mathbb{1}\right) \\ &= p\left(\mu^2 uqqu^* - \mathbb{1}\right) \\ &= p\left(u(\mu q)^2 u^* - \mathbb{1}\right) \\ &= p\left(\mu^2 uq^2 u^* - \mathbb{1}\right) \\ &= \mu^2 uq^2 u^* - \mathbb{1} \\ &= \mu^2 (uq)(uq)^* - \mathbb{1} \\ &= \mu^2 xx^* - \mathbb{1}. \end{aligned}$$

Therefore, C*-relation-removing Tietze transformations have

$$\begin{aligned} \mathcal{I} &\cong_{\mathbf{1C^*}} \left\langle \begin{array}{c} (x,\lambda), \\ (q,\lambda),(u,\lambda\mu) \end{array} \middle| \begin{array}{c} \mu^2 xx^* \geq \mathbb{1}, q = (x^*x)^{\frac{1}{2}}, \\ u = \mu x \left(p\left(\mu \left(x^*x\right)^{\frac{1}{2}} - \mathbb{1}\right) + \mathbb{1}\right)^{-1}, \\ \mathbb{1} \leq \mu q, u^*u = uu^* = \mathbb{1}, x = uq \end{array} \right\rangle_{\mathbf{1C^*}} \\ &\cong_{\mathbf{1C^*}} \left\langle \begin{array}{c} (x,\lambda), \\ (q,\lambda),(u,\lambda\mu) \end{array} \middle| \begin{array}{c} q = (x^*x)^{\frac{1}{2}}, \\ u = \mu x \left(p\left(\mu \left(x^*x\right)^{\frac{1}{2}} - \mathbb{1}\right) + \mathbb{1}\right)^{-1}, \\ \mathbb{1} \leq \mu q, u^*u = uu^* = \mathbb{1}, x = uq \end{array} \right\rangle_{\mathbf{1C^*}} \\ &\cong_{\mathbf{1C^*}} \left\langle \begin{array}{c} (x,\lambda), \\ (q,\lambda),(u,\lambda\mu) \end{array} \middle| \begin{array}{c} u = \mu x \left(p\left(\mu \left(x^*x\right)^{\frac{1}{2}} - \mathbb{1}\right) + \mathbb{1}\right)^{-1}, \\ \mathbb{1} \leq \mu q, u^*u = uu^* = \mathbb{1}, x = uq \end{array} \right\rangle_{\mathbf{1C^*}} \\ &\cong_{\mathbf{1C^*}} \left\langle \begin{array}{c} (x,\lambda), \\ (q,\lambda),(u,\lambda\mu) \end{array} \middle| \mathbb{1} \leq \mu q, u^*u = uu^* = \mathbb{1}, x = uq \right\rangle_{\mathbf{1C^*}}. \end{aligned}$$

As

$$1 = \|\mathbb{1}\|_{\mathcal{I}} = \|u^*u\|_{\mathcal{I}} = \|u\|_{\mathcal{I}}^2,$$

the following results.

$$\|x\|_{\mathcal{I}} = \|uq\|_{\mathcal{I}} \leq \|u\|_{\mathcal{I}}\|q\|_{\mathcal{I}} \leq \lambda$$

Thus, a generator-removing Tietze transformation and use of the unital free product yield

$$\begin{aligned}
\mathcal{I} &\cong_{\mathbf{1C^*}} \langle (q,\lambda),(u,\lambda\mu)\,|\,\mathbb{1} \leq \mu q, u^*u = uu^* = \mathbb{1}\rangle_{\mathbf{1C^*}} \\
&\cong_{\mathbf{1C^*}} \langle (q,\lambda)\,|\,\mathbb{1} \leq \mu q\rangle_{\mathbf{1C^*}} *_{\mathbb{C}} \langle (u,\lambda\mu)\,|\,u^*u = uu^* = \mathbb{1}\rangle_{\mathbf{1C^*}} \\
&\cong_{\mathbf{1C^*}} C\left[\frac{1}{\mu},\lambda\right] *_{\mathbb{C}} C(\mathbb{T}) \\
&\cong_{\mathbf{1C^*}} \begin{cases} C(\mathbb{T}), & \lambda\mu = 1, \\ C[0,1] *_{\mathbb{C}} C(\mathbb{T}), & \lambda\mu > 1, \end{cases}
\end{aligned}$$

recalling Examples 3.5.3 and 3.11.5.

In summary,

$$\mathcal{I} \cong_{\mathbf{1C^*}} \begin{cases} \mathbb{0}, & \lambda\mu < 1, \\ C(\mathbb{T}), & \lambda\mu = 1, \\ C\,[0,1] *_{\mathbb{C}} C(\mathbb{T}), & \lambda\mu > 1. \end{cases}$$

This resembles the result in the commutative cases, exchanging the types of coproducts. Again, the polar decomposition is demonstrated, splitting the generator into its positive and unitary parts. However, these two components need not commute, necessitating the free product between them.

3.11.4 C*-algebra of a One-Sided Invertible

Following the Tietze calculations as the previous example, one can ignore the use of either of the C*-relations $\mu^2 xx^* \geq \mathbb{1}$ or $\mu^2 x^*x \geq \mathbb{1}$ to yield the following isomorphisms.

Example 3.11.9 (A single left- or right-invertible). For $\lambda, \mu \in [0, \infty)$,

$$\left\langle (x, \lambda) \,\middle|\, \mu^2 x^* x \geq \mathbb{1} \right\rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} \begin{cases} \mathbb{0}, & \lambda\mu < 1, \\ \mathcal{T}, & \lambda\mu = 1, \\ C\,[0,1] *_{\mathbb{C}} \mathcal{T}, & \lambda\mu > 1, \end{cases}$$

where $\mathcal{T}$ denotes the Toeplitz algebra from Example 3.5.4. Similarly,

$$\left\langle (x, \lambda) \,\middle|\, \mu^2 x x^* \geq \mathbb{1} \right\rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} \begin{cases} \mathbb{0}, & \lambda\mu < 1, \\ \mathcal{T}, & \lambda\mu = 1, \\ C\,[0,1] *_{\mathbb{C}} \mathcal{T}, & \lambda\mu > 1. \end{cases}$$

The characterizations of a left, right, or true invertible heavily depended on the ability to demonstrate the positive part of the generator invertible. This allowed the partially isometric part to be isolated within the C*-algebra, where it could be then manipulated.

3.12 Example: Idempotency

Example 3.5.5 considered a projection. In this section, one considers a general idempotent. Specifically, let

$$\mathcal{A} := \left\langle (x, \lambda) \,\middle|\, x = x^2 \right\rangle_{\mathbf{1C^*}},$$

the unital C*-algebra of a single idempotent element of norm at most λ. Since x is not assumed to be normal, this algebra is likely not to be commutative. To classify this algebra, attention turns to unital *-representations of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$.

Let $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a unital *-representation of $\mathcal{A}$. Then, $\pi(x)^2 = \pi(x^2) = \pi(x)$ and $\|\pi(x)\|_{\mathcal{B}(\mathcal{H})} \leq \|x\|_{\mathcal{A}} \leq \lambda$.

Let $E \in \mathcal{B}(\mathcal{H})$ satisfy that $E^2 = E$ and $\|E\|_{\mathcal{B}(\mathcal{H})} \leq \lambda$. Then, by Theorem 3.3.2, there is a unique unital *-homomorphism $\rho_E : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ such that $\rho_E(x) = E$.

Thus, unital *-representations of $\mathcal{A}$ on $\mathcal{H}$ are in one-to-one correspondence with idempotents of norm at most λ in $\mathcal{B}(\mathcal{H})$.

3.12.1 Properties of Idempotents

Consider an operator $E \in \mathcal{B}(\mathcal{H})$ satisfying $E^2 = E$ and $\|E\|_{\mathcal{B}(\mathcal{H})} \leq \lambda$. To describe these types of operators, recall the following definition and well-known results.

Definition (Complementary subspaces, [9]). Two closed subspaces $\mathcal{M}, \mathcal{N} \subseteq \mathcal{H}$ are *complementary* if $\mathcal{M} + \mathcal{N} = \mathcal{H}$ and $\mathcal{M} \cap \mathcal{N} = \{0\}$.

Proposition 3.12.1 (Major Properties of Idempotents, [9]). *Consider $E \in \mathcal{B}(\mathcal{H})$.*

1. *E is idempotent iff $\mathbb{1} - E$ is idempotent.*

2. *If E is idempotent,* $\operatorname{ran}(E) = \ker(\mathbb{1} - E)$. *In particular,* $\operatorname{ran}(E)$ *is closed.*

3. *If E is idempotent,* $\operatorname{ran}(E)$ *and* $\ker(E)$ *are complementary.*

Theorem 3.12.2 (Specifying Kernel and Range, [9]). *For two complementary closed subspaces $\mathcal{M}, \mathcal{N} \subseteq \mathcal{H}$, there exists a unique idempotent $E \in \mathcal{B}(\mathcal{H})$ such that* $\operatorname{ran}(E) = \mathcal{M}$ *and* $\ker(E) = \mathcal{N}$. *In particular, $E(m + n) = m$ for all $m \in \mathcal{M}$ and $n \in \mathcal{N}$.*

Thus, idempotents are in one-to-one correspondence to pairs of complementary closed subspaces. However, these two statements have no connection to the norm of the idempotent operator. To complete the description of idempotents, recall the following definition.

Definition (Dixmier Angle, [12])**.** Given two subspaces, $\mathcal{M}, \mathcal{N} \subseteq \mathcal{H}$, the *Dixmier angle* or *minimum angle* between $\mathcal{M}$ and $\mathcal{N}$ is

$$\theta(\mathcal{M}, \mathcal{N}) := \arccos\left(\sup\left\{|\langle m, n\rangle_{\mathcal{H}}| : m \in \mathcal{M}, n \in \mathcal{N}, \|m\|_{\mathcal{H}} = \|n\|_{\mathcal{H}} = 1\right\}\right).$$

This is one of many notions of an "angle" in operator theoretic literature, but in particular, this notion is intimately related to the orthogonal projections onto each subspace.

Proposition 3.12.3 (Norm of the Product of Two Projections, [11])**.** *Given two projections $P, Q \in \mathcal{B}(\mathcal{H})$, let $\mathcal{M} := \operatorname{ran}(P)$ and $\mathcal{N} := \operatorname{ran}(Q)$. Then,*

$$\|PQ\|_{\mathcal{B}(\mathcal{H})} = \cos(\theta(\mathcal{M}, \mathcal{N})).$$

This norm of a product is then related to the norm of an idempotent in the following way. Let $P_{\mathcal{K}} : \mathcal{H} \to \mathcal{K}$ be the orthogonal projection of $\mathcal{H}$ onto a closed subspace $\mathcal{K}$ and $\mathbb{1}_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}$ the identity map on $\mathcal{K}$.

Theorem 3.12.4 (Norm of an Idempotent, [26])**.** *Given a nonzero idempotent operator $E \in \mathcal{B}(\mathcal{H})$, let $\mathcal{M} := \operatorname{ran}(E)$ and $\mathcal{N} := \ker(E)$. Then,*

$$\|E\|_{\mathcal{B}(\mathcal{H})} = \frac{1}{\sqrt{1 - \|P_{\mathcal{M}} P_{\mathcal{N}}\|^2_{\mathcal{B}(\mathcal{H})}}} = \csc\left(\theta(\mathcal{M}, \mathcal{N})\right).$$

3.12.2 Irreducible Idempotent Operators

With the general facts of idempotents at the ready, consider an irreducible idempotent operator $E \in \mathcal{B}(\mathcal{H})$. That is, E has no reducing subspace. By Theorem 3.12.2, E is determined uniquely by $\mathcal{M} := \text{ran}(E)$ and $\mathcal{N} := \ker(E)$.

First, there are two trivial cases. If $\mathcal{M} = \{0\}$, $E = 0$, meaning $\dim(\mathcal{H}) = 0$. If $\mathcal{N} = \{0\}$, $E = \mathbb{1}$, meaning $\dim(\mathcal{H}) = 1$.

Consider when $\mathcal{M}, \mathcal{N} \neq \{0\}$. Recall that the operator matrix

$$V := \begin{bmatrix} P_{\mathcal{M}} \\ P_{\mathcal{M}^\perp} \end{bmatrix}$$

is a unitary from $\mathcal{H}$ to $\mathcal{M} \oplus \mathcal{M}^\perp$. Then, defining $A := P_{\mathcal{M}} E P^*_{\mathcal{M}^\perp}$, conjugation by V yields

$$E \sim_V \begin{bmatrix} \mathbb{1}_{\mathcal{M}} & A \\ 0 & 0 \end{bmatrix}.$$

If an operator matrix $T \in \mathcal{B}(\mathcal{M} \oplus \mathcal{M}^\perp)$ commutes with VEV^*, observe that

$$\begin{aligned} (VEV^*)T &= T(VEV^*), \\ \begin{bmatrix} \mathbb{1}_{\mathcal{M}} & A \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X & Y \\ Z & W \end{bmatrix} &= \begin{bmatrix} X & Y \\ Z & W \end{bmatrix} \begin{bmatrix} \mathbb{1}_{\mathcal{M}} & A \\ 0 & 0 \end{bmatrix}, \\ \begin{bmatrix} X + AZ & Y + AW \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} X & XA \\ Z & ZA \end{bmatrix}, \end{aligned}$$

forcing $Z = 0$ and $Y = XA - AW$. If T is a projection, $XA = AW$, and X, W are projections.

Consider now the operator A. Using the polar decomposition, $A = UP$, where $U : \mathcal{M}^\perp \to \mathcal{M}$ is a partial isometry and $P : \mathcal{M}^\perp \to \mathcal{M}^\perp$ is positive such that $\ker(U) = \ker(P)$. Observe that the operator matrix

$$Q := \begin{bmatrix} UU^* & 0 \\ 0 & U^*U \end{bmatrix}$$

is a projection, and

$$UU^*A = UU^*UP = UP = A = UP = UPU^*U = AU^*U.$$

Hence, Q commutes with VEV^*, meaning $\operatorname{ran}(Q)$ is a reducing subspace of VEV^*. Thus, either $\operatorname{ran}(Q) = \{0\}$ or $\mathcal{M} \oplus \mathcal{M}^\perp$.

If $\operatorname{ran}(Q) = \mathcal{M} \oplus \mathcal{M}^\perp$, then $UU^* = \mathbb{1}_{\mathcal{M}}$ and $U^*U = \mathbb{1}_{\mathcal{M}^\perp}$. Therefore, U is a unitary, meaning the matrix

$$\hat{U} := \begin{bmatrix} U^* & 0 \\ 0 & \mathbb{1}_{\mathcal{M}^\perp} \end{bmatrix}$$

is a unitary from $\mathcal{M} \oplus \mathcal{M}^\perp$ to $\mathcal{M}^\perp \oplus \mathcal{M}^\perp$. In this case, conjugation by $\hat{U}$ yields

$$E \sim_{\hat{U}V} \begin{bmatrix} \mathbb{1}_{\mathcal{M}^\perp} & P \\ 0 & 0 \end{bmatrix}.$$

Given any projection $R \in \mathcal{B}(\mathcal{M}^\perp)$, then the operator

$$\hat{R} := \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}$$

commutes with $\hat{U}VEV^*\hat{U}^*$ if and only if $RP = PR$, when $\operatorname{ran}(R)$ reduces P. By the

Gelfand theory, P is irreducible if and only if $\dim(\mathcal{M}^\perp) = 1$. Hence,

$$E \sim_{\hat{U}V} \begin{bmatrix} 1 & \mu \\ 0 & 0 \end{bmatrix} \in M_2$$

for some $\mu \in [0, \infty)$. Notice that this matrix is irreducible if and only if $\mu \neq 0$.

If $\operatorname{ran}(Q) = \{0\}$, then $UU^* = 0$. Hence, $A = 0$, which resolves to the $\mu = 0$ case above.

Hence, the irreducible idempotent operators are precisely

$$\begin{bmatrix} 1 & \mu \\ 0 & 0 \end{bmatrix} \in M_2$$

for $\mu \neq 0$ and $1 \in \mathbb{C}$. Notice that the irreducible idempotents are at most 2-dimensional, not unlike the irreducible representations of Pedersen's C*-algebra of two projections, presented in [32].

3.12.3 Connection to Pedersen's Two-Projection Algebra

As demonstrated in Proposition 3.12.1 and Theorem 3.12.2, an idempotent is intimately tied to its kernel and range. Moreover, Proposition 3.12.3 and Theorem 3.12.4 reinforce this connection via the norms of the idempotent and the orthogonal projections, both connected to the Dixmier angle.

There are also algebraic connections between the two. In particular, an idempotent operator can be reconstructed from its kernel and range projections in the following way.

Theorem 3.12.5 (Formula for an Idempotent, [39])**.** *Given an idempotent operator $E \in \mathcal{B}(\mathcal{H})$, let $R, K \in \mathcal{B}(\mathcal{H})$ be the orthogonal projections onto its range and kernel,* *respectively. Then,*

$$E = (\mathbb{1} - RKR)^{-1}(R - RK).$$

In the reverse direction, the range projection can be recovered from the idempotent in a similar fashion.

Proposition 3.12.6 (Formula for the Range Projection, [10])**.** *Given an idempotent operator $E \in \mathcal{B}(\mathcal{H})$, let $R \in \mathcal{B}(\mathcal{H})$ be the orthogonal projection onto its range. Then,*

$$R = EE^* \left(\mathbb{1} + (E - E^*)^* (E - E^*)\right)^{-1}.$$

By Proposition 3.12.1, the kernel projection can be obtained by applying this result to $\mathbb{1} - E$. Using these two formulae, Tietze transformations will be used to characterize the unital C*-algebra of a single idempotent.

Example 3.12.7 (An idempotent). Let

$$\mathcal{A} := \left\langle (x, \lambda) \,\middle|\, x = x^2 \right\rangle_{\mathbf{1C^*}}.$$

To make use of the results about idempotent operators, let $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be the universal representation of this algebra. Then, $\pi(x)$ is an idempotent operator.

First, consider the trivial case when $\lambda < 1$. In this case, Proposition 3.12.4 states that $\pi(x) = 0$. Hence, $x = 0$, and $\mathcal{A} \cong_{\mathbf{1C^*}} \mathbb{C}$.

Otherwise, consider when $\lambda \geq 1$. Note that $x \neq 0$ as there are nontrivial representations given in the previous subsection. In this case, the algebra will be rewritten completely in terms of two projections, the kernel and range of x. To that end, observe that

$$\mathbb{1} \;\leq\; \mathbb{1} + (x - x^*)^* (x - x^*)$$

so $(\mathbb{1} + (x - x^*)^* (x - x^*))^{-1} \in C^*(\mathbb{1}, x)$. Define new generators

$$r := xx^* (\mathbb{1} + (x - x^*)^* (x - x^*))^{-1}$$

and

$$k := (\mathbb{1} - x)(\mathbb{1} - x)^* (\mathbb{1} + (x^* - x)^* (x^* - x))^{-1}.$$

By Proposition 3.12.6, $\pi(r)$ and $\pi(k)$ are its range and kernel projections of $\pi(x)$, and Theorem 3.12.4 gives that

$$\begin{aligned} \lambda &\geq \|x\|_{\mathcal{A}} \\ &= \|\pi(x)\|_{\mathcal{B}(\mathcal{H})} \\ &= \frac{1}{\sqrt{1 - \|\pi(rk)\|_{\mathcal{B}(\mathcal{H})}^2}} \\ &= \frac{1}{\sqrt{1 - \|rk\|_{\mathcal{A}}^2}}. \end{aligned}$$

Thus,

$$\begin{aligned} \lambda^{-1} &\leq \sqrt{1 - \|rk\|_{\mathcal{A}}^2}, \\ \lambda^{-2} &\leq 1 - \|rk\|_{\mathcal{A}}^2, \\ \|rk\|_{\mathcal{A}}^2 &\leq 1 - \lambda^{-2}, \\ \|rk\|_{\mathcal{A}} &\leq \sqrt{1 - \lambda^{-2}}. \end{aligned}$$

Also, Theorem 3.12.5 states that

$$\pi(x) = (\pi(\mathbb{1} - rkr))^{-1} \pi(r - rk)$$

However, note that $\mathbb{1} - rkr$ may not be invertible before quotienting. To incorporate this C*-relation, define $f_\lambda : [0,1] \to \mathbb{C}$ by

$$f_\lambda(\nu) := \begin{cases} \nu, & 0 \leq \nu \leq \sqrt{1-\lambda^{-2}}, \\ \dfrac{\sqrt{1-\lambda^{-2}}}{\sqrt{1-\lambda^{-2}}-1}(\nu - 1), & \sqrt{1-\lambda^{-2}} < \mu \leq 1, \end{cases}$$

a continuous function. By the continous functional calculus in $\mathcal{A}$,

$$\begin{aligned} \left(\mathbb{1} - f_\lambda\left(rk^*kr^*\right)\right)^{-1}(r - rk) &= \left(\mathbb{1} - rk^*kr^*\right)^{-1}(r - rk) \\ &= \left(\mathbb{1} - rk^2r\right)^{-1}(r - rk) \\ &= \left(\mathbb{1} - rkr\right)^{-1}(r - rk), \end{aligned}$$

the former of which exists for any unital C*-algebra elements r, k with norms bounded by 1. Further, as π is faithful,

$$\left(\mathbb{1} - f_\lambda\left(rk^*kr^*\right)\right)^{-1}(r - rk) = \left(\mathbb{1} - rkr\right)^{-1}(r - rk) = x.$$

Generator-adding Tietze transformations yield

$$\begin{aligned} \mathcal{A} &\cong_{\mathbf{1C^*}} \left\langle \begin{array}{c} (x,\lambda), \\ (r,1) \end{array} \middle| \begin{array}{c} x = x^2, \\ r = xx^*\left(\mathbb{1} + (x - x^*)^*(x - x^*)\right)^{-1} \end{array} \right\rangle_{\mathbf{1C^*}} \\ &\cong_{\mathbf{1C^*}} \left\langle \begin{array}{c} (x,\lambda), \\ (r,1),(k,1) \end{array} \middle| \begin{array}{c} x = x^2, \\ r = xx^*\left(\mathbb{1} + (x - x^*)^*(x - x^*)\right)^{-1}, \\ k = (\mathbb{1} - x)(\mathbb{1} - x)^*\left(\mathbb{1} + (x^* - x)^*(x^* - x)\right)^{-1} \end{array} \right\rangle_{\mathbf{1C^*}}, \end{aligned}$$

and C*-relation-adding Tietze transformations give

$$\mathcal{A} \cong_{\mathbf{1C^*}} \left\langle \begin{array}{c} (x,\lambda), \\ (r,1),(k,1) \end{array} \middle| \begin{array}{c} x = x^2, r^2 = r \\ r = xx^* \left(\mathbb{1} + (x-x^*)^* (x-x^*)\right)^{-1}, \\ k = (\mathbb{1}-x)(\mathbb{1}-x)^* \left(\mathbb{1} + (x^*-x)^* (x^*-x)\right)^{-1} \end{array} \right\rangle_{\mathbf{1C^*}}$$

$$\cong_{\mathbf{1C^*}} \left\langle \begin{array}{c} (x,\lambda), \\ (r,1),(k,1) \end{array} \middle| \begin{array}{c} x = x^2, r^2 = r^* = r \\ r = xx^* \left(\mathbb{1} + (x-x^*)^* (x-x^*)\right)^{-1}, \\ k = (\mathbb{1}-x)(\mathbb{1}-x)^* \left(\mathbb{1} + (x^*-x)^* (x^*-x)\right)^{-1} \end{array} \right\rangle_{\mathbf{1C^*}}$$

$$\cong_{\mathbf{1C^*}} \left\langle \begin{array}{c} (x,\lambda), \\ (r,1),(k,1) \end{array} \middle| \begin{array}{c} x = x^2, r^2 = r^* = r, k^2 = k \\ r = xx^* \left(\mathbb{1} + (x-x^*)^* (x-x^*)\right)^{-1}, \\ k = (\mathbb{1}-x)(\mathbb{1}-x)^* \left(\mathbb{1} + (x^*-x)^* (x^*-x)\right)^{-1} \end{array} \right\rangle_{\mathbf{1C^*}}$$

$$\cong_{\mathbf{1C^*}} \left\langle \begin{array}{c} (x,\lambda), \\ (r,1),(k,1) \end{array} \middle| \begin{array}{c} x = x^2, r^2 = r^* = r, k^2 = k^* = k \\ r = xx^* \left(\mathbb{1} + (x-x^*)^* (x-x^*)\right)^{-1}, \\ k = (\mathbb{1}-x)(\mathbb{1}-x)^* \left(\mathbb{1} + (x^*-x)^* (x^*-x)\right)^{-1} \end{array} \right\rangle_{\mathbf{1C^*}}$$

$$\cong_{\mathbf{1C^*}} \left\langle \begin{array}{c} (x,\lambda), \\ (r,1),(k,1) \end{array} \middle| \begin{array}{c} x = x^2, r^2 = r^* = r, k^2 = k^* = k, \|rk\| \leq \sqrt{1-\lambda^{-2}} \\ r = xx^* \left(\mathbb{1} + (x-x^*)^* (x-x^*)\right)^{-1}, \\ k = (\mathbb{1}-x)(\mathbb{1}-x)^* \left(\mathbb{1} + (x^*-x)^* (x^*-x)\right)^{-1} \end{array} \right\rangle_{\mathbf{1C^*}}$$

$$\cong_{\mathbf{1C^*}} \left\langle \begin{array}{c} (x,\lambda), \\ (r,1),(k,1) \end{array} \middle| \begin{array}{c} x = x^2, r^2 = r^* = r, k^2 = k^* = k, \|rk\| \leq \sqrt{1-\lambda^{-2}} \\ r = xx^* \left(\mathbb{1} + (x-x^*)^* (x-x^*)\right)^{-1}, \\ k = (\mathbb{1}-x)(\mathbb{1}-x)^* \left(\mathbb{1} + (x^*-x)^* (x^*-x)\right)^{-1}, \\ x = \left(\mathbb{1} - f_\lambda\left(rk^*kr^*\right)\right)^{-1}(r-rk) \end{array} \right\rangle_{\mathbf{1C^*}} .$$

Working in reverse, $\pi(r), \pi(k)$ are projections in $\mathcal{B}(\mathcal{H})$, whose ranges are complementary by Proposition 3.12.3. By Theorem 3.12.5,

$$\begin{aligned}\pi(x) &= \pi\left((\mathbb{1}-rkr)^{-1}(r-rk)\right)\\ &= \pi\left(\left(\mathbb{1}-rk^2r\right)^{-1}(r-rk)\right)\\ &= \pi\left((\mathbb{1}-rk^*kr^*)^{-1}(r-rk)\right)\\ &= \pi\left((\mathbb{1}-f_\lambda(rk^*kr^*))^{-1}(r-rk)\right),\end{aligned}$$

is the unique idempotent with range $\pi(r)$ and kernel $\pi(k)$. Proposition 3.12.6 then requires that

$$\pi(r) = \pi\left(xx^*\left(\mathbb{1}+(x-x^*)^*(x-x^*)\right)^{-1}\right)$$

and

$$\pi(k) = \pi\left((\mathbb{1}-x)(\mathbb{1}-x)^*\left(\mathbb{1}+(x^*-x)^*(x^*-x)\right)^{-1}\right).$$

Since π is faithful, C*-relation-removing Tietze transformations reveal

$$\begin{aligned}\mathcal{A} &\cong_{\mathbf{1C^*}} \left\langle \begin{matrix}(x,\lambda),\\ (r,1),(k,1)\end{matrix} \;\middle|\; \begin{matrix} r^2=r^*=r,\, k^2=k^*=k,\, \|rk\|\le\sqrt{1-\lambda^{-2}}\\ r = xx^*\left(\mathbb{1}+(x-x^*)^*(x-x^*)\right)^{-1},\\ k=(\mathbb{1}-x)(\mathbb{1}-x)^*\left(\mathbb{1}+(x^*-x)^*(x^*-x)\right)^{-1},\\ x=(\mathbb{1}-f_\lambda(rk^*kr^*))^{-1}(r-rk)\end{matrix}\right\rangle_{\mathbf{1C^*}}\\ &\cong_{\mathbf{1C^*}} \left\langle \begin{matrix}(x,\lambda),\\ (r,1),(k,1)\end{matrix} \;\middle|\; \begin{matrix} r^2=r^*=r,\, k^2=k^*=k,\, \|rk\|\le\sqrt{1-\lambda^{-2}}\\ k=(\mathbb{1}-x)(\mathbb{1}-x)^*\left(\mathbb{1}+(x^*-x)^*(x^*-x)\right)^{-1},\\ x=(\mathbb{1}-f_\lambda(rk^*kr^*))^{-1}(r-rk)\end{matrix}\right\rangle_{\mathbf{1C^*}}\\ &\cong_{\mathbf{1C^*}} \left\langle \begin{matrix}(x,\lambda),\\ (r,1),(k,1)\end{matrix} \;\middle|\; \begin{matrix} r^2=r^*=r,\, k^2=k^*=k,\, \|rk\|\le\sqrt{1-\lambda^{-2}}\\ x=(\mathbb{1}-f_\lambda(rk^*kr^*))^{-1}(r-rk)\end{matrix}\right\rangle_{\mathbf{1C^*}}.\end{aligned}$$

Lastly, Theorem 3.12.4 ensures that

$$\begin{aligned} \sqrt{1-\lambda^{-2}} &\geq \|rk\|_{\mathcal{A}} \\ &= \|\pi(rk)\|_{\mathcal{B}(\mathcal{H})} \\ &= \sqrt{1-\|\pi(x)\|_{\mathcal{B}(\mathcal{H})}^{-2}} \\ &= \sqrt{1-\|x\|_{\mathcal{A}}^{-2}} \end{aligned}$$

so $\|x\|_{\mathcal{A}} \leq \lambda$. A final generator-removing Tietze transformation gives

$$\mathcal{A} \cong_{\mathbf{1C^*}} \left\langle (r,1),(k,1) \,\middle|\, r^2 = r^* = r, k^2 = k^* = k, \|rk\| \leq \sqrt{1-\lambda^{-2}} \right\rangle_{\mathbf{1C^*}}.$$

Now that $\mathcal{A}$ has been written as a C*-algebra of two projections, the result of [32] is invoked. Specifically,

$$\mathcal{A} \cong_{\mathbf{1C^*}} \begin{bmatrix} C(X) & C_0(X \setminus \{0,1\}) \\ C_0(X \setminus \{0,1\}) & C(X) \end{bmatrix},$$

where $X := \sigma_{\mathcal{A}}(rkr)$. Since

$$\|rkr\|_{\mathcal{A}} = \left\|rk^2r\right\|_{\mathcal{A}} = \|rkk^*r^*\|_{\mathcal{A}} = \|rk\|_{\mathcal{A}}^2 \leq 1-\lambda^{-2}$$

and $rkr \geq 0$, $X \subseteq \left[0, 1-\lambda^{-2}\right]$. For $\alpha \in \left[0, 1-\lambda^{-2}\right]$, let $\mu := \sqrt{\dfrac{\alpha}{1-\alpha}}$,

$$r_\mu := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$k_\mu := \begin{bmatrix} \dfrac{\mu^2}{\mu^2+1} & \dfrac{-\mu}{\mu^2+1} \\ \dfrac{-\mu}{\mu^2+1} & \dfrac{1}{\mu^2+1} \end{bmatrix}$$

in M_2. A routine arithmetic check shows that both r_μ and k_μ are projections, the range and kernel projections of the idempotent matrix

$$E_\mu := \begin{bmatrix} 1 & \mu \\ 0 & 0 \end{bmatrix}.$$

Note that

$$\|E_\mu\|_{M_2} = \sqrt{1+\mu^2}$$

so by Theorem 3.12.4,

$$\|r_\mu k_\mu\|_{M_2} = \frac{\mu}{\sqrt{\mu^2+1}} = \sqrt{\alpha} \leq \sqrt{1-\lambda^{-2}}.$$

By Theorem 3.3.2, there is a unique $\phi_\mu : \mathcal{A} \to M_2$ such that $\phi_\mu(r) = r_\mu$ and $\phi_\mu(k) = k_\mu$. Observe that

$$\phi_\mu(rkr) = r_\mu k_\mu r_\mu = \begin{bmatrix} \dfrac{\mu^2}{\mu^2+1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}$$

so $\sigma_{\mathcal{A}}(pqp) \supseteq \sigma_{M_2}(r_\mu k_\mu r_\mu) = \{0, \alpha\}$. Therefore, $\sigma_{\mathcal{A}}(rkr) = \left[0, 1-\lambda^{-2}\right]$. Hence,

$$\mathcal{A} \cong_{\mathbf{1C^*}} \begin{bmatrix} C\left[0,1-\lambda^{-2}\right] & C_0\left(0,1-\lambda^{-2}\right] \\ C_0\left(0,1-\lambda^{-2}\right] & C\left[0,1-\lambda^{-2}\right] \end{bmatrix}$$

$$\cong_{\mathbf{1C^*}} \begin{cases} \mathbb{C}\oplus\mathbb{C}, & \lambda = 1, \\ \begin{bmatrix} C[0,1] & C_0(0,1] \\ C_0(0,1] & C[0,1] \end{bmatrix}, & \lambda > 1, \end{cases}$$

as $\left[0,1-\lambda^{-2}\right] \cong_{\mathbf{Top}} [0,1]$ for all $\lambda > 1$.

In summary,

$$\mathcal{A} \cong_{\mathbf{1C^*}} \begin{cases} \mathbb{C}, & \lambda < 1, \\ \mathbb{C}\oplus\mathbb{C}, & \lambda = 1, \\ \begin{bmatrix} C[0,1] & C_0(0,1] \\ C_0(0,1] & C[0,1] \end{bmatrix}, & \lambda > 1. \end{cases}$$

3.13 Example: Meet and Join of Projections

In Section 3.12, a formula for an idempotent was given in terms of its kernel and range projections. This formula guarantees that in the unital C*-algebra generated by these two projections, the idempotent will arise. Further, a set of C*-relations was given to ensure that the meet of two projections was trivial, the norm of their product strictly below 1.

Since there is a way to trivialize the meet of two projections, is there a way to directly manipulate the meet or join? That is, does a formula exist in terms of the two projections involved and C*-algebraic operations on them for the meet and join?

Unfortunately, this is not so, and this is supported by initial intuition. On Hilbert space, the usual means to compute the meet or join of two projections is to use an infimum or a supremum. Considering C*-algebras as non-commutative analogs of continuous function algebras, this immediately seems questionable as infima and suprema are not continuous operations.

To demonstrate this fact, first recall the characterization of the unital C*-algebra of two projections. Let $\mathcal{F} := \langle (a,1),(b,1)|\emptyset\rangle_{\mathbf{1C^*}}$,

$$\mathcal{P} := \left\langle (a,1),(b,1)|a=a^2=a^*, b=b^2=b^*\right\rangle_{\mathbf{1C^*}},$$

and $\varphi : \mathcal{F} \to \mathcal{P}$ the quotient map. From Example 3.10.2,

$$\mathcal{P} \cong_{\mathbf{1C^*}} \mathcal{C} := \begin{bmatrix} C[0,1] & C_0(0,1) \\ C_0(0,1) & C[0,1] \end{bmatrix},$$

and an isomorphism $\phi : \mathcal{P} \to \mathcal{C}$ is given by

$$a \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad b \mapsto \begin{bmatrix} \lambda & \left(\lambda - \lambda^2\right)^{\frac{1}{2}} \\ \left(\lambda - \lambda^2\right)^{\frac{1}{2}} & 1-\lambda \end{bmatrix}.$$

Let $p, q : [0,1] \to M_2$ by $p(\lambda) := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $q(\lambda) := \begin{bmatrix} \lambda & \left(\lambda - \lambda^2\right)^{\frac{1}{2}} \\ \left(\lambda - \lambda^2\right)^{\frac{1}{2}} & 1-\lambda \end{bmatrix}$ so $\phi(a) = p$ and $\phi(b) = q$.

Next, observe that domain restrictions of these matrix-valued functions yield nonconstant meets and joins. Fix $\alpha \in [0,1)$. Let $\mathcal{C}_\alpha := C\left([0,\alpha] \cup \{1\}, M_2\right)$ and

$z_\alpha, w_\alpha : [0,\alpha] \cup \{1\} \to M_2$ by

$$z_\alpha(\lambda) := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad w_\alpha(\lambda) := \begin{bmatrix} \lambda & \left(\lambda - \lambda^2\right)^{\frac{1}{2}} \\ \left(\lambda - \lambda^2\right)^{\frac{1}{2}} & 1 - \lambda \end{bmatrix}.$$

Notice that $z_\alpha, w_\alpha \in \mathcal{C}_\alpha$ and are projections. For $\lambda \in [0,\alpha]$, $z_\alpha(\lambda)$ and $w_\alpha(\lambda)$ are 1-dimensional, non-colinear projections so $z_\alpha(\lambda) \wedge_{M_2} w_\alpha(\lambda) = 0$ and $z_\alpha(\lambda) \vee_{M_2} w_\alpha(\lambda) = \mathbb{1}$. Also, $z_\alpha(1) = w_\alpha(1)$ so

$$z_\alpha(1) \vee_{M_2} w_\alpha(1) = z_\alpha(1) \wedge_{M_2} w_\alpha(1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

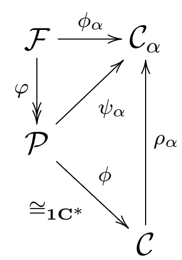
Define $m_\alpha, j_\alpha : [0,\alpha] \cup \{1\} \to M_2$ by

$$m_\alpha(\lambda) := \begin{cases} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, & \lambda \in [0,\alpha], \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, & \lambda = 1, \end{cases}, \quad j_\alpha(\lambda) := \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \lambda \in [0,\alpha], \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, & \lambda = 1, \end{cases}.$$

Notice that $m_\alpha, j_\alpha \in \mathcal{C}_\alpha$, $m_\alpha = z_\alpha \wedge_{\mathcal{C}_\alpha} w_\alpha$, and $j_\alpha = z_\alpha \vee_{\mathcal{C}_\alpha} w_\alpha$.

Lastly, the universal properties of $\mathcal{P}$ and $\mathcal{F}$ are used to bind these different C*-algebras together. By Theorem 3.2.1, there is a unique unital *-homomorphism $\phi_\alpha : \mathcal{F} \to \mathcal{C}_\alpha$ by $\phi_\alpha(a) := z_\alpha$ and $\phi_\alpha(b) := w_\alpha$. Also, $a^2 - a, a^* - a, b^2 - b, b^* - b \in \ker(\phi_\alpha)$ so by the universal property of the quotient, there is a unique $\psi_\alpha : \mathcal{P} \to \mathcal{C}_\alpha$ such that $\psi_\alpha \circ \varphi = \phi_\alpha$. Let $\rho_\alpha : \mathcal{C} \to \mathcal{C}_\alpha$ by domain restriction, a well-known unital

*-homomorphism.



Observe that

$$(\rho_\alpha \circ \phi)(a) = \rho_\alpha(p) = z_\alpha = \phi_\alpha(a)$$

and

$$(\rho_\alpha \circ \phi)(b) = \rho_\alpha(q) = w_\alpha = \phi_\alpha(b)$$

so by Theorem 3.3.2, $\rho_\alpha \circ \phi = \phi_\alpha$.

Now, the nonexistence of a "universal" meet or join can be shown.

Example 3.13.1 (Failure of Meet and Join). For purposes of contradiction, assume that there is $m \in \mathcal{F}$ such that for all unital *-homomorphisms $\Phi : \mathcal{F} \to \mathcal{A}$ satisfying that $\Phi(a)$ and $\Phi(b)$ are projections, $\Phi(m) = \Phi(a) \wedge_{\mathcal{A}} \Phi(b)$. Then,

$$\phi_\alpha(m) = \phi_\alpha(a) \wedge_{\mathcal{C}_\alpha} \phi_\alpha(b) = z_\alpha \wedge_{\mathcal{C}_\alpha} w_\alpha = m_\alpha$$

for all $\alpha \in [0, 1)$. Also,

$$\phi_\alpha(m) = (\psi_\alpha \circ \varphi)(m) = (\rho_\alpha \circ \phi \circ \varphi)(m)$$

so for all $\alpha \in [0, 1)$,

$$(\phi \circ \varphi)(m)(\alpha) = (\rho_\alpha \circ \phi \circ \varphi)(m)(\alpha) = m_\alpha(\alpha) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$(\phi \circ \varphi)(m)(1) = (\rho_\alpha \circ \phi \circ \varphi)(m)(1) = m_\alpha(1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, $(\phi \circ \varphi)(m)$ is not continuous, contradicting that $\mathcal{C}$ consisted of continuous functions. Hence, m cannot exist.

Similarly, assume that there is $j \in \mathcal{F}$ such that for all unital *-homomorphisms $\Phi : \mathcal{F} \to \mathcal{A}$ satisfying $\Phi(a)$ and $\Phi(b)$ are projections, $\Phi(j) = \Phi(a) \vee_{\mathcal{A}} \Phi(b)$. Then,

$$\phi_\alpha(j) = \phi_\alpha(a) \vee_{\mathcal{C}_\alpha} \phi_\alpha(b) = z_\alpha \vee_{\mathcal{C}_\alpha} w_\alpha = j_\alpha$$

for all $\alpha \in [0, 1)$. Also,

$$\phi_\alpha(j) = (\psi_\alpha \circ \varphi)(j) = (\rho_\alpha \circ \phi \circ \varphi)(j)$$

so for all $\alpha \in [0, 1)$,

$$(\phi \circ \varphi)(j)(\alpha) = (\rho_\alpha \circ \phi \circ \varphi)(j)(\alpha) = j_\alpha(\alpha) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$(\phi \circ \varphi)(j)(1) = (\rho_\alpha \circ \phi \circ \varphi)(j)(1) = j_\alpha(1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, $(\phi \circ \varphi)(j)$ is not continuous, contradicting that $\mathcal{C}$ consisted of continuous functions. Hence, j cannot exist.

Notice that this lack of continuity is arising within the primitive ideal space $[0, 1]$ of $\mathcal{C}$ as demonstrated in [32]. Similarly in Example 3.11.4, the failure to characterize the condition "x has an inverse" arose due to the topology of the spectrum. This

Unital C*-algebra	Presentation	Section
$\mathbb{O}$	$\langle (x,0)\,\vert 0 = \mathbb{1}\rangle_{\mathbf{1C^*}}$	3.5.3
$\mathbb{C}$	$\langle (x,0)\,\vert \emptyset\rangle_{\mathbf{1C^*}}$	3.5.1
$C[0,1]$	$\langle (x,1)\,\vert x = x^*\rangle_{\mathbf{1C^*}}$	3.5.2
$C(\mathbb{T})$	$\langle (x,1)\,\vert x^*x = xx^* = \mathbb{1}\rangle_{\mathbf{1C^*}}$	3.5.3
$\mathcal{T}$	$\langle (x,1)\,\vert x^*x = \mathbb{1}\rangle_{\mathbf{1C^*}}$	3.5.4

Table 3.1: Some Finitely Presented Unital C*-algebras

suggests that failure of existence for C*-relations may be directly connected to a topological issue with the spectrum of a particular C*-algebra.

3.14 Property: Generation & Separability in $\mathbf{1C^*}$

Since each unital C*-algebra has a presentation by Example 3.3.1, one can ask if there is a "simplest" presentation that yields that algebra. In particular, one notion of simplicity for a presentation is control on the number of generators and relations.

All of the examples seen thus far have been finitely presented, many shown in Table 3.1. Those which are not shown are combinations or quotients of these. In fact, all of them have a presentation with a single generator. In a sense, these are some of the most foundational C*-algebras due to the Jordan and polar decompositions.

However, given a unital C*-algebra, how simple can a presentation for it be? Can one control the number of generators or C*-relations used?

In actuality, this question of minimal generation can be used to characterize a very commonly assumed property of C*-algebras, topological separability. The proof is precisely the same reasoning stated in [4], modified appropriately to the crutched set context.

Proposition 3.14.1. *Given a unital C*-algebra $\mathcal{A}$, $\mathcal{A}$ is separable if and only if $\mathcal{A}$ is countably generated in* $\mathbf{1C^*}$.

Proof. ($\Rightarrow$) Assume that $\mathcal{A}$ is separable. Then, there is a countable, dense $S \subseteq \mathcal{A}$. Define $\phi : S \to \mathcal{A}$ by $\phi(s) := s$, the usual inclusion. By Corollary 3.2.2, there is a unique unital *-homomorphism $\hat{\phi} : \langle S, f_\phi | \emptyset \rangle_{\mathbf{1C^*}} \to \mathcal{A}$ such that $\hat{\phi}(s) = s$, where $f_\phi(s) = \|s\|_{\mathcal{A}}$. Thus, $S \subseteq \operatorname{ran}\left(\hat{\phi}\right)$, but note that *-homomorphisms are contractive and, therefore, have closed range. Then, $\hat{\phi}$ is surjective. Letting $K := \ker\left(\hat{\phi}\right)$,

$$\mathcal{A} \cong_{\mathbf{1C^*}} \langle S, f_\phi | K \rangle_{\mathbf{1C^*}}.$$

($\Leftarrow$) Assume that $\mathcal{A} = \langle S, f | R \rangle_{\mathbf{1C^*}}$ for some countable set S. Let $\mathcal{F} := \langle S, f | \emptyset \rangle_{\mathbf{1C^*}}$ and $q : \mathcal{F} \to \mathcal{A}$ the quotient map. By the construction of $\mathcal{F}$ in Section 3.1, non-commuting unital *-polynomials $(\mathbb{Q} + \imath\mathbb{Q})[S_f]$ are norm-dense in $\mathcal{F}$, and by a standard counting argument, $(\mathbb{Q} + \imath\mathbb{Q})[S_f]$ is countable. Hence, $\mathcal{F}$ is separable. As q is contractive and surjective, the image of $(\mathbb{Q} + \imath\mathbb{Q})[S_f]$ is countable and dense in $\mathcal{A}$. $\square$

With this characterization, any inseparable unital C*-algebra cannot be realized with a countable number of generators. Similarly, there are unital C*-algebras which are separable but cannot be realized with a finite number of generators, even commutative ones. To show this, the following characterization is proven.

Proposition 3.14.2. *Let X be a compact, Hausdorff space.*

1. *$C(X)$ is finitely generated in $\mathbf{1C^*}$ iff there is a continuous, one-to-one function $\alpha : X \to \prod_{j=1}^{n} \overline{\mathbb{D}}$ for some $n \in \mathbb{N}$.*
2. *$C(X)$ is countably generated in $\mathbf{1C^*}$ iff there is a continuous, one-to-one function $\alpha : X \to \prod_{j\in\mathbb{N}} \overline{\mathbb{D}}$.*

Proof. 1. ($\Rightarrow$) Assume that $C(X)$ is finitely generated. Then, there is a finite crutched set $(S,f) = (x_j,\lambda_j)_{j=1}^n$ and set of C*-relations R on (S,f) such that $\langle (x_1,\lambda_1),\ldots,(x_n,\lambda_n)|R\rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} C(X)$. Let $\mathcal{F} := \langle S,f|\emptyset\rangle_{\mathbf{1C^*}}$ and $q : \mathcal{F} \to C(X)$ be the quotient map. Observe that for each $1 \leq j,k \leq n$, $x_jx_k - x_kx_j, x_jx_k^* - x_k^*x_j \in \ker(q)$. Let

$$\mathcal{A} := \left\langle (x_j,\lambda_j)_{j=1}^n \left| x_jx_k = x_kx_j, x_jx_k^* = x_k^*x_j \forall 1 \leq j,k \leq n \right.\right\rangle_{\mathbf{1C^*}}.$$

Recall from Theorems 3.4.1 and 3.4.4,

$$\mathcal{A} \cong_{\mathbf{C1C^*}} \mathrm{Ab}_1(\mathcal{F}) \cong_{\mathbf{C1C^*}} C\left(\prod_{\{j:\lambda_j>0\}} \overline{\mathbb{D}}\right).$$

By the universal property of the quotient, there is a unique unital *-homomorphism $\hat{q} : \mathcal{A} \to C(X)$ such that $\hat{q}(x_j) = q(x_j)$. Further, as q was surjective, so is $\hat{q}$. Thus, applying the maximal ideal space functor $\Delta : \mathbf{C1C^*} \to \mathbf{Comp}$,

$$X \xrightarrow{\Delta(\hat{q})} \Delta(\mathcal{A}).$$

Letting $\phi : \prod_{\{j:\lambda_j>0\}} \overline{\mathbb{D}} \to \Delta(\mathcal{A})$ be the canonical homeomorphism, $\alpha := \phi^{-1} \circ \Delta(\hat{q})$ is a continuous embedding.

($\Leftarrow$) Assume that there is a one-to-one, continuous function $\alpha : X \to \prod_{j=1}^n \overline{\mathbb{D}}$. Application of the functor $C : \mathbf{Comp} \to \mathbf{C1C^*}$ yields the following.

$$C\left(\prod_{j=1}^n \overline{\mathbb{D}}\right) \xrightarrow{C(\alpha)} C(X)$$

Note that

$$\mathcal{B} := \langle (x_1, 1), \ldots, (x_n, 1) | x_j x_k = x_k x_j, x_j x_k^* = x_k^* x_j \forall 1 \leq j, k \leq n \rangle_{\mathbf{1C^*}}$$

is a presentation for the domain of $C(\alpha)$ by Theorems 3.4.1 and 3.4.4. Let $\mathcal{G} := \langle (x_1, 1), \ldots, (x_n, 1) | \emptyset \rangle_{\mathbf{1C^*}}$, $\rho : \mathcal{G} \to \mathcal{B}$ be the quotient map, and $\Gamma_{\mathcal{B}} : \mathcal{B} \to C\left(\prod_{j=1}^{n} \overline{\mathbb{D}}\right)$ be the Gelfand *-isomorphism. Then, $C(\alpha) \circ \Gamma_{\mathcal{B}} \circ \rho : \mathcal{F} \to C(X)$ is surjective. Letting $K := \ker\left(C(\alpha) \circ \Gamma_{\mathcal{B}} \circ \rho\right)$,

$$C(X) \cong_{\mathbf{1C^*}} \left\langle (x_1, 1), \ldots, (x_n, 1) \middle| K \right\rangle_{\mathbf{1C^*}}.$$

2. An identical argument proves the equivalence for countably generated case.

□

Recall the following sequence of homeomorphisms:

$$\prod_{j \in \mathbb{N}} \overline{\mathbb{D}} \cong_{\mathbf{Comp}} \prod_{j \in \mathbb{N}} [0, 1]^2 \cong_{\mathbf{Comp}} \prod_{j \in \mathbb{N} \uplus \mathbb{N}} [0, 1] \cong_{\mathbf{Comp}} \prod_{j \in \mathbb{N}} [0, 1].$$

Observe that this product is metrizable, the core of Urysohn's metrization theorem. Combining Urysohn's metrization theorem with the above characterization gives a quick proof to the following well-known result.

Corollary 3.14.3. *For a compact, Hausdorff space X, the following are equivalent.*

1. *$C(X)$ is countably generated in $\mathbf{1C^*}$.*

2. *$C(X)$ is separable.*

3. *X is metrizable.*

With these characterizations, the following examples show the distinctions between the countability of generators.

Example 3.14.4 (Uncountable versus countable). By Corollary 3.14.3, $C\left(\prod_{\lambda\in\mathbb{R}}[0,1]\right)$ is not countably nor finitely generated as $\prod_{\lambda\in\mathbb{R}}[0,1]$ is known to be non-metrizable.

The author would like to thank Dr. Susan Hermiller for the argument using embeddings of balls in the example below.

Example 3.14.5 (Countable versus finite). Consider then $C\left(\prod_{t\in\mathbb{N}}[0,1]\right)$, which is countably generated by Corollary 3.14.3. Let $Z := \prod_{t\in\mathbb{N}}[0,1]$ and $Z_n := \prod_{t=1}^{n}[0,1]$. Recall that there is a natural embedding $\beta_n : Z_n \to Z$ by

$$\beta_n(f)(k) := \begin{cases} f(k), & 1 \leq k \leq n, \\ 0 & k > n, \end{cases}$$

for each $n \in \mathbb{N}$. Notice that each β_n is one-to-one and continuous. Similarly, the maps $\gamma_n^m : Z_n \to Z_m$ by

$$\gamma_n^m(f)(k) := \begin{cases} f(k), & 1 \leq k \leq n, \\ 0 & m \geq k > n, \end{cases}$$

for $m \geq n$ are also one-to-one and continuous.

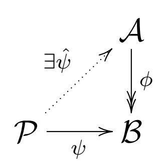
For purposes of contradiction, assume that there is an embedding $\alpha : Z \to Z_n$ for some $n \in \mathbb{N}$. Then, $\gamma_n^{2n} \circ \alpha \circ \beta_{4n}$ would be a one-to-one map, embedding Z_{4n} into Z_{2n}. However, this is known to be impossible to embed a dimension $2n$-ball into a n-ball in this way. Thus, Z cannot be embedded into any Z_n. Therefore, $C(Z)$ is countably generated, but not finitely generated by Proposition 3.14.2.

To consider countably or finitely related C*-algebras, one must consider generation of the kernel of the quotient map from a scaled-free unital C*-algebra. Since the kernel is generally non-unital, this discussion will be set aside for now.

3.15 Property: Projectivity & Liftability in $\mathbf{1C}^*$

Proposition 3.14.1 demonstrated that one property of unital C*-algebras can be captured completely by a quality of its presentation. In this section, another such property, a type of projectivity, is characterized in terms of a property of the C*-relations in play, accompanied by motivating examples.

To be clear, the projectivity being characterized here is projectivity with respect to the class of surjective, unital *-homomorphisms in $\mathbf{1C}^*$. Explicitly, a unital C*-algebra $\mathcal{P}$ is projective in this sense if given any unital *-homomorphism $\psi : \mathcal{P} \to \mathcal{B}$ and any surjective unital *-homomorphism $\phi : \mathcal{A} \to \mathcal{B}$, there is a unital *-homomorphism $\tilde{\psi} : \mathcal{P} \to \mathcal{A}$ such that $\psi = \phi \circ \hat{\psi}$. This is shown with the commutative diagram below in $\mathbf{1C}^*$.



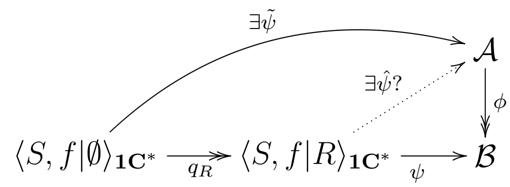
Be aware also that the map factorization above need not be unique like a universal property would require.

Observe that in the diagram above, $\mathcal{B} \cong_{\mathbf{1C}^*} \mathcal{A}/\ker(\phi)$, with ϕ acting as the quotient map. Thus, to test this flavor of projectivity, one need only consider a unital C*-algebra $\mathcal{A}$, an norm-closed, two-sided ideal J in $\mathcal{A}$, and the quotient map $q : \mathcal{A} \to \mathcal{A}/J$.

Recall that Proposition 3.2.6 stated that for any crutched set (S, f), $\langle S, f | \emptyset \rangle_{\mathbf{1C}^*}$

is projective with respect to all surjections in $\mathbf{1C^*}$. Further, Example 3.3.1 implies that $\mathbf{1C^*}$ has enough projectives with respect to surjections in $\mathbf{1C^*}$.

Let R be C*-relations on (S, f) and $q_R : \langle S, f|\emptyset\rangle_{\mathbf{1C^*}} \to \langle S, f|R\rangle_{\mathbf{1C^*}}$ be the quotient map. Given a unital C*-algebras $\mathcal{A}$ and $\mathcal{B}$, let $\phi : \mathcal{A} \to \mathcal{B}$ be a surjective unital *-homomorphism and $\psi : \langle S, f|R\rangle_{\mathbf{1C^*}} \to \mathcal{B}$ a unital *-homomorphism. By Proposition 3.2.6, there is a unital *-homomorphism $\tilde{\psi} : \langle S, f|\emptyset\rangle_{\mathbf{1C^*}} \to \mathcal{A}$ such that $\phi \circ \tilde{\psi} = \psi \circ q_R$. However, when can one find a $\hat{\psi} : \langle S, f|R\rangle_{\mathbf{1C^*}} \to \mathcal{A}$ such that $\phi \circ \hat{\psi} = \psi$?



If one could guarantee a factorization of $\tilde{\psi}$ via q_R, this would always occur. In [27], the notion of "liftable relations" is discussed and shown to be the solution to this question in the existing presentation theory. Here, the author makes this definition precise for the crutched set situation.

Definition. For a crutched set (S, f), a set of C*-relations R on (S, f) is *liftable* in $\mathbf{1C^*}$ if for any unital C*-algebras $\mathcal{A}, \mathcal{B}$, any surjective unital *-homomorphism $\phi : \mathcal{A} \to \mathcal{B}$, and any unital *-homomorphism $\rho : \langle S, f|\emptyset\rangle_{\mathbf{1C^*}} \to \mathcal{B}$ such that $R \subseteq \ker(\rho)$, there is a unital *-homomorphism $\hat{\rho} : \langle S, f|\emptyset\rangle_{\mathbf{1C^*}} \to \mathcal{A}$ such that $\phi \circ \hat{\rho} = \rho$ and $R \subseteq \ker(\hat{\rho})$. The map $\hat{\rho}$ is called a *lift* of ρ along ϕ.

Much like Theorem 3.3.2, this definition is verbose, but the notion behind it is intuitive. Given a choice of elements in $\mathcal{B}$ where C*-relations R "evaluate" to 0, one can find lifts in $\mathcal{A}$ along ϕ for each element that together also "evaluate" R to 0. Notice that content of the definition is indeed in the ability to find $\hat{\rho}$ such that $R \subseteq \ker(\hat{\rho})$ since Proposition 3.2.6 guarantees existence of a map from $\langle S, f|\emptyset\rangle_{\mathbf{1C^*}}$ to

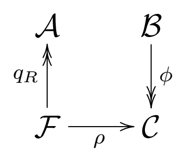
$\mathcal{A}$ completing the triangle.

Mirroring [27], liftability of C*-relations is the correct notion for a given presentation to be projective relative to all surjections in $\mathbf{1C^*}$.

Proposition 3.15.1. *Given a crutched set (S, f), C*-relations R on (S, f) are liftable in $\mathbf{1C^*}$ if and only if $\langle S, f | R \rangle_{\mathbf{1C^*}}$ is projective relative to all surjections in $\mathbf{1C^*}$.*

Proof. Let $\mathcal{F} := \langle S, f | \emptyset \rangle_{\mathbf{1C^*}}$, $\mathcal{A} := \langle S, f | R \rangle_{\mathbf{1C^*}}$, and $q_R : \mathcal{F} \to \mathcal{A}$ the quotient map.

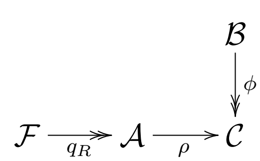
($\Leftarrow$) Let $\mathcal{B}$ and $\mathcal{C}$ be unital C*-algebras and $\phi : \mathcal{B} \to \mathcal{C}$ a surjective unital *-homomorphism. Given a unital *-homomorphism $\rho : \mathcal{F} \to \mathcal{C}$ with $R \subseteq \ker(\rho)$, consider the diagram below.



Then, $J_R \subseteq \ker(\rho)$ so by the universal property of the quotient, there is a unique unital *-homomorphism $\hat{\rho} : \mathcal{A} \to \mathcal{C}$ such that $\rho = \hat{\rho} \circ q_R$. As $\mathcal{A}$ is projective with respect to surjections, there is a *-homomorphism $\psi : \mathcal{A} \to \mathcal{B}$ such that $\hat{\rho} = \phi \circ \psi$. Hence, $\pi := \psi \circ q_R : \mathcal{F} \to \mathcal{A}$ is a unital *-homomorphism satisfying $R \subseteq \ker(q_R) \subseteq \ker(\pi)$. Also,

$$\phi \circ \pi = \phi \circ \psi \circ q_R = \hat{\rho} \circ q_R = \rho.$$

($\Rightarrow$) Let $\mathcal{B}$ and $\mathcal{C}$ be unital C*-algebras and $\phi : \mathcal{B} \to \mathcal{C}$ a surjective unital *-homomorphism. Given a unital *-homomorphism $\rho : \mathcal{A} \to \mathcal{C}$, consider the following diagram.



As R is liftable, there is a unital *-homomorphism $\psi : \mathcal{F} \to \mathcal{B}$ such that $\phi \circ \psi = \rho \circ q_R$

and $R \subseteq \ker(\psi)$. By the universal property of the quotient, there is a unique unital *-homomorphism $\hat{\psi} : \mathcal{A} \to \mathcal{B}$ such that $\psi = \hat{\psi} \circ q_R$. Observe that

$$\phi \circ \hat{\psi} \circ q_R = \phi \circ \psi = \rho \circ q_R$$

so by Theorem 3.3.2, $\phi \circ \hat{\psi} = \rho$.

□

Given this characterization, one can consider the examples already available. The results of this section's examples are summarized in Table 3.2.

Example 3.15.2 ($\mathbb{C}$). Recall that $\mathbb{C} \cong_{\mathbf{1C^*}} \langle (x, 0) \,|\, \emptyset \rangle_{\mathbf{1C^*}}$ so $\mathbb{C}$ is projective relative to all surjections in $\mathbf{1C^*}$ by Proposition 3.2.6.

Example 3.15.3 (An idempotent). For $\lambda \geq 1$, the unital C*-algebra

$$\mathcal{A}_\lambda := \left\langle (x, \lambda) \,\middle|\, x = x^2 \right\rangle_{\mathbf{1C^*}}$$

is not projective relative to all surjections in $\mathbf{1C^*}$. To show this, consider the unital *-homomorphism $q : C[0,1] \to \mathbb{C} \oplus \mathbb{C}$ by $q(f) := f(0) \oplus f(1)$. Let $\mathcal{F}_\lambda := \langle (x, \lambda) | \emptyset \rangle_{\mathbf{1C^*}}$. Observe that $0 \oplus 1$ is an idempotent in $\mathbb{C} \oplus \mathbb{C}$ of norm 1 so there is a unique unital *-homomorphism $\rho : \mathcal{F}_\lambda \to \mathbb{C} \oplus \mathbb{C}$ by $\rho(x) := 0 \oplus 1$. Also, $x - x^2 \in \ker(\rho)$.

Observe that all $f \in q^{-1}(0 \oplus 1)$ satisfy $f(0) = 0$ and $f(1) = 1$ so as each f is continuous, $\operatorname{ran}(f) \supseteq [0,1]$ by the Intermediate Value Theorem. However, given any idempotent $g \in C[0,1]$, $\operatorname{ran}(g) = \sigma_{C[0,1]}(g) \subseteq \{0,1\}$. Since g is continuous and $[0,1]$ connected, $\operatorname{ran}(g) = \{0\}$ or $\{1\}$. Hence, the additive and multiplicative identities are the only idempotents in $C[0,1]$, and neither of these is a pre-image of $0 \oplus 1$.

Thus, there is no choice of idempotent in $C[0,1]$ in the pre-image of $0 \oplus 1$ so there is no lift of ρ along q. Hence, $\{x - x^2\}$ is not a set of liftable C*-relations on $\{(x, \lambda)\}$.

By Proposition 3.15.1, $\mathcal{A}_\lambda$ is not projective with respect to all surjections in $\mathbf{1C}^*$.

Example 3.15.4 (Self-adjoint, Positive). For $\lambda \geq 0$, the unital C*-algebra

$$\mathcal{B}_\lambda := \langle (x, \lambda) \,|x = x^* \rangle_{\mathbf{1C}^*} \cong_{\mathbf{1C}^*} C[-\lambda, \lambda]$$

is projective relative to all surjections in $\mathbf{1C}^*$. To show this, let $\mathcal{F}_\lambda := \langle (x, \lambda) | \emptyset \rangle_{\mathbf{1C}^*}$, $\mathcal{A}$ and $\mathcal{B}$ be unital C*-algebras, and $\phi : \mathcal{A} \to \mathcal{B}$. Consider a unital *-homomorphism from $\rho : \mathcal{F}_\lambda \to \mathcal{B}$ such that $x - x^* \in \ker(\rho)$.

By Theorem 3.2.6, there is a unital *-homomorphism $\psi : \mathcal{F}_\lambda \to \mathcal{A}$ such that $\rho = \phi \circ \psi$. Let $a := \Re(\psi(x))$. Then,

$$\begin{aligned}
\phi(a) &= \frac{1}{2}(\phi \circ \psi)(x) + \frac{1}{2}(\phi \circ \psi)(x^*) \\
&= \frac{1}{2}\rho(x) + \frac{1}{2}\rho(x^*) \\
&= \frac{1}{2}\rho(x) + \frac{1}{2}\rho((x^* - x) + x) \\
&= \frac{1}{2}\rho(x) + \frac{1}{2}\rho(x^* - x) + \frac{1}{2}\rho(x) \\
&= \rho(x) + \frac{1}{2}(0) \\
&= \rho(x)
\end{aligned}$$

and

$$\begin{aligned}
\|a\|_{\mathcal{A}} &\leq \frac{1}{2}\|\psi(x)\|_{\mathcal{A}} + \frac{1}{2}\|\psi(x)^*\|_{\mathcal{A}} \\
&= \frac{1}{2}\|\psi(x)\|_{\mathcal{A}} + \frac{1}{2}\|\psi(x)\|_{\mathcal{A}} \\
&= \|\psi(x)\|_{\mathcal{A}} \\
&\leq \|x\|_{\mathcal{F}_\lambda} \\
&\leq \lambda.
\end{aligned}$$

Defining $\varphi : \{(x,\lambda)\} \to F^{\mathbf{CSet_1}}_{\mathbf{1C^*}}\mathcal{A}$ by $\varphi(x) := a$, φ is a constrictive map. By Theorem 3.2.1, there is a unique unital *-homomorphism $\hat{\varphi} : \mathcal{F}_\lambda \to \mathcal{A}$ such that $\varphi(x) = a$. Further, note that

$$\varphi(x - x^*) = a - a^* = a - a = 0.$$

Hence, $x - x^* \in \ker(\hat{\varphi})$ so $\{x - x^*\}$ is liftable on $\{(x,\lambda)\}$. By Proposition 3.15.1, $\mathcal{B}_\lambda$ is projective with respect to all surjections in $\mathbf{1C^*}$.

Also, observe that $\langle (x,\lambda) | x \geq 0 \rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} C[0,\lambda] \cong_{\mathbf{1C^*}} C[-\lambda,\lambda]$ as $[0,\lambda] \cong_{\mathbf{Top}} [-\lambda,\lambda]$. Thus, by Proposition 3.15.1, $\{x - p(\Re(x))\}$ is liftable on $\{(x,\lambda)\}$, where $p : \mathbb{R} \to \mathbb{R}$ by

$$p(\mu) := \begin{cases} 0, & \mu < 0, \\ \mu, & \mu \geq 0. \end{cases}$$

As summarized in Proposition B.4.8, a unital C*-algebra $\mathcal{P}$ being projective with respect to all surjections in $\mathbf{1C^*}$ implies that its abelianization $\mathrm{Ab}_1(\mathcal{P})$ is projective with respect to all surjections in $\mathbf{C1C^*}$. In turn, the maximal ideal space $\Delta(\mathrm{Ab}_1(\mathcal{P}))$ must be injective with respect to all one-to-one maps in **Comp**, an *absolute retract*. As such, many consider projectivity with respect to surjections in $\mathbf{1C^*}$ to be the non-commutative analog of absolute retracts, like [27].

Example 3.15.5 (Unitary, Isometry, Coisometry). For $\lambda \geq 1$,

$$\langle (x,\lambda) \,|\, x^*x = xx^* = \mathbb{1} \rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} C(\mathbb{T})$$

is not projective with respect to all surjections in $\mathbf{C1C^*}$, let alone $\mathbf{1C^*}$. Assuming to the contrary, then $\mathbb{T}$ would be an absolute retract. Hence, as $\mathbb{T} \subset \overline{\mathbb{D}}$, $\mathbb{T}$ would be a retract of $\overline{\mathbb{D}}$. However, this is well-known to be false by an argument by fundamental groups.

Extending this, the Toeplitz algebra has the following presentations.

$$\mathcal{T} \cong_{\mathbf{1C^*}} \langle (x,\lambda) \,|\, x^*x = \mathbb{1} \rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} \langle (x,\lambda) \,|\, xx^* = \mathbb{1} \rangle_{\mathbf{1C^*}}$$

It too is not projective with respect to all surjections in $\mathbf{1C^*}$ since

$$\mathrm{Ab}_1(\mathcal{T}) \cong_{\mathbf{C1C^*}} C(\mathbb{T})$$

by Theorem 3.4.4 and Proposition B.4.8. By Proposition 3.15.1, $\{x^*x - \mathbb{1}\}$, $\{xx^* - \mathbb{1}\}$, and $\{x^*x - \mathbb{1}, xx^* - \mathbb{1}\}$ are not liftable on $\{(x,\lambda)\}$.

Example 3.15.6 (Finite-Dimensional, Commutative). For $n \geq 2$, the unital C*-algebra $\mathbb{C}^n$ is not projective with respect to all surjections in $\mathbf{C1C^*}$, let alone $\mathbf{1C^*}$. To show this, notice that $\Delta\left(\mathbb{C}^n\right) \cong_{\mathbf{Comp}} [n] := \{1, \ldots, n\}$, a finite discrete space. Let $\alpha : [2] \to [n]$ by $\alpha(1) := 1$ and $\alpha(2) := 2$. This function is automatically continuous as both $[n]$ and $[2]$ are discrete. Consider then the following diagram in $\mathbf{Comp}$,

$$\begin{array}{ccc} [n] & & \\ {\scriptstyle\alpha}\uparrow & & \\ [2] & \underset{\iota}{\rightarrowtail} & [1,2] \end{array}$$

where $\iota : [2] \to [1,2]$ is the inclusion of the two point space $[2] := \{1,2\}$ into the continuous interval $[1,2]$. If $[n]$ were injective with respect to one-to-one functions in $\mathbf{Comp}$, there would be a continuous map $\hat{\alpha} : [1,2] \to [n]$ extending α. Hence, $\mathrm{ran}\left(\hat{\alpha}\right) \supseteq \{1,2\}$, but as $[1,2]$ is connected, this is not possible. Thus, $[n]$ does not have this type of injectivity.

Similarly, $\mathbb{0}$ is not projective with respect to all surjections in $\mathbf{C1C^*}$ or $\mathbf{1C^*}$. In this case, note that $\Delta(\mathbb{O}) = \emptyset$. Consider the following diagram in **Comp**,

$$\begin{array}{ccc} \emptyset & & \\ \uparrow {\scriptstyle id_{\emptyset}} & & \\ \emptyset & \xrightarrow[\mathbf{0}_{[1]}]{} & [1] \end{array}$$

where $\mathbf{0}_{[1]} : \emptyset \to [1]$ is the empty function into the one point space. Since there are no functions from $[1]$ to $\emptyset$, continuous or otherwise, $\emptyset$ cannot have this type of injectivity.

Thus, $\mathbb{C}$ is the only commutative finite-dimensional C*-algebra that is projective with respect to all surjections in $\mathbf{1C}^*$.

Example 3.15.7 (Normality). Given a crutched set (S, f), consider the unital C*-algebra

$$\mathcal{A} := \langle S, f \,| st = ts, s^*t = ts^* \forall s, t \in S \rangle_{\mathbf{1C}^*} \cong_{\mathbf{1C}^*} C\left(\prod_{s \notin f^{-1}(0)} \overline{\mathbb{D}}\right).$$

In its native category $\mathbf{C1C}^*$, $\mathcal{A}$ is projective with respect to surjections found there. This follows directly from Proposition 3.2.6, Theorem 3.4.4, and Proposition B.4.8, which give

$$\mathcal{A} \cong_{\mathbf{1C}^*} \mathrm{Ab}_1 \left(\langle S, f | \emptyset \rangle_{\mathbf{1C}^*}\right).$$

As a result, $\prod_{\lambda \in \Lambda} \overline{\mathbb{D}}$ is an absolute retract for all index sets Λ.

However, $\mathcal{A}$ is not projective relative to all surjections in $\mathbf{1C}^*$ unless f is constant 0, in which case $\mathcal{A} \cong_{\mathbf{1C}^*} \mathbb{C}$. Assume that there is $s_0 \in S$ such that $f(s_0) \neq 0$. Let

$$\mathcal{T} := C^*(T) \subset \mathcal{B}\left(\ell^2\right)$$

denote the Toeplitz algebra, generated on the unilateral shift $T \in \mathcal{B}\left(\ell^2\right)$. Consider the unital *-homomorphism $q : \mathcal{T} \to C(\mathbb{T})$ given by $q(T) := id_{\mathbb{T}}$. Recall that $\ker(q) = \mathcal{K}\left(\ell^2\right)$, the compact operators. Define $\varphi : (S, f) \to F_{\mathbf{1C^*}}^{\mathbf{CSet}_1}(C(\mathbb{T}))$ by

$$\varphi(s) := \begin{cases} f(s_0)\, id_{\mathbb{T}}, & s = s_0, \\ 0, & s \neq s_0, \end{cases}$$

which is constrictive. By Theorem 3.2.1, there is a unique unital *-homomorphism $\hat{\varphi} : \langle S, f | \emptyset \rangle_{\mathbf{1C^*}} \to C(\mathbb{T})$ such that $\hat{\varphi}(s) = \varphi(s)$ for all $s \in S$. Further, $st - ts, s^*t - ts^* \in \ker(\hat{\varphi})$ for all $s, t \in S$.

In [9], the Fredholm index of T is -1 so there is no $K \in \mathcal{K}\left(\ell^2\right)$ such that $T - K$ is normal. Likewise, there is no such K such that $f(s_0)\, T - K$ is normal. As a result, there are no normal operators in $q^{-1}(f(s_0)\, id_{\mathbb{T}})$. Thus, there is no lift of $\hat{\varphi}$ along q. Hence, $\{st - ts, st^* - t^*s : s, t \in S\}$ is not liftable on (S, f) when f is not identically 0. This was also shown in the existing presentation theory with a different operator in [27].

What the above example has shown is that knowledge of the category in question must be clear when discussing types of projectivity.

The remaining examples are combinations of the previous ones, either by free product or tensor product. Note that both of these constructions are the coproduct in either $\mathbf{1C^*}$ or $\mathbf{C1C^*}$, and there is an abstract connection between projectives and coproducts. That is, coproducts of projectives are once again projective, stated dually in Proposition A.4.2.

Example 3.15.8. Given that $C[0, 1]$ is projective with respect to surjections in $\mathbf{C1C^*}$, then

$$\begin{aligned} C\left(\overline{\mathbb{D}}\right) &\cong_{\mathbf{C1C^*}} C\left([0,1]^2\right) \\ &\cong_{\mathbf{C1C^*}} C[0,1] \otimes C[0,1] \\ &\cong_{\mathbf{C1C^*}} C[0,1] \coprod^{\mathbf{C1C^*}} C[0,1] \end{aligned}$$

is projective with respect to surjections in $\mathbf{C1C^*}$ also. Similarly, since $C[0,1]$ is projective with respect to surjections in $\mathbf{1C^*}$,

$$\begin{aligned} \langle (x,1),(y,1) | x \geq 0, y \geq 0 \rangle_{\mathbf{1C^*}} &\cong_{\mathbf{1C^*}} \langle (x,1) | x \geq 0 \rangle_{\mathbf{1C^*}} *_{\mathbb{C}} \langle (y,1) | y \geq 0 \rangle_{\mathbf{1C^*}} \\ &\cong_{\mathbf{1C^*}} C[0,1] *_{\mathbb{C}} C[0,1] \\ &\cong_{\mathbf{1C^*}} C[0,1] \coprod^{\mathbf{1C^*}} C[0,1] \end{aligned}$$

is projective with respect to surjections in $\mathbf{1C^*}$ too.

The above example demonstrates that clarity in the coproduct used is needed since $C\left(\overline{\mathbb{D}}\right)$ is not projective with respective to surjections in $\mathbf{1C^*}$.

However, the reverse is not always true in general; coproducts which are projective need not have projective factors. Fortunately, this is not the case in either $\mathbf{1C^*}$ or $\mathbf{C1C^*}$. Proposition B.3.5 yields that a product of compact Hausdorff spaces is an absolute retract if and only if each factor space was initially. Dually, this gains the projectivity result for $\mathbf{C1C^*}$. Proposition B.7.2 determines the analogous result for $\mathbf{1C^*}$.

Using these results, the projective properties of all the remaining examples can be determined.

Example 3.15.9 (Self-adjoint and normal invertibles). Recall that for $\lambda\mu > 1$,

$$\left[-\lambda, \frac{-1}{\mu}\right] \cup \left[\frac{1}{\mu}, \lambda\right] \cong_{\mathbf{Comp}} \left[\frac{1}{\mu}, \lambda\right] \prod^{\mathbf{Comp}} \{-1, 1\}$$

and

$$A_{\frac{1}{\mu},\lambda} \cong_{\mathbf{Comp}} \left[\frac{1}{\mu},\lambda\right] \coprod^{\mathbf{Comp}} \mathbb{T}.$$

Thus, $\left[-\lambda,\frac{-1}{\mu}\right] \cup \left[\frac{1}{\mu},\lambda\right]$ and $A_{\frac{1}{\mu},\lambda}$ are not injective relative to all monomorphisms in **Comp** by Proposition B.3.5 as $\{-1,1\}$ and $\mathbb{T}$ are not.

Symmetrically,

$$C\left(\left[-\lambda,\frac{-1}{\mu}\right] \cup \left[\frac{1}{\mu},\lambda\right]\right) \cong_{\mathbf{1C^*}} C\left[\frac{1}{\mu},\lambda\right] \otimes C(\{-1,1\})$$

and

$$C\left(A_{\frac{1}{\mu},\lambda}\right) \cong_{\mathbf{1C^*}} C\left[\frac{1}{\mu},\lambda\right] \otimes C(\mathbb{T}).$$

are not projective relative to all epimorphisms in $\mathbf{C1C^*}$ since $C(\{-1,1\})$ and $C(\mathbb{T})$ are not. Since epimorphisms in $\mathbf{C1C^*}$ are surjections in $\mathbf{1C^*}$, neither of these algebras can be projective to all surjections in $\mathbf{1C^*}$.

Example 3.15.10 (Pedersen's two-projection algebra, [32]). As $\mathbb{C}^2$ is not projective with respect to surjections in $\mathbf{1C^*}$,

$$\mathbb{C}^2 *_{\mathbb{C}} \mathbb{C}^2 \cong_{\mathbf{1C^*}} \begin{bmatrix} C[0,1] & C_0(0,1) \\ C_0(0,1) & C[0,1] \end{bmatrix}$$

is not projective in this sense either.

Example 3.15.11 (Non-commutative invertible algebras). Recall that for $\lambda\mu > 1$,

$$\left\langle (x,\lambda) \,\middle|\, \mu^2 x^* x \geq \mathbb{1}, \mu^2 x x^* \geq \mathbb{1} \right\rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} C[0,1] *_{\mathbb{C}} C(\mathbb{T})$$

Table 3.2: Projectivity for Current Examples in $\mathbf{1C}^*$

Unital C*-algebra	Surjections in $\mathbf{1C}^*$	Surjections in $\mathbf{C1C}^*$	Example
$\mathbb{C}$	Yes	Yes	3.5.1
$\mathbb{C}^n, n \in \mathbb{W} \setminus \{1\}$	No	No	3.15.6
$C[0,1]$	Yes	Yes	3.5.2
$C(\mathbb{T})$	No	No	3.5.3
$C([-2,-1] \cup [1,2])$	No	No	3.11.6
$C\left(A_{1,2}\right)$	No	No	3.11.7
$C\left(\prod_{\lambda\in\Lambda} \overline{\mathbb{D}}\right), \Lambda \neq \emptyset$	No	Yes	3.15.7
$\mathcal{T}$	No	-	3.5.4
$\begin{bmatrix} C[0,1] & C_0(0,1] \\ C_0(0,1] & C[0,1] \end{bmatrix}$	No	-	3.12.7
$\begin{bmatrix} C[0,1] & C_0(0,1) \\ C_0(0,1) & C[0,1] \end{bmatrix}$	No	-	3.10.2
$C[0,1] *_{\mathbb{C}} C(\mathbb{T})$	No	-	3.11.8
$C[0,1] *_{\mathbb{C}} \mathcal{T}$	No	-	3.11.9

and

$$\left\langle (x,\lambda) \,\middle|\, \mu^2 x^* x \geq \mathbb{1} \right\rangle_{\mathbf{1C}^*} \cong_{\mathbf{1C}^*} \left\langle (x,\lambda) \,\middle|\, \mu^2 x x^* \geq \mathbb{1} \right\rangle_{\mathbf{1C}^*} \cong_{\mathbf{1C}^*} C[0,1] *_{\mathbb{C}} \mathcal{T}.$$

Hence, neither of these coproducts can be projective relative to all surjections in $\mathbf{1C}^*$ since $C(\mathbb{T})$ and $\mathcal{T}$ are not.

Observe also that liftability of C*-relations can change if the crutch function changes. Recall the following examples:

$$\left\langle (x,\lambda) | x^2 = x^* = x \right\rangle_{\mathbf{1C}^*} \cong_{\mathbf{1C}^*} \begin{cases} \mathbb{C}, & 0 \leq \lambda < 1, \\ \mathbb{C} \oplus \mathbb{C}, & \lambda \geq 1, \end{cases}$$

$$\langle (x,\lambda) | x^* x = x x^*, \sin(x) = 0 \rangle_{\mathbf{1C}^*} \cong_{\mathbf{1C}^*} \mathbb{C}^{2n+1},$$

$$\langle (x,\lambda) | x^*x = xx^*, \|\exp(x)\| \leq \mu \rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} \begin{cases} \mathbb{O}, & \mu = 0 \text{ or } \ln(\mu) < -\lambda, \\ \mathbb{C}, & \ln(\mu) = -\lambda \text{ or } \lambda = 0 \leq \ln(\mu), \\ C\left(\overline{\mathbb{D}}\right), & \text{otherwise}, \end{cases}$$

$$\left\langle (x,\lambda) \left| \mu x \geq \mathbb{1} \right.\right\rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} \begin{cases} \mathbb{O}, & \lambda\mu < 1, \\ \mathbb{C}, & \lambda\mu = 1, \\ C[0,1], & \lambda\mu > 1, \end{cases}$$

$$\left\langle (x,\lambda) \left| x = x^2 \right.\right\rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} \begin{cases} \mathbb{C}, & \lambda < 1, \\ \mathbb{C} \oplus \mathbb{C}, & \lambda = 1, \\ \begin{bmatrix} C[0,1] & C_0(0,1] \\ C_0(0,1] & C[0,1] \end{bmatrix}, & \lambda > 1, \end{cases}$$

$$\left\langle S, f \left| st = ts, s^*t = ts^* \forall s,t \in S \right.\right\rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} C\left(\prod_{s \notin f^{-1}(0)} \overline{\mathbb{D}} \right).$$

In each of these, there is a transition where the presented unital C*-algebra passes from projective to not projective, relative to surjections in $\mathbf{1C^*}$. In fact, the normal-exponential algebras above alternate as λ changes.

Changes such as this will be examined more in detail in Section 3.16.

3.16 A Bifurcation Theory for Crutch Functions in $\mathbf{1C}^*$

As noted from Section 3.1, the crutch function plays a pivotal role in the construction of a unital C*-algebra. Proposition 3.2.4 showed that the scaled-free C*-algebra was unique up to the zero set of its crutch function. Also, each example presentation in this chapter has given rise to multiple distinct algebras, depending on the crutch function chosen prior to construction.

Prior works have considered changing norm bounds on generators before, such as [15], [16], and [17]. However, in most cases, the bounds tend to 0, rather than being allowed to grow as seen in the preceding examples.

This closing section of Chapter 3 considers this "bifurcating" behavior of isomorphism classes arising from the choice of crutch function.

First, one must be careful about what it means for two presentations to be the "same up to crutch function". Intuitively, this would mean that the set of generators and C*-relations are left alone while the crutch function is allowed to change, as shown in all the examples done thus far.

However, there is an issue with existence for certain C*-relations, specifically those built from the functional calculi. The following example describes such a situation.

Example 3.16.1 (Existence of a C*-relation). Recall that the power series $\displaystyle\sum_{j=1}^{\infty} \mu^j$ only converges if $|\mu| < 1$. For $\lambda \in [0, \infty)$, let $\mathcal{F}_\lambda := \langle (x, \lambda) | \emptyset \rangle_{\mathbf{1C}^*}$. For $\mu \in D_\lambda$, there is a unique unital *-homomorphism $\pi_\mu : \mathcal{F}_\lambda \to \mathbb{C}$ by $\pi_\mu(x) := \mu$. For $n > m \in \mathbb{N}$ and

$\lambda > 1$,

$$\begin{aligned}\left\|\sum_{j=m+1}^{n} x^j\right\|_{\mathcal{F}_\lambda} &\geq \left|\pi_\lambda\left(\sum_{j=m+1}^{n} x^j\right)\right| = \left|\sum_{j=m+1}^{n} \lambda^j\right| = \sum_{j=m+1}^{n} \lambda^j \\ &= \frac{\lambda^{n+1}-\lambda^{m+1}}{\lambda-1} \geq \frac{\lambda^{m+2}-\lambda^{m+1}}{\lambda-1} = \frac{\lambda^{m+1}(\lambda-1)}{\lambda-1} \\ &= \lambda^{m+1} > 1.\end{aligned}$$

Thus, $\sum_{j=1}^{\infty} x^j$ cannot exist in $\mathcal{F}_\lambda$ for $\lambda > 1$. However, for $\lambda < 1$,

$$\begin{aligned}\left\|\sum_{j=m+1}^{n} x^j\right\|_{\mathcal{F}_\lambda} &\leq \sum_{j=m+1}^{n} \|x\|_{\mathcal{F}_\lambda}^j \leq \sum_{j=m+1}^{n} \lambda^j \\ &\leq \sum_{j=m+1}^{\infty} \lambda^j = \frac{\lambda^{m+1}}{1-\lambda},\end{aligned}$$

making this sequence Cauchy. Thus, $\sum_{j=1}^{\infty} x^j$ exists in $\mathcal{F}_\lambda$ for $\lambda < 1$.

To avoid this problem, one can fix a working environment to construct the C*-relations and then consider moving away from that setting. This will be done by fixing a set of generators and relating crutch functions on it, thus connecting scaled-free C*-algebras on those generators. Next, this relationship will be used to move C*-relations on a fixed crutched set to other scaled-free C*-algebras, tying together variants of the same presentation.

To begin, fix a set S and partially order the crutch functions on S using the usual product order on $[0,\infty)^S$. That is, given crutch functions f and g, $g \leq f$ if $g(s) \leq f(s)$ for all $s \in S$. For $g \leq f$, the map $\phi_g^f : (S, f) \to (S, g)$ by $\phi_g^f(s) := s$ is a constriction. Further, if $h \leq g \leq f$,

$$\left(\phi_h^g \circ \phi_g^f\right)(s) = \phi_h^g(s) = s = \phi_h^f(s)$$

for all $s \in S$ so $\phi_h^f = \phi_h^g \circ \phi_g^f$. Further, $\phi_f^f = id_{(S,f)}$, making $\left((S,f), \phi_g^f\right)$ an inverse system in $\mathbf{CSet}_1$. Diagrammatically, this is summarized in the commutative diagram below for crutch functions $h \leq g \leq f$ on S.

$$\begin{array}{ccc} (S,f) & \xrightarrow{\phi_g^f} & (S,g) \\ & \searrow^{\phi_h^f} & \downarrow{\scriptstyle \phi_h^g} \\ & & (S,h) \end{array}$$

Applying 1C*Alg, $\left(\langle S, f|\emptyset\rangle_{\mathbf{1C}^*}, \rho_g^f\right)$ is also an inverse system in $\mathbf{1C}^*$, where $\rho_g^f :=$ 1C*Alg$\left(\phi_g^f\right)$ associates a generator to its counterpart in the target algebra. This is summarized in the commutative diagram below for crutch functions $h \leq g \leq f$ on S.

$$\begin{array}{ccc} \langle S, f|\emptyset\rangle_{\mathbf{1C}^*} & \xrightarrow{\rho_g^f} & \langle S, g|\emptyset\rangle_{\mathbf{1C}^*} \\ & \searrow^{\rho_h^f} & \downarrow{\scriptstyle \rho_h^g} \\ & & \langle S, h|\emptyset\rangle_{\mathbf{1C}^*} \end{array}$$

Fix a crutch function f on S and a set of C*-relations R_f on (S,f). For $g \leq f$, define $R_g := \rho_g^f(R_f)$, $\mathcal{A}_g := \langle S, g|R_g\rangle_{\mathbf{1C}^*}$, and let $q_g : \langle S, g|\emptyset\rangle_{\mathbf{1C}^*} \to \mathcal{A}_g$ be the quotient map. For $h \leq g \leq f$, observe that

$$\rho_h^g(R_g) = \left(\rho_h^g \circ \rho_g^f\right)(R_f) = \rho_h^f(R_f) = R_h$$

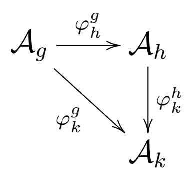
so $R_g \subseteq \ker\left(q_h \circ \rho_h^g\right)$. By the universal property of the quotient, there is a unique unital *-homomorphism $\varphi_h^g : \mathcal{A}_g \to \mathcal{A}_h$ such that $\varphi_h^g \circ q_g = q_h \circ \rho_h^g$. For $k \leq h \leq g \leq f$,

$$\varphi_k^h \circ \varphi_h^g \circ q_g = \varphi_k^h \circ q_h \circ \rho_h^g = q_k \circ \rho_k^h \circ \rho_h^g = q_k \circ \rho_k^g = \varphi_k^g \circ q_g$$

so by the universal property of the quotient, $\varphi_k^h \circ \varphi_h^g = \varphi_k^g$. Further,

$$\varphi_g^g \circ q_g = q_g \circ \rho_g^g = q_g \circ id_{\langle S,g|\emptyset\rangle_{\mathbf{1C^*}}} = q_g = id_{\mathcal{A}_g} \circ q_g$$

so by the universal property of the quotient, $\varphi_g^g = id_{\mathcal{A}_g}$. Thus, $(\mathcal{A}_g, \varphi_h^g)$ is an inverse system in $\mathbf{1C^*}$. This is summarized in the commutative diagram below for $k \leq h \leq g \leq f$.



This is precisely the notion desired for presentations to be "the same up to crutch function" as the following example demonstrates.

Example 3.16.2. Fix $S := \{x\}$ and $f : S \to [0, \infty)$ by $f(x) := \frac{1}{2}$. From Example 3.16.1, $\sum_{j=1}^{\infty} x^j \in \langle (x, \lambda) | \emptyset \rangle_{\mathbf{1C^*}}$ for $\lambda < 1$ so let $R_f := \left\{ \sum_{j=1}^{\infty} x^j \right\}$. For $g \leq f$, $R_g = \left\{ \sum_{j=1}^{\infty} x^j \right\}$ as ρ_g^f merely associates generators. Thus, letting $\lambda_g := g(x)$,

$$\mathcal{A}_g = \left\langle (x, \lambda_g) \middle| \sum_{j=1}^{\infty} x^j = 0 \right\rangle_{\mathbf{1C^*}}$$

for all $\lambda_g \leq \frac{1}{2}$.

Some C*-relations exist regardless of how large the crutched value is allowed to grow. In particular, those C*-relations determined by entire functions in the analytic functional calculus have this property. Examples of this situation have been the main focus of this chapter. As done in those examples, one can fix an arbitrary crutch function to construct the relationship above, but the arbitrary choice and universal property of Theorem 3.3.2 allow one to start anywhere in the inverse system.

Example 3.16.3. Fix $S := \{x\}$ and $f : S \to [0, \infty)$. Letting $\lambda := f(x)$, $\sin(x) \in \langle (x, \lambda) | \emptyset \rangle_{\mathbf{1C^*}}$ so let $R_f := \{\sin(x)\}$. For $g \le f$, $R_g = \{\sin(x)\}$ also. Thus, letting $\lambda_g := g(x)$,

$$\mathcal{A}_g = \langle (x, \lambda_g) \left| \sin(x) = 0 \right\rangle_{\mathbf{1C^*}} .$$

Assume that $e \ge f$. Letting $\lambda_e := e(x)$, $\sin(x) \in \langle (x, \lambda_e) \left| \emptyset \right\rangle_{\mathbf{1C^*}}$. Letting $R_e := \{\sin(x)\}$, note that $\rho_f^e (R_e) = \{\sin(x)\} = R_f$, avoiding ambiguity.

Next, with the issue of existence in mind, one can consider a visual representation of this inverse system, a *bifurcation diagram.*

Specifically, let R_f be a set of C*-relations on a crutched set (S, f). For each $g \le f$, the construction above yields an associated unital C*-algebra $\mathcal{A}_g$. Thinking of g as a point in $[0, \infty)^S$, one can consider the set of crutch functions which yield algebras isomorphic to $\mathcal{A}_g$.

Definition. For a set of C*-relations R_f on a crutched set (S, f) and a unital C*-algebra $\mathcal{A}$, the *class set* for $\mathcal{A}$ relative to (S, f) and R_f is given by

$$\Sigma^{\mathbf{1C^*}} (\mathcal{A} : S, f, R_f) := \left\{ g \in [0, \infty)^S : g \le f, \mathcal{A}_g \cong_{\mathbf{1C^*}} \mathcal{A} \right\} .$$

When $\mathrm{card}(S) \le 3$, these sets can be drawn on conventional axes, labeling the sets appropriately.

Example 3.16.4 (A normal element). Fix $S := \{x\}$, $f : S \to [0, \infty)$, and $R_f :=$

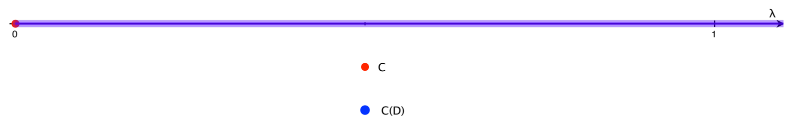


Figure 3.2: Bifurcation Diagram for $\{x^*x - xx^*\}$, $\lambda_f = 1.1$

$\{x^*x - xx^*\}$. Let $\lambda_f := f(x)$. Example 3.5.1 shows

$$\langle (x,\lambda)\,|x^*x = xx^*\rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} \begin{cases} \mathbb{C}, & \lambda = 0, \\ C\left(\overline{\mathbb{D}}\right), & \lambda > 0. \end{cases}$$

Thus,

$$\Sigma^{\mathbf{1C^*}}\left(\mathcal{A} : S, f, R_f\right) = \begin{cases} \{0\}, & \mathcal{A} \cong_{\mathbf{1C^*}} \mathbb{C}, \\ \left(0, \lambda_f\right], & \mathcal{A} \cong_{\mathbf{1C^*}} C\left(\overline{\mathbb{D}}\right), \\ \emptyset, & \text{otherwise.} \end{cases}$$

Graphing these sets on a 1-dimensional axis yields Figure 3.2 for $\lambda_f = 1.1$.

Example 3.16.5 (Sine and normality). Fix $S := \{x\}$, $f : S \to [0,\infty)$, and $R_f := \{\sin(x), x^*x - xx^*\}$. Let $\lambda_f := f(x)$. Example 3.6.1 shows

$$\langle (x,\lambda)\,|x^*x = xx^*, \sin(x)\rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} \mathbb{C}^{2n+1}, \pi n \le \lambda < \pi(n+1)$$

for $n \in \mathbb{W}$. Thus,

$$\Sigma^{\mathbf{1C^*}}\left(\mathcal{A} : S, f, R_f\right) = \begin{cases} [\pi n, \pi(n+1)) \cap \left[0, \lambda_f\right], & \mathcal{A} \cong_{\mathbf{1C^*}} \mathbb{C}^{2n+1}, n \in \mathbb{W}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Graphing these sets on a 1-dimensional axis yields Figure 3.3 for $\lambda_f = 7.1$.

Example 3.16.6 (An idempotent element). Fix $S := \{x\}$, $f : S \to [0,\infty)$, and $R_f :=$

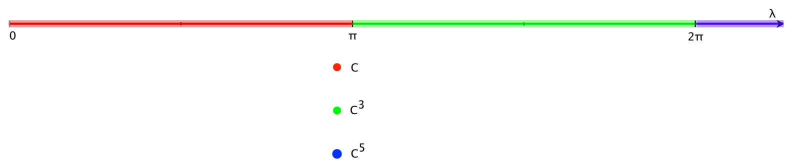


Figure 3.3: Bifurcation Diagram for $\{x^*x - xx^*, \sin(x)\}$, $\lambda_f = 7.1$

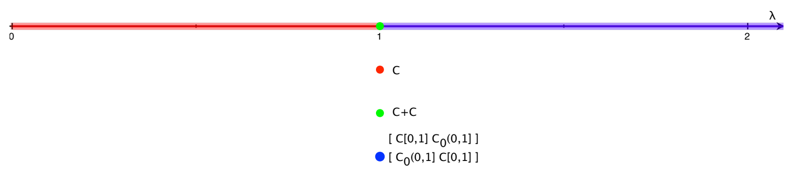


Figure 3.4: Bifurcation Diagram for $\left\{x - x^2\right\}$, $\lambda_f = 2.1$

$\left\{x - x^2\right\}$. Let $\lambda_f := f(x)$. Example 3.12.7 shows

$$\left\langle (x, \lambda) \,\middle|\, x = x^2 \right\rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} \begin{cases} \mathbb{C}, & \lambda < 1, \\ \mathbb{C} \oplus \mathbb{C}, & \lambda = 1, \\ \begin{bmatrix} C[0,1] & C_0(0,1] \\ C_0(0,1] & C[0,1] \end{bmatrix}, & \lambda > 1. \end{cases}$$

Thus,

$$\Sigma^{\mathbf{1C^*}}\left(\mathcal{A} : S, f, R_f\right) = \begin{cases} [0,1) \cap [0, \lambda_f]\,, & \mathcal{A} \cong_{\mathbf{1C^*}} \mathbb{C}, \\ \{1\} \cap [0, \lambda_f]\,, & \mathcal{A} \cong_{\mathbf{1C^*}} \mathbb{C} \oplus \mathbb{C}, \\ (1, \infty) \cap [0, \lambda_f]\,, & \mathcal{A} \cong_{\mathbf{1C^*}} \begin{bmatrix} C[0,1] & C_0(0,1] \\ C_0(0,1] & C[0,1] \end{bmatrix}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Graphing these sets on a 1-dimensional axis yields Figure 3.4 for $\lambda_f = 2.1$.

Example 3.16.7 (Invertibility). Fix $S := \{x, y\}$, $f : S \to [0, \infty)$, and $R_f := \{xy - \mathbb{1}, yx - \mathbb{1}\}$. Let $\lambda_f := f(x)$ and $\mu_f := f(y)$. From Example 3.11.8, the following Tietze transformations show

$$\langle (x,\lambda),(y,\mu) \,|\, xy = yx = \mathbb{1} \rangle_{\mathbf{1C^*}}$$

$$\cong_{\mathbf{1C^*}} \left\langle (x,\lambda),(y,\mu) \,\middle|\, \begin{array}{c} xy = yx = \mathbb{1}, \\ \mathbb{1} \le \mu^2 x^* x \end{array} \right\rangle_{\mathbf{1C^*}}$$

$$\cong_{\mathbf{1C^*}} \left\langle (x,\lambda),(y,\mu) \,\middle|\, \begin{array}{c} xy = yx = \mathbb{1}, \\ \mathbb{1} \le \mu^2 x^* x, \mathbb{1} \le \mu^2 x x^*, \\ y = \mu^2 \left(p \left(\mu \left(x^* x \right)^{\frac{1}{2}} - 1 \right) + 1 \right)^{-2} x^* \end{array} \right\rangle_{\mathbf{1C^*}}$$

$$\cong_{\mathbf{1C^*}} \left\langle (x,\lambda),(y,\mu) \,\middle|\, \begin{array}{c} yx = \mathbb{1}, \\ \mathbb{1} \le \mu^2 x^* x, \mathbb{1} \le \mu^2 x x^*, \\ y = \mu^2 \left(p \left(\mu \left(x^* x \right)^{\frac{1}{2}} - 1 \right) + 1 \right)^{-2} x^* \end{array} \right\rangle_{\mathbf{1C^*}}$$

$$\cong_{\mathbf{1C^*}} \left\langle (x,\lambda),(y,\mu) \,\middle|\, \begin{array}{c} \mathbb{1} \le \mu^2 x^* x, \mathbb{1} \le \mu^2 x x^*, \\ y = \mu^2 \left(p \left(\mu \left(x^* x \right)^{\frac{1}{2}} - 1 \right) + 1 \right)^{-2} x^* \end{array} \right\rangle_{\mathbf{1C^*}}$$

$$\cong_{\mathbf{1C^*}} \left\langle (x,\lambda) \,\middle|\, \mathbb{1} \le \mu^2 x^* x, \mathbb{1} \le \mu^2 x x^* \right\rangle_{\mathbf{1C^*}}$$

$$\cong_{\mathbf{1C^*}} \begin{cases} \mathbb{0}, & \lambda\mu < 1, \\ C(\mathbb{T}), & \lambda\mu = 1, \\ C[0,1] *_{\mathbb{C}} C(\mathbb{T}), & \lambda\mu > 1. \end{cases}$$

Thus,

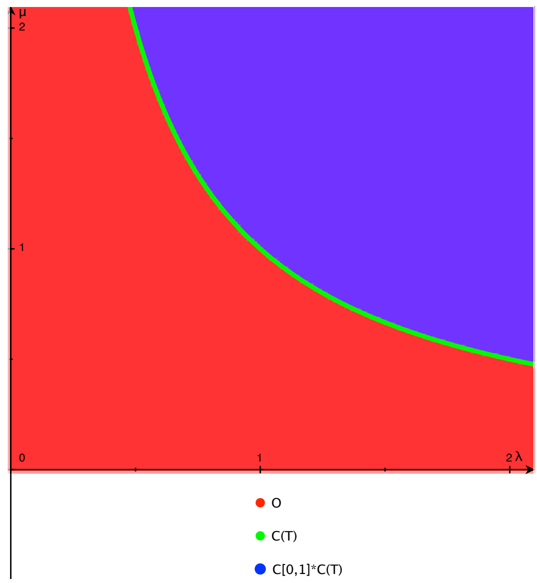


Figure 3.5: Bifurcation Diagram for $\{xy - \mathbb{1}, yx - \mathbb{1}\}$, $\lambda_f = \mu_f = 2.1$

$$\Sigma^{\mathbf{1C^*}}(\mathcal{A} : S, f, R_f)$$

$$= \begin{cases} \{(\lambda,\mu) : \lambda\mu < 1\} \cap ([0,\lambda_f] \times [0,\mu_f]), & \mathcal{A} \cong_{\mathbf{1C^*}} \mathbb{O}, \\ \{(\lambda,\mu) : \lambda\mu = 1\} \cap ([0,\lambda_f] \times [0,\mu_f]), & \mathcal{A} \cong_{\mathbf{1C^*}} C(\mathbb{T}), \\ \{(\lambda,\mu) : \lambda\mu > 1\} \cap ([0,\lambda_f] \times [0,\mu_f]), & \mathcal{A} \cong_{\mathbf{1C^*}} C[0,1] *_{\mathbb{C}} C(\mathbb{T}), \\ \emptyset, & \text{otherwise.} \end{cases}$$

Graphing these sets on a 2-dimensional axis yields Figure 3.5 for $\lambda_f = \mu_f = 2.1$.

Example 3.16.8 (Exponential and normality)*.* Fix $S := \{x, y\}$, $f : S \to [0, \infty)$, and $R_f := \{x^*x - xx^*, \exp(x) - y\}$. Let $\lambda_f := f(x)$ and $\mu_f := f(y)$. Example 3.8.2 and the following Tietze transformations show

$$\left\langle \begin{array}{c} (x,\lambda), \\ (y,\mu) \end{array} \middle| \begin{array}{c} x^*x = xx^*, \\ y = \exp(x) \end{array} \right\rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} \left\langle \begin{array}{c} (x,\lambda), \\ (y,\mu) \end{array} \middle| \begin{array}{c} x^*x = xx^*, y = \exp(x), \\ \|\exp(x)\| \leq \mu \end{array} \right\rangle_{\mathbf{1C^*}}$$

$$\cong_{\mathbf{1C^*}} \left\langle (x,\lambda) \middle| \begin{array}{c} x^*x = xx^*, \\ \|\exp(x)\| \leq \mu \end{array} \right\rangle_{\mathbf{1C^*}}$$

$$\cong_{\mathbf{1C^*}} \begin{cases} \mathbb{O}, & \mu = 0 \text{ or } \ln(\mu) < -\lambda, \\ \mathbb{C}, & \ln(\mu) = -\lambda \text{ or } \lambda = 0 \leq \ln(\mu), \\ C\left(\overline{\mathbb{D}}\right), & \text{otherwise.} \end{cases}$$

Thus,

$$\Sigma^{\mathbf{1C^*}}\left(\mathcal{A} : S, f, R_f\right)$$

$$= \begin{cases} \{(\lambda,\mu) : \mu < \exp(-\lambda)\} \cap ([0,\lambda_f] \times [0,\mu_f]), & \mathcal{A} \cong_{\mathbf{1C^*}} \mathbb{O}, \\ (\{(\lambda,\exp(-\lambda)), (0,\mu) : \mu \geq 1, \lambda \geq 0\}) \cap ([0,\lambda_f] \times [0,\mu_f]), & \mathcal{A} \cong_{\mathbf{1C^*}} \mathbb{C}, \\ \{(\lambda,\mu) : \mu > \exp(-\lambda), \lambda > 0\} \cap ([0,\lambda_f] \times [0,\mu_f]), & \mathcal{A} \cong_{\mathbf{1C^*}} C\left(\overline{\mathbb{D}}\right), \\ \emptyset, & \text{otherwise.} \end{cases}$$

Graphing these sets on a 2-dimensional axis yields Figure 3.6 for $\lambda_f = \mu_f = 2.1$.

With this notion, some natural questions arise. First, for a fixed set of C*-relations, how many distinct isomorphism classes are possible? In particular, how many distinct isomorphism classes are possible for a finite set of generators, specifically a single generator? From Section 3.6, a single generator can yield a countably many distinct unital C*-algebras. Further, one can yield any finite cardinality.

Example 3.16.9 (Finitely many isomorphism classes). Fix $n \in \mathbb{N}$. Define $r : \mathbb{C} \to \mathbb{C}$ by $r(\mu) := \prod_{j=0}^{n-1}(\mu - j)$, a polynomial with finitely many distinct zeroes. For $\lambda \geq 0$,

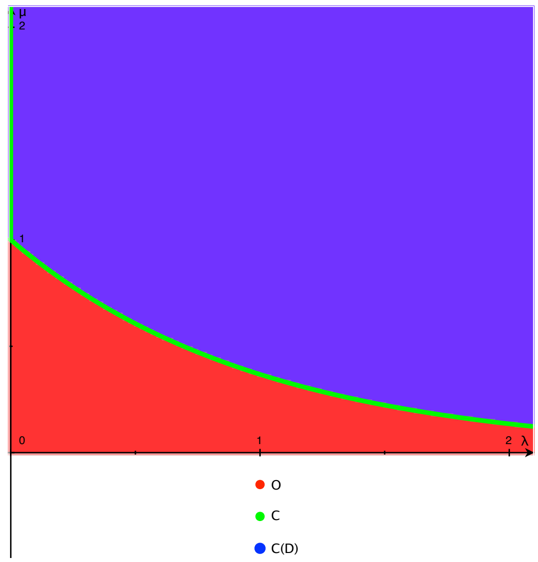


Figure 3.6: Bifurcation Diagram for $\{x^*x - xx^*, \exp(x) - y\}$, $\lambda_f = \mu_f = 2.1$

$r(x) \in \langle (x, \lambda) | \emptyset \rangle_{\mathbf{1C^*}}$ so consider

$$\mathcal{A} := \langle (x, \lambda) \, | x^*x = xx^*, r(x) = 0 \rangle_{\mathbf{1C^*}} .$$

Note that x is normal. In this case,

$$g_{r(x)}(\mu) = r(\mu).$$

Hence, $g^{-1}_{r(x)}(0) = \{0, \ldots, n-1\} \cap D_\lambda$. By Theorem 3.4.4,

$$\langle (x, \lambda) \, | x^*x = xx^*, r(x) = 0 \rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} \begin{cases} \mathbb{C}^j, & j-1 \leq \lambda < j, j = 1, \ldots, n-1, \\ \mathbb{C}^n, & n-1 \leq \lambda, \end{cases}$$

displaying n distinct isomorphism classes.

Whether or not a single generator can yield an uncountable number of pairwise

Table 3.3: Examples of Tenuous C*-algebras

Unital C*-algebra	Crutched Set	C*-relations
$\mathbb{C}$	$\{(x,1)\}$	$\{x^*x - xx^*\}$
$\mathbb{C}$	$\{(x,2),(y,2)\}$	$\{x^*x - xx^*, \exp(x) - y\}$
$\mathbb{C}\oplus\mathbb{C}$	$\{(x,2)\}$	$\{x - x^2\}$
$\mathbb{C}\oplus\mathbb{C}$	$\{(x,2),(y,2)\}$	$\{x - x^*, xy - \mathbb{1}, yx - \mathbb{1}\}$
$C(\mathbb{T})$	$\{(x,2),(y,2)\}$	$\{xy - \mathbb{1}, yx - \mathbb{1}\}$

non-isomorphic unital C*-algebras has not been determined at the time of this work.

Next, observe that in Figures 3.2, 3.4, 3.5, and 3.6 display isomorphism classes that could be termed "unstable". With very small changes in the crutch function, the behavior of the algebra can radically change. Equipping $[0,\infty)^S$ with the product topology, one can make the following definition.

Definition. Given a set R_f of C*-relations on a crutched set (S,f), a unital C*-algebra $\mathcal{A}$ is *tenuous* for R_f if the associated class set $\Sigma^{\mathbf{1C^*}}(\mathcal{A} : S, f, R_f)$ is nonempty and has empty interior.

As with the definitions of crutched set and constriction, use of the term "unstable" is avoided as "stable" is already in use in the context of the existing presentation theory detailed in [27]. What this definition states is that there are crutch functions $g \leq f$ such that $\langle S, g \,|R_g\rangle_{\mathbf{1C^*}} \cong_{\mathbf{1C^*}} \mathcal{A}$, but for a particular g, there is no ball around it so that all functions in the ball also yield $\mathcal{A}$. Table 3.3 shows some tenuous examples in this chapter.

One particular goal in this vein is to uncover criteria on a set of C*-relations determining when the resulting C*-algbras are tenuous relative to it. At first glance, many of these tenuous cases arise from a C*-relation involving the identity, such as invertibility or the power series for exp. However, normality and idempotency break this intuition as they are *-polynomials without reference to the identity. Also, the power series for cos involves the identity, but the algebras demonstrated in Section 3.6 are not tenuous. At the moment, this notion of tenuousness is nebulous but interesting. At the time of this work, no necessary nor sufficient criteria for tenuousness relative to a set of C*-relations have been determined.

As mentioned in Sections 3.1 and 3.5, the construction of the scaled-free unital C*-algebra and the presentation theory can be mirrored in other normed algebraic categories. There, a notion of this bifurcation behavior may well exist and yield more interesting cases for consideration.

Chapter 4

Non-Unital Category C*

This chapter adapts the constructions and results of Chapter 3 for general C*-algebras, developing a comparable presentation theory.

Also, with this general theory in play, the relationship between the presentation theory built in [19] can be determined, the present work subsuming and extending the results of [19] in Theorem 4.3.3.

4.1 The Modified Construction for C*

As done in Section 3.1, a scaled-free construction can be accomplished for the category of C*-algebras with *-homomorphisms. The arguments here will be nearly identical to those of Section 3.1 so for brevity, only an outline of the construction and statements of the main results will be given here.

A version of the construction was done previously in Section 1.3 of [19]. However, this presentation of the material explicitly carried the universal maps of both free *-semigroup and free *-algebra constructions throughout each result. The present work aims to streamline the construction for C*-algebras, moving directly from the original crutched set to the constructed algebra.

To begin, let $\mathbf{C}^*$ denote the category of C*-algebras and *-homomorphisms. Explicitly, $\mathrm{Ob}(\mathbf{C}^*)$ is the class of all C*-algebras, and for $\mathcal{A}, \mathcal{B} \in \mathrm{Ob}(\mathbf{C}^*)$, $\mathbf{C}^*(\mathcal{A}, \mathcal{B})$ is the set of all *-homomorphisms from $\mathcal{A}$ to $\mathcal{B}$.

As in Example 2.1.1, every $\mathcal{A} \in \mathrm{Ob}(\mathbf{C}^*)$ is a set with a nonnegative function $f_{\mathcal{A}} : \mathcal{A} \to [0, \infty)$ by $f_{\mathcal{A}}(a) := \|a\|_{\mathcal{A}}$. Thus, there is a natural forgetful map to $\mathrm{Ob}(\mathbf{CSet}_1)$, where one regards $\mathcal{A}$ as a crutched set $(\mathcal{A}, f_{\mathcal{A}})$, ignoring all structure except the norm function. Similarly, given $\mathcal{A}, \mathcal{B} \in \mathrm{Ob}(\mathbf{C}^*)$ and $\phi \in \mathbf{C}^*(\mathcal{A}, \mathcal{B})$, ϕ is firstly a function from $\mathcal{A}$ to $\mathcal{B}$, but it is a standard fact that $\|\phi(a)\|_{\mathcal{B}} \leq \|a\|_{\mathcal{A}}$ for all $a \in \mathcal{A}$. Hence, $\phi \in \mathbf{CSet}_1((\mathcal{A}, f_{\mathcal{A}}), (\mathcal{B}, f_{\mathcal{B}}))$ as in Example 2.1.9. One can quickly check that these two associations define a functor $F_{\mathbf{C}^*}^{\mathbf{CSet}_1} : \mathbf{C}^* \to \mathbf{CSet}_1$, where one ignores all data from $\mathbf{C}^*$ save the set and norm.

Now, fix (S, f) from $\mathrm{Ob}(\mathbf{CSet}_1)$, thought of as a set of generators normed by their values under f. The objective is to build a reflection of (S, f) along $F_{\mathbf{C}^*}^{\mathbf{CSet}_1}$. First, the norm structure of a C*-algebra will force any element crutched by 0 to be the zero element, so these elements are removed. Let $S_f := S \setminus f^{-1}(0)$.

Next, the adjoint structure will be encoded. Let $S_{f,*} := S_f \uplus S_f := \{0, 1\} \times S_f$, the disjoint union of S_f with itself. The original set S_f is identified with $\{0\} \times S_f$ while elements of $\{1\} \times S_f$ are denoted s^*, formal adjoints of elements in S_f. As such, it is standard to consider $S_{f,*} := S_f \cup \{s^* : s \in S_f\}$.

To encode the multiplicative structure, let $H_{S,f}$ be the set of all *nonempty* finite sequences of elements from $S_{f,*}$, thought of as non-commuting monomials. Specifically, the empty list is *not* included in $H_{S,f}$. Under concatenation of lists, $H_{S,f}$ is naturally a semigroup. However, it also has a natural involution by reversing order and swapping presence/absence of the *. Hence, $H_{S,f}$ is a *-semigroup, the free *-semigroup on S_f.

For additive structure, let $B_{S,f}$ be the set of all functions from $H_{S,f}$ to $\mathbb{C}$ whose support is finite, thought of as non-commuting polynomials with coefficients from $\mathbb{C}$. Under point-wise addition and scalar multiplication, $B_{S,f}$ is naturally a $\mathbb{C}$-vector space. Further, each function can be written uniquely as a $\mathbb{C}$-linear sum of functions with singleton support and value 1, denoted δ_l for each $l \in H_{S,f}$.

Vector multiplication is determined by the usual polynomial formula, described explicitly in Section 3.1. Similarly, the adjoint operation is determined in an equally natural way from Section 3.1. Under these operations, it is a standard exercise to show $B_{S,f}$ to be an involutive $\mathbb{C}$-algebra, the free *-algebra over $\mathbb{C}$ on S_f. This *-algebra and its properties were detailed in Sections 1.3.3-4 of [19].

To continue the construction, one must norm $B_{S,f}$. The following faithful *-representation is constructed just as in Lemma 3.1.1.

Lemma 4.1.1. *There exist a Hilbert space $\mathcal{H}$ and a *-homomorphism $\pi_0 : B_{S,f} \to \mathcal{B}(\mathcal{H})$, which is one-to-one and satisfies $\|\pi_0(\delta_s)\|_{\mathcal{B}(\mathcal{H})} = f(s)$ for all $s \in S_f$.*

This result is a refinement of Theorem 1.3.6.1 from [19].

Lemma 4.1.2. *For each $a \in B_{S,f}$, define*

$$\mathscr{T}_a := \left\{ \|\pi(a)\|_{\mathcal{B}} : \begin{array}{c} \mathcal{B} \text{ a } C^*\text{-algebra}, \\ \pi : B_{S,f} \to \mathcal{B} \text{ a *-homomorphism}, \\ \|\pi(\delta_s)\|_{\mathcal{B}} \leq f(s) \forall s \in S_f \end{array} \right\}.$$

and $\tau_{S,f} : B_{S,f} \to [0, \infty)$ by $\tau_{S,f}(a) := \sup \mathscr{T}_a$. Then, $\tau_{S,f}$ is a sub-multiplicative norm on $B_{S,f}$ satisfying the C^*-property.

Thus, $B_{S,f}$ is a *-algebra over $\mathbb{C}$ with a C*-norm. Therefore, the completion, denoted $\mathcal{B}_{S,f}$, is a C*-algebra, residing in $\mathrm{Ob}(\mathbf{C}^*)$. As such, one can consider $F_{\mathbf{C}^*}^{\mathbf{CSet}_1} \mathcal{B}_{S,f}$, this algebra with only its norm. There is a canonical association $\theta_{S,f} : S \to \mathcal{B}_{S,f}$ by

$$\theta_{S,f}(s) := \begin{cases} \delta_s, & s \in S_f, \\ 0, & s \notin S_f. \end{cases}$$

The C*-algebra $\mathcal{B}_{S,f}$ equipped with $\theta_{S,f}$ is a candidate for the reflection of (S, f) along $F_{\mathbf{C^*}}^{\mathbf{CSet}_1}$. This result is a refinement of Theorem 1.3.7.1 in [19].

Lemma 4.1.3. *The function $\theta_{S,f}$ is constrictive from (S, f) to $F_{\mathbf{C^*}}^{\mathbf{CSet}_1}\mathcal{B}_{S,f}$.*

Theorem 4.1.4. *The C*-algebra $\mathcal{B}_{S,f}$ equipped with $\theta_{S,f}$ is a reflection of (S, f) along $F_{\mathbf{C^*}}^{\mathbf{CSet}_1}$.*

Further, since (S, f) was arbitrary, Proposition A.5.1 states that there is a unique functor $\mathrm{C}^*\mathrm{Alg} : \mathbf{CSet}_1 \to \mathbf{C}^*$ such that $\mathrm{C}^*\mathrm{Alg}(S, f) = \mathcal{B}_{S,f}$, and $\mathrm{C}^*\mathrm{Alg} \dashv F_{\mathbf{C^*}}^{\mathbf{CSet}_1}$ by Theorem A.5.2.

4.2 Properties of the Functor $\mathrm{C}^*\mathrm{Alg}$

Since the constructions of Sections 3.1 and 4.1 are very similar, the properties of the resulting C*-algebras are closely related. The arguments for $\mathrm{C}^*\mathrm{Alg}$ are nearly identical as those for $1\mathrm{C}^*\mathrm{Alg}$ so for brevity, only a summary of the results will be given here. However, the key result of this section is Theorem 4.2.6, which demonstrates that the unital scaled-free C*-algebra of Section 3.1 is precisely the unitization of the C*-algebra constructed in Section 4.1. This, in turn, gives an immediate proof of projectivity in Proposition 4.2.7.

First is the explicit universal property of the adjoint pair $\mathrm{C}^*\mathrm{Alg} \dashv F_{\mathbf{C^*}}^{\mathbf{CSet}_1}$. Theorem 1.3.7.1 in [19] gives the same result.

Theorem 4.2.1 (Explicit Universal Property of $\mathrm{C}^*\mathrm{Alg} \dashv F_{\mathbf{C^*}}^{\mathbf{CSet_1}}$)**.** *Let* (S,f) *be a crutched set and* $\mathcal{B}$ *be a C*-algebra. Then for any constrictive map* $\phi : (S,f) \to F_{\mathbf{C^*}}^{\mathbf{CSet_1}}\mathcal{B}$*, there is a unique *-homomorphism* $\hat{\phi} : \mathrm{C}^*\mathrm{Alg}(S,f) \to \mathcal{B}$ *such that* $\hat{\phi}\left(\theta_{S,f}(s)\right) = \phi(s)$ *for all* $s \in S$*.*

Similarly, there are comparable "norm-stealing" and scaled-free forms.

Corollary 4.2.2 (Norm-Stealing Form)**.** *Let* S *be a set and* $\mathcal{B}$ *be a C*-algebra. For any function* $\phi : S \to \mathcal{B}$*, define* $f_\phi : S \to [0,\infty)$ *by* $f_\phi(s) := \|\phi(s)\|_{\mathcal{B}}$*. Then, there is a unique *-homomorphism* $\hat{\phi} : \mathrm{C}^*\mathrm{Alg}\left(S, f_\phi\right) \to \mathcal{B}$ *such that* $\hat{\phi}\left(\theta_{S,f_\phi}(s)\right) = \phi(s)$ *for all* $s \in S$*.*

Corollary 4.2.3 (Scaled-Free Mapping Property Form)**.** *Let* (S,f) *be a crutched set and* $\mathcal{B}$ *be a C*-algebra. Then, for any function* $\phi : S \to \mathcal{B}$*, there is a unique *-homomorphism* $\hat{\phi} : \mathrm{C}^*\mathrm{Alg}(S,f) \to \mathcal{B}$ *such that for all* $s \in S$*,*

$$\|\phi(s)\|_{\mathcal{B}} \cdot \hat{\phi}\left(\theta_{S,f}(s)\right) = f(s) \cdot \phi(s).$$

For this reason, the C*-algebra $\mathrm{C}^*\mathrm{Alg}(S,f)$ is termed the *scaled-free C*-algebra on* (S,f). The analogous uniqueness result also appears. This result generalizes Conclusion 4.1.2.9 in [19], which considers only strictly positive f.

Proposition 4.2.4 (Uniqueness of $\mathrm{C}^*\mathrm{Alg}(S,f)$)**.** *Given a crutched set* (S,f)*, let* $\mathbf{1}_{S_f} : S_f \to \{1\}$ *be the constant function. Then,* $\mathrm{C}^*\mathrm{Alg}(S,f) \cong_{\mathbf{1C^*}} \mathrm{C}^*\mathrm{Alg}\left(S_f, \mathbf{1}_{S_f}\right)$*.*

As stated before, Section 1.3 of [19] forms the non-unital algebra of contractions in a similar way to Section 4.1 of the present work. Section 4.1.2 of [19] holds a comparable analysis of the structure of this object. However, while the initial formulation in Section 1.1 of [19] mentions the forgetful functor and the adjoint situation, the categorical properties are not exploited in the work.

Since $\mathrm{C}^*\mathrm{Alg}$ has been shown to be a left adjoint functor in Theorem 3.1.4, it preserves all categorical colimits by Proposition A.5.4. A fundamental type of colimit is the coproduct. As shown in [5], $\mathbf{C}^*$ has all coproducts, namely the free product. As such, for an index set I and C*-algebras $(\mathcal{A}_i)_{i\in I}$, their free product will be denoted $\coprod_{i\in I}^{\mathbf{C}^*} \mathcal{A}_i$.

In regard to notation, the free product is usually denoted by "$*$". The "$\coprod$" notation will be used interchangeably with the "$*$" notation, but preference will be given to the "$\coprod$" with arbitrary index sets.

Recall that Proposition 2.2.9 described the "disjoint union" crutched set, which gave a canonical decomposition of a crutched set into singleton crutched sets. Combining this characterization with Proposition 4.2.4, the following canonical form is taken.

Corollary 4.2.5. *Given a crutched set* (S, f)*,*

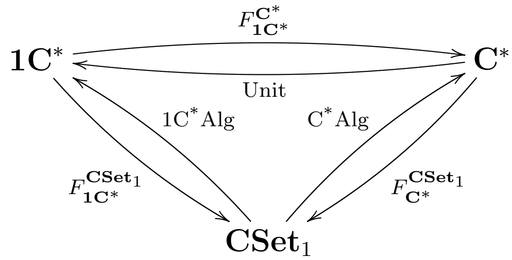
$$\mathrm{C}^*\mathrm{Alg}(S, f) \cong_{\mathbf{C}^*} \coprod_{s\in S_f}^{\mathbf{C}^*} \mathrm{C}^*\mathrm{Alg}\left(\{(s, f(s))\}\right) \cong_{\mathbf{C}^*} \coprod_{s\in S_f}^{\mathbf{C}^*} \mathrm{C}^*\mathrm{Alg}\left(\{(s, 1)\}\right).$$

In the case $\mathrm{card}(S) = 2$ and $f(s) > 0$ for all $s \in S$, this result can be stated in the traditional notation as

$$\mathrm{C}^*\mathrm{Alg}(S, f) \cong_{\mathbf{C}^*} \mathrm{C}^*\mathrm{Alg}\left(\{(s_1, 1)\}\right) * \mathrm{C}^*\mathrm{Alg}\left(\{(s_2, 1)\}\right).$$

Decompositions and characterizations such as this will be used extensively in the remainder of this chapter, particularly Sections 3.9.8 and 4.8.

Moreover, observe the following diagram of categories and functors,



where $F^{\mathbf{C^*}}_{\mathbf{1C^*}} : \mathbf{1C^*} \to \mathbf{C^*}$ denotes the forgetful functor from $\mathbf{1C^*}$ to $\mathbf{C^*}$, and Unit : $\mathbf{C^*} \to \mathbf{1C^*}$ denotes its left adjoint, the unitization as described in Section B.5. A quick check shows that the outer triangle commutes. That is,

$$F^{\mathbf{CSet}_1}_{\mathbf{C^*}} F^{\mathbf{C^*}}_{\mathbf{1C^*}} = F^{\mathbf{CSet}_1}_{\mathbf{1C^*}}.$$

However, Propostion A.5.3 shows that

$$\mathrm{Unit}\,\mathrm{C}^*\mathrm{Alg} \dashv F^{\mathbf{CSet}_1}_{\mathbf{1C^*}}.$$

Since a left adjoint is composed of reflections, the universal property of the reflection yields the following fact.

Theorem 4.2.6. *Given a crutched set* (S, f)*,*

$$\mathrm{Unit}\left(\mathrm{C}^*\mathrm{Alg}(S, f)\right) \cong_{\mathbf{1C^*}} 1\mathrm{C}^*\mathrm{Alg}(S, f).$$

Moreover, $\mathrm{C}^*\mathrm{Alg}(S, f)$ *is* $\mathbf{C^*}$*-isomorphic to the ideal in* $1\mathrm{C}^*\mathrm{Alg}(S, f)$ *generated by* $\eta_{S,f}(S)$ *via the universal *-homomorphism built from* $\eta_{S,f}$ *itself.*

That is, the inner triangle commutes up to isomorphism in $\mathbf{1C^*}$. This is a natural expectation for the unitization and gives a very close connection between the unital theory of Chapter 3 to the general theory of the current chapter.

This relationship will be used extensively in the coming presentation theory and its examples. As such, the isomorphism will be explicitly demonstrated.

Proof. For a crutched set (S, f), let $\mathcal{F} := \mathrm{C}^*\mathrm{Alg}(S, f)$, $\mathcal{G} := 1\mathrm{C}^*\mathrm{Alg}(S, f)$, and $\mathcal{A} := \mathrm{Unit}(\mathcal{F})$. Observe the following diagram in $\mathbf{CSet}_1$,

$$\begin{array}{ccccc} (S,f) & \xrightarrow{\theta_{S,f}} & F_{\mathbf{C}^*}^{\mathbf{CSet}_1}\mathcal{F} & \xrightarrow{F_{\mathbf{C}^*}^{\mathbf{CSet}_1}\iota_{\mathcal{F}}} & F_{\mathbf{1C}^*}^{\mathbf{CSet}_1}\mathcal{A} \\ & \searrow^{\eta_{S,f}} & & & \\ & & F_{\mathbf{1C}^*}^{\mathbf{CSet}_1}\mathcal{G} & & \end{array}$$

where $\iota_{\mathcal{F}} : \mathcal{F} \to \mathcal{A}$ is the inclusion into the unitization from Theorem B.5.2 and $\eta_{S,f} : S \to \mathcal{G}$ the inclusion of generators from Theorem 3.1.4.

Notice that $\eta_{S,f}$ is a constrictive map. By Theorem 4.2.1, there is a unique *-homomorphism $j_{S,f} : \mathcal{F} \to \mathcal{G}$ such that $j_{S,f} \circ \theta_{S,f} = \eta_{S,f}$, just associating the generators. By Theorem B.5.2, there is a unique unital *-homomorphism $\tilde{j}_{S,f} : \mathcal{A} \to \mathcal{G}$ such that $\tilde{j}_{S,f} \circ \iota_{\mathcal{F}} = j_{S,f}$.

Likewise, $F_{\mathbf{C}^*}^{\mathbf{CSet}_1}\iota_{\mathcal{F}} \circ \theta_{S,f}$ is a constrictive map. By Theorem 3.2.1, there is a unique unital *-homomorphism $k_{S,f} : \mathcal{G} \to \mathcal{A}$ such that $k_{S,f} \circ \eta_{S,f} = \iota_{\mathcal{F}} \circ \theta_{S,f}$, associating generators for $\mathcal{G}$ to the generators for $\mathcal{F}$ embedded in $\mathcal{A}$.

Observe that,

$$\begin{aligned} \tilde{j}_{S,f} \circ k_{S,f} \circ \eta_{S,f} &= \tilde{j}_{S,f} \circ \iota_{\mathcal{F}} \circ \theta_{S,f} \\ &= j_{S,f} \circ \theta_{S,f} \\ &= \eta_{S,f} \end{aligned}$$

so by Theorem 3.2.1, $\tilde{j}_{S,f} \circ k_{S,f} = id_{\mathcal{G}}$.

Similarly,

$$\begin{aligned} k_{S,f} \circ \tilde{j}_{S,f} \circ \iota_{\mathcal{F}} \circ \theta_{S,f} &= k_{S,f} \circ j_{S,f} \circ \theta_{S,f} \\ &= k_{S,f} \circ \eta_{S,f} \\ &= \iota_{\mathcal{F}} \circ \theta_{S,f} \end{aligned}$$

so by Theorem 4.2.1, $k_{S,f} \circ \tilde{j}_{S,f} \circ \iota_{\mathcal{F}} = \iota_{\mathcal{F}}$. By Theorem B.5.2, $k_{S,f} \circ \tilde{j}_{S,f} = id_{\mathcal{A}}$.

As in Theorem B.5.3, $j_{S,f}$ is one-to-one, $\operatorname{ran}(j_{S,f})$ is an ideal in $\mathcal{G}$, and $\mathcal{G}/\operatorname{ran}(j_{S,f}) \cong_{\mathbf{1C^*}} \mathbb{C}$. Specifically, let J_S be the ideal generated by $\eta_{S,f}(S)$ in $\mathcal{G}$. Notice that $\eta_{S,f}(S) \subseteq \operatorname{ran}(j_{S,f})$ so $J_S \subseteq \operatorname{ran}(j_{S,f})$. However, by construction, $\theta_{S,f}(S)$ generates $\mathcal{F}$ so $(j_{S,f} \circ \theta_{S,f})(S) = \eta_{S,f}(S)$ generates $\operatorname{ran}(j_{S,f})$. Thus, $\operatorname{ran}(j_{S,f}) \subseteq J_S$.

In conclusion, $\operatorname{ran}(j_{S,f})$ is the ideal generated by $\eta_{S,f}(S)$ in $\mathcal{G}$.

□

The map $j_{S,f}$ within this proof will be key in the coming sections, where this unitization result is extended to an entire presentation theory for general C*-algebras. Lastly, Theorem 4.2.6 combined with Propositions 3.2.6 and B.5.4 gives a quick proof of projectivity of $\mathrm{C}^*\mathrm{Alg}(S, f)$.

Proposition 4.2.7. *Given a crutched set (S, f), $\mathrm{C}^*\mathrm{Alg}(S, f)$ is projective with respect to all surjections in $\mathbf{C^*}$.*

4.3 Presentations for C* & Gerbracht Revisited

As done in Section 3.3, every C*-algebra has a scaled-free C*-algebra which quotients onto it.

Example 4.3.1. Given a C*-algebra $\mathcal{B}$, let $S := \mathcal{B}$, the underlying set of $\mathcal{B}$, and $f : S \to [0, \infty)$ by $f(s) := \|s\|_{\mathcal{B}}$. Define $\phi : S \to \mathcal{B}$ by $\phi(s) := s$, the identity map. Trivially, ϕ is a constriction from (S, f) to $F_{\mathbf{C^*}}^{\mathbf{CSet_1}}\mathcal{B}$. By Theorem 4.2.1, there is a unique *-homomorphism $\hat{\phi} : \mathrm{C}^*\mathrm{Alg}(S, f) \to \mathcal{B}$ such that $\hat{\phi}(\theta_{S,f}(s)) = \phi(s)$ for all $s \in S$. Then, for all $b \in \mathcal{B}$, $b = \hat{\phi}(\theta_{S,f}(b))$. Hence, $\hat{\phi}$ is surjective.

Thus, the following definitions are made, reflecting both the unital case as well as the general algebraic case.

Definition. For a crutched set (S, f), a *non-unital C*-relation* on (S, f) is an element of $\mathrm{C}^*\mathrm{Alg}(S, f)$. An element of $\theta_{S,f}(S)$ itself is called a *generator*.

Recall from Section 3.3 that a *(unital) C*-relation* was an element of $1\mathrm{C}^*\mathrm{Alg}(S, f)$. By Theorem 4.2.6, $j_{S,f}$ embeds $\mathrm{C}^*\mathrm{Alg}(S, f)$ into $1\mathrm{C}^*\mathrm{Alg}(S, f)$ so one may regard $\mathrm{C}^*\mathrm{Alg}(S, f)$ as the ideal generated by $\eta_{S,f}(S)$ within $1\mathrm{C}^*\mathrm{Alg}(S, f)$. Thus, every non-unital C*-relation may be regarded as a C*-relation as defined in Section 3.3. When needed, the distinction between these two types of C*-relations will be explicitly stated.

Definition. For a crutched set (S, f) and non-unital C*-relations $R \subseteq \mathrm{C}^*\mathrm{Alg}(S, f)$ on (S, f), let J_R be the two-sided, norm-closed ideal generated by R in $\mathrm{C}^*\mathrm{Alg}(S, f)$. Then, the *C*-algebra presented on* (S, f) *subject to* R is

$$\langle S, f | R \rangle_{\mathbf{C}^*} := \mathrm{C}^*\mathrm{Alg}(S, f)/J_R,$$

the quotient C*-algebra of $\mathrm{C}^*\mathrm{Alg}(S, f)$ by J_R.

By Example 4.3.1, every C*-algebra has a presentation in this sense. In parallel to the algebraic notion of presentation, the following definitions describe how a particular C*-algebra was formed.

Definition. Let $\mathcal{A}$ be a C*-algebra.

1. $\mathcal{A}$ is *finitely generated* in $\mathbf{C}^*$ if there is a crutched set (S, f) and non-unital C*-relations R on (S, f) such that $\text{card}(S) < \aleph_0$ and $\mathcal{A} \cong_{\mathbf{C}^*} \langle S, f|R\rangle_{\mathbf{C}^*}$.

2. $\mathcal{A}$ is *finitely related* in $\mathbf{C}^*$ if there is a crutched set (S, f) and non-unital C*-relations R on (S, f) such that $\text{card}(R) < \aleph_0$ and $\mathcal{A} \cong_{\mathbf{C}^*} \langle S, f|R\rangle_{\mathbf{C}^*}$.

3. $\mathcal{A}$ is *finitely presented* in $\mathbf{C}^*$ if there is a crutched set (S, f) and non-unital C*-relations R on (S, f) such that $\text{card}(S), \text{card}(R) < \aleph_0$ and $\mathcal{A} \cong_{\mathbf{C}^*} \langle S, f|R\rangle_{\mathbf{C}^*}$.

Analogously, one also defines *countably generated*, *countably related*, and *countably presented* by easing the strict inequality on the cardinalities to allow equality. As with the unital presentation theory of Section 3.3, the following conventions are taken:

1. Elements of $s \in S$ are associated to their images $[\theta_{S,f}(s)] \in \langle S, f|R\rangle_{\mathbf{C}^*}$;

2. C*-relations $r \in R$ will be written equationally, $r = 0$, when appropriate;

3. The scaled-free C*-algebra $\mathrm{C}^*\mathrm{Alg}(S, f)$ will be written as $\langle S, f|\emptyset\rangle_{\mathbf{C}^*}$;

4. For a finite set $S = \{s_1, \ldots, s_n\}$, let $\lambda_j := f(s_j)$ and use the notation

$$\langle (s_1, \lambda_1), \ldots, (s_n, \lambda_n) \,|R\rangle_{\mathbf{C}^*} := \langle S, f|R\rangle_{\mathbf{C}^*}.$$

As a presentation is built out of universal constructions, specifically the adjoint functor $\mathrm{C}^*\mathrm{Alg}$ and the C*-quotient, it satisfies a universal property. The proof is in complete analogy to Theorem 3.3.2

Theorem 4.3.2 (Universal Property of a Presentation)**.** *Let R be non-unital C*-relations on (S, f) and $\mathcal{B}$ a C*-algebra. Let $\phi : (S, f) \to F_{\mathbf{C}^*}^{\mathbf{CSet_1}}\mathcal{B}$ be a constriction and $\hat{\phi} : \langle S, f|\emptyset\rangle_{\mathbf{C}^*} \to \mathcal{B}$ the *-homomorphism guaranteed by Theorem 4.2.1. If $R \subseteq$*

$\ker\left(\hat{\phi}\right)$*, then there is a unique *-homomorphism* $\tilde{\phi} : \langle S, f|R\rangle_{\mathbf{C}^*} \to \mathcal{B}$ *such that* $\tilde{\phi}(s) = \phi(s)$.

As cited throughout this work, a very similar presentation theory for C*-algebras was constructed in [19]. With the non-unital presentation theory of the present work defined, the comparison with the work of [19] can be made explicit and formal.

First, recall the definitions and theorems used in [19]. Given a set M, let $\mathbb{C}^*\langle M\rangle$ stand for the free *-algebra over $\mathbb{C}$ on M. Let $\eta_1 : M \to \mathbb{C}^*\langle M\rangle$ be the map $\eta_1(m) := m$, associating the generators to their images in $\mathbb{C}^*\langle M\rangle$.

Elements of $\mathbb{C}^*\langle M\rangle$ will be termed here **-algebraic relations* on M. Given a set of *-algebraic relations $R \subseteq \mathbb{C}^*\langle M\rangle$, let K_R be the two-sided *-ideal generated by R in $\mathbb{C}^*\langle M\rangle$. Then,

$$\mathbb{C}^*\langle M, R\rangle := \mathbb{C}^*\langle M\rangle / K_R,$$

the quotient *-algebra of $\mathbb{C}^*\langle M\rangle$ by K_R. Let $\pi_R : \mathbb{C}^*\langle M\rangle \to \mathbb{C}^*\langle M, R\rangle$ be the quotient map. Section 2.1 of [19] constructs a presentation theory for *-algebras over $\mathbb{C}$, providing a Tietze transformation theorem in Proposition 2.1.2.11.

Let $\mu : M \to [0, \infty)$ be a nonnegative-valued function on M and define

$$S_{R,\mu} := \left\{ \rho : \mathbb{C}^*\langle M, R\rangle \to [0,\infty) : \begin{array}{c} \rho \text{ is a C*-semi-norm on } \mathbb{C}^*\langle M, R\rangle, \\ \left(\rho \circ \pi_R \circ \eta_1\right)(m) \leq \mu(m) \forall m \in M \end{array} \right\},$$

the set of all C*-semi-norms on $\mathbb{C}^*\langle M, R\rangle$ bounded by μ. Let

$$N_{R,\mu} := \bigcap_{\rho \in S_{R,\mu}} \rho^{-1}(0),$$

the set of all elements in $\mathbb{C}^*\langle M, R\rangle$ annihilated by all C*-semi-norms in $S_{R,\mu}$. This is naturally a *-ideal of $\mathbb{C}^*\langle M, R\rangle$ so let $\eta_{2,R,\mu} : \mathbb{C}^*\langle M, R\rangle \to \mathbb{C}^*\langle M, R\rangle / N_{R,\mu}$ be the quotient map.

Proposition 1.4.11 of [19] shows that the function on $\mathbb{C}^*\langle M, R\rangle / N_{R,\mu}$ defined by

$$\|\pi_R(x) + N_{R,\mu}\|_{\sup} := \sup\left\{\inf\left\{\rho\left(\pi_R(x) + z\right) : z \in N_{R,\mu}\right\} : \rho \in S_{R,\mu}\right\}$$

is a C*-norm. Let $C^*\langle M, R, \mu\rangle$ denote the completion of $\mathbb{C}^*\langle M, R\rangle / N_{R,\mu}$ in this norm and $\eta_3 : \mathbb{C}^*\langle M, R\rangle / N_{R,\mu} \to C^*\langle M, R, \mu\rangle$ the inclusion of the dense subalgebra.

Lastly, given any *-algebra B over $\mathbb{C}$ and function $\sigma : M \to B$, the universal property of the free *-algebra $\mathbb{C}^*\langle M\rangle$ guarantees a unique *-homomorphism $\hat{\sigma} : \mathbb{C}^*\langle M\rangle \to B$ such that $\hat{\sigma} \circ \eta_1 = \sigma$. For any $r \in \mathbb{C}^*\langle M\rangle$, define

$$\hat{r}(\sigma) := \hat{\sigma}(r),$$

the evaluation of r under the universal map $\hat{\sigma}$.

Part 4 of Proposition 1.4.12 in [19] gives the universal property of $C^*\langle M, R, \mu\rangle$ in terms of of this notation. For comparison to the theory of the present work, this result will be stated in the terminology of the present work.

Theorem (1.4.12, part 4, [19])**.** *If $\mathcal{B}$ is a C*-algebra, then any constriction $\sigma : (M, \mu) \to F_{\mathbf{C^*}}^{\mathbf{CSet_1}}\mathcal{B}$ which satisfies $\hat{r}(\sigma) = 0$ for all $r \in R$ can be extended to a unique *-homomorphism $\Theta_2 : C^*\langle M, R, \mu\rangle \to \mathcal{B}$, so that*

$$\Theta_2 \circ \eta_3 \circ \eta_{2,R,\mu} \circ \pi_R \circ \eta_1 = \sigma.$$

Part 1 of Proposition 2.2.5 in [19] contains the following norm result for generators.

Theorem (2.2.5, part 1, [19])**.** *For all* $m \in M$*,*

$$\left\| \left(\eta_3 \circ \eta_{2,R,\mu} \circ \pi_R \circ \eta_1\right)(m) \right\|_{C^*\langle M,R,\mu\rangle} \leq \mu(m).$$

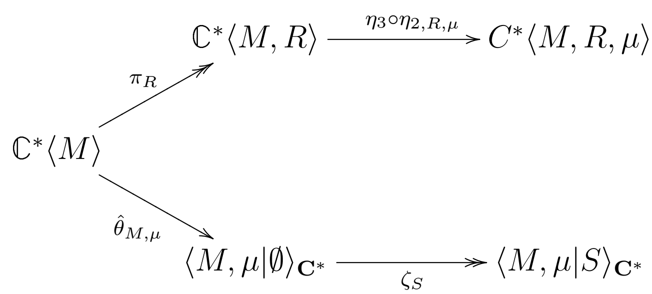
Fix a crutched set (M, μ) and *-algebraic relations R on M. The above construction gives a C*-algebra $C^*\langle M, R, \mu\rangle$. The objective here is to build a presentation in the theory of the current work which serves the same role.

To that end, let $\theta_{M,\mu} : (M, \mu) \to \langle M, \mu | \emptyset\rangle_{\mathbf{C^*}}$ be the association of generators for the scaled-free C*-algebra on (M, μ) from Theorem 4.1.4. By the universal property of the free *-algebra, there is a unique *-homomorphism $\hat{\theta}_{M,\mu} : \mathbb{C}^*\langle M\rangle \to \langle M, \mu|\emptyset\rangle_{\mathbf{C^*}}$ such that $\hat{\theta}_{M,\mu} \circ \eta_1 = \theta_{M,\mu}$.

Let $S := \hat{\theta}_{M,\mu}(R)$, the image of the *-algebraic relations R under $\hat{\theta}_{M,\mu}$. Then, S is a set of non-unital C*-relations on (M, μ) so one can form $\langle M, \mu|S\rangle_{\mathbf{C^*}}$. Let $\zeta_S : \langle M, \mu|\emptyset\rangle_{\mathbf{C^*}} \to \langle M, \mu|S\rangle_{\mathbf{C^*}}$ be the quotient map. Observe that a restatement of Theorem 4.3.2 is the following universal property:

> Given a C*-algebra $\mathcal{B}$ and a constriction $\phi : (M, \mu) \to F^{\mathbf{CSet_1}}_{\mathbf{C^*}}\mathcal{B}$, let $\hat{\phi} : \langle M, \mu|\emptyset\rangle_{\mathbf{C^*}} \to \mathcal{B}$ be the *-homomorphism guaranteed by Theorem 4.2.1. If $S \subseteq \ker\left(\hat{\phi}\right)$, then there exists a unique *-homomorphism $\tilde{\phi}$: $\langle M, \mu|S\rangle_{\mathbf{C^*}} \to \mathcal{B}$ such that $\tilde{\phi} \circ \zeta_S \circ \theta_{M,\mu} = \phi$.

Viewing the constructions of $C^*\langle M, R, \mu\rangle$ and $\langle M, \mu|S\rangle_{\mathbf{C^*}}$, the fundamental difference between them is the order in which certain universal constructions are done. With $C^*\langle M, R, \mu\rangle$, $\mathbb{C}^*\langle M\rangle$ is quotiented by K_R, normed by $\|\cdot\|_{\sup}$, and then completed. However, with $\langle M, \mu|S\rangle_{\mathbf{C^*}}$, $\mathbb{C}^*\langle M\rangle$ is normed by $\tau_{M,\mu}$, completed, and then quotiented by J_S. Diagrammatically, these processes are shown below in the category of *-algebras and *-homomorphisms.



The question here is if the resulting C*-algebras are isomorphic in $\mathbf{C}^*$. Does the order of these processes matter?

In actuality, the order does not matter. These two C*-algebras are indeed isomorphic in $\mathbf{C}^*$.

Theorem 4.3.3. *Given a crutched set* (M, μ) *and *-algebraic relations* R *on* M*,*

$$C^*\langle M, R, \mu\rangle \cong_{\mathbf{C}^*} \left\langle M, \mu \middle| \hat{\theta}_{M,\mu}(R) \right\rangle_{\mathbf{C}^*}.$$

Proof. Let $\mathcal{A} := C^*\langle M, R, \mu\rangle$ and $\mathcal{B} := \langle M, \mu | S\rangle_{\mathbf{C}^*}$.

By Proposition 2.2.5 in [19],

$$\eta_3 \circ \eta_{2,R,\mu} \circ \pi_R \circ \eta_1 : (M, \mu) \to F_{\mathbf{C}^*}^{\mathbf{CSet}_1} \mathcal{A}$$

is a constrictive map. By Theorem 4.2.1, there exists a unique *-homomorphism $\phi : \langle M, \mu | \emptyset\rangle_{\mathbf{C}^*} \to \mathcal{B}$ such that

$$\eta_3 \circ \eta_{2,R,\mu} \circ \pi_R \circ \eta_1 = \phi \circ \theta_{M,\mu}.$$

Observe that

$$\begin{aligned}\phi \circ \hat{\theta}_{M,\mu} \circ \eta_1 &= \phi \circ \theta_{M,\mu} \\ &= \eta_3 \circ \eta_{2,R,\mu} \circ \pi_R \circ \eta_1\end{aligned}$$

so by the universal property of $\mathbb{C}^*\langle M\rangle$, $\phi \circ \hat{\theta}_{M,\mu} = \eta_3 \circ \eta_{2,R,\mu} \circ \pi_R$. For all $r \in R$,

$$\begin{aligned}\left(\phi \circ \hat{\theta}_{M,\mu}\right)(r) &= (\eta_3 \circ \eta_{2,R,\mu} \circ \pi_R)(r) \\ &= (\eta_3 \circ \eta_{2,R,\mu})(0) \\ &= 0.\end{aligned}$$

Thus, $S \subseteq \ker(\phi)$ so by Theorem 4.3.2, there exists a unique *-homomorphism $\tilde{\phi} : \mathcal{B} \to \mathcal{A}$ such that

$$\tilde{\phi} \circ \zeta_S \circ \theta_{M,\mu} = \eta_3 \circ \eta_{2,R,\mu} \circ \pi_R \circ \eta_1.$$

By the universal property of $\mathbb{C}^*\langle M\rangle$, there is a unique *-homomorphism $\psi : \mathbb{C}^*\langle M\rangle \to \mathcal{B}$ such that $\zeta_S \circ \theta_{M,\mu} = \psi \circ \eta_1$. Note that

$$\begin{aligned}\psi \circ \eta_1 &= \zeta_S \circ \theta_{M,\mu} \\ &= \zeta_S \circ \hat{\theta}_{M,\mu} \circ \eta_1\end{aligned}$$

so by the universal property of $\mathbb{C}^*\langle M\rangle$, $\psi = \zeta_S \circ \hat{\theta}_{M,\mu}$. For all $r \in R$,

$$\begin{aligned}\hat{r}\left(\zeta_S \circ \hat{\theta}_{M,\mu}\right) &= \left(\zeta_S \circ \hat{\theta}_{M,\mu}\right)(r) \\ &= 0.\end{aligned}$$

By Lemma 4.1.3 and the contractivity of *-homomorphisms between C*-algebras, observe that for all $m \in M$,

$$\begin{aligned}\|(\zeta_S \circ \theta_{M,\mu})(m)\|_{\mathcal{B}} &\leq \|\theta_{M,\mu}(m)\|_{\langle M,\mu|\emptyset\rangle_{\mathbf{C}^*}} \\ &\leq \mu(m)\end{aligned}$$

so $\zeta_S \circ \theta_{M,\mu} : (M, \mu) \to F_{\mathbf{C^*}}^{\mathbf{CSet}_1}\mathcal{B}$ is constrictive. By Theorem 1.4.12 of [19], there exists a unique *-homomorphism $\tilde{\psi} : \mathcal{A} \to \mathcal{B}$ such that

$$\tilde{\psi} \circ \eta_3 \circ \eta_{2,R,\mu} \circ \pi_R \circ \eta_1 = \zeta_S \circ \theta_{M,\mu}.$$

Observe that

$$\begin{aligned} \tilde{\psi} \circ \tilde{\phi} \circ \zeta_S \circ \theta_{M,\mu} &= \tilde{\psi} \circ \eta_3 \circ \eta_{2,R,\mu} \circ \pi_R \circ \eta_1 \\ &= \zeta_S \circ \theta_{M,\mu}. \end{aligned}$$

By Theorem 4.3.2, $\tilde{\psi} \circ \tilde{\phi} = id_{\mathcal{B}}$. Similarly,

$$\begin{aligned} \tilde{\phi} \circ \tilde{\psi} \circ \eta_3 \circ \eta_{2,R,\mu} \circ \pi_R \circ \eta_1 &= \tilde{\phi} \circ \zeta_S \circ \theta_{M,\mu} \\ &= \eta_3 \circ \eta_{2,R,\mu} \circ \pi_R \circ \eta_1. \end{aligned}$$

By Theorem 1.4.12 of [19], $\tilde{\phi} \circ \tilde{\psi} = id_{\mathcal{A}}$.

□

This result shows that the work of [19] is recaptured and extended by the notion of C*-relations. In particular, the work of [19] cannot account for “analytic” relations such as $\sin(x) = 0$ or “continuous” relations such as $x \geq 0$. Many of the results stated in [19], particularly when building crossed products and extensions, necessitated the action be restricted to the image of the free *-algebra, which is only norm-dense in the presented C*-algebra.

The present work has removed these restrictions, allowing more elements to be considered “relations” for manipulation. As shown in Chapter 3, this has allowed characterization of several functional analytic notions, particularly by use of the functional calculus. The remainder of this chapter will be dedicated not only to mimicking the unital results of Chapter 3, but also to extending the constructions of [19] to the case of non-unital C*-relations.

4.4 Construction: Unitization & Presentations in $\mathbf{1C}^*$

Before computing examples, two familiar constructions are characterized first to make these calculations easier to manage. A well-known construction to many is the unitization, a canonical way of making an algebra unital. In Section B.5, the unitization is realized as a left adjoint functor to a natural forgetful functor.

As illustrated in Theorem 4.2.6, there is a tight relationship between the scaled-free C*-algebra and the unital scaled-free C*-algebra: the latter is the unitization of the former. Said another way, for any crutched set (S, f),

$$\mathrm{Unit}\left(\langle S, f|\emptyset\rangle_{\mathbf{C}^*}\right) \cong_{\mathbf{1C}^*} \langle S, f|\emptyset\rangle_{\mathbf{1C}^*}.$$

One would like that this is true for any set of non-unital C*-relations R on (S, f). Indeed, this fact does still hold in the following way. Let $j_{S,f} : \langle S, f|\emptyset\rangle_{\mathbf{C}^*} \to \langle S, f|\emptyset\rangle_{\mathbf{1C}^*}$ be the inclusion devised in Theorem 4.2.6.

Theorem 4.4.1. *Given a crutched set (S, f) and non-unital C*-relations R on (S, f),*

$$\mathrm{Unit}\left(\langle S, f|R\rangle_{\mathbf{C}^*}\right) \cong_{\mathbf{1C}^*} \left\langle S, f\left|j_{S,f}(R)\right.\right\rangle_{\mathbf{1C}^*}.$$

Moreover, $\langle S, f|R\rangle_{\mathbf{C}^*}$ is $\mathbf{C}^*$-isomorphic to the ideal generated by $(q_R \circ \eta_{S,f})(S)$ in $\left\langle S, f\left|j_{S,f}(R)\right.\right\rangle_{\mathbf{1C}^*}$, where $q_R : \langle S, f|\emptyset\rangle_{\mathbf{1C}^*} \to \left\langle S, f\left|j_{S,f}(R)\right.\right\rangle_{\mathbf{1C}^*}$ is the quotient map.

Proof. Let $\mathcal{F} := \langle S, f|\emptyset\rangle_{\mathbf{C}^*}$, $\mathcal{G} := \langle S, f\,|\emptyset\rangle_{\mathbf{1C}^*}$, and $j_{S,f} : \mathcal{F} \to \mathcal{G}$ be as in Theorem

4.2.6. Let $\mathcal{A} := \langle S, f \,|R\rangle_{\mathbf{C^*}}$ and $\zeta_R : \mathcal{F} \to \mathcal{A}$ be the quotient map. Likewise, let $\mathcal{B} := \langle S, f \,|j_{S,f}(R)\rangle_{\mathbf{1C^*}}$ and $q_R : \mathcal{G} \to \mathcal{B}$ be the quotient map. The diagram in $\mathbf{C}^*$ below illustrates this setup.

$$\begin{array}{ccc} \mathcal{F} & \overset{j_{S,f}}{\rightarrowtail} & \mathcal{G} \\ {\scriptstyle \zeta_R}\downarrow & & \downarrow{\scriptstyle q_R} \\ \mathcal{A} & & \mathcal{B} \end{array}$$

Observe that for all $r \in R$, $(q_R \circ j_{S,f})(r) = 0$ so by Theorem 4.3.2, there is a unique $j_{S,f,R} : \mathcal{A} \to \mathcal{B}$ such that $j_{S,f,R} \circ \zeta_R = q_R \circ j_{S,f}$. The unital C*-algebra $\mathcal{B}$ equipped with $j_{S,f,R}$ is a candidate for the unitization of $\mathcal{A}$.

To check the universal property, let $\mathcal{C}$ be a unital C*-algebra and $\phi : \mathcal{A} \to \mathcal{C}$ a *-homomorphism.

$$\begin{array}{ccc} \mathcal{F} & \overset{j_{S,f}}{\rightarrowtail} & \mathcal{G} \\ {\scriptstyle \zeta_R}\downarrow & & \downarrow{\scriptstyle q_R} \\ \mathcal{A} & \underset{j_{S,f,R}}{\longrightarrow} & \mathcal{B} \\ {\scriptstyle \phi}\downarrow & & \\ \mathcal{C} & & \end{array}$$

By Theorem 4.2.6, there is a unique unital *-homomorphism $\hat{\phi} : \mathcal{G} \to \mathcal{C}$ such that $\hat{\phi} \circ j_{S,f} = \phi \circ \zeta_R$. For all $r \in R$,

$$\begin{aligned} \hat{\phi}\left(j_{S,f}(r)\right) &= (\phi \circ \zeta_R)(r) \\ &= \phi(0) \\ &= 0 \end{aligned}$$

so by Theorem 3.3.2, there is a unique unital *-homomorphism $\tilde{\phi} : \mathcal{B} \to \mathcal{C}$ such that

$\tilde{\phi} \circ q_R = \hat{\phi}$. Observe that

$$\begin{aligned}\tilde{\phi} \circ j_{S,f,R} \circ \zeta_R &= \tilde{\phi} \circ q_R \circ j_{S,f} \\ &= \hat{\phi} \circ j_{S,f} \\ &= \phi \circ \zeta_R.\end{aligned}$$

By Theorem 4.3.2, $\tilde{\phi} \circ j_{S,f,R} = \phi$.

Assume there was $\psi : \mathcal{B} \to \mathcal{C}$ such that $\psi \circ j_{S,f,R} = \phi$. Then,

$$\begin{aligned}\phi \circ \zeta_R &= \psi \circ j_{S,f,R} \circ \zeta_R \\ &= \psi \circ q_R \circ j_{S,f}\end{aligned}$$

By Theorem 4.2.6, $\psi \circ q_R = \hat{\phi} = \tilde{\phi} \circ q_R$. Therefore, $\psi = \tilde{\phi}$ by Theorem 3.3.2.

By Theorem B.5.3, $j_{S,f,R}$ is one-to-one, $\operatorname{ran}(j_{S,f,R})$ is an ideal in $\mathcal{B}$, and

$$\mathcal{B}/\operatorname{ran}(j_{S,f,R}) \cong_{\mathbf{1C^*}} \mathbb{C}.$$

Following a similar proof as in Theorem 4.2.6, $\operatorname{ran}(j_{S,f,R})$ is the ideal generated by $(q_R \circ \eta_{S,f})(S)$ in $\mathcal{B}$.

$\square$

This result formalizes the natural intuition. The unital presentation is constructed from the images of the generators and the unit. Therefore, the non-unital version should be, and is, the C*-algebra built from the generators without the unit's involvement.

Symmetrically, this intuition gives a natural way to think of the unitization. The non-unital presentation is constructed from the set of generators so the unitization should be formed by appending a new generator and relations enforcing that it acts as the unit. The following construction was previously considered in Section 3.1 of [19] for *-algebraic relations. Here, the same rationale is used for non-unital C*-relations.

To begin, let $(S_1, f_1) := (S, f)\coprod^{\mathbf{CSet}_1}\{(u,1)\}$ be the disjoint union described in Proposition 2.2.9 and $\rho_1 : (S,f) \to (S_1, f_1)$ the canonical inclusion of (S,f). Applying the functor $\mathrm{C}^*\mathrm{Alg}$, $\hat{\rho}_1 := \mathrm{C}^*\mathrm{Alg}\left(\rho_1\right)$ maps $\langle S, f|\emptyset\rangle_{\mathbf{C}^*}$ to $\langle S_1, f_1|\emptyset\rangle_{\mathbf{C}^*}$ by association of generators. Define the set of non-unital C*-relations on (S_1, f_1) by

$$\hat{R} := \hat{\rho}_1(R) \cup \{su - s, us - s, s^*u - s^*, us^* - s^* : s \in S_1\},$$

encoding the non-unital C*-relations R on (S,f) as well as a trivial action of u on (S_1, f_1). The presentation $\left\langle S_1, f_1 \middle| \hat{R} \right\rangle_{\mathbf{C}^*}$ is yet another way of representing the unitization, an extension of Proposition 3.1.2 of [19].

Theorem 4.4.2. *Given a crutched set (S,f) and non-unital C*-relations R on (S,f),*

$$\mathrm{Unit}\left(\langle S, f|R\rangle_{\mathbf{C}^*}\right) \cong_{\mathbf{1C}^*} \left\langle S_1, f_1 \left| \hat{\rho}_1(R) \cup \{su - s, us - s, s^*u - s^*, us^* - s^* : s \in S_1\}\right.\right\rangle_{\mathbf{C}^*}$$

Proof. Let $\mathcal{F} := \langle S, f|\emptyset\rangle_{\mathbf{C}^*}$, $\mathcal{A} := \langle S, f|R\rangle_{\mathbf{C}^*}$, and $\zeta_R : \mathcal{F} \to \mathcal{A}$ be the quotient map. Likewise, let $\mathcal{H} := \langle S_1, f_1|\emptyset\rangle_{\mathbf{C}^*}$, $\mathcal{B} := \left\langle S_1, f_1 \middle| \hat{R} \right\rangle_{\mathbf{C}^*}$, and $\zeta_{\hat{R}} : \mathcal{H} \to \mathcal{B}$ be the quotient map. Lastly, let $\hat{\rho}_1 : \mathcal{F} \to \mathcal{H}$ be the mapping determined by inclusion of generators described above. Visually, this situation is described in the diagram below.

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\hat{\rho}_1} & \mathcal{H} \\ \downarrow{\scriptstyle \zeta_R} & & \downarrow{\scriptstyle \zeta_{\hat{R}}} \\ \mathcal{A} & & \mathcal{B} \end{array}$$

Observe that for all $r \in R$, $\left(\zeta_{\hat{R}} \circ \hat{\rho}_1\right)(r) = 0$ so by Theorem 4.3.2, there is a unique *-homomorphism $\iota : \mathcal{A} \to \mathcal{B}$ such that $\iota \circ \zeta_R = \zeta_{\hat{R}} \circ \hat{\rho}_1$.

In $\mathcal{B}$, note that $su = us = u$ and $s^*u = us^* = s^*$ for all $s \in S_1$. Thus, for any *-polynomial p in S_1, $pu = pu = p$ also. Since the *-polynomials are norm-dense in $\mathcal{B}$, for any $b \in \mathcal{B}$, there is a sequence $(p_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}$ of *-polynomials such that $\lim\limits_{n\to\infty} p_n = b$. Therefore,

$$\begin{aligned} bu &= \lim_{n\to\infty} p_n u \\ &= \lim_{n\to\infty} p_n \\ &= b \end{aligned}$$

and similarly, $ub = b$. Hence, u is a unit for $\mathcal{B}$. The unital C*-algebra $\mathcal{B}$ equipped with ι is a candidate for the unitization of $\mathcal{A}$.

To check the universal property, let $\mathcal{C}$ be a unital C*-algebra and $\phi : \mathcal{A} \to \mathcal{C}$ be a *-homomorphism.

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\hat{\rho}_1} & \mathcal{H} \\ {\scriptstyle \zeta_R}\downarrow & & \downarrow{\scriptstyle \zeta_{\hat{R}}} \\ \mathcal{A} & \xrightarrow[\iota]{} & \mathcal{B} \\ {\scriptstyle \phi}\downarrow & & \\ \mathcal{C} & & \end{array}$$

Define $\alpha : S_1 \to F_{\mathbf{C^*}}^{\mathbf{CSet_1}}\mathcal{C}$ by

$$\alpha(x) := \begin{cases} (\phi \circ \zeta_R \circ \theta_{S,f})(x), & x \in S, \\ \mathbb{1}_{\mathcal{C}}, & x = u. \end{cases}$$

By the contractivity of *-homomorphisms between C*-algebras and Lemma 4.1.3, for all $x \in S$,

$$\begin{aligned} \left\|(\phi \circ \zeta_R \circ \theta_{S,f})(x)\right\|_{\mathcal{C}} &\leq \left\|(\zeta_R \circ \theta_{S,f})(x)\right\|_{\mathcal{A}} \\ &\leq \left\|\theta_{S,f}(x)\right\|_{\mathcal{F}} \\ &\leq f(x). \end{aligned}$$

Also, $\|\mathbb{1}_{\mathcal{C}}\|_{\mathcal{C}} \leq 1$ so by Theorem 4.2.1, there is a unique *-homomorphism $\hat{\phi} : \mathcal{H} \to \mathcal{C}$

such that $\hat{\phi} \circ \theta_{S_1,f_1} = \alpha$. Observe that for all $s \in S$,

$$\left(\hat{\phi} \circ \hat{\rho}_1 \circ \theta_{S,f}\right)(s) \;=\; \left(\phi \circ \zeta_R \circ \theta_{S,f}\right)(s)$$

so $\hat{\phi} \circ \hat{\rho}_1 \circ \theta_{S,f} = \phi \circ \zeta_R \circ \theta_{S,f}$. By Theorem 4.2.1, $\hat{\phi} \circ \hat{\rho}_1 = \phi \circ \zeta_R$. For all $r \in R$,

$$\begin{aligned}\left(\hat{\phi} \circ \hat{\rho}_1\right)(r) &= \left(\phi \circ \zeta_R\right)(r) \\ &= \phi(0) \\ &= 0.\end{aligned}$$

Also,

$$\begin{aligned}\left(\hat{\phi} \circ \theta_{S_1,f_1}\right)(u) &= \alpha(u) \\ &= \mathbb{1}_{\mathcal{C}}.\end{aligned}$$

Thus,

$$\begin{aligned}\hat{\phi}(su - s) &= \hat{\phi}(s)\mathbb{1}_{\mathcal{C}} - \hat{\phi}(s) \\ &= \hat{\phi}(s) - \hat{\phi}(s) \\ &= 0,\end{aligned}$$

and likewise,

$$\hat{\phi}(us - s) = \hat{\phi}\left(us^* - s\right) = \hat{\phi}\left(s^* u - s\right) = 0.$$

By Theorem 4.3.2, there is a unique *-homomorphism $\tilde{\phi} : \mathcal{B} \to \mathcal{C}$ such that $\tilde{\phi} \circ \zeta_{\hat{R}} = \hat{\phi}$. Note that

$$\begin{aligned}\tilde{\phi} \circ \iota \circ \zeta_R &= \tilde{\phi} \circ \zeta_{\hat{R}} \circ \hat{\rho}_1 \\ &= \hat{\phi} \circ \hat{\rho}_1 \\ &= \phi \circ \zeta_R\end{aligned}$$

so by Theorem 4.3.2, $\tilde{\phi} \circ \iota = \phi$. Also, $\tilde{\phi}(u) = \mathbb{1}_{\mathcal{C}}$.

Assume $\psi : \mathcal{B} \to \mathcal{C}$ is a unital *-homomorphism such that $\psi \circ \iota = \phi$. Then, for all

$s \in S$,

$$\begin{aligned}(\psi \circ \zeta_{\hat{R}} \circ \theta_{S_1,f_1})(s) &= (\psi \circ \zeta_{\hat{R}} \circ \hat{\rho}_1 \circ \theta_{S,f})(s) \\ &= (\psi \circ \iota \circ \zeta_R \circ \theta_{S,f})(s) \\ &= (\phi \circ \zeta_R \circ \theta_{S,f})(s) \\ &= \alpha(s)\end{aligned}$$

and

$$(\psi \circ \zeta_{\hat{R}} \circ \theta_{S_1,f_1})(u) = \mathbb{1}_{\mathcal{C}} = \alpha(u)$$

so by Theorem 4.2.1, $\psi \circ \zeta_{\hat{R}} = \hat{\phi} = \tilde{\phi} \circ \zeta_{\hat{R}}$. Finally, $\psi = \tilde{\phi}$ by Theorem 4.3.2.

□

4.5 Construction: Abelianization for $\mathbf{C}^*$

The next construction is the abelianization, just as discussed in Sections 3.4 and B.4. For notation, let $\mathbf{CC}^*$ denote the category of commutative C*-algebras with *-homomorphisms and $\mathrm{Ab} : \mathbf{C}^* \to \mathbf{CC}^*$ the abelianization functor.

For the remainder of this section, fix a crutched set (S, f) and non-unital C*-relations R on (S, f). Composing a presentation for $\mathrm{Ab}\left(\langle S, f | R\rangle_{\mathbf{C}^*}\right)$ is straightforward and natural, merely forcing the generators and their adjoints to commute. The proof is identical to Proposition 3.4.1. The result was proven in Proposition 3.2.2 of [19] in the case of *-algebraic relations.

Theorem 4.5.1. *Given a crutched set (S, f) and non-unital C*-relations R on (S, f),*

$$\mathrm{Ab}\left(\langle S, f | R\rangle_{\mathbf{C}^*}\right) \cong_{\mathbf{CC}^*} \langle S, f \,| R \cup \{st - ts, st^* - t^*s : s, t \in S\}\rangle_{\mathbf{C}^*}.$$

However, as in Theorem 3.4.4, this can be significantly improved. First, observe the following isomorphisms from Proposition 3.4.1, Theorem 4.4.1, and Theorem

4.5.1.

$$\begin{aligned}\mathrm{Unit}\left(\mathrm{Ab}\left(\langle S,f|R\rangle_{\mathbf{C}^*}\right)\right) &\cong_{\mathbf{C1C}^*} \mathrm{Unit}\left(\left\langle S,f\,|R\cup\{st-ts,st^*-t^*s:s,t\in S\}\right\rangle_{\mathbf{C}^*}\right)\\ &\cong_{\mathbf{C1C}^*} \left\langle S,f\,|j_{S,f}(R)\cup\{st-ts,st^*-t^*s:s,t\in S\}\right\rangle_{\mathbf{1C}^*}\\ &\cong_{\mathbf{C1C}^*} \mathrm{Ab}_1\left(\left\langle S,f\,|j_{S,f}(R)\right\rangle_{\mathbf{1C}^*}\right)\\ &\cong_{\mathbf{C1C}^*} \mathrm{Ab}_1\left(\mathrm{Unit}\left(\left\langle S,f\,|R\right\rangle_{\mathbf{C}^*}\right)\right)\end{aligned}$$

This gives a proof of the functorial relationship between unitization and abelianization in Theorem B.6.1 by means of formal manipulation. However, the penultimate form in the above calculation allows direct use of Theorem 3.4.4 to concretely realize the unitized algebra.

As in Section 3.4, let $\mathcal{F} := \langle S,f|\emptyset\rangle_{\mathbf{1C}^*}$,

$$X := \prod_{s\in S}^{\mathbf{Comp}} D_{f(s)},$$

and for $\vec{x}\in X$, $\hat{\phi}_{\vec{x}}:\mathcal{F}\to\mathbb{C}$ the unique unital *-homomorphism such that $\hat{\phi}_{\vec{x}}(s) = \pi_s(\vec{x})$ for all $s\in S$. To simplify notation, define $T := j_{S,f}(R)$, the non-unital C*-relations considered in the unital scaled-free C*-algebra $\mathcal{F}$ as in Theorem 4.4.1. For $t\in T$, let $g_t : X\to\mathbb{C}$ by $g_t(\vec{x}) := \hat{\phi}_{\vec{x}}(t)$ as in Lemma 3.4.3. Applying Theorem 3.4.4,

$$\mathrm{Unit}\left(\mathrm{Ab}\left(\langle S,f|R\rangle_{\mathbf{C}^*}\right)\right) \cong_{\mathbf{C1C}^*} \mathrm{Ab}_1\left(\langle S,f\,|T\rangle_{\mathbf{1C}^*}\right) \cong_{\mathbf{C1C}^*} C\left(\bigcap_{t\in T} g_t^{-1}(0)\right).$$

This characterization yields the following more general result.

Theorem 4.5.2. *For a crutched set (S,f) and non-unital C*-relations R on (S,f),*

$$\mathrm{Ab}\left(\langle S,f|R\rangle_{\mathbf{C}^*}\right) \cong_{\mathbf{CC}^*} C_0\left(\left(\bigcap_{t\in T} g_t^{-1}(0)\right)\setminus\left\{\vec{0}\right\}\right).$$

Proof. Let $X_T := \bigcap_{t \in T} g_t^{-1}(0)$. Notice that for all $s \in S$,

$$\hat{\phi}_{\vec{0}}(s) = 0$$

so $S \subseteq \ker\left(\hat{\phi}_{\vec{0}}\right)$. By Theorem 4.2.6, $\operatorname{ran}\left(j_{S,f}\right)$ is the maximal ideal generated by S in $\mathcal{F}$, meaning $\operatorname{ran}\left(j_{S,f}\right) = \ker\left(\hat{\phi}_{\vec{0}}\right)$. Hence, for all $r \in R$,

$$\begin{aligned} g_{j_{S,f}(r)}\left(\vec{0}\right) &= \left(\hat{\phi}_{\vec{0}} \circ j_{S,f}\right)(r) \\ &= 0, \end{aligned}$$

giving $\vec{0} \in X_T$.

To handle notation, let

$$\hat{R} := R \cup \{st - ts, st^* - t^*s : s, t \in S\},$$

$\mathcal{A} := \left\langle S, f \middle| \hat{R} \right\rangle_{\mathbf{C}^*}$, $\mathcal{B} := \left\langle S, f \middle| j_{S,f}\left(\hat{R}\right) \right\rangle_{\mathbf{1C}^*}$, and $j_{S,f,\hat{R}} : \mathcal{A} \to \mathcal{B}$ be the unitization map from Theorem 4.4.1. From Theorem 3.4.4, recall the Gelfand isomorphism $\tilde{\psi} : \mathcal{B} \to C\left(X_T\right)$ by $\tilde{\psi}(s) = \rho_s$, where $\rho_s : X_T \to D_{f(s)}$ is the coordinate projection map.

Consider the evaluation map $\epsilon_{\vec{0}} : C\left(X_T\right) \to \mathbb{C}$ by $\epsilon_{\vec{0}}(a) := a\left(\vec{0}\right)$, a well-known unital *-homomorphism. Observe that for all $s \in S$,

$$\epsilon_{\vec{0}}\left(\rho_s\right) = \rho_s\left(\vec{0}\right) = 0.$$

Therefore, $\tilde{\psi}\left(j_{S,f,\hat{R}}(s)\right) = \rho_s \in \ker\left(\epsilon_{\vec{0}}\right)$ for all $s \in S$. By Theorems 3.4.4 and 4.4.1,

$$\tilde{\psi}\left(\operatorname{ran}\left(j_{S,f,\hat{R}}\right)\right) \subseteq \ker\left(\epsilon_{\vec{0}}\right).$$

Applying $\tilde{\psi}^{-1}$,

$$\operatorname{ran}\left(j_{S,f,\hat{R}}\right) \subseteq \tilde{\psi}^{-1}\left(\ker\left(\epsilon_{\vec{0}}\right)\right).$$

Since $\operatorname{ran}\left(j_{S,f,\hat{R}}\right)$ is a maximal ideal in $\mathcal{B}$ by Theorem 4.4.1, this is equality. Thus,

$$\mathcal{A} \cong_{\mathbf{C}^*} \operatorname{ran}\left(j_{S,f,\hat{R}}\right) \cong_{\mathbf{C}^*} \ker\left(\epsilon_{\vec{0}}\right) = \left\{a \in C\left(X_T\right) : a\left(\vec{0}\right) = 0\right\} \cong_{\mathbf{C}^*} C_0\left(X_T \setminus \left\{\vec{0}\right\}\right).$$

□

Consequently, if $\langle S, f | R\rangle_{\mathbf{C}^*}$ was commutative originally, it is completely described by this theorem.

4.6 Examples from 1C* Reviewed

With Theorems 4.4.1 and 4.5.2, several of the examples of Chapter 3 with non-unital C*-relations will be reconsidered, characterizing the non-unital version of each.

Example 4.6.1 (A normal element, [4]). For $\lambda \geq 0$, consider

$$\langle (x, \lambda) | x^* x = x x^* \rangle_{\mathbf{C}^*}.$$

Note that x is normal and generates this algebra, so it is commutative. In this case, $t = x^* x - x x^*$ so

$$g_t(\mu) = \overline{\mu}\mu - \mu\overline{\mu} = 0.$$

Hence, $g_t^{-1}(0) = D_\lambda$. By Proposition 4.5.2,

$$\langle (x, \lambda) | x^* x = x x^* \rangle_{\mathbf{C}^*} \cong_{\mathbf{C}^*} C_0\left(D_\lambda \setminus \{0\}\right) \cong_{\mathbf{C}^*} \begin{cases} \mathbb{0}, & \lambda = 0, \\ C_0\left(\overline{\mathbb{D}} \setminus \{0\}\right), & \lambda > 0, \end{cases}$$

since $D_\lambda \setminus \{0\} \cong_{\mathbf{Top}} D_1 \setminus \{0\} = \overline{\mathbb{D}} \setminus \{0\}$ for all $\lambda > 0$.

Example 4.6.2 (A self-adjoint element, [4]). For $\lambda \geq 0$, consider

$$\langle (x, \lambda) | x^* = x \rangle_{\mathbf{C^*}}.$$

Then, $xx^* = x^2 = x^*x$ so x is normal and generates this algebra. Hence, it is commutative. In this case, $t = x - x^*$ so

$$g_t(\mu) = \mu - \overline{\mu} = 2\imath\Im(\mu)$$

Hence, $g_t^{-1}(0) = [-\lambda, \lambda]$. By Proposition 4.5.2,

$$\langle (x, \lambda) | x^* = x \rangle_{\mathbf{C^*}} \cong_{\mathbf{C^*}} C_0\left([-\lambda, 0) \cup (0, \lambda]\right) \cong_{\mathbf{C^*}} \begin{cases} \mathbb{O}, & \lambda = 0, \\ C_0\left([-1, 0) \cup (0, 1]\right), & \lambda > 0, \end{cases}$$

since $[-\lambda, 0) \cup (0, \lambda] \cong_{\mathbf{Top}} [-1, 0) \cup (0, 1]$ for all $\lambda > 0$.

Example 4.6.3 (A projection). For $\lambda \geq 0$, consider

$$\langle (x, \lambda) | x^2 = x^* = x \rangle_{\mathbf{C^*}}.$$

Then, x is normal and generates this algebra, so it is commutative. In this case, $t_1 = x^2 - x$ and $t_2 = x - x^*$ so

$$g_{t_1}(\mu) = \mu^2 - \mu$$

and

$$g_{t_2}(\mu) = 2\imath\Im(\mu).$$

Hence, $g_{t_1}^{-1}(0) = \{0,1\} \cap D_\lambda$ and $g_{t_2}^{-1}(0) = [-\lambda, \lambda]$. By Proposition 4.5.2,

$$\left\langle (x,\lambda) | x^2 = x^* = x \right\rangle_{\mathbf{C}^*} \cong_{\mathbf{C}^*} C_0\left(\{1\} \cap [-\lambda,\lambda]\right) \cong_{\mathbf{C}^*} \begin{cases} \mathbb{0}, & 0 \leq \lambda < 1, \\ \mathbb{C}, & \lambda \geq 1. \end{cases}$$

Example 4.6.4 (Normality and Sine). For $\lambda \geq 0$, consider

$$\langle (x,\lambda) | x^*x = xx^*, \sin(x) = 0 \rangle_{\mathbf{C}^*}.$$

Then, x is normal and generates this algebra, so it is commutative. In this case, $t_1 = x^*x - xx^*$ and $t_2 = \sin(x)$ so

$$g_{t_1}(\mu) = 0$$

and

$$g_{t_2}(\mu) = \sin(\mu).$$

Hence, $g_{t_2}^{-1}(0) = \{\pi n : n \in \mathbb{Z}\} \cap D_\lambda$. By Proposition 4.5.2,

$$\langle (x,\lambda) | x^*x = xx^*, \sin(x) = 0 \rangle_{\mathbf{C}^*} \cong_{\mathbf{C}^*} C_0\left(\{\pi n : n \in \mathbb{Z} \setminus \{0\}\} \cap D_\lambda\right) \cong_{\mathbf{C}^*} \mathbb{C}^{2n},$$

for each $\pi n \leq \lambda < \pi(n+1)$ and $n \in \mathbb{W}$.

Recall from elementary analysis that for $f \in C\left(D_\lambda\right)$, $f(0) = 0$ if and only if there is a sequence of *-polynomials $(p_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} p_n = f$ in the supremum norm, each with constant term 0. Thus, recalling the function $p : \mathbb{R} \to \mathbb{R}$ by

$$p(\mu) := \begin{cases} 0, & \mu < 0, \\ \mu, & \mu \geq 0, \end{cases}$$

$p\left(\Re(a)\right) \in C^*(a)$ for any C*-algebra element a. Using Proposition 3.7.1, one can consider the C*-algebra generated by a single positive element. Recall that "$x \geq 0$" is used as shorthand for "$p\left(\Re(x)\right) - x = 0$".

Example 4.6.5 (A positive element, [4]). For $\lambda \geq 0$, consider

$$\langle (x, \lambda) | x \geq 0 \rangle_{\mathbf{C^*}}.$$

Then, x is normal and generates this algebra, so it is commutative. In this case, $t = p\left(\Re(x)\right) - x$ so

$$g_t(\mu) = p\left(\Re(\mu)\right) - \mu.$$

Hence, $g_t^{-1}(0) = [0, \lambda]$. By Proposition 4.5.2,

$$\langle (x, \lambda) | x \geq 0 \rangle_{\mathbf{C^*}} \cong_{\mathbf{C^*}} C_0(0, \lambda] \cong_{\mathbf{C^*}} \begin{cases} \mathbb{O}, & \lambda = 0, \\ C_0(0, 1], & \lambda > 0, \end{cases}$$

since $(0, \lambda] \cong_{\mathbf{Top}} (0, 1]$ for all $\lambda > 0$.

In a similar way,

$$p\left(\lambda^2 a^* a - (a^* a)^2\right) - \lambda^2 a^* a + (a^* a)^2 \in C^*(a)$$

for any scalar $\lambda \geq 0$ and C*-algebra element a. Using Proposition 3.8.1, one can consider attaching norm-bounds onto non-unital C*-algebra elements. Recall that "$\|a\| \leq \lambda$" is used as shorthand for

$$\text{“}p\left(\lambda^2 a^* a - (a^* a)^2\right) - \lambda^2 a^* a + (a^* a)^2 = 0\text{”}.$$

Example 4.6.6. For $\lambda, \mu \in [0, \infty)$, consider the algebra

$$\mathcal{A}_{\lambda,\mu} := \langle (x, \lambda) | x^* x = x x^*, \|\Im(x)\| \leq \mu \rangle_{\mathbf{C}^*}.$$

Then, x is normal and generates this algebra, so it is commutative. In this case, $t_1 = x^* x - x x^*$ and

$$\begin{aligned} t_2 &= p\left(\mu^2 \Im(x)^* \Im(x) - (\Im(x)^* \Im(x))^2\right) - \mu^2 \Im(x)^* \Im(x) + (\Im(x)^* \Im(x))^2 \\ &= p\left(\mu^2 \Im(x)^2 - \Im(x)^4\right) - \mu^2 \Im(x)^2 + \Im(x)^4 \end{aligned}$$

so $g_{t_1}(\nu) = 0$ and

$$g_{t_2}(\nu) = p\left(\mu^2 \Im(\nu)^2 - \Im(\nu)^4\right) - \mu^2 \Im(\nu)^2 + \Im(\nu)^4$$

Note that $g_{t_2}(\nu) = 0$ whenever $\mu \geq |\Im(\nu)|$. Hence, $g_{t_2}^{-1}(0) = \Im^{-1}(D_\mu) \cap D_\lambda$. By Theorem 4.5.2, $\mathcal{A}_{\lambda,\mu} \cong_{\mathbf{C}^*} C_0\left((D_\lambda \setminus \{0\}) \cap \Im^{-1}(D_\mu)\right)$.

Interpreting the spectrum,

$$\begin{aligned} \sigma_{\mathrm{Unit}\left(\mathcal{A}_{\lambda,\mu}\right)}(x) &= \Im^{-1}(D_\mu) \cap D_\lambda \\ &= \{\nu \in \mathbb{C} : |\Im(\nu)| \leq \mu\} \cap D_\lambda, \end{aligned}$$

the intersection of a disc and an infinite bar. Thus, there are only the following situations.

1. If $\lambda = 0$, the disc is degenerate, meaning the intersection is a singleton.

2. If $\lambda > 0 = \mu$, the bar has width zero, meaning the intersection is the interval $[-\lambda, \lambda]$.

3. If $0 < \lambda \leq \mu$, the bar envelopes the disc, meaning the intersection is the disc.

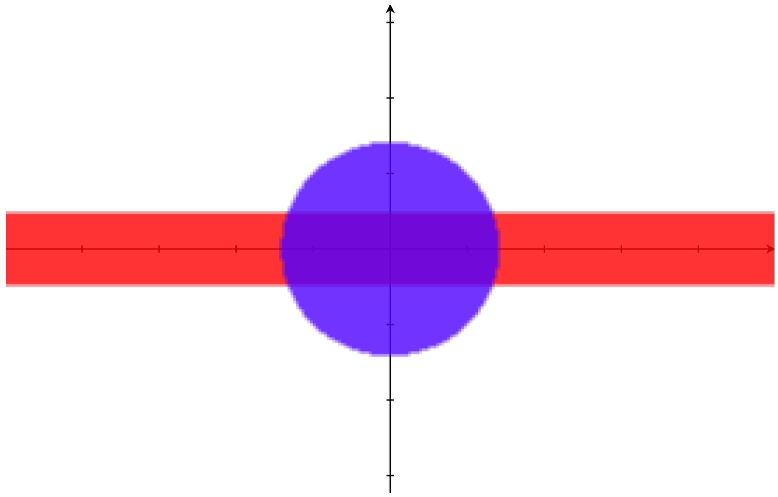


Figure 4.1: Intersection of a Disc and an Infinite Bar

4. In all other cases, the intersection is a full section of the disc, which is homeomorphic to a disc.

In summary,

$$\mathcal{A}_{\lambda,\mu} \cong_{\mathbf{1C^*}} \begin{cases} \mathbb{O}, & \lambda = 0, \\ C_0\left([-1,0) \cup (0,1]\right), & \lambda > 0 = \mu, \\ C_0\left(\overline{\mathbb{D}} \setminus \{0\}\right), & \text{otherwise.} \end{cases}$$

Example 4.6.7 (Arbitrarily many normal operators)*.* For a crutched set (S, f), consider the C*-algebra

$$\langle S, f \,| xy = yx, xy^* = y^*x \forall x, y \in S \rangle_{\mathbf{C^*}}.$$

By Theorems 4.5.1 and 4.5.2,

$$\begin{aligned}\langle S, f \,|xy = yx, xy^* = y^*x \forall x, y \in S\rangle_{\mathbf{C}^*} &\cong_{\mathbf{C}^*} \mathrm{Ab}\left(\langle S, f \,|\emptyset\rangle_{\mathbf{C}^*}\right) \\ &\cong_{\mathbf{C}^*} C_0\left(\left(\prod_{s \notin f^{-1}(0)} \overline{\mathbb{D}}\right) \setminus \left\{\vec{0}\right\}\right).\end{aligned}$$

The remaining two examples are non-commutative so Proposition 4.5.2 does not apply. Nevertheless, Theorem 4.4.1 still yields a characterization.

Example 4.6.8 (An idempotent). For $\lambda \geq 0$, consider

$$\mathcal{A}_\lambda := \left\langle (x, \lambda) \,\middle|\, x = x^2 \right\rangle_{\mathbf{C}^*}.$$

From Example 3.12.7, the unital C*-algebras

$$\mathcal{B}_\lambda := \left\langle (x, \lambda) \,\middle|\, x = x^2 \right\rangle_{\mathbf{1C}^*}$$

were characterized. For $\lambda < 1$, $\mathcal{B}_\lambda \cong_{\mathbf{1C}^*} \mathbb{C}$ and $x = 0$. By Theorem 4.4.1, $\mathcal{A}_\lambda \cong_{\mathbf{1C}^*} \mathbb{O}$.

For $\lambda \geq 1$,

$$\mathcal{B}_\lambda \cong_{\mathbf{1C}^*} \begin{bmatrix} C\left[0, 1 - \lambda^{-2}\right] & C_0\left(0, 1 - \lambda^{-2}\right] \\ C_0\left(0, 1 - \lambda^{-2}\right] & C\left[0, 1 - \lambda^{-2}\right] \end{bmatrix},$$

with the generator x identified with the matrix-valued function

$$f_x(\mu) := \begin{bmatrix} 1 & \mu \\ 0 & 0 \end{bmatrix}.$$

Using the association above, define $\phi : \mathcal{B}_\lambda \to \mathbb{C}$ by

$$\phi(f) := \begin{bmatrix} 0 & 1 \end{bmatrix} f(0) \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

stripping the 2,2 entry of the matrix $f(0)$. As ϕ is conjugation by $\begin{bmatrix} 0 & 1 \end{bmatrix}$, ϕ is readily $\mathbb{C}$-linear and *-preserving. For multiplicativity, let $g, h \in \mathcal{B}_\lambda$. Denote the coordinate functions $g_{i,j}$ and $h_{i,j}$ and observe that

$$\begin{aligned}
\phi(g \cdot h) &= \begin{bmatrix} 0 & 1 \end{bmatrix} (g \cdot h)(0) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \end{bmatrix} g(0)h(0) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} g_{11}(0)h_{11}(0) + g_{12}(0)h_{21}(0) & g_{11}(0)h_{12}(0) + g_{12}(0)h_{22}(0) \\ g_{21}(0)h_{11}(0) + g_{22}(0)h_{21}(0) & g_{21}(0)h_{12}(0) + g_{22}(0)h_{22}(0) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} g_{11}(0)h_{11}(0) & 0 \\ 0 & g_{22}(0)h_{22}(0) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= g_{22}(0)h_{22}(0) \\
&= \phi(g)\phi(h).
\end{aligned}$$

Further, $\phi(\mathbb{1}) = 1$ and $\phi(f_x) = 0$. By Theorem 4.4.1,

$$\begin{aligned} \mathcal{A}_\lambda &\cong_{\mathbf{C}^*} \ker(\phi) \\ &= \{f \in \mathcal{B}_\lambda : f_{2,2}(0) = 0\} \\ &\cong_{\mathbf{C}^*} \begin{bmatrix} C\left[0, 1-\lambda^{-2}\right] & C_0\left(0, 1-\lambda^{-2}\right] \\ C_0\left(0, 1-\lambda^{-2}\right] & C_0\left(0, 1-\lambda^{-2}\right] \end{bmatrix}. \end{aligned}$$

In summary,

$$\mathcal{A}_\lambda \cong_{\mathbf{C}^*} \begin{cases} \mathbb{O}, & \lambda < 1, \\ \mathbb{C}, & \lambda = 1, \\ \begin{bmatrix} C[0,1] & C_0(0,1] \\ C_0(0,1] & C_0(0,1] \end{bmatrix}, & \lambda > 1. \end{cases}$$

Example 4.6.9 (Pedersen's C*-algebra of two projections, [32]). Consider the C*-algebra

$$\mathcal{A} := \left\langle (p,1),(q,1) | p = p^* = p^2, q = q^* = q^2 \right\rangle_{\mathbf{C}^*}.$$

From Example 3.10.2,

$$\mathcal{B} := \left\langle (p,1),(q,1) | p = p^* = p^2, q = q^* = q^2 \right\rangle_{\mathbf{1C}^*} \cong_{\mathbf{1C}^*} \begin{bmatrix} C[0,1] & C_0(0,1) \\ C_0(0,1) & C[0,1] \end{bmatrix},$$

associating the generators p and q to the matrix-valued functions

$$f_p(\mu) := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$f_q(\mu) := \begin{bmatrix} \mu & \sqrt{\mu-\mu^2} \\ \sqrt{\mu-\mu^2} & 1-\mu \end{bmatrix},$$

respectively. Using the association above, define $\psi : \mathcal{B} \to \mathbb{C}$ by

$$\psi(f) := \begin{bmatrix} 0 & 1 \end{bmatrix} f(1) \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

stripping the 2,2 entry of the matrix $f(1)$. Similar to the previous example, this is a unital *-homomorphism such that $\psi(f_p) = \psi(f_q) = 0$. By Theorem 4.4.1,

$$\begin{aligned} \mathcal{A} &\cong_{\mathbf{C}^*} \ker(\psi) \\ &= \{f \in \mathcal{B} : f_{2,2}(1) = 0\} \\ &\cong_{\mathbf{C}^*} \begin{bmatrix} C[0,1] & C_0(0,1) \\ C_0(0,1) & C_0[0,1) \end{bmatrix}. \end{aligned}$$

4.7 Tietze Transformations for $\mathbf{C}^*$

Like Section 3.9, Tietze transformations can be done for presentations in $\mathbf{C}^*$ just as in $\mathbf{1C}^*$. In particular, the following transformations can be mimicked from Section 3.9:

- adding an unnecessary C*-relation,
- removing an unnecessary C*-relation,
- adding an unnecessary generator,
- removing an unnecessary generator,

- scaling generators and adjusting the C*-relations as needed,
- setting the crutch-value of a generator to the actual norm value.

For each manipulation, the construction and the proof of the corresponding isomorphism is nearly identical to the $\mathbf{1C}^*$ case. As such, the explicit proofs of these will be suppressed for this section.

Further, like the $\mathbf{1C}^*$ case, there is a Tietze theorem regarding presentations of the same C*-algebra. The proof is also nearly identical to the $\mathbf{1C}^*$ case, but the situation will be briefly summarized to state the results clearly.

For $j = 1, 2$, fix a crutched set (S_j, f_j) and a set of non-unital C*-relations R_j on (S_j, f_j). Define $\mathcal{F}_j := \langle S_j, f_j | \emptyset \rangle_{\mathbf{C}^*}$, $\mathcal{A}_j := \langle S_j, f_j | R_j \rangle_{\mathbf{C}^*}$, and $q_j : \mathcal{F}_j \to \mathcal{A}_j$ the quotient map.

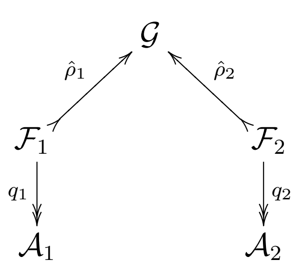
Define

$$(T, g) := (S_1, f_2) \coprod^{\mathbf{CSet}_1} (S_2, f_2)$$

to be the disjoint union described in Proposition 2.2.9, $\rho_j : (S_j, f_j) \to (T, g)$ the canonical inclusions for $j = 1, 2$, and $\mathcal{G} := \langle T, g | \emptyset \rangle_{\mathbf{C}^*}$. Theorem 4.1.4 and Proposition A.5.1 state that

$$\mathcal{G} \cong_{\mathbf{C}^*} \mathcal{F}_1 \coprod^{\mathbf{C}^*} \mathcal{F}_2 \cong_{\mathbf{C}^*} \mathcal{F}_1 * \mathcal{F}_2$$

with connecting maps $\hat{\rho}_j := \mathrm{C}^*\mathrm{Alg}\,(\rho_j)$ for $j = 1, 2$.



Lemma 4.7.1. *Given the notation above, assume $\Theta_j : \mathcal{G} \to \mathcal{F}_j$ is a *-homomorphism*

satisfying that $\Theta_j \circ \hat{\rho}_j = id_{\mathcal{F}_j}$*. Then,* $\ker(q_j \circ \Theta_j)$ *is the norm-closed, two-sided ideal* $\mathcal{J}_j$ *generated by* $\hat{\rho}_j(R_j) \cup \{s - (\hat{\rho}_j \circ \Theta_j)(s) : s \in S_{3-j}\}$ *in* $\mathcal{G}$*.*

Theorem 4.7.2 (Tietze Theorem for $\mathbf{C}^*$)**.** $\mathcal{A}_1 \cong_{\mathbf{C}^*} \mathcal{A}_2$ *if and only if there is a sequence of four Tietze transformations changing the presentation of* $\mathcal{A}_1$ *into the presentation for* $\mathcal{A}_2$*.*

Analogously, there is also a notion of *elementary Tietze transformation*, where only one aspect of the presentation is changed.

Corollary 4.7.3. *Given C*-algebras* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *are finitely presented in* $\mathbf{C}^*$, $\mathcal{A}_1 \cong_{\mathbf{C}^*} \mathcal{A}_2$ *if and only if there is a finite sequence of elementary Tietze transformations changing the presentation of* $\mathcal{A}_1$ *into the presentation for* $\mathcal{A}_2$*.*

Section 2.4 of [19] also considered Tietze transformations for C*-algebras using only *-algebraic relations. However, the proofs in Section 3.9 of the present work, and their non-unital analogs in this present section, make use of the properties of quotients in $\mathbf{C}^*$ and $\mathbf{1C}^*$. This approach bypasses the category of *-algebras over $\mathbb{C}$ altogether, providing shorter proofs of each isomorphism.

Further, the use of C*-relations has provided the Tietze theorems for both $\mathbf{C}^*$ and $\mathbf{1C}^*$, which may not have been achievable with only *-algebraic relations as conjectured in Remark 2.4.1.13 of [19].

Lastly, note also that these Tietze theorems relied upon the projectivity in Propositions 3.2.6 and 4.2.7, as well as the coproduct decompositions. This allowed the scaled-free mapping property to be used to built the retractions needed for the main results. However, in a category of normed objects where kernels are not necessarily proximinal, this may not be possible.

4.8 Construction: General Free Products

In [5], the existence of a free product of C*-algebras is assured, and Proposition 3.3.2 and Corollary 3.3.3 of [19] demonstrate its construction in the presentation theory of that work. This section gives a formal manipulation result similar to Theorem 3.10.1, which extends Proposition 3.3.2 of [19]. As the proofs of this section are nearly identical to those of Section 3.10, the initial setup will be shown, but the details of the proof will be suppressed.

As in Section 4.2, the free product is usually denoted by "$*$". The "$\coprod$" notation will be used interchangeably with the "$*$" notation, but preference will be given to the "$\coprod$" with arbitrary index sets.

To begin, let Γ be an index set, $(S_\gamma, f_\gamma)_{\gamma\in\Gamma}$ be crutched sets, and R_γ non-unital C*-relations on (S_γ, f_γ) for each γ. Define $\mathcal{F}_\gamma := \langle S_\gamma, f_\gamma | \emptyset\rangle_{\mathbf{C}^*}$, $\mathcal{A}_\gamma := \langle S_\gamma, f_\gamma | R_\gamma\rangle_{\mathbf{C}^*}$, and $q_\gamma : \mathcal{F}_\gamma \to \mathcal{A}_\gamma$ the quotient map for each γ. The $\mathcal{A}_\gamma$ will be the C*-algebras to merge.

Let

$$(S, f) := \coprod_{\gamma\in\Gamma}^{\mathbf{CSet}_1} (S_\gamma, f_\gamma),$$

the disjoint union described in Proposition 2.2.9, $\rho_\gamma : (S_\gamma, f_\gamma) \to (S, f)$ the canonical inclusions for each γ, and $\mathcal{F} := \langle S, f | \emptyset\rangle_{\mathbf{C}^*}$. Theorem 4.1.4 and Proposition A.5.1 state that

$$\mathcal{F} \cong_{\mathbf{C}^*} \coprod_{\gamma\in\Gamma}^{\mathbf{C}^*} \mathcal{F}_\gamma$$

with connecting maps $\hat{\rho}_\gamma := \mathrm{C}^*\mathrm{Alg}(\rho_\gamma)$. Let

$$R := \bigcup_{\gamma\in\Gamma} \hat{\rho}_\gamma (R_\gamma),$$

grouping all the C*-relations of the $\mathcal{A}_\gamma$ into one subset within the larger algebra $\mathcal{F}$.

Define $\mathcal{A} := \langle S, f|R\rangle_{\mathbf{C}^*}$, a candidate for the free product of the $\mathcal{A}_\gamma$, and $q : \mathcal{F} \to \mathcal{A}$ the quotient map. To create the connecting maps, fix $\gamma \in \Gamma$ and consider the following diagram in $\mathbf{C}^*$.

$$\begin{array}{ccc} \mathcal{F}_\gamma & \overset{q_\gamma}{\twoheadrightarrow} & \mathcal{A}_\gamma \\ {\scriptstyle \hat{\rho}_\gamma}\downarrow & & \\ \mathcal{F} & \underset{q}{\twoheadrightarrow} & \mathcal{A} \end{array}$$

Given $r \in R_\gamma$, observe that $\hat{\rho}_\gamma(r) \in R$ so $(q \circ \hat{\rho})(r) = 0$ by design. Thus, by the universal property of the quotient, there is a unique *-homomorphism $k_\gamma : \mathcal{A}_\gamma \to \mathcal{A}$ such that $k_\gamma \circ q_\gamma = q \circ \hat{\rho}_\gamma$.

Proof analogous to Theorem 3.10.1 shows that the $\mathcal{A}$ equipped with $(k_\gamma)_{\gamma\in\Gamma}$ is the coproduct of the $\mathcal{A}_\gamma$. This extends Proposition 3.3.2 in [19] to non-unital C*-relations.

Theorem 4.8.1. *The C*-algebra $\mathcal{A}$ equipped with *-homomorphisms $k_\gamma : \mathcal{A}_\gamma \to \mathcal{A}$ is a coproduct of $(\mathcal{A}_\gamma)_{\gamma\in\Gamma}$ in $\mathbf{C}^*$.*

In summary,

$$\langle S, f|R\rangle_{\mathbf{C}^*} \cong_{\mathbf{C}^*} \coprod_{\gamma\in\Gamma}^{\mathbf{C}^*} \langle S_\gamma, f_\gamma|R_\gamma\rangle_{\mathbf{C}^*}.$$

In the case $\Gamma = \{1, 2\}$, this result can be stated in the traditional notation as

$$\langle S, f|R\rangle_{\mathbf{C}^*} \cong_{\mathbf{C}^*} \langle S_1, f_1|R_1\rangle_{\mathbf{C}^*} * \langle S_2, f_2|R_2\rangle_{\mathbf{C}^*}.$$

As a concrete example, consider the free product in $\mathbf{C}^*$ of $\mathbb{C}$ with itself.

Example 4.8.2 (Pedersen's C*-algebra of two projections, [32]). Consider again the C*-algebra

$$\left\langle (p, 1), (q, 1) | p = p^* = p^2, q = q^* = q^2 \right\rangle_{\mathbf{C}^*}.$$

Observe that in this case, $S = \{p, q\}$, $f : S \to [0, \infty)$ by $f(p) = f(q) = 1$, and $R = \left\{p - p^*, p - p^2, q - q^*, q - q^2\right\}$.

Let $S_1 := \{p\}$, $f_1 : S_1 \to [0, \infty)$ by $f_1(p) := 1$, and $R_1 := \left\{p - p^*, p - p^2\right\}$. Likewise, let $S_2 := \{q\}$, $f_2 : S_2 \to [0, \infty)$ by $f_2(q) := 1$, and $R_2 := \left\{q - q^*, q - q^2\right\}$. By Example 4.6.3, the C*-algebras $\langle S_j, f_j | R_j \rangle_{\mathbf{C}^*} \cong_{\mathbf{C}^*} \mathbb{C}$ for $j = 1, 2$.

Letting $\rho_j : (S_j, f_j) \to (S, f)$ be the inclusions for $j = 1, 2$, observe that $R = \bigcup_{j=1}^{2} \mathrm{C}^*\mathrm{Alg}\,(\rho_j)\,(R_j)$. Thus, the above result states that

$$\begin{aligned} \mathbb{C} * \mathbb{C} &\cong_{\mathbf{C}^*} \langle S_1, f_1 | R_1 \rangle_{\mathbf{C}^*} * \langle S_2, f_2 | R_2 \rangle_{\mathbf{C}^*} \\ &\cong_{\mathbf{C}^*} \langle S, f | R \rangle_{\mathbf{C}^*} . \\ &\cong_{\mathbf{C}^*} \begin{bmatrix} C[0,1] & C_0(0,1) \\ C_0(0,1) & C_0[0,1) \end{bmatrix} \end{aligned}$$

from Example 4.6.9.

4.9 Property: Generation & Separability in $\mathbf{C}^*$

As in Section 3.14, control on the number of generators used in building a C*-algebra has a connection to its topological structure. The following proposition is a direct translation of Proposition 3.14.1, and its proof is nearly identical.

Proposition 4.9.1. *Given a C*-algebra $\mathcal{A}$, $\mathcal{A}$ is separable if and only if $\mathcal{A}$ is countably generated in* $\mathbf{C}^*$.

Combining this with Proposition 3.14.1 and Theorem 4.4.1 gives the following equivalence.

Corollary 4.9.2. *Given a C*-algebra $\mathcal{A}$, the following are equivalent.*

1. *$\mathcal{A}$ is separable.*

2. *$\mathcal{A}$ is countably generated in* $\mathbf{C}^*$.

3. $\mathrm{Unit}(\mathcal{A})$ *is countably generated in* $\mathbf{1C}^*$.

4. $\mathrm{Unit}(\mathcal{A})$ *is separable.*

Proof. ($1 \Leftrightarrow 2$) This is the content of Proposition 4.9.1.

($3 \Leftrightarrow 4$) This is the content of Proposition 3.14.1.

($2 \Rightarrow 3$) By Theorem 4.4.1, if $\mathcal{A} \cong_{\mathbf{C}^*} \langle S, f|R\rangle_{\mathbf{C}^*}$ for a crutched set (S, f) and non-unital C*-relations R on (S, f) with S countable, then $\mathrm{Unit}(\mathcal{A}) \cong_{\mathbf{1C}^*} \langle S, f\,|j_{S,f}(R)\rangle_{\mathbf{1C}^*}$.

($4 \Rightarrow 1$) Consider the map $\pi_1 : \mathrm{Unit}(\mathcal{A}) \to \mathcal{A}$ by $\pi_1(a, \lambda) := a$, the projection onto the first coordinate. A quick check shows that this map is linear, and $\ker(\pi_1) = \{0\} \times \mathbb{C}$, which is closed. Hence, π_1 is continuous and, therefore, bounded.

Assuming that there is a countable set $T \subseteq \mathrm{Unit}(\mathcal{A})$ which is dense in $\mathrm{Unit}(\mathcal{A})$, consider the image $\pi_1(T) \subseteq \mathcal{A}$. Given $a \in \mathcal{A}$ and $\epsilon > 0$, there is $(b, \lambda) \in T$ such that

$$\|(b, \lambda) - (a, 0)\|_{\mathrm{Unit}(\mathcal{A})} < \frac{\epsilon}{\|\pi_1\| + 1}.$$

Then,

$$\begin{aligned} \|b - a\|_{\mathcal{A}} &\leq \|\pi_1\|\, \|(b, \lambda) - (a, 0)\|_{\mathrm{Unit}(\mathcal{A})} \\ &< \epsilon. \end{aligned}$$

Thus, $\pi_1(T)$ is countable and dense in $\mathcal{A}$.

□

Corollary 4.9.3. *Given a unital C*-algebra $\mathcal{A}$, the following are equivalent.*

1. *$\mathcal{A}$ is separable.*

2. $\mathcal{A}$ is countably generated in $\mathbf{1C^*}$.

3. $\mathcal{A}$ is countably generated in $\mathbf{C^*}$.

Adapting the proof of Theorem 4.4.2 gives the following third equivalence.

Proposition 4.9.4. *Given a unital C*-algebra $\mathcal{A}$, $\mathcal{A}$ is finitely generated in $\mathbf{1C^*}$ if and only if $\mathcal{A}$ is finitely generated in $\mathbf{C^*}$.*

Proof. ($\Leftarrow$) Assume that $\mathcal{A} \cong_{\mathbf{C^*}} \langle S, f|R\rangle_{\mathbf{C^*}}$ for some crutched set (S, f) and non-unital C*-relations R with S finite. Let $\mathcal{F} := \langle S, f|\emptyset\rangle_{\mathbf{C^*}}$, $\mathcal{G} := \langle S, f|\emptyset\rangle_{\mathbf{1C^*}}$, and $j_{S,f} : \mathcal{F} \to \mathcal{G}$ from Theorem 4.2.6. Let $q_R : \mathcal{F} \to \mathcal{A}$ be the quotient map. Then, the following diagram exists in $\mathbf{C^*}$.

$$\begin{array}{ccc} \mathcal{F} & \overset{j_{S,f}}{\rightarrowtail} & \mathcal{G} \\ & {\scriptstyle q_R}\searrow & \\ & & \mathcal{A} \end{array}$$

From Theorem 4.2.6, there is a unique unital *-homomorphism $\phi : \mathcal{G} \to \mathcal{A}$ such that $\phi \circ j_{S,f} = q_R$. For all $s \in S$,

$$\phi(s) = (\phi \circ j_{S,f})(s) = q_R(s)$$

so $q_R(S) \subseteq \operatorname{ran}(\phi)$. Therefore, $\mathcal{A} = \operatorname{ran}(\phi)$, meaning $\mathcal{A} \cong_{\mathbf{1C^*}} \langle S, f| \ker(\phi)\rangle_{\mathbf{1C^*}}$.

($\Rightarrow$) Assume that $\mathcal{A} \cong_{\mathbf{1C^*}} \langle S, f|R\rangle_{\mathbf{1C^*}}$ for some crutched set (S, f) and unital C*-relations R with S finite. Let

$$\mathcal{H} := \left\langle (S, f) \coprod^{\mathbf{CSet_1}} \{(u, 1)\} | \emptyset \right\rangle_{\mathbf{C^*}}$$

as in Theorem 4.4.2. Defining $\psi : S \uplus \{u\} \to \mathcal{A}$ by

$$\psi(t) := \begin{cases} t, & t \in S, \\ \mathbb{1}_{\mathcal{A}}, & t = u, \end{cases}$$

ψ is constrictive from $(S,f)\coprod^{\mathbf{CSet}_1}\{(u,1)\}$ to $F_{\mathbf{C}^*}^{\mathbf{CSet}_1}\mathcal{A}$. By Theorem 4.2.1, there is a unique *-homomorphism $\hat{\psi} : \mathcal{H} \to \mathcal{A}$ such that $\hat{\psi}(t) = \psi(t)$ for all $t \in S \uplus \{u\}$. Observe that for all $s \in S$,

$$s = \psi(s) = \hat{\psi}(s)$$

and

$$\mathbb{1}_{\mathcal{A}} = \psi(u) = \hat{\psi}(u).$$

Thus, $\mathcal{A} = \operatorname{ran}\left(\hat{\psi}\right)$, meaning $\mathcal{A} \cong_{\mathbf{C}^*} \left\langle (S,f)\coprod^{\mathbf{CSet}_1}\{(u,1)\} \middle| \ker\left(\hat{\psi}\right) \right\rangle_{\mathbf{C}^*}$.

□

With these equivalences, the terms "finitely generated" and "countably generated" will no longer be qualified as occurring in $\mathbf{1C}^*$ or $\mathbf{C}^*$.

4.10 Property: Projectivity & Liftability in $\mathbf{C}^*$

As in Section 3.15, a type of projectivity for $\mathbf{C}^*$ can be characterized. Here, the projectivity used will be with respect to surjective *-homomorphisms in $\mathbf{C}^*$. The characterization below is proven in a way nearly identical to the unital case of Proposition 3.15.1. As such, the proofs will be suppressed for brevity.

Definition. For a crutched set (S,f), a set of non-unital C*-relations R on (S,f) is *liftable* in $\mathbf{C}^*$ if for any C*-algebras $\mathcal{A}, \mathcal{B}$, any surjective *-homomorphism $\phi : \mathcal{A} \to \mathcal{B}$, and any *-homomorphism $\rho : \langle S, f | \emptyset \rangle_{\mathbf{C}^*} \to \mathcal{B}$ such that $R \subseteq \ker(\rho)$, there is a *-

homomorphism $\hat{\rho} : \langle S, f|\emptyset\rangle_{\mathbf{1C^*}} \to \mathcal{A}$ such that $\phi \circ \hat{\rho} = \rho$ and $R \subseteq \ker(\hat{\rho})$. The map $\hat{\rho}$ is called a *lift* of ρ along ϕ.

Proposition 4.10.1. *Given a crutched set (S, f), non-unital C*-relations R on (S, f) are liftable in $\mathbf{C^*}$ if and only if $\langle S, f|R\rangle_{\mathbf{C^*}}$ is projective relative to all surjections in $\mathbf{C^*}$.*

However, combining this result with Propositions 3.15.1 and B.5.4, as well as Theorem 4.4.1, gives the following equivalence.

Theorem 4.10.2. *Given a crutched set (S, f), let R be a set of non-unital C*-relations on (S, f) and $j_{S,f} : \langle S, f|\emptyset\rangle_{\mathbf{C^*}} \to \langle S, f|\emptyset\rangle_{\mathbf{1C^*}}$ be as in Theorem 4.2.6. Then, the following are equivalent.*

1. *R is liftable in $\mathbf{C^*}$.*
2. *$\langle S, f|R\rangle_{\mathbf{C^*}}$ is projective with respect to surjections in $\mathbf{C^*}$.*
3. *$\langle S, f\,|j_{S,f}(R)\rangle_{\mathbf{1C^*}}$ is projective with respect to surjections in $\mathbf{1C^*}$.*
4. *$j_{S,f}(R)$ is liftable in $\mathbf{1C^*}$.*

If one associates $\langle S, f|\emptyset\rangle_{\mathbf{C^*}}$ to $\operatorname{ran}(j_{S,f})$, this states that "liftability" for a set of non-unital C*-relations is does not depend on the choice of category. Thus, the distinction can be blurred. With this equivalence, the projectivity of many of the current examples can now be determined.

Example 4.10.3 (Examples from Section 4.6). Recall the following isomorphisms in $\mathbf{1C^*}$ from Section 4.6.

$$\mathrm{Unit}\left(C_0\left([-1,0)\cup(0,1]\right)\right) \cong_{\mathbf{1C^*}} C[-1,1]$$

$$\mathrm{Unit}\left(\mathbb{O}\right) \cong_{\mathbf{1C^*}} \mathbb{C}$$

$$\mathrm{Unit}\left(C_0(0,1]\right) \cong_{\mathbf{1C^*}} C[0,1]$$

$$\mathrm{Unit}\left(C_0\left(\left(\prod_{\lambda\in\Lambda}\overline{\mathbb{D}}\right)\setminus\{\vec{0}\}\right)\right) \cong_{\mathbf{1C^*}} C\left(\prod_{\lambda\in\Lambda}\overline{\mathbb{D}}\right)$$

$$\mathrm{Unit}\left(\begin{bmatrix} C[0,1] & C_0(0,1] \\ C_0(0,1] & C_0(0,1] \end{bmatrix}\right) \cong_{\mathbf{1C^*}} \begin{bmatrix} C[0,1] & C_0(0,1] \\ C_0(0,1] & C[0,1] \end{bmatrix}$$

$$\mathrm{Unit}\left(\begin{bmatrix} C[0,1] & C_0(0,1) \\ C_0(0,1) & C_0[0,1) \end{bmatrix}\right) \cong_{\mathbf{1C^*}} \begin{bmatrix} C[0,1] & C_0(0,1) \\ C_0(0,1) & C[0,1] \end{bmatrix}$$

Thus, the projectivity of the unital C*-algebra on the right-hand side determines the projectivity of any "pre-unitization" of it via Table 3.2 and Proposition B.5.4. Similarly, Proposition B.5.6 determines projectivity for $\mathbf{CC^*}$ as well in the commutative cases.

By Proposition B.1.7, no nonzero examples of Chapter 3 can be projective relative to all surjections in $\mathbf{C^*}$ or $\mathbf{CC^*}$. The summary of these projectivity results is displayed in Table 4.1.

4.11 A Bifurcation Theory for Crutch Functions in $\mathbf{C^*}$

To conclude this chapter regarding $\mathbf{C^*}$, formal definitions for a bifurcation theory of crutch functions is given for this category's presentations, reflecting the case for

Table 4.1: Projectivity Relative to Surjections for Current Examples in $\mathbf{C}^*$

C*-algebra	$\mathbf{C}^*$	$\mathbf{CC}^*$	$\mathbf{1C}^*$	$\mathbf{C1C}^*$	Example
$\mathbb{O}$	Yes	Yes	No	No	3.15.6
$\mathbb{C}$	No	No	Yes	Yes	3.5.1
$\mathbb{C}^n, n \geq 2$	No	No	No	No	3.15.6
$C[0,1]$	No	No	Yes	Yes	3.5.2
$C(\mathbb{T})$	No	No	No	No	3.5.3
$C([-2,-1] \cup [1,2])$	No	No	No	No	3.11.6
$C(A_{1,2})$	No	No	No	No	3.11.7
$C\left(\prod_{\lambda\in\Lambda} \overline{\mathbb{D}}\right), \Lambda \neq \emptyset$	No	No	No	Yes	3.15.7
$\mathcal{T}$	No	-	No	-	3.5.4
$\begin{bmatrix} C[0,1] & C_0(0,1] \\ C_0(0,1] & C[0,1] \end{bmatrix}$	No	-	No	-	3.12.7
$\begin{bmatrix} C[0,1] & C_0(0,1) \\ C_0(0,1) & C[0,1] \end{bmatrix}$	No	-	No	-	3.10.2
$C[0,1] *_{\mathbb{C}} C(\mathbb{T})$	No	-	No	-	3.11.8
$C[0,1] *_{\mathbb{C}} \mathcal{T}$	No	-	No	-	3.11.9
$C_0([-1,0) \cup (0,1])$	Yes	Yes	-	-	4.6.2
$C_0(0,1]$	Yes	Yes	-	-	4.6.5
$C_0\left(\left(\prod_{\lambda\in\Lambda} \overline{\mathbb{D}}\right) \setminus \left\{\vec{0}\right\}\right), \Lambda \neq \emptyset$	No	Yes	-	-	4.6.7
$\begin{bmatrix} C[0,1] & C_0(0,1] \\ C_0(0,1] & C_0(0,1] \end{bmatrix}$	No	-	-	-	4.6.8
$\begin{bmatrix} C[0,1] & C_0(0,1) \\ C_0(0,1) & C_0[0,1) \end{bmatrix}$	No	-	-	-	4.6.9

$\mathbf{1C}^*$ in Section 3.16. First, one regards a summary of the construction of the inverse system of similar presentations.

Fix a set S and partially order the crutch functions on S using the usual product order on $[0,\infty)^S$. As in Section 3.16, the inverse system $\left((S,f),\phi_g^f\right)$ results, where $\phi_g^f : (S,f) \to (S,g)$ by $\phi_g^f(s) := s$ for $g \leq f$.

Applying $\mathrm{C}^*\mathrm{Alg}$, $\left(\langle S,f|\emptyset\rangle_{\mathbf{C}^*},\rho_g^f\right)$ is also an inverse system in $\mathbf{C}^*$, where $\rho_g^f := \mathrm{C}^*\mathrm{Alg}\left(\phi_g^f\right)$. This is summarized in the commutative diagram below for crutch functions $h \leq g \leq f$ on S.

$$\begin{array}{ccc} \langle S,f|\emptyset\rangle_{\mathbf{1C}^*} & \xrightarrow{\rho_g^f} & \langle S,g|\emptyset\rangle_{\mathbf{1C}^*} \\ & \searrow^{\rho_h^f} & \downarrow \rho_h^g \\ & & \langle S,h|\emptyset\rangle_{\mathbf{1C}^*} \end{array}$$

Fix a crutch function f on S and a set of non-unital C*-relations R_f on (S,f). For $g \leq f$, define $R_g := \rho_g^f(R_f)$, $\mathcal{A}_g := \langle S,g|R_g\rangle_{\mathbf{C}^*}$, and let $q_g : \langle S,g|\emptyset\rangle_{\mathbf{C}^*} \to \mathcal{A}_g$ be the quotient map. For $h \leq g \leq f$, observe that $R_g \subseteq \ker\left(q_h \circ \rho_h^g\right)$ by design. By the universal property of the quotient, there is a unique *-homomorphism $\varphi_h^g : \mathcal{A}_g \to \mathcal{A}_h$ such that $\varphi_h^g \circ q_g = q_h \circ \rho_h^g$. As in the unital case, $\varphi_k^h \circ \varphi_h^g = \varphi_k^g$ and $\varphi_g^g = id_{\mathcal{A}_g}$. Thus, $(\mathcal{A}_g,\varphi_h^g)$ is an inverse system in $\mathbf{C}^*$. This is summarized in the commutative diagram below for $k \leq h \leq g \leq f$.

$$\begin{array}{ccc} \mathcal{A}_g & \xrightarrow{\varphi_h^g} & \mathcal{A}_h \\ & \searrow^{\varphi_k^g} & \downarrow \varphi_k^h \\ & & \mathcal{A}_k \end{array}$$

With this inverse system constructed in $\mathbf{C}^*$, the definition for "class set" and "tenuous" can be made. Recall that $[0,\infty)^S$ is equipped with the product topology here.

Definition. For a set of non-unital C*-relations R_f on a crutched set (S,f) and a

C*-algebra $\mathcal{A}$, the *class set* for $\mathcal{A}$ relative to (S,f) and R_f is given by

$$\Sigma^{\mathbf{C}^*}\left(\mathcal{A}:S,f,R_f\right) := \left\{g\in[0,\infty)^S : g\leq f, \mathcal{A}_g \cong_{\mathbf{C}^*} \mathcal{A}\right\}.$$

Definition. Given a set R_f of non-unital C*-relations on a crutched set (S,f), a C*-algebra $\mathcal{A}$ is *tenuous* for R_f if the associated class set $\Sigma^{\mathbf{C}^*}\left(\mathcal{A}:S,f,R_f\right)$ is nonempty and has empty interior.

Observe that while these C*-relations are non-unital, there are still tenuous cases.

Example 4.11.1 (A normal element). Fix $S:=\{x\}$, $f:S\to[0,\infty)$, and $R_f:=\{x^*x-xx^*\}$. Let $\lambda_f:=f(x)$. Example 3.5.1 shows

$$\left\langle (x,\lambda)\,|x^*x=xx^*\right\rangle_{\mathbf{C}^*} \cong_{\mathbf{C}^*} \begin{cases} \mathbb{O}, & \lambda=0,\\ C_0\left(\overline{\mathbb{D}}\setminus\{0\}\right), & \lambda>0.\end{cases}$$

Thus,

$$\Sigma^{\mathbf{C}^*}\left(\mathcal{A}:S,f,R_f\right) = \begin{cases} \{0\}, & \mathcal{A}\cong_{\mathbf{C}^*}\mathbb{O},\\ (0,\lambda_f], & \mathcal{A}\cong_{\mathbf{C}^*} C_0\left(\overline{\mathbb{D}}\setminus\{0\}\right),\\ \emptyset, & \text{otherwise.}\end{cases}$$

Example 4.11.2 (An idempotent element). Fix $S:=\{x\}$, $f:S\to[0,\infty)$, and $R_f:=\left\{x-x^2\right\}$. Let $\lambda_f:=f(x)$. Section 3.12 shows

$$\left\langle (x,\lambda)\,\middle|x=x^2\right\rangle_{\mathbf{C}^*} \cong_{\mathbf{C}^*} \begin{cases} \mathbb{O}, & \lambda<1,\\ \mathbb{C}, & \lambda=1,\\ \begin{bmatrix} C[0,1] & C_0(0,1]\\ C_0(0,1] & C_0(0,1]\end{bmatrix}, & \lambda>1.\end{cases}$$

Thus,

$$\Sigma^{\mathbf{C}^*}(\mathcal{A} : S, f, R_f) = \begin{cases} [0,1) \cap [0, \lambda_f], & \mathcal{A} \cong_{\mathbf{C}^*} \mathbb{O}, \\ \{1\} \cap [0, \lambda_f], & \mathcal{A} \cong_{\mathbf{C}^*} \mathbb{C}, \\ (1, \infty) \cap [0, \lambda_f], & \mathcal{A} \cong_{\mathbf{C}^*} \begin{bmatrix} C[0,1] & C_0(0,1] \\ C_0(0,1] & C_0(0,1] \end{bmatrix}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

As in the unital case, no necessary nor sufficient criteria have been devised for detecting tenuousness of a C*-algebra relative to some set of non-unital C*-relations. This idea seems very interesting, but very young. Further study may reveal new insight into this behavior.

Appendix A

Categorical Preliminaries

This chapter briefly covers the central topics in category theory necessary to this body of work. In all sections, the material will be introduced summarily, stating results without proof. Full treatments of these topics can be found in most resources on the subject such as [1] or [6].

A.1 Definitions

To begin, recall the axioms of a category.

Definition. A *category* $\mathscr{C}$ consists of the following data

1. a collection $\mathrm{Ob}(\mathscr{C})$,

2. for all $A, B \in \mathrm{Ob}(\mathscr{C})$, there is a collection $\mathscr{C}(A, B)$,

3. for all $A, B, C \in \mathrm{Ob}(\mathscr{C})$, there is a composition law mapping

$$\circ : \mathscr{C}(B, C) \times \mathscr{C}(A, B) \to \mathscr{C}(A, C)$$

subject to the axioms for all $A, B, C, D \in \text{Ob}(\mathscr{C})$, $f \in \mathscr{C}(A,B)$, $g \in \mathscr{C}(B,C)$, and $h \in \mathscr{C}(C,D)$,

1. there is $id_B \in \mathscr{C}(B,B)$, $id_B \circ f = f$ and $g \circ id_B = g$,

2. $h \circ (g \circ f) = (h \circ g) \circ f$.

Each member in $\text{Ob}(\mathscr{C})$ is an *object* of $\mathscr{C}$, and $\text{Ob}(\mathscr{C})$ is the *object collection* of $\mathscr{C}$. For objects A, B, each $f \in \mathscr{C}(A,B)$ is a *morphism* of $\mathscr{C}$ between A and B, or *$\mathscr{C}$-morphism.* For f, the *domain* of f is A, sometimes denoted $\text{dom}(f)$. Dually, the *codomain* of f is B, sometimes denoted $\text{codom}(f)$.

For most considerations of this work, $\text{Ob}(\mathscr{C})$ will be a class and all morphism collections $\mathscr{C}(A,B)$ will be sets. Table A.1 has the descriptions of all the categories which will be used in this work.

Table A.1: Symbols for Categories

Category	Objects	Morphisms
Set	sets	functions
$\mathbf{CSet}_1$	crutched sets	constrictive functions
$\mathbf{CSet}_\infty$	crutched sets	bounded functions
Top	topological spaces	continuous functions
Comp	compact, T_2 topological spaces	continuous functions
$R\mathbf{1Alg}$	unital R-algebras[1]	unital R-algebra homomorphisms
$\mathbb{F}\mathbf{NVec}_1$	normed $\mathbb{F}$-vector spaces[2]	contractive $\mathbb{F}$-linear transformations
$\mathbb{F}\mathbf{Ban}_\infty$	$\mathbb{F}$-Banach spaces	bounded $\mathbb{F}$-linear transformations
$\mathbf{C}^*$	C*-algebras	*-homomorphisms
$\mathbf{1C}^*$	unital C*-algebras	unital *-homomorphisms
$\mathbf{CC}^*$	commutative C*-algebras	*-homomorphisms
$\mathbf{C1C}^*$	commutative, unital C*-algebras	unital *-homomorphisms

[1] R is a fixed commutative, unital ring.
[2] $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$

With a new type of mathematical object defined, mappings preserving structure are defined.

Definition. Given categories $\mathscr{D}, \mathscr{C}$, a *covariant functor* F from $\mathscr{D}$ to $\mathscr{C}$ consists of the following data

1. a mapping F_1 from $\mathrm{Ob}(\mathscr{D})$ to $\mathrm{Ob}(\mathscr{C})$,

2. for each $A, B \in \mathrm{Ob}(\mathscr{D})$, a mapping $F_{A,B}$ from $\mathscr{D}(A, B)$ to $\mathscr{C}(F_1 A, F_1 B)$,

subject to the axioms for all $A, B, C \in \mathrm{Ob}(\mathscr{D})$, $f \in \mathscr{D}(A, B)$, and $g \in \mathscr{D}(B, C)$,

1. $F_{A,A}(id_A) = id_{F_1 A}$,

2. $F_{A,C}(g \circ f) = F_{B,C}(g) \circ F_{A,B}(f)$.

This is denoted $F : \mathscr{D} \to \mathscr{C}$. The *domain* of F is $\mathscr{D}$, the *codomain* $\mathscr{C}$.

In essence, a functor is a homomorphism of categories. Similarly, one defines a *contravariant functor* in the same manner, but the composition preservation is reversed. That is,

$$F_{A,B} : \mathscr{D}(A, B) \to \mathscr{C}(F_1 B, F_1 A)$$

and

$$F_{A,C}(g \circ f) = F_{A,B}(f) \circ F_{B,C}(g).$$

Since contravariance can be recovered by means of considering the opposite category, the category with its composition law reversed, all functors will be assumed covariant for the remainder of this appendix.

A common practice is to leave the mappings of a functor unlabeled, using the same symbol for the mappings on the objects and morphisms. Usually, context can determine which mapping is in use without much issue.

Functors also have their own composition.

Definition. Let $\mathscr{A}, \mathscr{B}, \mathscr{C}$ be categories, and let $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{C}$ be functors. Define the *composition* of G on F, GF, in the following way for each $A, B \in \mathscr{A}$, and $f \in \mathscr{A}(A, B)$,

1. $GF(A) := G(F(A))$,
2. $GF(f) := G(F(f))$.

A quick check shows that this is a new functor, the *composition* of G on F.

Mimicking topology, there is a notion of "homotopy" between two parallel functors.

Definition. Given two functors $F, G : \mathscr{D} \to \mathscr{C}$, a *natural transformation* α consists of the following data

- for each $D \in \mathrm{Ob}(\mathscr{D})$, $\alpha_D \in \mathscr{C}(FD, GD)$

subject to the following axiom for all $D, E \in \mathrm{Ob}(\mathscr{D})$, and $f \in \mathscr{D}(D, E)$,

- $\alpha_E \circ Ff = Gf \circ \alpha_D$.

This is denoted $\alpha : F \to G$. The *domain* of α is F, the *codomain* G.

Pictorially, this can be described with the commutative diagram below for each pair of objects $D, E \in \mathrm{Ob}(\mathscr{D})$ and $f \in \mathscr{D}(D, E)$.

$$\begin{array}{ccc} FD & \xrightarrow{Ff} & FE \\ {\scriptstyle \alpha_D}\downarrow & & \downarrow{\scriptstyle \alpha_E} \\ GD & \xrightarrow{Gf} & GE \end{array}$$

In the spirit of homotopy, there are similarly two standard ways of combining natural transformations to gain new ones.

The first is analogous to homotopic equivalence of functions between two fixed topological spaces.

Definition. Let $\mathscr{D}, \mathscr{C}$ be fixed categories, $F, G, H : \mathscr{D} \to \mathscr{C}$ be functors, and $\alpha : F \to G$ and $\beta : G \to H$ be natural transformations. Define the *composition* of β on α, $\beta \circ \alpha$, by the following data

- for each $D \in \mathrm{Ob}(\mathscr{D})$, $(\beta \circ \alpha)_D := \beta_D \circ \alpha_D$.

Pictorially, this would be as follows for each D, E in $\mathrm{Ob}(\mathscr{D})$ and $f \in \mathscr{D}(D, E)$.

$$\begin{array}{ccc} FD & \xrightarrow{Ff} & FE \\ {\scriptstyle \alpha_D}\downarrow & & \downarrow{\scriptstyle \alpha_E} \\ GD & \xrightarrow{Gf} & GE \\ {\scriptstyle \beta_D}\downarrow & & \downarrow{\scriptstyle \beta_E} \\ HD & \xrightarrow{Hf} & HE \end{array}$$

Since the two squares commute, the outer rectangle commutes, making $\beta \circ \alpha$ a new natural transformation.

The second is analogous to the construction of the fundamental group of a topological space.

Definition. Given categories $\mathscr{C}, \mathscr{D}, \mathscr{E}$, let $F, G : \mathscr{C} \to \mathscr{D}$ and $H, K : \mathscr{D} \to \mathscr{E}$ be functors. Also, let $\alpha : F \to G$ and $\beta : H \to K$ be natural transformations. Define the *Godement product* of β on α, $\beta * \alpha$, by the following data.

- for each $D \in \mathrm{Ob}(\mathscr{D})$, $(\beta * \alpha)_D := \beta_{GD} \circ H\alpha_D$.

Observe the following diagram for $D, E \in \text{Ob}(\mathscr{C})$ and arrows $f \in \mathscr{C}(D, E)$.

$$\begin{array}{ccc} HFD & \xrightarrow{HFf} & HFE \\ {\scriptstyle H\alpha_D}\downarrow & & \downarrow{\scriptstyle H\alpha_E} \\ HGD & \xrightarrow{HGf} & HGE \\ {\scriptstyle \beta_{GD}}\downarrow & & \downarrow{\scriptstyle \beta_{GE}} \\ KGD & \xrightarrow{KGf} & KGE \end{array}$$

Since α is a natural transformation and H a functor, the top square commutes. As β is a natural transformation, the bottom square commutes, forcing commutativity of the outer rectangle. Hence, $\beta * \alpha$ is a natural transformation from $H \circ F$ to $K \circ G$. Equivalently, one can define $(\beta * \alpha)_D := K\alpha_D \circ \beta_{FD}$.

A.2 Types of Morphisms

With the basic definitions in hand, consider first the different types of morphisms for a fixed category $\mathscr{C}$. The first type is one of the most fundamental of mathematical ideas, distinction between objects.

Definition. Given objects $A, B \in \text{Ob}(\mathscr{C})$, $f \in \mathscr{C}(A, B)$ is an *isomorphism* in $\mathscr{C}$ if there is $g \in \mathscr{C}(B, A)$ such that $g \circ f = id_A$ and $f \circ g = id_B$. The objects A and B are *isomorphic* in $\mathscr{C}$, or *$\mathscr{C}$-isomorphic.*

There are weakenings of this notion which have uses as well.

Definition. Given objects $A, B \in \text{Ob}(\mathscr{C})$, let $f \in \mathscr{C}(A, B)$ and $g \in \mathscr{C}(B, A)$. If $g \circ f = id_A$, then f is a *section* of g and g a *retraction* of f. In this case, A is a *retract* of B.

In this definition, f is "left-cancelable". That is, given any $h, j \in \mathscr{C}(C, A)$ such that $f \circ h = f \circ j$, composition on the left by g forces $h = j$. Similarly, g is "right-cancelable". These two notions motivate the next definitions.

Definition. Given objects $A, B \in \mathrm{Ob}(\mathscr{C})$, $f \in \mathscr{C}(A, B)$ is a *monomorphism* if for all $C \in \mathrm{Ob}(\mathscr{C})$ and $g, h \in \mathscr{C}(C, A)$ such that $f \circ g = f \circ h$, then $g = h$. In this case, f is *monic*.

Dually, $f \in \mathscr{C}(A, B)$ is an *epimorphism* if for all $C \in \mathrm{Ob}(\mathscr{C})$ and $g, h \in \mathscr{C}(A, C)$ such that $g \circ f = h \circ f$, then $g = h$. In this case, f is *epic*.

From the definitions, several basic results follow quickly.

Proposition A.2.1 (Hierarchy of Morphisms, [6]). *Given $A, B, C \in \mathrm{Ob}(\mathscr{C})$, let $f \in \mathscr{C}(A, B)$ and $g \in \mathscr{C}(B, C)$.*

1. *id_A is an isomorphism.*
2. *If f, g are isomorphisms, $g \circ f$ is an isomorphism.*
3. *If f, g are sections, $g \circ f$ is a section.*
4. *If f, g are retractions, $g \circ f$ is a retraction.*
5. *If f, g are monic, $g \circ f$ is monic.*
6. *If f, g are epic, $g \circ f$ is epic.*
7. *If f is an isomorphism, it is both a section and a retraction.*
8. *If f is a section, it is monic.*
9. *If f is a retraction, it is epic.*
10. *If $g \circ f$ is monic, f is monic.*

11. If $g \circ f$ is epic, g is epic.

12. If f is an epic section, f is an isomorphism.

13. If f is an monic retraction, f is an isomorphism.

The primary example of these definitions is in **Set**, where there are the following characterizations for a function $f : X \to Y$.

- f is monic in **Set** iff f is one-to-one,
- f is epic in **Set** iff f is onto,
- f is an isomorphism in **Set** iff f is one-to-one and onto.

However, while these definitions are closely related and, indeed, coincide for **Set** and other categories, there are examples where these are all distinct. In particular, there are examples of epimorphisms which are not onto and monomorphisms which are not one-to-one. One should take care the meaning of these words in a particular category of study.

A.3 Limiting Processes

Limiting processes are the usual means by which a universal construction is built. The primary idea is that given an existing diagram in a fixed category $\mathscr{C}$, the limit construct provides a universal means either to enter or exit that diagram. To describe this, the following definitions are made.

Definition. Given a functor $F : \mathscr{D} \to \mathscr{C}$, a *cone* over F consists of the following data

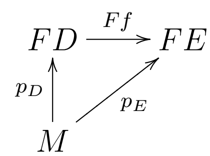
- an object $M \in \mathrm{Ob}(\mathscr{C})$,

- for each $D \in \mathrm{Ob}(\mathscr{D})$, $p_D \in \mathscr{C}(M, FD)$,

subject to the following axiom for all $D, E \in \mathrm{Ob}(\mathscr{D})$ and $f \in \mathscr{D}(D, E)$,

- $p_E = Ff \circ p_D$.

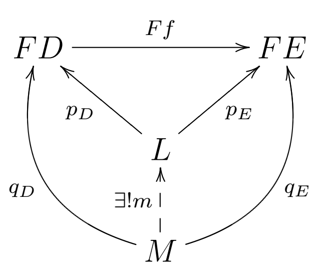
Pictorially, this can be described with the following commutative diagram for all $D, E \in \mathrm{Ob}(\mathscr{D})$ and $f \in \mathscr{D}(D, E)$.



The idea of a cone is that the domain category $\mathscr{D}$ encodes the framework of a diagram, letting the homomorphism properties of the functor carry the commutativity forward. With this notion, one can define the universal object.

Definition. Given a functor $F : \mathscr{D} \to \mathscr{C}$, a *(categorical) limit* of F is a cone (L, p_D) on F such that for every cone (M, q_D) on F, there exists a unique morphism $m \in \mathscr{C}(M, L)$ such that for every $D \in \mathrm{Ob}(\mathscr{D})$, $q_D = p_D \circ m$.

Pictorially, this can be described with the following characteristic diagram for all $D, E \in \mathrm{Ob}(\mathscr{D})$ and $f \in \mathscr{D}(D, E)$.



As a categorical limit has a universal property, it is a standard exercise to show that one is unique up to isomorphism. As such, it is customary to choose a particular representation, designated here by $\lim_{\mathscr{D}} FD$.

While this is a very broad and esoteric definition, there are three particular examples of this structure that are of particular interest.

Example A.3.1 (Products, [6]). Given a set I, it may be considered as what is called a *discrete category* $\mathscr{I}$, where the objects are the elements of the set and the only arrows are identities. Formally, this means

$$\mathrm{Ob}(\mathscr{I}) = I,$$

$$\mathscr{I}(i,j) = \begin{cases} \{id_i\}, & i = j \\ \emptyset, & i \neq j \end{cases}.$$

A functor $F : \mathscr{I} \to \mathscr{C}$ is, therefore, just an assignment of an object Fi for each $i \in I$. A cone (M, q_i) over F is merely a collection of morphisms $q_i \in \mathscr{C}(M, Fi)$ since there are no connecting maps between the objects of $\mathscr{I}$. Then, the categorical limit $\lim_{\mathscr{I}} Fi$ must satisfy the following universal property for any other cone,

$$\begin{array}{ccc} \lim_{\mathscr{I}} Fi & \xrightarrow{p_i} & Fi \\ \uparrow{\scriptstyle \exists! m} & \nearrow{\scriptstyle q_i} & \\ M & & \end{array}$$

which is the defining property of $\prod_{i \in I}^{\mathscr{C}} Fi$.

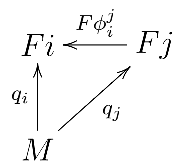
Example A.3.2 (Inverse Limits, [25]). Given a directed poset $(I, \leq)$, it may be considered as a category $\mathscr{I}$, where the objects are the elements of I and the arrows are determined by the order. Formally, this means

$$\mathrm{Ob}(\mathscr{I}) = I$$

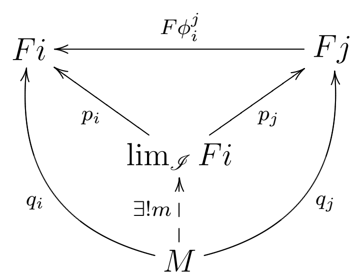
$$\mathscr{I}(j,i) = \begin{cases} \emptyset, & i \not\leq j, \\ \{id_i\}, & i = j, \\ \{\phi_i^j\}, & i \leq j, \end{cases}$$

which encodes the **reverse** ordering as arrows in a graph with the composition law $\phi_i^j \circ \phi_j^k = \phi_i^k$ for all $i \leq j \leq k$.

A functor $F : \mathscr{I} \to \mathscr{C}$ then is a selection of objects $(Fi)_{i \in I}$ and morphisms $F\phi_i^j \in \mathscr{C}(Fj, Fi)$ such that $F\phi_i^j \circ F\phi_j^k = F\phi_i^k$ and $F\phi_i^i = id_{Fi}$ for all $i \leq j \leq k$, an *inverse system*. A cone (M, q_i) over F satisfies the following commutative diagram for $i \leq j$.

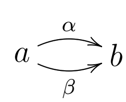


Then, the categorical limit $\lim\limits_{\mathscr{I}} Fi$ must satisfy the following universal property for any other cone,



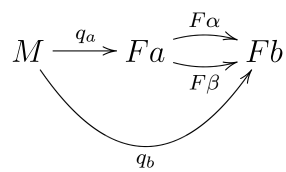
which is the defining property for $\varprojlim^{\mathscr{C}}(Fi, F\phi_i^j)$.

Example A.3.3 (Equalizers, [6]). Consider the category $\mathscr{K}$ defined by the following graph.

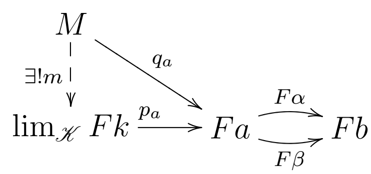


A functor $F : \mathscr{K} \to \mathscr{C}$ is merely a choice of objects Fa, Fb and maps $F\alpha, F\beta \in \mathscr{C}(Fa, Fb)$. A cone (M, q_a, q_b) is then a choice of morphism $q_a \in \mathscr{C}(M, Fa)$ such that

$F\alpha \circ q_a = F\beta \circ q_a = q_b$. This is described by the following commutative diagram.



Then, the categorical limit $\lim\limits_{\mathscr{K}} Fk$ must satisfy the following universal property for any other cone.



Such a universal structure is called an *equalizer* of α, β, denoted here by $\mathrm{Eq}_{\mathscr{C}}(\alpha, \beta)$.

This process is very abstract, but the above examples do reinforce the heuristic for understanding them. For an existing commutative diagram in $\mathscr{C}$, a cone is essentially an object with morphisms that connect to each object in the original diagram, where each new triangle commutes. The categorical limit is such a cone with the property that any other cone *must* factor through it uniquely. In short, if one wants to connect an object into the existing diagram, one can only do it one way, and it is through the limit object. The category $\mathscr{D}$ can be thought of as an indexing structure for the original diagram.

Not unlike metric structures, there is a notion of "completeness", where all limits of a certain kind exist within a given category.

Definition. A category $\mathscr{D}$ is *small* if $\mathrm{Ob}(\mathscr{D})$ is a set.

Definition. A category $\mathscr{D}$ is *(small) complete* if for all small categories $\mathscr{D}$ and functors $F : \mathscr{D} \to \mathscr{C}$, $\lim\limits_{\mathscr{D}} FD$ exists.

All the examples given above were "small" limits as their domain categories were small. Interestingly, when a category has arbitrary products and equalizers, one may construct all other small categorical limits from these two processes. This also gives a criterion for testing when a functor preserves all limits.

Theorem A.3.4 (Categorical Completeness, [6])**.** *A category $\mathscr{C}$ is complete if and only if it has all equalizers and arbitrary products.*

Corollary A.3.5 (Preservation of Limits, [6])**.** *A functor preserves categorical limits if and only if it preserves equalizers and arbitrary products.*

Sometimes, a category is not small complete, but all limit processes from small categories with finitely many objects may be accomplished. Many times, this is sufficient so the following definitions and results are made.

Definition. A small category $\mathscr{D}$ is *finite* if $\mathrm{Ob}(\mathscr{D})$ is a finite set.

Definition. A category $\mathscr{D}$ is *finitely complete* if for all finite categories $\mathscr{D}$ and functors $F : \mathscr{D} \to \mathscr{C}$, $\lim\limits_{\mathscr{D}} FD$ exists.

Theorem A.3.6 (Finite Categorical Completeness, [6])**.** *A category $\mathscr{C}$ is finitely complete if and only if it has all equalizers and finite products.*

Corollary A.3.7 (Preservation of Finite Limits, [6])**.** *A functor preserves finite categorical limits if and only if it preserves equalizers and finite products.*

Categorical limits were universal means of entering a diagram, and its dual notion is the universal means of exiting a diagram, a *colimit*. All the results for colimits are completely analogous to those for limits so they will not be restated, left to the reader.

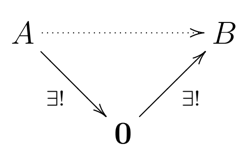
To close this section, there are two particular universal constructions that are usually of particular interest.

Definition. A product of an empty family of objects in $\mathscr{C}$ is a *terminal* object in $\mathscr{C}$, $\mathbf{1}$. A coproduct of an empty family is an *initial object* in $\mathscr{C}$, $\mathbf{0}$.

Equivalently, $\mathbf{1}$ is an object such that for all $A \in \mathrm{Ob}(\mathscr{C})$, $\mathrm{card}(\mathscr{C}(A, \mathbf{1})) = 1$. Dually, $\mathbf{0}$ is an object such that for all $A \in \mathrm{Ob}(\mathscr{C})$, $\mathrm{card}(\mathscr{C}(\mathbf{0}, A)) = 1$. In many algebraic categories, these two notions coincide in the singleton set $\{0\}$, leading to the following definition.

Definition. A *zero* object is an object which is both initial and terminal, usually denoted $\mathbf{0}$.

In the case that a category $\mathscr{C}$ has a zero object, each pair of objects has a morphism between them, the *zero morphism*. Explicitly, this morphism is the composition of a morphism into $\mathbf{0}$ and a morphism from $\mathbf{0}$. This situation is described in the following diagram for each $A, B \in \mathrm{Ob}(\mathscr{C})$.



Moreover, because of the uniqueness properties surrounding $\mathbf{0}$, there is only one morphism with a factorization of this sort.

The above notation and terminology are influenced by the case of **Set**, where the following characterizations exist.

- A product of a family of sets is their Cartesian product. A terminal object is a singleton set.

- A coproduct of a family of sets is their disjoint union. The only initial object is the empty set.

- If $f, g : X \to Y$, then

$$\mathrm{Eq}_{\mathbf{Set}}(f, g) \cong_{\mathbf{Set}} \{x \in X : f(x) = g(x)\}.$$

  Let $R := \{(f(x), g(x)) \in Y^2 : x \in X\}$ and $\sim$ the equivalence relation on Y generated by R. Then,

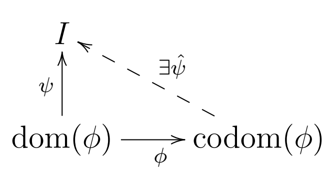
$$\mathrm{Coeq}_{\mathbf{Set}}(f, g) \cong_{\mathbf{Set}} Y/\sim .$$

A.4 Projectivity and Injectivity

While the last section dealt with universal map factorization properties, the notions of projectivity and injectivity are weakenings of the universal notions. To describe these concepts, fix a category $\mathscr{C}$ and a class of morphisms Φ. Generally, the maps in Φ are "extensions" for injectivity and "quotients" for projectivity.

Definition. For a class of morphisms Φ, an object I is Φ-*injective* if for all $\phi \in \Phi$ and $\psi \in \mathscr{C}(\mathrm{dom}(\phi), I)$, there is $\hat{\psi} \in \mathscr{C}(\mathrm{codom}(\phi), I)$ such that $\hat{\psi} \circ \phi = \psi$.

Pictorially, this can be described with the following characteristic diagram for each $\phi \in \Phi$ and $\psi \in \mathscr{C}(\mathrm{dom}(\phi), I)$.



Being a Φ-injective object equates to the ability to factoring any morphism into the object by any morphism in Φ. However, note that this property is not universal, no requirement for uniqueness of factorization.

Definition. The category $\mathscr{C}$ *has enough* Φ*-injective objects* if for any object A, there is a Φ-injective object I and morphism $\phi \in \mathscr{C}(A, I)$ from Φ.

Intuitively, $\mathscr{C}$ has "enough injectives" when one can "embed" any object into an "injective". In this situation, one would then want a "minimal injective", or an "injective envelope".

Definition. A morphism $\phi \in \Phi$ is Φ*-essential* if when a morphism α has $\mathrm{dom}(\alpha) = \mathrm{codom}(\phi)$ and $\alpha \circ \phi \in \Phi$, then $\alpha \in \Phi$.

Conceptually, ϕ is an "essential embedding" if when a composition of $\alpha \circ \phi$ is an "embedding" again, α must be an "embedding". This is the notion of "minimality".

Definition. For an object A, a Φ*-injective envelope/hull* of A is a Φ-injective object I and Φ-essential morphism $\phi \in \mathscr{C}(A, I)$.

Though Φ-injectivity is not a universal property, a Φ-injective envelope is unique up to isomorphism, implemented by morphisms in Φ.

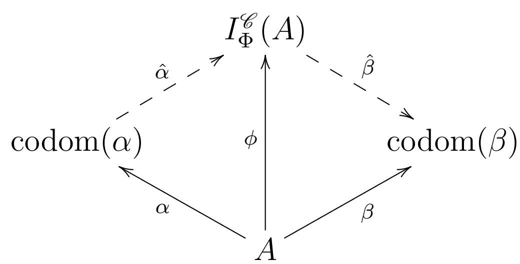
Proposition A.4.1 (Uniqueness of the Injective Envelope, [1])**.** *Suppose that* (I, ϕ) *and* (J, ψ) *are* Φ*-injective envelopes of an object* A*. Then, there is an isomorphism* $\alpha \in \Phi$ *with* $\mathrm{dom}(\alpha) = I$*,* $\mathrm{codom}(\alpha) = J$*, and* $\alpha \circ \phi = \psi$*. Further,* $\alpha^{-1} \in \Phi$*.*

Since this structure is unique up to isomorphism, it is common practice to choose a particular representation of it, denoted in these notes as $I_{\Phi}^{\mathscr{C}}(A)$. Intuitively, this structure is a "minimal injective extension", but it is also characterized as a "maximal essential extension".

Explicitly, this property is as follows. Let A be an object in $\mathscr{C}$ with Φ-injective envelope $(I_\Phi^{\mathscr{C}}(A), \phi)$. Suppose that α is a Φ-essential morphism with $\operatorname{dom}(\alpha) = A$. As $I_\Phi^{\mathscr{C}}(A)$ is Φ-injective, there is $\hat{\alpha} \in \mathscr{C}(\operatorname{codom}(\alpha), I_\Phi^{\mathscr{C}}(A))$ such that $\hat{\alpha} \circ \alpha = \phi$. As α is Φ-essential, $\hat{\alpha} \in \Phi$. That is, $I_\Phi^{\mathscr{C}}(A)$ is a "maximal essential embedding".

Similarly, suppose that β is a morphism with $\operatorname{dom}(\beta) = A$ and $\operatorname{codom}(\beta)$ Φ-injective. As $\operatorname{codom}(\beta)$ is Φ-injective, there is $\hat{\beta} \in \mathscr{C}(I_\Phi^{\mathscr{C}}(A), \operatorname{codom}(\beta))$ such that $\hat{\beta} \circ \phi = \beta$. As ϕ is Φ-essential, $\hat{\beta} \in \Phi$. That is, $I_\Phi^{\mathscr{C}}(A)$ is a "minimal injective embedding".

Pictorially, this can be summarized with the following commutative diagram for the α and β described above.



While this factorization property results in a unique object, the factorizations of the maps need not be unique.

As it happens, the notion of injectivity works well with products.

Proposition A.4.2 (Products of Injectives, [6]). *Given a class of morphisms Φ and index set J, suppose that $(I_j)_{j \in J}$ are Φ-injective and $\prod_{j \in J}^{\mathscr{C}} I_j$ exists. Then, $\prod_{j \in J}^{\mathscr{C}} I_j$ is Φ-injective.*

Many algebraic textbooks will state this as "if and only if". In most algebraic settings, this is true as one can use the zero object to connect any two objects, hence isolating the factors of the product.

Proposition A.4.3 (Products of Injectives II, [6])**.** *Given a class of morphisms* Φ *and index set* J*, suppose that* $(I_j)_{j\in J}$ *have a product* $\prod_{j\in J}^{\mathscr{C}} I_j$*. If* $\mathscr{C}$ *has a zero object,* $\prod_{j\in J}^{\mathscr{C}} I_j$ *is* Φ*-injective if and only if* I_j *is* Φ*-injective for all* $j \in J$*.*

However, this is not always possible in a more general setting.

Since a product of Φ-injectives is again Φ-injective, one has the following fact about the injective envelopes. If $(A_j)_{j\in J}$ are objects with Φ-injective envelopes $I_\Phi^{\mathscr{C}}(A_j)$, suppose the products $\prod_{j\in J}^{\mathscr{C}} A_j$ and $\prod_{j\in J}^{\mathscr{C}} I_\Phi^{\mathscr{C}}(A_j)$ exist. If $I_\Phi^{\mathscr{C}}\left(\prod_{j\in J}^{\mathscr{C}} A_j\right)$ exists, then there is a $\phi \in \Phi$ with $\text{dom}(\phi) = I_\Phi^{\mathscr{C}}\left(\prod_{j\in J}^{\mathscr{C}} A_j\right)$ and $\text{codom}(\phi) = \prod_{j\in J}^{\mathscr{C}} I_\Phi^{\mathscr{C}}(A_j)$. That is, provided all appropriate envelopes and products exist, one can "embed" the envelope of the product into the product of the factor envelopes. However, this is not always an isomorphism.

This concept of Φ-injective envelope may be dualized into a concept of "projective cover", which is studied for **Comp** in [23]. All results for projective objects are analogous to those for injective objects so they will not be restated, left to the reader.

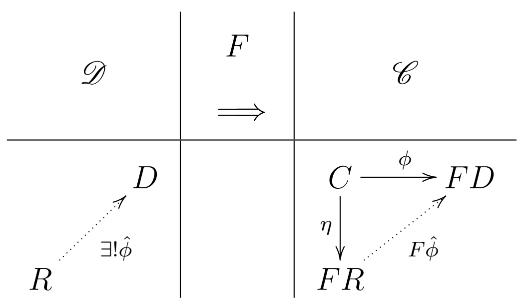
The usual choices of Φ in almost all settings is the class of monomorphisms for injectivity, and the class of epimorphisms for projectivity. In **Set**, all objects are projective with respect to epimorphisms, and only the empty set fails to be injective with respect to monomorphisms. As such, these notions are usually used in other contexts, where fewer objects have this property. Many sources discuss projectivity and injectivity, sometimes using different Φ classes. A few of these are as follows: [2], [7], [20], [23], [24], [25], [37], and [41].

A.5 Adjoint Functors

This section considers the categorical notion of a pair of adjoint functors, a powerful and ubiquitous concept in category theory. To begin, one considers the notion of a reflection, encapsulating the universal nature of an adjoint functor.

Definition. Given a functor $F : \mathscr{D} \to \mathscr{C}$ and $C \in \mathrm{Ob}(\mathscr{C})$, a *reflection* of C along F is an object $R \in \mathrm{Ob}(\mathscr{D})$ and a morphism $\eta \in \mathscr{C}(C, FR)$ such that for any $D \in \mathrm{Ob}(\mathscr{D})$ and $\phi \in \mathscr{D}(C, FD)$, there is a unique $\hat{\phi} \in \mathscr{C}(R, D)$ such that $F\hat{\phi} \circ \eta = \phi$.

Pictorially, this property can be described with the commutative diagrams below for each $D \in \mathrm{Ob}(\mathscr{D})$ and $\phi \in \mathscr{D}(C, FD)$.



As a reflection has a universal property, it is a standard exercise to show that one is unique up to isomorphism.

Conceptually, a reflection (R, η) of C along F is a sort of "universal pre-image" of C. Much like universal limiting processes, if one wishes to map from C into the image of F, one *must* factor through the image of the reflection. However, unlike limits, the factorization originates in the *domain* category, not the codomain category where C is.

Now, if every object in $\mathscr{C}$ has a reflection along F, one has a quite powerful statement, the defining notion of the left adjoint.

Proposition A.5.1 (Existence of a Left Adjoint, [6]). *Given a functor $F : \mathscr{D} \to \mathscr{C}$, assume that for every $C \in \mathrm{Ob}(\mathscr{C})$, C has a reflection (R_C, η_C) along F. Then, there is a unique functor $R : \mathscr{C} \to \mathscr{D}$ such that*

1. *$RC = R_C$ for all $C \in \mathrm{Ob}(\mathscr{C})$,*

2. *the class η_C is a natural transformation from $id_{\mathscr{C}}$ to $F \circ R$.*

Observe that item 2 above is formally very similar to homotopy equivalence of topological spaces. Notice that the choice of reflections is arbitrary, meaning any selection yields the same result. However, much like universal objects, it is a standard exercise to show that any choice in the proposition above is unique up to an invertible natural transformation. As such, one makes the following definition.

Definition. Given a functor $F : \mathscr{D} \to \mathscr{C}$, a functor $L : \mathscr{C} \to \mathscr{D}$ is a *left adjoint* to F if there is a natural transformation $\eta : id_{\mathscr{C}} \to F \circ L$ such that for all $C \in \mathrm{Ob}(\mathscr{C})$, (LC, η_C) is a reflection of C along F.

Dually, one can define a coreflection and a right adjoint in a similar fashion. As it happens, these two notions are not only dual to one another, but intimately tied in the following way.

Theorem A.5.2 (Adjoint Functor Pairs, [6]). *Given two functors $R : \mathscr{D} \to \mathscr{C}$ and $L : \mathscr{C} \to \mathscr{D}$, the following are equivalent.*

1. *L is a left adjoint to R,*

2. *there are natural transformations $\eta : id_{\mathscr{C}} \to R \circ L$ and $\epsilon : L \circ R \to id_{\mathscr{D}}$ such that*

$$(id_R * \epsilon) \circ (\eta * id_R) = id_R \quad \textit{and} \quad (\epsilon * id_L) \circ (id_L * \eta) = id_L,$$

3. for each $D \in \mathrm{Ob}(\mathscr{D})$ *and* $C \in \mathrm{Ob}(\mathscr{C})$*, there exist bijections*

$$\theta_{D,C} : \mathscr{D}(LC, D) \to \mathscr{C}(C, RD)$$

which are natural in both D *and* C*,*

4. R *is a right adjoint to* L*.*

Thus, when an adjoint functor appears, it is one of a pair. In this situation, the relationship is notated as $L \dashv R$. Further, Criterion 3 is formally similar to the definition of adjoint operators on Hilbert space within inner products, motivating the nomenclature.

Pleasantly, adjoint functors also work very well with compositions and limiting processes.

Proposition A.5.3. *Let* $F : \mathscr{A} \to \mathscr{B}$ *and* $H : \mathscr{B} \to \mathscr{C}$ *be functors. Suppose* $G \dashv F$ *and* $K \dashv H$*. Then,* $GK \dashv HF$*.*

Proposition A.5.4. *Given* $L \dashv R$*, then* R *preserves all categorical limits and* L *all categorical colimits.*

A more powerful connection between two categories is when the natural transformations ϵ and η are invertible. In this case, $L \dashv R$ and $R \dashv L$ so Proposition A.5.4 guarantees that both preserve limits and colimits, among most other properties. Thus, the following definitions are made.

Definition. A functor $R : \mathscr{D} \to \mathscr{C}$ is an *equivalence of categories* if R has a left adjoint $L : \mathscr{C} \to \mathscr{D}$ such that the natural transformations η and ϵ of Theorem A.5.2 are both invertible. In this situation, $\mathscr{D}$ and $\mathscr{C}$ are *equivalent.* If $\mathscr{C}$ is equivalent to $\mathscr{D}^{op}$, then $\mathscr{C}$ is *dual* to $\mathscr{D}$.

Appendix B

A Categorical Library for C*-algebras

This chapter briefly covers the application of the topics from Appendix A to categories of C*-algebras, which will be used in this body of work. As most of these results are well-known, the material will be introduced summarily, stating most results without proof. Many of these topics can be readily reconstructed from base principles. For the more complex notions, full treatments can be found in resources on the subject such as [10], [27], or [40].

While these many of these results and constructions are well-known, indeed even elementary, the author is unaware of a similar functorial treatment of these ideas. This perspective, though nonstandard, yields some useful computational results with immediate applicability to the current work of Chapters 3 and 4. Here are two motivating examples.

Example. Let Ab be the abelianization functor constructed in Section B.4. Given two C*-algebras $\mathcal{A}$ and $\mathcal{B}$, let their free product be denoted by $\mathcal{A} * \mathcal{B}$. From the functorial characterization in Section B.4,

$$\mathrm{Ab}(\mathcal{A} * \mathcal{B}) \cong \mathrm{Ab}(\mathcal{A}) \otimes \mathrm{Ab}(\mathcal{B}),$$

the tensor product of the resulting commutative C*-algebras, via *-homomorphism.

Example. Let Unit be the unitization functor constructed in Section B.5. From the functorial characterization in Section B.5.2,

$$\mathrm{Unit}(\mathcal{A} * \mathcal{B}) \cong \mathrm{Unit}(\mathcal{A}) *_{\mathbb{C}} \mathrm{Unit}(\mathcal{B}),$$

the free product amalgamated along the identities, via *-homomorphism.

The proofs of these two results, as well as a bevy more, are immediate from the general notions of Appendix A with the characterizations of universal constructions within this appendix. More sample results are given in Section B.6.

To accomplish this task, the first sections will introduce particular categories of C*-algebras, focusing on their principle properties and constructions. Sections B.4 and B.5 develop the abelianization and unitization functors, respectively, to move between and relate these categories. Lastly, these ideas are summarized and given application to a non-immediate result in Section B.7.2.

B.1 The Category $\mathbf{C}^*$

This section considers the category of C*-algebras with *-homomorphisms, denoted $\mathbf{C}^*$. In particular, the standard universal constructions can be done in this category. Equalizers, coequalizers, and products are done much like they would be done in **Set** or $\mathbb{C}$**1Alg**. The proofs of these are left to the reader.

Proposition B.1.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras and $\phi, \psi : \mathcal{A} \to \mathcal{B}$ be *-homomorphisms. Let*

$$\mathcal{K} := \{a \in \mathcal{A} : \phi(a) = \psi(a)\}$$

with the inherited operations from $\mathcal{A}$ and $\iota : \mathcal{K} \to \mathcal{A}$ by $\iota(a) := a$. Then, K equipped with ι is an equalizer of ϕ and ψ.

Proposition B.1.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras and $\phi, \psi : \mathcal{A} \to \mathcal{B}$ be *-homomorphisms. Let J be the two-sided, norm-closed ideal in $\mathcal{B}$ generated by the set*

$$\{\phi(a) - \psi(a) : a \in \mathcal{A}\},$$

$\mathcal{Q} := \mathcal{B}/J$ with inherited operations from $\mathcal{B}$, and $q : \mathcal{B} \to \mathcal{Q}$ the quotient map. Then, $\mathcal{Q}$ equipped with q is a coequalizer of ϕ and ψ.

Proposition B.1.3. *For an index set I, let $\mathcal{A}_i$ be a C*-algebra for $i \in I$. Define*

$$\mathcal{P} := \left\{\vec{a} \in \mathbf{Set}\left(I, \bigcup_{i \in I} \mathcal{A}_i\right) : \vec{a}(i) \in \mathcal{A}_i \forall i \in I, \sup\left\{\|\vec{a}(i)\|_{\mathcal{A}_i} : i \in I\right\} < \infty\right\}$$

with point-wise operations, $\|\cdot\|_{\mathcal{P}} : \mathcal{P} \to [0, \infty)$ by $\|\vec{a}\|_{\mathcal{P}} := \sup\left\{\|\vec{a}(i)\|_{\mathcal{A}_i} : i \in I\right\}$, and $\pi_i : \mathcal{P} \to \mathcal{A}_i$ by $\pi_i(\vec{a}) := \vec{a}(i)$. Then, $\mathcal{P}$ equipped with $(\pi_i)_{i \in I}$ is a product of $(\mathcal{A}_i)$.

However, the coproduct, known as the free product, is more subtle than these more elementary constructions. In the work of [5], the existence of a free product of C*-algebras is assured.

Theorem B.1.4 (Existence of the Free Product, [5])**.** *For an index set I, let $\mathcal{A}_i$ be a C*-algebra for $i \in I$. Then, $\coprod_{i \in I}^{\mathbf{C^*}} \mathcal{A}_i$ exists.*

The traditional symbol in C*-algebraic literature for the free product is "$*$", though the common symbol for a coproduct in category theory is "$\coprod$". The "$\coprod$" notation will be used interchangeably with the "$*$" notation, but preference will be given to the "$\coprod$" with arbitrary index sets.

Together, these four constructions yield that all limit and colimit processes can be done in $\mathbf{C}^*$.

Corollary B.1.5. *The category* $\mathbf{C}^*$ *is categorically complete and cocomplete.*

Consequently, all direct and inverse limit processes may be done in this category. Keep in mind that this is *not* the ring-theoretic direct or inverse limit, nor the topological *-algebraic variant of [35], but rather their analogues in the category $\mathbf{C}^*$. The direct limit is well-known and studied in numerous texts, including [27]. The inverse limit in $\mathbf{C}^*$ is lesser known, but exists, satisfying the appropriate universal property. One means of realizing it as follows.

Example B.1.6 (Inverse Limits in $\mathbf{C}^*$)*.* Given a poset $(P, \leq)$ and an inverse system $\left(\mathcal{A}_p, \phi_p^q\right)_{p,q\in P}$ in $\mathbf{C}^*$, the inverse limit can be realized by first forming the ℓ^∞-sum of the factors, the product in $\mathbf{C}^*$. Next, one restricts to the sequences $(a_p)_{p\in P}$, which satisfy the condition $\phi_p^q(a_q) = a_p$ for all $p \leq q$.

Further, an empty product yields a terminal object, the zero algebra $\mathbb{0} := \{0\}$. Likewise, the empty coproduct an initial object, also $\mathbb{0}$, meaning this is a zero object in the categorical sense. Naturally, the categorical zero morphisms are precisely the constant map to 0.

Projectivity relative to the class of all surjections in $\mathbf{C}^*$ is well-established in sources such as [27]. As demonstrated in Lemma 10.1.6 in [27], a C*-algebra which is projective with respect to surjections in $\mathbf{C}^*$ must be *contractible* in $\mathbf{C}^*$. Specifically, this is defined as follows, one of several equivalent definitions.

Definition. Given a C*-algebra $\mathcal{A}$, let $C_0([0,\infty),\mathcal{A})$ be the *cone* over $\mathcal{A}$, all continuous functions from $[0,\infty)$ to $\mathcal{A}$ which vanish at ∞. Let $\pi_{\mathcal{A},0} : C_0([0,\infty),\mathcal{A}) \to \mathcal{A}$ by $\pi_{\mathcal{A},0}(f) := f(0)$, evaluation at 0, a well-known *-homomorphism. $\mathcal{A}$ is *contractible* in $\mathbf{C^*}$ if $\pi_{\mathcal{A},0}$ is a retraction in $\mathbf{C^*}$.

Here, the notion is qualified by "in $\mathbf{C^*}$" since there is a broader notion of contractibility in **Top**. Specifically, the C*-algebra $\mathcal{A}$ is indeed null-homotopic to a singleton set, but that singleton set is purposefully chosen to be $\mathbb{O}$. Further, it is required that each stage of the homotopy be a *-homomorphism.

The proof of Lemma 10.1.6 of [27] follows directly from consideration of the following diagram in $\mathbf{C^*}$.

$$\begin{array}{ccc} & & C_0([0,\infty),\mathcal{A}) \\ & & \downarrow{\scriptstyle \pi_{\mathcal{A},0}} \\ \mathcal{A} & \xrightarrow[id_{\mathcal{A}}]{\cong_{\mathbf{C^*}}} & \mathcal{A} \end{array}$$

That is, if $\mathcal{A}$ is projective with respect to all surjections in $\mathbf{C^*}$, specifically $\pi_{\mathcal{A},0}$, there must be a *-homomorphism from completing this triangle, making $\pi_{\mathcal{A},0}$ a retraction in $\mathbf{C^*}$.

However, no nontrivial unital C*-algebra can ever satisfy this criterion.

Proposition B.1.7. *A unital C*-algebra is contractible in* $\mathbf{C^*}$ *if and only if* $\mathcal{A} \cong_{\mathbf{1C^*}} \mathbb{O}$.

This fact immediately destroys any possibility for a nontrivial unital C*-algebra to be projective relative to all surjections in $\mathbf{C^*}$. Since $\mathbf{C^*}$ has a zero object, one can invoke Proposition A.4.3 to yield the following result for free products.

Proposition B.1.8. *Given an index set I and C*-algebras $(\mathcal{P}_i)_{i\in I}$, then* $\coprod_{i\in I}^{\mathbf{C^*}} \mathcal{P}_i$ *is projective relative to all surjections in* $\mathbf{C^*}$ *if and only if each $\mathcal{P}_i$ is also for all $i \in I$.*

B.2 The Category $\mathbf{1C}^*$

This section considers the category of unital C*-algebras with unital *-homomorphisms, denoted $\mathbf{1C}^*$. Note that the zero algebra, $\mathbb{0}$, will be considered as a *unital* C*-algebra for the purposes of this work. Specifically, it will be thought of as the unique unital C*-algebra where $0 = \mathbb{1}$, or equivalently, $C(\emptyset)$, continuous functions on the empty topological space.

Just as in $\mathbf{C}^*$, the standard universal constructions can be done in this category. Equalizers, coequalizers, and products are computed just as they would are in $\mathbf{C}^*$, described in the previous section.

However, the coproduct, the *unital* free product, is similar to the free product in $\mathbf{C}^*$, but has a notable distinction. Here, the coproduct includes amalgamation of the identities of the factors. In [40], the free product with amalgamation of identities is shown to be the coproduct in $\mathbf{1C}^*$, satisfying the appropriate mapping property.

Theorem B.2.1 (Existence of the Unital Free Product, [40])**.** *For an index set I, let $\mathcal{A}_i$ be a unital C*-algebra for $i \in I$. Then, $\coprod_{i\in I}^{\mathbf{1C}^*} \mathcal{A}_i$ exists.*

Again, the usual notation for the unital free product is "$*_{\mathbb{C}}$", indicating the merger of the identities. Here too, the "$\coprod$" notation will be used interchangeably with the "$*_{\mathbb{C}}$" notation, but preference will be given to the "$\coprod$" with arbitrary index sets.

Together, these four constructions yield that all limit and colimit processes can be done in $\mathbf{1C}^*$.

Corollary B.2.2. *The category $\mathbf{1C}^*$ is categorically complete and cocomplete.*

Consequently, all direct and inverse limit processes may be done in this category, performed much like they were in $\mathbf{C}^*$. Further, an empty product yields a terminal object, $\mathbb{O}$. Likewise, the empty coproduct an initial object, the complex field $\mathbb{C}$. Since $\mathbb{C} \not\cong_{\mathbf{1C^*}} \mathbb{O}$, this category has no zero object in the categorical sense.

Projectivity relative to the class of all surjections in $\mathbf{1C^*}$ is also well-established in sources such as [27]. While $\mathbf{1C^*}$ has distinct initial and terminal objects, a version of Proposition A.4.3 holds. The proof of this result will be given as the author has no knowledge of its existence in the literature. However, the current proof requires the machinery of the remaining sections so it will be set aside until Section B.7.2.

B.3 The Categories C1C* & Comp

This section considers the category of commutative unital C*-algebras with unital *-homomorphisms, denoted $\mathbf{C1C^*}$. Again, the zero algebra, $\mathbb{O}$, will be considered as a *unital* C*-algebra for the purposes of this work. This category is well-known to be dual to the category of compact, Hausdorff topological spaces and continuous maps, denoted by **Comp**. To summarize this relationship, recall the following contravariant functors. Notice that both are of the form $\mathrm{Hom}(-, A)$.

Let $C : \mathbf{Comp} \to \mathbf{C1C^*}$ be defined by the following two maps:

- for $X \in \mathrm{Ob}(\mathbf{Comp})$, $C(X)$ is the continuous functions from X into $\mathbb{C}$,
- for $X, Y \in \mathrm{Ob}(\mathbf{Comp})$ and $f \in \mathbf{Comp}(X, Y)$, $C(f) : C(Y) \to C(X)$ by $C(f)(g) := g \circ f$.

Similarly, let $\Delta : \mathbf{C1C^*} \to \mathbf{Comp}$ be defined by the following two maps:

- for $\mathcal{A} \in \mathrm{Ob}(\mathbf{C1C^*})$, $\Delta(\mathcal{A})$ is the set of all nonzero multiplicative, linear functionals on $\mathcal{A}$ equipped with the weak-* topology, the *maximal ideal space*,

- for $\mathcal{A}, \mathcal{B} \in \mathrm{Ob}(\mathbf{C1C^*})$ and $\phi \in \mathbf{C1C^*}(\mathcal{A}, \mathcal{B})$, $\Delta(\phi) : \Delta(\mathcal{B}) \to \Delta(\mathcal{A})$ by $\Delta(\phi)(\psi) := \psi \circ \phi$.

Theorem B.3.1 (Gelfand-Naimark Theorem, [10]). *For $\mathcal{A} \in \mathrm{Ob}(\mathbf{C1C^*})$, define the function $\Gamma_{\mathcal{A}} : \mathcal{A} \to C(\Delta(\mathcal{A}))$ by $\Gamma_{\mathcal{A}}(a)(\phi) := \phi(a)$, the Gelfand transform. Then, $\Gamma_{\mathcal{A}}$ is an isomorphism in $\mathbf{C1C^*}$.*

The following result is usually a standard exercise, but is an important part of the story between $\mathbf{C1C^*}$ and **Comp**.

Theorem B.3.2. *For $X \in \mathrm{Ob}(\mathbf{Comp})$, define the function $\Phi_X : X \to \Delta(C(X))$ by $\Phi_X(x)(f) := f(x)$, the evaluation map at $x \in X$. Then, Φ_X is an isomorphism in* **Comp**.

A quick check shows that the following two diagrams commute in their respective categories.

$$
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\Gamma_{\mathcal{A}}} & C(\Delta(\mathcal{A})) \\
{\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle C(\Delta(\phi))} \\
\mathcal{B} & \xrightarrow{\Gamma_{\mathcal{B}}} & C(\Delta(\mathcal{B}))
\end{array}
\qquad
\begin{array}{ccc}
X & \xrightarrow{\Phi_X} & \Delta(C(X)) \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \Delta(C(f))} \\
Y & \xrightarrow{\Phi_Y} & \Delta(C(Y))
\end{array}
$$

Letting $\Gamma := (\Gamma_{\mathcal{A}})_{\mathcal{A} \in \mathrm{Ob}(\mathbf{C1C^*})}$ and $\Phi := (\Phi_X)_{X \in \mathrm{Ob}(\mathbf{Comp})}$, $\Gamma : id_{\mathbf{C1C^*}} \to C\Delta$ and $\Phi : id_{\mathbf{Comp}} \to \Delta C$ are invertible natural transformations, stating that $\mathbf{C1C^*}$ is equivalent to $\mathbf{Comp}^{\mathrm{op}}$, the opposite category of **Comp**. Hence, $\mathbf{C1C^*}$ and **Comp** are dual to one another.

Equalizers, coequalizers, and products for $\mathbf{C1C^*}$ are computed just as they would are in $\mathbf{C^*}$ and $\mathbf{1C^*}$ and have natural association to dual notions in **Comp**. For the coproduct in $\mathbf{C1C^*}$, there are two ways of viewing the construction, a generalization of the tensor product or using the duality in **Comp**.

Theorem B.3.3 (Generalized Tensor Product for $\mathbf{C1C^*}$). *For an index set I, let $\mathcal{A}_i$*

be a commutative, unital C-algebra for $i \in I$. Let* $\pi_i : \prod_{i\in I}^{\mathbf{Comp}} \Delta(\mathcal{A}_i) \to \Delta(\mathcal{A}_i)$ *be the usual projection map. Then,*

$$C\left(\prod_{i\in I}^{\mathbf{Comp}} \Delta(\mathcal{A}_i)\right),$$

equipped with $\left(C(\pi_i) \circ \Gamma_{\mathcal{A}_i}\right)_{i\in I}$ *is a coproduct of* $(\mathcal{A}_i)_{i\in I}$*. If I is finite, then*

$$C\left(\prod_{i\in I}^{\mathbf{Comp}} \Delta(\mathcal{A}_i)\right) \cong_{\mathbf{C1C^*}} \otimes_{i\in I}\mathcal{A}_i.$$

These construction equivalences are summarized in Table B.1. Together, these four constructions yield that all limit and colimit processes can be done in $\mathbf{C1C^*}$.

Corollary B.3.4. *The categories* $\mathbf{C1C^*}$ *and* **Comp** *are categorically complete and cocomplete.*

Consequently, all direct and inverse limit processes may be done in this category, performed much like they were in $\mathbf{C^*}$. Further, an empty product yields a terminal object, $\mathbb{O}$. Likewise, the empty coproduct an initial object, $\mathbb{C}$. Since $\mathbb{C} \not\cong_{\mathbf{C1C^*}} \mathbb{O}$, this category has no zero object in the categorical sense.

Table B.1: Universal Constructions in $\mathbf{C1C^*}$ and **Comp**

Construction in $\mathbf{C1C^*}$	characterization	dual notion in **Comp**
equalizer	norm-closed, unital *-subalgebra	quotient space by a closed equivalence relation
coequalizer	quotient C*-algebra	closed subspace
product	ℓ^∞-direct sum	Stone-Čech compactification of disjoint union
coproduct	generalized tensor product	Cartesian product

Projectivity relative to the class of all surjections in **C1C*** is also well-established, as is its dual notion, injectivity relative to the class of all one-to-one maps in **Comp**. In [30], this form of injectivity is termed the *universal extension property* and is characterized in the notion of an absolute retract.

Definition. A normal topological space I is an *absolute retract* if for every normal space X and closed subspace F of X satisfying $F \cong_{\mathbf{Top}} I$, F is a retract of X in **Top**.

While **C1C*** has distinct initial and terminal objects, a version of Proposition A.4.3 holds. The proof of this result will be given as the author has no knowledge of its existence in the literature. First, the dual fact will be proven for **Comp**.

Proposition B.3.5. *Let I be an index set and $(X_i)_{i\in I}$ be compact, Hausdorff spaces. Then, $\prod_{i\in I}^{\mathbf{Comp}} X_i$ is injective with respect to all one-to-one maps in* **Comp** *if and only if each X_i is also.*

Proof. (⇐) This fact is true in general by Proposition A.4.2

(⇒) First, note that $X_i \neq \emptyset$ for all $i \in I$. If to the contrary,

$$\prod_{i\in I}^{\mathbf{Comp}} X_i \cong_{\mathbf{Comp}} \emptyset,$$

which is not injective with respect to one-to-one maps.

Fix $j \in I$. Let X, Y be compact, Hausdorff spaces and $\alpha : X \to Y$ a one-to-one, continuous function. Consider a continuous function $\phi : X \to X_j$. The following diagram exists in **Comp**,

$$\begin{array}{ccc} X_j & \xleftarrow{\pi_j} & \prod_{i\in I}^{\mathbf{Comp}} X_i \\ \uparrow{\scriptstyle \phi} & & \\ X & \xrightarrow[\alpha]{} & Y \end{array}$$

where π_j is the canonical projection onto X_j.

For $i \neq j$, choose $x_i \in X_i$ and define $\phi_i : X \to X_i$ by $\phi_i(x) := x_i$, a constant function. Thus, ϕ_i is continuous. Letting $\phi_j := \phi$, there is a unique continuous function $\hat{\phi} : X \to \prod_{i \in I}^{\mathbf{Comp}} X_i$ such that $\pi_i \circ \hat{\phi} = \phi_i$ for all $i \in I$. By assumption, the product is injective with respect to one-to-one maps so there is $\tilde{\phi} : Y \to \prod_{i \in I}^{\mathbf{Comp}} X_i$ such that $\tilde{\phi} \circ \alpha = \hat{\phi}$.

Define $\psi := \pi_j \circ \tilde{\phi}$. Observe that

$$\psi \circ \alpha = \pi_j \circ \tilde{\phi} \circ \alpha = \pi_j \circ \hat{\phi} = \phi_j = \phi.$$

Thus, X_j is injective with respect to α, and since α was arbitrary, X_j is injective with respect to all one-to-one maps in **Comp**.

□

Using the duality with $\mathbf{C1C}^*$, the following statement holds.

Proposition B.3.6. *Given an index set I and commutative, unital C*-algebras $(\mathcal{P}_i)_{i \in I}$, then $\coprod_{i \in I}^{\mathbf{C1C}^*} \mathcal{P}_i$ is projective relative to all surjections in $\mathbf{C1C}^*$ if and only if each $\mathcal{P}_i$ is also for all $i \in I$.*

B.4 The Abelianization Functors

This section considers a well-known means of making a C*-algebra commutative, the abelianization. Here, this construction will be realized as a reflection across a natural forgetful functor. Full detail will be given as the author is not aware of a similar treatment in the literature. To construct this functor, let $\mathbf{CC}^*$ denote the category of commutative C*-algebras with *-homomorphisms.

Given any $\mathcal{A} \in \mathrm{Ob}(\mathbf{CC}^*)$, $\mathcal{A} \in \mathrm{Ob}(\mathbf{C}^*)$ so there is a natural forgetful map, ignoring the commutativity in $\mathcal{A}$. Similarly, given any $\mathcal{A}, \mathcal{B} \in \mathrm{Ob}(\mathbf{CC}^*)$, $\mathbf{CC}^*(\mathcal{A}, \mathcal{B}) \subseteq \mathbf{C}^*(\mathcal{A}, \mathcal{B})$. One can quickly check that these two associations define a functor $F^{\mathbf{C}^*}_{\mathbf{CC}^*} : \mathbf{CC}^* \to \mathbf{C}^*$, where one ignores the commutativity of the objects. Keep in mind that this is essentially an inclusion of $\mathbf{CC}^*$ into $\mathbf{C}^*$, changing no structure along the way.

Now, fix $\mathcal{A} \in \mathrm{Ob}(\mathbf{C}^*)$. Let $J_{\mathcal{A}}$ be the norm-closed, two-sided ideal in $\mathcal{A}$ generated by the set of commutators

$$\{ab - ba : a, b \in \mathcal{A}\}.$$

Let $\hat{\mathcal{A}} := \mathcal{A}/J_{\mathcal{A}}$ and $q_{\mathcal{A}} : \mathcal{A} \to \hat{\mathcal{A}}$ be the quotient map. The pair $\left(\hat{\mathcal{A}}, q_{\mathcal{A}}\right)$ is a candidate for the reflection of $\mathcal{A}$ along $F^{\mathbf{C}^*}_{\mathbf{CC}^*}$.

Theorem B.4.1. *The pair* $\left(\hat{\mathcal{A}}, q_{\mathcal{A}}\right)$ *is a reflection of* $\mathcal{A}$ *along* $F^{\mathbf{C}^*}_{\mathbf{CC}^*}$.

Proof. To check the universal property, let $\mathcal{B} \in \mathrm{Ob}(\mathbf{CC}^*)$ and $\phi \in \mathbf{C}^*\left(\mathcal{A}, F^{\mathbf{C}^*}_{\mathbf{CC}^*}\mathcal{B}\right)$. Observe that for all $a, b \in \mathcal{A}$,

$$\begin{aligned} \phi(ab - ba) &= \phi(a)\phi(b) - \phi(b)\phi(a) \\ &= \phi(a)\phi(b) - \phi(a)\phi(b) \\ &= 0 \end{aligned}$$

since $\mathcal{B}$ is commutative. Thus, $\{ab - ba : a, b \in \mathcal{A}\} \subseteq \ker(\phi)$ so $J_{\mathcal{A}} \subseteq \ker(\phi)$. By the universal property of the quotient, there is a unique $\hat{\phi} \in \mathbf{C}^*\left(\hat{\mathcal{A}}, \mathcal{B}\right)$ such that $\hat{\phi} \circ q_{\mathcal{A}} = \phi$. Since $\hat{\mathcal{A}}$ is commutative, $\hat{\phi} \in \mathbf{CC}^*\left(\hat{\mathcal{A}}, \mathcal{B}\right)$.

□

Further, since $\mathcal{A}$ was arbitrary, Proposition A.5.1 states that there is a unique functor $\mathrm{Ab} : \mathbf{C}^* \to \mathbf{CC}^*$ such that $\mathrm{Ab}(\mathcal{A}) = \hat{\mathcal{A}}$, and $\mathrm{Ab} \dashv F^{\mathbf{C}^*}_{\mathbf{CC}^*}$ by Theorem A.5.2. The explicit universal property of this adjoint pair is as follows.

Theorem B.4.2 (Explicit Universal Property of $\mathrm{Ab} \dashv F^{\mathbf{C}^*}_{\mathbf{CC}^*}$)**.** *Let $\mathcal{A}$ be a C*-algebra and $\mathcal{B}$ a commutative C*-algebra. Given any *-homomorphism $\phi : \mathcal{A} \to \mathcal{B}$, there is a unique *-homomorphism $\hat{\phi} : \mathrm{Ab}(\mathcal{A}) \to \mathcal{B}$ such that $\hat{\phi} \circ q_{\mathcal{A}} = \phi$.*

If $\mathcal{A}$ had been commutative, notice that $J_{\mathcal{A}} = \mathbb{0}$ so $\mathcal{A} \cong_{\mathbf{CC}^*} \mathrm{Ab}(\mathcal{A})$. Hence, all of $\mathrm{Ob}(\mathbf{CC}^*)$ is in the range of Ab, and no commutative C*-algebra has its structure altered in the process.

The functor Ab also encodes information regarding the multiplicative linear functionals on $\mathcal{A}$.

Proposition B.4.3. *For a C*-algebra $\mathcal{A}$, the multiplicative linear functionals on $\mathcal{A}$ are in bijection with those on $\mathrm{Ab}(\mathcal{A})$.*

Proof. Given a *-homomorphism $\varphi : \mathrm{Ab}(\mathcal{A}) \to \mathbb{C}$, then $\varphi \circ q_{\mathcal{A}} : \mathcal{A} \to \mathbb{C}$ is a *-homomorphism. Given a *-homomorphism $\phi : \mathcal{A} \to \mathbb{C}$, then there is a unique *-homomorphism $\hat{\phi} : \mathrm{Ab}(\mathcal{A}) \to \mathbb{C}$ such that $\hat{\phi} \circ q_{\mathcal{A}} = \phi$ by Theorem B.4.2.

□

Further, Ab preserves projectivity with respect to surjections in the following sense.

Proposition B.4.4. *If a C*-algebra $\mathcal{P}$ is projective with respect to surjections in $\mathbf{C}^*$, then $\mathrm{Ab}(\mathcal{P})$ is projective with respect to surjections in $\mathbf{CC}^*$.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be a commutative C*-algebras and $q : \mathcal{A} \to \mathcal{B}$ a surjective *-homomorphism. Consider a *-homomorphism $\phi : \mathrm{Ab}(\mathcal{P}) \to \mathcal{B}$. The following diagram exists in $\mathbf{C}^*$.

$$\begin{array}{ccccc} & & & & \mathcal{A} \\ & & & & \big\downarrow q \\ \mathcal{P} & \xrightarrow[q_{\mathcal{P}}]{} & \mathrm{Ab}(\mathcal{P}) & \xrightarrow[\phi]{} & \mathcal{B} \end{array}$$

Since $\mathcal{P}$ is projective with respect to surjections in $\mathbf{C}^*$, there is a *-homomorphism $\hat{\phi} : \mathcal{P} \to \mathcal{A}$ such that $q \circ \hat{\phi} = \phi \circ q_{\mathcal{P}}$. Since $\mathcal{A}$ is commutative, there is a unique *-homomorphism $\tilde{\phi} : \mathrm{Ab}(\mathcal{P}) \to \mathcal{A}$ such that $\tilde{\phi} \circ q_{\mathcal{P}} = \hat{\phi}$. Observe that

$$q \circ \tilde{\phi} \circ q_{\mathcal{P}} = q \circ \hat{\phi} = \phi \circ q_{\mathcal{P}}$$

so by the Theorem B.4.2, $q \circ \tilde{\phi} = \phi$. Thus, $\mathrm{Ab}(\mathcal{P})$ is projective with respect to q, and since q was arbitrary, $\mathrm{Ab}(\mathcal{P})$ is projective with respect to all surjections in $\mathbf{CC}^*$.

$\square$

If $\mathcal{A}$ had been unital, notice that $\mathrm{Ab}(\mathcal{A})$ is also, and $q_{\mathcal{A}}$ would preserve the unit. This gives a second adjoint relationship between $\mathbf{1C}^*$ and $\mathbf{C1C}^*$. As before, there is a natural forgetful functor $F^{\mathbf{1C}^*}_{\mathbf{C1C}^*} : \mathbf{C1C}^* \to \mathbf{1C}^*$ by ignoring the commutativity in play.

Theorem B.4.5. *The pair* $\left(\hat{\mathcal{A}}, q_{\mathcal{A}}\right)$ *is a reflection of* $\mathcal{A}$ *along* $F^{\mathbf{1C}^*}_{\mathbf{C1C}^*}$.

Further, since $\mathcal{A}$ was arbitrary, Proposition A.5.1 states that there is a unique functor $\mathrm{Ab}_1 : \mathbf{1C}^* \to \mathbf{C1C}^*$ such that $\mathrm{Ab}_1(\mathcal{A}) = \hat{\mathcal{A}}$, and $\mathrm{Ab}_1 \dashv F^{\mathbf{1C}^*}_{\mathbf{C1C}^*}$ by Theorem A.5.2. This functor shares many properties with its non-unital counterpart, which are proved in an identical fashion. As such, these proofs will be omitted for brevity.

Theorem B.4.6 (Explicit Universal Property of $\mathrm{Ab}_1 \dashv F^{\mathbf{1C}^*}_{\mathbf{C1C}^*}$)**.** *Let* $\mathcal{A}$ *be a unital C*-algebra and* $\mathcal{B}$ *a commutative unital C*-algebra. Given any unital *-homomorphism* $\phi : \mathcal{A} \to \mathcal{B}$*, there is a unique unital *-homomorphism* $\hat{\phi} : \mathrm{Ab}_1(\mathcal{A}) \to \mathcal{B}$ *such that* $\hat{\phi} \circ q_{\mathcal{A}} = \phi$.

Proposition B.4.7. *For a unital C*-algebra* $\mathcal{A}$*, the nonzero multiplicative linear functionals on* $\mathcal{A}$ *are in bijection with those on* $\mathrm{Ab}_1(\mathcal{A})$*, which are in bijection to points in* $\Delta\left(\mathrm{Ab}_1(\mathcal{A})\right)$.

Proposition B.4.8. *If a unital C^*-algebra $\mathcal{P}$ is projective with respect to surjections in $\mathbf{1C}^*$, then $\mathrm{Ab}_1(\mathcal{P})$ is projective with respect to surjections in $\mathbf{C1C}^*$. That is, $\Delta\left(\mathrm{Ab}_1(\mathcal{P})\right)$ is an absolute retract.*

B.5 The Unitization Functors

This section considers a well-known means of making a C*-algebra unital, the unitization. Here, this construction will be realized as a reflection across a natural forgetful functor. Full detail will be given for results that the author has not seen in the literature.

Given any $\mathcal{A} \in \mathrm{Ob}(\mathbf{1C}^*)$, $\mathcal{A} \in \mathrm{Ob}(\mathbf{C}^*)$ so there is a natural forgetful map, ignoring the existence of the unit in $\mathcal{A}$. Similarly, given any $\mathcal{A}, \mathcal{B} \in \mathrm{Ob}(\mathbf{1C}^*)$, $\mathbf{1C}^*(\mathcal{A}, \mathcal{B}) \subseteq \mathbf{C}^*(\mathcal{A}, \mathcal{B})$. One can quickly check that these two associations define a functor $F^{\mathbf{C}^*}_{\mathbf{1C}^*} : \mathbf{1C}^* \to \mathbf{C}^*$, where one ignores the existence of a unit and the unit-preserving properties of the maps. Keep in mind that this is essentially an inclusion of $\mathbf{1C}^*$ into $\mathbf{C}^*$, changing no structure along the way.

Now, fix $\mathcal{A} \in \mathrm{Ob}(\mathbf{C}^*)$. Recalling Proposition I.1.3 in [10]. Let $\tilde{\mathcal{A}} := \mathcal{A} \times \mathbb{C}$, the cartesian product of $\mathcal{A}$ with $\mathbb{C}$, which will serve as the underlying set. Define the following operations for all $(a, \lambda), (b, \mu) \in \tilde{\mathcal{A}}$ and $\nu \in \mathbb{C}$:

$$\begin{aligned}
(a,\lambda) + (b,\mu) &:= (a+b, \lambda+\mu), \\
\nu \cdot (a,\lambda) &:= (\nu a, \nu\lambda), \\
(a,\lambda) \cdot (b,\mu) &:= (ab + \lambda b + \mu a, \lambda\mu), \\
(a,\lambda)^* &:= \left(a^*, \overline{\lambda}\right). \\
\rho(a,\lambda) &:= \sup\left\{\|ab + \lambda b\|_{\mathcal{A}} : b \in \mathcal{A}, \|b\|_{\mathcal{A}} \le 1\right\}
\end{aligned}$$

Under these operations, it is a standard exercise to show $\tilde{\mathcal{A}}$ to be an involutive $\mathbb{C}$-

algebra with unit $(0,1)$. Proposition I.1.3 in [10] states that ρ is a C*-norm and that $\tilde{A}$ is complete in this norm. Here, $\tilde{\mathcal{A}}$ is equipped with the norm ρ and regarded as a unital C*-algebra.

Further, there are two canonical maps for each of the two coordinates of $\tilde{\mathcal{A}}$. Define $\pi_2 : \tilde{\mathcal{A}} \to \mathbb{C}$ by $\pi_2(a,\lambda) := \lambda$, the projection onto the second coordinate. A quick check shows that this is a unital *-homomorphism. Likewise, define $\iota_{\mathcal{A}} : \mathcal{A} \to \tilde{\mathcal{A}}$ by $\iota_{\mathcal{A}}(a) := (a,0)$, the inclusion into the first coordinate. A similar check shows that this is a *-homomorphism. Thus, the following diagram exists in $\mathbf{C}^*$,

$$\mathbb{O} \xrightarrow{\mathbf{0}_{\mathbb{O},\mathcal{A}}} \mathcal{A} \xrightarrow{\iota_{\mathcal{A}}} \tilde{\mathcal{A}} \xrightarrow{\pi_2} \mathbb{C} \xrightarrow{\mathbf{0}_{\mathbb{C},\mathbb{O}}} \mathbb{O} \tag{B.1}$$

where $\mathbf{0}_{\mathcal{B},\mathcal{C}} : \mathcal{B} \to \mathcal{C}$ is the constant 0 map from $\mathcal{B}$ to $\mathcal{C}$. Observe that for all $a \in \mathcal{A}$,

$$(\pi_2 \circ \iota_{\mathcal{A}})(a) = \pi_2(a,0) = 0$$

so $\operatorname{ran}(\iota_{\mathcal{A}}) \subseteq \ker(\pi_2)$. Furthermore, if $(a,\lambda) \in \ker(\pi_2)$, $0 = \pi_2(a,\lambda) = \lambda$, meaning $\operatorname{ran}(\iota_{\mathcal{A}}) = \ker(\pi_2)$.

Also, there is a map $\iota_2 : \mathbb{C} \to \tilde{\mathcal{A}}$ by $\iota_2(\lambda) := (0,\lambda)$. Another check shows this to be a unital *-homomorphism, and for all $\lambda \in \mathbb{C}$,

$$(\pi_2 \circ \iota_2)(\lambda) = \pi_2(0,\lambda) = \lambda.$$

Therefore, $\pi_2 \circ \iota_2 = id_{\mathbb{C}}$. Thus, ι_2 is a section in $\mathbf{1C}^*$ and π_2 a retraction in $\mathbf{1C}^*$.

If this diagram is considered in the abelian category of $\mathbb{C}$-Banach spaces and contractive $\mathbb{C}$-linear functions, this is a *split short exact sequence* of $\mathbb{C}$-Banach spaces. However, this term will not be used here since $\mathbf{C}^*$ is not an abelian category. Specifically, there are monic maps in $\mathbf{C}^*$ which are not kernels. An example of such a monic map would be an inclusion of a C*-subalgebra which is not an ideal.

The pair $\left(\tilde{\mathcal{A}}, \iota_{\mathcal{A}}\right)$ is a candidate for the reflection of $\mathcal{A}$ along $F^{\mathbf{C}^*}_{\mathbf{1C}^*}$.

Theorem B.5.1. *The pair* $\left(\tilde{\mathcal{A}}, \iota_{\mathcal{A}}\right)$ *is a reflection of* $\mathcal{A}$ *along* $F^{\mathbf{C}^*}_{\mathbf{1C}^*}$.

Proof. To check the universal property, let $\mathcal{B} \in \mathrm{Ob}(\mathbf{1C}^*)$ and $\phi \in \mathbf{C}^*(\mathcal{A}, F^{\mathbf{C}^*}_{\mathbf{1C}^*}\mathcal{B})$. Define $\tilde{\phi} : \tilde{\mathcal{A}} \to \mathcal{B}$ by

$$\tilde{\phi}(a, \lambda) := \phi(a) + \lambda \mathbb{1}_{\mathcal{B}}.$$

A quick check shows that $\tilde{\phi} \in \mathbf{1C}^*\left(\tilde{\mathcal{A}}, \mathcal{B}\right)$. Further, for all $a \in \mathcal{A}$,

$$\left(F^{\mathbf{C}^*}_{\mathbf{1C}^*}\tilde{\phi} \circ \iota_{\mathcal{A}}\right)(a) = F^{\mathbf{C}^*}_{\mathbf{1C}^*}\tilde{\phi}(a, 0) = \tilde{\phi}(a, 0) = \phi(a)$$

so $F^{\mathbf{C}^*}_{\mathbf{1C}^*}\tilde{\phi} \circ \iota_{\mathcal{A}} = \phi$.

Assume that $\psi : \tilde{\mathcal{A}} \to \mathcal{B}$ satisfies that $F^{\mathbf{C}^*}_{\mathbf{1C}^*}\psi \circ \iota_{\mathcal{A}} = \phi$. Then, for all $(a, \lambda) \in \mathcal{A}$,

$$\begin{aligned}
\psi(a, \lambda) &= \psi((a, 0) + (0, \lambda)) \\
&= \psi(a, 0) + \lambda\psi(0, 1) \\
&= F^{\mathbf{C}^*}_{\mathbf{1C}^*}\psi(a, 0) + \lambda \mathbb{1}_{\mathcal{B}} \\
&= \left(F^{\mathbf{C}^*}_{\mathbf{1C}^*}\psi \circ \iota_{\mathcal{A}}\right)(a) + \lambda \mathbb{1}_{\mathcal{B}} \\
&= \phi(a) + \lambda \mathbb{1}_{\mathcal{B}} \\
&= \tilde{\phi}(a, \lambda).
\end{aligned}$$

Hence, $\psi = \tilde{\phi}$.

□

Further, since $\mathcal{A}$ was arbitrary, Proposition A.5.1 states that there is a unique functor $\mathrm{Unit} : \mathbf{C}^* \to \mathbf{1C}^*$ such that $\mathrm{Unit}(\mathcal{A}) = \tilde{\mathcal{A}}$, and $\mathrm{Unit} \dashv F^{\mathbf{C}^*}_{\mathbf{1C}^*}$ by Theorem A.5.2. The explicit universal property of this adjoint pair is as follows.

Theorem B.5.2 (Explicit Universal Property of Unit $\dashv F^{\mathbf{C}^*}_{\mathbf{1C}^*}$)**.** *Let $\mathcal{A}$ be a C*-algebra and $\mathcal{B}$ a unital C*-algebra. Given any *-homomorphism $\phi : \mathcal{A} \to \mathcal{B}$, there is a unique unital *-homomorphism $\tilde{\phi} : \mathrm{Unit}(\mathcal{A}) \to \mathcal{B}$ such that $\tilde{\phi} \circ \iota_{\mathcal{A}} = \phi$.*

Recall Diagram (B.1), including also the map ι_2.

$$\mathbb{O} \xrightarrow{\mathbf{0}_{\mathbb{O},\mathcal{A}}} \mathcal{A} \xrightarrow{\iota_{\mathcal{A}}} \tilde{\mathcal{A}} \underset{\iota_2}{\overset{\pi_2}{\rightleftarrows}} \mathbb{C} \xrightarrow{\mathbf{0}_{\mathbb{C},\mathbb{O}}} \mathbb{O}$$

This is the classical characterization of the unitization, and it can be shown to be equivalent to the universal characterization of Theorem B.5.2.

Theorem B.5.3. *Let $\mathcal{A}$ be a C*-algebra. Then, a unital C*-algebra $\mathcal{B}$ equipped with a *-homomorphism $\iota : \mathcal{A} \to \mathcal{B}$ is a reflection along $F^{\mathbf{C}^*}_{\mathbf{1C}^*}$ of $\mathcal{A}$ if and only if ι is one-to-one,* $\mathrm{ran}(\iota)$ *is a two-sided ideal in $\mathcal{B}$, and $\mathcal{B}/\,\mathrm{ran}(\iota) \cong_{\mathbf{1C}^*} \mathbb{C}$.*

Proof. ($\Rightarrow$) Assuming that $(\mathcal{B}, \iota)$ is a reflection along $F^{\mathbf{C}^*}_{\mathbf{1C}^*}$, then consider the following diagram in $\mathbf{C}^*$.

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\iota_{\mathcal{A}}} & F^{\mathbf{C}^*}_{\mathbf{1C}^*}\tilde{\mathcal{A}} \\ & \searrow^{\iota} & \\ & & F^{\mathbf{C}^*}_{\mathbf{1C}^*}\mathcal{B} \end{array}$$

By the universal property of the reflection, there is a unique unital *-homomorphism $\alpha : \tilde{\mathcal{A}} \to \mathcal{B}$ such that $\iota = \alpha \circ \iota_{\mathcal{A}}$. Symmetrically, there is a unique unital *-homomorphism $\beta : \mathcal{B} \to \tilde{\mathcal{A}}$ such that $\iota_{\mathcal{A}} = \beta \circ \iota$. Thus, for all $a \in \mathcal{A}$,

$$\begin{aligned} \iota(a) &= (\alpha \circ \iota_{\mathcal{A}})(a) \\ &= (\alpha \circ \beta \circ \iota)(a) \\ &= ((\alpha \circ \beta) \circ \iota)(a) \end{aligned}$$

and

$$\begin{aligned} \iota_{\mathcal{A}}(a) &= (\beta \circ \iota)(a) \\ &= (\beta \circ \alpha \circ \iota_{\mathcal{A}})(a) \\ &= ((\beta \circ \alpha) \circ \iota_{\mathcal{A}})(a). \end{aligned}$$

Hence, $(\alpha \circ \beta) \circ \iota = \iota$ and $(\beta \circ \alpha) \circ \iota_{\mathcal{A}} = \iota_{\mathcal{A}}$ so by the universal property of the reflection, $\alpha \circ \beta = id_{\mathcal{B}}$ and $\beta \circ \alpha = id_{\tilde{\mathcal{A}}}$.

Observe that since β and α are isomorphisms,

$$\ker(\iota) = \ker(\beta \circ \iota) = \ker(\iota_{\mathcal{A}}) = 0$$

and

$$\operatorname{ran}(\iota) = \operatorname{ran}(\alpha \circ \iota_{\mathcal{A}}) = \alpha(\operatorname{ran}(\iota_{\mathcal{A}})).$$

Thus, $\operatorname{ran}(\iota)$ is a two-sided ideal in $\mathcal{B}$ and ι one-to-one. Lastly, by the first isomorphism theorem,

$$\mathcal{B}/\operatorname{ran}(\iota) \cong_{\mathbf{1C^*}} \tilde{\mathcal{A}}/\operatorname{ran}(\iota_{\mathcal{A}}) \cong_{\mathbf{1C^*}} \mathbb{C}$$

($\Leftarrow$) Assuming the result, let $\pi : \mathcal{B} \to \mathcal{B}/\operatorname{ran}(\iota) \cong_{\mathbf{1C^*}} \mathbb{C}$ be the quotient map. There is a unique unital *-homomorphism $\alpha : \tilde{\mathcal{A}} \to \mathcal{B}$ such that $\iota = \alpha \circ \iota_{\mathcal{A}}$. Explicitly, $\alpha(a, \lambda) = \iota(a) + \lambda \mathbb{1}_{\mathcal{B}}$ from Theorem B.5.1. If $(a, \lambda) \in \ker(\alpha)$,

$$0 = \alpha(a, \lambda) = \iota(a) + \lambda \mathbb{1}_{\mathcal{B}}.$$

Then, $0 = \pi(\iota(a) + \lambda \mathbb{1}_{\mathcal{B}}) = \lambda \mathbb{1}_{\mathcal{B}/\operatorname{ran}(\iota)} \sim \lambda \in \mathbb{C}$, meaning $\lambda = 0$. Hence, $\iota(a) = 0$, forcing $a = 0$ as ι is one-to-one. Therefore, α is one-to-one.

Notice that $\mathcal{B} = \bigcup_{\lambda \in \mathbb{C}} (\lambda \mathbb{1}_{\mathcal{B}} + \operatorname{ran}(\iota)) \subseteq \operatorname{ran}(\alpha)$ by the first isomorphism theorem. Hence, α is an isomorphism in $\mathbf{C}^*$.

Given any unital C*-algebra $\mathcal{C}$ and *-homomorphism $\phi : \mathcal{A} \to \mathcal{C}$, there is a unique unital *-homomorphism $\tilde{\phi} : \tilde{\mathcal{A}} \to \mathcal{C}$ such that $\tilde{\phi} \circ \iota_{\mathcal{A}} = \phi$. Then,

$$\phi = \left(\tilde{\phi} \circ \alpha^{-1}\right) \circ (\alpha \circ \iota_{\mathcal{A}}) = \left(\tilde{\phi} \circ \alpha^{-1}\right) \circ \iota.$$

If there was a unital *-homomorphism $\psi : \mathcal{B} \to \mathcal{C}$ such that $\phi = \psi \circ \iota$, then

$$\phi = (\psi \circ \alpha) \circ \left(\alpha^{-1} \circ \iota\right) = (\psi \circ \alpha) \circ \iota_{\mathcal{A}}.$$

Therefore, $\psi \circ \alpha = \tilde{\phi}$, meaning $\psi = \tilde{\phi} \circ \alpha^{-1}$. Hence, $(\mathcal{B}, \iota)$ is a reflection of $\mathcal{A}$ along $F^{\mathbf{C}^*}_{\mathbf{1C}^*}$.

□

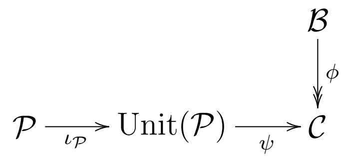
The usual way this is stated in [10] is that "$\mathcal{A}$ is an ideal of $\mathcal{B}$ of codimension 1". Since the universal notion of Theorem B.5.2 agrees with the classical notion, the term *unitization* will be used for either process interchangeably.

Notice that some unital C*-algebras can be shown not to be in the range of Unit. In particular, $\mathbb{O}$ has no ideals of codimension 1 so it cannot be obtained by unitizing another C*-algebra. Less trivially, the Calkin algebra, $\mathcal{B}\left(\ell^2\right)/\mathcal{K}\left(\ell^2\right)$, is simple, unital, and non-commutative. Thus, it cannot have a codimension 1 ideal and, therefore, cannot be obtained by unitizing another C*-algebra.

Also, unitization works well with projectivity, stated in [27].

Proposition B.5.4. *Given a C*-algebra $\mathcal{P}$, $\mathcal{P}$ is projective relative to all surjections in* $\mathbf{C}^*$ *if and only if* $\mathrm{Unit}(\mathcal{P})$ *is projective relative to all surjections in* $\mathbf{1C}^*$.

Proof. (⇒) Let $\mathcal{B}, \mathcal{C}$ be unital C*-algebras and $\phi : \mathcal{B} \to \mathcal{C}$ a surjective unital *-homomorphism. Given a unital *-homomorphism $\psi : \mathrm{Unit}(\mathcal{P}) \to \mathcal{C}$, the following diagram exists in $\mathbf{C}^*$.

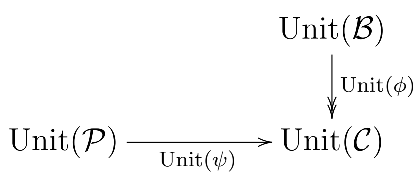


As $\mathcal{P}$ is projective relative to all surjections in $\mathbf{C}^*$, there is a *-homomorphism $\hat{\psi}$: $\mathcal{P} \to \mathcal{B}$ such that $\phi \circ \hat{\psi} = \psi \circ \iota_{\mathcal{P}}$. By Theorem B.5.2, there is a unique unital *-homomorphism $\tilde{\psi} : \text{Unit}(\mathcal{P}) \to \mathcal{B}$ such that $\tilde{\psi} \circ \iota_{\mathcal{P}} = \hat{\psi}$. Therefore,

$$\phi \circ \tilde{\psi} \circ \iota_{\mathcal{P}} = \phi \circ \hat{\psi} = \psi \circ \iota_{\mathcal{P}}$$

so by Theorem B.5.2, $\phi \circ \tilde{\psi} = \psi$.

($\Leftarrow$) Let $\mathcal{B}, \mathcal{C}$ be C*-algebras and $\phi : \mathcal{B} \to \mathcal{C}$ a surjective *-homomorphism. Given a *-homomorphism $\psi : \mathcal{P} \to \mathcal{C}$, application of Unit yields the following diagram in $\mathbf{1C}^*$.



As $\text{Unit}(\mathcal{P})$ is projective relative to all surjections in $\mathbf{1C}^*$, there is a unital *-homomorphism $\hat{\psi} : \text{Unit}(\mathcal{P}) \to \text{Unit}(\mathcal{B})$ such that $\text{Unit}(\phi) \circ \hat{\psi} = \text{Unit}(\psi)$.

Now, observe that by construction of the functor Unit,

$$\text{Unit}(\phi)(b, \lambda) = (\phi(b), \lambda)$$

and

$$\text{Unit}(\psi)(p, \lambda) = (\psi(p), \lambda).$$

Notice that for all $p \in \mathcal{P}$,

$$\begin{aligned}(\psi(p), 0) &= \text{Unit}(\psi)(p, 0) \\ &= \left(\text{Unit}(\phi) \circ \hat{\psi}\right)(p, 0)\end{aligned}$$

so $\hat{\psi}(p, 0) \in \text{Unit}(\phi)^{-1}(\psi(p), 0)$. Untangling this pre-image,

$$\begin{aligned}\text{Unit}(\phi)^{-1}(\psi(p), 0) &= \{(b, \lambda) \in \text{Unit}(\mathcal{B}) : \text{Unit}(\phi)(b, \lambda) = (\psi(p), 0)\} \\ &= \{(b, \lambda) \in \text{Unit}(\mathcal{B}) : (\phi(b), \lambda) = (\psi(p), 0)\} \\ &= \{(b, 0) \in \text{Unit}(\mathcal{B}) : \phi(b) = \psi(p)\},\end{aligned}$$

meaning $\text{ran}\left(\hat{\psi} \circ \iota_{\mathcal{P}}\right) \subseteq \text{ran}\left(\iota_{\mathcal{B}}\right)$.

Define $\tilde{\psi} := \left(\iota_{\mathcal{B}}|^{\text{ran}(\iota_{\mathcal{B}})}\right)^{-1} \circ \left(\hat{\psi} \circ \iota_{\mathcal{P}}\right)$. Then, for all $p \in \mathcal{P}$,

$$\begin{aligned}\left(\phi \circ \tilde{\psi}\right)(p) &= \left(\phi \circ \left(\iota_{\mathcal{B}}|^{\text{ran}(\iota_{\mathcal{B}})}\right)^{-1} \circ \left(\hat{\psi} \circ \iota_{\mathcal{P}}\right)\right)(p) \\ &= \left(\phi \circ \left(\iota_{\mathcal{B}}|^{\text{ran}(\iota_{\mathcal{B}})}\right)^{-1}\right)\left(\hat{\psi}(p, 0)\right) \\ &= \left(\left(\iota_{\mathcal{C}}|^{\text{ran}(\iota_{\mathcal{C}})}\right)^{-1} \circ \text{Unit}(\phi)\right)\left(\hat{\psi}(p, 0)\right) \\ &= \left(\iota_{\mathcal{C}}|^{\text{ran}(\iota_{\mathcal{C}})}\right)^{-1}(\text{Unit}(\psi)(p, 0)) \\ &= \left(\iota_{\mathcal{C}}|^{\text{ran}(\iota_{\mathcal{C}})}\right)^{-1}(\psi(p), 0) \\ &= \psi(p).\end{aligned}$$

Hence, $\phi \circ \tilde{\psi} = \psi$.

□

Further, observe that if $\mathcal{A}$ is commutative, $\tilde{\mathcal{A}}$ will also be commutative. This gives a second adjoint relationship between $\mathbf{CC}^*$ and $\mathbf{C1C}^*$. As before, there is a natural forgetful functor $F_{\mathbf{C1C}^*}^{\mathbf{CC}^*} : \mathbf{C1C}^* \to \mathbf{CC}^*$ by ignoring the existence of the unit and the unit-preserving properties of the maps.

Theorem B.5.5. *Given* $\mathcal{A} \in \mathrm{Ob}(\mathbf{CC}^*)$, $\left(\tilde{\mathcal{A}}, \iota_{\mathcal{A}}\right)$ *is a reflection along* $F^{\mathbf{CC}^*}_{\mathbf{C1C}^*}$.

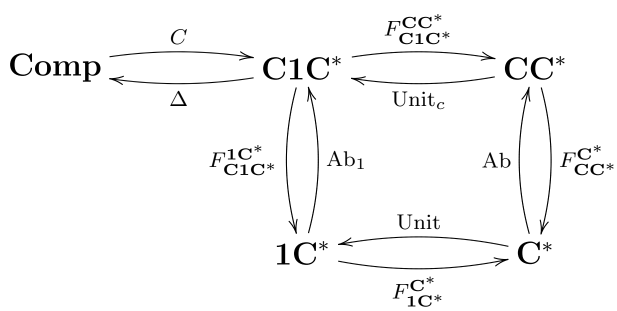
Further, since $\mathcal{A}$ was arbitrary, Proposition A.5.1 states that there is a unique functor $\mathrm{Unit}_c : \mathbf{CC}^* \to \mathbf{C1C}^*$ such that $\mathrm{Unit}_c(\mathcal{A}) = \tilde{\mathcal{A}}$, and $\mathrm{Unit}_c \dashv F^{\mathbf{CC}^*}_{\mathbf{C1C}^*}$ by Theorem A.5.2.

Also, appropriate restrictions yield the following projectivity result.

Proposition B.5.6. *Given a commutative C*-algebra* $\mathcal{P}$, $\mathcal{P}$ *is projective relative to all surjections in* $\mathbf{CC}^*$ *if and only if* $\mathrm{Unit}_c(\mathcal{P})$ *is projective relative to all surjections in* $\mathbf{C1C}^*$.

B.6 Summary: a Diagram of C*-algebraic Theory

To summarize the content of this appendix, consider the following diagram of categories and functors.



A quick check shows that the outer square commutes. That is,

$$F^{\mathbf{C}^*}_{\mathbf{CC}^*} F^{\mathbf{CC}^*}_{\mathbf{C1C}^*} = F^{\mathbf{C}^*}_{\mathbf{1C}^*} F^{\mathbf{1C}^*}_{\mathbf{C1C}^*} =: F^{\mathbf{C}^*}_{\mathbf{C1C}^*},$$

the forgetful functor from $\mathbf{C1C}^*$ to $\mathbf{C}^*$. From these functorial characterizations and their adjoint nature, several results follow immediately from the general content of Appendix A.

First, Proposition A.5.3 shows that

$$\mathrm{Unit}_c\,\mathrm{Ab} \dashv F^{\mathbf{C}^*}_{\mathbf{C1C}^*}$$

and

$$\mathrm{Ab}_1\,\mathrm{Unit} \dashv F^{\mathbf{C}^*}_{\mathbf{C1C}^*}.$$

Since a left adjoint is composed of reflections, the universal property of the reflection yields the following fact.

Theorem B.6.1. *Given a C^*-algebra $\mathcal{A}$,*

$$\mathrm{Unit}_c\left(\mathrm{Ab}(\mathcal{A})\right) \cong_{\mathbf{C1C}^*} \mathrm{Ab}_1\left(\mathrm{Unit}(\mathcal{A})\right).$$

That is, the inner square commutes up to isomorphism in $\mathbf{C1C}^*$.

By Proposition A.5.4, the following functors preserve all categorical colimits: Unit, Unit_c, Ab, Ab_1, C, and Δ. Two specific types of colimits are coproducts and direct limits, which give a bevy of results. Here are two examples of such results.

Corollary B.6.2. *Given an index set I and C^*-algebras $(\mathcal{A}_i)_{i\in I}$,*

$$\mathrm{Unit}\left(\coprod_{i\in I}^{\mathbf{C}^*} \mathcal{A}_i\right) \cong_{\mathbf{1C}^*} \coprod_{i\in I}^{\mathbf{1C}^*} \mathrm{Unit}\left(\mathcal{A}_i\right).$$

Corollary B.6.3. *Given a directed poset $(I,\leq)$, let $\left(\mathcal{A}_i,\phi^i_j\right)$ be a direct system in*

$\mathbf{C}^*$. *Then,*

$$\mathrm{Unit}\left(\lim_{\rightarrow}{}^{\mathbf{C}^*}\left(\mathcal{A}_i,\phi_j^i\right)\right)\cong_{\mathbf{1C}^*}\lim_{\rightarrow}{}^{\mathbf{1C}^*}\left(\mathrm{Unit}\left(\mathcal{A}_i\right),\mathrm{Unit}\left(\phi_j^i\right)\right).$$

Further, the notion of non-commutative geometry has been thought of as generalizing geometric notions in **Comp** to $\mathbf{1C}^*$. This could be described and studied via the composite functor $F^{\mathbf{1C}^*}_{\mathbf{C1C}^*}C : \mathbf{Comp} \to \mathbf{1C}^*$, which serves as a bridge between the two categories of study. For example, projectivity relative to surjections in $\mathbf{1C}^*$ is considered the "non-commutative analogue" of the absolute retract, as stated in [27]. This functor makes the connection more formal, giving results such as Proposition B.4.8.

B.7 Application: Free Products of Projectives in $\mathbf{1C}^*$

To close this appendix, an application of these categorical notions is demonstrated, which is not immediate from general principles. Specifically, the $\mathbf{1C}^*$ version of Proposition A.4.3 is proven. This result is not immediate as $\mathbf{1C}^*$ does not have a categorical zero object. The proof method is very closely related to that of Proposition B.3.5, and may be considered its non-commutative analogue.

To begin, the relationship between a unital free product and its multiplicative linear functionals is proven, which is very closely related to Proposition B.4.3. This will allow removal of certain degenerate cases from consideration.

Lemma B.7.1. *Let I be an index set and $(\mathcal{A}_i)_{i\in I}$ be unital C^*-algebras. Then, the nonzero multiplicative linear functionals on $\coprod_{i\in I}^{\mathbf{1C}^*}\mathcal{A}_i$ are in bijection to families of* *nonzero multiplicative linear functionals on each* $\mathcal{A}_i$*.*

Proof. For $i \in I$, let $\iota_i : \mathcal{A}_i \to \coprod_{i\in I}^{\mathbf{1C^*}} \mathcal{A}_i$ be the canonical inclusions into the unital free product. Given a unital *-homomorphism $\varphi : \coprod_{i\in I}^{\mathbf{1C^*}} \mathcal{A}_i \to \mathbb{C}$, then $\varphi \circ \iota_i : \mathcal{A}_i \to \mathbb{C}$ is a unital *-homomorphism. Thus, $(\varphi \circ \iota_i)_{i\in I}$ is a family of unital *-homomorphisms.

For $i \in I$, let $\phi_i : \mathcal{A}_i \to \mathbb{C}$ be unital *-homomorphisms. Then, there is a unique unital *-homomorphism $\hat{\phi} : \coprod_{i\in I}^{\mathbf{1C^*}} \mathcal{A}_i \to \mathbb{C}$ such that $\hat{\phi} \circ \iota_i = \phi_i$ by the universal property of the unital free product.

□

With this fact, the proof of the main result can proceed.

Proposition B.7.2. *Given an index set* I *and unital C*-algebras* $(\mathcal{P}_i)_{i\in I}$*, then* $\coprod_{i\in I}^{\mathbf{1C^*}} \mathcal{P}_i$ *is projective relative to all surjections in* $\mathbf{1C^*}$ *if and only if each* $\mathcal{P}_i$ *is also for all* $i \in I$*.*

Proof. (⇐) This fact is true in general by Proposition A.4.2

(⇒) Let $\mathcal{P} := \coprod_{i\in I}^{\mathbf{1C^*}} \mathcal{P}_i$ and $\iota_i : \mathcal{P}_i \to \mathcal{P}$ the canonical inclusions into the unital free product for $i \in I$. First, degenerate cases are removed from consideration. That is, each $\mathcal{P}_i$ will be shown to have a nonzero multiplicative linear functional. This will allow the factors of $\mathcal{P}$ to be separated and shown projective.

As $\mathcal{P}$ is projective relative to all surjections in $\mathbf{1C^*}$, Proposition B.4.8 states that $\Delta(\mathrm{Ab}_1(\mathcal{P}))$ is an absolute retract. Since $\emptyset$ is not an absolute retract, $\Delta(\mathrm{Ab}_1(\mathcal{P})) \neq \emptyset$. By Proposition B.4.7, there is a unital *-homomorphism $\varphi : \mathcal{P} \to \mathbb{C}$. By Lemma B.7.1, each $\mathcal{P}_i$ has a unital *-homomorphism $\eta_i := \varphi \circ \iota_i$.

Fix $j \in I$. To show $\mathcal{P}_j$ projective, let $\mathcal{A}$ and $\mathcal{B}$ be a unital C*-algebras and $q : \mathcal{A} \to \mathcal{B}$ a surjective, unital *-homomorphism. Consider a unital *-homomorphism

$\phi : \mathcal{P}_j \to \mathcal{B}$. The following diagram exists in $\mathbf{1C^*}$.

$$\begin{array}{ccc} \mathcal{P} & \xleftarrow{\iota_j} & \mathcal{P}_j \\ & & \downarrow \phi \\ \mathcal{A} & \xrightarrow[q]{} & \mathcal{B} \end{array}$$

Let $\psi : \mathbb{C} \to \mathcal{B}$ be the unique unital *-homomorphism given by $\psi(1) := \mathbb{1}_{\mathcal{B}}$. For $i \neq j$, define $\phi_i := \psi \circ \eta_i$. Letting $\phi_j := \phi$, there is a unique unital *-homomorphism $\hat{\phi} : \mathcal{P} \to \mathcal{B}$ such that $\hat{\phi} \circ \iota_i = \phi_i$ for all $i \in I$. By assumption, the coproduct is projective with respect to surjections so there is $\tilde{\phi} : \mathcal{P} \to \mathcal{A}$ such that $q \circ \tilde{\phi} = \hat{\phi}$.

Define $\theta := \tilde{\phi} \circ \iota_j$. Observe that

$$q \circ \theta = q \circ \tilde{\phi} \circ \iota_j = \hat{\phi} \circ \iota_j = \phi_j = \phi.$$

Thus, $\mathcal{P}_j$ is projective with respect to q, and since q was arbitrary, $\mathcal{P}_j$ is projective with respect to all surjections in $\mathbf{1C^*}$.

□

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