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Languages, geodesics, and HNN extensions

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LANGUAGES, GEODESICS, AND HNN EXTENSIONS

by

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The complexity of a geodesic language has connections to algebraic properties of the group. Gilman, Hermiller, Holt, and Rees show that a finitely generated group is virtually free if and only if its geodesic language is locally excluding for some finite inverse-closed generating set. The existence of such a correspondence and the result of Hermiller, Holt, and Rees that finitely generated abelian groups have piecewise excluding geodesic language for all finite inverse-closed generating sets motivated our work. We show that a finitely generated group with piecewise excluding geodesic language need not be abelian and give a class of infinite non-abelian groups which have piecewise excluding geodesic languages for certain generating sets. The quaternion group is shown to be the only non-abelian 2-generator group with piecewise excluding geodesic language for all finite inverse-closed generating sets. We also show that there are virtually abelian groups with geodesic languages which are not piecewise excluding for any finite inverse-closed generating set.

Autostackable groups were introduced by Brittenham, Hermiller, and Holt as a generalization of asynchronously automatic groups on prefix-closed normal forms and groups with finite convergent rewriting systems. Brittenham, Hermiller, and Johnson show that Stallings' non-FP_3 group, an HNN extension of a right-angled Artin group, is autostackable. We extend this autostackability result to a larger class of HNN extensions of right-angled Artin groups.

DEDICATION

To my parents, Maripat and Mark Franke.

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Chapter 1

Introduction

1.1 Piecewise Excluding Geodesic Languages

For a group G generated by a finite set X, Dehn's word problem asks if there exists an algorithm which determines whether or not a given word over $X \cup X^{-1}$ represents the trival element in G [8]. Dehn's word problem is known to be unsolvable in general [3]. But for certain classes of groups, such as groups with a computable geodesic language for some generating set, there are solutions to the word problem. There are two known classes of groups with regular geodesic language for all finite generating sets: word hyperbolic groups [9] and abelian groups [22, Theorems 4.4 and 4.1]. There are many known types of groups with regular geodesic language for some finite generating set: these include Coxeter groups [18], virtually abelian groups and geometically finite hyperbolic groups [22], Artin groups of finite type and more generally Garside groups [7], Artin groups of large type [16], and groups hyperbolic relative to virtually abelian subgroups [1]. The class of groups with regular geodesic language for some generating set is moreover closed under graph products [19]. Background, notation, and definitions relevant to this section and to Chapter 3 can be found in Chapter 2.

By considering more restrictive language classes than regular, it is possible to discover more properties of the underlying groups. In some cases, a characterization can be found. Gilman, Hermiller, Holt, and Rees show that a finitely generated group is virtually free if and only if its geodesic language is locally excluding for some finite symmetric (that is, inverse-closed) generating set [11, Theorem 1]. Hermiller, Holt, and Rees show that a finitely generated group is free abelian if and only if, for some finite symmetric generating set, it has piecewise excluding geodesic language where the excluded piecewise subwords all have length one [13, Theorem 3.2]. Our research is motivated by the existence of these correspondences and by the following implications.

Theorem 1.1. *[12, Proposition 6.2] Finitely generated abelian groups have piecewise excluding geodesic language for all finite symmetric generating sets.*

Theorem 1.2. *[12, Proposition 6.3] Finitely generated virtually abelian groups have piecewise testable geodesic language for some finite symmetric generating set.*

Cannon gives an example showing that a finitely generated virtually abelian group can have a non-regular geodesic language for some finite symmetric generating set [22]. A natural question to investigate is if Theorem 1.1 is a correspondence; that is, if groups with a piecewise excluding geodesic language for some generating set must be abelian. In Chapter 3, we show that a finitely generated group having piecewise excluding geodesic language does not imply that the group is abelian, even if the condition is strengthened to having piecewise excluding geodesic language for all finite symmetric generating sets.

Proposition 3.3. *Let K be a finitely generated abelian group, H a finite group, and G an extension of H by K: $1 \to H \to G \to K \to 1$. Then G has a piecewise excluding geodesic language for some finite symmetric generating set.*

Proposition 3.4. *The quaternion group, $Q_8 =< i, j, k \,|\, ijk^{-1}, jki^{-1}, kij^{-1}, i^4 >$, has piecewise excluding geodesic language for all finite symmetric generating sets.*

We show that the group Q_8 is a somewhat special 2-generator group and that the class of groups with piecewise excluding geodesic languages for all finite symmetric generating sets does not have nice closure properties.

Theorem 3.7. *The quaternion group, Q_8, is the only non-abelian 2-generator group with piecewise excluding geodesic language for all finite symmetric generating sets.*

Proposition 3.8. *The class of groups which have piecewise excluding geodesic languages for all finite symmetric generating sets is not closed under direct products.*

Recall that Theorem 1.2 shows that virtually abelian groups have piecewise testable geodesic language, a class which contains piecewise excluding geodesic languages. We show that the group property 'virtually abelain' also does not correspond to piecewise excluding geodesic language by exhibiting a family of virtually abelian groups which have, for any finite symmetric generating set A, a geodesic word containing both a generator and its inverse. By the proposition below, groups with a quotient isomorphic to a group in this family have, for any finite symmetric generating set, geodesic language which is not piecewise excluding.

Corollary 3.10. *There are finitely generated virtually abelian groups whose geodesic language is not piecewise excluding for any finite symmetric generating set.*

Proposition 3.11. *Let G be an extension $1 \to H \to G \xrightarrow{\pi} K \to 1$ of finitely generated groups H and K and let A be a finite symmetric generating set for G. If awa^{-1} is geodesic in K over the generating set $\pi(A)$ for some $a \in \pi(A)$ and $w \in \pi(A)^*$, then the geodesic language of G over A is not piecewise excluding.*

1.2 Autostackability of certain HNN extensions

Autostackable groups were introduced by Brittenham, Hermiller, and Holt as a generalization of asynchronously automatic groups with prefix-closed normal forms and groups with finite convergent rewriting systems [4, Theorem 4.1 and Corollary 5.4]. Autostackability of a group implies a finite presentation, a solution to the word problem, and a recursive algorithm for building van Kampen diagrams [4, Proposition 3.3]. Though by definition autostackability is a topological property of the Cayley graph along with a language theoretic restriction on this property, Brittenham, Hermiller, and Holt give a completely language theoretic characterization: a group is autostackable if and only if it admits a synchronously regular bounded convergent prefix-rewriting system [4, Theorem 5.3]. Background, notation, and definitions relevant to this section and to Chapter 4 can be found in Chapter 2.

Although both asynchronously automatic groups with prefix-closed normal forms and groups with finite convergent rewriting systems satisfy the homological finiteness condition FP_∞, Brittenham, Hermiller, and Johnson show that the class of autostackable groups includes a group not of type FP_3:

Theorem 1.3. *[5, Theorem 4.1] Stallings' non-*FP_3 *group* $< a, b, c, d, s \mid [a, c], [a, d], [b, c], [b, d], [s, ab^{-1}], [s, ac^{-1}], [s, ad^{-1}] >$ *is autostackable.*

Stallings' group is an HNN extension of a right-angled Artin group over its associated Bestvina-Brady group. The following two theorems are the closure results known to date for HNN extensions of autostackable groups.

Theorem 1.4. *[6, Theorem 3.5] Let* G *be a graph of groups over a finite connected graph* Λ *with at least one edge. If for each directed eddge* e *of* Λ *the vertex group* G_v *corresponding to the terminal vertex* $v = t(e)$ *of* e *is autostackable respecting the* *associated injective homomorphic image of the edge group G_e, then the fundamental group $\pi_1(G)$ is autostackable.*

Theorem 1.5. *[14, Corolary 4.6] Let H be an autostackable group over a symmetric generating set Z. Let $A \leq H$ be generated by a finite symmetric set $Y \subseteq Z$ with shortlex normal form set SL_A (with respect to some total ordering of Y), and let $\psi : A \to H$ be a monomorphism with $\psi(Y) \subseteq Z$. Suppose further that there are regular subsets $\mathcal{N}_{H/A}, \mathcal{N}_{H/\psi(A)} \subseteq \mathcal{N}_H$, each containing 1, representing transversals of these subgroups, and that for each $y \in Y$ and $\tilde{y} \in \psi(Y)$, the sets*

$$L_y = \{w \in \mathcal{N}_H \mid w =_H \mathsf{trans}_A(w)\mathsf{sub}_A(w) \text{ for some } \mathsf{trans}_A(w) \in \mathcal{N}_{H/A} \text{ and } \mathsf{sub}_A(w) \in SL_A \cap Y^*y\} \quad \text{and}$$

$$L'_{\tilde{y}} = \{w \in \mathcal{N}_H \mid w =_H \mathsf{trans}_{\psi(A)}(w)\mathsf{sub}_{\psi(A)}(w) \text{ for some } \mathsf{trans}_{\psi(A)}(w) \in \mathcal{N}_{H/\psi(A)} \text{ and } \mathsf{sub}_{\psi(A)}(w) \in \psi(SL_A) \cap \psi(Y)^*\tilde{y}\}$$

are also regular. Then the HNN extension $G = H*_\psi$ is autostackable.

The following is a stronger closure result for HNN extensions of algorithmically stackable groups (a weaker condition than autostackable).

Theorem 1.6. *[14, Corollary 4.7] Let H be an algorithmically stackable group, let $A, B \leq H$ be finitely generated, and let $\psi : A \to B$ be an isomorphism. Suppose further that the subgroup membership problem is decidable for the subgroups A and B in H. Then the HNN extension $G = H*_\psi$ is also algorithmically stackable.*

In Chapter 4, we show that the autostackability result in Theorem 1.3 can be extended to the following class of groups:

Definition 1.7. A *Stallings-like group* is an HNN extension $H*_{id_A}$ where $H = < a_1, ..., a_n \mid \{[a_i, a_j] \mid v_i \text{ is adjacent to } v_j \text{ in } \Lambda\} >$ is the right-angled Artin group associated to a connected finite simplicial graph Λ and A is the *Bestvina-Brady group* associated to Λ: $A = ker(\gamma)$ where $\gamma : H \to \mathbb{Z} =< x \,|\, >$ is defined by $\gamma(a_i) = x$ for all $i \in \{1, ..., n\}$.

Theorem 4.4. *Stallings-like groups are autostackable.*

Although Theorem 1.6 implies algorithmic stackability of Stallings-like groups, the closure results Theorem 1.4 and Theorem 1.5 do not imply autostackability of all Stallings-like groups for cannonical choices of graph of group decomposition and autostackable structure for right-angled Artin groups.

Remark 4.2. The graph of groups decomposition $H{*}_{id_A}$ for a Stallings-like group with the flag complex associated to Λ not simply-connected does not satisfy the hypotheses of Theorem 1.4.

Proposition 4.3. *Let Z be the generating set and let $\mathcal{N}_H$ be the set of normal forms for the right-angled Artin group $H =< a, b, c, d \,|\, [a, c], [a, d], [b, c], [b, d] >$ induced by the finite convergent rewriting system given by Hermiller and Meier in [15]. Let A be the Bestvina-Brady subgroup of H and let $\mathcal{N}_{H/A} = \{a^n \,|\, n \in \mathbb{Z}\}$, a transversal for A in $H{*}_{id_A}$. Let $L_y = \{w \in \mathcal{N}_H \,|\, w =_H \mathsf{trans}_A(w)\mathsf{sub}_A(w)$ for some $\mathsf{trans}_A(w) \in \mathcal{N}_{H/A}$ and $\mathsf{sub}_A(w) \in SL_A \cap Y^*y\}$, where $\mathsf{trans}_A(w)$ and $\mathsf{sub}_A(w)$ are the unique elements of the transversal $\mathcal{N}_{H/A}$ and of the shortlex representatives of A over Y, SL_A, respectively, such that $w =_H \mathsf{trans}_A(w)\mathsf{sub}_A(w)$. Then L_y is not regular for any generating set $Y \subseteq Z$ of A and any total ordering of Y.*

Chapter 2

Background

In this paper, all groups we consider are finitely generated and all generating sets are finite and symmetric (that is, inverse closed). We use the notation $[x, y]$ to mean the word $xyx^{-1}y^{-1}$. Let G be a group with generating set A. We denote the identity element of G by 1_G and use the notation $g =_G h$ to indicate that g and h are the same element of G. The smallest normal subgroup of G containing a set $\{x_1, ..., x_n\}$ is denoted by $< x_1, ..., x_n >^N$. A set of *normal forms* for G over A is a set N of words over A such that each element of G has a unique representative in N. The *Cayley graph* of G over A, denoted $\Gamma(G, A)$, is the directed graph with a vertex labeled g for each $g \in G$ and an edge labeled by a from g to ga for each $a \in A$ and each $g \in G$. The graph is endowed with a metric by making each edge isometric to the unit interval and using the induced Euclidean metric. A *geodesic word* in $\Gamma(G, A)$ is a word which labels a path of minimal length between two vertices in $\Gamma(G, A)$. The set of all finite length words over A, including the empty word, is denoted by A^*. A *language* L over A is a subset of A^*.

Definition 2.1. The *geodesic language* of G over A, denoted $\mathsf{Geo}(G, A)$, is the set of all geodesic words in $\Gamma(G, A)$.

For example, $F_2 =< x, y \mid\ >$ has $\mathsf{Geo}(F_2, \{x, y\}^{\pm 1}) = \{$freely reduced words over

$\{x,y\}^{\pm 1}\}$ and $\mathbb{Z}^2 =< a,b \mid [a,b] >$ has $\mathsf{Geo}(\mathbb{Z}^2, \{a,b\}^{\pm 1}) = \{$words over $\{a,b\}^{\pm 1}$ which do not contain both a generator and its inverse$\}$. See Appendix for illustrations of these Cayley graphs.

A language is *regular* if it can be built out of finite subsets of the alphabet using the operations of concatenation, union, intersection, complementation, and * (Kleene closure); such an expression for a language is called a *regular expression.* A language L is regular if and only if L can be recognized by a finite state automaton. For a reference on finite state automata and formal language theory, see [17].

The Pumping Lemma. *[17, Lemma 3.1] Let L be a regular language. Then there is a constant n such that for each word $z \in L$ of length at least n, we may write $z = uvw$ in such a way that uv has length at most n, v has length at least one, and for all $i \geq 0, uv^iw \in L$.*

Example 2.2. A virtually abelian group need not have regular geodesic language for every finite symmetric generating set. Cannon [22] exhibits the group $G = \mathbb{Z}^2 \rtimes {}^{\mathbb{Z}}\!/\!{}_{2\mathbb{Z}} =< a,b,t \mid [a,b], t^2, tatb^{-1} >$, which has regular geodesic language with the generating set $A = \{a,b,t\}^{\pm 1}$ but not with the generating set $B = \{a,d,c,t\}^{\pm 1}$, where $c =_G a^2$ and $d =_G ab$. See Appendix for illustrations of these Cayley graphs. The word tc^ntc^m is geodesic over the generating set B whenever $m < n$, but $tc^ntc^m =_G d^{2n}c^{m-n}$ so tc^ntc^m is not geodesic if $m \geq n$. By the Pumping Lemma, any regular language L containing the word tc^ntc^{n-1} must also contain the word tc^ntc^m for $m > n$ if $n-1$ is greater than the number of states in a minimal finite state automaton accepting L.

2.1 Language classes

The following three language class definitions can be found in [12]. A language L over an alphabet A is *locally excluding* if there is a finite set of words $F \subset A^*$

such that $w \in L$ if and only if w has no (contiguous) subword in F. For example, $F_2 =< x, y \,|\, >$ has locally excluding geodesic language $\mathsf{Geo}(F_2, \{x, y\}^{\pm 1})$ with the set of excluded subwords $\{xx^{-1}, x^{-1}x, yy^{-1}, y^{-1}y\}$. A language L over an alphabet A is *piecewise testable* if L is defined by a regular expression combining terms of the form $A^*a_1A^*a_2A^* \cdots A^*a_kA^*$, where $a_i \in A$, using the operations of concatenation, union, intersection, complementation, and * (Kleene closure). The string $a_1a_2 \cdots a_n \in A^*$ is called a *piecewise subword* of w if $w = w_0a_1w_1a_2 \cdots a_nw_n$ for some $w_i \in A^*$. A language L over an alphabet A is *piecewise excluding* if there is a finite set of words $F \subset A^*$ such that $w \in L$ if and only if w contains no piecewise subword in F. For example, $\mathbb{Z}^2 =< x, y \,|\, [x, y] >$ has piecewise excluding geodesic language with the set of excluded piecewise subwords $\{xx^{-1}, x^{-1}x, yy^{-1}, y^{-1}y\}$.

2.2 Synchronously regular languages

The following definitions and results can be found in [9]. Let \$ be a symbol not contained in the alphabet X and define $X_n = (X \cup \{\$\})^n \setminus \{(\$, ..., \$)\}$. For any n-tuple of words $u = (u_1, ..., u_n) \in (X^*)^n$, write $u_i = x_{i,1} \cdots x_{i,j_i}$ with each $x_{i,m} \in X$. Let $M = max\{j_1, ..., j_n\}$ and define $\tilde{u_i} = u\$^{M-j_i}$ so that each $\tilde{u_i}$ has length M. We can write $\tilde{u_i} = c_{i,1} \cdots c_{i,M}$ with each $c_{i,j} \in (X \cup \{\$\})$. The word $\mu(u) = (c_{1,1}, ..., c_{n,1}) \cdots (c_{1,M}, ..., c_{n,M})$ is the *padded word* over X_n induced by the n-tuple $(u_1, ..., u_n) \in (X^*)^n$. A subset $L \subseteq (X^*)^n$ is called *synchronously regular* if the padded extension set $\mu(L) = \{\mu(u) \mid u \in L\}$ of padded words associated to the elements of L is a regular language over the alphabet X_n.

Remark 2.3. The class of synchronously regular languages is closed under finite unions and intersections since the padded extension of a union (resp. intersection) is the union (resp. intersection) of the padded extensions [9].

Lemma 2.4. *[4, Lemma 2.3] The finite Cartesian product of regular languages over X is a synchronously regular language.*

Lemma 2.5. *[17, Theorem 3.6] If X is a finite set, $L \subset X^*$ is a regular language, and $w \in X^*$, then the quotient language $L/w = \{v \in X^* \mid vw \in L\}$ is also a regular language.*

2.3 Autostackability

The following definitions can be found in [5]; we follow their notation. Let G be a group with finite symmetric generating set X and let $\Gamma = \Gamma(G, X)$. Let $\vec{E}$ denote the set of directed edges in Γ and let $\vec{P}$ denote the set of directed paths in Γ. For each $g \in G$ and $x \in X$, let $e_{g,x}$ denote the directed edge with initial vertex g, terminal vertex gx, and label x. A *flow function* associated to a maximal tree $\mathcal{T}$ in Γ is a function $\Phi : \vec{E} \to \vec{P}$ satisfying:

(F1) for each $e \in \vec{E}$, the path $\Phi(e)$ has the same initial and terminal vertices as e,

(F2d) if the undirected edge underlying e lies in the tree $\mathcal{T}$, then $\Phi(e) = e$, and

(F2r) the transitive closure of $<_\Phi$ of the relation $<$ on $\vec{E}$, defined by $e' < e$ whenever e' lies in the path $\Phi(e)$ and the undirected edges underlying both e and e' do not lie in $\mathcal{T}$, is a strict well-founded partial ordering.

In other words, the map Φ fixes edges in the tree and describes a flow of the non-tree edges towards the tree; starting from a non-tree edge and iterating Φ finitely many times results in a path in the tree labeled by a word equal in G to the label of the initial edge [5]. The flow function is *bounded* if there is a constant k such that for every $e \in \vec{E}$, the path $\Phi(e)$ has length at most k.

Define $\mathsf{label} : \vec{P} \to X^*$ to be the function which maps each directed path to the word which labels it. For every $g \in G$, let y_g denote the label of the unique geodesic path in $\mathcal{T}$ from the identity to g. Let $\mathcal{N}_G = \{y_g \mid g \in G\}$ denote the set of these normal forms. Define $\mathsf{path} : \mathcal{N}_G \to \vec{P}$ by $\mathsf{path}(y_g, w)$ = the path in Γ starting at g and labeled by w. A group G is *algorithmically stackable* over X if there is a bounded flow function Φ on $\mathcal{T}$ for which the graph of the associated stacking map

$$graph(\phi) = \{(y_g, x, \mathsf{label}(\Phi(\mathsf{path}(y_g, x)))) \mid g \in G, x \in X\}$$

is decidable [14]. A group G is *autostackable* over X if there is a bounded flow function for which the graph of the associated stacking map is synchronously regular [4].

Brittenham, Hermiller and Susse define the condition of Theorem 1.4 as follows.

Definition 2.6. [6] A group H is *autostackable respecting a finitely generated subgroup* A if H has an autostackable structure with flow function Φ and maximal tree $\mathcal{T}$ on a generating set Z satisfying:

Subgroup closure: There is a finite symmetric generating set Y for A contained in Z such that $\mathcal{T}$ contains a spanning tree $\mathcal{T}_A$ for the subgraph $\Gamma(A, Y)$ of $\Gamma(H, Z)$, and for all $a \in A$ and $y \in Y$, $\mathsf{label}(\Phi(e_{a,y})) \in Y^*$.

A-translation invariance: The rest of $\mathcal{T}$ is an A-orbit of a transversal tree for A in H, and for all $a \in A, h \in H,$ and $z \in Z$ with $e_{h,z} \notin \Gamma(A, Y), \mathsf{label}(\Phi(e_{h,z})) = \mathsf{label}(\Phi(e_{ah,z}))$.

Note that this requires that H is autostackable.

2.4 Rewriting systems

The following definitions can be found in [4]. A *convergent prefix-rewriting system* for a group G consists of an alphabet X and a set of rules $R \subseteq X^* \times X^*$ such that G is presented as a monoid by $G = Mon < X \mid u = v$ whenever $(u, v) \in R >$ and the set of rewritings $\{uy \to vy \mid y \in A^*$ and $(u, v) \in R\}$ satisfy:

(termination) there is no infinite chain $w_0 \to w_1 \to w_2 \to \cdots$ of rewritings, and

(normal forms) each $g \in G$ is represented by exactly one irreducible word over X.

A prefix-rewriting system is *bounded* if X is finite and there is a constant k such that for each $(u, v) \in R$, there are words $x, t, w \in X^*$ such that $u = ws, v = wt$, and the length of s plus the length of t is at most k. A prefix-rewriting system is *synchronously regular* if X is finite and R is synchronously regular.

A *finite convergent rewriting system* for a group G is a finite set X and a finite set $R' \subseteq X^* \times X^*$ presenting G as a monoid such that the set of rewritings $\{xuy \to xvy \mid x, y \in X^* \text{ and } (u, v) \in R'\}$ satisfy the termination and normal form conditions above. Any finite convergent rewriting system R' has an associated synchronously regular bounded convergent prefix-rewriting system given by $R = \{(xu, xv) \mid x \in X^* \text{ and } (u, v) \in R'\}$ [4].

Theorem 2.7. *[4, Corollary 5.4] Groups with finite convergent rewriting systems are autostackable.*

2.5 Right-angled Artin groups

Given a finite simplicial graph Λ with vertices $v_i, ..., v_n$ such that each vertex v_i is labeled by a group G_i, the associated *graph product*, $G\Lambda$, is the quotient of the free product of the groups G_i by the relations that elements of vertex groups corresponding to adjacent vertices in Λ commute. If for each i, G_i is infinite cyclic with generator a_i, $G\Lambda$ is called a *right-angled Artin group* and is presented by $< a_1, ..., a_n \mid \{[a_i, a_j] \mid v_i \text{ is adjacent to } v_j \text{ in } \Lambda\} >$. When the underlying graph has no edges, the corresponding right-angled Artin group is the free group on n generators, F_n; when the underlying graph is complete, the corresponding right-angled Artin group is the free abelian group on n generators, $\mathbb{Z}^n$.

Example 2.8. $F_2 \times F_2 =< a, b, c, d \mid [a, c], [a, d], [b, c], [b, d] >$ is a right-angled Artin group with underlying graph Λ given in Figure 1.

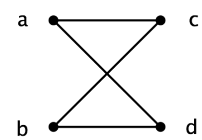


Figure 1.

Lemma 2.9. *[15, Theorem C] The graph product of finitely many groups which admit a convergent rewriting system admits a cannonical convergent rewriting system and, moreover, if the rewriting systems of the vertex groups are finite or regular then the system for the graph product is as well.*

Hermiller and Meier show this convergent rewriting system explicitly in [15]; as we will only need the case when the graph product is a right-angled Artin group, we provide a simplified version here. For the right-angled Artin group $\mathrm{G}\Lambda =< a_1, ..., a_n \mid \{[a_i, a_j] \mid v_i \text{ is adjacent to } v_j \text{ in } \Lambda\} >$, the finite convergent rewriting system is given by:

The alphabet is $Z = \{\alpha_1 \cdots \alpha_n \mid \alpha_i \in \{1, a_i, a_i^{-1}\}$ and $\{v_i \mid \alpha_i \neq 1\}$ is a nonempty set of vertices in a complete subgraph of $\Lambda\}$

and rules are $R = \{(\alpha_1 \cdots \alpha_{j-1} 1 \alpha_{j+1} \cdots \alpha_n)(\beta_1 \cdots a_j^{\epsilon} \cdots \beta_n) \to (\alpha_1 \cdots a_j^{\epsilon} \cdots \alpha_n)(\beta_1 \cdots \beta_{j-1} 1 \beta_{j+1} \cdots \beta_n)\} \cup$

$$\{(\alpha_1 \cdots a_j^{\epsilon} \cdots \alpha_n)(\beta_1 \cdots a_j^{-\epsilon} \cdots \beta_n) \to (\alpha_1 \cdots \alpha_{j-1} 1 \alpha_{j+1} \cdots \alpha_n)(\beta_1 \cdots \beta_{j-1} 1 \beta_{j+1} \cdots \beta_n)\},$$

with the assumptions that $\epsilon \in \{\pm 1\}$, that if $1 \cdots 1$ appears on the right hand side of a rule it is replaced with the empty word, and that a rule only occurs if the letters in the rule exist.

The Bestvina-Brady Theorem. *[2] Let Λ be a finite simplicial graph. Let L be the induced flag complex, let H be the associated right-angled Artin group, and let A be the corresponding Bestvina-Brady group. Then:*

(1) A is FP_{n+1} if and only if L is homologically n-connected.

(2) A is FP if and only if L is acyclic.

(3) A is finitely presented if and only if L is simply connected.

2.6 HNN extensions

Given groups $H =< Z \,|\, R >$ and $A =< Y >$ and injective homomorphisms $\alpha, \beta : A \hookrightarrow H$, the corresponding *HNN extension* is $< Z, s \,|\, R \cup \{s\alpha(y)s^{-1} = \beta(y) \,|\, y \in Y\} >$, denoted either $H*_A$ or $H*_{\beta\circ\alpha^{-1}}$.

In this presentation, s is called the *stable letter*. It is common to begin with $A \leq H$ and use a single homomorphism $\psi : A \hookrightarrow H$; in this case, the HNN extension is $H*_\psi =< Z, s \,|\, R \cup \{sys^{-1} = \psi(y) \,|\, y \in Y\} >$. Let $\mathcal{N}_A$ denote a set of normal forms for A over Y, let $\mathcal{N}_{\psi(A)}$ denote a set of normal forms for the cosets H/A with $1 \in \mathcal{N}_{H/A}$, and let $\mathcal{N}_{H/\psi(A)}$ denote a set of normal forms for the cosets $H/\psi(A)$ with $1 \in \mathcal{N}_{H/\psi(A)}$.

Definition 2.10. [20, page 181] The *Britton normal form set* for H is

$$\mathcal{N}_G = \{hs^{\epsilon_1}h_1s^{\epsilon_2}h_2...s^{\epsilon_n}h_n \,|\, n \geq 0, h \in \mathcal{N}_H \text{ and } \epsilon_i = \pm 1 \text{ for } 1 \leq i \leq n; \text{ if } \epsilon_i = 1 \text{ then } h_i \in \mathcal{N}_{H/A}, \text{ if } \epsilon_i = -1 \text{ then } h_i \in \mathcal{N}_{H/\psi(A)}, \text{ and if } \epsilon_i = -\epsilon_{i-1} \text{ then } h_i \neq 1_H\};$$

that is, $\mathcal{N}_G = \mathcal{N}_H(s^{-1}N_{H/\psi(A)} \cup s\mathcal{N}_{H/A})^* \setminus \cup_{\epsilon\in\{\pm 1\}} Z^* s^\epsilon s^{-\epsilon} Z^*$.

2.7 Autostackability of Stallings' non-FP3 group

The autostackible structure of Stallings' non-FP3 group G found in [5] is given below. The group $G = H*_{id_{ker(\gamma)}}$ where $H = F_2 \times F_2 =< a, b, c, d \mid [a, c], [a, d], [b, c],$

$[b, d] >$, the right-angled Artin group from Example 2.8, and $\gamma : H \to \mathbb{Z} =< x \,|\; >$ is defined by $\gamma(z) = x$ for all $z \in \{a, b, c, d\}$. In Theorem 1.3, G is shown to be autostackable with respect to the generating set $X = \{a, b, c, d, s\}^{\pm 1}$. The normal form set for this autostackable structure is

$$\mathcal{N}_G = \{hs^{\epsilon_1}a^{i_1}s^{\epsilon_2}a^{i_2}\cdots s^{\epsilon_n}a^{i_n} \mid h \in \mathcal{N}_H,\, n \geq 0, \epsilon_k \in \{\pm 1\} \text{ and } i_k \in \mathbb{Z} \text{ for all } k, \\ \text{and whenever } i_k = 0 \text{ then } \epsilon_k = \epsilon_{k+1}\},$$

where $\mathcal{N}_H = \{uv \,|\, u \in \{a^{\pm 1}, b^{\pm 1}\}^* \text{ and } v \in \{c^{\pm 1}, d^{\pm 1}\}^* \text{ are freely reduced}\}$.

The stacking map, $\phi : \mathcal{N}_G \times X \to X^*$, is defined by $\phi(y_g, x) =$

$$\begin{cases} x & \text{if either } y_g x \in \mathcal{N}_G \text{ or } y_{gx}x^{-1} \in \mathcal{N}_G \\ \mathsf{last}(y_g)^{-1}x\mathsf{last}(y_g) & \text{if } x \in \{a, b\}^{\pm 1},\ y_g \in Z^*, \text{ and } \mathsf{last}(y_g) \in \{c, d\}^{\pm 1} \\ \mathsf{last}(y_g)^{-1}x\mathsf{last}(y_g) & \text{if } x \in \{c, d\}^{\pm 1},\ y_g \notin Z^*, \text{ and } \mathsf{last}(y_g) \in \{a\}^{\pm 1} \\ c^{-\eta}xc^{\eta} & \text{if } x \in \{b\}^{\pm 1},\ y_g \notin Z^*,\ \eta \in \{\pm 1\}, \text{ and } \mathsf{last}(y_g) = a^{\eta} \\ \mathsf{last}(y_g)^{-1}xa^{-\eta}\mathsf{last}(y_g)a^{\eta} & \text{if } x \in \{b^{\eta}, c^{\eta}, d^{\eta}\} \text{ with } \eta \in \{\pm 1\} \text{ and } \mathsf{last}(y_g) \in \{s\}^{\pm 1} \end{cases}$$

for all $g \in G$ and $x \in X$, where $Z = \{a, b, c, d\}^{\pm 1}$, y_g denotes the normal form of g in $\mathcal{N}_G$, and $\mathsf{last}(w)$ is the last letter in the word w.

Chapter 3

Piecewise excluding geodesic languages

The following observation proved to be useful in showing particular geodesic languages were not piecewise excluding.

Lemma 3.1. *Let G be a group generated by a finite symmetric generating set A. If $\mathsf{Geo}(G, A)$ is piecewise excluding, then aa^{-1} must be an excluded piecewise subword for every $a \in A$ which does not represent the identity element of G.*

Proof. First note that if a or a^{-1} is excluded from $\mathsf{Geo}(G, A)$, then a must represent the identity of the group. If $a \neq_G 1_G$, then $a, a^{-1} \in \mathsf{Geo}(G, A)$. For any $a \in A$, $aa^{-1} \notin \mathsf{Geo}(G, A)$ since $aa^{-1} =_G 1_G$. In a piecewise excluding geodesic language, the only way to exclude the word aa^{-1} from the language without excluding a or a^{-1} is by excluding aa^{-1} as a piecewise subword. $\square$

This suggests something strong about commutativity and seems to be evidence in favor of the existence of a correspondence between abelian groups and piecewise excluding geodesic lanugages. But there are non-abelian groups which have piecewise excluding geodesic language for some generating sets.

Lemma 3.2. *All finite groups have a generating set whose geodesic language is piecewise excluding.*

Proof. Let G be a finite group and let $A = G \setminus \{1_G\}$. Then $\mathsf{Geo}(G, A)$ is piecewise excluding. In particular, $\mathsf{Geo}(G, A) = A^* \setminus \{A^*aA^*bA^* \mid a, b \in A\}$, which is the set of all words over A of length at most one. $\square$

Finite groups may also have piecewise excluding geodesic language for smaller generating sets. Consider $D_8 =< a, b, t \mid a^2, b^2, (ab)^4, ababt >$ and $A = \{a, b, t\}$. Then $\mathsf{Geo}(D_8, A) = A^* \setminus (\{A^*xA^*xA^* \mid x \in A\} \cup \{A^*xA^*yA^*zA^* \mid x, y, z \in A\})$, the set of all words over A of length at most two which do not contain duplicate letters. Note that with the generating set $B = \{a, b\}$, however, D_8 does not have piecewise excluding geodesic language since $aba \in \mathsf{Geo}(D_8, B)$. See Appendix for illustrations of these Cayley graphs.

Proposition 3.3. *Let K be a finitely generated abelian group, H a finite group, and G an extension of H by K: $1 \to H \to G \to K \to 1$. Then G has a piecewise excluding geodesic language for some finite symmetric generating set.*

Proof. Let the maps be $1 \to H \xrightarrow{\iota} G \xrightarrow{\pi} K \to 1$ and let A be a finite symmetric generating set for K with $1_K \notin A$. By Theorem 1.1 $\mathsf{Geo}(K, A)$ is piecewise excluding; let F be the finite set of excluded piecewise subwords. For each $a \in A$, choose a unique preimage under π, denoted $\bar{a}$, such that $\overline{a^{-1}} = \bar{a}^{-1}$. Let $\bar{A} = \{\bar{a} \mid a \in A\}$ and let $\bar{F} = \{\bar{a_1} \cdots \bar{a_n} \mid a_1 \cdots a_n \in F\}$. Then words over $\bar{A}$ are geodesic if and only if they have no piecewise subword in $\bar{F}$. Let $\bar{H} = \iota(H \setminus \{1_H\})$. Note that as no generators in $\bar{H}$ represent the identity element of G, words of length one over $\bar{H}$ are geodesic; because each non-identity element of H has a representative in $\bar{H}$, words of length two over $\bar{H}$ are not geodesic. Because $\iota(H)$ is a normal subgroup of G, for each $h \in H$ and each $a \in A$ there is an $h_a \in H$ such that $\bar{a}\iota(h)\bar{a}^{-1} =_G \iota(h_a)$. Suppose that $w \in (\bar{A} \cup \bar{H})^*$. Write $w = a_1h_1a_2h_2 \cdots a_nh_n$ where $a_i \in \bar{A}^*$ and $h_i \in \bar{H}$ for all $i \in \{1, ..., n\}$. Then $w =_G \tilde{h}a_1a_2 \cdots a_n$ where $\tilde{h} = (h_1)_{a_1}(h_2)_{a_1a_2} \cdots (h_n)_{a_1a_2\cdots a_n}$; that is, w is equal in G to a word in $(\bar{A} \cup \bar{H})^*$ with at most one element of $\bar{H}$ followed by $a_1 a_2 \cdots a_n$, the piecewise subword of w over $\bar{A}$. Therefore any word in $(\bar{A} \cup \bar{H})^*$ with more than one letter from $\bar{H}$ or containing a piecewise subword over $\bar{A}$ which has a piecewise subword in $\bar{F}$ is not geodesic. Thus $\mathsf{Geo}(G, \bar{A} \cup \bar{H})$ is the piecewise excluding language whose set of excluded piecewise subwords is $\bar{H}^2 \cup \bar{F}$. $\square$

For example, let $G = H \rtimes_\phi K$ where $K =< a \mid \ >$, $H =< t \mid t^3 >$, and $\phi(a)(t) = t^{-1}$. For simplicity, supress the natural injective homomorphisms and assume that $H, K \leq G$. Let $A = \{a, t\}^{\pm 1}$. Then $\mathsf{Geo}(G, A) = A^* \setminus ([\bigcup_{\epsilon \in \{\pm 1\}} A^* a^\epsilon A^* a^{-\epsilon} A^*] \cup [\bigcup_{\epsilon, \delta \in \{\pm 1\}} A^* t^\epsilon A^* t^\delta A^*])$ is piecewise excluding. See Appendix for an illustration of this Cayley graph.

Proposition 3.4. *The quaternion group, $Q_8 =< i, j, k \,|\, ijk^{-1}, jki^{-1}, kij^{-1}, i^4 >$, has piecewise excluding geodesic language for all finite symmetric generating sets.*

Proof. Because the center of the group is $\{1_{Q_8}, i^2\}$ and all other elements have order four, any set of elements of Q_8 containing at most one order four element (not including inverses) generates an abelian group. Hence any generating set for the non-abelian group Q_8 includes at least two order four elements which do not commute. Let A be a finite symmetric generating set for Q_8, and let a and b be two order four elements in A which do not commute. Then the eight words $1, a, a^{-1}, b, b^{-1}, a^2, ba$, and $b^{-1}a$ represent distinct elements of Q_8. The element 1 has order one, the element a^2 has order two, and no two of the remaining (order four) elements can be equal because that would contradict that a and b do not commute and both have order four. Thus, any word of length at least three is not geodesic. The language of geodesics is therefore piecewise excluding: the set of excluded piecewise subwords is the set of words of length three together with all words of length at most two that are not geodesic. $\square$

The following two lemmas are used in the proof of Theorem 3.7.

Lemma 3.5. *All proper quotients of the group* $G = {}^{\mathbb{Z}}/_{5\mathbb{Z}} \rtimes \mathbb{Z} =< a, x \mid a^5, xax^{-1}a^2 >$ *are either abelian or, for some finite symmetric generating set, have a geodesic language which is not piecewise excluding.*

Proof. First note that a set of normal forms for G over $A = \{a, x\}^{\pm 1}$ is $\{a^i x^n \mid i \in \{0, 1, 2, 3,\ 4\}, n \in \mathbb{Z}\}$. Observe that $xa =_G a^3x$, and so $x^n a =_G a^{3^n} x^n$ for all $n \in \mathbb{Z}$, and that $x^4 a =_G ax^4$. Any proper quotient H of G is isomorphic to ${}^{G}/_{<a^{i_1}x^{n_1}, a^{i_2}x^{n_2}, \ldots, a^{i_k}x^{n_k}>^N}$ for some $\{a^{i_j}x^{n_j}\}_{j=1}^{k}$ where for each $j \in \{1, ..., k\}$, $i_j \in \{0, 1, 2, 3, 4\}$, $n_j \in \mathbb{Z}$, and i_j, n_j are not both zero.

Case A: There is a $j \in \{1, ..., k\}$ *such that* $n_j = 0$.

In this case $i_j \in \{1, 2, 3, 4\}$, so $< a^{i_j} > =_G < a >$ is trivial in the quotient. Thus H is a quotient of $\mathbb{Z}$, and so H is abelian.

Case B: There is a $j \in \{1, ..., k\}$ *such that* $i_j = 0$ *and* $n_j \neq 0 \ (\mathsf{mod} 4)$.

In this case $a(x^{n_j})a^{-1} =_G ax^{n_j}a^4 =_G a^{1+4\cdot 3^{n_j}}x^{n_j} \in < x^{n_j} >^N$, which implies that $a \in < x^{n_j} >^N$ for all possible n_j. Thus H is a quotient of ${}^{\mathbb{Z}}/_{n_j\mathbb{Z}}$, and so H is abelian.

Case C: There is a $j \in \{1, ..., k\}$ *such that* $i_j \neq 0$ *and* $n_j \neq 0 \ (\mathsf{mod} 4)$.

In this case $x^{n_j}(a^{i_j}x^{n_j})x^{-n_j} =_G x^{n_j}a^{i_j} \in < a^{i_j}x^{n_j} >^N$, which implies that $(x^{n_j}a^{i_j})^{-1} =_G a^{-i_j}x^{-n_j} \in < a^{i_j}x^{n_j} >^N$. So $(a^{i_j}x^{n_j})(a^{-i_j}x^{-n_j}) =_G a^{i_j+(5-i_j)3^{n_j}} \in < a^{i_j}x^{n_j} >^N$. Note that $a^{i_j+(5-i_j)3^{n_j}}$ is a nontrivial element of ${}^{\mathbb{Z}}/_{5\mathbb{Z}}$ for any $i_j \neq 0$ and $n_j \neq 0 \ (\mathsf{mod} 4)$. Hence we have that $a \in < a^{i_j}x^{n_j} >^N$ in all subcases. Thus H is a quotient of ${}^{\mathbb{Z}}/_{n_j\mathbb{Z}}$, and so H is abelian.

Case D: There is a $j \in \{1, ..., k\}$ *such that* $i_j \neq 0$ *and* $n_j = 0 \ (\mathsf{mod} 4)$ *is nonzero.*

Note that in this case $n_j - 2 = 2 \ (\mathsf{mod} 4)$, so $x^{n_j-2}a^{i_j} =_G a^{4i_j}x^{n_j-2}$. Thus $x^{n_j-2}\ (a^{i_j}x^{n_j})x^{-(n_j-2)} =_G a^{-i_j}x^{n_j} \in < a^{i_j}x^{n_j} >^N$. This implies that $(a^{-i_j}x^{n_j})^{-1} =_G a^{i_j}x^{-n_j} \in < a^{i_j}x^{n_j} >^N$. Hence $(a^{i_j}x^{n_j})(a^{i_j}x^{-n_j}) =_G a^{2i_j} \in < a^{i_j}x^{n_j} >^N$. As a^{2i_j} is a nontrivial element of $\mathbb{Z}/5\mathbb{Z}$ for all $i_j \neq 0$, in all subcases we have that $a \in < a^{i_j} x^{n_j} >^N$. Thus H is a quotient of $\mathbb{Z}/n_j\mathbb{Z}$, and so H is abelian.

Case E: For every $j \in \{1, ..., k\}$, $i_j = 0$ and $n_j = 0 \ (\mathsf{mod} 4)$ is nonzero.

Note that we can simplify the quotient to $G/<x^{\mathsf{gcd}(n_1,...,n_k)}>^N$ in this case and that $\mathsf{gcd}(n_1, ..., n_k) \geq 4$. Consider the generating set $B = \{(ax), (xa)\}^{\pm 1}$. Observe that $(ax)^{-1} =_G a^3x^{-1}$ and $(a^3x^{-1})(xa) =_G a^{-1}$ so this is in fact a generating set for G, and thus for H as well. Note that $(ax)(xa)(ax)^{-1} =_G a^3x$. As a has order 5 in H, none of the generators in B are equal in H to a^3x. A word $w \in B^*$ of length two represents a group element $h \in G$ with a normal form w' over A that has an even power of x. So no words in B^* of length two can be equal in H to a^3x. Hence the word $(ax)(xa)(ax)^{-1}$ is geodesic in H. Therefore by Lemma 3.1, the geodesic language of H over B is not piecewise excluding. □

Lemma 3.6. *All proper quotients of the group $G = BS(1,2) = < a, t \,|\, tat^{-1}a^{-2} >$ are either abelian or, for some finite symmetric generating set, have a geodesic language which is not piecewise excluding.*

Proof. First note that a set of normal forms for G over $\{a,t\}^{\pm 1}$ is $\{t^{-i}a^nt^j \,|\, i, j \in (\mathbb{N} \cup 0), n \in \mathbb{Z}, \text{and } 2 \nmid n \text{ if both } i, j > 0\}$. Any proper quotient H of G is isomorphic to $G/<t^{-i_1}a^{n_1}t^{j_1}, t^{-i_2}a^{n_2}t^{j_2}, ..., t^{-i_m}a^{n_m}t^{j_m}>^N$ for some $\{t^{-i_k}a^{n_k}t^{j_k}\}_{k=1}^m$ where for each $k \in \{1, ..., m\}$, $i_k, j_k \in (\mathbb{N} \cup 0), n_k \in \mathbb{Z}$, i_k, j_k, n_k are not all zero, and $i_k \neq j_k$ whenever $n_k = 0$. Let H be a non-abelian proper quotient of G. We first show that H is a quotient of one of a specific collection of semi-direct products.

Case 1: Suppose there is an index k such that $n_k \neq 0$.

Note that $t^{-i_k}a^{n_k}t^{j_k} =_H 1_H$. If $i_k = j_k$ then $a^{n_k} =_H 1_H$. If $i_k \neq j_k$ then $t^{i_k - j_k} =_H a^{n_k}$, which is equal in G to $t^{-1}a^{2n_k}t$. Hence $t^{i_k - j_k} =_H a^{2n_k}$, and so $a^{n_k} =_H a^{2n_k}$, which implies that $a^{n_k} =_H 1_H$ and thus that $t^{i_k - j_k} =_H 1_H$.

If n_k is even, then $ta^{n_k/2} =_G a^{n_k}t =_H t$, which implies that $a^{n_k/2} =_H 1_H$. So we can continue cutting the known order of a in H in half until we are left with an odd number, call it n. Then $ta =_G a^2t$ implies the relations $ta^l =_H a^{2l}t$ for every integer l such that $1 \leq l < n/2$ and $ta^l =_H a^{2l-n}t$ for every integer l such that $n/2 < l < n$. These relations allow us to move $a^{\pm 1}$ past $t^{\pm 1}$ in either direction in any word.

Thus H is a quotient of the semidirect product $\mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z} =< a, t \mid a^n, tat^{-1}a^{-2} >$ (if $i_k = j_k$) or of the semidirect product $\mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/|i_k-j_k|\mathbb{Z} =< a, t \mid a^n, t^{|i_k-j_k|}, tat^{-1}a^{-2} >$ (if $i_k \neq j_k$), with n odd in either case.

Case 2: Suppose there is an index k such that $n_k = 0$.

Then $i_k - j_k \neq 0$, so we may replace t^{-1} with $t^{|i_k-j_k|-1}$ in the relation $tat^{-1} =_G a^2$ and obtain the relations $t^{-1}a =_H a^{2^{|i_k-j_k|-1}}t^{-1}$ and $at =_H ta^{1-2^{|i_k-j_k|-1}}$. So $a =_G t^{-1}a^2t =_H a^{2^{|i_k-j_k|}}$, which implies that $a^{2^{|i_k-j_k|}-1} =_H 1_H$. Hence H is a quotient of the semidirect product $\mathbb{Z}/(2^{|i_k-j_k|}-1)\mathbb{Z} \rtimes \mathbb{Z}/|i_k-j_k|\mathbb{Z} =< a, t \mid a^{2^{|i_k-j_k|}-1}, t^{|i_k-j_k|}, tat^{-1}a^{-2} >$.

Because H is non-abelian, Cases 1 and 2 show that H is isomorphic to either a quotient of $\mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}$ with n odd and at least three or to a quotient of $\mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/|i-j|\mathbb{Z}$ with n odd and at least three and $|i - j| \geq 2$.

Note that if $|i - j| = 2$, then $a =_H t^2a =_H a^4t^2 =_H a^4$. So in this case $a^3 =_H 1_H$ and, moreover, H is isomorphic to a quotient of $S_3 =< x, y \mid x^2, y^2, (xy)^3 >$. This means that either H has a geodesic language which is not piecewise excluding or H is abelian: if $H \cong S_3$ then the word xyx^{-1} is geodesic over the generating set $\{x, y\}^{\pm 1}$; if H is a proper quotient of S_3 then H is abelian. Note that if $|i - j| = 3$, then $a =_H t^3a =_H a^8t^3 =_H a^8$. So in this case $a^7 =_H 1_H$ and, moreover, H is isomorphic to a quotient of $\mathbb{Z}/7\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z} =< a, t \mid a^7, t^3, tat^{-1}a^{-2} >$. This means that either H has a geodesic language which is not piecewise excluding or H is abelian: if $H \cong \mathbb{Z}/7\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z}$ then the word $(at)t(at)^{-1}$ is geodesic over the generating set $\{(at), t\}^{\pm 1}$; if H is a proper quotient of $\mathbb{Z}/7\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z}$ then H has order 1,3, or 7 and so is abelian.

What remains to be considered is when H is isomorphic to either $\mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}$ with n odd and at least three or to $\mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/|i-j|\mathbb{Z}$, with n odd and at least three and $|i-j| > 3$. Let $B = \{(at), (ta)\}^{\pm 1}$. Note that $(at)^{-1} =_H t^{-1}a^{n-1} =_G (a^{\frac{n-1}{2}})t^{-1}$ and that $(a^{\frac{n-1}{2}}t^{-1})(ta) =_G a^{\frac{n+1}{2}}$. So as $(a^{\frac{n+1}{2}})^2 =_H a$, the set B is a generating set for H. Consider the word $(at)(ta)(at)^{-1}$, which is equal in G to a^3t.

Note that a^3t cannot be equal in H to a single generator as $n \notin \{1, 2\}$ and $|i-j| \neq 2$: the element at is equal to in H to a^3t only if $a^2 =_H 1$; the element ta is equal in H to a^3t only if $a =_H 1$; the elements $(at)^{-1}$, $(ta)^{-1}$ are equal in H to a^3t only if $t^2 =_H 1$. Note also that a^3t cannot be equal in H to a word of length two in the generators as $|i-j| \notin \{1, 3\}$: the words $(at)(at)$, $(at)(ta)$, $(at)(ta)^{-1}$, $(ta)(ta)$, $(ta)(at)$, $(ta)(at)^{-1}$, $(at)^{-1}(ta)$, $(ta)^{-1}(at)$ are equal in H to a^3t only if $t =_H 1$; the words $(at)^{-1}(at)^{-1}$, $(at)^{-1}(ta)^{-1}$, $(ta)^{-1}(ta)^{-1}$, $(ta)^{-1}(at)^{-1}$ are equal in H to a^3t only if $t^3 =_H 1$.

Therefore the word $(at)(ta)(at)^{-1}$ is geodesic over B. By Lemma 3.1, the geodesic language of H over B is not piecewise excluding. $\square$

Theorem 3.7. *The quaternion group, Q_8, is the only non-abelian 2-generator group with piecewise excluding geodesic language for all finite symmetric generating sets.*

Proof. Consider a minimal symmetric generating set $\{a, b\}^{\pm 1}$ for a two generator non-abelian group G with piecewise excluding geodesic language for all finite symmetric generating sets. Because G is non-abelian, $aba^{-1} \notin \{1, a, a^{-1}, b\}$. Note that aba^{-1} is not geodesic by Lemma 3.1, so it must then be equal in G to either b^{-1} or to a product of two generators. It can be shown, by straight-forward computations, that ten of the sixteen choices for words of length two over $\{a, b\}^{\pm 1}$ also lead to contradictions if they are equal in G to aba^{-1}. For example, $aba^{-1} =_G ab^{-1}$ implies that $b^2 =_G a$, which contradicts the assumption that a and b do not commute. Thus a representative of aba^{-1}

must be in the set $\{b^{-1}, a^{-1}b, a^{-1}b^{-1}, ba, b^2, b^{-1}a, b^{-2}\}$. Similarly, the possibilites for representatives of bab^{-1} can be reduced to the set $\{a^{-1}, b^{-1}a, b^{-1}a^{-1}, ab, a^2, a^{-1}b, a^{-2}\}$.

Table 1 shows the group defined by only the two relations in each of the forty-nine possible pairs of choices for representaives of aba^{-1} (along the first row) and for representatives of bab^{-1} (along the first column). Note that by symmetry, the upper and lower diagonals are isomorphic groups. Most of the finite groups were found by entering the presentation into the GAP system and referencing the small group information within GAP; some (those listed below) required referencing groupprops.subwiki.org. We refer readers unfamiliar with GAP to [10]. The pairs $aba^{-1} =_G a^{-1}b^{-1}$ with $bab^{-1} =_G b^{-1}a^{-1}$ and $aba^{-1} =_G b^{-1}a$ with $bab^{-1} =_G a^{-1}b$ both returned the group [24:3], which is $SL_2(\mathbb{Z}/3\mathbb{Z})$. The pair $aba^{-1} =_G b^{-2}$ with $bab^{-1} =_G a^{-2}$ returned the group [27,4], which is $\mathbb{Z}/9\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z} =< x, y \,|\, x^9, y^3, yxy^{-1}x^{-4} >$. Notice that $aba^{-1} = a^{-1}b$ and $bab^{-1} = a^2$ are actually both the same relation, so this pair yields the group $BS(1, 2)$. Groups which were reported to be infinite were calculated by hand using Tietze transformations. For example,

$$< a, b \,|\, aba^{-1} = b^{-1}, bab^{-1} = b^{-1}a > \cong < a, b \,|\, ab = b^{-1}a, bab^{-1} = b^{-1}a >$$
$$\cong < a, b \,|\, ab = b^{-1}a, bab^{-1} = b^{-1}a, b^2a = ab > \cong < a, b \,|\, ab = bab^{-1}, b^2a = ab >$$
$$\cong < a, b \,|\, ab^2 = ba, b^2a = ab > \cong < a, b \,|\, ab^2 = ba, bab^2 = ab >$$
$$\cong < a, b \,|\, ab^2 = ba, ab^4 = ab > \cong < a, b \,|\, ab^2 = ba, b^3 = 1 > .$$

	b^{-1}	$a^{-1}b$	$a^{-1}b^{-1}$	ba	b^2	$b^{-1}a$	b^{-2}
a^{-1}	$Q_8^{\ddagger}$						
$b^{-1}a$	$\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}^{\dagger}$	1					
$b^{-1}a^{-1}$	$\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	1	$SL_2(\mathbb{Z}/3\mathbb{Z})^{\dagger}$				
ab	$\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}^{\dagger}$	1	1	1			
a^2	$\mathbb{Z}/2\mathbb{Z}$	$BS(1,2)^{\dagger}$	$S_3^{\dagger}$	$\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}^{\dagger}$	1		
$a^{-1}b$	$\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	1	$\mathbb{Z}/5\mathbb{Z}$	1	$S_3^{\dagger}$	$SL_2(\mathbb{Z}/3\mathbb{Z})^{\dagger}$	
a^{-2}	$\mathbb{Z}/6\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/6\mathbb{Z}$	$\mathbb{Z}/5\mathbb{Z} \rtimes_{\alpha} \mathbb{Z}^{\dagger}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/6\mathbb{Z}$	$\mathbb{Z}/9\mathbb{Z} \rtimes_{\beta} \mathbb{Z}/3\mathbb{Z}^{\dagger}$

Table 1.

The map α in the entry $\mathbb{Z}/5\mathbb{Z} \rtimes_{\alpha} \mathbb{Z}$ is defined by the generator of $\mathbb{Z}$ conjugating the generator of $\mathbb{Z}/5\mathbb{Z}$ to its square and the map β in the entry $\mathbb{Z}/9\mathbb{Z} \rtimes_{\beta} \mathbb{Z}/3\mathbb{Z}$ is defined by the generator of $\mathbb{Z}/3\mathbb{Z}$ conjugating the generator of $\mathbb{Z}/9\mathbb{Z}$ to its fourth power. The group $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}$ is the nontrivial semi-direct product.

The group G must be a quotient of one of the groups in the table. All quotients of abelian groups are abelian, so G cannot be a quotient of an abelian group in the table. The groups which are non-abelian but have a geodesic language which is not piecewise excluding for some finite symmetric generating set, demonstrated below, are denoted by a single dagger. We show below that all proper quotients of each of these groups are either abelian or have a geodesic language which is not piecewise excluding for some finite symmetric generating set. In each case of a geodesic language which is not piecewise excluding, a check of all words of length at most two against a set of normal forms shows that the given length three word is geodesic. Cayley graphs illustrating these cases can be found in the Appendix.

The group $S_3 =< a, b \,|\, a^2, b^2, (ab)^3 >$ with the generating set $A = \{a, b\}^{\pm 1}$ has $aba^{-1} \in \mathsf{Geo}(S_3, A)$. So by Lemma 3.1 $\mathsf{Geo}(S_3, A)$ is not piecewise excluding. The only proper quotients of S_3 are abelian. Thus G cannot be a quotient of S_3.

The group $SL_2(\mathbb{Z}/3\mathbb{Z}) =< a, b \,|\, a^6, b^4, ab^{-1}ab^{-1}ab >$ with the generating set $A = \{a, b\}^{\pm 1}$ has $bab^{-1} \in \mathsf{Geo}(SL_2(\mathbb{Z}/3\mathbb{Z}), A)$. So by Lemma 3.1 $\mathsf{Geo}(SL_2(\mathbb{Z}/3\mathbb{Z}), A)$ is not piecewise excluding. The only proper quotients of $SL_2(\mathbb{Z}/3\mathbb{Z})$ are quotients of A_4 and quotients of $\mathbb{Z}/3\mathbb{Z}$ (see groupprops.subwiki.org). The group $A_4 =< a, b \,|\, a^3, b^2, (ab)^3 >$ with the generating set $B = \{a, b\}^{\pm 1}$ has $bab^{-1} \in \mathsf{Geo}(A_4, B)$. By Lemma 3.1 $\mathsf{Geo}(A_4, B)$ is not piecewise excluding. The only proper quotients of A_4 are abelian. Thus G cannot be a quotient of $SL_2(\mathbb{Z}/3\mathbb{Z})$.

The group $\mathbb{Z}/9\mathbb{Z} \rtimes_\beta \mathbb{Z}/3\mathbb{Z} =< x, y \,|\, x^9, y^3, yxy^{-1}x^{-4} >$ with the generating set $A = \{x, y\}^{\pm 1}$ has $xyx^{-1} \in \mathsf{Geo}(\mathbb{Z}/9\mathbb{Z} \rtimes_\beta \mathbb{Z}/3\mathbb{Z}, A)$. By Lemma 3.1 $\mathsf{Geo}(\mathbb{Z}/9\mathbb{Z} \rtimes_\beta \mathbb{Z}/3\mathbb{Z}, A)$ is not piecewise excluding. As nontrivial proper subgroups of $\mathbb{Z}/9\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z}$ have order either 3 or 3^2, proper nontrivial quotients of $\mathbb{Z}/9\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z}$ have order either 3^2 or 3 and thus are abelian. Hence G cannot be a quotient of $\mathbb{Z}/9\mathbb{Z} \rtimes_\beta \mathbb{Z}/3\mathbb{Z}$.

Lemma 3.6 shows that the group $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z} =< a, x \,|\, a^3, xax^{-1}a > \cong BS(1,2)/<a^3>^N$ and all its proper quotients are either abelian or have a finite symmetric generating set with geodesic language which is not piecewise excluding. Thus G cannot be a quotient of $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}$.

The group $\mathbb{Z}/5\mathbb{Z} \rtimes_\alpha \mathbb{Z} =< a, x \,|\, a^5, xax^{-1}a^2 >$ with the generating set $A = \{x, y\}^{\pm 1}$, where $y =_G x^3 a$, has $yxy^{-1} \in \mathsf{Geo}(\mathbb{Z}/5\mathbb{Z} \rtimes_\alpha \mathbb{Z}, A)$. By Lemma 3.1 $\mathsf{Geo}(\mathbb{Z}/5\mathbb{Z} \rtimes_\alpha \mathbb{Z}, A)$ is not piecewise excluding. Lemma 3.5 shows that all proper quotients of $\mathbb{Z}/5\mathbb{Z} \rtimes_\alpha \mathbb{Z}$ are either abelian or have a finite symmetric generating set with geodesic language which is not piecewise excluding. Thus G cannot be a quotient of $\mathbb{Z}/5\mathbb{Z} \rtimes_\alpha \mathbb{Z}$.

The group $BS(1, 2) =< a, t \,|\, tat^{-1}a^{-2} >$ with the generating set $A = \{a, t\}^{\pm 1}$ has $t^{-1}at \in \mathsf{Geo}(BS(1, 2), A)$. By Lemma 3.1 $\mathsf{Geo}(BS(1, 2), A)$ is not piecewise excluding. Lemma 3.6 shows that all proper quotients of $BS(1, 2)$ are either abelian or have a finite symmetric generating set with geodesic language which is not piecewise excluding. Thus G cannot be a quotient of $BS(1,2)$.

The quaternion group, Q_8, denoted by a double dagger, has piecewise excluding geodesic language for all finite symmetric generating sets by Proposition 3.4. The only proper quotients of Q_8 are abelian. Therefore, as all other possibilities lead to contradictions of our assumptions, the group G must be isomorphic to Q_8. $\square$

The class of groups with piecewise excluding geodesic languages for all finite symmetric generating sets does not even have one of the nicest closure properties one might hope for.

Proposition 3.8. *The class of groups which have piecewise excluding geodesic languages for all finite symmetric generating sets is not closed under direct products.*

Proof. Consider the generating set $A = \{i_1, j_1k_2, i_2, k_2\}^{\pm 1}$ for the group $G = Q_8 \times Q_8$, where i, j, k are as in the generating set for $Q_8 =< i, j, k \,|\, ijk^{-1}, jki^{-1}, kij^{-1}, i^4 >$ and the subscripts denote to which copy of Q_8 each belongs. Consider the element $g = i_1(j_1k_2)i_1^{-1} =_G i_1^2 j_1 k_2$. Note that $g \notin A$. If $g =_G ab$ for some $a, b \in A$, then exactly one of a, b must be $(j_1k_2)^{\pm 1}$ or $(k_2)^{\pm 1}$ and the other must be $i_1^{\pm 1}$ so that the projection into the second copy of Q_8 is k_2. But that forces the projection into the first copy of Q_8 to be one of $i_1^{\pm 1}, k_1^{\pm 1}$. Hence g cannot be written with fewer than three generators, and so $i_1(j_1k_2)i_1^{-1} \in \mathsf{Geo}(G, A)$. Thus $\mathsf{Geo}(G, A)$ is not piecewise excluding by Lemma 3.1. $\square$

Proposition 3.9. *Let $\mathbb{Z}^n =< x_i, ...x_n \,|\, [x_i, x_j]$ whenever $i \neq j >$ and let $G = \mathbb{Z}^n \rtimes_\phi \mathbb{Z}/2\mathbb{Z}$ for some $n \in \mathbb{N}$ with either (1) $\phi(x_i) = x_i^{-1}$ for some $i \in \{1, ..., n\}$ and $\phi(x_k) = x_k$ for all $k \in \{1, ..., n\} \setminus \{i\}$ or (2) $\phi(x_i) = x_j$ for some $i, j \in \{1, ..., n\}$ with $i \neq j$ and $\phi(x_k) = x_k$ for all $k \in \{1, ..., n\} \setminus \{i, j\}$. Then for every finite symmetric generating set of A of G, the geodesic language of G over A is not piecewise excluding. Moreover, there is a geodesic word over A containing both a generator and its inverse.*

Proof. Let $B = \{x_1, ..., x_n, y\}^{\pm 1}$, where $\mathbb{Z}/2\mathbb{Z} =< y >$, and let $N = \{x_1^{m_1} \cdots x_n^{m_n} y^{\epsilon} \mid m_i \in \mathbb{Z}$ for all $i \in \{1, ..., n\}$ and $\epsilon \in \{0, 1\}\}$, a set of normal forms for G over B. Let A be any finite symmetric generating set for G. For every word $w \in A^*$, let $\rho_N(w)$ be the unique word in N such that $\rho_N(w) =_G w$.

Case 1: $\phi(x_i) = x_i^{-1}$ for some $i \in \{1, ..., n\}$ and $\phi(x_h) = x_h$ for all $h \in \{1, ..., n\} \setminus \{i\}$.
Subcase A: Suppose there is a generator $\alpha \in A$ such that $\rho_N(\alpha) = x_1^{m_1} \cdots x_i^{m} \cdots x_n^{m_n}$ for some $m \in \mathbb{Z} \setminus \{0\}$.

Let $a \in A$ be the generator with $\rho_N(a) = x_1^{m_1} \cdots x_i^{m} \cdots x_n^{m_n}$ such that m is maximal. Note that $m > 0$ because A is symmetric and that there must be at least one generator in $\beta \in A$ such that $\rho_N(\beta) = x_1^{k_1} \cdots x_n^{k_n} y$. Let $b \in A$ be the generator with $\rho_N(b) = x_1^{k_1} \cdots x_i^{k} \cdots x_n^{k_n} y$ such that k is maximal. Suppose that aba^{-1} is not geodesic. Observe that $\rho_N(aba^{-1}) = x_1^{k_1} \cdots x_i^{2m+k} \cdots x_n^{k_n} y$.

subsubcase i: The word $aba^{-1} =_G \gamma$ for some $\gamma \in A$. Because $m > 0$ implies that $2m + k > k$, no generator $\gamma \in A$ with $\rho_N(\gamma) = x_1^{k_1} \cdots x_i^{2m+k} \cdots x_n^{k_n} y$ exists by maximality of k.

subsubcase ii: The word $aba^{-1} =_G \delta\zeta$ for some $\delta, \zeta \in A$ with $\rho_N(\delta) = x_1^{m'_1} \cdots x_i^{p} \cdots x_n^{m'_n}$ and $\rho_N(\zeta) = x_1^{k'_1} \cdots x_i^{q} \cdots x_n^{k'_n} y$. Then $p + q = 2m + k$. But the pair of inequalites $p \leq m$ and $q \leq k$ imply that $p + q \leq m + k < 2m + k$. Thus no such pair of generators $\delta, \zeta \in A$ exists.

subsubcase iii: The word $aba^{-1} =_G \zeta\delta$ for some $\delta, \zeta \in A$ with $\rho_N(\delta) = x_1^{m'_1} \cdots x_i^{p} \cdots x_n^{m'_n}$ and $\rho_N(\zeta) = x_1^{k'_1} \cdots x_i^{q} \cdots x_n^{k'_n} y$. Then $q - p = 2m + k$. But if $p \geq 0$, then $q - p \leq q \leq k < 2m + k$; if $p < 0$, then $|p| \leq m$ implies that $q - p \leq k + m < 2m + k$. Thus no such pair of generators $\delta, \zeta \in A$ exists.

Hence aba^{-1} is geodesic over A.

Subcase B: Suppose there is no generator $\alpha \in A$ such that $\rho_N(\alpha) = x_1^{m_1} \cdots x_i^{m} \cdots x_n^{m_n}$

for some $m \in \mathbb{Z} \setminus \{0\}$.

Let $a \in A$ be the generator with $\rho_N(a) = x_1^{m_1} \cdots x_i^{m} \cdots x_n^{m_n} y$ such that m is maximal. Let $b \in A$ be the generator with $\rho_N(b) = x_1^{k_1} \cdots x_i^{k} \cdots x_n^{k_n} y$ such that k is minimal. Note that $k \neq m$, as otherwise A generates only elements of G with the power of x_i in normal form either 0 or m. Suppose that aba^{-1} is not geodesic. Observe that $\rho_N(aba^{-1}) = x_1^{k_1} \cdots \ x_i^{2m-k} \cdots x_n^{k_n} y$. Let $\delta, \zeta \in A$ where $\rho_N(\delta) = x_1^{m'_1} \cdots x_i^{p} \cdots x_n^{m'_n} y^{\epsilon_1}$ and $\rho_N(\zeta) = x_1^{k'_1} \cdots x_i^{q} \cdots x_n^{k'_n} y^{\epsilon_2}$. Note that $\rho_N(\delta\zeta) = x_1^{m'_1+k'_1} \cdots x_i^{p+(-1)^{\epsilon_1} q} \cdots x_n^{m'_n+k'_n} y^{\epsilon_1+\epsilon_2 (\mathsf{mod}\, 2)}$ and that if $\epsilon_1 \neq 0$ [or $\epsilon_2 \neq 0$], then $p = 0$ [$q = 0$]. So $p + (-1)^{\epsilon_1} q \leq m$ whenever $\epsilon_1 + \epsilon_2 (\mathsf{mod}\, 2) \neq 0$. If aba^{-1} is equal in G to a generator $\gamma \in A$ with $\rho_N(\gamma) = x_1^{k_1} \cdots x_i^{2m-k} \cdots x_n^{k_n} y$ or $aba^{-1} =_G \delta\zeta$, we have a contradiction to our choices of a and b since $k < m$ implies that $2m - k > m$. Hence aba^{-1} is geodesic over A.

<u>Case 2</u>: $\phi(x_i) = x_j$ for some $i, j \in \{1, ..., n\}$ with $i \neq j$ and $\phi(x_h) = x_h$ for all $h \in \{1, ..., n\} \setminus \{i, j\}$.

Subcase A: Suppose there is a generator $\alpha \in A$ such that $\rho_N(\alpha) = x_1^{m_1} \cdots x_i^{m} \cdots x_j^{k} \cdots x_n^{m_n}$ for some $m \neq k \in \mathbb{Z}$.

Let $c \in A$ be the generator with $\rho_N(c) = x_1^{m_1} \cdots x_i^{m_0} \cdots x_j^{k_0} \cdots x_n^{m_n}$ such that $|m_0 - k_0|$ maximal. Note that $|m_0 - k_0| > 0$ by the assumption of this subcase and that there must be at least one generator $\beta \in A$ such that $\rho_N(\beta) = x_1^{k_1} \cdots x_n^{k_n} y$. Let $b \in A$ be the generator with $\rho_N(b) = x_1^{k_1} \cdots x_i^{p} \cdots x_j^{q} \cdots x_n^{k_n} y$ such that $|p - q|$ is maximal. If the signs of $m_0 - k_0$ and $p - q$ agree or if $p = q$, let $a = c$; otherwise, let $a = c^{-1}$. Let $m, k \in \mathbb{Z}$ be such that $\rho_N(a) =_G x_1^{m_1} \cdots x_i^{m} \cdots x_j^{k} \cdots x_n^{m_n}$. Suppose that aba^{-1} is not geodesic. Observe that $\rho_N(aba^{-1}) = x_1^{k_1} \cdots x_i^{m+p-k} \cdots x_j^{k+q-m} \cdots x_n^{k_n} y$. Becuase the signs of $m - k$ and $p - q$ do not disagree, the difference in the powers of x_i and x_j in $\rho_N(aba^{-1})$ is $|(m+p-k) - (k+q-m)| = |2(m-k) + (p-q)| = 2|m-k| + |p-q|$.

subsubcase i: The word $aba^{-1} =_G \gamma$ *for some* $\gamma \in A$. Because $|m-k| > 0$ implies that $2|m-k| + |p-q| > |p-q|$, no generator $\gamma \in A$ with $\rho_N(\gamma) = x_1^{k_1} \cdots x_i^{m+p-k} \cdots x_j^{k+q-m} \cdots x_n^{k_n} y$ exists by our choice of b.

subsubcase ii: The word $aba^{-1} =_G \delta\zeta$ *for some* $\delta, \zeta \in A$ *with* $\rho_N(\delta) = x_1^{m'_1} \cdots x_i^l \cdots x_j^r \cdots x_n^{m'_n}$ *and* $\rho_N(\zeta) = x_1^{k'_1} \cdots x_i^s \cdots x_j^t \cdots x_n^{k'_n} y$. Then $l+s = m+p-k$ and $r+t = k+q-m$. But the pair of inequalities $|l-r| \leq |m-k|$ and $|s-t| \leq |p-q|$ imply that $|(l+s)-(r+t)| \leq |m-k| + |p-q| < 2|m-k| + |p-q|$. Thus no such pair of generators $\delta, \zeta \in A$ exists.

subsubcase iii: The word $aba^{-1} =_G \zeta\delta$ *for some* $\delta, \zeta \in A$ *with* $\rho_N(\delta) = x_1^{m'_1} \cdots x_i^l \cdots x_j^r \cdots x_n^{m'_n}$ *and* $\rho_N(\zeta) = x_1^{k'_1} \cdots x_i^s \cdots x_j^t \cdots x_n^{k'_n} y$. Then $r+s = m+p-k$ and $l+t = k+q-m$. But the pair of inequalities $|l-r| \leq |m-k|$ and $|s-t| \leq |p-q|$ imply that $|(r+s)-(l+t)| \leq |m-k| + |p-q| < 2|m-k| + |p-q|$. Thus no such pair of generators $\delta, \zeta \in A$ exists.

Hence aba^{-1} is geodesic over A.

Subcase B: Suppose there is no generator $\alpha \in A$ *such that* $\rho_N(\alpha) = x_1^{m_1} \cdots x_i^m \cdots x_j^k \cdots x_n^{m_n}$ *for some* $m \neq k \in \mathbb{Z}$.

Let $a \in A$ be the generator with $\rho_N(a) = x_1^{m_1} \cdots x_i^m \cdots x_j^k \cdots x_n^{m_n} y$ such that $|m-k|$ is maximal. Let $b \in A$ be the generator with $\rho_N(b) = x_1^{k_1} \cdots x_i^p \cdots x_j^q \cdots x_n^{k_n} y$ such that $|p-q|$ is minimal. Note that $|m-k| \neq |p-q|$ as otherwise A would only generate elements of G with even differences in powers of x_i and x_j in normal forms without a y: the product $(x_1^{m'_1} \cdots x_i^c \cdots x_j^d \cdots x_n^{m'_n} y)(x_1^{k'_1} \cdots x_i^e \cdots x_j^f \cdots x_n^{k'_n} y) =_G x_1^{m'_1+k'_1} \cdots x_i^{c+f} \cdots x_j^{d+e} \cdots x_n^{m'_n+k'_n}$; if $|c-d| = |e-f|$ then $|(c+f)-(d+e)| = |(c-d)-(e-f)|$, which is either 0 or $2|c-d|$. Suppose that aba^{-1} is not geodesic. Observe that $\rho_N(aba^{-1}) = x_1^{k_1} \cdots x_i^{m+q-k} \cdots x_j^{k+p-m} \cdots x_n^{k_n} y$. Because $|m-k| > |q-p|$ implies that $(m-k)+(q-p)$ has the same sign as that of $m-k$, the difference in the powers of x_i and x_j in $\rho_N(aba^{-1})$ is $|(m+q-k)-(k+p-m)| = |(m-k)+(q-p)+(m-k)| = |(m-k)+$

$(q-p)|+|m-k| > |m-k|$. The word aba^{-1} must either be equal in G to a generator $\gamma \in A$ with $\rho_N(\gamma) = x_1^{k_1} \cdots x_i^{m+q-k} \cdots x_j^{k+p-m} \cdots x_n^{k_n} y$ or to a product of generators $\delta, \zeta \in A$ with $\rho_N(\delta) = x_1^{l_1} \cdots x_i^t \cdots x_j^t \cdots x_n^{l_n}$ and $\rho_N(\zeta) = x_1^{l_1'} \cdots x_i^r \cdots x_j^s \cdots x_n^{l_n'} y$ such that $t+r = m+q-k$ and $t+s = k+p-m$ (note that $\delta\zeta =_G \zeta\delta$). By our choice of a, the largest possible difference in the powers of x_i and x_j in $\rho_N(\gamma)$, $\rho_N(\delta\zeta)$, or $\rho_N(\zeta\delta)$ is $|m-k|$. Therefore neither such a generator γ nor such a pair of generators δ, ζ exists. Hence aba^{-1} is geodesic over A.

Thus $aba^{-1} \in \mathsf{Geo}(G, A)$ for the generators a and b defined in each subcase. By Lemma 3.1 $\mathsf{Geo}(G, A)$ cannot be piecewise excluding. $\square$

Corollary 3.10. *There are finitely generated virtually abelian groups whose geodesic language is not piecewise excluding for any finite symmetric generating set.*

Proposition 3.11. *Let G be an extension $1 \to H \to G \xrightarrow{\pi} K \to 1$ of finitely generated groups H and K and let A be a finite symmetric generating set for G. If awa^{-1} is geodesic in K over the generating set $\pi(A)$ for some $a \in \pi(A)$ and $w \in \pi(A)^*$, then the geodesic language of G over A is not piecewise excluding.*

Proof. Let $ab_1b_2 \cdots b_na^{-1} \in \mathsf{Geo}(K, \pi(A))$ for some $a \in \pi(A)$, $b_1, ..., b_n \in \pi(A)$. For each $x \in \pi(A)$, choose a unique preimage under π in A, denoted $\bar{x}$, such that $\overline{x^{-1}} = \bar{x}^{-1}$. If $\bar{a}\bar{b}_1\bar{b}_2 \cdots \bar{b}_n\bar{a}^{-1} \notin \mathsf{Geo}(G, A)$, then there is a word of length at most $n+1$ over A equal in G to $\bar{a}\bar{b}_1\bar{b}_2 \cdots \bar{b}_n\bar{a}^{-1}$, say it is $x_1x_2 \cdots x_k$. Then $\pi(x_1x_2 \cdots x_k) =_K \pi(x_1)\pi(x_2) \cdots \pi(x_k) =_K ab_1b_2 \cdots b_na^{-1}$, which implies that $ab_1b_2 \cdots b_na^{-1}$ is not geodesic, giving a contradiction. Thus $\bar{a}\bar{b}_1\bar{b}_2 \cdots \bar{b}_n\bar{a}^{-1}$ must be geodesic, and so by Lemma 3.1, $\mathsf{Geo}(G, A)$ cannot be piecewise excluding. $\square$

Corollary 3.12. *Let $\mathbb{Z}^n =< x_i, ...x_n \,|\, [x_i, x_j]$ whenever $i \neq j >$ and let $G = \mathbb{Z}^n \rtimes_\phi \mathbb{Z}/2\mathbb{Z}$ for some $n \in \mathbb{N}$ with either $\phi(x_i) = x_i^{-1}$ for some $i \in \{1, ..., n\}$ and $\phi(x_k) = x_k$ for all*

$k \in \{1, ..., n\} \setminus \{i\}$ or $\phi(x_i) = x_j$ for some $i, j \in \{1, ..., n\}$ with $i \neq j$ and $\phi(x_k) = x_k$ for all $k \in \{1, ..., n\} \setminus \{i, j\}$. Then any group with a quotient isomorphic to G has a geodesic language which is not piecewise excluding for any finite symmetric generating set.

Chapter 4

Autostackability of certain HNN extensions

We first observe that the closure results for HNN extensions of autostackable groups known to date do not, with cannonical choices for the graph of groups decomposition and the autostackable structure, imply autostackability of Stallings-like groups.

Remark 4.1. Right-angled Artin groups are autostackable by Lemma 2.9 and Theorem 2.7, so they are algorithmically stackable. A result of Meier and VanWyk [21] shows that the Bestvina-Brady subgroup is finitely generated when the underlying graph is connected. Because the subgroup membership problem for the Bestvina-Brady subgroup is decidable (by checking if the exponent sum of a word is 0), Theorem 1.6 proves that Stallings-like groups are algorithmically stackable.

Remark 4.2. The graph of groups decomposition $H*_{id_A}$ for a Stallings-like group with the flag complex associated to Λ not simply-connected does not satisfy the hypotheses of Theorem 1.4.

Proof. In order for H to be autostackable respecting A (see Definition 2.6), the group A itself must be autostackable, which implies that A must be finitely presented [4]. But when the flag complex associated to Λ is not simply connected, the group A is not finitely presented by The Bestvina-Brady Theorem [2]. $\square$

Proposition 4.3. *Let Z be the generating set and let $\mathcal{N}_H$ be the set of normal forms for the right-angled Artin group $H =< a, b, c, d \,|\, [a, c], [a, d], [b, c], [b, d] >$ induced by the finite convergent rewriting system given by Hermiller and Meier in [15]. Let A be the Bestvina-Brady subgroup of H and let $\mathcal{N}_{H/A} = \{a^n \,|\, n \in \mathbb{Z}\}$, a transversal for A in $H*_{id_A}$. Let $L_y = \{w \in \mathcal{N}_H \,|\, w =_H \mathsf{trans}_A(w)\mathsf{sub}_A(w)$ for some $\mathsf{trans}_A(w) \in \mathcal{N}_{H/A}$ and $\mathsf{sub}_A(w) \in SL_A \cap Y^*y\}$, where $\mathsf{trans}_A(w)$ and $\mathsf{sub}_A(w)$ are the unique elements of the transversal $\mathcal{N}_{H/A}$ and of the shortlex representatives of A over Y, SL_A, respectively, such that $w =_H \mathsf{trans}_A(w)\mathsf{sub}_A(w)$. Then L_y is not regular for any generating set $Y \subseteq Z$ of A and any total ordering of Y.*

Proof. The explicit rules of the finite convergent rewriting system for H can be found in section 2.5; the generating set is $Z = \{a, b, c, d, (ac), (ad), (bc), (bd), (ac^{-1}), (ad^{-1}), (bc^{-1}), (bd^{-1})\}^{\pm 1}$, where the parentheses indicate a single letter which is equal in H to the word contained inside. Let $Y = \{(ac^{-1}), (ad^{-1}), (bc^{-1}), (bd^{-1})\}^{\pm 1}$ and the ordering on Y be $(ac^{-1}) < (a^{-1}c) < (ad^{-1}) < (a^{-1}d) < (bc^{-1}) < (b^{-1}c) < (bd^{-1}) < (b^{-1}d)$.

We will show that the word $v_{n,m} = (bd^{-1})^n(ac^{-1})^m \in \mathcal{N}_H$, where $n, m \in (\mathbb{N} \cup 0)$, is in $L_{(a^{-1}c)}$ if and only if $m \leq n$, which proves that $L_{(a^{-1}c)}$ is not a regular language by the Pumping Lemma. First note that $\mathsf{trans}_A(v_{n,m}) = a^{2m}$ for all $n, m \in (\mathbb{N} \cup 0)$, so $\mathsf{sub}_A(v_{n,m}) =_H a^{-2m}b^n a^m d^{-n} c^m$. Let g denote an undetermined element of $\{a, b\}$ and let h denote an undetermined element of $\{c, d\}$; we will refer to these options as fillers. For example, to add an a to a word over Y, we may either use either (ac^{-1}) or (ad^{-1}); we denote this unmade choice by (ah^{-1}).

We first compute a lower bound on the length of $\mathsf{sub}_A(v_{n,m})$. Because the length of the subword of $\mathsf{sub}_A(v_{n,m})$ over $\{a, b\}^{\pm 1}$ is $3m+n$, the length of $\mathsf{sub}_A(v_{n,m})$ must be at least $3m+n$ as no letter of Y is equal in H to a word over $\{a, b, c, d\}^{\pm 1}$ containing both an $a^{\pm 1}$ and a $b^{\pm 1}$. The constraint of using only elements of Y, which have exponent sum zero over $\{a,b,c,d\}^{\pm 1}$ guarantees that another $\mathsf{min}\{m,n\}$ letters must be used to write $\mathsf{sub}_A(v_{n,m})$ over Y. We can overlap the subwords of $\mathsf{sub}_A(v_{n,m})$ over $\{a,b\}^{\pm 1}$ and over $\{c,d\}^{\pm 1}$ in two ways: each d^{-1} can be paired with either a b or an a, in which case we need an extra c^m at the end of the word since c and d do not commute; or each c can be paired with an a^{-1}, in which case we need an extra d^{-n} at the beginning of the word. The final $\mathsf{min}\{m,n\}$ necessary letters come from the requirement that $\mathsf{sub}_A(v_{n,m})$ must freely reduce to $a^{-2m}b^n a^m d^{-n} c^{-m}$. We consider the maximum possible cancellation in the fillers by looking at the exponent sum of all occurances of each type of filler (a nonzero exponent sum means that we need at least that many extra letters as all fillers must cancel if we are to have the correct word in H). In the first pairing option, for example $(a^{-1}h)^{2m}(bd^{-1})^n(ah^{-1})^m(g^{-1}c)^m$, the fillers labeled h have an exponent sum m and the fillers labeled g have an exponent sum $-m$. In the second pairing option, for example $(gd^{-1})^n(a^{-1}c)^m(a^{-1}h)^m(bh^{-1})^n(ah^{-1})^m$, the fillers labeled g have an exponent sum of $-n$ and the fillers labeled h have an exponent sum of n. Thus, a lower bound for the length of $\mathsf{sub}_A(v_{n,m})$ over Y is $3m+n+2\mathsf{min}\{m,n\}$.

Before we move on to cases, we eliminate construction choices for $\mathsf{sub}_A(v_{n,m})$ that will increase length or lexicographic weight unnecessarily. Shuffles of either pairing option, such as those containing the subword $(a^{-1}h)(gd^{-1})(a^{-1}h)$, create unnecessary length: to avoid having a b between two a^{-1}'s (so that we have the correct word in H), we need g to be an a; both occurences of h need to be a c so that we avoid having a d^{-1} between two c's; but then we have a generator followed by its inverse. Choosing filler options from $\{b,d\}^{\pm 1}$ when we could have chosen from $\{a,c\}^{\pm 1}$ instead increases the shortlex weight.

Now we find $\mathsf{sub}_A(v_{n,m})$, the shortlex least word over Y which is equal in H to $a^{-2m}b^n a^m d^{-n} c^m$.

Case A: Suppose that $m > n$.

We pair c^m with a^{-m} to minimize length. So in this case, we have the template $(gd^{-1})^n(a^{-1}c)^m(a^{-1}h)^m(bh^{-1})^n(ah^{-1})^m$. We choose the lexicographically least possible options for the fillers to get $(ad^{-1})^n(a^{-1}c)^m(a^{-1}c)^m(bc^{-1})^n(ac^{-1})^m$. To make our word equal in H to $a^{-2m}b^na^md^{-n}c^{-m}$, we add $(a^{-1}c)^n$ in the middle, for a resulting word of length $3m+3n$. So when $m > n$,

$$\mathsf{sub}_A(v_{n,m}) = (ad^{-1})^n(a^{-1}c)^{2m+n}(bc^{-1})^n(ac^{-1})^m$$

Case B: Suppose that $m \leq n$.

We pair d^{-n} with some of b^na^m to minimize length. So in this case, we have the template $(a^{-1}h)^{2m}(bh^{-1})^n(ah^{-1})^m(gd^{-1})^n(g^{-1}c)^m$, where the d^{-1}'s can be paired with either the b's or a's. We choose c as the filler option for the fillers paired with each a^{-1}. Then we need to cancel these extra $2m$ c's before the $(gd^{-1})^n$.

subcase i: $n < 2m$. Since $m \leq n$ implies that $2m - n \leq m$, we currently have $(a^{-1}c)^{2m}(bc^{-1})^n(ac^{-1})^{2m-n}(ah^{-1})^{n-m}(gd^{-1})^n\ (g^{-1}c)^m$. So we now pair $n-m$ of the $(gd^{-1})^n$ with $(ah^{-1})^{n-m}$ and then choose the lexicographically least options for the remaining fillers. We then have the word $(a^{-1}c)^{2m}(bc^{-1})^n(ac^{-1})^{2m-n}\ (ad^{-1})^n(a^{-1}c)^m$, which is equal in H to $a^{-2m}b^na^m\ d^{-n}c^{-m}$ and has length $5m+n$. So when $m \leq n < 2m$,

$$\mathsf{sub}_A(v_{n,m}) = (a^{-1}c)^{2m}(bc^{-1})^n(ac^{-1})^{2m-n}(ad^{-1})^n(a^{-1}c)^m$$

subcase ii: $n \geq 2m$. We use the first $2m$ of the fillers in $(bh^{-1})^n$ to get to $(a^{-1}c)^{2m}(bc^{-1})^{2m}(bd^{-1})^{n-2m}(gd^{-1})^{2m}(ah^{-1})^m(g^{-1}c)^m$. So we now pair $(ah^{-1})^m$ with m of the $(gd^{-1})^{2m}$ and then choose the lexicographically least options for the remaining fillers. We then have the word $(a^{-1}c)^{2m}(bc^{-1})^{2m}\ (bd^{-1})^{n-2m}(ad^{-1})^{2m}(a^{-1}c)^m$, which is equal in H to $a^{-2m}b^na^md^{-n}c^{-m}$ and has length $5m+n$. So when $n \geq 2m$

$$\mathsf{sub}_A(v_{n,m}) = (a^{-1}c)^{2m}(bc^{-1})^{2m}(bd^{-1})^{n-2m}(ad^{-1})^{2m}(a^{-1}c)^m$$

Because in the cases where $m \leq n$, the word $(bd^{-1})^n(ac^{-1})^m \in L_{(ac^{-1})}$ and $sub_A(w)$ is unique, the word $(bd^{-1})^n(ac^{-1})^m \notin L_{(a^{-1}c)}$.

Note that if we had instead chosen to use the smaller generating set $Y = \{(ac^{-1}), (ad^{-1}), (bc^{-1})\}^{\pm 1}$, cases A and B would still show that $L_{(a^{-1}c)}$ is not regular. If we had used a different ordering on Y or another generating set, a similar argument could be made to show that L_z is not regular for some $z \in \{(ac^{-1}), (ad^{-1}), (bc^{-1}), (bd^{-1})\}^{\pm 1}$.

□

Theorem 4.4. *Stallings-like groups are autostackable.*

Proof. This proof loosely follows that of [5, Theorem 4.1]. Let G be a Stallings-like group; that is, G is an HNN extension $H*_{id_A}$ where H is the right-angled Artin group associated to a connected finite simplicial graph Λ and A is the *Bestvina-Brady group* associated to Λ. Consider the finite convergent rewriting system for H given by Hermiller and Meier in [15] (see section 2.5 for details). Let Z be the set of generators and let R be the rules defined by this rewriting system. Let $\mathcal{N}_H$ be the set of irreducible words over R, a set of normal forms for H. Choose a designated vertex v_0 in Λ; let $a \in Z$ denote its corresponding generator. Then

$$\mathcal{N}_G = \{hs^{\epsilon_1}a^{i_1}s^{\epsilon_2}a^{i_2}\cdots s^{\epsilon_m}a^{i_m} \mid h \in \mathcal{N}_H,\ \epsilon_k \in \{\pm 1\} \text{ and } i_k \in \mathbb{Z} \text{ for all } k \text{ and}$$
$$\text{whenever } i_k = 0 \text{ then } \epsilon_k = \epsilon_{k+1}\}$$

is the Britton normal form set for the HNN extension G over the generating set $X = Z \cup \{s\}^{\pm 1}$. Let $\Gamma = \Gamma(G, X)$, with sets $\vec{E}$ and $\vec{P}$ of directed edges and paths, respectively, and let $\mathcal{T}$ be the tree in Γ corresponding to the set of normal forms $\mathcal{N}_G$.

G is stackable:

We first establish all the notation we will use to prove this. Because Λ is connected, we may define $M = \max\{\mathsf{d}_\Lambda(v_0, v) \mid v \in V(\Lambda)\}$ and $D_k = \{v \mid \mathsf{d}_\Lambda(v_0, v) = k\}$ for each $1 \leq k \leq M$, where the distance $\mathsf{d}_\Lambda(v, v')$ is the minimum number of edges in a path in Λ between v and v'. We define $\mathsf{adj}_\Lambda : V(\Lambda) \setminus \{v_0\} \to V(\Lambda)$ as follows. For each $v \in D_1$, define $\mathsf{adj}_\Lambda(v) = v_0$; for $2 \leq k \leq M$ and for each $v \in D_k$, let $\mathsf{adj}_\Lambda(v)$ be a vertex adjacent to v which is also an element of D_{k-1}. Let $\mathsf{adj}_Z : \{a_1, ..., a_n\} \setminus \{a\} \to \{a_1, ..., a_n\}$ be defined by $\mathsf{adj}_Z(a_i) = a_j$ where $\mathsf{adj}_\Gamma(v_i) = v_j$.

For all $g \in G$, let y_g represent the normal form of g in $\mathcal{N}_G$. For each $z, z' \in Z$ with $zz' \notin \mathcal{N}_H$, let $\overline{zz'}$ be the result of applying the one applicable rewriting rule from R which changes the leftmost entries of z and z'. For each $u \in Z^*$, let $\mathsf{r}(u)$ be the minimum number of rewrites under R required to put u into normal form. For each $w \in X^*$, let $\mathsf{n}_s(w)$ denote the number of occurences of s and s^{-1} in w, let $\mathsf{suff}_a(w)$ be the length of the maximal suffix of w over $\{a\}^{\pm 1}$, and let $l(w)$ be the length of w over X. For each nonempty word $w \in X^*$, let $\mathsf{first}(w)$ be the first letter in w, and let $\mathsf{last}(w)$ be the last letter in w. For each generator $z \in Z$, where $z =_H a_1^{\epsilon_1} \cdots a_n^{\epsilon_n}$ and $\epsilon_i \in \{-1, 0, 1\}$ for each $i \in \{1, ..., n\}$, let $\mathsf{wt}(z)$ be the number of indices where $\epsilon_i \neq 0$ and let $\mathsf{break}(z)$ be the word $a_1^{\epsilon_1} \cdots a_n^{\epsilon_n}$, where any appearence of a_i^0 is replaced by the empty word.

Define $\phi : \mathcal{N}_G \times X \to X^*$ by, for all $g \in G$ and $x \in X$, $\phi(y_g, x) =$

$$
\begin{cases}
x & \text{if either } y_g x \in \mathcal{N}_G \text{ or } y_{gx} x^{-1} \in \mathcal{N}_G \\
\mathsf{last}(y_g)^{-1}\overline{\mathsf{last}(y_g)x} & \text{if } x \in Z,\ y_g \in Z^*, \text{ and } y_g x, y_{gx}x^{-1} \notin \mathcal{N}_H \\
\mathsf{break}(x) & \text{if } y_g \notin Z^* \text{ and } x \in Z \text{ with } \mathsf{wt}(x) > 1 \\
\mathsf{adj}_Z(x)^{-\eta} x \mathsf{adj}_Z(x)^{\eta} & \text{if } y_g \notin Z^*,\ x = a_i^{\pm 1} \text{ where } v_i \in D_k \text{ for some } k \geq 1, \text{ and} \\
 & \qquad \mathsf{last}(y_g) = a^{\eta} \text{ where } \eta \in \{\pm 1\} \\
\mathsf{last}(y_g)^{-1} x a^{-\eta} \mathsf{last}(y_g) a^{\eta} & \text{if } x = a_i^{\eta} \text{ where } \eta \in \{\pm 1\} \text{ and } v_i \in D_k \text{ for some } k \geq 1, \\
 & \qquad \text{and } \mathsf{last}(y_g) \in \{s\}^{\pm 1}.
\end{cases}
$$

Note that these cases are disjoint, so ϕ is well-defined. In all cases which do not appear explicitly, either $y_g x$ or $y_g x^{-1}$ lies in $\mathcal{N}_G$.

Let $\Phi : \vec{E} \to \vec{P}$ be defined by $\Phi(e_{g,x}) = \mathsf{path}(y_g, \phi(y_g, x))$. Note that properties (F1) and (F2d) hold for Φ by our definition. To prove (F2r) also holds, we use the following function $\Psi : \vec{E} \to \mathbb{N}^4$. If $e_{g,x}$ lies in $\mathcal{T}$, define $\Psi(e_{g,x}) = (0,0,0,0)$; otherwise (note that this implies that $x \in Z$) define $\Psi(e_{g,x}) =$

$$\begin{cases} (0,0,0,\mathsf{r}(y_g x)) & \text{if } y_g \in Z^* \text{ and } y_g x, y_{gx} x^{-1} \notin \mathcal{N}_H \\ (\mathsf{n}_s(y_g), 1, l(\mathsf{suff}_a(y_g)), k-1) & \text{if } y_g \notin Z^* \text{ and } x \in Z \text{ with } \mathsf{wt}(x) > 1 \text{ where, if } x =_H \\ & \qquad a_1^{\epsilon_1} \cdots a_n^{\epsilon_n},\ k = \max\{m \mid v_i \in D_m \text{ and } \epsilon_i \neq 0\} \\ (\mathsf{n}_s(y_g), 0, l(\mathsf{suff}_a(y_g)), k-1) & \text{if } y_g \notin Z^*,\ x = a_i^{\pm 1}, \text{ and } v_i \in D_k \text{ for some } k \geq 1. \end{cases}$$

Let $<_{\mathbb{N}^4}$ denote the lexicographic ordering on $\mathbb{N}^4$ obtained from the standard ordering on $\mathbb{N}$, a well-founded strict partial ordering. To prove (F2r), it suffices to show that $e' <_\Phi e$ implies that $\Psi(e') <_{\mathbb{N}^4} \Psi(e)$. Let $e_{g,x} \in \vec{E}$ be an edge whose underlying undirected edge does not lie in $\mathcal{T}$.

Case 1: Suppose that $y_g \in Z^*$ and $y_g x, y_{gx} x^{-1} \notin \mathcal{N}_H$.

In this case $\Psi(e_{g,x}) = (0,0,0,\mathsf{r}(y_g x))$ and the path $\Phi(e_{g,x})$ contains two or three edges, depending on the length of $\overline{\mathsf{last}(y_g)x}$. In the subcase of three edges: $e_1 = e_{g,\mathsf{last}(y_g)^{-1}}$, $e_2 = e_{g\mathsf{last}(y_g)^{-1},\mathsf{first}(\overline{\mathsf{last}(y_g)x})}$, and $e_3 = e_{g\mathsf{last}(y_g)^{-1}\mathsf{first}(\overline{\mathsf{last}(y_g)x}),\mathsf{last}(\overline{\mathsf{last}(y_g)x})}$; in the subcase of two edges: $e_1' = e_{g,\mathsf{last}(y_g)^{-1}} = e_1$ and $e_2' = e_{g\mathsf{last}(y_g)^{-1},\overline{\mathsf{last}(y_g)x}}$. In both subcases, e_1 lies in $\mathcal{T}$.

As y_g is in normal form, the only place a rewriting rule can be applied is to $\mathsf{last}(y_g)x$, so there is a fixed number of rewrites that need to occur to put $y_g x$ into normal form (push everything as far left as possible in each coordinate). So in the first subcase, $\mathsf{r}(y_{g\mathsf{last}(y_g)^{-1}}\mathsf{first}(\overline{\mathsf{last}(y_g)x})) < \mathsf{r}(y_g x)$ and $\mathsf{r}(y_{g\mathsf{last}(y_g)^{-1}\mathsf{first}(\overline{\mathsf{last}(y_g)x})}\mathsf{last}(\overline{\mathsf{last}(y_g)x})) < \mathsf{r}(y_g x)$; in the second subcase $\mathsf{r}(y_{g\mathsf{last}(y_g)^{-1}}\overline{\mathsf{last}(y_g)x}) = \mathsf{r}(y_g x) - 1$. Hence each edge e_i in the path $\Phi(e_{g,x})$ which is not in $\mathcal{T}$ has $\Psi(e) <_{\mathbb{N}^4} \Psi(e_{g,x})$.

Case 2: Suppose that $y_g \notin Z^*$ *and* $x \in Z$ *with* $\mathsf{wt}(x) > 1$.

In this case $\Psi(e_{g,x}) = (\mathsf{n}_s(y_g), 1, l(\mathsf{suff}_a(y_g)), k-1)$ where, if $x =_H a_1^{\epsilon_1} \cdots a_n^{\epsilon_n}$, $k = \mathsf{max}\{m \mid v_i \in D_m \text{ and } \epsilon_i \neq 0\}$ and the path $\Phi(e_{g,x})$ contains $\mathsf{wt}(x)$ edges. Note that $2 \leq \mathsf{wt}(x) \leq n$, so $\Phi(e_{g,x})$ has length at most n. Each edge e_i in the path $\Phi(e_{g,x})$ has $\Psi(e_i)$ with first entry $\mathsf{n}_s(y_g)$ and second entry 0. Hence each edge e_i in $\Phi(e_{g,x})$ which is not in $\mathcal{T}$ has $\Psi(e_i) <_{\mathbb{N}^4} \Psi(e_{g,x})$.

Case 3: Suppose that $y_g \notin Z^*$, $x = a_i^{\pm 1}$ *with* $v_i \in D_k$ *for some* $k \geq 1$, *and* $\mathsf{last}(y_g) = a^\eta$ *where* $\eta \in \{\pm 1\}$.

In this case $\Psi(e_{g,x}) = (\mathsf{n}_s(y_g), 0, l(\mathsf{suff}_a(y_g)), k-1)$ and the path $\Phi(e_{g,x})$ is $e_1e_2e_3$ where $e_1 = e_{g,\mathsf{adj}_Z(x)^{-\eta}}$, $e_2 = e_{g\mathsf{adj}_Z(x)^{-\eta},x}$, and $e_3 = e_{g\mathsf{adj}_Z(x)^{-\eta}x,\mathsf{adj}_Z(x)^\eta}$. If $k = 1$, then e_1 lies in $\mathcal{T}$. If $k > 1$, then $\Psi(e_1) = (\mathsf{n}_s(y_g), 0, l(\mathsf{suff}_a(y_g)), k-2)$ as $v_i \in D_k$ and $\mathsf{adj}_\Lambda(v_i) \in D_{k-1}$.

As the suffix over $\{a\}^{\pm 1}$ of $y_{ws^\epsilon a^{\eta m} z^{-\eta}}$ for any $w \in X^*$ and $z \in Z$ with $\mathsf{wt}(z) = 1$ is $a^{\eta(m-1)}$, then $\Psi(e_2) = (\mathsf{n}_s(y_g), 0, l(\mathsf{suff}_a(y_g)) - 1, k-1)$. Let $x = a_i^\epsilon$. If $\epsilon = \eta$, we have that $\Psi(e_3) = (\mathsf{n}_s(y_g), 0, l(\mathsf{suff}_a(y_g)), k-2)$; otherwise $\Psi(e_3) = (\mathsf{n}_s(y_g), 0, l(\mathsf{suff}_a(y_g)) - 2, k-2)$. Hence each edge e_i in $\Phi(e_{g,x})$ which is not in $\mathcal{T}$ has $\Psi(e) <_{\mathbb{N}^4} \Psi(e_{g,x})$.

Case 4: Suppose that $x = a_i^\eta$ *where* $\eta \in \{\pm 1\}$ *with* $v_i \in D_k$ *for some* $k \geq 1$ *and* $\mathsf{last}(y_g) \in \{s\}^{\pm 1}$.

In this case $\Psi(e_{g,x}) = (\mathsf{n}_s(y_g), 0, l(\mathsf{suff}_a(y_g)), k-1)$ and the path $\Phi(e_{g,x})$ is $e_1e_2e_3e_4e_5$ where $e_1 = e_{g,\mathsf{last}(y_g)^{-1}}$, $e_2 = e_{g\mathsf{last}(y_g)^{-1},x}$, $e_3 = e_{g\mathsf{last}(y_g)^{-1}x,a^{-\eta}}$, $e_4 = e_{g\mathsf{last}(y_g)^{-1}xa^{-\eta},\mathsf{last}(y_g)}$, and $e_5 = e_{g\mathsf{last}(y_g)^{-1}xa^{-\eta}\mathsf{last}(y_g),a^\eta}$. Note that e_1 lies in $\mathcal{T}$. Because $\mathsf{last}(y_g) \in \{s\}^{\pm 1}$, the number $\mathsf{n}_s(y_{g\mathsf{last}(y_g)^{-1}}) = \mathsf{n}_s(y_g) - 1$. So $\Psi(e_i)$ has first entry $\mathsf{n}_s(y_g) - 1$ for $i \in \{2,3,4\}$. Since $\mathsf{last}(y_{g\mathsf{last}(y_g)^{-1}xa^{-\eta}\mathsf{last}(y_g)}) \in \{s^{\pm 1}\}$ and the edge e_5 is labeled by a^η, the edge e_5 lies in $\mathcal{T}$. Hence each edge e_i in $\Phi(e_{g,x})$ which is not in $\mathcal{T}$ has $\Psi(e) <_{\mathbb{N}^4} \Psi(e_{g,x})$.

Thus Φ is a flow function with bounding constant $\max\{5, n\}$.

G is autostackable:

The function ϕ defined earlier in the proof is the stacking function associated to the bounded flow function Φ. The graph of the stacking map is

$\mathsf{graph}(\phi) = \{(y_g, x, \phi(y_g, x) \mid g \in G, x \in X\} =$

$(\bigcup_{x \in X} L_x \times \{x\} \times \{x\}) \quad \cup$

$(\bigcup_{x \in Z} L'_x \times \{x\} \times \{\mathsf{last}(y_g)^{-1}\overline{\mathsf{last}(y_g)x}\}) \quad \cup$

$(\bigcup_{\{x \in Z \,|\, \mathsf{wt}(x) > 1\}} L''_x \times \{x\} \times \{\mathsf{break}(x)\}) \quad \cup$

$(\bigcup_{k \in \{1,\ldots,M\}, \eta \in \{\pm 1\}, x \in \{(1,\ldots,1,a_i^{\pm 1},1,\ldots,1) \in Z \,|\, v_i \in D_k\}} L_{k,\eta,x} \times \{x\} \times \{\mathsf{adj}_Z(x)^{-\eta} x \mathsf{adj}_Z(x)^\eta\}) \;\cup$

$(\bigcup_{k \in \{1,\ldots,M\}, \eta \in \{\pm 1\}, x \in \{(1,\ldots,1,a_i^{\eta},1,\ldots,1) \in Z \,|\, v_i \in D_k\}, z \in \{s^{\pm 1}\}} L_{k,\eta,x,z} \times \{x\} \times \{z^{-1} x a^{-\eta} z a^\eta\})$

where

$L_x = \{y_g \in \mathcal{N}_G \,|\, y_g x \in \mathcal{N}_G \text{ or } y_{gx} x^{-1} \in \mathcal{N}_G\}$,

$L'_x = \{y_g \in \mathcal{N}_G \,|\, y_g \in Z^* \text{ and } y_g x, y_{gx} x^{-1} \notin \mathcal{N}_H\}$,

$L''_x = \{y_g \in \mathcal{N}_G \,|\, y_g \notin Z^*\}$,

$L_{k,\eta,x} = \{y_g \in \mathcal{N}_G \,|\, y_g \notin Z^* \text{ and } \mathsf{last}(y_g) = a^\eta\}$, and

$L_{k,\eta,x,z} = \{y_g \in \mathcal{N}_G \,|\, \mathsf{last}(y_g) = z\}$.

By Remark 2.3 and Lemma 2.4, it suffices to show that each of L_x, L'_x, L''_x, $L_{k,\eta,x}$, and $L_{k,\eta,x,z}$ is regular in order to prove $\mathsf{graph}(\phi)$ is synchronously regular. We can write the set of normal forms as $\mathcal{N}_G = X^* \setminus X^* M X^*$ where M is the regular language defined by $\{xx^{-1} \mid x \in X\} \cup \{h \mid h \in Z^* \setminus \mathcal{N}_H\} \cup s^{\pm 1}(\{a\}^* \cup \{a^{-1}\}^*)\{z \mid z \in Z \setminus \{a, a^{-1}\}\}$. Then

$L_x = (\mathcal{N}_G/x) \cup (\mathcal{N}_G \cap X^* x^{-1})$,

$L'_x = (\mathcal{N}_G \cap Z^*) \setminus L_x$,

$L''_x = \mathcal{N}_G \setminus Z^*$,

$L_{k,\eta,x} = (\mathcal{N}_G \cap X^* a^\eta) \setminus Z^*$, and

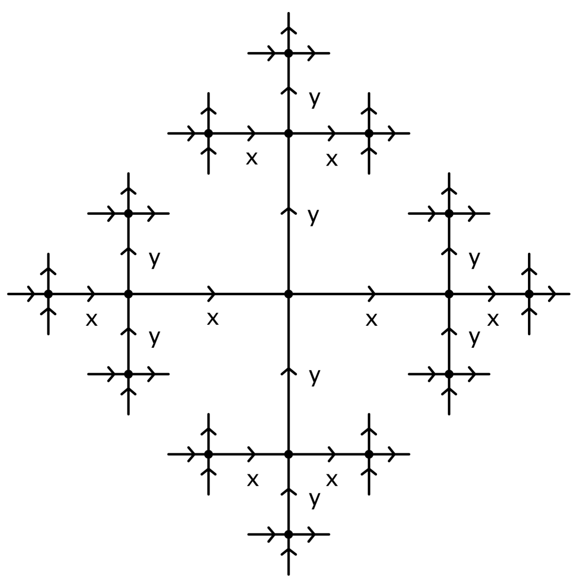
$L_{k,\eta,x,z} = \mathcal{N}_G \cap X^*\{z\}$.

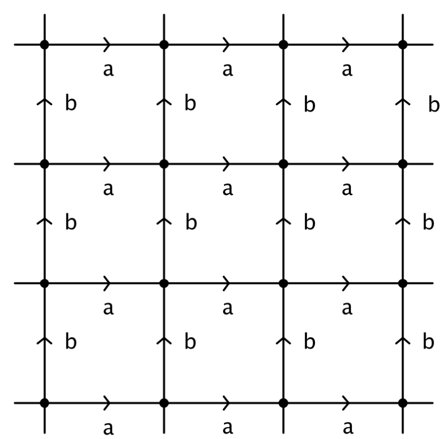
Hence, by Lemma 2.5 and closure properties of regular languages, each language is regular. $\square$

Appendix

Let $F_2 =< x, y \mid \ >$ and $\mathbb{Z}^2 =< a, b \mid [a, b] >$. Then $\mathsf{Geo}(F_2, \{x, y\}^{\pm}1) = \{$freely reduced words over $\{x, y\}^{\pm 1}\}$ and $\mathsf{Geo}(\mathbb{Z}^2, \{a, b\}^{\pm 1}) = \{$words over $\{a, b\}^{\pm 1}$ which do not contain both a generator and its inverse$\}$.

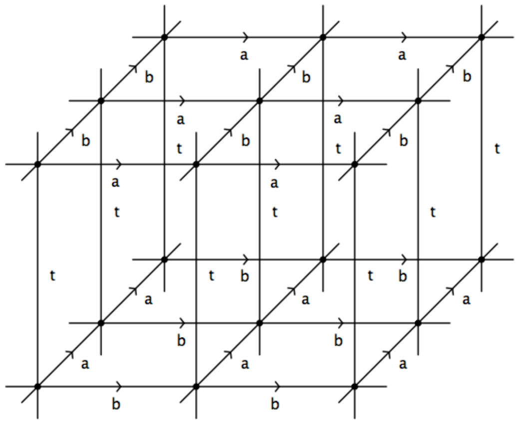
$\Gamma(F_2, \{x, y\}^{\pm 1})$: $\Gamma(\mathbb{Z}^2, \{a, b\}^{\pm 1})$:



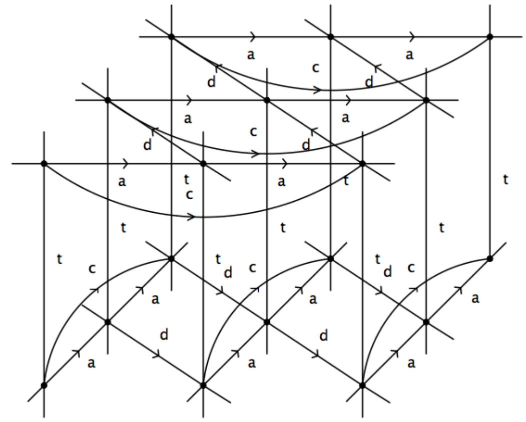


The group $\mathbb{Z}^2 \rtimes {}^{\mathbb{Z}}\!/\!{}_{2\mathbb{Z}} =< a, b, t \mid [a, b], t^2, tatb^{-1} >$ has regular geodesic language with the generating set $\{a, b, t\}^{\pm 1}$ but not with the generating set $\{a, d, c, t\}^{\pm 1}$, where $c =_G a^2$ and $d =_G ab$.

$\Gamma(\mathbb{Z}/2\mathbb{Z}, \{a, b, t\}^{\pm 1})$:

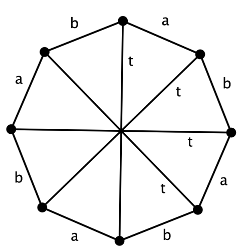


$\Gamma(\mathbb{Z}/2\mathbb{Z}, \{a, c, d, t\}^{\pm 1})$:

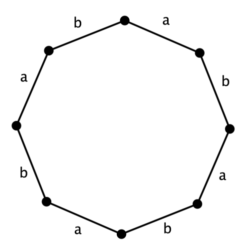


The group $D_8 =< a, b, t \mid a^2, b^2, (ab)^4, ababt >$ with generating set $A = \{a, b, t\}$ has $\mathsf{Geo}(D_8, A) = A^* \setminus (\{A^*xA^*xA^* \mid x \in A\} \cup \{A^*xA^*yA^*zA^* \mid x, y, z \in A\})$; with generating set $B = \{a, b\}$, the word $aba^{-1} \in \mathsf{Geo}(D_8, B)$.

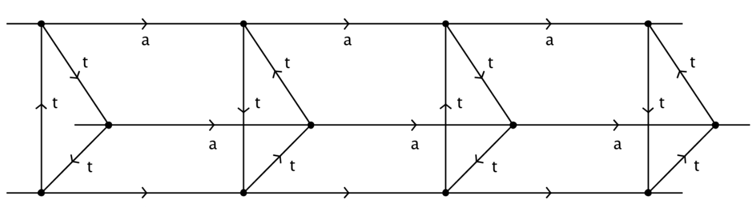
$\Gamma(D_8, A)$:



$\Gamma(D_8, B)$:

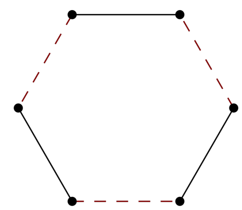


The group $G =< a, t \mid t^3 >$ with generating set $A = \{a, t\}^{\pm 1}$ has $\mathsf{Geo}(G, A) = A^* \setminus ([\bigcup_{\epsilon \in \{\pm 1\}} A^*a^{\epsilon}A^*a^{-\epsilon}A^*] \cup [\bigcup_{\epsilon, \delta \in \{\pm 1\}} A^*t^{\epsilon}A^*t^{\delta}A^*])$. $\Gamma(G, \{a, t\}^{\pm 1})$:

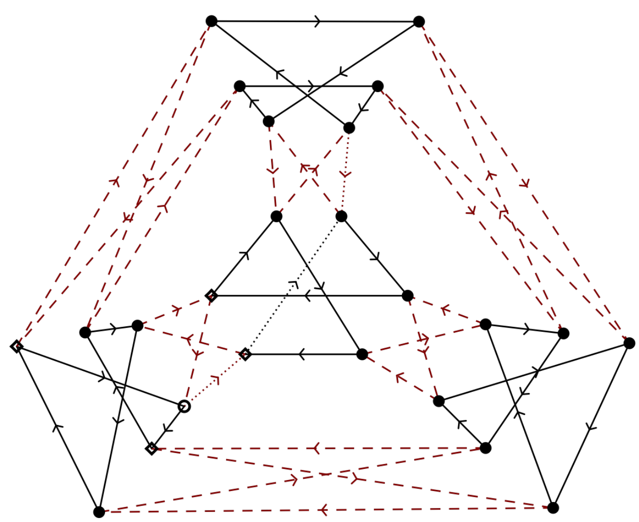


In each Cayley graph below, the edges labeled by a and y are solid and the edges labeled by b, x, and t are dashed. In all but the graph for S_3, a path with label equal to the stated geodesic word is dashed; the initial vertex of the path is denoted with an open circle and its adjacent vertices are denoted by open squares.

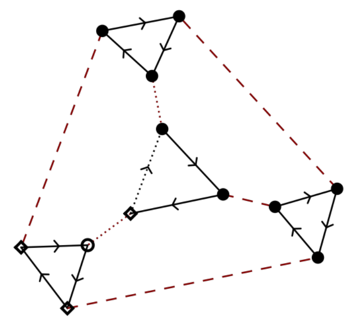
The group $S_3 =< a, b \mid a^2, b^2, (ab)^3 >$ with generating set $A = \{a, b\}^{\pm 1}$ has $aba^{-1} \in \mathsf{Geo}(S_3, A)$. $\Gamma(S_3, \{a, b\}^{\pm 1})$:



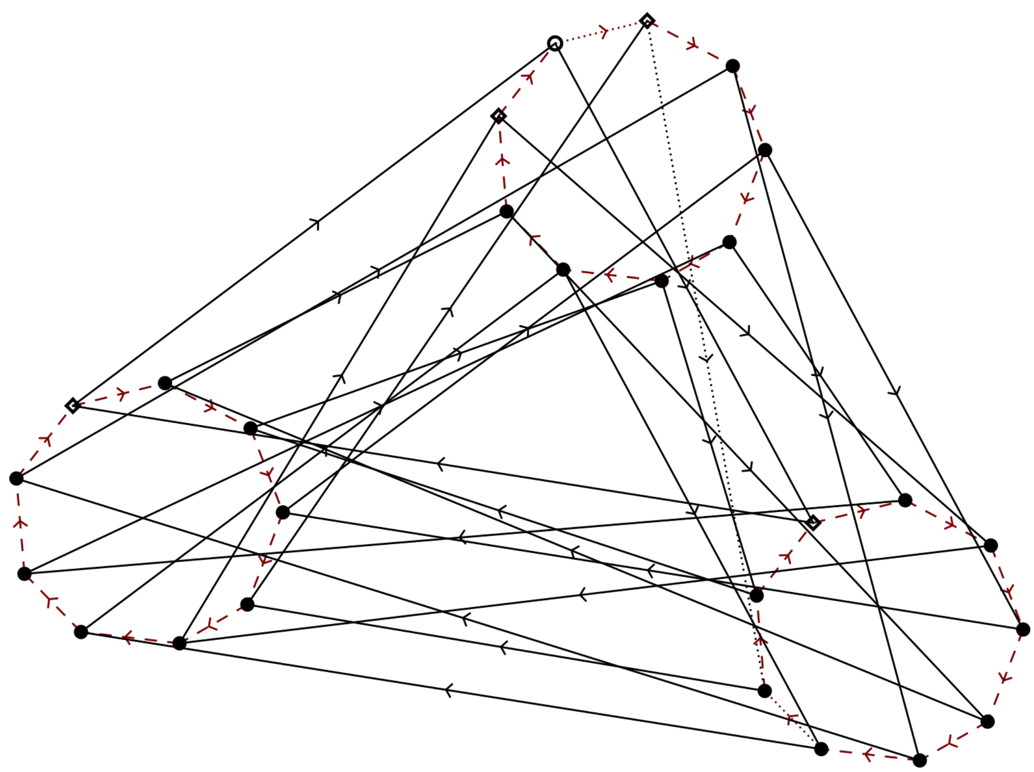
The group $SL_2(\mathbb{Z}/3\mathbb{Z}) =< a, b \mid a^6, b^4, ab^{-1}ab^{-1}ab >$ with generating set $A = \{a, b\}^{\pm 1}$ has $bab^{-1} \in \mathsf{Geo}(SL_2(\mathbb{Z}/3\mathbb{Z}), A)$. $\Gamma(SL_2(\mathbb{Z}/3\mathbb{Z}), \{a, b\}^{\pm 1})$:



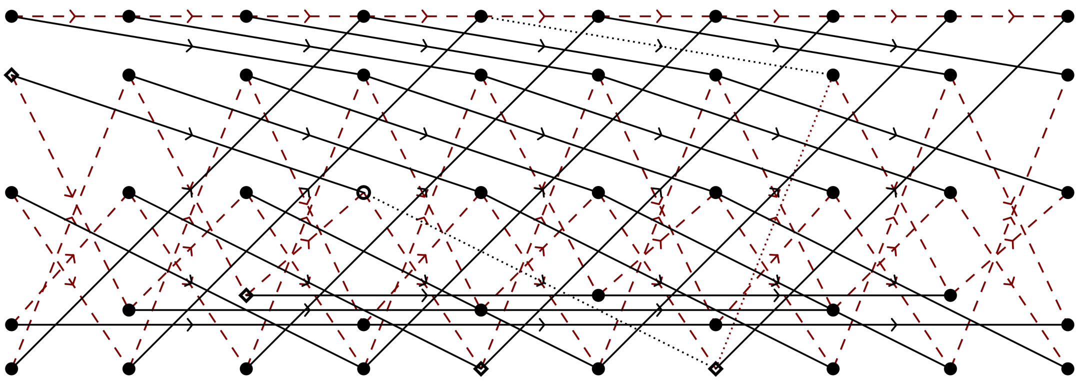
The group $A_4 =< a, b \,|\, a^3, b^2, (ab)^3 >$ with generating set $B = \{a, b\}^{\pm 1}$ has $bab^{-1} \in$ $\mathsf{Geo}(A_4, B)$. $\Gamma(A_4, \{a, b\}^{\pm 1})$:



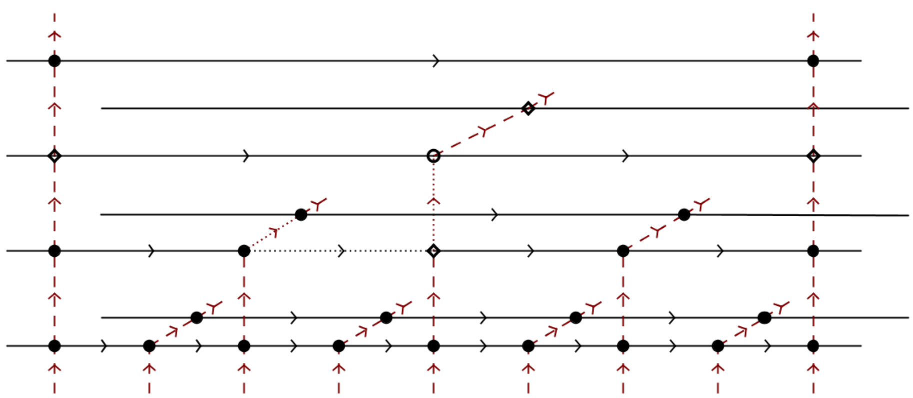
The group $\mathbb{Z}/9\mathbb{Z} \rtimes_\beta \mathbb{Z}/3\mathbb{Z} =< x, y \,|\, x^9, y^3, yxy^{-1}x^{-4} >$ with generating set $A = \{x, y\}^{\pm 1}$ has $xyx^{-1} \in \mathsf{Geo}(\mathbb{Z}/9\mathbb{Z} \rtimes_\beta \mathbb{Z}/3\mathbb{Z}, A)$. $\Gamma(\mathbb{Z}/9\mathbb{Z} \rtimes_\beta \mathbb{Z}/3\mathbb{Z}, \{x, y\}^{\pm 1})$:



The group $\mathbb{Z}/5\mathbb{Z} \rtimes_\alpha \mathbb{Z} =< a, x \,|\, a^5, xax^{-1}a^2 >$ with generating set $A = \{x, y\}^{\pm 1}$, where $y =_G x^3 a$, has $yxy^{-1} \in \mathsf{Geo}(\mathbb{Z}/5\mathbb{Z} \rtimes_\alpha \mathbb{Z}, A)$. $\Gamma(\mathbb{Z}/5\mathbb{Z} \rtimes_\alpha \mathbb{Z}, \{x, y\}^{\pm 1})$:



The group $BS(1,2) =< a, t \,|\, tat^{-1}a^{-2} >$ with generating set $A = \{a, t\}^{\pm 1}$ has $t^{-1}at \in \mathsf{Geo}(BS(1,2), A)$. $\Gamma(BS(1,2), \{a, t\}^{\pm 1})$:



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