7-2015

Local and Nonlocal Models in Thin-Plate and Bridge Dynamics

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LOCAL AND NONLOCAL MODELS IN THIN-PLATE AND BRIDGE DYNAMICS

by

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A DISSERTATION

Presented to the Faculty of

The Graduate College at the University of Nebraska

In Partial Fulfilment of Requirements

For the Degree of Doctor of Philosophy

Major: Mathematics

Under the Supervision of Professors Petronela Radu and Daniel Toundykov

Lincoln, Nebraska

July, 2015

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University of Nebraska, 2015

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This thesis explores several models in continuum mechanics from both local and nonlocal perspectives. The first portion settles a conjecture proposed by Filippo Gazzola and his collaborators on the finite-time blow-up for a class of fourth-order differential equations modeling suspension bridges. Under suitable assumptions on the nonlinearity and the initial data, a finite-time blowup is demonstrated as a result of rapid oscillations with geometrically growing amplitudes. The second section introduces a nonlocal peridynamic (integral) generalization of the biharmonic operator. Its action converges to that of the classical biharmonic as the radius of nonlocal interactions—the "horizon"—tends to zero. For the corresponding steady state problem, which represents a peridynamic analog of a hinged or clamped plate under load, the existence and uniqueness are shown. By utilizing a compactness result devised by Jean Bourgain, Haïm Brezis, and Petru Mironescu and employing a method developed by Qiang Du and Tadele Mengesha, it is demonstrated that as the horizon tends to zero, the solutions of the nonlocal boundary value problems converge strongly in L^2 to the solutions of the corresponding classical elliptic problems.

DEDICATION

I would like to dedicate this work to my grandparents Irene and Sam Trageser. I wish you two could have been here to see its completion.

ACKNOWLEDGMENTS

My journey towards my doctorate has been an amazing experience. It was not always a pleasant experience, but throughout it all I have always had amazing people there to support and guide me.

I would like to first thank my advisers Petronela Radu and Daniel Toundykov. I could not have asked for better mentors. Their unending support and insights have made the present work possible and given me a strong foundation on which to build my career. I have no doubt a large part of my success is due to them and for that I am eternally grateful.

I met my wife Yu shortly before starting graduate school. Because of her, I can truthfully say it has been some of the best times of my life and I know the best is yet to come. I would like to sincerely thank her for her boundless support and encouragement. Her organizational skills turned a mess of deadlines and scheduling into a smooth process culminating in my degree and a research postdoctoral position. When I first began graduate school I had no idea where I would end up, but with her at my side I know anything is possible. I cannot thank her enough for helping me and always being there when things got rough.

I would like to thank my family. Firstly, my parents Marty and Jerry whose guidance, faith, and encouragement have made me the person I am today. They have always put me first (sorry Chris, but I am the first born) and been an incredible influence in my life. I really appreciate all of the sacrifices they have selflessly made for me. I would also like to thank my brother Chris whose advice and support is always fantastic.

Next I would like to thank my committee members: George Avalos, Florin Bobaru, Mikil Foss, Marta Lewicka, and Mohammad Rammaha for their comments and suggestions. They are an amazing group of people and I am lucky to have had all of them on my committee.

My career in mathematics did not begin in graduate school and I definitely need to thank Griff Elder, Steve From, and Andrzej Roslanowski. They were the first people to open me up to the beautiful world of mathematics and for that I am truly grateful.

I would also like to thank the UNL office staff (Liz, Lori, Tom, Rex, and Marilyn) for all their help with paperwork, scheduling, advice, and making graduate school a little less stressful.

Lastly, I would like to thank my friends from graduate school. I will only mention a couple, but I am so appreciative of all of my friends and am so lucky to have their friendship. First I want to thank Ethan Twisdale, whose lousy ping pong skills were always there to serve as a pickup when I felt stressed. In all seriousness though, he is a great friend and was an immense help getting through this last year. Next, I would like to thank my academic sister Pei Pei (Daniel's second favorite student) for all of her advice and encouragement.

Contents

Chapter 1

Introduction

Thin plates and curved shells are ubiquitous components in many natural and man-made structures. For example, bridges, ship hulls, domes, membranes, and airfoils can all be described and analyzed with plate/shell theory. This exposition will investigate certain aspects of thin-plate dynamics from both "local" (classical, or differential) and "nonlocal" (interal) perspectives. Respectively, we begin our investigation with the local perspective in Section 2.1 which provides an overview of the literature and a brief introduction to the rich theory surrounding suspension bridge modeling. Our journey in the local setting continues in Chapter 2, where we delve into the analysis of a specific fourth-order equation describing bridge oscillations. Various invariants, energies, and technical estimates related to the equation are developed, culminating with the proof of a previously open conjecture proposed by Filippo Gazzola pertaining to blowup of traveling wave solutions of the equation.

No discussion of suspension bridges (or many other thin-plate systems) would be complete without mentioning material failure, an inevitable occurrence in many structures. Unfortunately, modeling the formation and propagation of fractures remains a challenging area of research; however, a recently developed theory called peridynamics provides a

method of modeling material fatigue and system dynamics within a single unified solution by utilizing nonlocal operators. In this nonlocal setting, points are allowed to "interact" with one another over a finite distance called the horizon. Chapter 3 starts with a review of the literature in section 3.1 and then introduces a nonlocal formulation of the biharmonic operator Δ^2 which appears in many formulations in thin-plate theory. Then theorems describing conditions for the nonlocal Laplacian to be Lipschitz continuous and the nonlocal biharmonic to be L^2 integrable are presented. Next, nonlocal formulations of a fourth order elliptic equation with hinged and clamped boundary conditions will be introduced and analyzed with well-posedness results proven. We conclude the chapter with the main nonlocal results, Theorems 3.7.6 and 3.7.3, which demonstrate a powerful connection between the local and nonlocal theory; specifically, that solutions to the nonlocal hinged and clamped systems converge strongly in L^2 to the weak solutions of their local analogues when the horizon converges to 0.

Chapter 2

Finite time blow-up in nonlinear suspension bridge models

In this chapter we will look at the local theory of plate dynamics, particularly with reference to suspension bridge oscillations. We begin in Section 2.1 with an overview of the literature and background information. We then proceed to Section 2.2 where we present our main result, Theorem 2.2.1, regarding equation (2.1.2): a sufficient condition for finite time blow-up. Section 2.2 also includes a few observations as well as open problems. For the reader's convenience, Section 2.4 presents a comprehensive list of constants and energy functions used throughout the chapter. In Section 2.5 we prove several lemmas about the energies introduced in Section 2.4, while Section 2.6 contains some results concerning the shape of the graphs of solutions to equation (2.1.2). We conclude our local analysis with Section 2.7 where we derive additional estimates for solutions of (2.1.2) and complete the proof of Theorem 2.2.1.

2.1 Local models and literature overview

The topic of suspension bridges is a celebrated area of applied mathematics filled with engineering marvels, as well as many dramatic events. One of the most notorious disasters is the Tacoma Narrows Bridge collapse of 1940. The collapse of the bridge had been previously explained by a resonance-like effect produced by a wind of under 80km/h [2]; however, in recent literature (e.g. [17], [26]) it has been demonstrated that resonance theory does not accurately describe these vibrational patterns. The phenomenon of self-amplifying oscillations in bridge dynamics is now recognized to be far more complex than originally believed; see the wonderful historical overview of existing theories in [14], [26]. To explain these dynamics, several models have been proposed. The following model based on the Euler-Bernoulli beam equation was introduced by Lazer and McKenna in [27] and investigated further in [26]:

$$u_{tt} + u_{xxxx} + \gamma u^+ = W(x, t), \quad x \in (0, L), \ t > 0. \tag{2.1.1}$$

In the above model u denotes vertical displacement, $L > 0$ is the length of the bridge, $u^+ = \max\{u, 0\}$, γu^+ represents the force from the cables treated as springs with a one-sided restoring force, and W accounts for additional forces such as weight and wind. In [28] McKenna and Walter investigated traveling waves for this model. After some normalization, traveling wave solutions to (2.1.1) necessarily satisfy

$$w''''(t) + kw''(t) + f(w(t)) = 0 \tag{2.1.2}$$

with $f(s) = (s+1)^+ - 1$. In [10], a smooth analog of this nonlinearity given by $f(t) = e^t - 1$ was considered.

Recent research (e.g. [14]) shows that the stability of a bridge can be critically affected

by torsional oscillations, in particular, their interactions with vertical displacements. Consequently, a one-dimensional model in the above interpretation does not accurately describe large oscillations, as twisting effects would not be taken into account. Indeed, traveling wave solutions corresponding to (2.1.2) are global when $f(s) \in \text{Lip}_{\text{loc}}(\mathbb{R})$, $f(s)s > 0$ for $s \neq 0$, and $f(s)$ has at most linear growth **either** as $s \to +\infty$ or as $s \to -\infty$, which was shown in [5].

Observations of actual bridge oscillations (Millennium Bridge [4]) and collapses (Tacoma Narrows Bridge [13]) reinforce the idea that torsional and vertical oscillations in suspension bridges are coupled. To model this interaction, [11] introduced a second unknown function to measure potentially unbounded torsional effects. Subsequently, it was suggested in [16] that the coupling mechanism be incorporated into a one-dimensional model by allowing the forcing term f to take arbitrarily large negative values. In this new model *p*ositive values of w correspond to vertical oscillations, while negative ones describe torsional deformations. A suitable function f would necessarily be sign preserving, e.g. $f(s) = s^3 + s$. For an extensive overview of the theory for ODEs of the form (2.1.2) see the book [33] by Peletier and Troy.

The study of traveling waves for (2.1.1) when vertical and torsional oscillations are unbounded has been a challenging open problem. The current chapter investigates finite time blow-up of solutions to equation (2.1.2) when f is a locally Lipschitz function unbounded as $|t| \to \infty$.

In their paper [17] Gazzola and Pavani offered an innovative proof of blow-up for the case $k \leq 0$; this range for k corresponds to models of beams in tension where $-k \geq 0$ represents the tension [18]. The scenario $k > 0$ corresponds to traveling wave solutions

$$u(t, x) = w(x + ct)$$

of the Euler-Bernoulli equation, with $k = c^2$ where c is the speed of the wave. Numerical evidence strongly supports the blow-up for $k > 0$, as presented in [17]. This case, however, remained open until now since positive values of the parameter k critically alter some intrinsic invariants associated with the ODE. We settle the blow-up conjecture when

- $k \in (0, 2)$ for a large class of nonlinear functions; see Theorem 2.2.1.

- $k \geq 2$ for scaled versions of nonlinearities satisfying the hypothesis of Theorem 2.2.1; see Corollary 2.2.3.

Moreover, we cover the case $k \leq 0$ with an alternate proof that requires less regularity on f and w. For all ranges of k, the assumptions on f are satisfied for power-type nonlinearities. We mention that the splitting into the cases $k < 2$ and $k \geq 2$ happens for technical reasons in our proof; however, it may be related to the fact that the corresponding linearized system for (2.1.2) has 2-dimensional stable and unstable manifolds for $|k| < 2$ and has purely imaginary eigenvalues when $k \geq 2$, see [5, Prop. 20].

The approach for all cases is inspired by the remarkable strategy developed in [17]; however, major challenges had to be overcome to accommodate $k > 0$:

- First, most of the energy functionals used in [17] are not convex for $k > 0$, yet this ingredient is critical in understanding the behavior of solutions. We introduce new energy functionals adapted to (2.1.2) and take advantage of their convexity and monotonicity properties to describe the behavior of solutions.

- An essential feature in the proof of the blow-up in [17] for $k \leq 0$ was the ability to ensure the existence of exactly one inflection point between consecutive zeros of the function. The same analysis does not extend to $k > 0$ so our proof allows multiple inflection points on an interval of one sign. Numerical evidence seems to indicate

that for sufficiently large energy there is eventually only one inflection point between neighboring zeros; however, verifying this conjecture is still an open problem.

- The intricate employment of test functions from [17] could not be reproduced for $k > 0$. Instead, we rely on geometric features of energy functions to show the growth and blow-up of solutions. This approach also allows us to handle less regular solutions, since we do not need to differentiate the ODE multiple times.

The theoretical work described above corroborates preliminary numerical computations that we have performed; together they prompt the following remarks with physical implications:

- For fourth order ODEs, the blow-up phenomenon, although oscillatory, seems to be driven not by frequency (as in resonance for second-order ODEs), but rather by the amplitude of the oscillations. Nonmonotone blow-up in finite time cannot be reproduced with a time-dependent external forcing that matches the frequency of the system. Instead, the mechanism is based on a transversal displacement inducing a torsional oscillation and vice versa. This "dual-excitation" process leads to a finite-time blow-up of the traveling wave solution.

- The forcing term $f(u)$ considered in this work is *defocusing* from the perspective of the hyperbolic or Petrovsky dynamics. However, in the context of traveling wave formation this restoring force has the opposite effect and induces a locally unbounded profile.

Our results are applicable to the study of some partial differential equations, e.g. Zakharov-Kuznetsov equations [24] and biharmonic coercive equations [15, Section 3].

2.2 Main blow-up result

Assume that f satisfies the regularity conditions:

$$f \in C^1(\mathbb{R}) \text{ and there exists a } \kappa_1 \in \mathbb{R} \text{ such that } f' \geq \kappa_1. \tag{2.2.1}$$

We also impose that f satisfies the following growth condition: there exist constants

$$p > q \geq 1, \quad \alpha \geq 0, \quad \text{and} \quad 0 < \rho \leq \beta$$

such that

$$\rho|s|^{p+1} \leq sf(s) \leq \alpha|s|^{q+1} + \beta|s|^{p+1} \quad \forall s \in \mathbb{R}. \tag{2.2.2}$$

Furthermore, let

$$F(s) := \int_0^s f(\tau)d\tau.$$

Theorem 2.2.1. *Assume f satisfies (2.2.1) and (2.2.2). Let $k < 2$,*

$$a = \rho\left(\frac{2}{\rho(p+1)}\right)^{\frac{p+1}{p-1}} - \left(\frac{2}{\rho(p+1)}\right)^{\frac{2}{p-1}}, \quad \gamma_2 = \frac{\alpha(p+1) + \beta(q+1)}{(q+1)(p+1)\rho},$$

$$\mu_3 \in \begin{cases} \left(0, \dfrac{2-k}{k}\right), & k \in (0,2) \\[2mm] (0,\infty), & k \leq 0 \end{cases}, \quad c \in \begin{cases} \left(0, \dfrac{2-(1+k)\mu_3}{\frac{k}{2}+k\gamma_2}\right), & k \in (0,2) \\[2mm] (0,\infty), & k \leq 0. \end{cases}$$

If $w(t)$ is a local solution to (2.1.2) that satisfies

$$\frac{k}{2}w'(0)^2 + w'(0)w'''(0) + F(w(0)) - \frac{1}{2}w''(0)^2 > \frac{\alpha}{q+1} - \frac{a}{c}, \tag{2.2.3}$$

then w blows up in finite time for $t > 0$.

Remark 2.2.2 (Conditions on the initial data). The condition (2.2.3) is in the spirit of [17, Eqn. (12)]; however, here we define it in terms of the energy E (introduced below in (2.4.2)), which is constant in time and therefore is a more natural invariant for the problem, obtained in fact through the classical energy multiplier w' applied to (2.1.2). Instead [17, see (12) and p. 20] employs a different non-constant convex invariant to characterize the initial data. Global existence for other data sets is under investigation; numerical evidence suggests that nontrivial solutions may exist globally in time. For example, see Figure 2.1 where the solution does not satisfy (2.2.3) and appears to be stable and doubly-periodic.
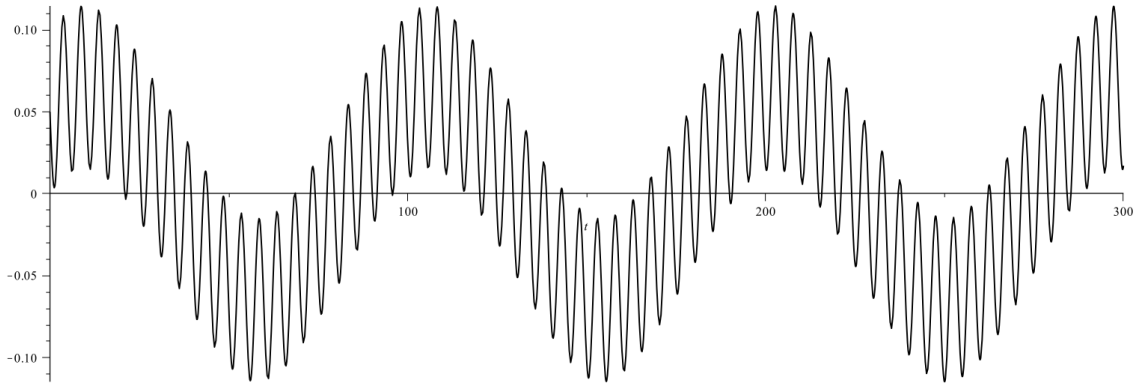
Figure 2.1: Numerically obtained solution of (2.1.2) for:
$k = 1.6$, $f(t) = t^3$, and $[w(0), w'(0), w''(0), w'''(0)] = [0.05, -0.06, 0, 0.1]$. These initial conditions *do not* satisfy the sufficient assumption (2.2.3) for blow-up.

Using linear transformations we are able to prove blow-up results for $k \in \mathbb{R}$ provided the nonlinearity f satisfies certain conditions.

Corollary 2.2.3. *Let $c_1, c_2 \neq 0$ and $c_3, r \in \mathbb{R}$ be such that $r/c_2^2 < 2$. Let g be chosen so that*

$$f(s) := \frac{1}{c_1 c_2^4} g\left(c_1 \left[s + c_3\right]\right) \tag{2.2.4}$$

where f satisfies conditions (2.2.1) *and* (2.2.2). *Let a, γ_2, μ_2, and c be defined as in Theorem*

2.2.1. Suppose u solves

$$u'''' + ru'' + g(u) = 0 \qquad (2.2.5)$$

with initial conditions satisfying

$$\frac{r}{2c_1^2 c_2^4} u'(0)^2 + \frac{1}{c_1^2 c_2^4} u'(0)u'''(0) + F\left(\frac{1}{c_1}u(0) - c_3\right) - \frac{1}{2c_1 c_2^4} u''(0)^2$$

$$> \frac{\alpha}{q+1} - \frac{a}{c}. \qquad (2.2.6)$$

Then u blows up in finite time on the right-maximal existence interval $[0, \omega)$ if $c_2 > 0$, and on the left-maximal interval $(-\omega, 0]$ if $c_2 < 0$.

(See Example 2.2.4 below for an application.)

Proof. Let

$$w(t) := \frac{1}{c_1}u\left(\frac{t}{c_2}\right) - c_3$$

so that $u(t) = c_1 (w(c_2 t) + c_3)$. Rewrite equation (2.2.5) in terms of w:

$$\begin{aligned}
0 =& u''''(t) + ru''(t) + g(w(t)) \\
 =& c_1 c_2^4 w''''(c_2 t) + rc_1 c_2^2 w''(c_2 t) + g(c_1 [w(c_2 t) + c_3]) \\
 =& c_1 c_2^4 w''''(c_2 t) + rc_1 c_2^2 w''(c_2 t) + c_1 c_2^4 f(w(c_2 t)) \quad \text{using (2.2.4)}.
\end{aligned}$$

Dividing by $c_1 c_2^4$ results in

$$0 = w''''(c_2 t) + \frac{r}{c_2^2} w''(c_2 t) + f(w(c_2 t)), \quad t \geq 0.$$

Set $k = r/c_2^2 < 2$. If u satisfies (2.2.5) on some neighborhood $(-C, C)$ of 0, then for $\tau = c_2 t$, the function w satisfies

$$w''''(\tau) + kw''(\tau) + f(w(\tau)) = 0$$

on the interval $(-C/|c_2|, C/|c_2|)$. The restriction on the initial conditions of u from equation (2.2.6) implies

$$\frac{k}{2}w'(0)^2 + w'(0)w'''(0) + F(w(0)) - \frac{1}{2}w''(0)^2 > \frac{\alpha}{q+1} - \frac{a}{c}.$$

The conclusion now follows from Theorem 2.2.1. □

Example 2.2.4. Consider a function $w(t)$ satisfying

$$u''''(t) + ru''(t) + |u(t)|^b u(t) = 0$$

with $r > 0$ and $b > 0$. If we pick $c_1 = 1, c_2 = \sqrt{r}, c_3 = 0$, notice from (2.2.4) we have that

$$f(s) := \frac{1}{r^2}g(s) = \frac{1}{r^2}|s|^b s$$

satisfies conditions (2.2.1) and (2.2.2) and $r/c_2^2 = 1 < 2$. Corollary 2.2.3 now gives a set of initial conditions where u will blow up. In particular, when

$$\frac{1}{2r}u'(0)^2 + \frac{1}{r^2}u'(0)u'''(0) + \frac{|u(0)|^{b+2}}{r^2(b+2)} - \frac{1}{2r^2}u''(0)^2 > \frac{\alpha}{q+1} - \frac{a}{c}.$$

Corollary 2.2.3 can also be used to tackle some nonlinearities not satisfying conditions (2.2.1) and (2.2.2). The next example will look at the nonlinearity $f(w) = w^3 - 3w^2 + 4w - 2$ which certainly doesn't satisfy (2.2.2).

Example 2.2.5. Consider a function $u(t)$ satisfying

$$u''''(t) + ru''(t) + u^3 - 3u^2 + 4u - 2 = 0$$

where $r > 0$. Set $c_1 = 1, c_2 = \sqrt{r}, c_3 = 1$. Then

$$f(s) := \frac{1}{r^2} g(s + 1) = \frac{1}{r^2}(s^3 + s)$$

satisfies conditions (2.2.1) and (2.2.2). Corollary 2.2.3 now prescribes initial conditions for which u will blow up in finite time. In particular, u blows up when

$$\frac{1}{2r} u'(0)^2 + \frac{1}{r^2} u'(0)u'''(0) + \frac{1}{r^2}\left(\frac{(u(0) - 1)^3}{3} + \frac{(u(0) - 1)^2}{2}\right) - \frac{1}{2r^2}u''(0)^2$$
$$> \frac{\alpha}{q + 1} - \frac{a}{c}.$$

2.3 Outline of the proof to Theorem 2.2.1

The proof of the main Theorem 2.2.1 relies on three main components:

- The first step, to show that global solutions to (2.1.2) cannot be eventually of one sign, has been settled for $k \geq 0$ in [5] under suitable conditions on f. In [17] an oscillation result was proven for $k \leq 0$. The only contribution to this step by the present dissertation is a modified proof of the oscillation result for $k \leq 0$ which holds under relaxed differentiability assumptions on f.

- Inspired by the approach in [17], we introduce certain energy functions in Sections 2.4, which are studied in Section 2.5. We use the energy functions to prove in Section 2.6 that extremum values for the solution grow at least geometrically fast.

- Using the results of Section 2.6 we show in Section 2.7 that distances between consecutive zeros of a solution w form a summable sequence, therefore $|w|$ necessarily blows up in finite time. To accomplish this we employ the following properties, derived in Section 2.6, of a solution of (2.1.2) satisfying the hypothesis of Theorem

2.2.1:

(W1) w changes sign at zeros and is never eventually of one sign.

(W2) On an interval of one sign $[z_i, z_{i+1}]$ where $w(z_i) = w(z_{i+1}) = 0$, there is exactly one extremum m_i.

(W3) On $[z_i, m_i]$ and $[m_i, z_{i+1}]$, $|w|$ is nondecreasing and nonincreasing respectively.

(W4) On $[m_i, z_{i+1}]$ we know w is concave down on an interval of positivity and concave up on an interval of negativity.

For an example of a numerically obtained solution see Figure 2.2. Notice how the peaks increase in magnitude while the zeros converge to a limit point.
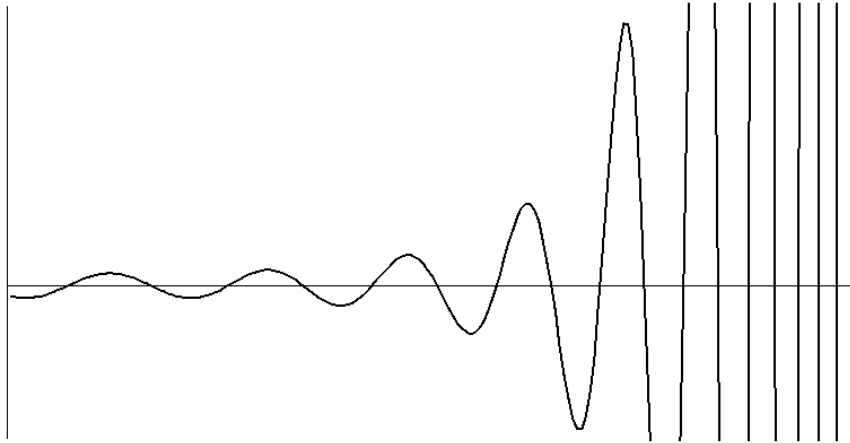
Figure 2.2: Blow-up through oscillations. This numerically obtained function was rescaled vertically to exhibit more peaks.

The plot in Figure 2.3 exhibits the behavior between consecutive zeros on a positivity interval and the geometric features (W1)–(W4) of the solution.
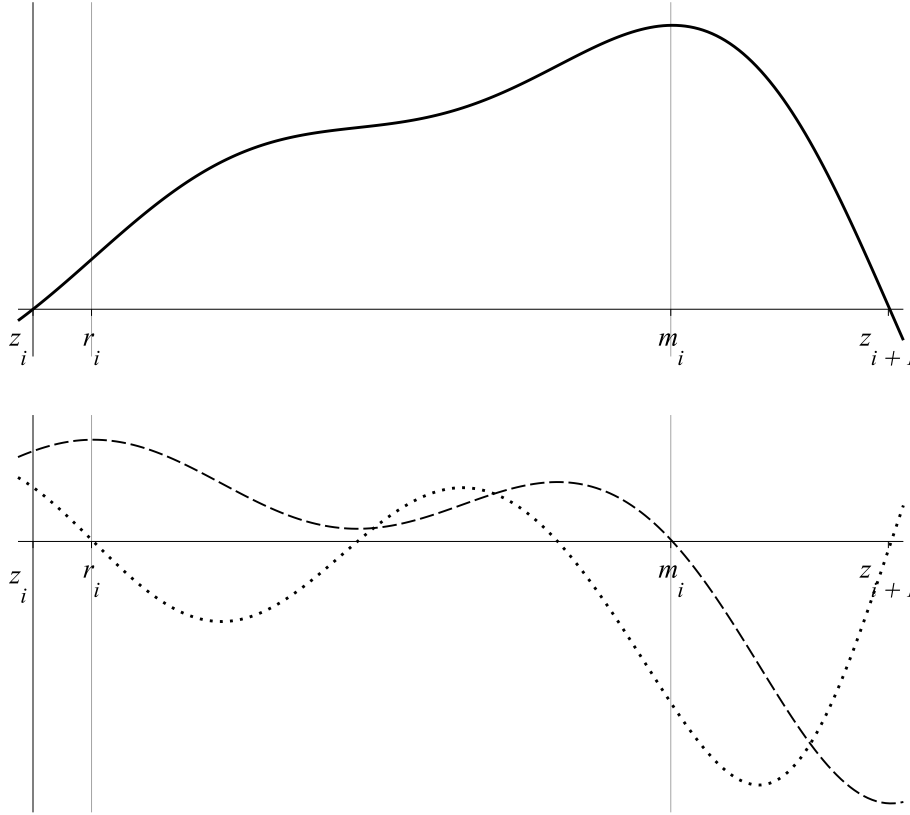
Figure 2.3: Numerically obtained solution of (2.1.2) for:
$k = 3.5$, $f(t) = t^5$, and $[w(0), w'(0), w''(0), w'''(0)] = [0, 0.5, 0.3, -0.6]$.
z_i, z_{i+1} : consecutive zeros of the solution.
m_i : unique extremum on the interval of positivity.
r_i : first inflection point after the zero z_i.
———: $w(t)$
− − −: $w'(t)$
$\cdots\cdots$: $w''(t)$

2.4   Summary of constants and energy functions

In this section we summarize for easy reference all parameters and energy functions that

will be used to prove the main theorem. Recall from condition (2.2.2) that ρ, p, and q were

constants used to quantify the growth of f. If we let a be the minimum of the polynomial

$\rho|x|^{p+1} - x^2$ then

$$a := \rho \left(\frac{2}{\rho(p+1)} \right)^{\frac{p+1}{p-1}} - \left(\frac{2}{\rho(p+1)} \right)^{\frac{2}{p-1}}.$$

Parameters $\gamma_1, \gamma_2, \zeta_1, \zeta_2$ will describe the growth of $F(s) := \int_0^s f(\tau)d\tau$ in (2.5.2), Lemma 3.7.1, and they are defined by

$$\gamma_1 := \frac{\rho}{(p+1)(\alpha+\beta)} \qquad \zeta_1 := -\gamma_1\alpha$$

$$\gamma_2 := \frac{\alpha}{(q+1)\rho} + \frac{\beta}{(p+1)\rho} \qquad \zeta_2 := \frac{\alpha}{q+1}.$$

The set of constants to follow will be used to define suitable energy functions associated with solution w. We begin with μ_3 which will be a free parameter in the interval

$$\mu_3 \in \begin{cases} \left(0, \dfrac{2-k}{k} \right), & k \in (0, 2) \\[2ex] (0, \infty), & k \le 0. \end{cases}$$

The constant c will be used to determine admissible initial energies for blow-up results. It can be set to any value in

$$c \in \begin{cases} \left(0, \dfrac{2-(1+\mu_3)k}{\frac{k}{2} + k\gamma_2} \right), & k \in (0, 2) \\[2ex] (0, \infty), & k \le 0. \end{cases}$$

In terms of the above parameters we define:

$$\mu_1 := 2 - k\left(\frac{c}{2} + c\gamma_2 + 1 + \mu_3\right) \qquad \alpha_1 := \frac{1}{2}(2 - k(\mu_3 + c\gamma_2 + 1))$$

$$\mu_2 := \mu_3 + c\left(\gamma_2 + \frac{3}{2}\right) \qquad\qquad \alpha_2 := \frac{c}{2} + \mu_3 + c\gamma_2 + 1 \qquad (2.4.1)$$

$$\alpha_3 := \mu_3 + c\gamma_2 + 1.$$

It will be shown that these constants are positive. Recall that κ_1 was a lower bound on f' (from (2.2.1)). Using it we define

$$\kappa_2 := \min\left\{\frac{|\mu_2|}{|k| + 1}, \frac{|\mu_1|}{|\kappa_1| + 1}\right\}$$

We now introduce the energy functions (and compute some of their derivatives) that will aid in exhibiting the blow-up mechanism of solutions to (2.1.2). These functionals inspired by similar constructs in [17] are able to detect the blow-up of w through their properties (e.g. convexity for G and H):

$$E(t) := \frac{k}{2}w'(t)^2 + w'(t)w'''(t) + F(w(t)) - \frac{1}{2}w''(t)^2$$

$$A(t) := w(t)f(w(t)) + w''(t)^2 + 2w(t)w''(t).$$

$$(2.4.2)$$

It will be shown that the above energy E remains invariant during the life-time of solutions; E and A together will be used to establish the growth and convexity properties of the following functional G, which will be one of the primary tools for showing the blow-up of

w:

$$G(t) := \alpha_1 w(t)^2 + \alpha_2 w'(t)^2 - \alpha_3 w(t) w''(t)$$

$$G'(t) = 2\alpha_1 w(t) w'(t) + (2\alpha_2 - \alpha_3) w'(t) w''(t) - \alpha_3 w(t) w'''(t)$$

$$G''(t) = 2\alpha_1 \left(w(t) w''(t) + w'(t)^2 \right) + 2 (\alpha_2 - \alpha_3) \left(w'(t) w'''(t) + w''(t)^2 \right)$$

$$+ \alpha_3 \left(w''(t)^2 - w(t) w''''(t) \right).$$

(2.4.3)

In addition the following two functionals will be employed to prove the geometric growth of the extremum values of *w*. (Φ was previously introduced in [17]):

$$\Phi(t) := \frac{1}{2} w''(t)^2 + F(w(t))$$

$$H(t) := G(t) + \kappa_2 \Phi(t)$$

$$= \alpha_1 w(t)^2 + \alpha_2 w'(t)^2 - \alpha_3 w(t) w''(t) + \frac{\kappa_2}{2} w''(t)^2 + \kappa_2 F(w(t)).$$

(2.4.4)

2.5 Convexity of G and H

This section proves several properties of the energy functions introduced in (2.4.2)–(2.4.4), in particular the strict convexity of G and H. We begin by showing that E is conserved in time.

Lemma 2.5.1. *If w is a solution to equation (2.1.2) then $E(t) = E(0)$ for all t in the interval of existence.*

Proof. Assume *w* is a solution of equation (2.1.2), then (suppressing "(*t*)")

$$\frac{dE}{dt} = kw'w'' + w'w'''' + w''w''' + f(w)w' - w''w'''$$

$$= w\left(kw'' + w'''' + f(w) \right) = 0.$$

□

The next lemma introduces and derives a lower bound for the function A. A will be used later to provide a lower bound on G''.

Lemma 2.5.2. *Assume f satisfies condition (2.2.2) and*

$$A(t) := w(t)f(w(t)) + w''(t)^2 + 2w(t)w''(t).$$

Then

$$A(t) \geq a := \rho \left(\frac{2}{\rho(p+1)} \right)^{\frac{p+1}{p-1}} - \left(\frac{2}{\rho(p+1)} \right)^{\frac{2}{p-1}}$$

for all t in the interval of existence.

Proof. We estimate directly:

$$A = wf(w) + (w'')^2 + 2ww'' \geq \rho|w|^{p+1} + (w'')^2 + 2ww'' \quad \text{(condition (2.2.2))}$$

$$= \rho|w|^{p+1} - w^2 + w^2 + (w'')^2 + 2ww''$$

$$\geq \rho|w|^{p+1} - w^2 \quad \text{(Young's inequality)}$$

$$\geq a.$$

The last line follows by minimizing $\rho|x|^{p+1} - x^2$ with $x \in \mathbb{R}$. □

The next result describes the growth of F and will be used later to show the convexity of the energy G.

Lemma 2.5.3 (Growth of F). *Assume f satisfies condition (2.2.2) (with parameters α, β, p, q). Let $F(s) := \int_0^s f(\tau)d\tau$, and*

$$\gamma_1 = \frac{\rho}{(p+1)(\alpha+\beta)}, \quad \zeta_1 = -\gamma_1\alpha, \quad \gamma_2 = \frac{\alpha(p+1)+\beta(q+1)}{(q+1)(p+1)\rho}, \quad \zeta_2 = \frac{\alpha}{q+1},$$

then for all $s \in \mathbb{R}$

$$\frac{\rho}{p+1}|s|^{p+1} \leq F(s) \leq \frac{\alpha}{q+1}|s|^{q+1} + \frac{\beta}{p+1}|s|^{p+1}. \qquad (2.5.1)$$

Consequently, by (2.2.2) for all t in the interval of existence

$$\gamma_1 w(t)f(w(t)) + \zeta_1 \leq F(w(t)) \leq \gamma_2 w(t)f(w(t)) + \zeta_2. \qquad (2.5.2)$$

Proof. Recall that $p > q$. Then for all s (note that both $s \geq 0$ and $s < 0$ yield the same result via lower bound in (2.2.2)) we have

$$
\begin{aligned}
F(s) = \int_0^s f(\tau)d\tau &\geq \int_0^s \rho\tau|\tau|^{p-1}d\tau \\
&= \frac{\rho}{p+1}|s|^{p+1} = \gamma_1\left(\alpha + \beta\right)|s|^{p+1} \\
&= \gamma_1\alpha\left(1 + |s|^{p+1}\right) + \gamma_1\beta|s|^{p+1} - \gamma_1\alpha \\
&\geq \gamma_1\alpha|s|^{q+1} + \gamma_1\beta|s|^{p+1} + \zeta_1 \\
&\geq \gamma_1 sf(s) + \zeta_1.
\end{aligned}
$$

We find the upper bound in a similar manner (again both $s \leq 0$ and $s > 0$ yield the same

inequality using the upper bound in (2.2.2)):

$$
\begin{aligned}
F(s) = \int_0^s f(\tau)d\tau &\le \int_0^s \alpha\tau|\tau|^{q-1} + \beta\tau|\tau|^{p-1}d\tau \\
&= \frac{\alpha}{q+1}|s|^{q+1} + \frac{\beta}{p+1}|s|^{p+1} \\
&\le \frac{\alpha}{q+1}\left(1 + |s|^{p+1}\right) + \frac{\beta}{p+1}|s|^{p+1} \\
&= \frac{\alpha}{q+1} + \left(\frac{\alpha}{q+1} + \frac{\beta}{p+1}\right)|s|^{p+1} \\
&= \zeta_2 + \gamma_2\rho|s|^{p+1} \\
&\le \zeta_2 + \gamma_2 s f(s).
\end{aligned}
$$

$\square$

Recall from Section 2.4 the definition of the energy function G:

$$
G(t) = \alpha_1 w(t)^2 + \alpha_2 w'(t)^2 - \alpha_3 w(t)w''(t)
$$

where coefficients α_i satisfy

$$
\alpha_1 = \frac{1}{2}(2 - k(\underbrace{\mu_3 + c\gamma_2 + 1}_{\alpha_3})), \quad \alpha_2 = \frac{c}{2} + \mu_3 + c\gamma_2 + 1.
$$

The next lemma will show that by placing certain restrictions on the initial energy we can ensure that G'' is bounded below by a positive constant.

Lemma 2.5.4 (Strict convexity of G). *Assume w is a local solution to equation (2.1.2) and f satisfies condition (2.2.2). Let $\varepsilon > 0$, $k < 2$,*

$$
\mu_3 \in \begin{cases} \left(0, \dfrac{2-k}{k}\right), & k \in (0,2) \\[2ex] (0, \infty), & k \le 0 \end{cases}, \quad c \in \begin{cases} \left(0, \dfrac{2 - (1+\mu_3)k}{\frac{k}{2} + k\gamma_2}\right), & k \in (0,2) \\[2ex] (0, \infty), & k \le 0 \end{cases},
$$

$$\mu_1 = 2 - k\left(\frac{c}{2} + \underbrace{c\gamma_2 + 1 + \mu_3}_{\alpha_3}\right), \quad \mu_2 = \mu_3 + c\gamma_2 + \frac{3c}{2}.$$

Then $\mu_1, \mu_2, \mu_3, \alpha_1, \alpha_2, \alpha_3 > 0$ and if $E(0)$ satisfies $cE(0) > \zeta_2 c - a$ then for some $\varepsilon > 0$ we

have

$$G'' \geq \varepsilon + \mu_1(w')^2 + \mu_2(w'')^2 + \mu_3 w f(w) \tag{2.5.3}$$

on the interval of existence of the solution.

Proof. First let us verify the positivity of the constants. If $k \leq 0$ then all above constants

are trivially positive; therefore, let us consider $k \in (0, 2)$. It is clear that μ_3 and c exist and

are positive. Because $\gamma_2 > 0$ we have $\mu_2, \alpha_2, \alpha_3 > \mu_3 > 0$. Next, we focus on μ_1. Recall c

satisfies

$$0 < c < \frac{2 - (1 + \mu_3)k}{\frac{k}{2} + k\gamma_2}$$

so that

$$\frac{ck}{2} + ck\gamma_2 < 2 - k - \mu_3 k$$

and

$$0 < 2 - \frac{ck}{2} - ck\gamma_2 - k - \mu_3 k = \mu_1.$$

Since $2\alpha_1 = 2 - k\alpha_3 = \mu_1 + \frac{ck}{2}$ we have that $\alpha_1 > 0$ as well.

Now we will establish a lower bound on G''. Recall from (2.4.3) that

$$G'' = 2\alpha_1\left(ww'' + (w')^2\right) + 2(\alpha_2 - \alpha_3)\left(w'w''' + (w'')^2\right) + \alpha_3\left((w'')^2 - ww''''\right).$$

Since $cE(0) > \zeta_2 c - a$ pick $\varepsilon > 0$ such that $cE(0) \geq \zeta_2 c - a + \varepsilon$. Utilizing the lower bound

on the energy function $A(t)$ from Lemma 2.5.2, we obtain

$$c\zeta_2 + \varepsilon$$

$$\leq cE(0) + a$$

$$\leq cE(t) + A(t)$$

$$= \frac{ck}{2}(w')^2 + cw'w''' + \left[cF(w)\right] - \frac{c}{2}(w'')^2 + wf(w) + (w'')^2 + 2ww''$$

$$\leq \frac{ck}{2}(w')^2 + cw'w''' + \left[\underbrace{c\gamma_2 wf(w) + c\zeta_2}_{\text{by (2.5.2)}}\right] - \frac{c}{2}(w'')^2 + wf(w) + (w'')^2 + 2ww''.$$

Subtracting $c\zeta_2$ from each side, and adding and subtracting $\mu_3 wf(w)$ results in

$$\varepsilon \leq \frac{ck}{2}(w')^2 + cw'w''' + c\gamma_2 wf(w) - \frac{c}{2}(w'')^2 + wf(w) + (w'')^2 + 2ww''$$

$$\underbrace{+\mu_3 wf(w) - \mu_3 wf(w)}_{=0}$$

$$= \frac{ck}{2}(w')^2 + cw'w''' \underbrace{-ck\gamma_2 ww'' - c\gamma_2 ww''''}_{c\gamma_2 wf(w)=c\gamma_2 w(-kw''-w'''')} - \frac{c}{2}(w'')^2 \underbrace{-kww'' - ww''''}_{wf(w)=w(-kw''-w'''')}$$

$$+ (w'')^2 + 2ww'' \underbrace{-\mu_3 kww'' - \mu_3 ww''''}_{\mu_3 wf(w)=\mu_3 w(-kw''-w'''')} -\mu_3 wf(w).$$

Combining like terms and using arithmetic manipulations yields

$$
\begin{aligned}
\varepsilon \leq & \frac{ck}{2}(w')^2 + cw'w''' + (-ck\gamma_2 - k + 2 - \mu_3 k)\, ww'' - \underbrace{(c\gamma_2 + 1 + \mu_3)}_{\alpha_3}\, ww'''' \\
& + \left(1 - \frac{c}{2}\right)(w'')^2 - \mu_3 w f(w) \\
= & \frac{ck}{2}(w')^2 + cw'w''' + (-ck\gamma_2 - k + 2 - \mu_3 k)\, ww'' - \alpha_3 ww'''' + \left(1 - \frac{c}{2}\right)(w'')^2 \\
& - \mu_3 w f(w) + c\underbrace{\left((w'')^2 - (w'')^2\right)}_{=0} + (-ck\gamma_2 - k + 2 - \mu_3 k)\underbrace{\left((w')^2 - (w')^2\right)}_{=0} \\
& + \alpha_3 \underbrace{\left((w'')^2 - (w'')^2\right)}_{=0} \\
= & c(w'w''' + (w'')^2) + (-ck\gamma_2 - k + 2 - \mu_3 k)(ww'' + (w')^2) + \alpha_3((w'')^2 - ww'''') \\
& + \left(\frac{ck}{2} + ck\gamma_2 + k - 2 + \mu_3 k\right)(w')^2 + \left(1 - c - c\gamma_2 - \frac{c}{2} - 1 - \mu_3\right)(w'')^2 \\
& - \mu_3 w f(w) \\
= & 2\left(\alpha_2 - \alpha_3\right)\left(w'w''' + (w'')^2\right) + 2\alpha_1\left(ww'' + (w')^2\right) + \alpha_3\left((w'')^2 - ww''''\right) \\
& - \mu_1(w')^2 - \mu_2(w'')^2 - \mu_3 w f(w) \\
= & G'' - \mu_1(w')^2 - \mu_2(w'')^2 - \mu_3 w f(w)\,.
\end{aligned}
$$

We conclude

$$
G'' \geq \varepsilon + \mu_1(w')^2 + \mu_2(w'')^2 + \mu_3 w f(w) \geq \varepsilon\,.
$$

$\square$

The proof of Theorem 2.2.1 will rely on the fact that G is convex and therefore we will frequently appeal to the following condition on the initial energy $E(0)$ (note also that since since $c, \zeta_2 \geq 0$ and $a \leq 0$ we infer $E(0) > 0$):

$$
cE(0) > \zeta_2 c - a > 0 \tag{2.5.4}
$$

which is precisely the condition (2.2.3) in the hypothesis of Theorem 2.2.1.

The next result will show that H is convex.

Lemma 2.5.5 (Strict convexity of H). *Let w be a nontrivial local solution to equation (2.1.2) with $k < 2$. Assume f satisfies conditions (2.2.1) and $E(0)$ satisfies (2.5.4). Let*

$$\Phi(t) := \frac{1}{2}w''(t)^2 + F(w(t)) \quad and \quad H(t) := G(t) + \kappa_2\Phi(t)$$

where $\kappa_2 > 0$ is given by

$$\kappa_2 := \min\left\{\frac{|\mu_2|}{|k| + 1}, \frac{|\mu_1|}{|\kappa_1| + 1}\right\} \tag{2.5.5}$$

Then $H''(t) \geq \varepsilon$, for some $\varepsilon > 0$.

Proof. By condition (2.2.1) we know $f' > \kappa_1$. We differentiate Φ twice to find

$$\Phi''(t) = (w''')^2 - k(w'')^2 + f'(w)(w')^2 \geq -k(w'')^2 + \kappa_1(w')^2.$$

Recall from Lemma 2.5.4 that

$$G'' \geq \mu_1(w')^2 + \mu_2(w'')^2 + \varepsilon$$

since $\mu_3 w f(w) \geq 0$. Thus,

$$
\begin{aligned}
H'' &= \kappa_2\Phi'' + G'' \\
&\geq -k\kappa_2(w'')^2 + \kappa_2\kappa_1(w')^2 + \mu_1(w')^2 + \mu_2(w'')^2 + \varepsilon \\
&\geq -\left|\frac{\mu_2}{|k| + 1}k\right|(w'')^2 - \left|\frac{\mu_1}{|\kappa_1| + 1}\kappa_1\right|(w')^2 + \mu_1(w')^2 + \mu_2(w'')^2 + \varepsilon \\
&\geq \varepsilon.
\end{aligned}
$$

□

Now that we have convexity results for G and H we can infer several interesting properties of w that will be discussed in the next section.

2.6   Growth of G and behavior of w

The convexity results in Section 2.5 prove that both G and H are eventually strictly increasing. This fact leads to several interesting properties of hypothetical *global* solutions to (2.1.2). As Section 2.7 will subsequently demonstrate, these conditions lead to a contradiction implying that global solutions to (2.1.2) cannot exist. The first result we prove is that $G(t)$ grows exponentially for t sufficiently large.

Lemma 2.6.1. *Assume f satisfies (2.2.2), $k < 2$, w is a global solution to (2.1.2), and $E(0)$ satisfies (2.5.4). For some $T \geq 0$, there exist constants $\theta > 0$, $C_1 > 0$, and $C_2 \in \mathbb{R}$ such that*

$$G(t) \geq C_1 e^{\theta t} + C_2 \quad for\ all \quad t \geq T .$$

Proof. Since $p > 1, \mu_3, \varepsilon > 0$, there exists $C_3 > 0$ such that

$$C_3 \left(\mu_3 \rho |w|^{p+1} + \varepsilon \right) \geq \left(\alpha_1 + \frac{\alpha_3}{2} \right) w^2.$$

Recall $\mu_1, \mu_2, \mu_3 > 0$. Find $C_4 > 0$ such that

$$C_4 \geq \max \left\{ C_3, \frac{\alpha_2}{\mu_1}, \frac{\alpha_3}{2\mu_2} \right\} .$$

Then

$$C_4 G'' \geq C_4 \varepsilon + C_4 \mu_3 w f(w) + C_4 \mu_1 (w')^2 + C_4 \mu_2 (w'')^2$$

$$\geq C_4 (\varepsilon + \underbrace{\mu_3 \rho |w|^{p+1}}_{\text{condition (2.2.2)}}) + C_4 \mu_1 (w')^2 + C_4 \mu_2 (w'')^2$$

$$\geq \left(\alpha_1 + \frac{\alpha_3}{2} \right) w^2 + \alpha_2 (w')^2 + \frac{\alpha_3}{2} (w'')^2$$

$$= \alpha_1 w^2 + \alpha_2 (w')^2 + \alpha_3 \left(\frac{w^2}{2} + \frac{(w'')^2}{2} \right)$$

$$\geq \alpha_1 w^2 + \alpha_2 (w')^2 - \alpha_3 w w''$$

$$= G.$$

Because G'' is strictly positive by Lemma 3.7.1, we can find a time $T > 0$ such that $G(t) \geq 0$ and $G'(t) \geq 1$ for $t \geq T$. Then for $t \geq T$ we know G is bounded below by the solution to the initial value problem:

$$C_4 u''(t) = u(t) \quad \text{for} \quad t \geq T, \quad \text{with} \quad u(T) = 0, u'(t) = 1.$$

Hence for $t \geq T$,

$$G(t) \geq C_1 e^{\theta t} + C_2.$$

$\square$

The next result provides a lower exponential bound for the growth of $w(t)$ at extrema when t is taken sufficiently large.

Lemma 2.6.2 (Growth of w at the extrema). *Assume w is a nontrivial global solution to equation (2.1.2), f satisfies condition (2.2.2), and $E(0)$ satisfies (2.5.4). There exists a $T \geq 0$ and positive constants C, r such that for any local extremum $m \geq T$ of w we have*

$$|w(m)| \geq C e^{rm}.$$

Proof. Since $w'(m) = 0$ we have

$$E(m) = F(w(m)) - \frac{1}{2}w''(m)^2 = E(0)$$

$$G(m) = \alpha_1 w(m)^2 - \alpha_3 w(m)w''(m).$$

Solving for $|w''(m)|$ in the energy equation yields

$$|w''(m)| = \sqrt{2F(w(m)) - 2E(0)}.$$

Notice that at a local extremum m we always have $w(m)w''(m) \le 0$ and consequently,

$$G(m) = \alpha_1 w(m)^2 + \alpha_3 |w(m)| \sqrt{2F(w(m)) - 2E(0)}.$$

Thus by bound (2.5.1) on F from Lemma 3.7.1, and using $p > q$

$$G(m) \le \alpha_1 w(m)^2 + \alpha_3 |w(m)| \sqrt{\frac{2\alpha}{q+1}|w(m)|^{q+1} + \frac{2\beta}{p+1}|w(m)|^{p+1}} \tag{2.6.1}$$

$$\le c_1 + c_2 |w(m)|^{\frac{p+3}{2}}.$$

where $c_1, c_2 > 0$ are sufficiently large. By Lemma 2.6.1 we know there is a T such that for $m \ge T$ we have $G(m) \ge C_1 e^{\theta m} + C_2$ for some constants $C_1, C_2 > 0$. Consequently,

$$|w(m)| \ge \left(\frac{C_1 e^{\theta m} + C_2 - c_1}{c_2}\right)^{\frac{2}{p+3}}$$

and for sufficiently large T (and consequently large m),

$$|w(m)| \ge \left(\frac{C_1}{2c_2}\right)^{\frac{2}{p+3}} e^{\frac{2\theta}{p+3}m}. \tag{2.6.2}$$

$\square$

We will discover that w is never eventually of one sign. This coupled with the above lemma allows us to conclude that w is unbounded, at least asymptotically. Next we will look at the value of $G(t)$ at an extremum, m, of w. The next Lemma will show that for m sufficiently large, $G(m)$ is comparable to $|w(m)|^{\frac{p+3}{2}}$.

Lemma 2.6.3 (Bounds on $G(m)$). *Assume f satisfies condition (2.2.2). If w is a nontrivial global solution to (2.1.2) with initial conditions satisfying (2.5.4), then there exists non-negative constants T, C_1, and C_2, such that if m is a local extremum of w and $m \geq T$, then*

$$C_1|w(m)|^{\frac{p+3}{2}} \leq G(m) \leq C_2|w(m)|^{\frac{p+3}{2}}.$$

Proof. The upper bound follows as in the proof of Lemma 2.6.2, from (2.6.1) and the fact that $w(m)$ is eventually large, as dictated by (2.6.2).

For the lower bound, again, since m is an extremum, we know $w(m)w''(m) \leq 0$ and

$$E(m) = F(w(m)) - \frac{1}{2}w''(m)^2 \quad \Leftrightarrow \quad |w''(m)| = \sqrt{2F(w(m)) - 2E(0)}. \tag{2.6.3}$$

Thus for a sufficiently large m,

$$\begin{aligned}
G(m) &= \alpha_1 w(m)^2 - \alpha_3 w(m)w''(m) \\
&= \alpha_1 w(m)^2 + \alpha_3|w(m)|\sqrt{2F(w(m)) - 2E(0)} \quad \text{(Equation (2.6.3))} \\
&\geq \alpha_1 w(m)^2 + \alpha_3|w(m)|\sqrt{\frac{2\rho}{p+1}|w(m)|^{p+1} - 2E(0)} \quad \text{(Lemma 3.7.1)} \\
&\geq \alpha_1 w(m)^2 + \alpha_3|w(m)|\sqrt{\frac{\rho}{p+1}|w(m)|^{p+1}} \quad \text{(use Lemma (2.6.2) and large m)} \\
&\geq \alpha_3\left(\frac{\rho}{p+1}\right)^{\frac{1}{2}}|w(m)|^{\frac{p+3}{2}} \quad \text{(recall $p > 1$)} \\
&= C_2|w(m)|^{\frac{p+3}{2}}.
\end{aligned}$$

$\square$

2.6.1 Oscillatory behavior of w

Following the approach of [17], the proof of the main result is based on the fact that w cannot remain of one sign and exist globally.

The following result comes from [5, Thm. 4]. For our purposes, it states that if $k \geq 0$ and f satisfies conditions (2.2.1) and (2.2.2), then nontrivial global solutions to (2.1.2) are never eventually of one sign.

Theorem 2.6.4 (Oscillations of w when $k \geq 0$. [5, Thm. 4]). *Let $k \geq 0$ and suppose*

$$f \in Lip_{loc}(\mathbb{R}), \quad f(t)t > 0 \text{ for every } t \in \mathbb{R} \backslash \{0\}.$$

If w is a nontrivial global solution to (2.1.2), then $w(t)$ changes sign infinitely many times as $t \to \infty$ and as $t \to -\infty$.

First we need the following lemma that will also come in handy later on when proving some geometric properties of the graph of w.

Lemma 2.6.5 (Decreasing $|w|$ after an extremum). *Assume f satisfies (2.2.1) and (2.2.2), $k < 2$, and w is a nontrivial global solution to (2.1.2) with initial conditions satisfying (2.5.4). Let (z_i, z_{i+1}) be an interval where w is of one sign. There exists a $T \geq 0$ such that if $z_i \geq T$, $m_i \in (z_i, z_{i+1})$ is a zero of w', and $|w|$ is strictly decreasing on some interval $(m_i, m_i + \delta) \subset (m_i, z_{i+1})$ with $\delta > 0$, then $w' \neq 0$ on $(m_i, z_{i+1}]$.*

Proof. To the contrary, assume $n_i \geq m_i + \delta$ is the next point where $w'(n_i) = 0$ in $(m_i, z_{i+1}]$. By assumption $|w(n_i)| < |w(m_i)|$. If in addition $w(n_i) = 0$, then the energy satisfies $E(n_i) = -\frac{1}{2}w''(n_i)^2 \leq 0$ contradicting (2.5.4) which requires $E(0) > 0$. Thus $w(n_i) \neq 0$, in particular

$n_i < z_{i+1}$. Since $|w|$ is decreasing till n_i, we know n_i cannot be a point of maximum of $|w|$ and consequently

$$w(n_i)w''(n_i) \geq 0 \,.$$

Now recall,

$$G(n_i) \;=\; \alpha_1 w(n_i)^2 - \alpha_3 w(n_i)w''(n_i)$$

$$G(m_i) \;=\; \alpha_1 w(m_i)^2 - \alpha_3 w(m_i)w''(m_i) \,.$$

Provided m_i is sufficiently large, Lemma 2.5.4 implies $G(n_i) > G(m_i)$; hence,

$$\alpha_1 w(n_i)^2 - \alpha_3 w(n_i)w''(n_i) \;>\; \alpha_1 w^2(m_i) - \alpha_3 w(m_i)w''(m_i).$$

This implies that

$$0 \;>\; \alpha_1 w^2(n_i) - \alpha_1 w^2(m_i) \;>\; \alpha_3 w(n_i)w''(n_i) - \alpha_3 w(m_i)w''(m_i) \,.$$

It follows that $w(m_i)w''(m_i) > w(n_i)w''(n_i)$. However, $w(m_i)w''(m_i) \leq 0$ and so $w(n_i)w''(n_i) < 0$ contradicting $w(n_i)w''(n_i) \geq 0$. We conclude $w' \neq 0$ on $(m_i, z_{i+1}]$. $\qquad\square$

The oscillation result for the case $k \leq 0$ was proven in [17] using the assumption that f is twice continuously differentiable away from 0 and imposing some restrictions on f''. Using the ideas in [5, 17] we provide a modified proof (see 4.1) of this result by requiring assumptions on only one derivative of f, consequently requiring less regularity on w.

Lemma 2.6.6 (Oscillations of w when $k \leq 0$, cf. [17])**.** *Assume f satisfies conditions* (2.2.1) *and* (2.2.2)*, $k \leq 0$, and w is a global solution of* (2.1.2) *with initial conditions satisfying* (2.5.4)*. Then w is never eventually of one sign.*

Proof. See 4.1. □

2.6.2 Shape of the graph and geometric growth

In the last section we learned that provided certain conditions were met, a solution w of (2.1.2) is never eventually of one sign. In this section we establish the behavior of w between consecutive zeros. We also gather previous results here to give a general picture of w in a region of one sign. In the first lemma we discover only one extremum can exist on an interval of one sign.

Lemma 2.6.7 (Single extremum on an interval of one sign). *Assume f satisfies conditions (2.2.1) and (2.2.2) and w is a nontrivial global solution to equation (2.1.2) with initial conditions satisfying (2.5.4). There exists a $T \geq 0$ such that if $w(z_i) = w(z_{i+1}) = 0$ for $z_i \geq T$, and (z_i, z_{i+1}) is an interval where w is of one sign, then there exists exactly one local maximum of $|w|$ on $[z_i, z_{i+1}]$; moreover, $|w|$ is increasing on $[z_i, m_i]$ and decreasing on $[m_i, z_{i+1}]$.*

Proof. Either by Lemma 2.6.5, or just observing that $G'(z_i) = \alpha_2 w'(z_i)^2$ and G' is eventually positive, we know that $w'(z_i) \neq 0$. So $|w|$ is increasing on some interval $[z_i, z_i + \varepsilon]$. Let m_i be the first place in $[z_i, z_{i+1}]$ such that $|w|$ is not increasing on some interval $[m_i, m_i + \varepsilon)$. If $|w|$ is decreasing on some $[m_i, m_i + \delta)$, $\delta \leq \varepsilon$, then we are done by Lemma 2.6.5. We know w is not constant anywhere as solutions are unique, in particular, by continuity there cannot be a dense subset of zeros of w'. We also know that we can not have a sequence of isolated points $\{n_j\}$ converging to m_i from the right with $w'(n_j) = 0$ unless $|w|$ is increasing on each interval $[n_{j+1}, n_j]$, as for j large that would contradict the assumption that $|w|$ is non-increasing $[m_i, m_i + \varepsilon)$. We conclude there is exactly one local maximum of $|w|$ on $[z_i, z_{i+1}]$ and the monotone behavior before and after m_i on $[z_i, z_{i+1}]$ follows. □

There are several consequences of Lemmas 2.6.5 and 2.6.7 that are of use to us. Provided T is sufficiently large and the hypotheses of the two lemmas hold, we know

- w changes sign at zeros.

- w has exactly one extremum, m_i, on an interval $[z_i, z_{i+1}]$ where $w(z_i) = w(z_{i+1}) = 0$ and w is of one sign on (z_i, z_{i+1}).

- $|w|$ is nondecreasing on $[z_i, m_i)$ and decreasing on $(m_i, z_i]$.

To simplify things, we introduce some notation. Any global nontrivial solutions to (2.1.2) satisfying the hypothesis of Theorem 2.6.4, will have infinitely many zeros, extrema, and inflection points, denoted as follows:

- $\mathcal{Z} := \{z_i\}$ will be the zeros of w with $z_i < z_{i+1}$.

- $\mathcal{M} := \{m_i\}$ will be the extremas of w with $m_i \in (z_i, z_{i+1})$.

- $\mathcal{R} = \{r_i\}$ where r_i is the smallest number in $[z_i, m_i]$ such that $w''(r_i) = 0$.

The next result will show that not only the sequence $\{|w(m_i)|\}$ is unbounded, but eventually it grows geometrically.

Lemma 2.6.8 (Geometric growth of w at extrema). *Assume f satisfies conditions (2.2.1) and (2.2.2) and w is a nontrivial global solution to equation (2.1.2) with initial conditions satisfying (2.5.4). There exists a $T \geq 0$ such that any subsequence of $\{|w(m_i)| : m_i \in \mathcal{M}, m_i \geq T\}$ is bounded below by the sequence $\left\{\left(\frac{4}{3}\right)^{\frac{i}{p+1}} : i \in \mathbb{N}\right\}$.*

Proof. The argument was motivated by the approach in [17, Step 6, p. 25]. Let ℓ_i be the last inflection point between m_{i-1} and m_i, possibly m_i itself (one must exist because one m is a local minimum, while the other is a local maximum). Let T be sufficiently large so that the consequences of Lemma 2.6.3 are valid and Lemma 2.5.5 implies $H'(t) > 0, H(t) > 0$.

Recalling

$$F(w) - E = -\frac{k}{2}(w')^2 - w'w''' + \frac{1}{2}(w'')^2$$

and $w'(m_{i-1}) = 0$ results in

$$\kappa_2\left(2F(w(m_{i-1})) - E(m_{i-1})\right) = \kappa_2\left[F(w(m_{i-1})) + \frac{1}{2}(w''(m_{i-1}))^2\right].$$

At any extremum m_{i-1} we know $w(m_{i-1})w''(m_{i-1}) \le 0$ and $w'(m_{i-1}) = 0$; therefore,

$$\kappa_2\left[2F(w(m_{i-1})) - E(m_{i-1})\right] \le \kappa_2\left[F(w(m_{i-1})) + \frac{1}{2}(w''(m_i))^2\right]$$

$$- \alpha_3 w(m_{i-1})w''(m_{i-1}) + \alpha_1 w^2(m_{i-1})$$

$$= H(m_{i-1}) \tag{2.6.4}$$

$$\le H(\ell_i) \quad \text{(Lemma 2.5.5)}$$

$$= \alpha_1 w(\ell_i)^2 + \alpha_2 w'(\ell_i)^2 + \kappa_2 F(w(\ell_i)).$$

By Lemma 2.6.5, on $[\ell_i, m_i)$ we know $f(w)$ and w' have the same sign, thus $F(w(t))$ and $w^2(t)$ are both nondecreasing; as a result from (2.6.4) we have

$$\kappa_2\left[2F(w(m_{i-1})) - E(m_{i-1})\right] \le \alpha_1 w(m_i)^2 + \alpha_2 w'(\ell_i)^2 + \kappa_2 F(w(m_i)). \tag{2.6.5}$$

By Lemma 2.6.3 (comparing growth of $G(m_i)$ and $w(m_i)$),

$$\alpha_2 w'(\ell_i)^2 \le \alpha_1 w(\ell_i)^2 + \alpha_2 w'(\ell_i)^2 = G(\ell_i) \le G(m_i) \le C_1 |w(m_i)|^{\frac{p+3}{2}}. \tag{2.6.6}$$

Since $p > 1$ we have $p + 1 > \frac{p+3}{2} > 2$. Furthermore $|w(m_i)| \to \infty$ by Lemma 2.6.2. Also recall the growth estimate

$$F(s) \ge \frac{\rho}{p+1}|s|^{p+1}$$

from Lemma 3.7.1. These facts, equation (2.6.6), and T taken sufficiently large show that $F(w(m_i))$ is of higher order than any of the terms appearing in (2.6.6). Hence, choosing a specific constant, const $= \kappa_2/2$, (which affects the choice of T) we get from (2.6.6)

$$\alpha_1 w(m_i)^2 + \alpha_2 w'(\ell_i)^2 + \kappa_2 E(0) < \alpha_1 w(m_i)^2 + C_1 |w(m_i)|^{\frac{p+3}{2}} + \kappa_2 E(0)$$
$$< \frac{\kappa_2}{2} F(w(m_i)). \tag{2.6.7}$$

Recall that $E(m_{i-1}) = E(0)$ by Lemma 2.5.2, so

$$2\kappa_2 F(w(m_{i-1})) \overset{(2.6.5)}{\leq} \kappa_2 E(m_{i-1}) + \alpha_1 w(m_i)^2 + \alpha_2 w'(\ell_i)^2 + \kappa_2 F(w(m_i))$$
$$\overset{(2.6.7)}{\leq} \frac{\kappa_2}{2} F(w(m_i)) + \kappa_2 F(w(m_i)).$$

Since κ_2, defined in (2.5.5), is positive, we conclude

$$\frac{4}{3} F(w(m_{i-1})) \leq F(w(m_i)).$$

Reindex m_i, so that $m_0 > T$. Then $\left(\frac{4}{3}\right)^i F(w(m_0)) \leq F(w(m_i))$. From (2.5.1) we see that for $|s|$ large there is a constant (dependent on α, β, p, q) such that

$$F(s) \leq C|s|^{p+1}.$$

Hence

$$\frac{F(w(m_0))}{C} \left(\frac{4}{3}\right)^{\frac{i}{p+1}} \leq |w(m_i)|.$$

We may assume that $F(w(m_0)) > C$ and drop the coefficient on the left, which yields the desired result. $\qquad \square$

The next result will show that $|w|$ is strictly concave on $[m_i, z_{i+1}]$ when m_i is taken adequately large.

Lemma 2.6.9 (Concave behavior after an extremum). *Assume f satisfies conditions (2.2.1) and (2.2.2) and w is a nontrivial global solution to equation (2.1.2) with initial conditions satisfying (2.5.4). There exists a $T \geq 0$ such that if (z_i, z_{i+1}) is an interval where w is of one sign and $z_i \geq T$, then there are no inflection points after the extremum point m_i; therefore, $|w|$ is strictly concave on $[m_i, z_{i+1}]$.*

Proof. Take T to be large enough for the consequences of Lemma 2.6.5 to hold. To the contrary, assume $n \in (m_i, z_{i+1}]$ is the next inflection point of $|w|$ on (z_i, z_{i+1}) after m_i. If we appeal to Lemma 2.5.4 and T is sufficiently large so that $G'(t) > 0$ for all $t \geq T$ we have

$$0 < G'(n) = 2\alpha_1 w(n) w'(n) - \alpha_3 w(n) w'''(n)$$

$$\implies \quad 2\alpha_1 w(n) w'(n) > \alpha_3 w(n) w'''(n).$$

Thus $2\alpha_1 w'(n) > \alpha_3 w'''(n)$ on an interval of positivity (hence, $2\alpha_1 w'(n) < \alpha_3 w'''(n)$ on an interval of negativity). We conclude $w'''(n) < 0$ on an interval of positivity ($w'''(n) > 0$ on an interval of negativity); however, this implies, for an interval of positivity, $w'' < 0$ ($w'' > 0$ on an interval of negativity) on some interval $(n, n + \varepsilon')$ with $\varepsilon' > 0$. This contradicts though the fact that n is the next inflection point after m. $\qquad\square$

The next result describes the slope of w at its zeros. In particular, the lemma shows that $|w'(z_i)|$ grows geometrically when z_0 is taken sufficiently large.

Lemma 2.6.10 (Geometric growth of w' at zeros of w). *Assume w is a global solution to equation (2.1.2), f satisfies conditions (2.2.1) and (2.2.2), and the initial conditions satisfy (2.5.4). There exists a $T > 0$ such that if $z_0 \geq T$, then*

$$w'(z_i)^2 \geq \frac{C_1}{\alpha_2} \left(\frac{4}{3}\right)^{\frac{(p+3)i}{2(p+1)}},$$

where $\{z_i\}$ is a sequence of consecutive zeros of w.

Proof. Pick T sufficiently large so that $G'(t) > 0$ for $t \geq T$ (Lemma 2.5.4) and the conse-quences of Lemmas 2.6.3 and 2.6.8 are valid. Then

$$\alpha_2 w'(z_i)^2 = G(z_i) \geq G(m_{i-1}) \geq C_1 |w(m_{i-1})|^{\frac{p+3}{2}} \geq C_1 \left(\frac{4}{3}\right)^{\frac{(p+3)i}{2(p+1)}}.$$

$\square$

Recall that $r_i \in \mathcal{R}$ denotes the first zero of w'' in the interval $[z_i, m_i]$. In the proof of the Theorem 2.2.1 it will be necessary for $w(r_i)$ to be large. The next lemma will show that provided the inflection point is large enough, this is indeed the case.

Lemma 2.6.11 (Geometric growth of w at inflection points)**.** *Assume w is a global solu-tion to equation (2.1.2), $k < 2$, f satisfies conditions (2.2.1) and (2.2.2), and the initial conditions satisfy (2.5.4). There exists a $T \geq 0$ so that if $z_0 \in \mathcal{Z}$ and $z_0 \geq T$ then*

$$|w(r_i)| \geq \left(\frac{\alpha_1}{\alpha_3}\right)^{\frac{1}{p+1}} \frac{C_1}{\alpha_2} \left(\frac{4}{3}\right)^{\frac{2(p+3)}{(p+1)^2} i}$$

for $r_i \in \mathcal{R}$, and for each $i \geq 0$.

Proof. Let T be chosen so that $G'(t) > 0$ for $t \geq T$ (Lemma 2.5.4) and the conclusions of Lemmas 2.6.9 and 2.6.10 hold. Let $\delta = \frac{1}{2} \min \{\mu_1, \mu_2\}$. By Lemma 2.5.4,

$$\frac{d}{dt} \left(G' - \delta (w')^2\right) = G'' - 2\delta w' w''$$

$$\geq \varepsilon + \mu_1 (w')^2 + \mu_2 (w'')^2 + \mu_3 w f(w) - \mu_1 (w')^2 - \mu_2 (w'')^2$$

$$= \varepsilon + \mu_3 w f(w)$$

$$\geq \varepsilon.$$

For some $t_0 \geq T$ we have $w'(t_0) = 0$ by Theorem 2.6.4 and Lemma 2.6.6. Since $G'(t) \geq 0$

for $t \geq t_0$,

$$G'(t) > \delta w'(t)^2, \quad \forall t \geq t_0. \tag{2.6.8}$$

Henceforth without loss of generality let $T = t_0$. Next notice

$$G'(r_i) = 2\alpha_1 w(r_i)w'(r_i) - \alpha_3 w(r_i)w'''(r_i).$$

Recall $w(r_i) \neq 0$ as otherwise $G'(r_i) = 0$ contradicting Lemma 2.5.4. Solving for $w'''(r_i)$ yields

$$w'''(r_i) = \frac{2\alpha_1}{\alpha_3}w'(r_i) - \frac{G'(r_i)}{\alpha_3 w(r_i)}.$$

Substitute the last identity into the definition of the energy function $E(t)$ and recall $E(t) = E(0)$ by Lemma 2.5.1:

$$\frac{k}{2}w'(r_i)^2 + F(w(r_i)) + \frac{2\alpha_1}{\alpha_3}w'(r_i)^2 - \frac{w'(r_i)G'(r_i)}{\alpha_3 w(r_i)} - E(0) = 0.$$

Hence multiplication by $w(r_i)$ results in

$$F(w(r_i))w(r_i) - \frac{1}{\alpha_3}w'(r_i)G'(r_i) + \left(\frac{2\alpha_1}{\alpha_3} + \frac{k}{2}\right)w'(r_i)^2 w(r_i) - E(0)w(r_i) = 0.$$

Assume r_i is on an interval of positivity. Consequently, we have

$$F(w(r_i))w(r_i) + \left[\left(\frac{2\alpha_1}{\alpha_3} + \frac{|k|}{2}\right)w'(r_i)^2 - E(0)\right]w(r_i) - \frac{1}{\alpha_3}w'(r_i)G'(r_i) \geq 0. \tag{2.6.9}$$

To simplify the subsequent analysis introduce a shorthand:

$$x := w(r_i), \quad u := \left(\frac{2\alpha_1}{\alpha_3} + \frac{|k|}{2}\right)w'(r_i)^2 - E(0), \quad v := \frac{1}{\alpha_3}w'(r_i)G'(r_i).$$

From Lemma 2.6.9 we have $w'(r_i) \geq w'(z_i) \geq 0$, then by Lemma 2.6.10 it follows that if $r_i > z_i \geq T$ for T sufficiently, large then $u > 0$. Then we may rewrite equation (2.6.9) as

$$xF(x) + ux - v \geq 0, \quad u > 0. \tag{2.6.10}$$

By equation (2.6.8) we know that

$$v = \frac{1}{\alpha_3} w'(r_i) G'(r_i) \geq \frac{\delta}{\alpha_3} w'(r_i)^3.$$

Since $u^{\frac{3}{2}}$ has cubic growth in $w'(r_i)$, and the latter can be assumed sufficiently large, then there exists a constant $y > 0$ such that for all $r_i \geq T$

$$yu^{\frac{3}{2}} < \frac{1}{\alpha_3} w'(r_i) G'(r_i) = v.$$

Consequently, by (2.6.10)

$$xF(x) + ux - yu^{\frac{3}{2}} > 0.$$

Recalling the estimate on F from Lemma 3.7.1 we obtain

$$\left(\frac{\beta}{p+1} |x|^{p+1} + \frac{\alpha}{q+1} |x|^{q+1} \right) x + ux - yu^{\frac{3}{2}} > 0.$$

We claim that $x > u^{\frac{1}{p+1}}$. To the contrary, suppose $x \leq u^{\frac{1}{p+1}}$, then

$$\begin{aligned}
&\left(\frac{\beta}{p+1} |x|^{p+1} + \frac{\alpha}{q+1} |x|^{q+1} \right) x + ux - yu^{\frac{3}{2}} \\
&\leq \frac{\beta}{p+1} u^{\frac{p+2}{p+1}} + \frac{\alpha}{q+1} u^{\frac{q+2}{p+1}} + u^{\frac{p+2}{p+1}} - yu^{\frac{3}{2}}.
\end{aligned} \tag{2.6.11}$$

From condition (2.2.2), we have

$$0 < \frac{q+2}{p+1} < \frac{p+2}{p+1} < \frac{3}{2},$$

hence the right side of (2.6.11) can not be positive for large u. This yields a contradiction and so we assume on an interval of positivity with $r_i \geq T$ that

$$x = w(r_i) \geq u^{\frac{1}{p+1}} = \left(\left(\frac{2\alpha_1}{\alpha_3} + \frac{|k|}{2}\right) w'(r_i)^2 - E(0)\right)^{\frac{1}{p+1}}. \tag{2.6.12}$$

By Lemma 2.6.10 we know that for T large

$$\frac{\alpha_1}{\alpha_3} w'(r_i)^2 - E(0) \geq 0.$$

Consequently,

$$w(r_i) \geq \left(\frac{\alpha_1}{\alpha_3}\right)^{\frac{1}{p+1}} w'(r_i)^{\frac{2}{p+1}}. \tag{2.6.13}$$

By Lemma 2.6.9 we know that w is convex on $[w_i, r_i)$; consequently,

$$\left(\frac{\alpha_1}{\alpha_3}\right)^{\frac{1}{p+1}} w'(r_i)^{\frac{2}{p+1}} \geq \left(\frac{\alpha_1}{\alpha_3}\right)^{\frac{1}{p+1}} w'(z_i)^{\frac{2}{p+1}}. \tag{2.6.14}$$

Lastly, Lemma 2.6.10 gives us

$$\left(\frac{\alpha_1}{\alpha_3}\right)^{\frac{1}{p+1}} w'(z_i)^{\frac{2}{p+1}} \geq \left(\frac{\alpha_1}{\alpha_3}\right)^{\frac{1}{p+1}} \left(\frac{C_1}{\alpha_2}\right)^{\frac{1}{p+1}} \left(\frac{4}{3}\right)^{\frac{(p+3)}{2(p+1)^2}i}.$$

Combining (2.6.12), (2.6.13), and (2.6.14) results in

$$w(r_i) \geq \left(\frac{\alpha_1 C_1}{\alpha_3 \alpha_2}\right)^{\frac{1}{p+1}} \left(\frac{4}{3}\right)^{\frac{(p+3)}{2(p+1)^2}i}.$$

A similar proof works if r_i is on an interval of negativity. □

2.7 Estimating distances between zeros

In this section we will show the distances between consecutive zeros of w are bounded above by a summable geometric series. This observation along with the fact

$$\lim_{i \to \infty} |w(m_i)| = \infty$$

from Lemma 2.6.8 will give finite time blow-up for solutions of (2.1.2) provided the assumptions of Theorem 2.2.1 are satisfied.

Begin with the following auxiliary result for G similar to the one used back in Lemma 2.6.6 about oscillations of w for $k \leq 0$.

Lemma 2.7.1. *Assume w is a global solution to equation (2.1.2), f satisfies (2.2.1) and (2.2.2), and the initial conditions satisfy (2.5.4). Let $\lambda > 0$ be such that*

$$\frac{(p+1)(1+\lambda)}{p-\lambda} < 2 \quad and \quad 2(1+\lambda) < p+1 \tag{2.7.1}$$

(λ exists since $p > 1$). The set of points t in $[0, \infty)$ where w exists and

$$|w'|^{2(1+\lambda)}(t) < 2w(t)f(w(t))$$

has finite measure (the factor 2 is just for convenience).

Proof. From the definition of G it follows that whenever $|w'|^{2(1+\lambda)} < 2wf(w)$ we have

$$G^{1+\lambda}$$

$$\lesssim |(w')|^{2(1+\lambda)} + |w|^{2(1+\lambda)} + |ww''|^{1+\lambda} + 1$$

$$\lesssim |w'|^{2(1+\lambda)} + |w|^{2(1+\lambda)} + |w|^{p+1} + |w''|^{\frac{(p+1)(1+\lambda)}{p-\lambda}} + 1 \quad \text{(Young's Inequality)}$$

$$\lesssim |w'|^{2(1+\lambda)} + |w|^{p+1} + |w''|^2 + 1 \quad \text{(recall } p + 1 \geq 2(1 + \lambda))$$

$$\lesssim wf(w) + |w''|^2 + 1 \quad \text{(condition (2.2.2) and } |w'|^{2(1+\lambda)} < \tfrac{1}{2}wf(w)).$$

From Lemma 2.5.4 we have

$$G'' \geq \mu_1(w')^2 + \mu_2(w'')^2 + \mu_3 wf(w) + \varepsilon$$

$$\geq \mu_1(w')^2 + \mu_2(w'')^2 + \mu_3 \rho |w|^{p+1} + \varepsilon$$

and so for some constant $\beta > 0$ we have

$$\beta G''(t) > G(t)^{1+\lambda} \tag{2.7.2}$$

on sets where

$$|w'(t)|^{2(1+\lambda)} < 2w(t)f(w).$$

Let U be the union of all (open) sets where $|w'(t)|^{2(1+\lambda)} < 2w(t)f(w(t))$, then on U the estimate (2.7.2) holds. Any solution u to the differential inequalities $u' > \varepsilon > 0$ and $\beta u''(t) > u(t)^{1+\lambda}$ has only finite existence time. For T sufficiently large we know that $G' > \varepsilon > 0$ on $[T, \infty)$ by Lemma 2.5.4. Since G satisfies (2.7.2) on the set U, and G remains strictly increasing outside U, we conclude that $|U| < \infty$, otherwise, G blows up in finite time. □

Remark 2.7.2. The proof for Theorem 2.2.1 simplifies drastically in both cases, $k > 0$ and

$k \leq 0$, if one shows inequality (2.7.2) holds on the entire real line. This inequality reduces to showing

$$(w')^{2(1+\lambda)} \lesssim (w')^2 + (w'')^2 + wf(w).$$

Numerical evidence supports this conjecture, however, we were unable to prove it at this point.

The following lemma was motivated by [17, pp. 22–24] and will show

$$\sum |m_i - z_{i+1}| < \infty.$$

Lemma 2.7.3. *Assume w is a global solution to equation (2.1.2), f satisfies conditions (2.2.1) and (2.2.2), and the initial values satisfy (2.5.4). There exists a T sufficiently large such that if $z_0 \in \mathcal{Z}$ and $z_0 \geq T$ then*

$$|m_i - z_{i+1}| < \frac{C}{\sqrt{w(m_i)}} \leq C\left(\frac{3}{4}\right)^{\frac{p-1}{4(p+1)}i}$$

where $i \in \mathbb{N}$ and $C = C(k, \rho, p)$.

Proof. Let T be large enough for the consequences of Lemmas 2.6.8 and 2.6.9 to be valid and Lemma 2.5.4 implies $G'(t) > 0$ for $t \geq T$. For convenience reindex the set of zeros $\mathcal{Z}$ so that $z_0 \geq T$. Since $w(t - m_i)$ is a solution to equation (2.1.2) if and only if $w(t)$ is, we may assume $m_i = 0$. Integrating $w'''' + kw'' + f(w) = 0$ four times with respect to t results in

$$w(t) = t^3\frac{w'''(0)}{6} + t^2\frac{w''(0)}{2} + \left(t + \frac{t^3k}{6}\right)w'(0) + \left(1 + \frac{t^2k}{2}\right)w(0)$$

$$- k\int_0^t\int_0^{t_3} w(t_2)dt_2dt_3 - \int_0^t\int_0^{t_3}\int_0^{t_2}\int_0^{t_1} f(w(t_0))dt_0dt_1dt_2dt_3. \quad (2.7.3)$$

Suppose that $[z_i, z_{i+1}]$ is an interval of positivity of w. Since $m_i = 0$ is a maximum we know that $w'(0) = 0$ and $w''(0) \le 0$. Additionally, by Lemma 2.5.4 we know $G'(0) = -\alpha_3 w(0) w'''(0) \ge 0$. It follows that $w'''(0) \le 0$. Applying this to equation (2.7.3) yields

$$w(t) \le \left(1 + \frac{t^2 k}{2}\right) w(0) - k \int_0^t \int_0^{t_3} w(t_2) dt_2 dt_3 - \int_0^t \int_0^{t_3} \int_0^{t_2} \int_0^{t_1} f(w(t_0)) dt_0 dt_1 dt_2 dt_3.$$

Recall from condition 2.2.2, that $f(w) \ge \rho w^p$ since we are on an interval of positivity. Thus

$$w(t) \le \left(1 + \frac{t^2 |k|}{2}\right) w(0) + |k| \int_0^t \int_0^{t_3} w(t_2) dt_2 dt_3 - \rho \int_0^t \int_0^{t_3} \int_0^{t_2} \int_0^{t_1} w(t_0)^p dt_0 dt_1 dt_2 dt_3.$$

By Lemmas 2.6.7 and 2.6.9 we know w is decreasing and concave on $[0, z_{i+1})$. Consequently, for any $\delta \in (0, z_{i+1})$ and $t \in [0, z_{i+1}]$ we know that

$$w(t) \ge \ell(t) := w(0) - \frac{w(0)}{\delta} t$$

and $w(t) \le w(0)$. We conclude

$$w(t) \le \left(1 + \frac{t^2 |k|}{2}\right) w(0) + |k| w(0) \int_0^t \int_0^{t_3} dt_2 dt_3 - \int_0^t \int_0^{t_3} \int_0^{t_2} \int_0^{t_1} \rho \ell(t_0)^p dt_0 dt_1 dt_2 dt_3.$$

Evaluating at $t = \delta$ gives us

$$w(\delta) \le \frac{w(0)}{24 + 6p} \left(24 + 6p + \delta^2 (6|k|p + 24|k|) - w(0)^{p-1} \rho \delta^4\right).$$

Since $w(\delta) \ge 0$, we have

$$0 \le 24 + 6p + \delta^2 (6|k|p + 24|k|) - w(0)^{p-1} \rho \delta^4 .$$

Consequently,

$$\delta^2 \leq \frac{(6|k|p + 24|k|) + \sqrt{(6|k|p + 24|k|)^2 + 4\rho(24 + 6p)w(0)^{p-1}}}{2\rho w(0)^{p-1}}.$$

This inequality along with Lemma 2.6.8 (about $w(m_i)$ being eventually large, whence here we may assume that for $w(0)$) implies that

$$\delta \leq \frac{C}{w(0)^{\frac{p-1}{4}}}, \quad \text{for some } C = C(k, p, \rho) > 0.$$

Note that C is not dependent on m_i. This estimate holds uniformly for every $\delta \in [m_i, z_{i+1})$, hence

$$|m_i - z_{i+1}| \leq \frac{C}{w(0)^{\frac{p-1}{4}}} = \frac{C}{w(m_i)^{\frac{p-1}{4}}}.$$

The result follows by recalling $|w(m_i)| \geq \left(\frac{4}{3}\right)^{\frac{i}{p+1}}$ from Lemma 2.6.8. A similar proof works for an interval of negativity. $\square$

The next and final lemma will show that $\sum |z_i - m_i| < \infty$.

Lemma 2.7.4. *Assume w is a global solution to equation (2.1.2), f satisfies conditions (2.2.1) and (2.2.2), and the initial values satisfy (2.5.4). There exists a $T \geq 0$ such that if $z_i \geq T$, then*

$$\sum_{i=1}^{\infty} |z_i - m_i| < \infty.$$

Proof. Let T be sufficiently large so that $G' > 0$ for $t \geq T$ (Lemma 2.5.4) and the consequences of Lemmas 2.6.7, 2.6.10, and 2.6.11 are valid. For convenience reindex the set of zeros $\mathcal{Z}$ so that $z_0 \geq T$. Recall $w(t)f(w(t)) \geq \rho|w|^{p+1}$ by condition (2.2.2). Let $\lambda > 0$ be chosen as in Lemma 2.7.1. By Lemma 2.7.1 we know the total measure of sets where

$$|w'(t)|^{2(1+\lambda)} < 2w(t)f(w(t))$$

is finite so we will consider subsets of $[z_i, m_i]$ where

$$|w'(t)|^{2(1+\lambda)} > w(t)f(w(t)) \geq \rho|w(t)|^{p+1} .$$

The sets where these (strict) inequalities hold forms an open cover of the interval of existence of w. Let us focus on a subinterval $[z_i, m_i]$ and assume it resides in an interval of positivity.

We know the inequality $|w'|^{2(1+\lambda)} > \rho|w|^{p+1}$ is equivalent to

$$w' > \rho^{\frac{1}{2(1+\lambda)}} w^{\frac{p+1}{2(1+\lambda)}}$$

since w, w' are nonnegative.

By Lemma 2.6.11 we know that $\xi_i \leq r_i$ where r_i is the first inflection point on $[z_i, m_i]$. Using the estimate on $w(r_i)$ from Lemma 2.6.11 we see that there exists $\xi_i \in (z_i, r_i]$ such that

$$w(\xi_i) = v := \left(\frac{\alpha_1 C_1}{\alpha_2 \alpha_3}\right)^{\frac{1}{p+1}} \left(\frac{4}{3}\right)^{\frac{(p+3)}{2(p+1)^2}i} .$$

We will now give a bound on the measure of sets contained in $[\xi_i, m_i]$ where $w' > \rho^{\frac{1}{2(1+\lambda)}} w^{\frac{p+1}{2(1+\lambda)}}$. Consider the initial value problem

$$\begin{cases} u' = \rho^{\frac{1}{2(1+\lambda)}} u^{\nu+1} \\ u(\xi_i) = u_{\xi_i} \end{cases} \quad \text{for} \quad \nu := \frac{p+1}{2(1+\lambda)} - 1 > 0 .$$

Solving the above equation for u results in

$$u(t) = \left(-\nu\rho^{\frac{1}{2(1+\lambda)}} t + \nu\rho^{\frac{1}{2(1+\lambda)}} \xi_i + \frac{1}{u_{\xi_i}^\nu}\right)^{-\frac{1}{\nu}} = \left(\nu\rho^{\frac{1}{2(1+\lambda)}} (\xi_i - t) + \frac{1}{u_{\xi_i}^\nu}\right)^{-\frac{1}{\nu}} .$$

It is clear that the solution u to the above initial value problem is only defined for $t <$

$\xi_i + \left(u_{\xi_i}^\gamma \nu \rho^{\frac{1}{2(1+\lambda)}}\right)^{-1}$. Thus, using the definition of u_{ξ_i}, we see that the existence time for u in the interval $[\xi_i, m_i]$ is no more than

$$\left(u_{\xi_i}^\gamma \phi \rho^{\frac{1}{2(1+\lambda)}}\right)^{-1} \le C(\alpha_1, \alpha_2, \phi, \rho, k)\left(\frac{4}{3}\right)^{-\frac{\phi(p+3)}{2(p+1)^2}i} \quad \text{with} \quad \phi > 0.$$

By Lemma 2.6.5 we know w monotonically increases to $w(m_i)$ on the first part $[z_i, m_i]$ of the positivity interval $[z_i, z_{i+1}]$. Thus the existence time in $[\xi_i, m_i]$ for w where $|w'|^{2(1+\lambda)} > \rho|w|^{p+1}$ is no larger than the existence time for u; therefore, an upper bound on the measure of sets where $|w'|^{2(1+\lambda)} > \rho|w|^{p+1}$ in $[\xi_i, m_i]$ is given by $C(\alpha_1, \alpha_2, \phi, \rho, k)\left(\frac{4}{3}\right)^{-\frac{\phi(p+3)}{2(p+1)^2}i}$. We conclude

$$\sum_{i=1}^\infty |\xi_i - m_i| < \infty.$$

Now let us consider the interval $[z_i, \xi_i]$. Since $\xi_i \le r_i$ we know that w is convex on the interval. Thus $w(\xi_i) \ge (\xi_i - z_i)w'(z_j)$. Appealing to Lemma 2.6.10, we obtain

$$w'(z_j) \ge \left(\frac{C_1}{\alpha_2}\right)^{\frac{1}{2}}\left(\frac{4}{3}\right)^{\frac{(p+3)i}{4(p+1)}}.$$

Consequently ,

$$|\xi_i - z_i| < w(\xi_i)\left(w'(z_j)\right)^{-1}$$
$$\le \left(\frac{\alpha_1 C_1}{\alpha_2 \alpha_3}\right)^{\frac{1}{p+1}}\left(\frac{4}{3}\right)^{\frac{(p+3)}{2(p+1)^2}i}\left(\frac{\alpha_2}{C_1}\right)^{\frac{1}{2}}\left(\frac{3}{4}\right)^{\frac{(p+3)i}{4(p+1)}}$$
$$= \left(\frac{\alpha_1 C_1}{\alpha_2 \alpha_3}\right)^{\frac{1}{p+1}}\left(\frac{\alpha_2}{C_1}\right)^{\frac{1}{2}}\left(\frac{3}{4}\right)^{\left(\frac{p+3}{4(p+1)} - \frac{p+3}{2(p+1)^2}\right)i}.$$

Since $p > 1$ we know $\frac{p+3}{4(p+1)} - \frac{p+3}{2(p+1)^2} > 0$ and so

$$\sum_{i=1}^\infty |\xi_i - z_i| < \infty.$$

□

Now we are in position to finish the proof of Theorem 2.2.1.

Theorem 2.2.1. We will show that no global solution exists. To the contrary, assume w is a global solution to (2.1.2) with initial energy satisfying (2.2.3). By Theorem 2.6.4 and Lemma 2.6.6 we know that w must change sign infinitely many times as $t \to \infty$. If $\{m_i\}$ are the extrema of w, Lemma 2.6.8 implies $\{|w(m_i)|\} \to \infty$ as $i \to \infty$. Lemmas 2.7.3 and 2.7.4 imply the zeros of w, hence m_i's, have a limit point m_∞. We conclude $w(t)$ is unbounded in a neighborhood of m_∞. So w cannot be extended past m_∞. □

Chapter 3

A nonlocal biharmonic operator and its connection with the classical analogue

In this chapter we investigate the nonlocal theory of plate dynamics by analyzing nonlocal formulations of elliptic equations. The benefit of this approach is a reduced requirement on the regularity of solutions which allows fracture propagation and other discontinuities to be modeled. With this goal in mind, we begin our journey into the nonlocal theory with a literature overview in Section 3.1. Then the nonlocal framework is outlined in Section 3.2. Following this, necessary conditions for L^2 integrability of the nonlocal Laplacian and nonlocal biharmonic, as well as Hölder-continuity of the nonlocal Laplacian are proven. In Section 3.3 we define the nonlocal spaces in which we prove our well-posedness results. Section 3.4 reviews critical compactness results that are necessary for the proof of our main theorems. Section 3.5 addresses the connection between the local and nonlocal operators, by proving convergence results (and rates of convergence results) as the size of the support of the kernel shrinks to zero. Section 3.6 presents the well-posedness proof for the nonlocal steady state problem with hinged and clamped boundary conditions. Finally, our main results lie in Section 3.7 where it is proven that when the horizon approaches zero, solutions

of the nonlocal clamped and hinged steady state problems converge strongly in L^2 to the weak solutions of their local analogues.

3.1   Nonlocal background and literature overview

Classical models of continuum mechanics give rise to fourth-order elliptic PDEs describing transversal deformations of thin plates, shells or beams, possibly in coupling with additional equations quantifying shear forces [36, 38]. Under some regularity conditions on the boundary, solutions to fourth-order elliptic boundary value problem generally acquire four orders of weak differentiability with respect to the regularity of the interior forcing term. In two dimensions, systems with square integrable forcing such systems typically possess weak solutions that have at least $H^2 \subset W^{1,\infty}$ Sobolev regularity. In particular, such solutions (in the 2D case) are necessarily continuous which makes it non-trivial to account for irregularities, such as cracks. On the other hand, prime examples of plates structures are suspension bridges, where the dynamic formation of cracks and their evolution is of great interest, whereas discontinuities corresponding to damage preclude the inclusion of smoothness assumptions on the solutions.

A proposed paradigm for investigating less regular solutions is to replace the classical operators of elasticity theory with suitable approximations that replace derivatives with singular integral operators. This approach is prompted by physical considerations, such as describing the stress at a point via the cumulative interaction with points from its neighborhood; this interaction is often captured through a suitable integral kernel.

Nonlocal versions of the classical Laplace operator have been investigated in various settings and phenomena: nonlocal diffusion [3], population and swarm models [9, 30], and image processing [19]. Recently, this nonlocal operator was used in the peridynamic theory developed by Silling [37] to describe the evolution of damage in solids. The relaxed

regularity conditions permit fractures to be represented in the solution of the system rather than being considered through separate ad-hoc frameworks. Within the context of the state-based formulation of peridynamics, nonlocal models for beams and plates were developed in [31, 32]; however, the nonlocality in these models resembles more the nonlocal Laplacian structure (see [6, 12, 20]). Higher order nonlocal operators have been considered; however, much of the work has been on time-asymptotics as seen in [3]. In [22] a composition of a local Laplacian with a nonlocal Laplacian was investigated. By replacing differential operators with integral operators it is possible to have well-defined solutions with discontinuities (in fact without any Sobolev regularity).

Motivated by these developments we introduce a nonlocal version of the biharmonic operator, obtained by iterating the nonlocal Laplacian. We show that solutions of the nonlocal biharmonic equation with nonlocal analogues of hinged or clamped boundary conditions require minimal integrability assumptions; moreover, in the limit they recover the classical weak solutions to the corresponding elliptic fourth-order problem.

As previous works have also demonstrated, the nonlocal setting offers an alternative way to study problems in a weak formulation. In contrast with the classical framework which often considers *regularized solutions* to a problem and passes to the limit in a weaker topology to obtain a *less regular* solution, in the nonlocal framework the investigation starts in a weaker topology, and then as the support of the kernel shrinks the corresponding weak solutions converge (in the weak topology) to a *more regular* solution. The diagram below stresses this idea:

Modeling approach	Convergence
Local/Classical	regularized approximations $\to$ weak solution
Nonlocal	L^2 approximations $\to$ weak solution

Finally, the results presented here do not rely on the scalar setting, so they are transferrable to the vectorial framework as introduced in [12].

Boundary conditions. Although a nonlocal form of the biharmonic operator appears natural, the form of the boundary conditions (BC) is more delicate. Two types of homogeneous BC are fundamental in plate systems: hinged ($u = 0$, $\Delta u = 0$ on the boundary of the domain) and clamped ($u = 0$, $\frac{\partial u}{\partial \nu} = 0$). Since nonlocal operators are associated with collar-type constraints—imposed on sets of positive Lebesgue measure that surround the domain—we need to find appropriate nonlocal generalizations that converge to the classical conditions in the limit as the approximations improve. To our knowledge, this is the first work that deals with integral approximations of higher order elliptic operators with first and second order boundary conditions. One of the features of the nonlocal approach is that approximations and their boundary conditions can be formulated for very rough domains. If the domain has C^1 regularity (in fact, some relaxation might be possible, see Remark 3.4.4) then these nonlocal solutions converge in L^2 to some function as the "interaction horizon" decreases. The limit can be shown to be a distributional solution to the original PDE. Naturally, this can be identified with weak (or strong) solutions of the classical elliptic problem only if the domain possesses some additional smoothness C^2 (or C^4).

Some of the main tools used in local theories are compactness results (such as Gagliardo-Nirenberg-Sobolev embedding theorems), estimates obtained through Poincaré-type inequalities, and in parabolic/elliptic theory one often uses the gain in the smoothness of solutions. For nonlocal problems in the case of operators with weakly (i.e. integrable) singular kernels, however, these methods do not apply. Whereas solutions to the Poisson problem gain two degrees of regularity over the forcing term, in the nonlocal scenario there is no improvement in regularity. A nonlocal version of Poincaré's inequality exists, however, it does not yield higher L^p integrability for the solution from bounds of the nonlocal gradient. In addition, the embedding theorems do not hold, unless the kernel exhibits a strong (i.e. non-integrable) singularity. One of the key tools is the result of of Bourgain, Brezis, and Mironescu [7] that exhibits compactness when the kernel is "almost-integrable". Such

compactness theorems have been used and further developed by Du and Mengesha [29] (which is also an inspiration for our work), as well as by Ignat and his collaborators [23].

3.1.1 Contributions to the nonlocal theory

The contributions of this exposition to the development of nonlocal theories and understanding the connections between local and nonlocal models are as follows:

- we introduce a higher-order nonlocal analogue to the biharmonic operator and we formulate nonlocal equivalents of clamped and hinged boundary conditions; we establish well-posedness of solutions to these nonlocal boundary values problems supplemented with nonlocal BCs;

- under certain assumptions on the kernel of the integral operator (which include singular and weakly singular cases) we prove several properties for the nonlocal operators such as: L^2 integrability as well as Hölder continuity of the nonlocal Laplacian applied to a sufficiently regular function (see Proposition 3.2.12, respectively, Theorem 3.2.13);

- we show L^2 strong convergence of the sequence of nonlocal solutions to the local one as the radius of the support of the kernel in the integral operator goes to zero.

3.2   Background

This section contains definitions of several fundamental integral operators and associated function spaces that will be central to our work. Henceforth, Ω will denote an open bounded connected subset of $\mathbb{R}^d$; in some results, we will specialize to $d = 2$ as the more interesting case due to the integrability conditions and because of its connection to the thin plate theory. As we will see in the sequel the existence results for solutions to the nonlocal problems
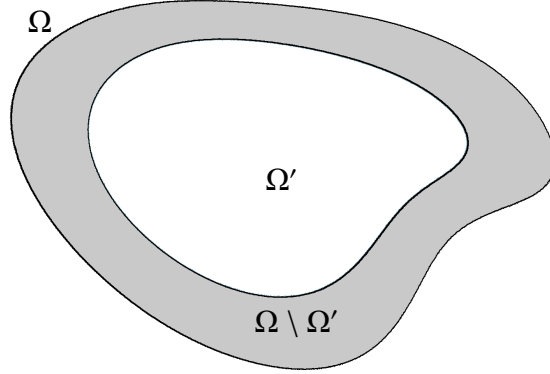
Figure 3.1: The nonlocal domain Ω' with its collar boundary $\Omega \setminus \Omega'$.

defined below do not require any regularity conditions on Ω. However, in order to establish connections between nonlocal and classical systems, we will need impose some smoothness conditions on the boundary. The open subdomain Ω' will be compactly contained in Ω (see Figure 3.1 for an example of the domain Ω' with its boundary $\Omega \setminus \Omega'$).

3.2.1 Operators

As in [12], we introduce several nonlocal peridynamic operators.

Definition 3.2.1 (Nonlocal divergence). For a function $\boldsymbol{\nu} : \Omega \times \Omega \rightarrow \mathbb{R}^k$ and an antisymmetric vector-valued kernel $\alpha : \Omega \times \Omega \rightarrow \mathbb{R}^k$, the nonlocal divergence operator $\mathcal{D}_\alpha$ is a function-valued mapping whose image $\mathcal{D}_\alpha[\boldsymbol{\nu}]$ is defined by $\mathcal{D}_\alpha[\boldsymbol{\nu}] : \Omega \rightarrow \mathbb{R}$

$$\mathcal{D}_\alpha[\boldsymbol{\nu}](\mathbf{x}) := \int_\Omega (\boldsymbol{\nu}(\mathbf{x}, \mathbf{y}) + \boldsymbol{\nu}(\mathbf{y}, \mathbf{x})) \cdot \alpha(\mathbf{x}, \mathbf{y}) d\mathbf{y} \quad \text{for} \quad \mathbf{x} \in \Omega.$$

Definition 3.2.2 (Nonlocal two-point gradient). Given a function $u(\mathbf{x}) : \mathbb{R}^d \rightarrow \mathbb{R}$, the formal adjoint of $\mathcal{D}_\alpha$ is the nonlocal two-point gradient operator $\mathcal{G}_\alpha : u \mapsto \mathcal{G}_\alpha$ where

$\mathcal{G}_\alpha : \Omega \times \Omega \to \Omega$ is given by

$$\mathcal{G}_\alpha[u](\mathbf{x}, \mathbf{y}) = (u(\mathbf{y}) - u(\mathbf{x}))\alpha(\mathbf{x}, \mathbf{y}) \quad \text{for} \quad (\mathbf{x}, \mathbf{y}) \in \Omega \times \Omega.$$

Definition 3.2.3 (Nonlocal normal derivative). For a function $\nu : \Omega \times \Omega \to \mathbb{R}^k$ and an antisymmetric vector-valued function $\alpha : \Omega \times \Omega \to \mathbb{R}^k$, the nonlocal normal operator is a mapping $\mathcal{N}_\alpha : \nu \mapsto \mathcal{N}_\alpha[\nu]$ where $\mathcal{N}_\alpha[\nu] : \Omega \to \mathbb{R}$ is given by

$$\mathcal{N}_\alpha[\nu](\mathbf{x}) := \int_{\Omega \backslash \Omega'} (\nu(\mathbf{x}, \mathbf{y}) + \nu(\mathbf{y}, \mathbf{x})) \cdot \alpha(\mathbf{x}, \mathbf{y}) d\mathbf{y} \quad \text{for} \quad \mathbf{x} \in \text{int}(\Omega'). \tag{3.2.1}$$

Definition 3.2.4 (Nonlocal Laplacian). Let $u : \Omega \to \mathbb{R}$ and $\mu = \alpha^2$ where $\alpha : \Omega \times \Omega \to \mathbb{R}^k$ is an antisymmetric vector-valued function. The nonlocal Laplace operator is defined by:

$$\mathcal{L}_\alpha[u](\mathbf{x}) := \mathcal{D}_\alpha[\mathcal{G}_\alpha[u]] = 2 \int_\Omega (u(\mathbf{y}) - u(\mathbf{x}))\mu(\mathbf{x}, \mathbf{y}) d\mathbf{y} \quad \text{for} \quad \mathbf{x} \in \Omega.$$

It was shown in [12, Prop. 5.4], that if α^2 is formally replaced by distributional application of $\frac{1}{2}\Delta_\mathbf{y}\delta(\mathbf{y} - \mathbf{x})$, then $\mathcal{L}_\alpha$ can be identified, in the sense of distributions, with the Laplace operator $\Delta_\mathbf{x}$.

Following the above framework we define the nonlocal biharmonic operator:

Definition 3.2.5 (Nonlocal biharmonic). Let $\alpha : \Omega \times \Omega \to \mathbb{R}^k$ be an antisymmetric vector valued function and $u : \Omega \to \mathbb{R}$. We define the nonlocal biharmonic by

$$\mathcal{B}_\alpha[u](\mathbf{x}) = \mathcal{L}_\alpha[\mathcal{L}_\alpha[u]], \quad \text{for} \quad \mathbf{x} \in \Omega. \tag{3.2.2}$$

3.2.2 Continuity and integrability

Let us recall and prove several results regarding the integrability and continuity for some of the above nonlocal operators.

First let us recall a nonlocal version of the "integration by parts" theorem, a simple consequence of the fact that the integrand is antisymmetric:

Proposition 3.2.6 (Nonlocal integration by parts [12])**.** *Let $\Omega \subset \mathbb{R}^d$ be open, $u, w : \Omega \to \mathbb{R}$, $\alpha : \Omega \times \Omega \to \mathbb{R}^k$ be antisymmetric, and $\boldsymbol{v} : \Omega \times \Omega \to \mathbb{R}^k$. When $\mathcal{D}_\alpha v \in L^2(\Omega)$ and $\mathcal{G}_\alpha \in L^2(\Omega \times \Omega)$ we have*

$$\int_\Omega u(\mathbf{x}) \mathcal{D}_\alpha[\boldsymbol{v}] d\mathbf{x} = - \int_\Omega \int_\Omega \mathcal{G}_\alpha[u] \cdot \boldsymbol{v} d\mathbf{y} d\mathbf{x}. \tag{3.2.3}$$

As a special case, when $\mathcal{L}_\alpha[u], w \in L^2(\Omega)$ and $\mathcal{G}_\alpha[u], \mathcal{G}_\alpha[w] \in L^2(\Omega \times \Omega)$, we have

$$\int_\Omega \mathcal{L}_\alpha[u] w d\mathbf{x} = - \int_\Omega \int_\Omega \mathcal{G}_\alpha[u] \mathcal{G}_\alpha[w] d\mathbf{y} d\mathbf{x}. \tag{3.2.4}$$

As remarked subsequently, the identity (3.2.4) *by definition applies to weak nonlocal Laplacian introduced below in Definition 3.3.3.*

Remark 3.2.7. The terms in "nonlocal" integration by parts already incorporate the information from the collar of the domain, thus boundary terms do not explicitly appear in (3.2.3) or (3.2.4).

The next result provides us with an inequality that gives an upper bound for the nonlocal gradient in terms of its classical counterpart:

Theorem 3.2.8 (c.f. [7, Thm. 1])**.** *Let Ω be a bounded Lipschitz domain. Suppose $f \in$*

$W^{1,p}(\Omega), 1 \le p < \infty$ *and let* $\xi \in L^1(\mathbb{R}^d), \xi \ge 0$. *Then*

$$\int_\Omega \int_\Omega \frac{|f(\mathbf{x}) - f(\mathbf{y})|^p}{|\mathbf{x} - \mathbf{y}|^p} \xi(\mathbf{x} - \mathbf{y}) d\mathbf{x} d\mathbf{y} \le C \|f\|_{W^{1,p}(\Omega)}^p \|\xi\|_{L^1(\Omega)},$$

where C depends only on p and Ω.

Remark 3.2.9. The result [7, Thm. 1] focuses on smooth domains, but the proof only requires the function in question to have a $W^{1,p}(\mathbb{R}^n)$ extension. So it suffices for Ω to satisfy a strong locally Lipschitz condition, or equivalently be bounded and Lipschitz (see, for example, [1, p. 83 and Thm. 5.24 on p. 154]).

We will need the following assumption, first introduced in [7], on the family of kernels used in our nonlocal formulations.

Assumption 3.2.10 (Kernel α). *For* $\delta > 0$ *let* ρ_δ *be a radial compactly-supported mollifier satisfying*

$$\rho_\delta : C^\infty(\mathbb{R}^+; \mathbb{R}^+), \quad \int_{\mathbb{R}^d} \rho_\delta(|\mathbf{x}|) d\mathbf{x} = 1, \quad \text{supp}(\rho_\delta) \subset [0, \delta). \tag{3.2.5}$$

Define

$$\alpha_\delta(\mathbf{x}, \mathbf{y}) := \frac{\sqrt{\rho_\delta(|\mathbf{x} - \mathbf{y}|)}}{|\mathbf{x} - \mathbf{y}|^2}(\mathbf{x} - \mathbf{y}). \tag{3.2.6}$$

Henceforth when if δ *is held constant in the context, we will often drop subscript temporarily denoting*

$$\alpha := \alpha_\delta$$

whenever there is no confusion.

With Assumption 3.2.10 placed on α, we will study the conditions that must be placed on the function u so that $\mathcal{L}_\alpha[u] \in L^2(\Omega)$ and $\mathcal{B}_\alpha[u] \in L^2(\Omega)$. In particular, note that $\mathcal{L}_\alpha$ is formally quadratic in α which means that the kernel μ in Definition 3.2.4 would not be

integrable for domains $\Omega \subset \mathbb{R}^2$. The next few propositions will provide sufficient conditions which ensure that these functions are well-defined.

Proposition 3.2.11. *Suppose $\rho \in C_c^\infty(\mathbb{R}^d)$ and $d \geq 2$. If $\Omega \subset \mathbb{R}^d$ is bounded and $a > 0$, then the following mapping is continuous on $\mathbb{R}^d$:*

$$F(\mathbf{x}) := \int_\Omega \frac{\rho(|\mathbf{y} - \mathbf{x}|)}{|\mathbf{y} - \mathbf{x}|^{2-a}} d\mathbf{y}.$$

Proof. Fix $\mathbf{x} \in \mathbb{R}^d$. Note that the result clearly holds for $a \geq 2$, so assume $0 < a < 2$. Since $|\mathbf{y} - \mathbf{x}|^{-(2-a)} \in L^1(\mathbb{R}^d)$ for $a > 0$ and $d \geq 2$, there exists an $r > 0$ such that

$$\int_{B(\mathbf{x},r)} \frac{1}{|\mathbf{y} - \mathbf{x}|^{2-a}} d\mathbf{y} < \frac{\varepsilon}{3\|\rho\|_{L^\infty}}. \tag{3.2.7}$$

Since ρ is smooth, clearly

$$\mathbf{y} \mapsto \kappa(\mathbf{x}, \mathbf{y}) := \frac{\rho(|\mathbf{y} - \mathbf{x}|)}{|\mathbf{y} - \mathbf{x}|^{2-a}} \in C\left(\mathbb{R}^d \setminus B\left(\mathbf{x}, \frac{r}{2}\right)\right).$$

Thus there exists a $\delta \in \left(0, \frac{r}{2}\right)$ such that whenever $\mathbf{x}' \in B(\mathbf{x}, \delta)$ we have

$$|\kappa(\mathbf{x}, \mathbf{y}) - \kappa(\mathbf{x}', \mathbf{y})| < \frac{\varepsilon}{3|\Omega|} \quad \text{for all} \quad \mathbf{y} \in \Omega \setminus B(\mathbf{x}, r).$$

Then

$$|F(\mathbf{x}) - F(\mathbf{x}')| = \left| \int_{\Omega} [\kappa(\mathbf{x}, \mathbf{y}) - \kappa(\mathbf{x}', \mathbf{y})] d\mathbf{y} \right|$$

$$\leq \left| \int_{\Omega \setminus B(\mathbf{x}, \delta)} [\kappa(\mathbf{x}, \mathbf{y}) - \kappa(\mathbf{x}', \mathbf{y})] d\mathbf{y} \right|$$

$$+ \|\rho\|_{L^{\infty}} \int_{B(\mathbf{x}, \delta)} \frac{1}{|\mathbf{y} - \mathbf{x}|^{2-a}} d\mathbf{y} + \|\rho\|_{L^{\infty}} \int_{B(\mathbf{x}, \delta)} \frac{1}{|\mathbf{y} - \mathbf{x}'|^{2-a}} d\mathbf{y}$$

$$\overset{(3.2.7)}{\leq} \frac{\varepsilon}{3|\Omega|} \int_{\Omega \setminus B(\mathbf{x}, \delta)} d\mathbf{y} + \|\rho\|_{L^{\infty}} \frac{\varepsilon}{3\|\rho\|_{L^{\infty}}} + \|\rho\|_{L^{\infty}} \int_{B(\mathbf{x}', \delta)} \frac{1}{|\mathbf{y} - \mathbf{x}'|^{2-a}} d\mathbf{y}$$

$$\overset{(3.2.7)}{\leq} \varepsilon.$$

The second to last line follows by noting that integrating $\frac{1}{|\mathbf{y} - \mathbf{x}'|^{2-a}}$ in terms of $\mathbf{y}$ over a ball centered at $\mathbf{x}'$ will be larger than integrating over a ball of the same radius centered at any other point since the singularity is at $\mathbf{y} = \mathbf{x}'$. It follows that $F \in C(\mathbb{R}^d)$. $\qquad \square$

Proposition 3.2.12 (L^2-integrability of the nonlocal Laplacian). *Suppose α satisfies Assumption 3.2.10 and $d \geq 2$. If $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain and $u \in W^{1,p}(\Omega)$ with $p > 2$ then $\mathcal{L}_\alpha[u] \in L^2(\Omega)$.*

Proof. For a shorthand let

$$\kappa(\mathbf{x}, \mathbf{y}) := \frac{\rho(|\mathbf{x} - \mathbf{y}|)}{|\mathbf{y} - \mathbf{x}|} \quad \text{and} \quad q(\mathbf{x}, \mathbf{y}) := \frac{u(\mathbf{y}) - u(\mathbf{x})}{|\mathbf{y} - \mathbf{x}|}.$$

Also, henceforth, if $\psi(\mathbf{x}, \mathbf{y})$ is a function on $\Omega \times \Omega$ then $L^q(\Omega, \mathbf{x})$ or $L^q(\Omega, \mathbf{y})$ will denote the L^q norm of the $\mathbf{y}$- or $\mathbf{x}$-section of ψ respectively.

By Hölder's inequality, for any $\varepsilon \in (0, 1)$ we have

$$\|\mathcal{L}_\alpha[\phi]\|_{L^2}^2 = \int_{\Omega} \left| \int_{\Omega} \frac{u(\mathbf{y}) - u(\mathbf{x})}{|\mathbf{y} - \mathbf{x}|^2} \rho(\mathbf{x}, \mathbf{y}) d\mathbf{y} \right|^2 d\mathbf{x}$$

$$\leq \int_{\Omega} \left(\|q(\mathbf{x}, \mathbf{y})\|_{L^p(\Omega, \mathbf{y})} \|\kappa(\mathbf{x}, \mathbf{y})\|_{L^{p^*}(\Omega, \mathbf{y})} \right)^2 d\mathbf{x}.$$

where $p = \frac{2-\varepsilon}{1-\varepsilon}$ and p^* is the Hölder conjugate of p.

Another application of Hölder's inequality to the integral in $\mathbf{x}$ yields:

$$
\begin{aligned}
\|\mathcal{L}_\alpha[u]\|_{L^2}^2 &\leq \int_\Omega \|q(\mathbf{x}, \mathbf{y})\|_{L^p(\Omega,\mathbf{y})}^2 \, \|\kappa(\mathbf{x}, \mathbf{y})\|_{L^{p^*}(\Omega,\mathbf{y})}^2 \, d\mathbf{x} \\
&\leq \left\| \, \|q(\mathbf{x}, \mathbf{y})\|_{L^p(\Omega,\mathbf{y})}^2 \right\|_{L^{\frac{p}{2}}(\Omega,\mathbf{x})} \left\| \, \|\kappa(\mathbf{x}, \mathbf{y})\|_{L^{p^*}(\Omega,\mathbf{y})}^2 \right\|_{L^{\frac{p^*}{\varepsilon}}(\Omega,\mathbf{x})} \\
&= \left(\int_\Omega \int_\Omega |q(\mathbf{x}, \mathbf{y})|^p d\mathbf{y} d\mathbf{x} \right)^{\frac{2}{p}} \left[\int_\Omega \left(\int_\Omega \kappa(\mathbf{x}, \mathbf{y})^{p^*} d\mathbf{y} \right)^{\frac{2}{\varepsilon}} d\mathbf{x} \right]^{\frac{\varepsilon}{p^*}}.
\end{aligned}
$$

Applying Theorem 3.2.8 (for instance with $\xi = \chi_\Omega$) to the first integral factor to obtain:

$$
\|\mathcal{L}_\alpha[\phi]\|_{L^2(\Omega)}^2 \lesssim \left(\|u\|_{W^{1,p}}^p \right)^{\frac{2}{p}} \left[\int_\Omega \left(\int_\Omega \left(\frac{\rho(|\mathbf{x} - \mathbf{y}|)}{|\mathbf{y} - \mathbf{x}|} \right)^{p^*} d\mathbf{y} \right)^{\frac{2}{\varepsilon}} d\mathbf{x} \right]^{\frac{\varepsilon}{p^*}}.
$$

Proposition 3.2.11 implies that

$$
\int_\Omega \left(\frac{\rho(|\mathbf{x} - \mathbf{y}|)}{|\mathbf{y} - \mathbf{x}|} \right)^{p^*} d\mathbf{y}
$$

is bounded on $\overline{\Omega}$ (notice $p^* = 2 - \varepsilon$); consequently, $\|\mathcal{L}_\alpha[u]\|_{L^2(\Omega)} < \infty$ provided $u \in W^{1,p}(\Omega)$. As $\varepsilon \searrow 0$ the integrability index p tends to 2 from above, hence the condition $p > 2$.

$\square$

Theorem 3.2.13 (Hölder countinuity of the nonlocal Laplacian). *Suppose α satisfies Assumption 3.2.10, $\Omega \subset \mathbb{R}^d$ is a bounded open set, and $d \geq 2$. If $u \in C^2(\Omega) \cap W^{2,\infty}(\Omega)$, then for any $a \in \left(0, \frac{1}{2} \right)$ we have $\mathcal{L}_\alpha[u](\mathbf{x}) \in C^{0,a}(\Omega)$.*

Proof. From the assumption it follows that u is Lipschitz on Ω, hence $(u(\mathbf{y}) - \mathbf{y}(x))/|\mathbf{y} - \mathbf{x}|^2$ is integrable on Ω, provided the space dimension is $d \geq 2$. Utilizing the definition of the

nonlocal Laplacian, we have

$$
\begin{aligned}
&|\mathcal{L}_\alpha[u](\mathbf{x}) - \mathcal{L}_\alpha[u](\mathbf{x}')| \\
&\leq \left| \int_\Omega \frac{(u(\mathbf{y}) - u(\mathbf{x}))\rho(|\mathbf{y} - \mathbf{x}|)}{|\mathbf{y} - \mathbf{x}|^2} - \frac{(u(\mathbf{y}) - u(\mathbf{x}'))\rho(|\mathbf{y} - \mathbf{x}'|)}{|\mathbf{y} - \mathbf{x}'|^2} d\mathbf{y} \right| \\
&\leq \int_\Omega \frac{|u(\mathbf{y}) - u(\mathbf{x})||\rho(|\mathbf{x} - \mathbf{y}|) - \rho(|\mathbf{y} - \mathbf{x}'|)|}{|\mathbf{y} - \mathbf{x}|^2} d\mathbf{y} \\
&\quad + \int_\Omega \rho(|\mathbf{y} - \mathbf{x}'|) \left| \frac{u(\mathbf{y}) - u(\mathbf{x})}{|\mathbf{y} - \mathbf{x}|^2} - \frac{u(\mathbf{y}) - u(\mathbf{x}')}{|\mathbf{y} - \mathbf{x}'|^2} \right| d\mathbf{y} \leq \cdots
\end{aligned}
$$

Since both ρ and u are bounded and Lipschitz continuous and via $|\mathbf{x} - \mathbf{y}| - |\mathbf{x}' - \mathbf{y}| \leq |\mathbf{x} - \mathbf{x}'|$ we can estimate the above integrals as

$$
\cdots \leq M|\mathbf{x} - \mathbf{x}'| \int_\Omega \frac{1}{|\mathbf{y} - \mathbf{x}|} d\mathbf{y} + M \int_\Omega \left| \frac{u(\mathbf{y}) - u(\mathbf{x})}{|\mathbf{y} - \mathbf{x}|^2} - \frac{u(\mathbf{y}) - u(\mathbf{x}')}{|\mathbf{y} - \mathbf{x}'|^2} \right| d\mathbf{y}.
$$

Since $d \geq 2$, the first term is bounded by some $K|\mathbf{x} - \mathbf{x}'|$ for some constant K. It remains to estimate the second integral. Use the fact that a fortiori $u \in C^1(\Omega)$ with support compactly contained in Ω to define

$$
A(\mathbf{x}, \mathbf{y}) := \int_0^1 \nabla u(\lambda \mathbf{y} + (1 - \lambda)\mathbf{x}) d\lambda \quad \text{and} \quad \hat{\xi} := \frac{\xi}{|\xi|},
$$

for ξ a vector in $\mathbb{R}^d$. Then we have

$$
\begin{aligned}
&\int_\Omega \left| \frac{u(\mathbf{y}) - u(\mathbf{x})}{|\mathbf{y} - \mathbf{x}|^2} - \frac{u(\mathbf{y}) - u(\mathbf{x}')}{|\mathbf{y} - \mathbf{x}'|^2} \right| d\mathbf{y} = \int_\Omega \left| \frac{A(\mathbf{x}, \mathbf{y}) \cdot \widehat{\mathbf{y} - \mathbf{x}}}{|\mathbf{y} - \mathbf{x}|} - \frac{A(\mathbf{x}', \mathbf{y}) \cdot \widehat{\mathbf{y} - \mathbf{x}'}}{|\mathbf{y} - \mathbf{x}'|} \right| d\mathbf{y} \\
&\leq \int_\Omega \left| \frac{A(\mathbf{x}, \mathbf{y}) \cdot (\widehat{\mathbf{y} - \mathbf{x}} - \widehat{\mathbf{y} - \mathbf{x}'})}{|\mathbf{y} - \mathbf{x}|} \right| d\mathbf{y} \\
&\quad + \int_\Omega \left| \frac{\widehat{\mathbf{y} - \mathbf{x}'} \cdot (A(\mathbf{x}, \mathbf{y})|\mathbf{y} - \mathbf{x}'| - A(\mathbf{x}', \mathbf{y})|\mathbf{y} - \mathbf{x}|)}{|\mathbf{y} - \mathbf{x}||\mathbf{y} - \mathbf{x}'|} \right| d\mathbf{y} =: I_1 + I_2
\end{aligned}
\tag{3.2.8}
$$

Consider the resulting integrals I_1 and I_2 separately. Beginning with the more straighfor-

ward one I_2 we get, by adding and subtracting $\mathbf{y} \stackrel{\wedge}{-} \mathbf{x}' \cdot A(\mathbf{x}, \mathbf{y})|\mathbf{y} - \mathbf{x}|$ in the numerator:

$$
\begin{aligned}
I_2 &= \\
&= \int_\Omega \left| \frac{\mathbf{y} \stackrel{\wedge}{-} \mathbf{x}' \cdot (|\mathbf{y} - \mathbf{x}'| - |\mathbf{y} - \mathbf{x}|)A(\mathbf{x}, \mathbf{y})}{|\mathbf{y} - \mathbf{x}||\mathbf{y} - \mathbf{x}'|} \right| d\mathbf{y} + \int_\Omega \left| \frac{\mathbf{y} \stackrel{\wedge}{-} \mathbf{x}' \cdot (A(\mathbf{x}, \mathbf{y}) - A(\mathbf{x}', \mathbf{y}))}{|\mathbf{y} - \mathbf{x}'|} \right| d\mathbf{y} \\
&\leq \int_\Omega |A(\mathbf{x}, \mathbf{y})| \frac{\left\| |\mathbf{y} - \mathbf{x}'| - |\mathbf{y} - \mathbf{x}| \right\|^{1-a}}{|\mathbf{y} - \mathbf{x}| \, |\mathbf{y} - \mathbf{x}'|} \left\| |\mathbf{y} - \mathbf{x}'| - |\mathbf{y} - \mathbf{x}| \right\|^a d\mathbf{y} \\
&\quad + \int_\Omega \left| \frac{\mathbf{y} \stackrel{\wedge}{-} \mathbf{x}' \, (A(\mathbf{x}, \mathbf{y}) - A(\mathbf{x}', \mathbf{y}))}{|\mathbf{y} - \mathbf{x}'|} \right| d\mathbf{y} \\
&\leq |\mathbf{x} - \mathbf{x}'|^a \int_\Omega |A(\mathbf{x}, \mathbf{y})| \frac{2^{(1-\alpha)}(|\mathbf{y} - \mathbf{x}'|^{1-a} + |\mathbf{y} - \mathbf{x}|^{1-a})}{|\mathbf{y} - \mathbf{x}||\mathbf{y} - \mathbf{x}'|} d\mathbf{y} \\
&\quad + \int_\Omega \left| \frac{\mathbf{y} \stackrel{\wedge}{-} \mathbf{x}' \, (A(\mathbf{x}, \mathbf{y}) - A(\mathbf{x}', \mathbf{y}))}{|\mathbf{y} - \mathbf{x}'|} \right| d\mathbf{y} \\
&\leq M |\mathbf{x} - \mathbf{x}'|^a + M|\mathbf{x} - \mathbf{x}'| \int_\Omega \frac{1}{|\mathbf{y} - \mathbf{x}'|} d\mathbf{y}
\end{aligned}
$$

The last line follows by noting that ∇u is Lipschitz continuous on Ω. Thus the last integral I_2 in (3.2.8) is bounded by $K|\mathbf{x} - \mathbf{x}'|^a$. Now consider I_1:

$$
I_1 = \int_\Omega \frac{\left| A(\mathbf{x}, \mathbf{y}) \left(\mathbf{y} \stackrel{\wedge}{-} \mathbf{x} - \mathbf{y} \stackrel{\wedge}{-} \mathbf{x}' \right) \right|}{|\mathbf{y} - \mathbf{x}|} d\mathbf{y} \leq M \int_\Omega \frac{1}{|\mathbf{y} - \mathbf{x}|} \left| \mathbf{y} \stackrel{\wedge}{-} \mathbf{x} - \mathbf{y} \stackrel{\wedge}{-} \mathbf{x}' \right| d\mathbf{y}
$$

Notice that the difference of two normalized vectors is $\sqrt{2(1 - \cos(\theta))}$ where θ is the angle between them. Let's look at the triangle formed by $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{y}'$.
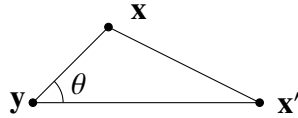
Without loss of generality assume that $(\mathbf{x}' - \mathbf{y}) = (x_1' - y_1)\hat{\mathbf{i}}$ with $y_1 < x_1'$; hence, $|\mathbf{y} - \mathbf{x}'| =$

$x_1' - y_1$. Then

$$\frac{1}{2}|\widehat{\mathbf{y} - \mathbf{x}} - \widehat{\mathbf{y} - \mathbf{x}'}|^2 = 1 - \cos(\theta)$$

$$= 1 - \frac{(\mathbf{x}' - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})}{|\mathbf{y} - \mathbf{x}'||\mathbf{y} - \mathbf{x}|} = 1 - \frac{x_1 - y_1}{|\mathbf{y} - \mathbf{x}|} = \frac{|\mathbf{y} - \mathbf{x}| - x_1 + y_1}{|\mathbf{y} - \mathbf{x}|}$$

$$\leq \frac{|\mathbf{y} - \mathbf{x}'| + |\mathbf{x}' - \mathbf{x}| - x_1 + y_1}{|\mathbf{y} - \mathbf{x}|}$$

$$= \frac{x_1' - y_1 + |\mathbf{x}' - \mathbf{x}| - x_1 + y_1}{|\mathbf{y} - \mathbf{x}|} \leq 2\frac{|\mathbf{x}' - \mathbf{x}|}{|\mathbf{y} - \mathbf{x}|}$$

Therefore

$$I_1 \leq M \int_\Omega \frac{2}{|\mathbf{y} - \mathbf{x}|} \sqrt{\frac{|\mathbf{x}' - \mathbf{x}|}{|\mathbf{y} - \mathbf{x}|}} \, d\mathbf{y} = 2M|\mathbf{x}' - \mathbf{x}|^{1/2} \int_\Omega \frac{d\mathbf{y}}{|\mathbf{y} - \mathbf{x}|^{3/2}}$$

Finally we conclude

$$|\mathcal{L}[u](\mathbf{x}) - \mathcal{L}[u](\mathbf{x}')| \leq K|\mathbf{x} - \mathbf{x}'|^a + N|\mathbf{x} - \mathbf{x}'|^{\frac{1}{2}}$$

where

$$N = 2M \int_\Omega \frac{d\mathbf{y}}{|\mathbf{y} - \mathbf{x}|^{3/2}} < \infty.$$

$\square$

Proposition 3.2.14 (Integrability of the nonlocal biharmonic)**.** *Let $\varepsilon \in (0, 1)$, $d \geq 2$ and $\Omega \subset \mathbb{R}^d$ be a bounded open set. Suppose $u \in W^{2,\infty}(\Omega) \cap C^2(\Omega)$. Furthermore, let $\rho \in C_c^\infty(\mathbb{R}^+)$. Then $\mathcal{B}_\alpha[u] \in L^2(\Omega)$.*

Notice for $\mathbf{x} \in \Omega$ we have

$$\mathcal{B}_\alpha[u](\mathbf{x}) = \int_\Omega \frac{(\mathcal{L}_\alpha[u](\mathbf{x}') - \mathcal{L}_\alpha[u](\mathbf{x}))\rho(|\mathbf{x}' - \mathbf{x}|)}{|\mathbf{x}' - \mathbf{x}|^2} d\mathbf{x}'$$

By appealing to Theorem 3.2.13 we know $\mathcal{L}_\alpha[u]$ is a-Hölder continuous on Ω for some $a > 0$ (in particular, at least $a \in (0, 1/2)$), and consequently there exists a constant M such that

$$|\mathcal{B}_\alpha[u](\mathbf{x})| \le M \int_\Omega \frac{1}{|\mathbf{x}' - \mathbf{x}|^{2-a}} d\mathbf{x}' \le M' \int_{\mathbb{R}^2} \frac{1}{|\mathbf{x}'|^{2-a}} d\mathbf{x}' < \infty.$$

Since $\Omega \subset \mathbb{R}^2$ is bounded we conclude $|\mathcal{B}[u](\mathbf{x})|$ is bounded on Ω. Thus $\mathcal{B}_\alpha[u] \in L^2(\Omega)$.

3.3   Nonlocal function spaces

This section will introduce various Hilbert spaces we will be working in for the formulation of our nonlocal problems later in the chapter. Following [29] we will utilize the functional space

$$\mathscr{H}_\alpha^1(\Omega) := \left\{ u \in L^2(\Omega) : \|\mathcal{G}_\alpha[u]\|_{L^2(\Omega \times \Omega)} < \infty \right\}. \tag{3.3.1}$$

Define the bilinear forms

$$(\!(u, w)\!)_1 = \int_\Omega \int_\Omega \mathcal{G}_\alpha[u] \mathcal{G}_\alpha[w] d\mathbf{x}' d\mathbf{x} \tag{3.3.2}$$

and

$$(u, w)_{\mathscr{H}_\alpha^1} = (u, w)_{L^2(\Omega)} + (\!(u, w)\!)_1 . \tag{3.3.3}$$

Note that if $|\alpha|^2$ is integrable, then $\mathscr{H}_\alpha^1(\Omega)$ is equivalent to $L^2(\Omega)$. However, under Assumption 3.2.10, this may not be the case when $\Omega \subset \mathbb{R}^2$.

Theorem 3.3.1 (c.f. [29, Thm 2.2]). *Assume α satisfies Assumption 3.2.10. Then, $\mathscr{H}_\alpha^1(\Omega)$ is a Hilbert space with inner product* (3.3.3).

Definition 3.3.2. For $\Omega' \subset\subset \Omega$, define $\mathscr{H}_0^1(\Omega')$ to be the closed subspace of functions

vanishing on $\Omega \setminus \Omega'$

$$\mathcal{H}^1_{\alpha,0}(\Omega') = \left\{u \in \mathcal{H}^1_\alpha(\Omega) : u = 0 \text{ a.e. in } \Omega \setminus \Omega'\right\}.$$

Definition 3.3.3. Let $u \in \mathcal{H}^1_\alpha(\Omega)$. We say $v \in L^2(\Omega)$ is the **nonlocal weak** Laplacian of u provided

$$-(\!(u, \phi)\!)_1 = \int_\Omega v\phi d\mathbf{x} \quad \forall \phi \in \mathcal{H}^1_\alpha(\Omega).$$

Proposition 3.3.4. *Let $\Omega \subset \mathbb{R}^d$ be open, $u : \Omega \to \mathbb{R}$, $\alpha : \Omega \times \Omega \to \mathbb{R}^k$ be antisymmetric. If $\mathcal{L}_\alpha[u] \in L^2(\Omega)$ and $\mathcal{G}_\alpha[u] \in L^2(\Omega^2)$ then the weak nonlocal Laplacian $\mathcal{L}^*_\alpha[u]$ and the nonlocal Laplacian $\mathcal{L}_\alpha[u]$ agree a.e. in Ω.*

Proof. Let $v \in C^\infty_c(\Omega)$. Then by Definition 3.3.3 and Proposition 3.2.6 we have

$$\int_\Omega \mathcal{L}^*[u]v d\mathbf{x} = -\int_\Omega \int_\Omega \mathcal{G}_\alpha[u]\mathcal{G}_\alpha[v]d\mathbf{y}d\mathbf{x} = \int_\Omega \mathcal{L}[u]v d\mathbf{x}.$$

Since this holds for any $v \in C^\infty_c(\Omega)$ we conclude $\mathcal{L}^*[u] = \mathcal{L}[u]$ a.e. in Ω. $\qquad\square$

Remark 3.3.5. For the remainder of the chapter, we will use the notation $\mathcal{L}_\alpha[u]$ to denote the weak Laplacian of the function u. When the distinction between weak and original definition is essential it will be indicated. To begin with, note that the integration by parts Proposition 3.2.6 holds for the nonlocal weak Laplacian simply by definition of the latter.

Definition 3.3.6. Let

$$\mathcal{H}^2_\alpha(\Omega) := \left\{u \in \mathcal{H}^1_\alpha(\Omega) : \mathcal{L}_\alpha[u] \in L^2(\Omega)\right\} \tag{3.3.4}$$

and defining

$$(\!(u, w)\!)_2 := \int_\Omega \int_\Omega \mathcal{L}_\alpha[u]\mathcal{L}_\alpha[w]d\mathbf{x}'d\mathbf{x} \tag{3.3.5}$$

introduce an inner product on $\mathscr{H}_\alpha^2$:

$$(u, w)_{\mathscr{H}_\alpha^2} = (u, w)_{\mathscr{H}_\alpha^1} + (\!(u, w)\!)_2 \,.$$

Proposition 3.3.7. *Suppose α satisfies Assumption 3.2.10. The space $\mathscr{H}_\alpha^1(\Omega)$ is a Hilbert space with inner product $(\cdot, \cdot)_{\mathscr{H}^1}$.*

Proof. All that remains to be proven is completeness. Let (u_n) be a Cauchy sequence in $\mathscr{H}_\alpha^1(\Omega)$. By definition, (u_n) is a Cauchy sequence in $L^2(\Omega)$ and consequently there exists a strong $L^2(\Omega)$-limit $u \in L^2(\Omega)$. Similarly, we also know $\mathcal{G}_\alpha[u_n]$ converges to some L^2-norm limit $\mathbf{v} \in L^2(\Omega \times \Omega)$. Consider the truncated kernel $\alpha_\tau = \alpha \chi_{[\frac{1}{\tau}, \infty)}$ for $\tau > 0$. For every $\tau > 0$ and $\boldsymbol{\phi} \in L^2(\Omega \times \Omega)$ we have by proposition 3.2.6,

$$\begin{aligned}
\lim_{n\to\infty} \left| \int_\Omega \int_\Omega (\mathcal{G}_{\alpha_\tau}[u_n] - \mathcal{G}_{\alpha_\tau}[u]) \cdot \boldsymbol{\phi} \, dy \, dx \right| &= \lim_{n\to\infty} \left| \int_\Omega \int_\Omega \mathcal{G}_{\alpha_\tau}[u_n - u] \cdot \boldsymbol{\phi} \, dy \, dx \right| \\
&= \lim_{n\to\infty} \left| \int_\Omega (u_n - u) \mathcal{D}_{\alpha_\tau}[\boldsymbol{\phi}] \, dx \right| \\
&\le \lim_{n\to\infty} \|u_n - u\|_{L^2} \|\mathcal{D}_{\alpha_\tau}[\boldsymbol{\phi}]\|_{L^2} \\
&= 0
\end{aligned}$$

Note that $\mathcal{D}_{\alpha_\tau}[\mathbf{v}] \in L^2(\Omega)$ since $(\mathbf{v}' + \mathbf{v})\alpha_\tau \in C_c^\infty(\Omega \times \Omega)$ for every $\tau > 0$. We conclude $\lim_{n\to\infty} \mathcal{G}_{\alpha_\tau}[u_n] = \mathcal{G}_{\alpha_\tau}[u]$ a.e. for $(\mathbf{x}, \mathbf{y}) \in \Omega \times \Omega$. Combining this with the fact that $\mathcal{G}_\alpha[u_n] \to \mathbf{v}$ in $L^2(\Omega \times \Omega)$ implies for every $\tau > 0$,

$$\mathcal{G}_{\alpha_\tau}[u] = \lim_{n\to\infty} \mathcal{G}_{\alpha_\tau}[u_n] = \lim_{n\to\infty} \mathcal{G}_\alpha[u_n] = \mathbf{v} \quad \text{a.e. in } \left\{ (\mathbf{x}, \mathbf{y}) \in \Omega \times \Omega : |\mathbf{x} - \mathbf{y}| \ge \frac{1}{\tau} \right\}.$$

From this we conclude $\mathcal{G}_\alpha[u] = \mathbf{v}$ a.e. in $\Omega \times \Omega$.

$\square$

Proposition 3.3.8. *The space $\mathscr{H}_\alpha^2(\Omega)$ is a Hilbert space with inner product $(\cdot, \cdot)_{\mathscr{H}_\alpha^2}$.*

Proof. Again we only need to verify completeness. Let (u_n) be a Cauchy sequence in $\mathcal{H}_\alpha^2(\Omega)$. Then (u_n), $(\mathcal{G}[u_n])$, and $(\mathcal{L}_\alpha(u_n))$ are Cauchy in L^2 and consequently have L^2 limits u, v, and w respectively. Using the same method as in Proposition 3.3.7, we have $v = \mathcal{G}[u_n]$ a.e. Let $\phi \in \mathcal{H}^1(\Omega)$.

Appealing to Proposition 3.2.6, Theorem 3.2.8, and Hölder's inequality, we obtain

$$\lim_{n\to\infty} \int_\Omega \int_\Omega \mathcal{G}_\alpha[u_n - u]\mathcal{G}_\alpha[\phi]d\mathbf{y}d\mathbf{x} \le \lim_{n\to\infty} \|\mathcal{G}_\alpha[u - u_n]\|_{L^2(\Omega\times\Omega)}\|\mathcal{G}_\alpha[\phi]\|_{L^2(\Omega\times\Omega)}$$

$$\le \lim_{n\to\infty} C\|\mathcal{G}_\alpha[u - u_n]\|_{L^2(\Omega\times\Omega)}\|\phi\|_{W^{1,2}(\Omega)}$$

$$= 0$$

and

$$\lim_{n\to\infty} \int_\Omega \int_\Omega (w - \mathcal{L}_\alpha[u_n])\phi d\mathbf{y}d\mathbf{x} \le \lim_{n\to\infty} \|w - \mathcal{L}_\alpha[u_n]\|_{L^2(\Omega)}\|\phi\|_{L^2(\Omega)}$$

$$= 0.$$

Putting this together, we obtain via Proposition 3.2.6

$$-\int_\Omega \int_\Omega \mathcal{G}_\alpha[u]\mathcal{G}_\alpha[\phi]d\mathbf{x} = -\lim_{n\to\infty} \int_\Omega \int_\Omega \mathcal{G}_\alpha[u_n]\mathcal{G}_\alpha[\phi]d\mathbf{x}$$

$$= \lim_{n\to\infty} \int_\Omega \mathcal{L}_\alpha[u_n]\phi d\mathbf{x} = \int_\Omega w\phi dx$$

Thus, $w = \mathcal{L}_\alpha[u]$ a.e. in Ω in the nonlocal weak sense and, consequently, $u \in \mathcal{H}_\alpha^2(\Omega)$. □

Finally, we define Hilbert spaces associated with the boundary conditions that we will consider.

Definition 3.3.9 (Nonlocal "hinged" and "clamped" spaces)**.** Let $\Omega'' \subset\subset \Omega'$ be an open

sub-domain of Ω'. Define respectively

$$\mathscr{H}^2_{\alpha,H}(\Omega', \Omega'') = \left\{ u \in \mathscr{H}^1_{\alpha,0}(\Omega') \cap \mathscr{H}^2_\alpha(\Omega) : \mathcal{L}_\alpha[u] = 0 \text{ a.e. on } \Omega' \setminus \Omega'' \right\} \qquad (3.3.6)$$

$$\mathscr{H}^2_{\alpha,C}(\Omega') = \left\{ u \in \mathscr{H}^1_{\alpha,0}(\Omega') \cap \mathscr{H}^2_\alpha(\Omega) : \mathcal{N}_\alpha[\mathcal{G}_\alpha[u]] = 0 \text{ a.e. on } \text{int}(\Omega') \right\} \qquad (3.3.7)$$

From the definition (3.2.1) of the nonlocal normal operator and the nonlocal two-point gradient (3.2.2) it follows that

$$\mathcal{N}_\alpha[\mathcal{G}_\alpha[u]](\mathbf{x}) = \int_{\Omega \setminus \Omega'} (u(\mathbf{y}) - u(\mathbf{x}))\mu(\mathbf{x}, \mathbf{y})d\mathbf{y} \quad \text{for} \quad \mathbf{x} \in \text{int}(\Omega') .$$

Moreover, because in the clamped space we also have $u(\mathbf{y}) = 0$ on $\Omega \setminus \Omega'$, then the identity $\mathcal{N}_\alpha[\mathcal{G}_\alpha[u]](\mathbf{x}) = 0$ actually reduces to

$$-u(\mathbf{x}) \int_{\Omega \setminus \Omega'} \mu(\mathbf{x}, \mathbf{y})d\mathbf{y} = 0 .$$

If we set $\Omega' = \Omega_\delta$ where

$$\Omega_\delta := \{\mathbf{x} \in \Omega : \text{dist}(\mathbf{x}, \partial\Omega) > \delta\}$$

and choose $\alpha = \alpha_\delta$ as in Assumption 3.2.10, then because $y \mapsto \mu(\mathbf{x}, \mathbf{y})$ is strictly positive and continuous on $\Omega \setminus \Omega'$ for any fixed $\mathbf{x} \in \text{int}\,\Omega_\delta$, we conclude that $u(\mathbf{x}) = 0$ a.e. in $\Omega \setminus \Omega_{2\delta}$. Thus we have an alternative representation for the nonlocal "clamped" space:

$$\mathscr{H}^2_{\alpha_\delta,C}(\Omega_\delta) = \mathscr{H}^1_{\alpha_{(2\delta)},0}(\Omega_\delta) \cap \mathscr{H}^2_{\alpha_\delta}(\Omega) . \qquad (3.3.8)$$

3.4 Compactness theorems

A key tool in subsequent analysis will be the following version of a nonlocal Poincaré inequality.

Theorem 3.4.1 ([34, Thm. 1.2])**.** *Assume Ω is a bounded domain of dimension $d = \dim \Omega \geq 2$ with Lipschitz boundary. Let (δ_n) be a sequence of positive numbers decreasing to 0. Let (ρ_{δ_n}) be a sequence of functions satisfying (3.2.5). There exists a constant $C_{p,d,\Omega}$ dependent on the domain, p (and also on the choice of the sequence of mollifiers ρ_{δ_n}) such that*

$$\|f - f_\Omega\|_{L^p(\Omega)}^p \leq C_{p,d,\Omega} \int_\Omega \int_\Omega \frac{|f(\mathbf{x}) - f(\mathbf{y})|^p}{|\mathbf{x} - \mathbf{y}|^p} \rho_{\delta_n}(|\mathbf{x} - \mathbf{y}|) d\mathbf{x} d\mathbf{y}$$

for every $f \in L^p(\Omega)$ with the convention that the right-hand side is $+\infty$ when diverges. Here f_Ω is the average value of f in Ω.

Remark 3.4.2. It should be noted that in [34], the result of Theorem 3.4.1 is extended to dimension $d = 1$; however, in that case, it is necessary to place an additional constraint on ρ_{δ_n}.

In the nonlocal setting we cannot appeal to the embedding and compactness methods of Sobolev theory. Instead, the crucial compactness result in this context will be provided by the following theorem of Brezis, Bourgain, and Mironescu:

Theorem 3.4.3 ([7, Thm. 4])**.** *Let Ω be a bounded domain of class C^1. Let (δ_n) be a sequence decreasing to 0. Suppose (f_n) is a sequence in $L^p(\Omega)$, $1 \leq p < \infty$, of functions satisfying the uniform estimate*

$$\int_\Omega \int_\Omega \frac{|f_n(\mathbf{x}) - f_n(\mathbf{y})|^p}{|\mathbf{x} - \mathbf{y}|^p} \rho_{\delta_n}(|\mathbf{x} - \mathbf{y}|) d\mathbf{x} d\mathbf{y} \leq C_0 \tag{3.4.1}$$

where (ρ_{δ_n}) *is a sequence of non-increasing mollifiers satisfying* (3.2.5). *If*

$$\int_\Omega f_n(\mathbf{x})d\mathbf{x} = 0 \quad \text{for all} \quad n, \tag{3.4.2}$$

then

(i) *the sequence* (f_n) *is relatively compact in* $L^p(\Omega)$, *so up to a subsequence we may assume* $f_n \to f$ *in* $L^p(\Omega)$

(ii) *if, in addition,* $1 < p < \infty$, *then* $f \in W^{1,p}(\Omega)$ *and* $\|\nabla f\|_{L^p(\Omega)} \leq K(p,d)C_0$ *for* K *dependent only on* p *and the dimension* d.

Remark 3.4.4. The compactness in L^p (part (i)) result of [7, Thm. 4] uses Riesz-Fréchet-Kolmogorov's theorem (e.g., see [8, Thm. IV.25, p. 72]) which establishes this compactness result on a set compactly contained within a given open domain. To get the conclusion on all of Ω, the proof of [7, Thm. 4] uses an extension of the functions f_n by reflection across the boundary of Ω; due to the monotonicity condition on ρ_{δ_n} such a reflection preserves the property (3.4.1). Thus, part (i) needs the C^1 regularity only to obtain an L^p-preserving extension by reflection. This result reduces to a change of variable theorem for the mapping that locally defines the boundary; this procedure could potentially be carried out under weaker boundary regularity conditions, for instance, see [21].

Recall that

$$\Omega_\delta := \{\mathbf{x} \in \Omega : \text{dist}(\mathbf{x}, \partial\Omega) > \delta\}$$

Below we prove a useful corollary to Theorem 3.4.3:

Corollary 3.4.5. *In Theorem 3.4.3 we can replace assumption* (3.4.2) *by the assertion that* (f_n) *are bounded in* $L^p(\Omega)$. *Moreover, if* $1 < p < \infty$ *and* $\text{supp } f_n \subset \overline{\Omega}_{\delta_n}$, *then* $f \in W_0^{1,p}(\Omega)$.

Proof. Let $a_n := \dfrac{1}{|\Omega|} \displaystyle\int_\Omega f_n(\mathbf{x})d\mathbf{x}$. Since Ω is bounded and (f_n) is bounded in $L^p(\Omega)$, then the scalar sequence (a_n) is bounded. Define

$$g_n(\mathbf{x}) := f_n(\mathbf{x}) - a_n \,.$$

Then each g_n obeys (3.4.1) and has zero average. By Theorem 3.4.3 we know $\{g_n\}$ is pre-compact in $L^p(\Omega)$. Because (a_n) is a bounded scalar sequence and Ω is bounded, then $\{f_n\}$ is also pre-compact in $L^p(\Omega)$.

Now to show $f \in W_0^{1,p}(\Omega)$. Recall supp $f_n \subset\subset \Omega_{\delta_n}$. Choose $\Omega_{\text{ext}} \subset \mathbb{R}^d$ open and bounded such that $\Omega \subset\subset \Omega_{\text{ext}}$. Let

$$\tilde{f}_n = \begin{cases} f_n, & \mathbf{x} \in \Omega \\ 0, & \mathbf{x} \in \Omega_{\text{ext}} \setminus \Omega \end{cases}$$

Notice that the integral in

$$\int_{\Omega_{\text{ext}}} \int_{\Omega_{\text{ext}}} \frac{|\tilde{f}_n(\mathbf{x}) - \tilde{f}_n(\mathbf{y})|^p}{|\mathbf{x} - \mathbf{y}|^p} \rho_{\delta_n}(|\mathbf{x} - \mathbf{y}|)d\mathbf{x}d\mathbf{y}$$

can be decomposed as

$$\left[\int_\Omega \int_\Omega + \int_{\Omega_{\text{ext}}\setminus\Omega} \int_\Omega + \int_\Omega \int_{\Omega_{\text{ext}}\setminus\Omega_{\delta_n}} \right](\ldots)d\mathbf{x}d\mathbf{y}$$

When $\mathbf{x} \in \Omega_{\text{ext}} \setminus \Omega$ then either $\mathbf{y} \in \Omega \setminus \Omega_{\delta_n}$ in which case $\tilde{f}_n(\mathbf{x}) - \tilde{f}_n(\mathbf{y}) = 0$ via the zero condition on the collar possibly excluding a set of measure zero, or the distance between $\mathbf{x}$ and $\mathbf{y}$ is at least δ whence $\rho_{\delta_n}(|\mathbf{x} - \mathbf{y}|) = 0$. So the second double integral above is 0. A

symmetric argument shows that the third one is zero as well. Conclude:

$$\int_{\Omega_{ext}} \int_{\Omega_{ext}} \frac{|\tilde{f}_n(\mathbf{x}) - \tilde{f}_n(\mathbf{y})|^p}{|\mathbf{x} - \mathbf{y}|^p} \rho_{\delta_n} d\mathbf{x} d\mathbf{y} = \int_{\Omega} \int_{\Omega} \frac{|f_n(\mathbf{x}) - f_n(\mathbf{y})|^p}{|\mathbf{x} - \mathbf{y}|^p} \rho_{\delta_n} d\mathbf{x} d\mathbf{y} \leq C_0$$

By Theorem 3.4.3 we know $\tilde{f} := \lim_{n \to \infty} \tilde{f}_n \in W^{1,p}(\Omega_{ext})$. Notice that $\tilde{f}$ is a $W^{1,2}$ extension of f to Ω_{ext} across $\partial\Omega$. The trace of $\tilde{f}$ on the (C^1) sub-manifold $\partial\Omega$ is then uniquely determined (e.g., [1, Thm. 5.36]) by treating it as the boundary of Ω and of $\Omega_{ext} \setminus \Omega$ respectively. Since $\tilde{f} = 0$ on $\Omega_{ext} \setminus \Omega$ we conclude $f = 0$ on $\partial\Omega$. $\qquad\square$

The preceding Corollary 3.4.5, in turn allows us to state two versions of Theorem 3.4.1 applicable to functions compactly supported in Ω:

Corollary 3.4.6 (Nonlocal Poincaré with zero shrinking collar for a sequence)**.** *Under the assumptions of Theorem 3.4.1, suppose (f_n) is a family of functions in $L^p(\Omega)$ such that* supp $f_n \subset \overline{\Omega}_{\delta_n}$. *Then there is a constant $C_{p,d,\Omega} > 0$ satisfying*

$$\|f_n\|_{L^p(\Omega)}^p \leq C_{p,d,\Omega} \int_{\Omega} \int_{\Omega} \frac{|f_n(\mathbf{x}) - f_n(\mathbf{y})|^p}{|\mathbf{x} - \mathbf{y}|^p} \rho_{\delta_n}(|\mathbf{x} - \mathbf{y}|) d\mathbf{x} d\mathbf{y}$$

for every n with the convention that the right-hand side is $+\infty$ when undefined.

Proof. Proceed by contradiction as in [34, p. 12]. Suppose we can extract a subsequence (re-indexed again by n) so that the candidates for "$C_{p,d,\Omega}$" diverge to $+\infty$. In particular, suppose a sequence of scalars (c_n) diverges to $+\infty$ and for every n

$$\|f_n\|_{L^p(\Omega)}^p \geq c_n \int_{\Omega} \int_{\Omega} \frac{|f(\mathbf{x}) - f(\mathbf{y})|^p}{|\mathbf{x} - \mathbf{y}|^p} \rho_{\delta_n}(|\mathbf{x} - \mathbf{y}|) d\mathbf{x} d\mathbf{y}$$

Re-normalizing both sides by $\|f_n\|_{L^p(\Omega)}^p$ we may just assume that $\|f_n\|_{L^p} = 1$ and

$$\lim_{n \to \infty} \int_{\Omega} \int_{\Omega} \frac{|f_n(\mathbf{x}) - f_n(\mathbf{y})|^p}{|\mathbf{x} - \mathbf{y}|^p} \rho_{\delta_n}(|\mathbf{x} - \mathbf{y}|) d\mathbf{x} d\mathbf{y} = 0 \,. \tag{3.4.3}$$

By Corollary 3.4.5 functions (f_n) converge strongly to f in $L^p(\Omega)$ with $\|f\|_{L^p(\Omega)} = 1$. Moreover, $f \in W_0^{1,p}(\Omega)$. In addition, by and the same Corollary (referring now to part (ii) of Theorem 3.4.3) we have from (3.4.3) that $\|\nabla f\|_{L^p(\Omega)} = 0$. Poincaré-Wirtinger's inequality now implies that $f = 0$ in $W_0^{1,p}(\Omega)$ thus contradicting the fact that $\|f\|_{L^p(\Omega)} = 1$. $\qquad \square$

Corollary 3.4.7 (Nonlocal Poincaré with zero collar). *Under the assumptions of Theorem 3.4.1, suppose if $\Omega_0 \subset\subset \Omega$. Then there is a constant $C_{p,d,\Omega,\Omega_0} > 0$ such that*

$$\|f\|_{L^p(\Omega)}^p \le C_{p,d,\Omega,\Omega_0} \int_\Omega \int_\Omega \frac{|f(\mathbf{x}) - f(\mathbf{y})|^p}{|\mathbf{x} - \mathbf{y}|^p} \rho_{\delta_n}(|\mathbf{x} - \mathbf{y}|) d\mathbf{x} d\mathbf{y}$$

for all $f \in L^p(\Omega)$ satisfying $\operatorname{supp} f \subset \Omega_0$

Proof. The proof by contradiction reduces to a construction of a sequence (f_n) in $L^p(\Omega)$ that violates Corollary 3.4.6. Since $\Omega_0 \subset\subset \Omega$ is fixed, then we may assume that each $\Omega_0 \subseteq \Omega_{\delta_n}$ for all n. $\qquad \square$

3.5 Convergence of the nonlocal operators

3.5.1 Scaled operators

With appropriate scaling, the nonlocal Laplace and biharmonic operators converge to their local analogues. Throughout this subsection suppose Assumption 3.2.10 holds for kernel α_δ and

$$\mu_\delta(\mathbf{x}, \mathbf{y}) := |\alpha_\delta(\mathbf{x}, \mathbf{y})|^2 = \frac{\rho_\delta(|\mathbf{x} - \mathbf{y}|)}{|\mathbf{x} - \mathbf{y}|^2}$$

Since this is a radial function we may write with a slight abuse of notation:

$$\mu_\delta(\mathbf{x}, \mathbf{y}) = \mu_\delta(|\mathbf{x} - \mathbf{y}|) \quad \text{and} \quad \mu_\delta(s) = \frac{\rho_\delta(s)}{s^2}.$$

Definition 3.5.1 (Scaling). Let

$$\pi_\delta(r) := \int_r^\delta s\mu_\delta(s)ds. \tag{3.5.1}$$

Let ω_{d-1} be the surface measure of unit sphere in $\mathbb{R}^d$ and define

$$C(\delta) := \frac{1}{2}\int_{B_\delta(0)}\pi(|\mathbf{z}|)d\mathbf{z} = \frac{1}{2}\omega_{d-1}\int_0^\delta \pi(r)r^{d-1}dr, \tag{3.5.2}$$

which is finite for $d \geq 2$.

Proposition 3.5.2. *Let $\delta > 0$ and $C(\delta)$ be given by (3.5.2). Then*

$$C(\delta) = 1/(2d)$$

Proof. By appealing to the definitions of $C(\delta), \pi_\delta(r), \mu_\delta(s)$ and changing the order of integration, we obtain

$$C(\delta) = \frac{\omega_{d-1}}{2}\int_0^\delta\int_r^\delta \frac{\rho_\delta(s)}{s}r^{d-1}dsdr = \frac{\omega_{d-1}}{2}\int_0^\delta\int_0^s r^{d-1}\frac{\rho_\delta(s)}{s}drds$$

$$= \frac{\omega_{d-1}}{2d}\int_0^\delta s^{d-1}\rho_\delta(s)ds$$

The result follows since by construction $\int_{\mathbb{R}^d}\rho_\delta(\mathbf{x})d\mathbf{x} = \int_{B_\delta(0)}\rho_\delta(\mathbf{x})d\mathbf{x} = 1$. $\qquad\square$

In light of the previous proposition we can write C rather than $C(\delta)$ in (3.5.2) and introduce

$$\sigma := C^{-1} = 2d. \tag{3.5.3}$$

Accordingly, we redefine the nonlocal Laplacian and biharmonic operators with the scaling

term:

$$\mathcal{L}_{\alpha_\delta} u(\mathbf{x}) \quad := \quad \sigma \int_\Omega [u(\mathbf{y}) - u(\mathbf{x})] \mu_\delta(\mathbf{x}, \mathbf{y}) d\mathbf{y} \tag{3.5.4}$$

$$\mathcal{B}_{\alpha_\delta} u(\mathbf{x}) \quad := \quad \mathcal{L}_{\alpha_\delta} \left[\mathcal{L}_{\alpha_\delta}[u](\mathbf{x}) \right]. \tag{3.5.5}$$

3.5.2 Pointwise convergence

We will show that the nonlocal operators approach uniformly their classical versions when acting on smooth functions as the peridynamic horizon δ goes to zero. The proofs of the following results were inspired by the strategy used in the upcoming paper [35].

Lemma 3.5.3. *Let* $\Omega \subset \mathbb{R}^{d \geq 2}$ *be bounded and open,* $u \in C^2(\Omega) \cap W^{1,\infty}(\Omega)$. *Further suppose* α_δ *satisfies Assumption 3.2.10. Then for any* $\mathbf{x} \in \Omega$ *and all* $\delta > 0$ *such that* $B_\delta(\mathbf{x}) \subset \Omega$,

$$\mathcal{L}_{\alpha_\delta}[u](\mathbf{x}) = \sigma \int_0^1 \int_{B_\delta(0)} s\Big(\Delta u(\mathbf{x} + s\mathbf{z}) - \Delta u(\mathbf{x})\Big) \pi(|\mathbf{z}|) d\mathbf{z} ds + \Delta u(\mathbf{x}).$$

Proof. We interpret $\mathcal{L}_{\alpha_\delta}$ as the weak nonlocal Laplacian, but since u is a fortiori in $W^{1,p}(\Omega)$ then $\mathcal{L}_\alpha[u] \in L^2(\Omega)$ by Proposition 3.2.12 and we may use the pointwise original definition according to Proposition 3.3.4. Because the support of μ_δ is contained in $B_\delta(\mathbf{x})$, we have

$$\begin{aligned}
\mathcal{L}_{\alpha_\delta}[u](\mathbf{x}) &= \sigma \int_{B_\delta(\mathbf{x})} [u(\mathbf{y}) - u(\mathbf{x})] \mu_\delta(\mathbf{x}, \mathbf{y}) d\mathbf{y} \\
&= \sigma \int_{B_\delta(\mathbf{x})} \int_0^1 \frac{d}{ds} \left[u(\mathbf{x} + s(\mathbf{y} - \mathbf{x})) \right] \mu_\delta(\mathbf{x}, \mathbf{y}) ds d\mathbf{y} \\
&= \sigma \int_{B_\delta(\mathbf{x})} \int_0^1 \nabla u(\mathbf{x} + s(\mathbf{y} - \mathbf{x})) \cdot (\mathbf{y} - \mathbf{x}) \mu_\delta(\mathbf{x}, \mathbf{y}) ds d\mathbf{y}.
\end{aligned}$$

Because u is Lipschitz and $(\mathbf{y} - \mathbf{x})\mu_\delta(\mathbf{x}, \mathbf{y})$ is integrable, then we can change the order of

integration and then apply substitution $\mathbf{z} = \mathbf{y} - \mathbf{x}$ to obtain

$$\mathcal{L}_{\alpha_\delta}[u](\mathbf{x}) = \sigma \int_0^1 \int_{B_\delta(0)} \nabla u(\mathbf{x} + s\mathbf{z}) \cdot [\mathbf{z}\mu_\delta(|\mathbf{z}|)] d\mathbf{z} ds.$$

With π given by (3.5.1) we know

$$\nabla_{\mathbf{z}} \pi(|\mathbf{z}|) = \pi'(|\mathbf{z}|)\frac{\mathbf{z}}{|\mathbf{z}|} = -\mu_\delta(|\mathbf{z}|)\mathbf{z}$$

and consequently,

$$\mathcal{L}_{\alpha_\delta}[u](\mathbf{x}) = -\sigma \int_0^1 \int_{B_\delta(0)} \nabla u(\mathbf{x} + s\mathbf{z}) \cdot \nabla_{\mathbf{z}} \pi(|\mathbf{z}|) d\mathbf{z} ds.$$

Since $\pi(\delta) = 0$, then integration by parts gives

$$\mathcal{L}_{\alpha_\delta}[u](\mathbf{x}) = \sigma \int_0^1 \int_{B_\delta(0)} \mathrm{div}_{\mathbf{z}}[\nabla u(\mathbf{x} + s\mathbf{z})]\pi(|\mathbf{z}|) d\mathbf{z} ds$$

$$= \sigma \int_0^1 \int_{B_\delta(0)} \Delta u(\mathbf{x} + s\mathbf{z}) \, s \, \pi(|\mathbf{z}|) d\mathbf{z} ds.$$

Let ω_{d-1} be the surface measure of the unit sphere in $\mathbb{R}^d$. Using the identity

$$\int_{B_\delta(0)} \pi(|\mathbf{z}|) d\mathbf{z} = \int_0^\delta \pi(r)\omega_{d-1} r^{d-1} dr$$

we may rewrite $\mathcal{L}_{\alpha_{\delta_n}}[u](\mathbf{x})$ in the desired form:

$$\mathcal{L}_{\alpha_\delta}[u](\mathbf{x}) = \sigma \int_0^1 \int_{B_\delta(0)} (\Delta u(\mathbf{x} + s\mathbf{z}) - \Delta u(\mathbf{x})) \, s\pi(|\mathbf{z}|) d\mathbf{z} ds$$

$$+ \sigma \Delta u(\mathbf{x})\frac{\omega_{d-1}}{2} \int_0^\delta \pi(r)r^{d-1} dr.$$

Then apply the fact that $\sigma = 2d$ and use the definition (3.5.2) along with Proposition 3.5.2

to establish that all constants in the right-most term cancel. $\qquad\square$

Theorem 3.5.4 (Convergence of the nonlocal Laplacian)**.** *Suppose* $\Omega \subset \mathbb{R}^{d \geq 2}$ *is bounded and open, and* α_δ *satisfies Assumption 3.2.10. Let* $u \in C^4(\Omega) \cap W^{4,\infty}(\Omega)$ *and* $\sigma = 2d$ *in (3.5.4). Then there is* $K(u, d) > 0$ *dependent only on u and n such that*

$$|\mathcal{L}_{\alpha_\delta}[u](\mathbf{x}) - \Delta u(\mathbf{x})| \leq K(u, d)\delta^2 \quad \textit{whenever } B_\delta(\mathbf{x}) \subset \Omega. \tag{3.5.6}$$

In addition

$$\sup_{\mathbf{x} \in \Omega_\delta} |\mathcal{L}_{\alpha_\delta}[u](\mathbf{x})| \leq \|\Delta u\|_{L^\infty(\Omega)} + K(u, d)\delta^2. \tag{3.5.7}$$

so as $\delta \to 0$

$$\chi_{\Omega_\delta} \mathcal{L}_{\alpha_{\delta_n}}[u] \to \Delta u \quad \textit{strongly in} \quad L^2(\Omega). \tag{3.5.8}$$

And, if $\mathrm{supp}\, u \subset\subset \Omega$ *with* $\delta < \frac{1}{2} \cdot \mathrm{dist}(\mathrm{supp}\, u, \partial\Omega)$ *then*

$$\sup_{\mathbf{x} \in \Omega} |\mathcal{L}_{\alpha_\delta}[u](\mathbf{x})| \leq \|\Delta u\|_{L^\infty(\Omega)} + K(u, d) \cdot \mathrm{dist}(\mathrm{supp}\, u, \partial\Omega)^2. \tag{3.5.9}$$

whence as $\delta \to 0$

$$\mathcal{L}_{\alpha_{\delta_n}}[u] \to \Delta u \quad \textit{strongly in} \quad L^2(\Omega). \tag{3.5.10}$$

Proof. Let $\delta > 0$ be sufficiently small so that $B_\delta(\mathbf{x}) \subset \Omega$. Then by Lemma 3.5.3

$$\mathcal{L}_{\alpha_\delta}[u](\mathbf{x}) = \sigma \int_0^1 \int_{B_\delta(0)} s[\Delta u(\mathbf{x} + s\mathbf{z}) - \Delta u(\mathbf{x})]\pi(|\mathbf{z}|)d\mathbf{z}ds + \Delta u(\mathbf{x}).$$

Thus,

$$\mathcal{L}_{\alpha_\delta}[u](\mathbf{x}) - \Delta u(\mathbf{x}) = \sigma \int_0^1 \int_{B_\delta(0)} s[\Delta u(\mathbf{x} + s\mathbf{z}) - \Delta u(\mathbf{x})]\pi(|\mathbf{z}|)d\mathbf{z}ds$$

We will let $P_i(s)$ be a polynomial of degree i in s, that will be chosen appropriately. Rewrite

$s = \frac{d}{ds}(\frac{s^2-1}{2}) =: P'_2(s)$ and integrate by parts in s using $\frac{d}{ds}(\Delta u(\mathbf{x} + s\mathbf{z})) = \Delta\nabla u(\mathbf{x} + s\mathbf{z}) \cdot \mathbf{z})$ in order to obtain

$$\mathcal{L}_{\alpha_\delta}[u](\mathbf{x}) - \Delta u(\mathbf{x}) = -\sigma \int_{B_\delta(0)} \int_0^1 P_2(s)[\Delta\nabla u(\mathbf{x} + s\mathbf{z})] \cdot \mathbf{z}\pi(|\mathbf{z}|)ds d\mathbf{z}$$

$$= -\sigma \int_{B_\delta(0)} \int_0^1 P_2(s)[\Delta\nabla u(\mathbf{x} + s\mathbf{z}) - \Delta\nabla u(\mathbf{x})] \cdot \mathbf{z}\pi(|\mathbf{z}|)ds d\mathbf{z}$$

where the last step follows from

$$\int_{B_\delta(0)} \mathbf{z}\pi(|\mathbf{z}|)d\mathbf{z} = 0.$$

Since fourth-order derivatives of u are bounded, then we know that $|\Delta\nabla u(\mathbf{x}+s\mathbf{z})-\Delta\nabla u(\mathbf{x})| \leq M(u)|s\mathbf{z}|$ for some constant $M(u)$. Thus, we obtain

$$|\mathcal{L}_{\alpha_\delta}[u](\mathbf{x}) - \Delta u(\mathbf{x})| \leq \sigma M(u) \int_{B_\delta(0)} \int_0^1 |P_2(s)s||\mathbf{z}|^2\pi(|\mathbf{z}|)ds d\mathbf{z}$$

$$\lesssim \sigma M(u)\frac{1}{2}\omega_{d-1} \int_0^\delta \rho^{n+1}\pi(\rho)d\rho.$$

Finally, use the fact that $\rho^{n+1} \leq \rho^{d-1}\delta^2$ for $\rho \in (0, \delta)$ and definition (3.5.2) along with Proposition 3.5.2 to obtain

$$|\mathcal{L}_{\alpha_\delta}[u](\mathbf{x}) - \Delta u(\mathbf{x})| \leq K(u, d)\delta^2\sigma C = K(u, d)\delta^2 . \tag{3.5.11}$$

For any $\mathbf{x} \in \Omega_\delta$, we know (3.5.11) holds since $B_\delta(x) \subset \Omega$ (note the same δ is valid all $\mathbf{x} \in \Omega_\delta$). Thus

$$\|\mathcal{L}_{\alpha_\delta}[u](\mathbf{x})\|_{L^\infty(\Omega_\delta)} \leq \|\Delta u\|_{L^\infty(\Omega_\delta)} + K(u, d)\delta^2 .$$

This supplies a uniform bound on $\chi_{\Omega_\delta}\mathcal{L}_{\alpha_{\delta_n}}[u]$, so the convergence result (3.5.8) follows

from (3.5.6) which applies on Ω_δ and the fact that the measure of $\Omega \setminus \Omega_\delta$ tends to 0.

To verify (3.5.9) take $\delta \leq \frac{1}{2} \cdot \text{dist}(\text{supp}\, u, \partial\Omega)$. When $\mathbf{x} \in \Omega \setminus \Omega_\delta$ notice that $\Delta u(\mathbf{x}) = \mathcal{L}_{\alpha_\delta}[u](\mathbf{x}) = 0$. Wheres for all $\mathbf{x} \in \Omega_\delta$ estimate (3.5.11) holds (note δ does not depend on $\mathbf{x}$ here). Thus for all $\delta \leq \frac{1}{2} \cdot d(\text{supp}\, u, \partial\Omega)$ we have

$$\sup_{\mathbf{x} \in \Omega} |\mathcal{L}_{\alpha_\delta}[u](\mathbf{x})| \leq \|\Delta u\|_{L^\infty(\Omega)} + K(u, d) \cdot d(\text{supp}\, u, \partial\Omega)^2.$$

The assertion of a compact support simplifies (3.5.8) to (3.5.10). $\qquad\qquad \square$

Theorem 3.5.5 (Convergence of the nonlocal biharmonic)**.** *Suppose $\Omega \subset \mathbb{R}^{d \geq 2}$ is bounded and open, $\mathbf{x} \in \Omega$, and α_δ satisfies Assumption 3.2.10. Let $u \in C^5(\Omega) \cap W^{5,\infty}(\Omega)$ and $\sigma = 2d$. Then there is a constant $K(u, d) > 0$ dependent only on u and d, such that*

$$|\mathcal{B}_{\alpha_\delta}[u](\mathbf{x}) - \Delta^2 u(\mathbf{x})| \leq K(u, d)\delta$$

whenever $B_\delta(\mathbf{x}) \subset \Omega$. Moreover, if $\text{supp}\, u \subset\subset \Omega$ and $\delta < \frac{1}{3} \cdot d(\text{supp}\, u, \partial\Omega)$ then the estimate is uniform in $\mathbf{x}$ and

$$\sup_{x \in \Omega} |\mathcal{B}_{\alpha_\delta}[u]| \leq \|\Delta^2 u\|_{L^\infty(\Omega)} + K(u, d) \cdot d(\text{supp}\, u, \partial\Omega)$$

Proof. Function u has sufficient regularity to expand the weak biharmonic according to its definition:

$$B_{\alpha_\delta}[u](\mathbf{x}) = \sigma \int_{B_\delta(\mathbf{x})} [\mathcal{L}_{\alpha_\delta}[u](\mathbf{y}) - \mathcal{L}_{\alpha_\delta}[u](\mathbf{x})] \, \mu_\delta(\mathbf{x}, \mathbf{y}) d\mathbf{y}.$$

Appealing to Lemma 3.5.3, and canceling σ with C, yields

$$
\begin{aligned}
B_{\alpha_\delta}[u](\mathbf{x}) = \sigma \int_{B_\delta(\mathbf{x})} \Bigg[&\Delta u(\mathbf{y}) - \Delta u(\mathbf{x}) \\
&+ \sigma \int_0^1 \int_{B_\delta(0)} s[\Delta u(\mathbf{y} + s\mathbf{z}) - \Delta u(\mathbf{y})]\pi(|\mathbf{z}|)d\mathbf{z}ds \\
&- \sigma \int_0^1 \int_{B_\delta(0)} s[\Delta u(\mathbf{x} + s\mathbf{z}) - \Delta u(\mathbf{x})]\pi(|\mathbf{z}|)d\mathbf{z}ds \Bigg] \mu_\delta(\mathbf{x}, \mathbf{y})d\mathbf{y}.
\end{aligned}
\tag{3.5.12}
$$

The first term in the above equation can be simplified using the definition of (scaled) non-local Laplacian $\mathcal{L}_{\alpha_\delta}$:

$$
\sigma \int_{B_\delta(\mathbf{x})} [\Delta u(\mathbf{y}) - \Delta u(\mathbf{x})]\mu_\delta(\mathbf{x}, \mathbf{y})d\mathbf{y} = \mathcal{L}_{\alpha_\delta}[\Delta u](\mathbf{x}).
$$

From Lemma 3.5.3, again, we obtain

$$
\mathcal{L}_{\alpha_\delta}[\Delta u](\mathbf{x}) = \boxed{\sigma \int_0^1 \int_{B_\delta(0)} s[\Delta^2 u(\mathbf{x} + s\mathbf{z}) - \Delta^2 u(\mathbf{x})]\pi(|\mathbf{z}|)d\mathbf{z}ds} + \Delta^2 u(\mathbf{x}).
$$

Substituting this back into (3.5.12) results in

$$
\begin{aligned}
&\mathcal{B}_{\alpha_\delta}[u](\mathbf{x}) - \Delta^2 u(\mathbf{x}) \\
&= \sigma \int_{B_\delta(\mathbf{x})} \Bigg[\sigma \int_0^1 \int_{B_\delta(0)} s[\Delta u(\mathbf{y} + s\mathbf{z}) - \Delta u(\mathbf{y})]\pi(|\mathbf{z}|)d\mathbf{z}ds \\
&\qquad\qquad - \sigma \int_0^1 \int_{B_\delta(0)} s[\Delta u(\mathbf{x} + s\mathbf{z}) - \Delta u(\mathbf{x})]\pi(|\mathbf{z}|)d\mathbf{z}ds \Bigg] \mu_\delta(\mathbf{x}, \mathbf{y})d\mathbf{y} \\
&+ \boxed{\sigma \int_0^1 \int_{B_\delta(0)} s[\Delta^2 u(\mathbf{x} + s\mathbf{z}) - \Delta^2 u(\mathbf{x})]\pi(|\mathbf{z}|)d\mathbf{z}ds}.
\end{aligned}
\tag{3.5.13}
$$

Demonstrating that the boxed term is of order δ is a simplified version of the argument necessary for the first two integrals (which incorporate the non-integrable kernel μ_δ). The rest of the proof will focus on the first two summands in (3.5.13).

For s introduce

$$F_s(\mathbf{x}) := \sigma \int_{B_\delta(0)} [\Delta u(\mathbf{x} + s\mathbf{z}) - \Delta u(\mathbf{x})] \pi(\mathbf{z}) d\mathbf{z}$$

and rewrite the first two terms on the right in (3.5.13) as $\int_0^1 s\mathcal{L}_\delta[F_s](\mathbf{x}) \, ds$. By Lemma 3.5.3

$$
\begin{aligned}
&\int_0^1 s\mathcal{L}_\delta[F_s](\mathbf{x}) \, ds \\
&= \int_0^1 s\sigma \int_0^1 \int_{B_\delta(0)} \tilde{s}\left(\Delta F_s(\mathbf{x} + \tilde{s}\tilde{\mathbf{z}}) - \Delta F_s(\mathbf{x})\right) \pi(|\tilde{\mathbf{z}}|) d\tilde{\mathbf{z}} d\tilde{s} + \Delta F_s(\mathbf{x}) ds
\end{aligned}
\tag{3.5.14}
$$

Recall $u \in C^5(\Omega) \cap W^{5,\infty}(\Omega)$, whence there exists $M > 0$ such that

$$\sup_{0 \le i \le 4} \left\| D^i \left(u(\mathbf{x} + s\mathbf{z}) - u(\mathbf{x}) \right) \right\|_{\mathbb{R}^{n^i}} \le M|s\mathbf{z}|.$$

Therefore,

$$
\begin{aligned}
|\Delta F_s(\mathbf{x})| &\le \sigma \int_{B_\delta(0)} \left| \Delta^2 u(\mathbf{x} + s\mathbf{z}) - \Delta^2 u(\mathbf{x}) \right| \pi(\mathbf{z}) d\mathbf{z} \\
&\le \sigma M \int_{B_\delta(0)} |s\mathbf{z}| \pi(\mathbf{z}) d\mathbf{z} \\
&\le \sigma M |s| \omega_{d-1} \int_{B_\delta(0)} r^d \pi(r) dr \qquad \text{(change of variables)} \\
&\le |s| \delta M \sigma \omega_{d-1} \int_0^\delta r^{d-1} \pi(r) dr \\
&\le 2|s| \delta M \qquad \text{(By definition of } \sigma \text{)}
\end{aligned}
\tag{3.5.15}
$$

By applying (3.5.15) to (3.5.14) we obtain the bound

$$\int_0^1 s\mathcal{L}_\delta[F_s](\mathbf{x})\,ds \le \int_0^1 s\sigma \int_0^1 \int_{B_\delta(0)} \tilde{s}\,(4|s|\delta M)\,\pi(|\tilde{\mathbf{z}}|)\,d\tilde{z}d\tilde{s} + 2|s|\delta M\,ds$$

$$\le M'\delta\left(\sigma \int_{B_\delta(0)} \pi(|\tilde{\mathbf{z}}|)\,d\tilde{\mathbf{z}} + 1\right)$$

$$\le 3M'\delta$$

Combining (3.5.13) with (3.5.2), we finally arrive at

$$|\mathcal{B}_{\alpha_\delta}[u](\mathbf{x}) - \Delta^2 u(\mathbf{x})| \le K(n,u)\cdot\delta \tag{3.5.16}$$

Now to verify the uniform bound. Suppose $\delta < \frac{1}{3}\cdot d(\operatorname{supp} u, \partial\Omega)$. First consider $\mathbf{x} \in \Omega\backslash\Omega_\delta$. Since $B_\delta(\mathbf{x}) \cap \operatorname{supp}(u) = \emptyset$ we have

$$\mathcal{L}_{\alpha_\delta}[u](\mathbf{x}) = \sigma \int_{B_\delta(\mathbf{x})} (u(\mathbf{y}) - u(\mathbf{x}))\mu(\mathbf{x},\mathbf{y})d\mathbf{y} = 0$$

Likewise, for any $\mathbf{y} \in B_\delta(\mathbf{x})$ we have $B_\delta(\mathbf{y}) \cap \operatorname{supp}(u) = \emptyset$. Consequently,

$$\mathcal{L}_{\alpha_\delta}[u](\mathbf{y}) = \sigma \int_{B_\delta(\mathbf{y})} (u(\mathbf{z}) - u(\mathbf{y}))\mu(\mathbf{y},\mathbf{z})d\mathbf{z} = 0$$

We conclude, for any $\mathbf{x} \in \Omega\backslash\Omega_\delta$ we have

$$\mathcal{B}_{\alpha_\delta}[u](\mathbf{x}) = \sigma \int_{B_\delta(\mathbf{x})} (\mathcal{L}_{\alpha_\delta}[u](\mathbf{y}) - \mathcal{L}_{\alpha_\delta}[u](\mathbf{x}))\mu(\mathbf{x},\mathbf{y})d\mathbf{y} = 0$$

Thus $\mathcal{B}_{\alpha_\delta}[u]$ and $\Delta^2 u$ agree on $\Omega\backslash\Omega_\delta$. Also notice that $\forall \mathbf{x} \in \Omega_\delta$ we have $B_\delta(\mathbf{x}) \subset \Omega$ and so (3.5.16) holds (note δ does not depend on $\mathbf{x}$ here). Thus for all $\delta < \frac{1}{2}\cdot d(\operatorname{supp} u, \partial\Omega)$ we have

$$\sup_{\mathbf{x}\in\Omega} |\mathcal{B}_{\alpha_\delta}[u]| \le \|\Delta^2 u\|_{L^\infty(\Omega)} + K(u,d)\cdot d(\operatorname{supp} u, \partial\Omega).$$

$\square$

3.6 Well-posedness of the nonlocal steady state problem

Throughout this section Ω is any bounded open set in $\mathbb{R}^d$ for $d \geq 2$. The first problem we will look at is the nonlocal elliptic biharmonic equation

$$\mathcal{B}_\alpha[u] = f \quad \text{in} \quad \Omega' \tag{3.6.1}$$

with nonlocal equivalent of hinged or clamped boundary conditions (Definition 3.3.9):

$$u \in \mathbf{H} = \mathcal{H}^2_{\alpha,H} \quad \text{or} \quad \mathcal{H}^2_{\alpha,C} \tag{3.6.2}$$

Note that both the spaces are topologized by the norm in $\mathcal{H}^2_\alpha$.

Proposition 3.6.1. *Suppose $f \in L^2(\Omega')$, and α satisfies Assumption 3.2.10. There exists a unique (weak) solution $u \in \mathbf{H}$ of the nonlocal PDE* (3.6.1).

Proof. We prove this result by using the Lax-Milgram Lemma. For $v \in \mathbf{H}$ the associated weak formulation is

$$\int_\Omega \mathcal{B}_\alpha[u]v \, d\mathbf{x} = \int_\Omega fv \, d\mathbf{x}.$$

Using the fact that $v = 0$ on $\Omega \setminus \Omega'$ (regardless of the definition of $\mathbf{H}$) and through repeated application of Proposition 3.2.6 we obtain

$$a[u, v] = \int_\Omega fv \, d\mathbf{x},$$

where

$$a[u, v] = \int_\Omega \mathcal{L}_\alpha[u]\mathcal{L}_\alpha[v]d\mathbf{x}.$$

This bilinear form is continuous since

$$|a[u,v]| \leq \|\mathcal{L}_\alpha[u]\|_{L^2(\Omega)}\|\mathcal{L}_\alpha[v]\|_{L^2(\Omega)} \leq \|u\|_{\mathscr{H}^2(\Omega)}\|v\|_{\mathscr{H}^2(\Omega)}.$$

It remains to show coercivity. Since all functions in $\mathbf{H}$ vanish on a fixed collar $\Omega \setminus \Omega'$, then by Corollary 3.4.7 there is $C > 0$ such that

$$\|u\|^2_{L^2(\Omega)} \leq C\|\mathcal{G}_\alpha[u]\|^2_{L^2(\Omega\times\Omega)}. \tag{3.6.3}$$

Integration by parts (Proposition 3.2.6) and Hölder's inequality yields

$$\|\mathcal{G}_\alpha[u]\|^2_{L^2(\Omega\times\Omega)} = -\int_\Omega u\mathcal{L}_\alpha[u]d\mathbf{x} \leq \|\mathcal{L}_\alpha[u]\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}. \tag{3.6.4}$$

By combining (3.6.3) and (3.6.4) we conclude that $\|u\|_{L^2(\Omega)} \leq C_1\|\mathcal{L}[u]\|_{L^2(\Omega)}$ and $\|\mathcal{G}_\alpha[u]\|_{L^2(\Omega\times\Omega)} \leq C\|\mathcal{L}[u]\|_{L^2(\Omega)}$. Thus, there is $c_2 > 0$ such that for all $u \in \mathbf{H}$

$$a(u,u) = \|\mathcal{L}[u]\|^2_{L^2(\Omega)} \geq c_2\|u\|^2_{\mathscr{H}^2(\Omega)}$$

Since $\mathbf{H}$ is a Hilbert space, by the Lax-Milgram theorem there exists a unique element $u \in \mathbf{H}$ satisfying

$$B[u] = f.$$

$\square$

3.7 Convergence results

In this section we will show that the solution to a nonlocal system approximates the corresponding solution of the classical elliptic boundary value problem as the horizon ap-

proaches 0. Such a study has been previously carried out for the Navier system in [29].

Before proceeding to the biharmonic operator we collect several results for the classical and nonlocal Poisson problems:

$$\begin{cases} \Delta v = f, & \mathbf{x} \in \Omega \\ v = 0, & \mathbf{x} \in \partial\Omega \end{cases} \tag{3.7.1}$$

and its nonlocal analogue:

$$\begin{cases} \mathcal{L}_{\alpha_{\delta_n}}[v_n] = f, & \mathbf{x} \in \Omega_{\tilde{\delta}_n} \\ v_n = 0, & \mathbf{x} \in \Omega \setminus \Omega_{\tilde{\delta}_n}, \quad \tilde{\delta}_n \geq \delta_n \,. \end{cases} \tag{3.7.2}$$

Note that for convenience we allow the zero Dirichlet data to be prescribed over a possibly thicker collar than the horizon of the Laplace operator. Recall that we are using the scaled (by $\sigma = 2d$) version of the Laplacian (3.5.4). The following result is an adjusted version of [29, Thm 5.4].

Theorem 3.7.1 (cf. [29, Thm 5.4]). *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be an open bounded set of class C^1. Suppose a sequence of positive scalars (δ_n) converges to 0 as $n \to \infty$. For any $f \in L^2(\Omega)$, the sequence of solutions $\{v_n\} \subset \mathcal{H}^1_{\alpha_{\delta_n},0}(\Omega_{\tilde{\delta}_n})$ to (3.7.2) converges strongly in $L^2(\Omega)$ to $v \in W^{1,2}_0(\Omega)$ where v is the weak (variational) solution of the Laplace equation (3.7.1). In addition, there is $C > 0$ so that for all n*

$$\|v_n\|_{L^2(\Omega)} + \|\mathcal{G}_{\alpha_{\delta_n}}[v_n]\|_{L^2(\Omega \times \Omega)} \leq C \|f\|_{L^2(\Omega_{\tilde{\delta}_n})} \,. \tag{3.7.3}$$

Moreover, if Ω is of class C^2 then the classical elliptic theory (see, for instance, [8, Thm. IX.25, p. 181]) gives $v \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$.

Proof. To invoke Theorem 3.4.3 we first establish bounds on $\|\mathcal{G}_{\alpha_{\delta_n}}[v_n]\|_{L^2(\Omega \times \Omega)}$ and on

$\|v_n\|_{L^2(\Omega)}$, independently of n. Apply Corollary 3.4.6, nonlocal integration by parts, and Hölder's inequality:

$$
\begin{aligned}
\|v_n\|_{L^2(\Omega)}^2 &\leq C\|\mathcal{G}_{\alpha_{\delta_n}}[v_n]\|_{L^2(\Omega\times\Omega)}^2 \\
&= -C\langle \mathcal{L}_{\alpha_{\delta_n}}[v_n], v_n\rangle_{L^2(\Omega)} \\
&= -C\langle \mathcal{L}_{\alpha_{\delta_n}}[v_n], v_n\rangle_{L^2(\Omega_{\tilde{\delta}_n})} \\
&= -C\langle f, v_n\rangle_{L^2(\Omega_{\tilde{\delta}_n})} \\
&\leq C\|f\|_{L^2(\Omega_{\tilde{\delta}_n})}\|v_n\|_{L^2(\Omega)},
\end{aligned}
\tag{3.7.4}
$$

where C is independent of n. Dividing by $\|v_n\|_{L^2(\Omega)}$ yields a bounds on the latter in terms of $\|f\|_{L^2(\Omega_{\tilde{\delta}_n})}$. That in turn gives a bound on the nonlocal gradient verifying (3.7.3) up to a minor adjustment of the constant.

We conclude by Corollary 3.4.5 that $\{v_n\}$ is relatively compact in $L^2(\Omega)$. We will show that every cluster point of $\{v_n\}$ solves the classical Poisson equation (3.7.1) which has a unique solution in $W_0^{1,2}(\Omega)$. Let (v_n) be a convergent subsequence. Consider a test function $\phi \in C_c^\infty(\Omega)$. We may assume that $\tilde{\delta}_n$ is small enough to ensure that $\operatorname{supp}\phi \subset \Omega_{\tilde{\delta}_n}$. Then via Proposition 3.2.6

$$
\int_\Omega v_n \mathcal{L}_{\alpha_{\delta_n}}[\phi]d\mathbf{x} = \int_\Omega \mathcal{L}_{\alpha_{\delta_n}}[v_n]\phi d\mathbf{x} = \int_{\Omega_{\tilde{\delta}_n}} \mathcal{L}_{\alpha_{\delta_n}}[v_n]\phi d\mathbf{x} = \int_\Omega f\phi d\mathbf{x}.
$$

Since $v_n \to v$ strongly in $L^2(\Omega)$, and by the result of Theorem 3.5.4 applied to the compactly supported function ϕ the limit as $n \to \infty$ gives

$$
\int_\Omega v\Delta\phi \, d\mathbf{x} = \int_\Omega f\phi d\mathbf{x}
$$

as $\delta_n \to 0$, verifying that v is the distributional solution of the Poisson problem. Since $v \in W_0^{1,2}(\Omega)$, then via integration by parts we conclude that v is the weak variational solution

to the Poisson problem. □

Now consider a family of equations

$$
\begin{cases}
\mathcal{L}_{\alpha_{\delta_n}}[v_n] = f_n, & \mathbf{x} \in \Omega_{\tilde{\delta}_n} \\
v_n = 0, & \mathbf{x} \in \Omega \setminus \Omega_{\tilde{\delta}_n}, \quad \tilde{\delta}_n \geq \delta_n
\end{cases}
\tag{3.7.5}
$$

where $v_n \in \mathcal{H}^1_{\alpha_{\delta_n},0}(\Omega_{\tilde{\delta}_n})$.

Corollary 3.7.2. *The result of Theorem 3.7.1 holds if in each nonlocal problem* (3.7.2) *and in* (3.7.3) *we replace f by $f_n \in L^2(\Omega)$ assuming that $f_n \to f$ in $L^2(\Omega)$.*

Proof. Since f_n are uniformly bounded in $L^2(\Omega)$ then as in (3.7.4) in the proof of Theorem 3.7.1 we conclude that $\{v_n\}$ is relatively compact in $L^2(\Omega)$. Let v be a cluster point of this sequence. Consider a test function $\phi \in C_c^\infty(\Omega)$. We may assume that $\operatorname{supp}\phi \subset \Omega_{\tilde{\delta}_n}$ for all n. By Proposition 3.2.6 we obtain

$$
\int_\Omega v_n \mathcal{L}_{\alpha_{\delta_n}}[\phi]d\mathbf{x} = \int_\Omega \mathcal{L}_{\alpha_{\delta_n}}[v_n]\phi d\mathbf{x} = \int_{\Omega_{\tilde{\delta}_n}} \mathcal{L}_{\alpha_{\delta_n}}[v_n]\phi d\mathbf{x} = \int_\Omega f_n\phi d\mathbf{x} \,.
$$

Since $f_n \to f$ and $v_n \to v$ in L^2, via Theorem 3.5.4 applied to ϕ conclude

$$
\int_\Omega v\Delta\phi d\mathbf{x} = \int_\Omega f\phi d\mathbf{x}
$$

From the $W_0^{1,2}(\Omega)$ regularity it follows that v is a weak solution of the Poisson problem (3.7.1). □

3.7.1   Convergence to the classical solution for the hinged problem

We turn to the elliptic problem for the nonlocal biharmonic and the analog of hinged boundary conditions. Note that second-order boundary condition will be applied to the extended
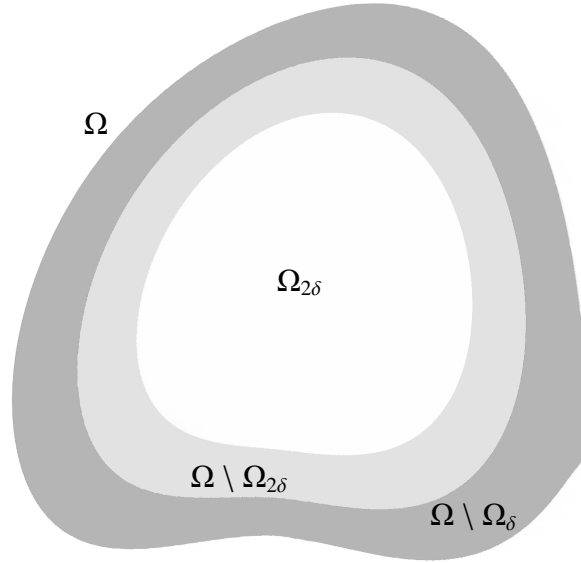
Figure 3.2: The nonlocal domain Ω with its collar boundaries $\Omega \setminus \Omega_{2\delta}$ and $\Omega \setminus \Omega_{\delta}$.

collar. In particular we consider this problem on the space $\mathscr{H}^2_{\alpha_{\delta_n}}(\Omega_\delta, \Omega_{2\delta})$ according to the definition (3.3.6).

Theorem 3.7.3. *Let $\Omega \subset \mathbb{R}^n$, $d \geq 2$, be a bounded domain of class C^2. Suppose sequence of positive scalars (δ_n) converges to 0 as $n \to \infty$. The solutions of the nonlocal hinged problems*

$$\begin{cases} \mathcal{B}_{\alpha_{\delta_n}}[u_n] = f, & \mathbf{x} \in \Omega_{2\delta_n} \\ u_n = 0, & \mathbf{x} \in \Omega \setminus \Omega_{\delta_n} \\ \mathcal{L}_{\alpha_{\delta_n}}[u_n] = 0, & \mathbf{x} \in \Omega_{\delta_n} \setminus \Omega_{2\delta_n} \end{cases} \qquad (3.7.6)$$

converge in $L^2(\Omega)$ to the weak (variational) solution $u \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ of

$$\begin{cases} \Delta^2 u = f, & \mathbf{x} \in \Omega \\ u = \Delta u = 0, & \mathbf{x} \in \partial\Omega \end{cases} \qquad (3.7.7)$$

as $n \to \infty$. If, in addition, Ω is of class C^4, then u is also in $W^{4,2}(\Omega)$.

Remark 3.7.4 (If $\mathcal{L}_{\alpha_{\delta_n}}[u_n]$ were also zero on $\Omega \setminus \Omega_{\delta_n}$). We conjecture that such an overlap would actually enforce u to be zero on the full extended collar $\Omega \setminus \Omega_{2\delta}$. In turn, by the same argument, that would imply $u = 0$ on $\Omega \setminus \Omega_{3\delta}$ which might contradict the data in the interior unless very special "f" is considered.

Proof. **Step 1.** Let χ_δ be the characteristic function of Ω_δ. Set

$$v_n(\mathbf{x}) := \chi_\delta(\mathbf{x})\mathcal{L}_{\alpha_{\delta_n}}[u_n](\mathbf{x}).$$

Due to the support of $\mu_{\delta_n}(\mathbf{x}, \cdot)$ the value of $\mathcal{B}_{\alpha_{\delta_n}}[u_n](\mathbf{x})$ for $\mathbf{x} \in \Omega_{2\delta_n}$ only depends on $\mathcal{L}_{\alpha_{\delta_n}}[u_n]$ in Ω_{δ_n}. Since in Ω_{δ_n} the functions v_n and $\mathcal{L}_{\alpha_{\delta_n}}[u_n]$ coincide (by definition) then

$$\mathcal{L}_{\alpha_{\delta_n}}[v_n](\mathbf{x}) = \mathcal{B}_{\alpha_{\delta_n}}[u_n](\mathbf{x}) = f(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in \Omega_{\delta_{2n}}.$$

When $\mathbf{x} \in \Omega \setminus \Omega_{\delta_n}$ then $\mathbf{v}_n = 0$, and if $\mathbf{x} \in \Omega_{\delta_n} \setminus \Omega_{2\delta_n}$, then $v_n(\mathbf{x}) = \mathcal{L}_{\alpha_{\delta_n}}[u_n](\mathbf{x}) = 0$ by the second boundary condition in (3.7.6). Hence v_n satisfies

$$\begin{cases} \mathcal{L}_{\alpha_{\delta_n}}[v_n] = f, & \mathbf{x} \in \Omega_{2\delta_n} \\ v_n = 0, & \mathbf{x} \in \Omega \setminus \Omega_{2\delta_n}. \end{cases}$$

By Theorem 3.7.1 (with $\tilde{\delta}_n = 2\delta_n \geq \delta_n$) we know that $v_n \to v$ in $L^2(\Omega)$ with $v \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$, where $\Delta v = f$.

Step 2. By definition of v_n, we also know that u_n solves

$$\begin{cases} \mathcal{L}_{\alpha_{\delta_n}}[u_n] = v_n, & \mathbf{x} \in \Omega_{\delta_n} \\ u_n = 0, & \mathbf{x} \in \Omega \setminus \Omega_{\delta_n}. \end{cases}$$

According to Corollary 3.7.2 (now $\tilde{\delta}_n = \delta_n$), $u_n \to u$ in $L^2(\Omega)$, where $u \in W_0^{1,2}(\Omega)$ and u is

the weak solution to $\Delta u = v$. Then it follows that

$$\Delta^2 u = \Delta v = f$$

in the sense of distributions. Since $u \in W_0^{1,2}(\Omega)$ and $\Delta u = v \in L^2(\Omega)$, then C^2 regularity of the domain ensures $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. This fact, along with $\Delta u \in W_0^{1,2}(\Omega)$ via integration by parts of the distributional identity $\Delta^2 u = f$ verifies that u is, in fact, a weak solution of (3.7.7).

Moreover, from the elliptic theory follows that if the boundary is of class C^4 then $u \in W^{4,2}(\Omega)$ [25, Sec 3A, pp. 282–284] (the latter, in fact, deals with smooth boundary, but it suffices to consider the order of boundary regularity which matches that of the operator, namely, C^4). □

3.7.2   Convergence to the classical solution for the clamped problem

The next result deals with a nonlocal version the clamped biharmonic problem. This time the boundary conditions are up to the first order, and are both prescribed on the "original" collar $\Omega \setminus \Omega_{\delta_n}$. As indicated earlier in the analysis of nonlocal clamped boundary conditions, this condition in fact affects the extended collar too (3.3.6) essentially forcing the solution to be zero on the extended collar.

Remark 3.7.5 (Distinction between nonlocal hinged and clamped). Note that $u = 0$ on $\Omega \setminus \Omega_{2\delta}$ does not provide $\mathcal{L}_{\alpha_{\delta_n}}[u_n] = 0$ on $\Omega_\delta \setminus \Omega_{2\delta}$ as imposed in the hinged case, since the nonlocal Laplacian on $\Omega_\delta \setminus \Omega_{2\delta}$ subdomain also draws information from inside $\Omega_{2\delta}$.

Had all boundary conditions in (3.7.6) were instead imposed on the same collar $\Omega \setminus \Omega_\delta$, then those conditions would have been a consequence of the nonlcoal clamped data on Ω_δ.

$$
\begin{cases}
\mathcal{B}_{\alpha_{\delta_n}}[u_n] = f, & \mathbf{x} \in \Omega_{2\delta} \\
u_n = \mathcal{N}_{\alpha_{\delta_n}}[\mathcal{G}_{\alpha_{\delta_n}}[u_n]] = 0, & \mathbf{x} \in \Omega \setminus \Omega_{\delta_n}, \quad (\text{equiv. } u_n = 0, \ \mathbf{x} \in \Omega \setminus \Omega_{2\delta})
\end{cases}
\tag{3.7.8}
$$

and its classical analog:

$$
\begin{cases}
\Delta^2 u = f, & \mathbf{x} \in \Omega \\
u = \dfrac{\partial u}{\partial v} = 0, & \mathbf{x} \in \partial\Omega \,.
\end{cases}
\tag{3.7.9}
$$

Theorem 3.7.6. *Let $\Omega \subset \mathbb{R}^n$, $d \geq 2$, be bounded open of class C^6. Suppose sequence of positive scalars (δ_n) converges to 0 as $n \to \infty$. For all $f \in L^2(\Omega)$, the sequence of solutions $\{u_n\} \subset \mathcal{H}^2_{\alpha_{\delta_n},C}(\Omega_{\delta_n})$ to (3.7.8) converges strongly in $L^2(\Omega)$ to $v \in W^{2,2}_0(\Omega)$, which is a weak solution of (3.7.9), as $n \to \infty$. Moreover u, in fact, is the regular solution to this elliptic problem: $u \in W^{4,2}(\Omega)$.*

Proof. **Step 1.** We plan to invoke Theorem 3.4.3 (Corollary 3.4.5), so we need to demonstrate an upper bound on $\|\mathcal{G}_{\alpha_{\delta_n}}[u_n]\|_{L^2(\Omega)}$, independent of n. Apply the Poincaré-type inequality of Corollary 3.4.6 and nonlocal integration by parts (Proposition 3.2.6, applied in the weak sense as indicated in Remark 3.3.5):

$$
\begin{aligned}
c\|u_n\|^2_{L^2(\Omega)} \leq \|\mathcal{G}_{\alpha_{\delta_n}}[u_n]\|^2_{L^2(\Omega\times\Omega)} &= -\int_\Omega \mathcal{L}_{\alpha_{\delta_n}}[u_n] u_n d\mathbf{x} \\
&\leq \|\mathcal{L}_{\alpha_{\delta_n}}[u_n]\|_{L^2(\Omega)} \|u_n\|_{L^2(\Omega)}
\end{aligned}
\tag{3.7.10}
$$

for some $c > 0$. Hence

$$
c\|u_n\|_{L^2(\Omega)} \leq \|\mathcal{L}_{\alpha_{\delta_n}}[u_n]\|_{L^2(\Omega)}
$$

$$
\|\mathcal{G}_{\alpha_{\delta_n}}[u_n]\|^2_{L^2(\Omega\times\Omega)} \leq \frac{1}{c}\|\mathcal{L}_{\alpha_{\delta_n}}[u_n]\|^2_{L^2(\Omega)} \,.
$$

Thus:

$$\|u_n\|^2_{L^2(\Omega)} \leq \frac{1}{c}\|\mathcal{G}_{\alpha_{\delta_n}}[u_n]\|^2_{L^2(\Omega\times\Omega)} \leq \frac{1}{c^2}\|\mathcal{L}_{\alpha_{\delta_n}}[u_n]\|^2_{L^2(\Omega)}$$
$$= \frac{1}{c^2}\left|\langle\mathcal{B}_{\alpha_{\delta_n}}[u_n], u_n\rangle_{L^2(\Omega)}\right| .$$

$$(3.7.11)$$

Since the clamped space (3.3.8) enforces zero data on the extended collar $\Omega \setminus \Omega_{2\delta}$ we continue the estimate (3.7.11)

$$\cdots \leq \frac{1}{c^2}\left|\langle\mathcal{B}_{\alpha_{\delta_n}}[u_n], u_n\rangle_{L^2(\Omega_{2\delta})}\right| = \frac{1}{c^2}\left|\langle f, u_n\rangle_{L^2(\Omega)}\right|$$
$$\leq \frac{1}{c^2}\|f\|_{L^2(\Omega)}\|u_n\|_{L^2(\Omega)}$$

$$(3.7.12)$$

where c is independent of n. Therefore, (u_n) is bounded in $L^2(\Omega)$ and so is $(\mathcal{G}_{\alpha_{\delta_n}}[u_n])$. By Corollary 3.4.5, $\{u_n\}$ is relatively compact in $L^2(\Omega)$ and if u is a cluster point of $\{u_n\}$, then $u \in W_0^{1,2}(\Omega)$.

It remains to show that any cluster point of $\{u_n\}$ is the unique weak solution of (3.7.9). Pick a test function $\phi \in C_c^\infty(\Omega)$. We may assume that δ_n is small enough so that $\text{supp}\,\phi \subset \Omega_{3\delta_n}$ (note the factor of 3 instead of just 2). By Proposition 3.2.6 we then have

$$\langle\mathcal{B}_{\alpha_{\delta_n}}[\phi], u_n\rangle_{L^2(\Omega)} = \langle\phi, \mathcal{B}_{\alpha_{\delta_n}}[u_n]\rangle_{L^2(\Omega)} = \langle\phi, f\rangle_{L^2(\Omega)}$$

Use the fact that $u_n \to u$ strongly in $L^2(\Omega)$ and Theorem 3.5.5 to conclude that

$$\int_\Omega \Delta^2\phi\, u\, d\mathbf{x} = \int \phi f d\mathbf{x}$$

as $n \to \infty$. This verifies that $u \in W_0^{1,2}(\Omega)$ is a distributional solution to

$$\Delta^2 u = f .$$

Step 2. Let's prove $u \in W^{2,2}(\Omega)$. Set $z_n = \mathcal{L}_{\alpha_{\delta_n}}[u_n]$. From (3.7.11)–(3.7.12)

$$\|z_n\|^2_{L^2(\Omega)} \le \|f\|^2_{L^2(\Omega)} \|u_n\|^2_{L^2(\Omega)} \,.$$

Since (u_n) is bounded in $L^2(\Omega)$ then so is (z_n). Conclude that (z_n) converges weakly to some z in $L^2(\Omega)$. At the same time, for $\phi \in C_c^\infty(\Omega)$ nonlocal integration by parts gives

$$\langle z_n, \phi \rangle_{L^2(\Omega)} = \langle u_n, \mathcal{L}_{\alpha_{\delta_n}}[\phi] \rangle_{L^2(\Omega)}$$

We know $u_n \to u$ strongly in $L^2(\Omega)$ and because ϕ is compactly supported, by Theorem 3.5.4 we also have $\mathcal{L}_{\alpha_{\delta_n}}[\phi] \to \Delta\phi$ strongly in $L^2(\Omega)$. And since z_n converges weakly then in the limit $n \to \infty$ this identity becomes

$$\langle z, \phi \rangle_{L^2(\Omega)} = \langle u, \Delta\phi \rangle_{L^2(\Omega)} \,.$$

So $z \in L^2(\Omega)$ is the distributional Laplacian of u. Since $u \in W_0^{1,2}(\Omega)$ and the domain Ω is smooth, then the elliptic regularity ensures that $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.

Step 3. To prove that u is the weak variational solution of the classical biharmonic problem it remains to establish that u has zero normal trace. For this purpose consider a family of test functions that belong to $C^4(\Omega) \cap W^{4,\infty}(\Omega)$, which will be used to apply Theorem 3.5.4, and whose boundary traces are dense in $L^2(\Gamma)$. For, instance, relying on 2D Sobolev embeddings, it suffices to consider $W^{6,2}(\Omega)$. If Ω is sufficiently smooth, e.g., of (uniform) C^6 regularity, then the traces of these functions are exactly the space $W^{5+\frac{1}{2},2}(\Gamma)$ which is dense in $L^2(\Gamma)$.

Fix such a test function ψ. We have just shown that $\Delta u \in L^2(\Omega)$ and that $z_n = \mathcal{L}_{\alpha_{\delta_n}}[u_n]$

converges to Δu weakly in $L^2(\Omega)$. Thus

$$\lim_{n\to\infty} \int_\Omega \mathcal{L}_{\alpha_{\delta_n}}[u_n]\psi \, d\mathbf{x} = \int_\Omega \Delta u\psi \, d\mathbf{x}. \tag{3.7.13}$$

The goal is to verify that

$$\int_\Omega \Delta u\phi \, d\mathbf{x} = \int_\Omega u\Delta\phi \, d\mathbf{x}$$

Invoke again nonlocal integration by parts:

$$\int_\Omega \mathcal{L}_{\alpha_{\delta_n}}[u_n](\mathbf{x})\psi(\mathbf{x})d\mathbf{x} = \int_\Omega u_n(\mathbf{x})\mathcal{L}_{\alpha_{\delta_n}}[\psi](\mathbf{x})d\mathbf{x}$$

$$= \left(\int_{\Omega_{\delta_n}} + \int_{\Omega\backslash\Omega_{\delta_n}}\right) u_n(\mathbf{x})\mathcal{L}_{\alpha_{\delta_n}}[\psi](\mathbf{x})d\mathbf{x}$$

The second of the two integrals vanishes since $u_n(\mathbf{x}) = 0$ on the collar $\Omega \setminus \Omega_\delta$. Whereas $\chi_{\Omega_{\delta_n}}\mathcal{L}_{\alpha_{\delta_n}}[\psi]$ converges strongly to $\Delta\psi$ by Theorem 3.5.4. Along with the strong convergence $u_n \to u$ in $L^2(\Omega)$ this argument gives

$$\lim_{n\to\infty} \int_\Omega \mathcal{L}_{\alpha_{\delta_n}}[u_n]\psi(\mathbf{x})d\mathbf{x} = \int_\Omega u\Delta\psi d\mathbf{x}.$$

In combination with (3.7.13) we conclude

$$\int_\Omega \Delta u\psi d\mathbf{x} = \int_\Omega u\Delta\psi d\mathbf{x}.$$

Since $u = 0$ on the boundary, this identity for every $\psi \in W^{6,2}(\Omega)$ implies

$$\frac{\partial u}{\partial \nu} = 0$$

because the traces of ψ are dense in $L^2(\Gamma)$. Thus u is a weak solution to the elliptic problem (3.7.9). The additional $W^{4,2}(\Omega)$ regularity follows from the same remarks as at the end at

the end of the proof of Theorem 3.7.3. □

Chapter 4

Appendix

4.1 Oscillations of w when $k \leq 0$

A critical component of the blowup result from Chapter 2 was knowledge that global so-lutions of (2.1.2) could never eventually be of one sign. The following lemma supporting this for the case $k \leq 0$ appeared without proof in Chapter 2. Below we provide the proof.

Lemma 2.6.6 *Assume f satisfies conditions* (2.2.1) *and* (2.2.2), $k \leq 0$, *and w is a global solution of* (2.1.2) *with initial conditions satisfying* (2.5.4). *Then w is never eventually of one sign.*

Proof. We will assume that w is eventually nonnegative. A similar proof will work in the case when w is eventually nonpositive. Suppose there exists a $T \geq 0$ such that $w(t) \geq 0$ for $t \geq T$. We will show that w blows up in finite time. Let T be sufficiently large so the consequences of Lemma 2.6.5 hold.

Step 1. We will show that $w' \geq 0$ for $t \geq T$. Suppose $w' < 0$ for some $t \geq T$. Lemma

2.6.5 implies $w' < 0$ as long as $w \geq 0$; thus $w' < 0$ for all $t \geq T$. Recall

$$
\begin{aligned}
G' &= 2\alpha_1 w w' + (2\alpha_2 - \alpha_3) w' w'' - \alpha_3 w w'' \\
E &= \frac{k}{2}(w')^2 + w w''' + F(w) - \frac{\alpha_3}{2}(w'')^2.
\end{aligned}
$$

From Lemma 2.5.2 we know $E(t)$ is constant. From Lemma 2.5.4 we know $G' \to \infty$ (monotonically for t large) as $t \to \infty$. We conclude

$$
G' + \alpha_3 E = 2\alpha_1 w w' + (2\alpha_2 - \alpha_3) w' w'' + \frac{k\alpha_3}{2}(w')^2 + \alpha_3 F(w) - \frac{\alpha_3}{2}(w'')^2
$$

$$
\to \infty \quad \text{(monotonically for } t \text{ large) as } t \to \infty.
$$

Since w is bounded, we know $F(w)$ is bounded. Thus

$$
2\alpha_1 w w' + (2\alpha_2 - \alpha_3) w' w'' + \frac{k\alpha_3}{2}(w')^2 - \frac{1}{2}(w'')^2 \to \infty \quad \text{as} \quad t \to \infty.
$$

Recall that $w \geq 0$, $w' < 0$, $k \leq 0$, consequently, dropping non-positive terms gives

$$
(2\alpha_2 - \alpha_3) w' w'' \to \infty.
$$

Since $\alpha_2 > \alpha_3$ by (3.2.11) and $w' < 0$, we must have w' or $w'' \to -\infty$ contradicting $w \geq 0$.

We conclude

$$
w, w' \geq 0 \quad \text{for} \quad t \in (T, \infty). \tag{4.1.1}
$$

Step 2. We claim that $w \to \infty$**.** Since w is monotone increasing ($w' \geq 0$), it suffices to prove that w is unbounded. To the contrary, assume it is bounded. From (2.1.2) and (2.2.2)

$$
\frac{d}{dt}(w''' + k w') = w'''' + k w'' = -f(w) \leq -\rho w^p
$$

it follows that for t large $w''' + kw'$ is strictly decreasing. Thus we may assume that T is, in addition, sufficiently large so that for some constant C

$$w''' < C - kw', \quad t \geq T. \tag{4.1.2}$$

Since $w \geq 0$ we have from the definition of the energy

$$E \leq \frac{k}{2}(w')^2 + w(C - kw') + F(w) - \frac{1}{2}(w'')^2.$$

By Lemma 2.5.4 we have $\lim_{t\to\infty}(G + E) = \infty$ so,

$$\alpha_1 w^2 + \alpha_2(w')^2 - \alpha_3 ww'' + \frac{k}{2}(w')^2 + w(C - kw') + F(w) - \frac{1}{2}(w'')^2$$

$$\geq G + E \to \infty \quad \text{as} \quad t \to \infty.$$

By our assumption that w is bounded, we have $\alpha_1 w^2$, Cw, $F(w)$, and $-\alpha_3 ww'' - \frac{1}{2}(w'')^2$ are all bounded above. Dropping these terms and the non-positive quantity $\frac{k}{2}(w')^2$ (note that $k \leq 0$) we arrive at

$$\lim_{t\to\infty} \left(\alpha_2(w')^2 + |k|ww' \right) = \infty.$$

Since $w \geq 0$ is bounded and $w' \geq 0$ we must have $\lim_{t\to\infty} w' = \infty$ contradicting w being bounded. Thus

$$w \to \infty \text{ monotonically on } (T, \infty). \tag{4.1.3}$$

Step 3. Positivity of w''. It follows from (2.2.2) and (4.1.3) that

$$-f(w) \to -\infty \quad \text{as} \quad t \to \infty.$$

Now let us look at the behavior of $w''(t)$. We will show that $w''(t) \geq 0$ for all t sufficiently large. To the contrary, suppose $w''(t) \leq 0$ for all $t \in (T_1, \infty)$ for some $T_1 \geq T$. From equation (2.1.2) and the assumption $k \leq 0$ we get

$$w'''' = -kw'' - f(w) \to -\infty \quad \text{as} \quad t \to \infty,$$

which implies that $w(t) \to -\infty$, contradicting our initial assertion that $w \geq 0$.

We conclude that w'' must be positive somewhere on (T, ∞). Assume b_1 is the next point where w'' changes sign, i.e. w'' is positive on a left neighborhood of b_1, negative on a right neighborhood and $w'''(b_1) \leq 0$. Let (b_1, b_2) be the subsequent maximal right-interval of negativity for w''. From equation (2.1.2) we obtain

$$w'''' = -kw'' - f(w) \leq -f(w) \quad \text{on} \quad (b_1, b_2).$$

Because we may assume that w is positive (in fact increasing), we conclude w'' is negative and strictly concave on (b_1, b_2), whence $b_2 = \infty$. But then we contradict the earlier observation that w'' can not remain negative on an interval of the form (T_1, ∞).

Thus, for T sufficiently large,

$$w, w', w'' \geq 0, \quad t \in (T, \infty). \tag{4.1.4}$$

Step 4. Finite-time blow-up. From equation (2.1.2) we have

$$\frac{d^2}{dt^2}(kw + w'') = kw'' + w'''' = -f(w)$$

$$\leq -\rho w^p \quad \text{(condition (2.2.2) for } |w| \text{ large).} \tag{4.1.5}$$

Recall from (4.1.3) that $w \to \infty$ monotonically. Combining this with (4.1.5) allows us to

find a $C > 0$ such that for sufficiently large t we have

$$\frac{d^2}{dt^2}(kw(t) + w''(t)) \leq -C < 0.$$

So for all large t, $kw + w'' < 0$. In other words, we may assume that T is large enough so that $-kw = |k|w > w''$ on (T, ∞).

To summarize: on (T, ∞) we have $w, w', w'' \geq 0$ and $|k|w > w''$. Consequently

$$G' = 2\alpha_1 ww' + 2\alpha_2 w'w'' - \alpha_3(w'w'' + ww''')$$

$$\leq (\alpha_1 + \alpha_2|k|)2w'w - \alpha_3(w'w'' + ww''').$$

For $t \geq T$ we obtain:

$$G(t) - G(T) = \int_T^t G' ds$$

$$\leq (\alpha_1 + \alpha_2|k|)w(t)^2 - \alpha_3 w(t)w''(t) + C_T.$$

(4.1.6)

Pick $\lambda > 0$ such that $\frac{(p+1)(1+\lambda)}{p-\lambda} < 2$ (equivalent to $\lambda < \frac{p-1}{p+3}$) and $2(1+\lambda) < p+1$. Since $p > 1$, such a λ exists. With $\lesssim$ below indicating omitted positive constant factors (independent of t or the solution) we have for $t \geq T$,

$$G^{1+\lambda} \leq \left((\alpha_1 + \alpha_2|k|)w^2 - \alpha_3 ww'' + C_T + G(T)\right)^{1+\lambda} \quad \text{(by (4.1.6))}$$

$$\lesssim \left(|w^2| + |ww''| + 1\right)^{1+\lambda}$$

$$\lesssim |w|^{2(1+\lambda)} + |ww''|^{1+\lambda} + 1$$

$$\lesssim |w|^{2(1+\lambda)} + |w|^{p+1} + |w''|^{\frac{(p+1)(1+\lambda)}{p-\lambda}} + 1 \quad \text{(by Young's inequality)}$$

$$\lesssim |w|^{p+1} + |w''|^2 + 1 \quad \left(\text{recall } p + 1 \geq 2(1+\lambda) \text{ and } \frac{(p+1)(1+\lambda)}{p-\lambda} < 2\right)$$

$$\lesssim wf(w) + |w''|^2 + 1 \quad \text{(by condition (2.2.2))}$$

$$\lesssim G'' \quad \text{(Lemma 2.5.4)}.$$

From Lemma 2.5.4 we know $G'' \geq \varepsilon > 0$, so G is positive and strictly increasing for t sufficiently large. This along with the fact $G^{1+\lambda} \lesssim G''$ for all $t \geq T$ implies G blows up in finite time. This in turn implies w blows up in finite time. $\qquad \square$

Bibliography

[1] R. Adams and J. Fournier. *Sobolev Spaces*. Academic Press, 2003.

[2] O. H. Ammann, T. von Kármán, and G. B. Woodruff. The failure of the Tacoma Narrows Bridge. *Federal Works Agency*, 1941.

[3] F. Andreu-Vaillo, J. M. Mazón, J. D. Rossi, and J. J. Toledo-Melero. *Nonlocal diffusion problems*, volume 165 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI; Real Sociedad Matemática Española, Madrid, 2010.

[4] Arup Coroporation. London millennium bridge opening, 2007. [Online; accessed 21-July-2014].

[5] E. Berchio, A. Ferrero, F. Gazzola, and P. Karageorgis. Qualitative behavior of global solutions to some nonlinear fourth order differential equations. *Journal of Differential Equations*, 251(10):2696 – 2727, 2011.

[6] F. Bobaru, Y. D. Ha, and W. Hu. Damage progression from impact in layered glass modeled with peridynamics. *Open Engineering*, pages 551–561, 2012.

[7] J. Bourgain, H. Brezis, and P. Mironescu. Another look at sobolev spaces. In *Optimal Control and Partial Differential Equations*, pages 439–455, 2001.

[8] H. Brezis. *Analyse fonctionnelle*. Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree]. Masson, Paris, 1983. Théorie et applications. [Theory and applications].

[9] C. Carrillo and P. Fife. Spatial effects in discrete generation population models. *Journal of Mathematical Biology*, 50(2):161–188, 2005.

[10] Y. Chen and P. J. McKenna. Traveling waves in a nonlinearly suspended beam: theoretical results and numerical observations. *J. Differential Equations*, 136(2):325–355, 1997.

[11] P. Drábek, G. Holubová, A. Matas, and P. Nečesal. Nonlinear models of suspension bridges: discussion of the results. *Appl. Math.*, 48(6):497–514, 2003. Mathematical and computer modeling in science and engineering.

[12] Q. Du, M. Gunzburger, R. B. Lehoucq, and K. Zhou. A nonlocal vector calculus, nonlocal volume-constrained problems, and nonlocal balance laws. *Mathematical Models and Methods in Applied Sciences*, 23(03):493–540, 2013.

[13] B. Elliott. Tacoma narrows bridge collapse (1940) (sound version). [Online; accessed 28-March-2014].

[14] F. Gazzola. Nonlinearity in oscillating bridges. *Electron. J. Differential Equations*, pages No. 211, 47, 2013.

[15] F. Gazzola and R. Pavani. Blow up oscillating solutions to some nonlinear fourth order differential equations. *Nonlinear Anal.*, 74(17):6696–6711, 2011.

[16] F. Gazzola and R. Pavani. Blow-up oscillating solutions to some nonlinear fourth order differential equations describing oscillations of suspension bridges. In F. Biondini

and D. Frangopol, editors, *Bridge Maintenance, Safety, Management, Resilience and Sustainability*, 2012.

[17] F. Gazzola and R. Pavani. Wide oscillation finite time blow up for solutions to nonlinear fourth order differential equations. *Archive for Rational Mechanics and Analysis*, 207(2):717–752, 2013.

[18] F. Gazzola and R. Pavani. The impact of nonlinear restoring forces acting on hinged elastic beams (preprint). 2014.

[19] G. Gilboa and S. Osher. Nonlocal operators with applications to image processing. *Multiscale Modeling & Simulation*, 7(3):1005–1028, 2009.

[20] Y. D. Ha and F. Bobaru. Characteristics of dynamic brittle fracture captured with peridynamics. *Engineering Fracture Mechanics*, 78(6):1156 – 1168, 2011.

[21] P. Hajłasz. Change of variables formula under minimal assumptions. *Colloq. Math.*, 64(1):93–101, 1993.

[22] J. Han. The Cauchy problem and steady state solutions for a nonlocal Cahn-Hilliard equation. *Electron. J. Differential Equations*, pages No. 113, 9 pp. (electronic), 2004.

[23] L. I. Ignat, T. I. Ignat, and D. Stancu-Dumitru. A compactness tool for the analysis of nonlocal evolution equations. *SIAM J. Math. Anal.*, 47(2):1330–1354, 2015.

[24] M. Inc. New solitary wave solutions with compact support and Jacobi elliptic function solutions for the nonlinearly dispersive Boussinesq equations. *Chaos Solitons Fractals*, 37(3):792–798, 2008.

[25] I. Lasiecka and R. Triggiani. *Control theory for partial differential equations: continuous and approximation theories. I*, volume 74 of *Encyclopedia of Mathematics and*

its Applications. Cambridge University Press, Cambridge, 2000. Abstract parabolic systems.

[26] A. Lazer and P. McKenna. Large-amplitude periodic oscillations in suspension bridges: Some new connections with nonlinear analysis. *SIAM Review*, 32(4):537–578, 1990.

[27] A. C. Lazer and P. J. McKenna. Large scale oscillatory behaviour in loaded asymmetric systems. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 4(3):243–274, 1987.

[28] P. J. McKenna and W. Walter. Travelling waves in a suspension bridge. *SIAM J. Appl. Math.*, 50(3):703–715, 1990.

[29] T. Mengesha and Q. Du. The bond-based peridynamic system with Dirichlet-type volume constraint. *Proc. Roy. Soc. Edinburgh Sect. A*, 144(1):161–186, 2014.

[30] A. Mogilner and L. Edelstein-Keshet. A non-local model for a swarm. *Journal of Mathematical Biology*, 38(6):534–570, 1999.

[31] J. O'Grady and J. Foster. Peridynamic beams: A non-ordinary, state-based model. *International Journal of Solids and Structures*, 51(18):3177 – 3183, 2014.

[32] J. O'Grady and J. Foster. Peridynamic plates and flat shells: A non-ordinary, state-based model. *International Journal of Solids and Structures*, 51(25–26):4572 – 4579, 2014.

[33] L. A. Peletier and W. C. Troy. *Spatial patterns*. Progress in Nonlinear Differential Equations and their Applications, 45. Birkhäuser Boston Inc., Boston, MA, 2001. Higher order models in physics and mechanics.

[34] A. C. Ponce. An estimate in the spirit of Poincaré's inequality. *J. Eur. Math. Soc. (JEMS)*, 6(1):1–15, 2004.

[35] P. Radu and M. Foss. Sobolev regularity for solutions to nonlocal systems with weakly singular kernels. Preprint., 2014.

[36] A. P. S. Selvadurai. *Partial differential equations in mechanics. 2*. Springer-Verlag, Berlin, 2000. The biharmonic equation, Poisson's equation.

[37] S. Silling. Reformulation of elasticity theory for discontinuities and long-range forces. *Journal of the Mechanics and Physics of Solids*, 48:175–209, 2000.

[38] P.-F. Yao. *Modeling and control in vibrational and structural dynamics*. Chapman & Hall/CRC Applied Mathematics and Nonlinear Science Series. CRC Press, Boca Raton, FL, 2011. A differential geometric approach.