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Stable local cohomology and cosupport

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STABLE LOCAL COHOMOLOGY AND COSUPPORT

by

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STABLE LOCAL COHOMOLOGY AND COSUPPORT

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This dissertation consists of two parts, both under the overarching theme of resolutions over a commutative Noetherian ring R. In particular, we use complete resolutions to study stable local cohomology and cotorsion-flat resolutions to investigate cosupport.

In Part I, we use complete (injective) resolutions to define a stable version of local cohomology. For a module having a complete injective resolution, we associate a stable local cohomology module; this gives a functor to the stable category of Gorenstein injective modules. We show that this functor behaves much like the usual local cohomology functor. When there is only one non-zero local cohomology module, we show there is a strong connection between that module and the stable local cohomology module; in fact, the latter gives a Gorenstein injective approximation of the former.

In Part II, we utilize minimal cotorsion-flat resolutions (both on the left and right) to compute cosupport. We first develop a criterion for a cotorsion-flat resolution to be minimal. For a module having an appropriately minimal resolution by cotorsion-flat modules, we show that its cosupport coincides with those primes "appearing" in such a resolution—much like the dual notion that minimal injective resolutions detect (small) support. This gives us a method to compute the cosupport of various modules, including all flat modules and all cotorsion modules. Moreover, if R is either a 1-dimensional domain that is not a complete local ring or any ring of the form $k[x, y]$

for an uncountable field k, we show that the cosupport of R is all of $\mathrm{Spec}(R)$, and consequently that the cosupport of a finitely generated module over such a ring is the same as its support.

DEDICATION

Lovingly dedicated to my wife, Hannah.

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Introduction

A central theme in the following dissertation is that of resolutions of modules over commutative Noetherian rings. Modules—the building blocks of commutative algebra—admit "presentations," a description of the module in terms of generators and relations among the generators. Hilbert, in the 1890s, had the brilliant idea of extending the notion of a presentation of a module to a *free resolution*. Roughly, a free resolution gives not just generators and relations for a module, but also relations among the relations, relations among these higher relations, and so on. In this way, a free resolution encodes a significant amount of information about a module. Continuing into the middle of the twentieth century, this formalism began to be applied to other algebraic structures as well, and along with it came the introduction of other types of resolutions. Recently, more exotic resolutions have become prevalent as a tool in commutative and homological algebra, and their full potential is still being realized.

If a *resolution* is an extended description of the relations in a module, then a *complete resolution* encodes the "stable" data in this description. A resolution of a module can still be hard to fully grasp (keep in mind, a resolution is often given by an infinite amount of data), and one option is to focus only on the stable properties of the resolution—this leads to the idea of a complete resolution. Part I of this document is devoted to developing a stable analogue of a classical invariant; as a result, we are better able to understand *local cohomology* modules—an invariant of modules that has been well-studied since Grothendieck's introduction of them in the 1960s.

Resolutions allow us to replace complicated modules by "nice" modules. Free modules have the "nicest" structure, but many other modules have well-understood structures as well, leading to other useful resolutions. One such class of modules is that of *cotorsion-flat* modules—these can be decomposed into components corresponding to the prime ideals of the ring. In Part II, we use *cotorsion-flat resolutions* to compute an invariant known as the *cosupport*, an invariant useful in understanding the stratification of derived categories and other algebraic systems.

Apart from the intrinsic interest of complete resolutions and cotorsion-flat resolutions, recent work has shown them to be useful tools. Complete resolutions have been used by Iyengar and Krause [IK06] to better understand an equivalence between the homotopy and derived categories of projective and injective modules, and Neeman [Nee08] and Murfet and Salarian [MS11] utilize this equivalence to extend a description of the homotopy category of projective modules to non-affine schemes. More recently, cotorsion-flat resolutions are used by Marley and Webb [MW16] over rings of prime characteristic to extend a result of Peskine and Szpiro to not-necessarily finitely generated modules. One goal of this work has been to explore other applications of these types of resolutions—both complete resolutions and cotorsion-flat resolutions—to local cohomology as well as to cosupport.

We take the study of resolutions of modules in two directions: Studying complete resolutions in order to develop *stable local cohomology* (a version of this work will appear in *Communications in Algebra*, see [Tho15]), and analyzing cotorsion-flat resolutions in order to compute the cosupport of modules over these rings. We now outline both of these motifs below, giving some motivation and historical context for each of these directions, as well as sketching out some of the main results.

Stable local cohomology

We first develop a stable version of local cohomology in Part I and investigate its connection to classical local cohomology. Local cohomology, introduced by Grothendieck in the early 1960s, has been extensively studied over the past 60 years. It has proven to be an incredibly useful tool in many areas, used in proving in various connectedness results in algebraic geometry to answering the question of how many generators an ideal has up to radical. As local cohomology modules are often not finitely generated, and are therefore quite "large" modules, one focus of this area of research has been to show what finiteness properties local cohomology modules do have. Substantial progress on this was made by Huneke and Sharp (in characteristic p) and Lyubeznik (in characteristic 0), but open questions remain in more general situations.

Given a module over a Gorenstein local ring, a high syzygy in a projective resolution is maximal Cohen-Macaulay. This leads to the construction of the stable derived category, which has been studied by Buchweitz [Buc86], Orlov [Orl04], Krause [Kra05], Avramov and Iyengar [AI13], Stevenson [Ste14], and others. On the other hand, taking high degree cosyzygies in an injective resolution over such a ring results in Gorenstein injective modules, giving an injective counterpart to the stable derived category.

One of our motivating questions was what the correct notion of local cohomology might be in the stable derived category. Building on ideas developed by Stevenson [Ste14], we propose a definition of stable local cohomology below. We prove a number of results showing that stable local cohomology behaves as one might expect (analogous to classical local cohomology). Also, in the case of only one non-vanishing local cohomology module, we are able to give a strong connection between stable and classical local cohomology.

Classically, local cohomology supported at an ideal $\mathfrak{a}$ of R is defined by taking an injective resolution, applying the $\mathfrak{a}$-torsion functor $\Gamma_{\mathfrak{a}}(-)$, and taking cohomology. The corresponding stable version of an injective resolution is a *complete injective resolution*. When R is Gorenstein, for any R-module M, there exists a complete injective resolution $M \to I \to U$, where U is an exact complex of injective modules (usually unbounded). However, $\Gamma_{\mathfrak{a}}(U)$ is an exact complex (due to Lipman [Lip02]), so the last natural step of taking cohomology would provide a degenerate definition. Rather, we consider syzygies of this complex. Since the syzygies are all the same up to translation in the stable category, we do not obtain a number of modules as in the classical sense; instead, we fix a particular syzygy, and have a single *stable local cohomology module*, $\Gamma_{\mathfrak{a}}^{\mathrm{stab}}(M) := Z^0\Gamma_{\mathfrak{a}}(U)$, for each R-module M and $\mathfrak{a} \subset R$. This module is an example of a *Gorenstein injective* module—a module that appears as a syzygy of an exact complex of injective R-modules (when R is Gorenstein). For a precise definition, see Definition 3.0.4.

In the case where only one local cohomology module is nonzero, i.e., $H_{\mathfrak{a}}^i(M) = 0$ for all $i \neq c$, we obtain a strong connection between stable and classical local cohomology. Our main result in this direction is the following theorem:

Theorem 1 (cf. Theorem 5.0.1)**.** *Let* $(R, \mathfrak{m})$ *be a Gorenstein local ring. Suppose* $M \neq 0$ *is an* R*-module where* $\operatorname{Gid}_R M = \operatorname{depth} M$ *and* $\mathfrak{a} \subseteq R$ *is an ideal satisfying* $c = \operatorname{depth}(\mathfrak{a}, M) = \operatorname{cd}(\mathfrak{a}, M)$*. Then* $\Gamma_{\mathfrak{a}}^{\mathrm{stab}}(\Omega^c M)$ *provides a Gorenstein injective approximation of* $H_{\mathfrak{a}}^c(M)$*.*

Here, $\operatorname{Gid}_R(M)$ is the Gorenstein injective dimension of M and $\Omega^c M$ is a cosyzygy in a minimal injective resolution of M. A *Gorenstein injective approximation* is the dual notion to a MCM approximation. The conclusion of the theorem, in particular, shows that in the stable category of Gorenstein injective modules, there is an isomorphism:

$$H^c_{\mathfrak{a}}(M) \simeq \Gamma^{\text{stab}}_{\mathfrak{a}}(\Omega^c M).$$

Computing cosupport

In Part II, we apply tools developed by Enochs [Eno84, Eno87, Eno89] and Xu [Xu96] to tackle the question of computing the cosupport of a module (or complex) in a commutative Noetherian ring. Benson, Iyengar, and Krause [BIK12] develop the notion of cosupport for an object in a triangulated category in order to classify the colocalizing subcategories of, for instance, the stable module category of a finite group. However, for modules over a commutative Noetherian ring, "cosupport seems hard to compute," even for the ring itself [BIK12]. We show that if a module over a commutative Noetherian ring has a minimal resolution by cotorsion-flat modules, then this resolution can detect the cosupport.

In 1984, Enochs showed that a flat module B also satisfying the property that $\operatorname{Ext}^1_R(F, B) = 0$, for every flat module F (i.e., B is also cotorsion), can be uniquely decomposed by the primes of R; in particular, $B \cong \prod_{\mathfrak{p} \in \operatorname{Spec}(R)} \widehat{R_{\mathfrak{p}}^{(X_{\mathfrak{p}})}}$, where $X_{\mathfrak{p}}$ is a (possibly infinite or empty) index set for each prime $\mathfrak{p}$. We refer to these modules as *cotorsion-flat* modules.

Our first goal in Part II is to develop a criterion for cotorsion-flat resolutions to be minimal. We show the following (see Theorem 8.3.4 for a precise statement):

Theorem 2 (cf. Theorem 8.3.4)**.** *Let R be a commutative Noetherian ring of finite Krull dimension. If B is a left cotorsion-flat resolution of a cotorsion module (or a right cotorsion-flat resolution of a flat module), then the following are equivalent:*

1. *For every $\mathfrak{p} \in \operatorname{Spec}(R)$, the complex $\operatorname{Hom}_R(R_{\mathfrak{p}}, B) \otimes_R R/\mathfrak{p}$ has zero differential;*

2. *The complex B is built (minimally) from flat covers (or respectively, from cotorsion envelopes);*

3. *B is* minimal*, in the sense that every self homotopy equivalence is an isomorphism.*

Our main result with regards to cosupport is that the cosupport of a module having a minimal cotorsion-flat resolution is the set of primes "appearing" in such a resolution, i.e., those primes $\mathfrak{p}$ for which $\widehat{R_\mathfrak{p}^{(X_\mathfrak{p})}}$ appears in the minimal cotorsion-flat resolution with $X_\mathfrak{p} \neq \varnothing$.

Theorem 3 (cf. Theorem 9.2.2)**.** *Let R be a commutative Noetherian ring of finite Krull dimension and M an R-module having a minimal cotorsion-flat resolution B. Then $\mathfrak{p}$ is in the cosupport of M if and only if $\mathfrak{p}$ appears in B.*

In particular, if M is either a cotorsion module or a flat module, then it has a readily accessible minimal cotorsion-flat resolution (see Chapter 8). With this, we compute the cosupport of some low dimensional rings.

Proposition 4 (cf. Propositions 9.5.4 and 9.5.2)**.** *Assume that R is either a 1-dimensional domain that is not complete local, or that $R = k[x, y]$ for any uncountable field k. Then $\operatorname{cosupp}_R R = \operatorname{Spec}(R)$.*

It would be interesting to find a larger class of rings for which the cosupport of the ring itself is all of $\operatorname{Spec}(R)$, or more generally, understand what property of the ring forces the cosupport to be all of $\operatorname{Spec}(R)$. A conjecture of Enochs [Eno89] would yield a class of (regular) rings of any finite Krull dimension with this property. More generally, it would be interesting to find a larger class of rings where $\operatorname{cosupp}_R(R)$ is closed. Apart from simple cases such as complete semi-local rings, to my knowledge this is the largest class of rings known to have closed cosupport.

In fact, the cosupport of a finitely generated module only depends on the cosupport of the ring and the support of the module (see Proposition 9.3.2). In particular, when the cosupport of R is closed, the cosupport of every finitely generated module over R is closed as well. Hence the following corollary generalizes the case of $R = \mathbb{Z}$ in [BIK12, Proposition 4.18].

Corollary 5 (see Corollary 9.5.5)**.** *Let R be as in Proposition 4 and M a complex of R-modules with H^*M finitely generated. Then* $\operatorname{cosupp}_R(M) = \operatorname{supp}_R(M)$.

Background

Throughout this work, unless otherwise stated, R is assumed to be a commutative and Noetherian ring with an identity. For some of the basic tools and notation we will use, refer to Chapter 1 and Chapter 7. Additionally, useful texts that will be referenced throughout containing much of the background material required, on commutative and (Gorenstein) homological algebra, include [AM69, BH98, Chr00, EJ00, ILL$^+$07, Mat89, Wei94, Xu96].

Part I

Stable local cohomology

Introduction to Part I

Let R be a Gorenstein local ring with Krull dimension d, $\mathfrak{a}$ an ideal in R, and M an R-module. Local cohomology of M supported at $\mathfrak{a}$ is computed by considering the $\mathfrak{a}$-torsion functor $\Gamma_{\mathfrak{a}}$ applied to an injective resolution of M. In a Gorenstein ring, every module has a complete injective resolution, so it is natural to ask what one obtains by applying $\Gamma_{\mathfrak{a}}$ to the complete injective resolution as opposed to the usual injective resolution. Applying $\Gamma_{\mathfrak{a}}$ to a complete injective resolution yields an acyclic complex, so taking cohomology yields nothing of interest. Instead, given an R-module M with a complete injective resolution U, we define a single module $\Gamma_{\mathfrak{a}}^{\text{stab}}(M)$ as the zeroeth syzygy of $\Gamma_{\mathfrak{a}}(U)$. In a Gorenstein ring, $\Gamma_{\mathfrak{a}}^{\text{stab}}(-) : \operatorname{Mod} R \to \underline{\text{GInj}}(R)$ defines a functor, where $\underline{\text{GInj}}(R)$ is the stable category of Gorenstein injective R-modules.

As a motivating example, we turn to maximal Cohen Macaulay (or MCM) modules over a hypersurface; recall that MCM modules correspond to matrix factorizations [Eis80]. For a local Gorenstein ring R, we have an induced triangulated functor $\Gamma_{\mathfrak{a}}^{\text{stab}}(-) : \underline{\text{MCM}}(R) \to \underline{\text{GInj}}(R)$, where $\underline{\text{MCM}}(R)$ is the stable category of MCM R-modules (see [Buc86]). Let $(S, \mathfrak{m})$ be a regular local ring, f a non-zerodivisor, $Q = S/(f)$, and $\mathfrak{m}$ the maximal ideal of Q. Then $\Gamma_{\mathfrak{a}}^{\text{stab}}(-) : \underline{\text{MCM}}(Q) \to \underline{\text{GInj}}(Q)$ induces a map $- \otimes_S \Gamma_{\mathfrak{a}}(D) : [\text{mf}(S, f)] \to [\text{IF}(S, f)]$, where D is a minimal injective resolution of S and $[\text{mf}(S, f)]$ and $[\text{IF}(S, f)]$ are the homotopy categories of finitely generated matrix factorizations and injective factorizations, respectively. For a MCM

Q-module M, there exists a corresponding matrix factorization $(S^r \underset{B}{\overset{A}{\rightleftarrows}} S^r)$, where $\operatorname{coker}(A) = M$. Then $\Gamma_{\mathfrak{a}}^{\mathrm{stab}}(M)$ can be computed by considering $(S^r \underset{B}{\overset{A}{\rightleftarrows}} S^r) \otimes_S \Gamma_{\mathfrak{a}}(D)$. When $\mathfrak{a} = \mathfrak{m}$, this is just $(E^r \underset{B}{\overset{A}{\rightleftarrows}} E^r)$, where E is the injective hull of $S/\mathfrak{m}$, and thus $\Gamma_{\mathfrak{m}}^{\mathrm{stab}}(M)$ is isomorphic to either $\ker(A : E^r \to E^r)$ or $\ker(B : E^r \to E^r)$ (depending on the parity of $\dim S$) in the stable category $\underline{\mathrm{GInj}}(Q)$ (i.e., isomorphic up to direct sums of injective modules). We describe this situation more generally in Proposition 4.0.6.

More generally for any Gorenstein ring R, we obtain a nice description of stable local cohomology at the maximal ideal. Classically, $H_{\mathfrak{m}}^d(M) \cong M \otimes_R E_R(R/\mathfrak{m})$ [ILL+07, Exercise 9.7]. If we let $\Omega_d^{\mathrm{cpr}} M$ be the d-th shift of M in $\underline{\mathrm{MCM}}(R)$, we can give a similar result stably (Proposition 3.1.1):

Proposition A. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring of Krull dimension d and M be a MCM R-module with no nonzero free summands. Then $\Gamma_{\mathfrak{m}}^{\mathrm{stab}}(M) \simeq \Omega_d^{\mathrm{cpr}} M \otimes E(R/\mathfrak{m})$, where $\simeq$ represents isomorphism in $\underline{\mathrm{GInj}}(R)$.*

Perhaps the next case of interest is a height $d-1$ prime ideal $\mathfrak{q}$ of R. In Proposition 3.2.4, we relate $\Gamma_{\mathfrak{m}}^{\mathrm{stab}}(M)$ and $\Gamma_{\mathfrak{q}}^{\mathrm{stab}}(M)$ in an exact triangle in $\underline{\mathrm{GInj}}(R)$:

$$\Gamma_{\mathfrak{m}}^{\mathrm{stab}}(M) \to \Gamma_{\mathfrak{q}}^{\mathrm{stab}}(M) \to \Gamma_{\mathfrak{q}}^{\mathrm{stab}}(M_{\mathfrak{q}}) \to .$$

Furthermore, we have (Proposition 3.3.2):

Proposition B. *Let R be a Gorenstein ring of dimension d, M any R-module, $\mathfrak{a}$ any ideal of R, and $x \in R$ any element. Set $\mathfrak{b} = (\mathfrak{a}, x)$. Then there exists a short exact sequence of R-modules*

$$0 \to \Gamma_{\mathfrak{b}}^{\mathrm{stab}}(M) \to \Gamma_{\mathfrak{a}}^{\mathrm{stab}}(M) \to \Gamma_{\mathfrak{a}}^{\mathrm{stab}}(M_x) \to 0.$$

If M is a MCM R-module, recall that $\operatorname{depth}(\mathfrak{a})$ and $\operatorname{cd}(\mathfrak{a})$ are the integers representing the first and last, respectively, degrees at which $H^i_{\mathfrak{a}}(M)$ is non-vanishing. In the case where $\operatorname{depth}(\mathfrak{a}) = \operatorname{cd}(\mathfrak{a})$, i.e., $H^i_{\mathfrak{a}}(M) = 0$ for all $i \neq \operatorname{depth}(\mathfrak{a})$, we are able to relate the stable local cohomology module and the one nonzero local cohomology module (see Theorem 5.0.1 for a more general statement). One instance where $\operatorname{depth}(\mathfrak{a}) = \operatorname{cd}(\mathfrak{a})$ is when $\mathfrak{a}$ is generated (up to radical) by a regular sequence.

Theorem C. *Let R be a Gorenstein local ring of Krull dimension d. Suppose $M \neq 0$ is a MCM R-module, such that $\mathfrak{a} \subset R$ is an ideal satisfying $c = \operatorname{depth}(\mathfrak{a}) = \operatorname{cd}(\mathfrak{a})$. Then there exists a short exact sequence*

$$0 \to H^c_{\mathfrak{a}}(M) \to \Gamma^{\text{stab}}_{\mathfrak{a}}(\Omega^c_{\text{inj}} M) \oplus E_R(H^c_{\mathfrak{a}}(M)) \to K \to 0,$$

where $\operatorname{id}_R K < \infty$. Moreover, when $0 \leq c \leq t-1$, we have $\operatorname{id}_R K = t - c - 1$ and when $c = t$, the sequence splits and $K \cong E_R(\Gamma^{\text{stab}}_{\mathfrak{a}}(\Omega^t_{\text{inj}} M))$.

Here $\Omega^c_{\text{inj}} M$ represents the c-th cosyzygy of M, i.e., if $M \to I$ is an injective resolution, then $\Omega^c_{\text{inj}} M = \ker(I^c \to I^{c+1})$.

In fact, the short exact sequence of Theorem C gives a Gorenstein injective approximation of $H^c_{\mathfrak{a}}(M)$, see Corollary 5.0.10. In particular, we have an isomorphism $H^c_{\mathfrak{a}}(M) \simeq \Gamma^{\text{stab}}_{\mathfrak{a}}(\Omega^c_{\text{inj}} M)$ in the stable category $\underline{\text{GInj}}(R)$.

We now give a brief outline of Part I. In Chapter 1, we set notation and review some basics of injective modules and Gorenstein homological algebra.

In Chapter 2, we explore alternative ways of constructing "stable" resolutions; we develop some of the constructions, based on much of the projective analogues found in [AM02]. One of the main goals of this section is Proposition 2.2.2, which gives a way to build complete injective resolutions from complete projective resolutions.

We define and build up the notion of stable local cohomology in Chapter 3. This theory builds (in a more concrete fashion) the composition of functors $Z^0\Gamma_{V(I)}I_\lambda Q_\rho$ in [Ste14]. Our definition appears at Definition 3.0.4. We also derive relations between stable local cohomology modules that are analogous to ones found in classical local cohomology theory; in particular, we prove Propositions A and B from above.

We explore the hypersurface case in Chapter 4. In order to compute some explicit stable local modules, we first show, for a regular local ring Q and non-zerodivisor f, there is an equivalence between the homotopy category of (not necessarily finitely generated) matrix factorizations $[\mathrm{MF}(Q,f)]$ and the homotopy category of injective factorizations $[\mathrm{IF}(Q,f)]$, that agrees with the equivalence between $\mathrm{K}_{\mathrm{ac}}(\operatorname{Prj} R)$ and $\mathrm{K}_{\mathrm{ac}}(\operatorname{Inj} R)$ given by [IK06].

In Chapter 5, we show there is a tight connection between stable local cohomology and classical local cohomology, at least in the case where there is only one nonzero local cohomology module. Our main result in this direction is Theorem 5.0.1, which we prove in this section (in particular, this proves Theorem C from above). In fact, the stable local cohomology module will give a Gorenstein injective approximation of $H^i_{\mathfrak{a}}(M)$, see Corollary 5.0.10.

Finally, we present some further directions and questions in Chapter 6.

Chapter 1

Basics

We first introduce notation for the categories we will be considering.

Notation 1.0.1. Let $\operatorname{Mod} R$ be the category of all R-modules and homomorphisms, $\mathrm{C}(\operatorname{Mod} R)$ denote the category of complexes of R-modules, and $\mathrm{K}(\operatorname{Mod} R)$ the associated homotopy category. Here, $\operatorname{Mod} R$ can be replaced with $\operatorname{Prj} R$ or $\operatorname{Inj} R$, representing the subcategories of projective modules or injective modules, respectively. If we only want to consider finitely generated modules, we will use lower case letters, namely $\operatorname{mod} R$ or $\operatorname{prj} R$. We often will want to consider the full subcategories of acyclic complexes, which we will denote by $\mathrm{K}_{\mathrm{ac}}(-)$.

When R is Gorenstein, denote by $\underline{\mathrm{MCM}}(R)$ the category with the same objects as $\mathrm{MCM}(R)$ (the category of finitely generated maximal Cohen-Macaulay R-modules), but with morphisms given by the following: if $M, N \in \underline{\mathrm{MCM}}(R)$, then

$$\operatorname{Hom}_{\underline{\mathrm{MCM}}(R)}(M,N) = \operatorname{Hom}_R(M,N)/\{f : M \to N | f \text{ factors through some } P \in \operatorname{prj} R\}.$$

We call $\underline{\mathrm{MCM}}(R)$ the *stable category of maximal Cohen-Macaulay R-modules*. Recall that in a Gorenstein ring, maximal Cohen-Macaulay (henceforth abbreviated

MCM) modules coincide with finitely generated Gorenstein projective R-modules [EJ00, Corollary 10.2.7].

Likewise, $\underline{\mathrm{GInj}}(R)$ denotes the *stable category of Gorenstein injective R-modules*, where objects are the same as in $\mathrm{GInj}(R)$ (the category of Gorenstein injective modules, whose definition we recall below), and we have factored the Hom sets by those maps that factor through an injective module.

We will use $\simeq$ to denote isomorphism in stable categories (whose context should be clear) or to denote a homotopy equivalence in $\mathrm{C}(\mathrm{Mod}\, R)$, and $\cong$ to denote isomorphism in $\mathrm{Mod}\, R$ (or in $\mathrm{C}(\mathrm{Mod}\, R)$).

1.1 Complexes, homotopies, dualizing complexes, $\Gamma_{\mathfrak{a}}(-)$, and injectives

We call C a *complex (of R-modules)* if C is a $\mathbb{Z}$-graded R-module with a differential ∂ such that $\partial^2 = 0$. We can either display our complexes homologically:

$$C = \cdots \to C_{i+1} \to C_i \to C_{i-1} \to \cdots$$

or cohomologically:

$$C = \cdots \to C^{i-1} \to C^i \to C^{i+1} \to \cdots$$

We say that a complex C is *bounded on the left (resp., right)* if $C_i = 0$ for $i \gg 0$ or $C^i = 0$ for $i \ll 0$ (resp., $C_i = 0$ for $i \ll 0$ or $C^i = 0$ for $i \gg 0$). For two complexes C and D, we define their tensor product $C \otimes_R D$ as the direct sum totalization of the obvious double complex and $\mathrm{Hom}_R(C, D)$ as the direct product totalization of the corresponding double complex (see [Wei94] 2.7.1 and 2.7.4, respectively).

For a complex C of R-modules, we denote by $\Sigma^i C$ the complex with $(\Sigma^i C)^n = C^{n+i}$ and differential $\partial^n_{\Sigma^i C} = (-1)^i \partial^{n+i}_C$. Given a complex C, set $Z^i(C) := \ker(C^i \to C^{i+1})$ and $\Omega_i(C) := \operatorname{coker}(C_{i+1} \to C_i)$.

The *truncation* of a complex C, denoted $C^{\geq i}$, is the complex where $(C^{\geq i})^j = \begin{cases} C^j, & j \geq i \\ 0, & j < i \end{cases}$. Similarly, we may use $C_{\geq i}$, $C^{\leq i}$, or $C_{\leq i}$.

If $f, g : C \to D$ are two chain maps, we use $f \sim g$ to denote the existence of a homotopy from f to g, i.e., there exists a cohomological degree -1 map $h : C \to D$ such that $f - g = \partial_D h + h \partial_C$. A complex C is *contractible* if $\mathrm{id}_C \sim 0_C$. A subcomplex A of C is *irrelevant* if A^i is a summand of C^i for each $i \in \mathbb{Z}$ and A is contractible.

We denote the *R-dual* of a complex C by $C^* := \operatorname{Hom}_R(C, R)$; similarly, for an R-module M, its R-dual is $M^* := \operatorname{Hom}_R(M, R)$.

A *dualizing complex* D for a ring R is a bounded complex of injective modules with finitely generated cohomology, and such that the natural homothety morphism $R \to \operatorname{Hom}_R(D, D)$ is a quasi-isomorphism. If D is a dualizing complex for a ring R, then R is Cohen Macaulay if and only if $H^i(D) = 0$ for $i \neq 0$ [ABŞ05, 1.4]. Furthermore, R is Gorenstein if and only if $H^i(D) = 0$ for $i \neq 0$ and $H^0(D) \cong R$ [ABŞ05, 1.5.7]. Refer to [Har66, Chapter V] (see also [IK06, Section 3]) for additional details about dualizing complexes; for instance, we may use the following facts without further comment:

1. A commutative Noetherian ring having a dualizing complex necessarily has finite Krull dimension. [Har66, Chapter V, Corollary 7.2]

2. If R is a quotient of a Gorenstein ring Q of finite Krull dimension, then R has a dualizing complex [Har66, Chapter V] (see also [Kaw02, Corollary 1.4]).

More precisely, if $Q \to I$ is a minimal injective resolution, then $\operatorname{Hom}_Q(R, I)$ is a dualizing complex for R.

3. If a Noetherian ring R has a dualizing complex, then R is a quotient of a Gorenstein ring of finite Krull dimension [Kaw02, Corollary 1.4].

When working in a Gorenstein ring R, the minimal injective resolution of R is a dualizing complex for R, which is unique up to isomorphism in $C(\operatorname{Mod} R)$. Because we can explicitly write out a minimal injective resolution of R, we will often assume D is a particular minimal injective resolution rather than just a dualizing complex for R.

For the remainder of this section, assume R is a commutative Noetherian ring. Recall that for an R-module M, the $\mathfrak{a}$-*torsion functor* $\Gamma_{\mathfrak{a}}(-)$ is defined as

$$\Gamma_{\mathfrak{a}}(M) = \{x \in M : \mathfrak{a}^n x = 0 \text{ for some } n\},$$

which yields a left exact functor [BS13, Lemma 1.1.6]. If I is an injective resolution of M, the *i-th local cohomology module with support in $\mathfrak{a}$ (or in $V(\mathfrak{a})$)* is $H^i_{\mathfrak{a}}(M) := H^i(\Gamma_{\mathfrak{a}}(I))$.

Recall that over a Noetherian ring R, we have a decomposition of injective R-modules, due to Matlis [Mat58]. In fact, there exists a bijection between prime ideals $\mathfrak{p}$ of $\operatorname{Spec}(R)$ and indecomposable injective modules $E(R/\mathfrak{p})$, where $E(R/\mathfrak{p}) = E_R(R/\mathfrak{p})$ denotes the injective hull of $R/\mathfrak{p}$ over R. In this way, every injective R-module J can be uniquely (up to isomorphism) expressed as $J \cong \bigoplus_{\mathfrak{p} \in \operatorname{Spec}(R)} E(R/\mathfrak{p})^{\alpha_{\mathfrak{p}}}$. The indecomposable injective module $E(R/\mathfrak{p})$ is $\mathfrak{p}$-torsion and $\mathfrak{p}$-local [Sha69, page 354]; a module M is $\mathfrak{p}$-*torsion* if for every $x \in M$, there exists $n \geq 1$ such that $\mathfrak{p}^n x = 0$ and M is $\mathfrak{p}$-*local* if for every $y \in R \backslash \mathfrak{p}$, multiplication by y on M is an automorphism.

For any prime ideal $\mathfrak{p}$ and any other ideal $\mathfrak{a}$, we have

$$\Gamma_{\mathfrak{a}}(E(R/\mathfrak{p})) = \begin{cases} E(R/\mathfrak{p}), & \mathfrak{p} \supseteq \mathfrak{a} \\ 0, & \mathfrak{p} \not\supseteq \mathfrak{a} \end{cases}.$$

To see this, suppose $\mathfrak{a} \not\subseteq \mathfrak{p}$. Then there exists $a \in \mathfrak{a} \backslash \mathfrak{p}$. For $x \in \Gamma_{\mathfrak{a}}(E(R/\mathfrak{p}))$, there exists n such that $a^n x = 0$, but a acts as an automorphism on $E(R/\mathfrak{p})$, hence $x = 0$. On the other hand, since $\Gamma_{\mathfrak{a}}(E(R/\mathfrak{p}) \subseteq E(R/\mathfrak{p})$ is clear, it is enough to show the other containment. If $\mathfrak{a} \subseteq \mathfrak{p}$, then for any $x \in E(R/\mathfrak{p})$, there exists n such that $\mathfrak{p}^n x = 0$, hence $\mathfrak{a}^n x = 0$, so $x \in \Gamma_{\mathfrak{a}}(E(R/\mathfrak{p}))$.

From this, it follows that if J is an injective R-module, then $\Gamma_{\mathfrak{a}}(J)$ is also injective. We also have [ILL+07, Theorem A.20]:

$$\mathrm{Hom}_R(R/\mathfrak{m}, E(R/\mathfrak{p})) = \begin{cases} R/\mathfrak{m}, & \text{if } \mathfrak{p} = \mathfrak{m} \\ 0, & \text{if } \mathfrak{p} \neq \mathfrak{m} \end{cases}.$$

As a last remark about the interplay between $\Gamma_{\mathfrak{a}}$ and injectives, we note that $E(\Gamma_{\mathfrak{a}}(M)) \cong \Gamma_{\mathfrak{a}}(E(M))$. To see this, as $\Gamma_{\mathfrak{a}}(-)$ is left exact, we know that $\Gamma_{\mathfrak{a}}(M) \to \Gamma_{\mathfrak{a}}(E(M))$ is an injection; we need only show it is essential and appeal to the uniqueness (up to isomorphism) of injective hulls. Let $N \subseteq \Gamma_{\mathfrak{a}}(E(M)) \subseteq E(M)$ be a nonzero submodule. As $M \to E(M)$ is essential, we immediately have $N \cap M \neq 0$, but need to show that $N \cap \Gamma_{\mathfrak{a}}(M) \neq 0$. In fact, take any $0 \neq x \in N \cap M$. Then $x \in N \subseteq \Gamma_{\mathfrak{a}}(E(M)) = \{y \in E(M) : \mathfrak{a}^t y = 0 \text{ for some } t \in \mathbb{N}\}$, so there exists $t \in \mathbb{N}$ such that $\mathfrak{a}^t x = 0$, implying that in fact $x \in \Gamma_{\mathfrak{a}}(N \cap M) = \Gamma_{\mathfrak{a}}(N) \cap \Gamma_{\mathfrak{a}}(M) \subseteq N \cap \Gamma_{\mathfrak{a}}(M)$. Hence the extension remains essential as claimed.

1.2 Gorenstein homological algebra

Gorenstein projective and Gorenstein injective modules (introduced and studied in [EJ95a], see definitions below) over a Gorenstein ring can be thought of as acting similar to projective and injective modules over a regular local ring. For instance, over a Gorenstein local ring R, all R-modules have both finite Gorenstein projective dimension and finite Gorenstein injective dimension [Chr00, 4.4.8 and 6.2.7]. We have an important inequality: The Gorenstein projective (Gorenstein injective) dimension of a module is always less than or equal to the projective (injective) dimension of a module, with equality holding if the projective (injective) dimension is finite [Hol04, "Important Note" and Proposition 2.27]. Immediately, we see that projective (injective) modules are Gorenstein projective (Gorenstein injective). For relevant definitions and basics for Gorenstein projective and Gorenstein injective modules, we will use primarily as references Enochs and Jenda's book [EJ00] and Christensen's book [Chr00].

Definition 1.2.1. [EJ00, Definition 10.1.1] An R-module M is said to be *Gorenstein injective* if and only if there is a (possibly unbounded) exact complex U of injective R-modules such that $M = Z^0(U)$ and such that for any injective R-module J, $\mathrm{Hom}_R(J, U)$ is exact.

We say M is *Gorenstein projective* if and only if there is a (possibly unbounded) exact complex T of projective R-modules such that $M = \Omega_0(T)$ and such that for any projective R-module P, $\mathrm{Hom}_R(T, P)$ is exact.

Definition 1.2.2. [EJ95a, Definition 1.1] Let M be an R-module. If $\phi : E \to M$ is a homomorphism where E is an injective R-module, then $\phi : E \to M$ is called an *injective precover* if $\mathrm{Hom}_R(J, E) \to \mathrm{Hom}_R(J, M) \to 0$ is exact for every injective module J.

We call $\phi : E \to M$ an *injective cover* if ϕ is an injective precover and whenever $f : E \to E$ is linear such that $\phi \circ f = \phi$ then f is an isomorphism of E.

We call a complex of the form

$$\cdots \to E_1 \to E_0 \to M \to 0$$

an *injective resolvent* of M if $E_0 \to M$, $E_1 \to \ker(E_0 \to M)$, $E_i \to \ker(E_{i-1} \to E_{i-2})$ for $i \geq 2$ are all injective precovers [EJ95a, Definition 1.3]. If these maps are all injective covers, we say the complex is a *minimal injective resolvent* of M. In this case, the complex is unique up to isomorphism [EJ95a, page 613]. In general, an injective resolvent is unique up to homotopy [EJ95a, page 613].

In general, injective (pre)covers are not necessarily surjective. For examples of injective (pre)covers, see [CEJ88]. However, we do have that an R-module M is Gorenstein injective if and only if its minimal injective resolvent is exact and $\operatorname{Ext}^i_R(J, M) = 0$ for $i \geq 1$ when J is any injective R-module [EJ95a, Corollary 2.4].

Finally, an R-module M is called *reduced* if it has no nonzero injective submodules [EJ00, page 241].

Chapter 2

Complete resolutions

We first introduce complete projective and complete injective resolutions. When R is Gorenstein, we briefly recall the construction of a minimal complete projective resolution of a MCM module (the situation of [AM02, Construction 3.6] which we will utilize) and more carefully go through the construction of a minimal complete injective resolution of any module (see [Nuc98, Section 7]). With these tools, our first goal will be to construct more computationally convenient complete injective resolutions for MCM modules.

2.1 Minimality and complete resolutions

For this section, let R be a commutative Noetherian ring. We essentially follow [CJ14] for definitions regarding complete resolutions.

Definition 2.1.1. An acyclic complex T of projective R-modules is called a *totally acyclic complex of projectives* if the complex $\operatorname{Hom}_R(T, Q)$ is acyclic for every projective R-module Q. An acyclic complex U of injective R-modules is called a *totally acyclic complex of injectives* if the complex $\operatorname{Hom}_R(J, U)$ is acyclic for every injective R-module

J. When context is clear, we often just refer to either such complex as *totally acyclic.*

Remark 2.1.2. If R is Gorenstein, a complex of projective (resp., injective) R-modules is totally acyclic if and only if it is acyclic [IK06, Corollary 5.5]. With this in mind, an R-module M is Gorenstein projective if and only if there exists an exact complex T of projective R-modules such that $\Omega_0(T) = M$; M is Gorenstein injective if and only if there exists an exact complex U of injective R-modules such that $Z^0(U) = M$.

Remark 2.1.3. If T is a totally acyclic complex of finitely generated projective modules, then $T \otimes I$ is acyclic for any injective module I. This follows as

$$T \otimes I \cong T \otimes \operatorname{Hom}_R(R, I) \xrightarrow{\cong} \operatorname{Hom}(\operatorname{Hom}(T, R), I),$$

where the last isomorphism follows from degree-wise isomorphisms given by [Ish65, Lemma 1.6]

2.1.1 Minimal complexes

Definition 2.1.4. [AM02] A complex C is *minimal* if each homotopy equivalence $\gamma : C \to C$ is an isomorphism.

An equivalent condition for minimality is given in:

Proposition 2.1.5. *[AM02, Proposition 1.7] Let C be a complex of R-modules. Then C is minimal if and only if each morphism $\gamma : C \to C$ homotopic to id_C is an isomorphism. Additionally, if C is minimal and A an irrelevant subcomplex, then $A = 0$.*

If $M \to I$ is an injective resolution such that I is minimal, then $M \to I$ is a *minimal injective resolution of M*. Similarly, if $P \to M$ is a projective resolution such that P is minimal, then $P \to M$ is a *minimal projective resolution of M*.

Remark 2.1.6. When C is a complex of finitely generated projectives over a local ring, Definition 2.1.4 is equivalent to the familiar notion of a minimal complex of free modules [AM02, Proposition 8.1]; when C is an injective resolution of some module, this notion of minimality is equivalent [AM02, Example 1.8] to the essential hull notion of minimality as in [ILL^{+}07, Remark 3.15]. More explicitly, any complex of injective modules U is minimal if and only if U^i is the injective hull of $\ker \partial_U^i$ for all $i \in \mathbb{Z}$ if and only if the result of applying $\mathrm{Hom}_R(R/\mathfrak{p}, -)_{\mathfrak{p}}$ to the morphism $\partial_U^i : U^i \to U^{i+1}$ gives the zero morphism for all $i \in \mathbb{Z}$ and all $\mathfrak{p} \in \mathrm{Spec}(R)$.

2.1.2 Complete projective resolutions

Definition 2.1.7. A *complete projective resolution* of an R-module M is a diagram

$$T \xrightarrow{\tau} P \xrightarrow{\pi} M,$$

where τ and π are chain maps, T is a totally acyclic complex of projective modules, $\pi : P \to M$ is a projective resolution, and $\tau_i : T_i \to P_i$ is an isomorphism for $i \gg 0$. Such a resolution is *minimal* if T and P are minimal complexes. Occasionally, we will refer to just the complex T as a complete projective resolution for M.

The following is a special case of [AM02, Construction 3.6].

Construction 2.1.8. [AM02, Construction 3.6] Given a MCM module M over a Noetherian commutative ring R, we construct its complete projective resolution as follows. Let $P \to M$ be a projective resolution with differential ∂^P. Let $L \to M^*$ be a projective resolution with differential ∂^L, recalling that $M^* = \mathrm{Hom}_R(M, R)$. Apply $(-)^* = \mathrm{Hom}_R(-, R)$ to $L \to M^*$ to obtain $M^{**} \to L^*$. Say $\zeta : M \to M^{**}$ is the canonical isomorphism, $\pi : P_0 \to M$ is the augmentation map, and $\iota : M^{**} \to (L_0)^*$.

Define

$$T_i = \begin{cases} P_i, & i \geq 0; \\ (L_{-i-1})^*, & i < 0; \end{cases} \quad \text{and } \partial_i^T = \begin{cases} \partial_i^P, & i > 0 \\ \iota \circ \zeta \circ \pi, & i = 0 \\ (\partial_{-i}^L)^*, & i < 0 \end{cases} \cdot$$

Then T is an acyclic complex of projectives and there exists a chain map $\tau : T \to P$, where $\tau_i = \mathrm{id}_{P_i}$ for $i \geq 0$.

If R is assumed to be Gorenstein local, then $T \to P \to M$ is easily checked to be a complete projective resolution of M. If, in addition, $P \to M$ and $L \to M^*$ are chosen minimally and M has no nonzero free summands, then $T \to P \to M$ is a minimal complete projective resolution.

2.1.3 Complete injective resolutions

Definition 2.1.9. A *complete injective resolution* of an R-module M is a diagram

$$M \xrightarrow{\iota} I \xrightarrow{\nu} U,$$

where ι and ν are chain maps, U is a totally acyclic complex of injective modules, $\iota : M \to I$ is an injective resolution, and $\nu^i : I^i \to U^i$ is an isomorphism for $i \gg 0$. A *minimal complete injective resolution* of M is such a resolution where I and U are minimal complexes. Occasionally, we will refer to just the complex U as a complete injective resolution for M.

Remark 2.1.10. For an R-module M, a complete injective resolution of M exists if and only if the Gorenstein injective dimension of M is finite [CJ14, 5.2]. Moreover, a local Cohen Macaulay ring R admitting a dualizing complex is Gorenstein if and only if every R-module has finite Gorenstein injective dimension [Chr00, Gorenstein

Theorem, GID Version 6.2.7]. For a local Cohen Macaulay ring R admitting a dualizing complex, every R-module has a complete injective resolution if and only if R is Gorenstein.

Lemma 2.1.11. *Suppose M and N are R-modules with complete injective resolutions, say $M \xrightarrow{\iota_M} I \xrightarrow{\rho_M} U$ and $N \xrightarrow{\iota_N} J \xrightarrow{\rho_N} V$, respectively. If $f : M \to N$ is a map, then there exist chain maps $\phi : I \to J$ and $\widetilde{\phi} : U \to V$ making the following diagram commute:*

$$\begin{array}{ccccc} M & \xrightarrow{\iota_M} & I & \xrightarrow{\rho_M} & U \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \widetilde{\phi}} \\ N & \xrightarrow{\iota_N} & J & \xrightarrow{\rho_N} & V. \end{array}$$

Moreover, ϕ and $\widetilde{\phi}$ are unique up to homotopy equivalence.

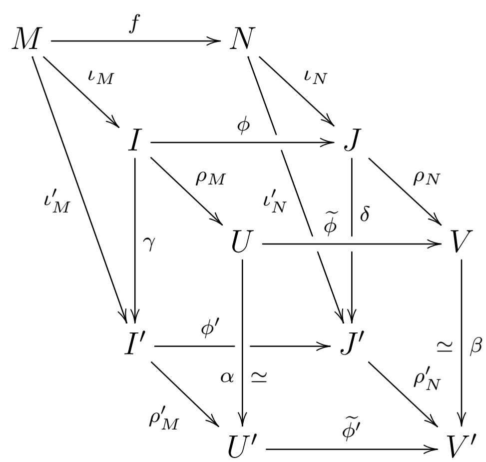
Proof. The chain map ϕ making the square on the left commute exists and is unique up to homotopy equivalence by [Wei94, Comparison Theorem 2.3.7]. The existence and uniqueness (up to homotopy equivalence) of $\widetilde{\phi}$ such that the square on the right also commutes follows from the Comparison Theorem for injective resolutions [Wei94, Comparison Theorem 2.3.7] and for injective resolvents [EJ00, page 169] applied to a high enough cosyzygy of $\phi : I \to J$. □

Lemma 2.1.12. *Suppose M and N are R-modules with complete injective resolutions. Suppose $M \xrightarrow{\iota_M} I \xrightarrow{\rho_M} U$ and $M \xrightarrow{\iota'_M} I' \xrightarrow{\rho'_M} U'$ are two choices of complete injective resolutions of M; similarly, suppose $N \xrightarrow{\iota_N} J \xrightarrow{\rho_N} V$ and $N \xrightarrow{\iota'_N} J' \xrightarrow{\rho'_N} V'$ are two choices of complete injective resolutions of N. If $f : M \to N$ is a map inducing maps as in Lemma 2.1.11, then the following square commutes up to homotopy equivalence*

$$\begin{array}{ccc} U & \xrightarrow{\widetilde{\phi}} & V \\ \simeq\downarrow{\scriptstyle \alpha} & & \simeq\downarrow{\scriptstyle \beta} \\ U' & \xrightarrow{\widetilde{\phi'}} & V' \end{array}$$

where α *and* β *are the homotopy equivalences induced by Lemma 2.1.11 applied to* id_M *and* id_N*, respectively.*

Proof. Lemma 2.1.11 yields the following diagram:



where $\gamma : I \to I'$ and $\alpha : U \to U'$ are the unique (up to homotopy) homotopy equivalences such that $\alpha\rho_M\iota_M = \rho'_M\iota'_M$ (and $\gamma\iota_M = \iota'_M$ and $\alpha\rho_M = \rho'_M\gamma$); $\delta : J \to J'$ and $\beta : V \to V'$ are the unique (up to homotopy) homotopy equivalences such that $\beta\rho_N\iota_N = \rho'_N\iota'_N$ (and $\delta\iota_N = \iota'_N$ and $\beta\rho_N = \rho'_N\delta$); $\widetilde{\phi}$ is the unique (up to homotopy) map such that $\widetilde{\phi}\rho_M\iota_M = \rho_N\iota_N f$ (and $\iota_N f = \phi\iota_M$ and $\rho_N\phi = \widetilde{\phi}\rho_M$); and $\widetilde{\phi}'$ is the unique (up to homotopy) map such that $\widetilde{\phi}'\rho'_M\iota'_M = \rho'_N\iota'_N f$ (and $\iota'_N f = \phi'\iota'_M$ and $\rho'_N\phi' = \widetilde{\phi}'\rho'_M$). Therefore we have that $\widetilde{\phi}'\alpha$ is the unique map (up to homotopy) such that $(\widetilde{\phi}'\alpha)\rho_M\iota_M = \rho'_M\iota'_N f$ (also making the intermediate diagrams commute with $\phi'\gamma$), and $\beta\widetilde{\phi}$ is the unique map (up to homotopy) such that $(\beta\widetilde{\phi})\rho_M\iota_M = \rho'_N\iota'_N f$ (also making the intermediate diagrams commute with $\delta\phi$). By the uniqueness of these maps, we then have that the front square commutes up to homotopy equivalence, i.e., $\widetilde{\phi}'\alpha \sim \beta\widetilde{\phi}$ (such that this agrees with the intermediate maps where $\phi'\gamma = \delta\phi$). $\square$

Proposition 2.1.13. *Let* R *be a Gorenstein ring and for each* R*-module* M*, choose a complete injective resolution* $M \to I \to U$*. Then there exists a covariant func-*

tor $\mathrm{CIR}(-) : \mathrm{Mod}\, R \to \mathrm{K}_{\mathrm{ac}}(\mathrm{Inj}\, R)$ *defined on objects by* $\mathrm{CIR}(M) = U$. *Moreover, this functor does not depend on the choice of complete injective resolution, up to a canonical natural isomorphism.*

Proof. By Lemma 2.1.11, we have that for any map $f : M \to N$ of R-modules, there exists a unique (up to homotopy equivalence) map $\mathrm{CIR}(f) : \mathrm{CIR}(M) \to \mathrm{CIR}(N)$, where clearly $\mathrm{CIR}(-)$ respects the identity map and compositions (by appealing to uniqueness given by Lemma 2.1.11).

Moreover, Lemma 2.1.12 shows that any two families of choices of complete injective resolutions for such a functor $\mathrm{CIR}(-)$ yield naturally isomorphic functors, where the canonical natural isomorphism is given by Lemma 2.1.12. □

Definition 2.1.14. If R is a Gorenstein local ring and M is an R-module with a minimal complete injective resolution $M \to I \to U$, we define $\mathrm{cir}(M) := U \in \mathrm{C}(\mathrm{Mod}\, R)$. By definition of minimality, $\mathrm{cir}(M)$ is defined uniquely up to isomorphism; however, considered as an assignment $\mathrm{Mod}\, R \to \mathrm{C}(\mathrm{Mod}\, R)$, $\mathrm{cir}(-)$ is *not* a functor since this isomorphism is non-canonical. As an object in $\mathrm{K}(\mathrm{Mod}\, R)$, however, $\mathrm{cir}(M) \simeq \mathrm{CIR}(M)$.

Remark 2.1.15. Recall that $\mathrm{CIR}(-)$ naturally factors through $\underline{\mathrm{GInj}}(R)$. By [Ste14, Proposition 4.7], there is an equivalence $K_{\mathrm{ac}}(\mathrm{Inj}\, R) \underset{\mathrm{CIR}(-)}{\overset{Z^0(-)}{\rightleftarrows}} \underline{\mathrm{GInj}}(R)$.

Remark 2.1.16. For an R-module M, Enochs and Jenda defined a "complete minimal injective resolution of M" to be the concatenation of the minimal injective resolvent $J \to M$ and minimal injective resolution $M \to I$ of M [EJ95a, Definition 1.8]. However, in a Gorenstein ring, this complex is acyclic if and only if M is Gorenstein injective [EJ95a, Corollary 2.3]. When R is Gorenstein and M is reduced and Gorenstein injective, this coincides with our notion of minimal complete injective resolution; when M is just Gorenstein injective (not necessarily reduced), the concatenation of the minimal injective resolvent and minimal injective resolution of M contains the minimal complete injective resolution (as we have defined) as a direct summand (but is not isomorphic to it in general).

For any R-module M, we now construct a minimal complete injective resolution of M.

Construction 2.1.17. Assume R is Gorenstein of dimension d and M is any R-module. Let $\iota : M \to I$ be a minimal injective resolution of M, with differential ∂_I on I. Fix the minimal integer $g \geq 0$ such that $\ker \partial_I^g$ is reduced Gorenstein injective; such a g exists and indeed is such that $g \leq d+1$ by [EJ00, Theorem 10.1.13]. Set $G = \ker \partial_I^g$ and $j : G \to I^g$ the canonical inclusion. Letting J be the minimal injective resolvent for G, which exists by [Eno81, Theorem 2.1], we have that the augmented complex

$$\cdots \xrightarrow{\partial_2^J} J_1 \xrightarrow{\partial_1^J} J_0 \xrightarrow{\pi} G \to 0,$$

is exact by [EJ95a, Corollary 2.4]. Define the following complex

$$U^i = \begin{cases} I^i, & \text{if } i \geq g; \\ J_{g-1-i}, & \text{if } i < g; \end{cases} \quad \text{and } \partial_U^i = \begin{cases} \partial_I^i, & \text{if } i \geq g \\ j \circ \pi, & \text{if } i = g-1 \\ \partial_{g-1-i}^J, & \text{if } i < g-1 \end{cases} \cdot$$

As J is an injective resolvent of G, we have that $\pi : U^{g-1} \to G$ is an injective precover, and so there exists a map $\nu^{g-1} : I^{g-1} \to U^{g-1}$ such that $\nu^{g-1}\pi$ agrees with the canonical surjection $I^{g-1} \to G$. The map ν^{g-1} restricts to a map $\ker \partial_I^{g-1} \to \ker \partial_U^{g-1}$, and then we induct, using that $U^{g-i} \to \ker(\partial_U^{g-i+1})$ are injective precovers for $i > 1$. Induction gives maps $\nu^i : I^i \to U^i$ for all $i < g$, making all of the squares commute in the following diagram, where we also set $\nu^i = \mathrm{id}_{I^i}$ for all $i \geq g$ and unlabeled maps are the obvious ones given above:

$$\begin{array}{ccccccccccccccccc}
\cdots & \longrightarrow & U^{-1} & \longrightarrow & U^{0} & \longrightarrow & \cdots & \longrightarrow & U^{g-2} & \longrightarrow & U^{g-1} & \longrightarrow & U^{g} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow{\scriptstyle \nu^0} & & & & \uparrow{\scriptstyle \nu^{g-2}} & & \uparrow{\scriptstyle \nu^{g-1}} & & \uparrow{\scriptstyle \nu^{g}} \; = & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & I^{0} & \longrightarrow & \cdots & \longrightarrow & I^{g-2} & \longrightarrow & I^{g-1} & \longrightarrow & I^{g} & \longrightarrow & \cdots
\end{array}$$

With this construction, U is an acyclic complex of injective modules with a map of complexes $\nu : I \to U$ such that ν^i is an isomorphism for $i \geq g$. As I and J were chosen minimally, it is easy to verify that U is also a minimal complex. To see this, note that because G is reduced, the proof of [EJ00, Proposition 10.1.11] shows that $Z^i(U) \to U^i$ is an essential injection for $i < g$. As R is a Gorenstein ring, we obtain for free that U is totally acyclic, see Remark 2.1.2. By assumption, $M \to I$ is an injective resolution, and by construction $\nu^i : I^i \to U^i$ is an isomorphism for $i \geq g$. Further, since I and J were chosen minimally, U is a minimal complex. Hence $M \xrightarrow{\iota} I \xrightarrow{\nu} U$ is a minimal complete injective resolution, with $\nu^i : I^i \to U^i$ an isomorphism for $i \geq g$.

Remark 2.1.18. We could alter this construction by not requiring I or J to minimal; in this case, we would not require $G = \ker \partial_I^g$ to be reduced (such a $g \leq d$ exists by [EJ00, Theorem 10.1.13]). Following the rest of the construction through verbatim, this gives a (not necessarily minimal) complete injective resolution of M.

Proposition 2.1.19. *Let M be an R-module. If U is a minimal complete injective resolution of M and V is any other complete injective resolution of M, then U appears (up to isomorphism) as a direct summand of V with a contractible complementary summand.*

Proof. There exists homotopy inverses $\alpha : U \to V$ and $\beta : V \to U$. The minimality of U implies [AM02, Proposition 1.7] that α is injective, β is surjective, $\ker \beta$ is contractible, and $V = \operatorname{im}\alpha \oplus \ker\beta$. □

2.2 Constructing complete injective resolutions

We now move to constructing more computationally useful complete injective resolutions of MCM modules, utilizing complete projective resolutions.

Remark 2.2.1. Complete projective resolutions are unique up to homotopy equivalence and a map of R-modules $M \to N$ induces a map (which is unique up to homotopy equivalence) between their complete projective resolutions [AM02, Lemma 5.3]. For each MCM R-module M, choose a complete projective resolution $T \to P \to M$ and set $\mathrm{CPR}(M) = T$; this yields a functor $\mathrm{CPR}(-) : \mathrm{MCM}(R) \to \mathrm{K_{ac}}(\operatorname{prj} R)$. An argument dual to Lemma 2.1.12 and Proposition 2.1.13 gives that the functor $\mathrm{CPR}(-)$ does not depend on the choice of complete projective resolution up to a canonical natural isomorphism. In fact, when R is Gorenstein, Buchweitz shows [Buc86, Theorem 4.4.1] that $\Omega_0(-) : \mathrm{K_{ac}}(\operatorname{prj} R) \to \underline{\mathrm{MCM}}(R)$ is an equivalence and it easily follows that $\mathrm{CPR}(-) : \underline{\mathrm{MCM}}(R) \to \mathrm{K_{ac}}(\operatorname{prj} R)$ gives an inverse equivalence. If $T \to P \to M$ is a minimal complete projective resolution, set $\operatorname{cpr}(M) = T \in \mathrm{C}(\operatorname{Mod} R)$; then $\operatorname{cpr}(-)$ is a well-defined assignment of a module to a complex, since minimality of T implies that it is unique up to (a non-canonical) isomorphism. Again, we caution that $\operatorname{cpr}(-)$ is *not* a functor since this isomorphism is non-canonical.

Proposition 2.2.2. *Let R be Gorenstein with $\dim(R) = d$, D a minimal injective resolution for R, and M a MCM R-module. If $T \xrightarrow{\tau} P \to M^*$ is a complete projective resolution of M^*, then $M \to \operatorname{Hom}_R(P, D) \xrightarrow{\operatorname{Hom}_R(\tau, D)} \operatorname{Hom}_R(T, D)$ is a complete injective resolution of M. In fact, $\operatorname{Hom}_R(\mathrm{CPR}((-)^*), D)$ and $\mathrm{CIR}(-)$ are naturally isomorphic functors $\mathrm{MCM}(R) \to \mathrm{K_{ac}}(\operatorname{Inj} R)$.*

Proof. Let M be any MCM R-module and set $\mathrm{CPR}(M^*) = T$. Then there exists a projective resolution P such that the diagram $T \xrightarrow{\tau} P \xrightarrow{\pi} M^*$ is a complete projective resolution of M^*, with τ_i an isomorphism for $i \geq g$, for some fixed integer g. Apply $\mathrm{Hom}_R(-, D)$ to this to obtain maps of complexes

$$\mathrm{Hom}_R(M^*, D) \xrightarrow{\mathrm{Hom}(\pi, D)} \mathrm{Hom}_R(P, D) \xrightarrow{\mathrm{Hom}(\tau, D)} \mathrm{Hom}_R(T, D).$$

As π is a quasi-isomorphism, so is $\mathrm{Hom}_R(\pi, D)$ by [Wei94, Lemma 10.7.3]. Next, applying a result of Ischebeck [BH98, Exercise 3.1.24] that says in a local ring positive Ext modules vanish for a MCM module against a finitely generated module of finite injective dimension, we obtain the map $\mathrm{Hom}_R(M^*, R) \to \mathrm{Hom}_R(M^*, D)$ induced by the quasi-isomorphism $R \to D$ is also a quasi-isomorphism. As M is MCM, $M \xrightarrow{\cong} \mathrm{Hom}_R(M^*, R)$, so this gives $M \to \mathrm{Hom}_R(M^*, D)$ is a quasi-isomorphism. Put $\iota : M \to \mathrm{Hom}_R(P, D)$ as the quasi-isomorphism defined by the composition of this quasi-isomorphism and $\mathrm{Hom}(\pi, D)$.

As D is a bounded complex of injective modules and $T \in \mathrm{K_{ac}}(\mathrm{prj}\, R)$, $\mathrm{Hom}_R(T, D)$ is an acyclic complex of injective modules. Also, $\mathrm{Hom}_R(P, D)$ is a complex of injective modules such that $\mathrm{Hom}_R(P, D)^i = 0$ for $i < 0$. As $\iota : M \to \mathrm{Hom}_R(P, D)$ is a quasi-isomorphism, we then have that $\iota : M \to \mathrm{Hom}_R(P, D)$ is an injective resolution. Recall that τ_i is an isomorphism for $i \geq g$, hence $\mathrm{Hom}(\tau, D)^i$ is an isomorphism for $i \geq g + d$. We then have that

$$M \xrightarrow{\iota} \mathrm{Hom}_R(P, D) \xrightarrow{\mathrm{Hom}(\tau, D)} \mathrm{Hom}_R(T, D)$$

is a complete injective resolution of M.

So, for any MCM R-module M, both $\mathrm{CIR}(M)$ and $\mathrm{Hom}_R(\mathrm{CPR}(M^*), D)$ are complete injective resolutions of M. Proposition 2.1.13 implies $\mathrm{CIR}(-)$ and $\mathrm{Hom}_R(\mathrm{CPR}((-)^*), D)$ are naturally isomorphic functors $\mathrm{MCM}(R) \to \mathrm{K_{ac}}(\mathrm{Inj}\, R)$. □

Lemma 2.2.3. *Let R be a Gorenstein local ring. For a MCM R-module M with no nonzero free summands, we have*

$$(\mathrm{cpr}(M^*))^* \cong \Sigma^1 \,\mathrm{cpr}(M),$$

in $\mathrm{C}(\mathrm{Mod}\, R)$.

Proof. Let $P \to M$ and $L \to M^*$ be minimal projective resolutions. Since M has no nonzero free summands, the concatenation of P and $\Sigma^{-1}L^*$ is a minimal complete projective resolution of M, hence isomorphic to $\mathrm{cpr}(M)$. Since P is also a minimal projective resolution of M^{**}, we have $\mathrm{cpr}(M^*)$ is the concatenation of L and $\Sigma^{-1}P^*$. Hence $((\mathrm{cpr}(M^*))^*)_{\geq 1} \cong P$ and $((\mathrm{cpr}(M^*)^*)_{\leq 0} \cong L^*$, therefore $(\mathrm{cpr}(M^*))^* \cong \Sigma^1 \,\mathrm{cpr}(M)$. □

Proposition 2.2.4. *Let R be local Gorenstein with* $\dim(R) = d$*, M a MCM R-module with no nonzero free summands, and D a minimal injective resolution for R. Then we have isomorphisms in* $\mathrm{C}(\mathrm{Mod}\, R)$

$$\mathrm{Hom}_R(\mathrm{cpr}(M^*), D) \cong \mathrm{cpr}(M^*)^* \otimes_R D \cong \Sigma^1 \,\mathrm{cpr}(M) \otimes_R D,$$

and therefore, these all give isomorphic complete injective resolutions of M.

Proof. Set $T = \mathrm{cpr}(M^*)$. By [Ish65, Lemma 1.1], [IK06, proof of Theorem 4.2], we can see that the map $T^* \otimes_R D \xrightarrow{\cong} \mathrm{Hom}_R(T, D)$ is an isomorphism, giving the first isomorphism. The second isomorphism follows from Lemma 2.2.3. Proposition 2.2.2 then shows that these all give complete injective resolutions of M. □

Remark 2.2.5. Although the isomorphisms in Lemma 2.2.3 and Proposition 2.2.4 take place in $\mathrm{C}(\mathrm{Mod}\, R)$, these are not natural in $\mathrm{C}(\mathrm{Mod}\, R)$. However, after passing to $\mathrm{K}(\mathrm{Mod}\, R)$, the isomorphisms become natural.

Let R be a Gorenstein ring, M a MCM R-module. The constructions of complete injective resolutions in Proposition 2.2.4 are not in general minimal, even though the complete projective resolutions are chosen minimally. To see this, consider the following:

Example 2.2.6. Consider the hypersurface $R = k[[x, y]]/(x^2 - y^2)$, where k is any algebraically closed field of characteristic not equal to 2 (this is an A_1 ADE singularity, see [LW12]). Let $\mathfrak{p} = (x + y)$. Note that this is a minimal prime ideal, since $R/\mathfrak{p} \cong k[[x]]$ and $\mathrm{ht}(\mathfrak{p}) = 0$. Over this ring, we consider the MCM R-module defined by $M = R/\mathfrak{p}$. We claim that the construction of the complete injective resolution of M given in Proposition 2.2.4 is *not* minimal.

Since $\dim(R) = 1$, we have the minimal injective resolution of R is isomorphic to $D = 0 \to E^0 \to E^1 \to 0$, where $E^i = \bigoplus_{\mathrm{ht}(\mathfrak{q})=i} E(R/\mathfrak{q})$.

Consider the complex

$$T = \qquad \cdots \longrightarrow R \xrightarrow{x+y} R \xrightarrow{x-y} \underset{\text{degree } 0}{R} \xrightarrow{x+y} R \longrightarrow \cdots ,$$

where we clearly have $T \cong \Sigma^1 \operatorname{cpr}(M)$. We show that $T \otimes_R D$ is not a minimal complex. As $T \otimes_R D$ is a complex of injectives, showing that it is not minimal is equivalent (by Remark 2.1.6) to showing that for some prime $\mathfrak{q}$, and some $i \in \mathbb{Z}$,

$$\mathrm{Hom}_{R_\mathfrak{q}}(\kappa(\mathfrak{q}), (T_i)_\mathfrak{q} \otimes_{R_\mathfrak{q}} D_\mathfrak{q}) \to \mathrm{Hom}_{R_\mathfrak{q}}(\kappa(\mathfrak{q}), (T_{i-1})_\mathfrak{q} \otimes_{R_\mathfrak{q}} D_\mathfrak{q})$$

is not the zero map. We consider the prime $\mathfrak{p} = (x + y)$. Note that $D_\mathfrak{p} = E(R/\mathfrak{p})$, which is a complex concentrated in degree 0. It will be enough to show that for some $i \in \mathbb{Z}$,

$$\mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), (T_i)_\mathfrak{p} \otimes_{R_\mathfrak{p}} E(R/\mathfrak{p})) \to \mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), (T_{i-1})_\mathfrak{p} \otimes_{R_\mathfrak{p}} E(R/\mathfrak{p}))$$

is not the zero map. Localizing the map $R \xrightarrow{x-y} R$ at $\mathfrak{p}$ gives an isomorphism $R_\mathfrak{p} \xrightarrow{\cong} R_\mathfrak{p}$, applying $- \otimes_{R_\mathfrak{p}} E(R/\mathfrak{p})$ preserves isomorphisms, hence $R_\mathfrak{p} \otimes_{R_\mathfrak{p}} E(R/\mathfrak{p}) \xrightarrow{\cong} R_\mathfrak{p} \otimes_{R_\mathfrak{p}} E(R/\mathfrak{p})$ is an isomorphism. Furthermore, $\mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), -)$ preserves isomorphisms, hence

$$\mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), R_\mathfrak{p} \otimes_{R_\mathfrak{p}} E(R/\mathfrak{p})) \xrightarrow{\cong} \mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), R_\mathfrak{p} \otimes_{R_\mathfrak{p}} E(R/\mathfrak{p}))$$

is an isomorphism. Therefore, $T \otimes D$ is not minimal.

Chapter 3

Stable local cohomology

Our goal for this chapter is to develop a stable notion of local cohomology. Recall that $\mathrm{K}(\operatorname{Inj} R)$ is the homotopy category of complexes of injective R-modules, and $\mathrm{K}_{\mathrm{ac}}(\operatorname{Inj} R)$ (respectively, $\mathrm{K}_{\mathrm{tac}}(\operatorname{Inj} R)$) is the subcategory of acyclic (respectively, totally acyclic) complexes. We first remark that both localization at an element and the $\mathfrak{a}$-torsion functor preserve totally acyclic complexes of injectives. This is known to the experts, but we include a proof for completeness. It is worth noting that when R is Gorenstein, $\mathrm{K}_{\mathrm{ac}}(\operatorname{Inj} R) = \mathrm{K}_{\mathrm{tac}}(\operatorname{Inj} R)$ [IK06, Corollary 5.5]. Furthermore, localization (at any multiplicatively closed set) preserves acyclic complexes of injectives, hence in a Gorenstein ring, preserves total acyclicity as well. Using this, the proof of [Lip02, Lemma 3.5.1] shows that the $\mathfrak{a}$-torsion functor preserves acyclic complexes of injectives as well, and therefore when R is Gorenstein, also preserves total acyclicity.

Lemma 3.0.1. *Let R be a commutative Noetherian ring. For an element $x \in R$ and ideal $\mathfrak{a} \subset R$, if $U \in \mathrm{K}_{\mathrm{tac}}(\operatorname{Inj} R)$, then $U_x \in \mathrm{K}_{\mathrm{tac}}(\operatorname{Inj} R)$ and $\Gamma_{\mathfrak{a}}(U) \in \mathrm{K}_{\mathrm{tac}}(\operatorname{Inj} R)$.*

Proof. We show first that $U_x \in \mathrm{K}_{\mathrm{tac}}(\operatorname{Inj} R)$. Note that $\mathrm{K}_{\mathrm{tac}}(\operatorname{Inj} R) \subseteq \mathrm{K}(\operatorname{Inj} R)$ is a localizing subcategory [IK06, Proposition 5.9]; in particular, $\mathrm{K}_{\mathrm{tac}}(\operatorname{Inj} R)$ is closed under coproducts. We recall that U_x can be described as the homotopy colimit of the the sequence $U \xrightarrow{x} U \xrightarrow{x} U \xrightarrow{x} \cdots$. In fact, since U is a complex of injective R-modules, we actually have a degree-wise split short exact sequence of complexes:

$$0 \to \oplus_{i\geq 0} U \to \oplus_{i\geq 0} U \to U_x \to 0,$$

where the first map sends $(u_0, u_1, u_2, ...) \mapsto (u_0, u_1 - xu_0, u_2 - xu_1, ...)$. For any injective R-module J, applying $\operatorname{Hom}_R(J, -)$ to this yields a short exact sequence. Since $\mathrm{K}_{\mathrm{tac}}(\operatorname{Inj} R)$ is closed under coproducts, $\operatorname{Hom}_R(J, \oplus_{i\geq 0} U)$ is acyclic. Hence $\operatorname{Hom}_R(J, U_x)$ is acyclic, and therefore $U_x \in \mathrm{K}_{\mathrm{tac}}(\operatorname{Inj} R)$ as well.

If $\mathfrak{a}$ is an ideal with generators $a_1, \ldots, a_n$, then we have an exact sequence (degree-wise split, as all terms are complexes of injectives):

$$0 \to \Gamma_{\mathfrak{a}}(U) \to U \to \oplus_{i=1}^{n} U_{a_i} \to \cdots \to U_{a_1 \cdots a_n} \to 0,$$

coming from the Čech complex on $\mathfrak{a}$. Again, for any injective R-module J, application of $\operatorname{Hom}_R(J, -)$ to this sequence will yield another exact sequence; a similar argument yields that $\Gamma_{\mathfrak{a}}(U) \in \mathrm{K}_{\mathrm{tac}}(\operatorname{Inj} R)$. □

Two consequences of this generalize results of [Saz04], removing the Gorenstein hypothesis. The first corollary generalizes [Saz04, Theorem 3.2] and the second generalizes [Saz04, Theorem 3.1].

Corollary 3.0.2. *Let R be a commutative Noetherian ring and $\mathfrak{a} \subset R$ an ideal. If $G \in \operatorname{GInj}(R)$, then $\Gamma_{\mathfrak{a}}(G) \in \operatorname{GInj}(R)$.*

Proof. Let G be a Gorenstein injective R-module. By definition, G is the zeroth syzygy of an acyclic complex U of injective modules. Since $\Gamma_{\mathfrak{a}}(-)$ is left exact,

$Z^0\Gamma_{\mathfrak{a}}(U) = \Gamma_{\mathfrak{a}}(Z^0 U)$, which coincides with $\Gamma_{\mathfrak{a}}(G)$ since $\Gamma_{\mathfrak{a}}(U)$ is totally acyclic by Lemma 3.0.1. Hence again by definition, $\Gamma_{\mathfrak{a}}(G)$ is Gorenstein injective. □

Corollary 3.0.3. *If R is a commutative Noetherian ring, G is a Gorenstein injective R-module, and $\mathfrak{a} \subset R$ is an ideal, then $H^i_{\mathfrak{a}}(G) = 0$ for $i > 0$.*

Proof. Since G is Gorenstein injective, it is the zeroth syzygy of a totally acyclic complex U of injective modules. Then $\Gamma_{\mathfrak{a}}(G)$ is the zeroth syzygy of the totally acyclic (by Lemma 3.0.1) complex $\Gamma_{\mathfrak{a}}(U)$ of injective modules. For $i > 0$, $H^i_{\mathfrak{a}}(G) = H^i(\Gamma_{\mathfrak{a}}(U^{\geq 0})) = H^i(\Gamma_{\mathfrak{a}}(U)) = 0$. □

We now come to the main definition of Part I:

Definition 3.0.4. Let R be a commutative Noetherian ring and M be an R-module that has a minimal complete injective resolution $M \to I \to U$. For an ideal $\mathfrak{a}$ of R, we define the *stable local cohomology module of M with respect to $\mathfrak{a}$* as

$$\Gamma^{\mathrm{stab}}_{\mathfrak{a}}(M) = Z^0(\Gamma_{\mathfrak{a}}(U)) \in \operatorname{Mod} R,$$

where $Z^0(-)$ represents taking the kernel of the map between the modules in cohomological degrees 0 and 1. Evidently then $\Gamma^{\mathrm{stab}}_{\mathfrak{a}}(M)$ is a Gorenstein injective R-module (by Lemma 3.0.1), and this module is unique up to a non-canonical isomorphism by the minimality of U. Because each homomorphism of R-modules induces a homomorphism of their complete injective resolutions, which is unique up to homotopy equivalence, Remark 2.1.15 shows that each homomorphism of R-modules $\phi : M \to M'$ induces a homomorphism in $\underline{\mathrm{GInj}}(R)$

$$\Gamma^{\mathrm{stab}}_{\mathfrak{a}}(\phi) : \Gamma^{\mathrm{stab}}_{\mathfrak{a}}(M) \to \Gamma^{\mathrm{stab}}_{\mathfrak{a}}(M'),$$

that is, $\Gamma_{\mathfrak{a}}^{\mathrm{stab}}(-)$ defines a functor $\operatorname{Mod} R \to \underline{\mathrm{GInj}}(R)$.

Remark 3.0.5. Since complete injective resolutions are unique up to homotopy (Lemma 2.1.11), we can equivalently define $\Gamma_{\mathfrak{a}}^{\mathrm{stab}}(M) = Z^0(\Gamma_{\mathfrak{a}}(\mathrm{CIR}(M))) \in \underline{\mathrm{GInj}}(R)$, which we may do without further comment.

Here are a few basic properties of stable local cohomology:

Proposition 3.0.6. *Let M be an R-module that has a complete injective resolution. Then*

1. *If $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$, then $\Gamma_{\mathfrak{a}}^{\mathrm{stab}}(M) \cong \Gamma_{\mathfrak{b}}^{\mathrm{stab}}(M)$.*
2. *Let $\{M_\lambda\}$ be a family of R-modules. Then*

$$\Gamma_{\mathfrak{a}}^{\mathrm{stab}}\left(\bigoplus_{\lambda} M_\lambda\right) \cong \bigoplus_{\lambda} \Gamma_{\mathfrak{a}}^{\mathrm{stab}}(M_\lambda).$$

3. *If $\operatorname{id}_R M < \infty$, then $\Gamma_{\mathfrak{a}}^{\mathrm{stab}}(M) = 0$. Conversely, if $\Gamma_{0}^{\mathrm{stab}}(M) = 0$, then $\operatorname{id}_R M < \infty$.*

Proof. (1) and (2) follow from [BS13, Exercise 1.1.3] and [ILL+07, Proposition 7.3], respectively.

For (3), if $\operatorname{id}_R M < \infty$ and $M \to I$ is an injective resolution, then $M \to I \to 0$ is a minimal complete injective resolution, hence $\Gamma_{\mathfrak{a}}^{\mathrm{stab}}(M) = 0$. Conversely, if $\Gamma_{0}^{\mathrm{stab}}(M) = 0$, then M has a minimal complete injective resolution of the form $M \to I \to 0$, and therefore $I^i = 0$ for $i \gg 0$, so $\operatorname{id}_R M < \infty$. □

If R is Gorenstein and $R \to S$ is a flat ring homomorphism, we have a change of rings result for stable local cohomology.

Proposition 3.0.7. *Let $R \to S$ be a ring homomorphism such that R is Gorenstein, S is flat as an R-module, M is any S-module having a complete S-injective resolution, and $\mathfrak{a} \subseteq R$ an ideal of R. Then*

$$\Gamma_{\mathfrak{a}}^{\mathrm{stab}}(M) \cong \Gamma_{\mathfrak{a}S}^{\mathrm{stab}}(M).$$

Proof. Recall that injective S-modules are injective as R-modules since S is a flat R-module. Then a complete injective resolution U of M as an S-module will coincide with a complete injective resolution of M as an R-module (we require R to be Gorenstein so that U will be totally acyclic over R), and the result follows by definition of stable local cohomology. □

Remark 3.0.8. We can remove the condition that R is Gorenstein in Proposition 3.0.7 if $R \to S$ is localization at any multiplicatively closed set, since localization takes injective R-modules to injective S-modules [BS13, Proposition 10.1.14]. In this case, for a totally acyclic complex of injective S-modules U and an injective R-module J, adjointness yields $\mathrm{Hom}_R(J, U) \cong \mathrm{Hom}_S(J \otimes S, U)$, so U is totally acyclic over R as well.

Before proceeding further, we consider a simple example.

Example 3.0.9. Let $R = \frac{k[[x]]}{(x^2)}$, where k is any field. Then R is a hypersurface with $\dim(R) = 0$, and so the projective and injective modules coincide. Set T as the complex of projective (and hence injective) modules R with all maps multiplication by x:

$$T := \cdots \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} \cdots .$$

Then $k \to T^{\geq 0} \to T$ (with the obvious maps) is a complete injective resolution of k.

In fact, T is minimal as in this case we have $R \cong E_R(k)$. We notice that

$$\Gamma^{\mathrm{stab}}_{(x)}(k) = Z^0\Gamma_{(x)}(T) = \ker(\Gamma_{(x)}(R) \xrightarrow{x} \Gamma_{(x)}(R)) = \ker(R \xrightarrow{x} R) = k.$$

On the other hand, $\Gamma^{\mathrm{stab}}_{(x)}(R) = 0$ since $\mathrm{id}_R R < \infty$.

A motivation for calling this stable local cohomology is that $\Gamma^{\mathrm{stab}}_{\mathfrak{a}}(-)$ is the composition of the stabilization functor $Z^0 \operatorname{CIR}(-)$ and the $\mathfrak{a}$-torsion functor. Notice that $Z^0 \operatorname{CIR}(-)$ is called the Gorenstein approximation functor in [Kra05].

Remark 3.0.10. Recall that $\simeq$ denotes an isomorphism in the stable category $\underline{\mathrm{GInj}}(R)$ and $\cong$ denotes an isomorphism in $\operatorname{Mod} R$. For Gorenstein injective modules M and N, we comment that $M \simeq N$ if and only if there exists (possibly zero) injective R-modules J_1 and J_2 such that $M \oplus J_1 \cong N \oplus J_2$. (In fact, if M and N are reduced Gorenstein injective modules, then $M \simeq N$ if and only if $M \cong N$.)

In general, if M is a module over a commutative Noetherian ring having a complete injective resolution, $\Gamma^{\mathrm{stab}}_{\mathfrak{a}}(M)$ can be difficult to compute. We will therefore mainly restrict ourselves to working in a Gorenstein ring R so that we may use the construction of a (minimal) complete injective resolution given earlier. Restricting further to MCM modules with no nonzero free summands will allow us to use the more accessible minimal complete projective resolution of M to obtain a complete injective resolution of M.

Lemma 3.0.11. *Let R be a commutative Noetherian ring, T be any complex of projectives, and D any complex of R-modules. Then*

$$T \otimes_R \Gamma_{\mathfrak{a}}(D) \xrightarrow{\cong} \Gamma_{\mathfrak{a}}(T \otimes_R D).$$

Proof. For a free R-module F and any other R-module M, it is clear that $F \otimes_R \Gamma_{\mathfrak{a}}(M) \xrightarrow{\cong} \Gamma_{\mathfrak{a}}(F \otimes_R M)$ since $\Gamma_{\mathfrak{a}}(-)$ commutes with arbitrary direct sums [ILL+07, Proposition 7.3]. Consequently, if P is any projective R-module, we have $P \otimes_R \Gamma_{\mathfrak{a}}(M) \xrightarrow{\cong} \Gamma_{\mathfrak{a}}(P \otimes_R M)$. For $i, j \in \mathbb{Z}$, T_i is a projective module and D_j an R-module, hence $T_i \otimes_R \Gamma_{\mathfrak{a}}(D_j) \xrightarrow{\cong} \Gamma_{\mathfrak{a}}(T_i \otimes_R D_j)$. We have a map of bicomplexes $T \otimes_R \Gamma_{\mathfrak{a}}(D) \to \Gamma_{\mathfrak{a}}(T \otimes_R D)$, which is an isomorphism in each bidegree; totalizing yields the desired result. □

Proposition 3.0.12. *Let R be a Gorenstein local ring of dimension d, D a minimal injective resolution for R, M a MCM R-module with no nonzero free summands, and $\mathfrak{a}$ an ideal of R. If $T := \operatorname{cpr}(M^*)$ and $S := \operatorname{cpr}(M)$, then*

$$Z^0\Gamma_{\mathfrak{a}}(T^* \otimes_R D) \cong Z^0\Gamma_{\mathfrak{a}}(\operatorname{Hom}_R(T, D)) \cong Z^1\Gamma_{\mathfrak{a}}(S \otimes_R D) \cong Z^1(S \otimes \Gamma_{\mathfrak{a}}(D)),$$

and all of these coincide with $\Gamma_{\mathfrak{a}}^{\text{stab}}(M)$ in $\underline{\operatorname{GInj}}(R)$.

Proof. The R-module isomorphisms follow since $\Sigma^1 S \cong T^*$ and $\operatorname{Hom}_R(T, D) \xrightarrow{\cong} T^* \otimes_R D$ by [Ish65, Lemma 1.1] and [IK06, proof of Theorem 4.2], and the last isomorphism is just an application of Lemma 3.0.11. It is therefore enough to show (2), which follows by Proposition 2.2.2. □

Notation 3.0.13. Suppose R is a Gorenstein ring and M is an R-module. If R is local and $T \xrightarrow{\tau} P \xrightarrow{\pi} M$ is a minimal complete projective resolution of M, we denote the *i-th stable syzygy* of M by

$$\Omega_i^{\text{cpr}}(M) := \operatorname{coker}(\tau_{i+1} : T_{i+1} \to T_i)$$

for all $i \in \mathbb{Z}$, and the *i-th syzygy* of M by

$$\Omega_i^{\text{prj}}(M) := \operatorname{coker}(\pi_{i+1} : P_{i+1} \to P_i)$$

for $i \geq 0$. In this case, if M is a MCM R-module, $\Omega_i^{\text{cpr}}(M) \simeq \Omega_i^{\text{prj}}(M)$ for $i \geq 0$ (isomorphic in $\underline{\text{MCM}}(R)$).

If R is not necessarily local and $M \xrightarrow{\iota} I \xrightarrow{\rho} U$ is a minimal complete injective resolution of M, we denote the *i-th stable cosyzygy* of M by

$$\Omega_{\text{cir}}^i(M) := \ker(\rho^i : U^i \to U^{i+1})$$

for all $i \in \mathbb{Z}$, and the *i-th cosyzygy* of M by

$$\Omega_{\text{inj}}^i(M) := \ker(\iota^i : I^i \to I^{i+1})$$

for $i \geq 0$. Here, when M is a Gorenstein injective R-module, $\Omega_{\text{cir}}^i(M) \simeq \Omega_{\text{inj}}^i(M)$ for $i \geq 0$ (isomorphic in $\underline{\text{GInj}}(R)$).

Translation functors on $\underline{\text{MCM}}(R)$ and $\underline{\text{GInj}}(R)$ are given by Ω_{-1}^{cpr} and Ω_{cir}^1, respectively, which agree with the translation functor endowed by the equivalences $K_{\text{ac}}(\text{prj}\, R) \underset{\text{CPR}(-)}{\overset{\Omega_0(-)}{\rightleftarrows}} \underline{\text{MCM}}(R)$ and $K_{\text{ac}}(\text{Inj}\, R) \underset{\text{CIR}(-)}{\overset{Z^0(-)}{\rightleftarrows}} \underline{\text{GInj}}(R)$. In their respective stable categories, note that $\Omega_0^{\text{cpr}}(-)$ and $\Omega_{\text{cir}}^0(-)$ are isomorphic to the identity functors. This agrees with the triangulation spelled out as in [Buc86, Theorem 4.4.1], where the inverse loop functor gives the shift functor on $\underline{\text{MCM}}(R)$, i.e., an exact triangle in $\underline{\text{MCM}}(R)$ has the form

$$L \to M \to N \to \Omega_{-1}^{\text{cpr}} L.$$

Proposition 3.0.14. *Let R be a local Gorenstein ring. As a functor between stable* *categories,* $\Gamma_{\mathfrak{a}}^{\mathrm{stab}}(-) : \underline{\mathrm{MCM}}(R) \to \underline{\mathrm{GInj}}(R)$ *is triangulated. Furthermore, for any MCM R-module M, we have an R-module isomorphism*

$$\Gamma_{\mathfrak{a}}^{\mathrm{stab}}(\Omega_{-i}^{\mathrm{cpr}}M) \cong \Omega_{\mathrm{cir}}^{i}\Gamma_{\mathfrak{a}}^{\mathrm{stab}}(M).$$

Proof. As $\Gamma_{\mathfrak{a}}^{\mathrm{stab}}(-) \simeq Z^0\Gamma_{\mathfrak{a}}(\mathrm{CIR}(-))$, it is enough to show $Z^0(-) : \mathrm{K_{ac}}(\mathrm{Inj}\, R) \to \underline{\mathrm{GInj}}(R)$, $\Gamma_{\mathfrak{a}}(-) : \mathrm{K_{ac}}(\mathrm{Inj}\, R) \to \mathrm{K_{ac}}(\mathrm{Inj}\, R)$, and $\mathrm{CIR}(-) : \underline{\mathrm{MCM}}(R) \to \mathrm{K_{ac}}(\mathrm{Inj}\, R)$ are triangulated functors. The first two functors are triangulated by [Ste14, Hap88] and [Lip09, 1.5.2], respectively. Recall that $\mathrm{CIR}(-)$ is naturally isomorphic to

$$\mathrm{Hom}_R(\mathrm{CPR}((-)^*), D)$$

(by Proposition 2.2.2), where D is a minimal injective resolution for R. Note that $(-)^*$ and $\mathrm{Hom}_R(-, D)$ are triangulated by [Lip09, 1.5.2 and 1.5.3], resp.), and by [Buc86, Theorem 4.4.1] we have $\mathrm{CPR}(-)$ is triangulated. Composing all of these pieces shows that $\Gamma_{\mathfrak{a}}^{\mathrm{stab}}(-) : \underline{\mathrm{MCM}}(R) \to \underline{\mathrm{GInj}}(R)$ is a triangulated functor.

For a MCM R-module M, this then gives for any $i \in \mathbb{Z}$ that $\Gamma_{\mathfrak{a}}^{\mathrm{stab}}(\Omega_{-i}^{\mathrm{cpr}}M) \simeq \Omega_{\mathrm{cir}}^{i}\Gamma_{\mathfrak{a}}^{\mathrm{stab}}(M)$, and as both of these modules are reduced, by Remark 3.0.10 we can conclude they are isomorphic as R-modules. □

Remark 3.0.15. Recall that an equivalent way of defining (classical) local cohomology is as a direct limit. We have a natural isomorphism [ILL$^+$07, Theorem 7.8]:

$$H_{\mathfrak{a}}^i(M) \cong \varinjlim \mathrm{Ext}_R^i(R/\mathfrak{a}^n, M).$$

There is a "stable" Ext module, namely $\widehat{\mathrm{Ext}}_R^i(M, N)$, which is defined by replacing M by a complete projective resolution, or equivalently, by replacing N by a complete injective resolution. It is natural to ask, then, why we would not define stable local cohomology in an analogous way, i.e., as $\varinjlim \widehat{\operatorname{Ext}}^i_R(R/\mathfrak{a}^n, M)$, or whether this is naturally isomorphic to the construction above. Quite simply, it's not: For an R-module M that has a complete injective resolution U,

$$\varinjlim \widehat{\operatorname{Ext}}^i_R(R/\mathfrak{a}^n, M) = 0$$

for all $i \in \mathbb{Z}$. Using the fact that $H^i(-)$ commutes with filtered limits, see [ILL+07, Theorem 4.33 and following comments], we have

$$\begin{aligned}
\varinjlim \widehat{\operatorname{Ext}}^i_R(R/\mathfrak{a}^n, M) &\cong \varinjlim H^i \operatorname{Hom}_R(R/\mathfrak{a}^n, U), \text{ by [CJ14, Theorem 5.4]}, \\
&\cong H^i \varinjlim \operatorname{Hom}_R(R/\mathfrak{a}^n, U) \\
&\cong H^i \Gamma_{\mathfrak{a}}(U) \\
&= 0,
\end{aligned}$$

where the last equality follows because $\Gamma_{\mathfrak{a}}(U)$ is acyclic (Lemma 3.0.1).

We now examine some of the special cases of Definition 3.0.4, which may shed some light on why this seems to be the best approach for such a definition. We will end the chapter with some relations among stable local cohomology modules that reflect analogous results in (classical) local cohomology.

3.1 Stable local cohomology at the maximal ideal

We consider first the extremal case of $\Gamma^{\text{stab}}_{\mathfrak{m}}(-)$, where $\mathfrak{m}$ is the maximal ideal of the d-dimensional local Gorenstein ring $(R, \mathfrak{m})$. Recall that in this case, for a MCM R-module M, $H^d_{\mathfrak{m}}(M) \cong M \otimes_R H^d_{\mathfrak{m}}(R) \cong M \otimes E_R(R/\mathfrak{m})$, and all other local cohomology modules vanish. In this case, $H^d_{\mathfrak{m}}(M)$ is a Gorenstein injective module [Saz04], and so is already stable in the sense we are looking for. Since $H^d_{\mathfrak{m}}(M)$ comes to us in degree d, we would therefore expect $H^d_{\mathfrak{m}}(M)$ to coincide with $\Omega^d_{\mathrm{cir}}\Gamma^{\mathrm{stab}}_{\mathfrak{m}}(M)$ (in $\underline{\mathrm{GInj}}(R)$).

We first find a more explicit computation for $\Gamma^{\mathrm{stab}}_{\mathfrak{m}}(M)$, for M a MCM R-module with no nonzero free summands.

Proposition 3.1.1. *Let $(R,\mathfrak{m})$ be a Gorenstein local ring of dimension d, M a MCM R-module with no nonzero free summands, and D a minimal injective resolution for R. Then for $i \in \mathbb{Z}$,*

$$\Gamma^{\mathrm{stab}}_{\mathfrak{m}}(\Omega^{\mathrm{cpr}}_{-i}M) \simeq \Omega^{\mathrm{cpr}}_{d-i}M \otimes E(R/\mathfrak{m}).$$

In particular, $\Gamma^{\mathrm{stab}}_{\mathfrak{m}}(M) \simeq \Omega^{\mathrm{cpr}}_{d}M \otimes E(R/\mathfrak{m})$.

Proof. Let M be MCM with no nonzero free summands and set $\mathrm{cpr}(M) = S$. Since M has no nonzero free summands, the remarks following Construction 2.1.8 show that $M \cong \Omega^0(S)$. Proposition 3.0.12 then yields:

$$\begin{aligned}\Gamma^{\mathrm{stab}}_{\mathfrak{m}}(M) &\simeq Z^1(S \otimes_R \Gamma_{\mathfrak{m}}(D))\\ &\cong Z^1(\cdots \to \underbrace{S_{d-1} \otimes E(R/\mathfrak{m})}_{\text{degree 1}} \to \underbrace{S_{d-2} \otimes E(R/\mathfrak{m})}_{\text{degree 2}} \to \cdots)\\ &\cong \Omega^{\mathrm{cpr}}_{d}M \otimes E(R/\mathfrak{m}),\end{aligned}$$

where the last isomorphism follows from S being totally acyclic, hence $S \otimes E(R/\mathfrak{m})$ is acyclic as well (by Remark 2.1.3). Finally, we remark that $\Omega^{\mathrm{cpr}}_{d}\Omega^{\mathrm{cpr}}_{-i}(M) \cong \Omega^{\mathrm{cpr}}_{d-i}(M)$ for $i \in \mathbb{Z}$, so

$$\Gamma^{\mathrm{stab}}_{\mathfrak{m}}(\Omega^{\mathrm{cpr}}_{-i}M) \simeq \Omega^{\mathrm{cpr}}_{d}\Omega^{\mathrm{cpr}}_{-i}M \otimes E(R/\mathfrak{m}) \cong \Omega^{\mathrm{cpr}}_{d-i}M \otimes E(R/\mathfrak{m}).$$

□

Remark 3.1.2. Now it's easy to see that $\Omega^d_{\mathrm{cir}}\Gamma^{\mathrm{stab}}_{\mathfrak{m}}(M)$ and $H^d_{\mathfrak{m}}(M)$ agree in the above setting. Let M be a MCM R-module with no nonzero free summands. Then

$$\begin{aligned}\Omega^d_{\mathrm{cir}}\Gamma^{\mathrm{stab}}_{\mathfrak{m}}(M) &\cong \Gamma^{\mathrm{stab}}_{\mathfrak{m}}(\Omega^{-d}_{\mathrm{cir}}M), \text{ by Proposition 3.0.14,}\\ &\simeq \Omega^{\mathrm{cpr}}_{d-d}M \otimes E(R/\mathfrak{m}), \text{ by Proposition 3.1.1,}\\ &\simeq M \otimes E(R/\mathfrak{m})\\ &\cong H^d_{\mathfrak{m}}(M),\end{aligned}$$

so stable and classical local cohomology do indeed coincide in $\underline{\mathrm{GInj}}(R)$ in this situation (as well as in more generality, see ahead to Corollary 5.0.8). In fact, $\Gamma^{\mathrm{stab}}_{\mathfrak{a}}(M)$ and $H^d_{\mathfrak{m}}(M)$ are isomorphic as R-modules if $H^d_{\mathfrak{m}}(M)$ is reduced.

3.2 Stable local cohomology at a height $d-1$ prime ideal

Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension d and $\mathfrak{q}$ a prime ideal of height $d-1$. Let M be a MCM R-module with no nonzero free summands. In what follows, $\otimes = \otimes_R$, unless otherwise specified. Let $T = \mathrm{cpr}(M)$. By Proposition 3.0.12 and Lemma 3.0.11, we have that $\Gamma^{\mathrm{stab}}_{\mathfrak{q}}(M) \simeq Z^1(T \otimes \Gamma_{\mathfrak{q}}(D))$, where D is a minimal injective resolution for R. Since $T \otimes D$ is not necessarily a minimal complete injective resolution for M (see Example 2.2.6), we will only consider $\Gamma^{\mathrm{stab}}_{\mathfrak{q}}(M) \in \underline{\mathrm{GInj}}(R)$. As R is Gorenstein, we have $\Gamma_{\mathfrak{q}}(D) \cong (\cdots 0 \to E(R/\mathfrak{q}) \xrightarrow{\partial} E(R/\mathfrak{m}) \to 0 \to \cdots)$, concentrated in degrees $d-1$ and d, with differential induced by that of D. Set τ as the differential on T. Then we have $T \otimes \Gamma_{\mathfrak{q}}(D)$ is the direct sum totalization of the following (commutative) double complex:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots \longrightarrow & & T_{d-1}\otimes E(R/\mathfrak{q}) & \xrightarrow{\tau_{d-1}\otimes 1} & T_{d-2}\otimes E(R/\mathfrak{q}) & \xrightarrow{\tau_{d-2}\otimes 1} & T_{d-3}\otimes E(R/\mathfrak{q}) & \longrightarrow \cdots & \\
 & & \downarrow{\scriptstyle 1\otimes\partial} & & \downarrow{\scriptstyle 1\otimes\partial} & & \downarrow{\scriptstyle 1\otimes\partial} & & \\
\cdots \longrightarrow & & T_{d-1}\otimes E(R/\mathfrak{m}) & \xrightarrow{\tau_{d-1}\otimes 1} & T_{d-2}\otimes E(R/\mathfrak{m}) & \xrightarrow{\tau_{d-2}\otimes 1} & T_{d-3}\otimes E(R/\mathfrak{m}) & \longrightarrow \cdots & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Note that T_i lives in cohomological degree $-i$, $E(R/\mathfrak{q})$ in degree $d-1$ and $E(R/\mathfrak{m})$ in degree d. So we get that $T \otimes \Gamma_{\mathfrak{q}}(D) =$

$$\cdots \to \underbrace{\begin{array}{c} T_d\otimes E(R/\mathfrak{m}) \\ \oplus \\ T_{d-1}\otimes E(R/\mathfrak{q}) \end{array}}_{\text{degree } 0} \xrightarrow{\begin{pmatrix} \tau_d\otimes 1 & 1\otimes\partial \\ 0 & \tau_{d-1}\otimes 1 \end{pmatrix}} \underbrace{\begin{array}{c} T_{d-1}\otimes E(R/\mathfrak{m}) \\ \oplus \\ T_{d-2}\otimes E(R/\mathfrak{q}) \end{array}}_{\text{degree } 1} \xrightarrow{\begin{pmatrix} \tau_{d-1}\otimes 1 & 1\otimes\partial \\ 0 & \tau_{d-2}\otimes 1 \end{pmatrix}} \underbrace{\begin{array}{c} T_{d-2}\otimes E(R/\mathfrak{m}) \\ \oplus \\ T_{d-3}\otimes E(R/\mathfrak{q}) \end{array}}_{\text{degree } 2} \to \cdots$$

Hence we have (with $\simeq$ representing isomorphism in $\underline{\text{GInj}}(R)$)

$$\Gamma_{\mathfrak{q}}^{\text{stab}}(M) \simeq \ker \left(\underbrace{\begin{array}{c} T_{d-1}\otimes E(R/\mathfrak{m}) \\ \oplus \\ T_{d-2}\otimes E(R/\mathfrak{q}) \end{array}}_{\text{degree } 1} \to \underbrace{\begin{array}{c} T_{d-2}\otimes E(R/\mathfrak{m}) \\ \oplus \\ T_{d-3}\otimes E(R/\mathfrak{q}) \end{array}}_{\text{degree } 2} \right),$$

and also a commuting diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & T_{d-1} \otimes E(R/\mathfrak{m}) & \xhookrightarrow{\binom{1}{0}} & \begin{array}{c} T_{d-1} \otimes E(R/\mathfrak{m}) \\ \oplus \\ T_{d-2} \otimes E(R/\mathfrak{q}) \end{array} & \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} & T_{d-2} \otimes E(R/\mathfrak{q}) & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle \tau_{d-1} \otimes 1} & & \Big\downarrow {\scriptstyle \begin{pmatrix} \tau_{d-1} \otimes 1 & 1 \otimes \partial \\ 0 & \tau_{d-2} \otimes 1 \end{pmatrix}} & & \Big\downarrow {\scriptstyle \tau_{d-2} \otimes 1} & & \\
0 & \longrightarrow & T_{d-2} \otimes E(R/\mathfrak{m}) & \xhookrightarrow{\binom{1}{0}} & \begin{array}{c} T_{d-2} \otimes E(R/\mathfrak{m}) \\ \oplus \\ T_{d-3} \otimes E(R/\mathfrak{q}) \end{array} & \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} & T_{d-3} \otimes E(R/\mathfrak{q}) & \longrightarrow & 0
\end{array}
$$

The snake lemma then provides an exact sequence relating the kernels and cokernels. For any injective module E, by [Mur13, Lemma 4.5], we have the kernel of $T_{d-i} \otimes E \to T_{d-i-1} \otimes E$ is $\Omega^{\mathrm{cpr}}_{d-i+1} M \otimes E$ and the cokernel of the same map is $\Omega^{\mathrm{cpr}}_{d-i-1} M \otimes E$. But then note that the connecting map in the above snake diagram is zero, hence we have an induced short exact sequence of R-modules:

$$
0 \to \Omega^{\mathrm{cpr}}_{d} M \otimes E(R/\mathfrak{m}) \to \ker \begin{pmatrix} \tau_{d-1} \otimes 1 & 1 \otimes \partial \\ 0 & \tau_{d-2} \otimes 1 \end{pmatrix} \to \Omega^{\mathrm{cpr}}_{d-1} M \otimes E(R/\mathfrak{q}) \to 0 \tag{3.2.1}
$$

(where these R-modules are occurring as the kernels of the vertical maps above).

In $\underline{\mathrm{GInj}}(R)$, the short exact sequence 3.2.1 of R-modules induces a distinguished triangle:

$$\Omega_d^{\text{cpr}} M \otimes E(R/\mathfrak{m}) \to \Gamma_{\mathfrak{q}}^{\text{stab}}(M) \to \Omega_{d-1}^{\text{cpr}} M \otimes E(R/\mathfrak{q}) \to \Omega_{\text{cir}}^1(\Omega_d^{\text{cpr}} M \otimes E(R/\mathfrak{m})). \tag{3.2.2}$$

Lemma 3.2.3. *Using notation from above, we have the following isomorphism in* $\underline{\text{GInj}}(R)$*:*

$$\Omega_{d-1}^{\text{cpr}} M \otimes E_R(R/\mathfrak{q}) \simeq \Gamma_{\mathfrak{q}}^{\text{stab}}(M_{\mathfrak{q}}).$$

Proof. Recall that $E_R(R/\mathfrak{q})$ is $\mathfrak{q}$-local, and so $E_R(R/\mathfrak{q}) \cong E_R(R/\mathfrak{q})_{\mathfrak{q}} \cong E_{R_{\mathfrak{q}}}(R_{\mathfrak{q}}/\mathfrak{q}R_{\mathfrak{q}})$, and so we have

$$\begin{aligned} \Omega_{d-1}^{\text{cpr}} M \otimes_R E_R(R/\mathfrak{q}) &\cong \Omega_{d-1}^{\text{cpr}} M \otimes_R R_{\mathfrak{q}} \otimes_{R_{\mathfrak{q}}} E_{R_{\mathfrak{q}}}(R_{\mathfrak{q}}/\mathfrak{q}R_{\mathfrak{q}}) \\ &\simeq \Omega_{d-1}^{\text{cpr}} M_{\mathfrak{q}} \otimes_{R_{\mathfrak{q}}} E_{R_{\mathfrak{q}}}(R_{\mathfrak{q}}/\mathfrak{q}R_{\mathfrak{q}}) \\ &\simeq \Gamma_{\mathfrak{q}R_{\mathfrak{q}}}^{\text{stab}}(M_{\mathfrak{q}}), \end{aligned}$$

where the last isomorphism in $\underline{\text{GInj}}(R)$ comes from applying Proposition 3.1.1 to the $(d-1)$-dimensional Gorenstein local ring $(R_{\mathfrak{q}}, \mathfrak{q}R_{\mathfrak{q}})$. Notationally, we usually just write this as $\Gamma_{\mathfrak{q}}^{\text{stab}}(M_{\mathfrak{q}})$ with the ideal $\mathfrak{q}$ here understood to be taken as an ideal of $R_{\mathfrak{q}}$ and $M_{\mathfrak{q}}$ considered as an $R_{\mathfrak{q}}$-module. □

Proposition 3.2.4. *Let* $(R, \mathfrak{m})$ *be a Gorenstein local ring of dimension* d*, with* $\mathfrak{q}$ *a prime of height* $d-1$*. Let* M *be a MCM* R*-module with no nonzero free summands. Then there exists a distinguished triangle in* $\underline{\text{GInj}}(R)$*:*

$$\Gamma_{\mathfrak{m}}^{\text{stab}}(M) \to \Gamma_{\mathfrak{q}}^{\text{stab}}(M) \to \Gamma_{\mathfrak{q}}^{\text{stab}}(M_{\mathfrak{q}}) \to \Omega_{\text{cir}}^1 \Gamma_{\mathfrak{m}}^{\text{stab}}(M).$$

Proof. Apply Proposition 3.1.1 and Lemma 3.2.3 to the distinguished triangle 3.2.2

to obtain the result. □

3.3 Short exact sequence in stable local cohomology

We now obtain a short exact sequence in stable local cohomology relating $\Gamma_{\mathfrak{a}}^{\text{stab}}(-)$ and $\Gamma_{(\mathfrak{a},x)}^{\text{stab}}(-)$ where $\mathfrak{a}$ is any ideal and $x \in R$ any element.

Remark 3.3.1. Localization at a multiplicatively closed set preserves (minimal) injective resolutions [Bas62, Corollary 1.3], and hence in a Gorenstein ring, by the remarks prior to Lemma 3.0.1, also (minimal) complete injective resolutions. Localization at an element preserves (minimal) complete injective resolutions in any commutative Noetherian ring (by Lemma 3.0.1).

Proposition 3.3.2. *Let R be a Gorenstein ring of dimension d, M any R-module, $\mathfrak{a}$ any ideal of R and $x \in R$ any element. Set $\mathfrak{b} = (\mathfrak{a}, x)$. Then there exists a short exact sequence of R-modules*

$$0 \to \Gamma_{\mathfrak{b}}^{\text{stab}}(M) \to \Gamma_{\mathfrak{a}}^{\text{stab}}(M) \to \Gamma_{\mathfrak{a}}^{\text{stab}}(M_x) \to 0.$$

Proof. Choose a minimal complete injective resolution $M \to I \to U$ of M. We then have an exact sequence of complexes (see remarks in [HT07] before Theorem 3.2):

$$0 \to \Gamma_x(U) \to U \to U_x \to 0.$$

Applying $\Gamma_{\mathfrak{a}}(-)$, truncating the resulting complexes at 0, and taking cohomology gives the desired short exact sequence (noting that U_x is a minimal complete injective resolution of M_x by Remark 3.3.1 and $\Gamma_{\mathfrak{a}} \circ \Gamma_x = \Gamma_{\mathfrak{b}}$). □

Corollary 3.3.3. *In $\underline{\mathrm{GInj}}(R)$, under the same hypotheses as Proposition 3.3.2, we have the following distinguished triangle:*

$$\Gamma_{\mathfrak{b}}^{\mathrm{stab}}(M) \to \Gamma_{\mathfrak{a}}^{\mathrm{stab}}(M) \to \Gamma_{\mathfrak{a}}^{\mathrm{stab}}(M_x) \to \Omega_{\mathrm{cir}}^{1}\Gamma_{\mathfrak{b}}^{\mathrm{stab}}(M).$$

3.4 Extension of Stevenson's functor

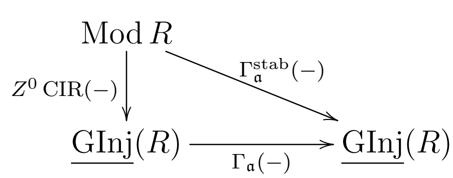
Let R be a Gorenstein ring. Greg Stevenson considers in [Ste14], for any ideal $\mathfrak{a} \subset R$,

$$\Gamma_{\mathfrak{a}}(-) : \mathrm{K}_{\mathrm{ac}}(\mathrm{Inj}\, R) \to \mathrm{K}_{\mathrm{ac}}(\mathrm{Inj}\, R),$$

which takes an acyclic complex of injectives U to an acyclic complex of injectives $\Gamma_{\mathfrak{a}}(U)$ where the degree i piece consists of those indecomposable injectives corresponding to primes in $V(\mathfrak{a})$, i.e., primes containing $\mathfrak{a}$ (although he uses the notation $\Gamma_{V(\mathfrak{a})}(-)$ for $\Gamma_{\mathfrak{a}}(-)$). Via the equivalence $\mathrm{K}_{\mathrm{ac}}(\mathrm{Inj}\, R) \to \underline{\mathrm{GInj}}(R)$ sending $X \mapsto Z^0(X)$, he considers $\Gamma_{\mathfrak{a}}(-)$ as a functor

$$\Gamma_{\mathfrak{a}}(-) : \underline{\mathrm{GInj}}(R) \to \underline{\mathrm{GInj}}(R),$$

i.e., for a Gorenstein injective module G with complete injective resolution U, $\Gamma_{\mathfrak{a}}(G) = Z^0\Gamma_{\mathfrak{a}}(U)$. The functor $\Gamma_{\mathfrak{a}}^{\mathrm{stab}}(-)$ is a lifting of this, such that the following diagram commutes:



i.e., for any R-module M,

$$\Gamma_{\mathfrak{a}}^{\mathrm{stab}}(M) = Z^0\Gamma_{\mathfrak{a}}(\mathrm{CIR}(M)) \cong \Gamma_{\mathfrak{a}}(Z^0\,\mathrm{CIR}(M)).$$

If G is a Gorenstein injective R-module, then $Z^0\,\mathrm{CIR}(G) \simeq G$, hence $\Gamma_{\mathfrak{a}}^{\mathrm{stab}}(G) \simeq \Gamma_{\mathfrak{a}}(G)$ in $\underline{\mathrm{GInj}}(R)$ (and $\Gamma_{\mathfrak{a}}^{\mathrm{stab}}(G) \cong \Gamma_{\mathfrak{a}}(G)$ if G is reduced Gorenstein injective).

Chapter 4

The hypersurface case

Let Q be a regular local ring, $f \in Q$ a non-zerodivisor and $R = Q/(f)$. Referring to [Wal14, DM13], we let $[\mathrm{LF}(Q, f)]$ denote the homotopy category of linear factorizations, and $[\mathrm{mf}(Q, f)]$, $[\mathrm{MF}(Q, f)]$, $[\mathrm{IF}(Q, f)]$ denote the full subcategories of finitely generated matrix factorizations, not necessarily finitely generated matrix factorizations, and injective factorizations, respectively. Dual to the notion of MCM modules being cokernels of finitely generated matrix factorizations [Eis80], Gorenstein injective modules appear as kernels of injective factorizations. More precisely, Walker proves the following (as this has not appeared publicly, we include his proof below):

Theorem 4.0.1. *[Wal14] For a regular ring Q and non-zerodivisor $f \in Q$, the functor*

$$\ker : [\mathrm{IF}(Q, f)] \to \underline{\mathrm{GInj}}(R)$$

(that sends an object $(I_1 \rightleftarrows I_0)$ of $\mathrm{IF}(Q, f)$ to $\ker(I_0 \to I_1)$) is an equivalence of triangulated categories, where $R = Q/(f)$.

Proof. Since an endomorphism of an injective module determined by a non-zerodivisor is surjective, the maps α and β in an injective factorization $(I_1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} I_0)$ are surjective. In particular, this yields a short exact sequence

$$0 \to \ker(\beta) \to I_0 \xrightarrow{\beta} I_1 \to 0$$

over Q. Since $fx = \alpha\beta x = 0$ for all $x \in \ker(\beta)$, $\ker(\beta)$ is an R-module. Then $\mathrm{id}_Q \ker(\beta) \leq 1$ implies, by [BM10, Theorem 4.2], that $\mathrm{Gid}_R \ker(\beta) \leq 0$, hence $\ker(\beta)$ is Gorenstein injective. We obtain a functor $\mathrm{IF}(Q, f) \to \underline{\mathrm{GInj}}(R)$. This functor sends the difference of homotopic maps of injective factorizations to a map that factors through an injective module (given by the homotopy), hence we have an induced functor $\ker : [\mathrm{IF}(Q, f)] \to \underline{\mathrm{GInj}}(R)$.

On the other hand, this functor factors though $\mathrm{K_{ac}}(\mathrm{Inj}\, R)$ in the following manner. For an injective Q-module I, define $I^R = \mathrm{Hom}_Q(R, I)$, clearly seen to be an injective R-module. Given a map $\alpha : I_1 \to I_0$ of injective Q-modules, let α^R denote the induced map of R-modules from I_1^R to I_0^R. Observe that I^R is a Q-submodule of I and α^R is the restriction of α. For $\mathbb{I} = (I_1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} I_0)$ in $\mathrm{IF}(Q, f)$,

$$\mathbb{I}^R := \left(\cdots \xrightarrow{\alpha^R} I_0^R \xrightarrow{\beta^R} I_1^R \xrightarrow{\alpha^R} I_0^R \xrightarrow{\beta^R} \cdots \right)$$

is an acyclic complex (since α and β are surjective). The assignment

$$\mathbb{I} \mapsto \mathbb{I}^R$$

yields a functor $\mathrm{IF}(Q, f) \to \mathrm{K_{ac}}(\mathrm{Inj}\, R)$, and it clearly preserves homotopies and hence induces a functor on the associated homotopy categories, $(-)^R : [\mathrm{IF}(Q, f)] \to \mathrm{K_{ac}}(\mathrm{Inj}\, R)$. The induced functor $(-)^R$ commutes with suspensions and mapping cones and hence is triangulated. Note that as $\ker(\beta)$ is an R-module, $\ker(\beta) = \ker(\beta^R)$. Given $\mathbb{I} = (\ I_1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} I_0\ ) \in [\mathrm{IF}(Q,f)]$,

$$\ker(\mathbb{I}) = \ker(\beta) = \ker(\beta^R) = Z^0(\mathbb{I}^R),$$

yielding a commutative diagram of functors, where $Z^0 : \mathrm{K}_{\mathrm{ac}}(\mathrm{Inj}\, R) \to \underline{\mathrm{GInj}}(R)$ is a triangulated equivalence by [Ste14, Proposition 4.7]:

$$\begin{array}{ccc} [\mathrm{IF}(Q,f)] & \xrightarrow{\quad\ker\quad} & \underline{\mathrm{GInj}}(R) \\ & {\scriptstyle (-)^R}\searrow \qquad \nearrow{\scriptstyle Z^0,\ \sim} & \\ & \mathrm{K}_{\mathrm{ac}}(\mathrm{Inj}\, R) & \end{array}$$

The triangulated structure on $\underline{\mathrm{GInj}}(R)$ is by definition taken to be inherited from $\mathrm{K}_{\mathrm{ac}}(\mathrm{Inj}\, R)$, and we therefore have that $\ker : [\mathrm{IF}(Q,f)] \to \underline{\mathrm{GInj}}(R)$ is a triangulated functor. It remains to show that ker is essentially surjective and fully faithful.

Given a Gorenstein injective R-module $M \neq 0$, it is straightforward from [BM10, Theorem 4.2] to see that $\mathrm{id}_Q M \leq 1$. There then exists a Q-injective resolution $0 \to M \to I_0 \xrightarrow{\beta} I_1 \to 0$ of M. Since multiplication by f on M is 0, there is a unique map $\alpha : I_1 \to I_0$ such that $\alpha\beta$ is multiplication by f on I_0. Note that $\beta\alpha\beta = f\beta$ and hence $f = \beta\alpha$ since β is surjective. Thus $(\ I_1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} I_0\ )$ is an object of $\mathrm{IF}(Q,f)$ with $\ker(\beta) = M$, hence ker is essentially surjective.

For the remainder of the proof, set $\mathbb{I} = (\ I_1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} I_0\ )$ and $\mathbb{I}' = (\ I_1' \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} I_0'\ )$. Suppose $g : \ker(\beta) \to \ker(\beta')$ is a morphism in $\underline{\mathrm{GInj}}(R)$. Then we may find maps $g_j : I_j \to I_j'$ for $j = 0, 1$ such that $\beta' g_0 = g_1 \beta$. An easy diagram chase shows that the g_j's also commute with the induced maps α, α', and hence the g_j's determine a morphism of linear factorizations from $\mathbb{I}$ to $\mathbb{I}'$ with $g_0|_{\ker(\beta)} = g$. This shows ker is a full functor.

Finally, suppose $h : \mathbb{I} \to \mathbb{I}'$ is a morphism such that $h : \ker(\beta) \to \ker(\beta')$ factors through an injective R-module, say J. We may find a Q-injective resolution $0 \to J \to E_0 \xrightarrow{\gamma} E_1 \to 0$ and construct an injective factorization $\mathbb{E} = (E_1 \underset{\gamma}{\overset{\delta}{\rightleftarrows}} E_0)$. By uniqueness up to homotopy equivalence of Q-injective resolutions, $h_j : I_j \to I'_j$ factors through E_j for $j = 0, 1$ (up to homotopy equivalence), and moreover, $h : \mathbb{I} \to \mathbb{I}'$ factors through $\mathbb{E}$ (up to homotopy equivalence). Next, setting $E = E_Q(J)$, we claim that

$$0 \to J \to E \xrightarrow{f} E \to 0$$

is also an injective resolution of J. Since f is a non-zerodivisor and E is an injective Q-module, $f : E \to E$ is onto. The only thing left to check is that $J = K := \ker(f : E \to E)$. We have $J \subseteq K$, since J is annihilated by f, and it is clear that K is an R-module. Given any nonzero R-submodule N of K, N is also a Q-submodule of E and hence, since $J \to E$ is essential, we have $N \cap J \neq 0$. This proves $J \to K$ is an essential extension of R-modules and hence, since J is injective, $J = K$. Set $\mathbb{E}' = (E \underset{f}{\overset{1}{\rightleftarrows}} E)$ as the corresponding injective factorization. But $\mathbb{E}'$ is contractible and $\mathbb{E}' \simeq \mathbb{E}$, so $h : \mathbb{I} \to \mathbb{I}'$ factors (in the homotopy category $[\mathrm{IF}(Q, f)]$) through a contractible object, hence h is null-homotopic, so ker is faithful. Therefore $\ker : [\mathrm{IF}(Q, f)] \to \underline{\mathrm{GInj}}(R)$ is an equivalence of triangulated categories. □

When Q and R are as above, and $M \in \mathrm{GPrj}(R)$ (or, in particular when M is MCM), we will compute $\Gamma^{\mathrm{stab}}_{\mathfrak{a}}(M)$ by utilizing the equivalence $[\mathrm{MF}(Q, f)] \to \underline{\mathrm{GPrj}}(R)$ given in Lemma 4.0.3. This yields a plethora of concrete examples of stable local cohomology of MCM modules over a hypersurface. Before proceeding, we need a few lemmas. Compare the following with [Swa, Proposition 23.6]:

Lemma 4.0.2. *Let S be a commutative Noetherian ring, M an S-module, and x*

a non-zerodivisor on S and on M. If $M \to I$ is a minimal injective resolution of M, then there is a canonical induced map $M/xM \to \Sigma^1 \operatorname{Hom}_S(S/xS, I)$ and it is a minimal injective resolution of M/xM.

Proof. Applying $\operatorname{Hom}_S(S/xS, -)$ to I, we obtain a short exact sequence of complexes:

$$0 \to \operatorname{Hom}_S(S/xS, I) \to I \xrightarrow{x} I \to 0.$$

Note that $\operatorname{Hom}_S(S/xS, I^0) = 0$ (otherwise, since $I^0 \cong E_R(M)$, we would have $(0 :_{E_R(M)} x) \cap M \neq 0$, contradicting x being a non-zerodivisor on M). The long exact sequence in cohomology yields a short exact sequence:

$$0 \to M \xrightarrow{x} M \to \operatorname{Ext}^1_S(S/xS, M) \to 0.$$

Therefore, we have a canonical injection $M/xM \cong \operatorname{Ext}^1_S(S/xS, M) \to \operatorname{Hom}_S(S/xS, I^1)$, which implies $M/xM \to \Sigma^1 \operatorname{Hom}_S(S/xS, I)$ is an injective resolution. Minimality of $\operatorname{Hom}_S(S/xS, I)$ follows by definition and minimality of I: for $i \geq 1$, if $0 \neq N \subseteq (0 :_{I^i} x) \subseteq I^i$, then $N \cap \ker(\partial^i_I) \neq 0$, hence $N \cap (0 :_{\ker(\partial^i_I)} x) \neq 0$, so the complex $\operatorname{Hom}_S(S/xS, I)$ is minimal as well. ☐

In particular, for a regular local ring Q, $f \in Q$ a non-zerodivisor, and $R = Q/(f)$, if $Q \to D_Q$ is a minimal injective resolution of Q, then $R \to \Sigma^1 \operatorname{Hom}_Q(R, D_Q)$ is a minimal injective resolution of R.

The following lemma extends the classical result that for a hypersurface $R = Q/(f)$, $\operatorname{coker} : [\operatorname{mf}(Q, f)] \to \underline{\operatorname{MCM}}(R)$ is an equivalence [Eis80, Corollary 6.3].

Lemma 4.0.3. *Let Q be a regular local ring, $f \in Q$ a non-zerodivisor, and $R =$*

$Q/(f)$. *Then*

$$\mathrm{coker} : [\mathrm{MF}(Q, f)] \to \underline{\mathrm{GPrj}}(R)$$

is an equivalence, where if $P = (\ P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0\ ) \in \mathrm{MF}(Q, f)$, *then* $\mathrm{coker}(P) = \mathrm{coker}(d_1)$.

Proof. We omit the proof, as it is completely analogous to the proof that $[\mathrm{mf}(Q, f)] \to \underline{\mathrm{MCM}}(R)$ is an equivalence [Orl04, proof of Proposition 3.7], except one needs the additional fact that a nonzero Gorenstein projective R-module G (not necessarily finitely generated) has $\mathrm{pd}_Q\, G = 1$ [BM10, Theorem 4.1]. □

Lemma 4.0.4. *Let* R *be a Gorenstein ring. Then* $\underline{\mathrm{GPrj}}(R) \underset{\Omega_0(-)}{\overset{\mathrm{CPR}(-)}{\rightleftarrows}} \mathrm{K_{ac}}(\mathrm{Prj}\, R)$ *is an equivalence.*

Proof. We mirror the proof Buchweitz gives for showing $\Omega_0 : \mathrm{K_{ac}}(\mathrm{prj}\, R) \to \underline{\mathrm{MCM}}(R)$ is an equivalence [Buc86, Theorem 4.4.1]. By definition, if $P \in \mathrm{K_{ac}}(\mathrm{Prj}\, R)$, $\Omega_0(P)$ is a Gorenstein projective R-module; conversely, given a Gorenstein projective R-module G, the definition implies there exists $P \in \mathrm{K_{ac}}(\mathrm{Prj}\, R)$ such that $\Omega_0(P) = G$, hence Ω_0 is an essentially surjective functor.

Showing Ω_0 is fully faithful follows from [AM02, Lemma 5.3]: If $S, T \in \mathrm{K_{ac}}(\mathrm{Prj}\, R)$ and $f : \Omega_0 S \to \Omega_0 T$ is any map, then there exists a unique up to homotopy map $\widetilde{f} : S \to T$ such that the diagram

$$\begin{array}{ccccc} S & \longrightarrow & S_{\geq 0} & \longrightarrow & \Omega_0 S \\ \downarrow{\scriptstyle \widetilde{f}} & & \downarrow{\scriptstyle \widetilde{f}_{\geq 0}} & & \downarrow{\scriptstyle f} \\ T & \longrightarrow & T_{\geq 0} & \longrightarrow & \Omega_0 T \end{array}$$

commutes up to homotopy, and further, if $f : \Omega_0 S \to \Omega_0 T$ is an isomorphism, then $\widetilde{f} : S \to T$ is a homotopy equivalence. Since $\widetilde{f}_{-1}|_{\Omega_0 S} = f$, Ω_0 is full. On the other hand if $\alpha, \beta : S \to T$ are two maps such that their restrictions to $\Omega_0 S$ agree in

$\underline{\mathrm{GInj}}(R)$, then [AM02, Lemma 5.3] implies α and β are homotopy equivalent, hence Ω_0 is faithful.

It is straightforward to see that $\mathrm{CPR}(-)$ gives an inverse equivalence to Ω_0. □

Proposition 4.0.5. *Let Q be a regular local ring, $f \in Q$ a non-zerodivisor, and $Q \to D_Q$ a minimal injective resolution. Then the functor*

$$- \otimes_Q D_Q : [\mathrm{MF}(Q, f)] \to [\mathrm{IF}(Q, f)]$$

is an equivalence of triangulated categories which agrees with the equivalence $- \otimes_R D_R : \mathrm{K}_{\mathrm{ac}}(\mathrm{Prj}\, R) \to \mathrm{K}_{\mathrm{ac}}(\mathrm{Inj}\, R)$ [IK06], where $R = Q/(f)$ and D_R is a minimal injective resolution of R.

Proof. Set $R = Q/(f)$. Composing the equivalence $\ker : [\mathrm{IF}(Q, f)] \to \underline{\mathrm{GInj}}(R)$ from Theorem 4.0.1 with the equivalence $\mathrm{CIR} : \underline{\mathrm{GInj}}(R) \to \mathrm{K}_{\mathrm{ac}}(\mathrm{Inj}\, R)$ [Ste14, Proposition 4.7], we have $\mathrm{Hom}_Q(R, -) : [\mathrm{IF}(Q, f)] \to \mathrm{K}_{\mathrm{ac}}(\mathrm{Inj}\, R)$, and hence $\Sigma^1 \mathrm{Hom}_Q(R, -) : [\mathrm{IF}(Q, f)] \to \mathrm{K}_{\mathrm{ac}}(\mathrm{Inj}\, R)$, is an equivalence. Lemmas 4.0.3 and 4.0.4 show that the composition $\mathrm{CPR} \circ \mathrm{coker}(-) : [\mathrm{MF}(Q, f)] \to \mathrm{K}_{\mathrm{ac}}(\mathrm{Prj}\, R)$ is an equivalence, and we can interpret the composition $\mathrm{CPR} \circ \mathrm{coker}(-)$ as naturally isomorphic to modding out by f and "unfolding" the injective factorization by forgetting the 2 periodicity, we denote this simply by $- \otimes_Q R$.

Setting D_R to be a minimal injective resolution of R, Iyengar and Krause show [IK06, Theorem 4.2] that $- \otimes_R D_R : \mathrm{K}_{\mathrm{ac}}(\mathrm{Prj}\, R) \to \mathrm{K}_{\mathrm{ac}}(\mathrm{Inj}\, R)$ is an equivalence. We therefore have the following diagram, where the horizontal functors implicitly involve a forgetting of the 2 periodicity:

$$\begin{array}{ccc} [\mathrm{MF}(Q,f)] & \xrightarrow[\sim]{-\otimes_Q R} & \mathrm{K_{ac}}(\operatorname{Prj} R) \\ \Big\downarrow {\scriptstyle -\otimes_Q D_Q} & & {\scriptstyle \sim}\Big\downarrow {\scriptstyle -\otimes_R D_R} \\ [\mathrm{IF}(Q,f)] & \xrightarrow[\Sigma^1 \operatorname{Hom}_Q(R,-)]{\sim} & \mathrm{K_{ac}}(\operatorname{Inj} R) \end{array}$$

We need only show this diagram commutes. Let $\mathbb{E} \in [\mathrm{MF}(Q,f)]$. Then

$$\begin{aligned} \mathbb{E} \otimes_Q R \otimes_R D_R &\simeq \mathbb{E} \otimes_Q D_R \\ &\simeq \mathbb{E} \otimes_Q \Sigma^1 \operatorname{Hom}_Q(R, D_Q), \text{ by Lemma 4.0.2,} \\ &\simeq \Sigma^1 \operatorname{Hom}_Q(R, D_Q \otimes_Q \mathbb{E}), \text{ since } \mathbb{E} \text{ is flat.} \end{aligned}$$

This shows the diagram commutes, and therefore $-\otimes_Q D_Q : [\mathrm{MF}(Q,f)] \to [\mathrm{IF}(Q,f)]$ is an equivalence. □

Now, for a Gorenstein projective module over a hypersurface, we can equivalently compute stable local cohomology by applying $\Gamma_{\mathfrak{a}}$ to the kernel of one of the maps of the corresponding injective factorization via this equivalence. More precisely, we have

Proposition 4.0.6. *Let Q be a regular local ring, $f \in Q$ a non-zerodivisor, and $R = Q/(f)$. If $M \in \underline{\mathrm{GPrj}}(R)$ is an R-module with corresponding matrix factorization $\mathbb{E} \in [\mathrm{MF}(Q,f)]$, we have*

$$\Gamma_{\mathfrak{a}}^{\mathrm{stab}}(M) \simeq \ker(\mathbb{E} \otimes_Q \Gamma_{\mathfrak{a}}(D_Q)) \in \underline{\mathrm{GInj}}(R),$$

where D_Q is a minimal injective resolution of Q.

Proof. For $M \in \underline{\mathrm{GPrj}}(R)$, Lemma 4.0.3 allows us to find $\mathbb{E} = \left(P_1 \underset{B}{\overset{A}{\rightleftarrows}} P_0 \right) \in$

[MF(Q,f)] with coker $A = M$. Then

$$\begin{aligned}
\Gamma^{\text{stab}}_{\mathfrak{a}}(M) &\simeq Z^1\Gamma_{\mathfrak{a}}(\mathrm{CPR}(M)\otimes_R D_R), \text{ by Proposition 3.0.12,}\\
&\simeq Z^1\Gamma_{\mathfrak{a}}(\mathbb{E}\otimes_Q R\otimes_R D_R)\\
&\simeq Z^1\Gamma_{\mathfrak{a}}(\Sigma^1 \operatorname{Hom}_Q(R,\mathbb{E}\otimes_Q D_Q)), \text{ by the proof of Proposition 4.0.5,}\\
&\simeq Z^0\Gamma_{\mathfrak{a}}(\operatorname{Hom}_Q(R,\mathbb{E}\otimes_Q D_Q)), \text{ by 2-periodicity,}\\
&\simeq \Gamma_{\mathfrak{a}} Z^0 \operatorname{Hom}_Q(R,\mathbb{E}\otimes_Q D_Q)\\
&\simeq \Gamma_{\mathfrak{a}} Z^0(\mathbb{E}\otimes_Q D_Q)\\
&\simeq \ker(\mathbb{E}\otimes_Q \Gamma_{\mathfrak{a}}(D_Q)).
\end{aligned}$$

□

In particular, if $(Q^r \underset{B}{\overset{A}{\rightleftarrows}} Q^r) \in [\mathrm{mf}(Q,f)]$ and $\mathrm{coker}(A) = M$ (i.e., M is MCM), Proposition 4.0.6 allows us to easily compute $\Gamma^{\text{stab}}_{\mathfrak{m}}(M)$, where we use $\mathfrak{m}$ to denote the maximal ideal of both R and Q. Note that $\Gamma_{\mathfrak{m}}(D_Q) \cong \Sigma^{-\dim Q} E_Q(Q/\mathfrak{m})$, and $(Q^r \underset{B}{\overset{A}{\rightleftarrows}} Q^r) \otimes_Q E_Q(Q/\mathfrak{m}) = (E_Q(Q/\mathfrak{m})^r \underset{B}{\overset{A}{\rightleftarrows}} E_Q(Q/\mathfrak{m})^r)$, hence

$$\begin{aligned}
\Gamma^{\text{stab}}_{\mathfrak{m}}(M) &\simeq Z^{\dim Q}(E_Q(Q/\mathfrak{m})^r \underset{B}{\overset{A}{\rightleftarrows}} E_Q(Q/\mathfrak{m})^r)\\
&\simeq \begin{cases} \ker(A : E_Q(Q/\mathfrak{m})^r \to E_Q(Q/\mathfrak{m})^r), & \text{if } \dim Q \text{ is odd,}\\ \ker(B : E_Q(Q/\mathfrak{m})^r \to E_Q(Q/\mathfrak{m})^r), & \text{if } \dim Q \text{ is even}\end{cases}\\
&\simeq \begin{cases} \ker(A : E_R(R/\mathfrak{m})^r \to E_R(R/\mathfrak{m})^r), & \text{if } \dim Q \text{ is odd,}\\ \ker(B : E_R(R/\mathfrak{m})^r \to E_R(R/\mathfrak{m})^r), & \text{if } \dim Q \text{ is even.}\end{cases}
\end{aligned}$$

Example 4.0.7. Consider the isolated singularity $R = \frac{k[[x,y]]}{(xy)}$, where k is a field of characteristic 0. Set $\mathfrak{m} = (x,y)R$, $E = E_R(R/\mathfrak{m})$, and $M = R/(x)$. We can see that

M is a MCM R-module coming from the matrix factorization $(\ k[[x,y]] \underset{y}{\overset{x}{\rightleftarrows}} k[[x,y]] \ )$, and that we have $\Omega_1^{\mathrm{prj}} M \cong R/(y)$. By Proposition 4.0.6 and the following remarks, we have

$$\Gamma_{\mathfrak{m}}^{\mathrm{stab}}(M) \simeq \ker(E \xrightarrow{y} E) \cong E/(y)E$$

and

$$\Gamma_{\mathfrak{m}}^{\mathrm{stab}}(\Omega_1^{\mathrm{prj}} M) \simeq \ker(E \xrightarrow{x} E) \cong E/(x)E.$$

(Alternatively, this can be seen by using Proposition 3.1.1.) In fact, as the complex

$$\cdots \xrightarrow{x} E \xrightarrow{y} E \xrightarrow{x} E \xrightarrow{y} \cdots$$

is minimal, $E/(y)E$ and $E/(x)E$ are reduced R-modules, hence we obtain isomorphisms as R-modules:

$$\Gamma_{\mathfrak{m}}^{\mathrm{stab}}(R/(x)) \cong E/(y)E \text{ and } \Gamma_{\mathfrak{m}}^{\mathrm{stab}}(R/(y)) \cong E/(x)E.$$

Even more explicitly, recall that we can describe E as the k-vector space spanned by $x^i y^j$ for $i, j \leq -1$, and with a natural R-module structure (for $x^m y^n \in R$ and $x^i y^j \in E$, $x^m y^n \cdot x^i y^j = x^{m+i} y^{n+j}$ if $m+i \leq -1$ and $n+j \leq -1$, and $= 0$ otherwise, see [Lyu93, proof of Proposition 2.3]). We write this as $k\langle x^i y^j \rangle_{i,j \leq -1}$. In this way, we can see that

$$\Gamma_{\mathfrak{m}}^{\mathrm{stab}}(R/(x)) \cong k\langle x^i y^{-1} \rangle_{i \leq -1} \text{ and } \Gamma_{\mathfrak{m}}^{\mathrm{stab}}(R/(y)) \cong k\langle x^{-1} y^j \rangle_{j \leq -1},$$

both given the R-module structure described above.

Chapter 5

A bridge between stable and classical local cohomology

Before stating and proving our main connection between stable local cohomology and classical local cohomology, we recall some definitions. For an ideal $\mathfrak{a} \subseteq R$ we define the $\mathfrak{a}$-*depth* of a (not necessarily finitely generated) module M to be $\operatorname{depth}(\mathfrak{a}, M) := \inf\{i | H^i_{\mathfrak{a}}(M) \neq 0\}$. By [FI03], $\operatorname{depth}(\mathfrak{a}, M)$ coincides with $\inf\{j | \operatorname{Ext}^j_R(R/\mathfrak{a}, M) \neq 0\}$. In particular, if $(R, \mathfrak{m})$ is a local ring, we say the *depth* of M is $\operatorname{depth}(M) := \operatorname{depth}(\mathfrak{m}, M)$. We also define the *cohomological dimension* of M at $\mathfrak{a}$ to be $\operatorname{cd}(\mathfrak{a}, M) := \sup\{i | H^i_{\mathfrak{a}}(M) \neq 0\}$. For $\mathfrak{p} \in \operatorname{Spec}(R)$, for convenience we set $\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. Finally, we define the *i-th Bass number of* M with respect to $\mathfrak{p} \in \operatorname{Spec}(R)$ as $\mu^i_R(\mathfrak{p}, M) = \dim_{\kappa(\mathfrak{p})} \operatorname{Ext}^i_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), M_{\mathfrak{p}})$, which coincides [BS13, 11.1.4 and 11.1.8] with the number of copies of $E(R/\mathfrak{p})$ in I^i, where I is a minimal injective resolution of M.

Theorem 5.0.1. *Let* $(R, \mathfrak{m})$ *be a Gorenstein local ring of Krull dimension* d. *Suppose* $M \neq 0$ *is a (not-necessarily finitely generated)* R*-module where* $\operatorname{Gid}_R M = \operatorname{depth} M$ *and* $\mathfrak{a} \subset R$ *is an ideal satisfying* $c = \operatorname{depth}(\mathfrak{a}, M) = \operatorname{cd}(\mathfrak{a}, M)$. *Set* $\operatorname{Gid}_R M = t$. *Then* *there exists a short exact sequence*

$$0 \to H^c_{\mathfrak{a}}(M) \to \Gamma^{\text{stab}}_{\mathfrak{a}}(\Omega^c_{\text{inj}} M) \oplus E_R(H^c_{\mathfrak{a}}(M)) \to K \to 0,$$

where $\operatorname{id}_R K < \infty$*. Moreover, when* $0 \leq c \leq t-1$*, we have* $\operatorname{id}_R K = t - c - 1$ *and when* $c = t$*, the sequence splits and* $K \cong E_R(\Gamma^{\text{stab}}_{\mathfrak{a}}(\Omega^t_{\text{inj}} M))$*.*

Remark 5.0.2. Assume $(R, \mathfrak{m})$, M, and $\mathfrak{a}$ are as in Theorem 5.0.1. The condition that $\operatorname{depth}(\mathfrak{a}, M) = \operatorname{cd}(\mathfrak{a}, M)$ occurs if and only if there is only one nonzero local cohomology module of M with support in $\mathfrak{a}$. Finitely generated modules M such that $\operatorname{depth}(\mathfrak{a}, M) = \operatorname{cd}(\mathfrak{a}, M)$ have been studied by others (e.g., [Zar15, Definition 2.2]) under the name of *relative Cohen-Macaulay modules with respect to* $\mathfrak{a}$. If M is finitely generated in addition to satisfying the hypotheses of the theorem, then M is MCM by [CFH06, Theorem II].

There do, however, exist non-finitely generated modules M satisfying the hypotheses of Theorem 5.0.1. For instance, certain cosyzygies in minimal injective resolutions of MCM modules satisfy the hypotheses of the theorem but are not finitely generated. To see this, let $N \neq 0$ be a (finitely generated) MCM R-module with minimal injective resolution I and $\mathfrak{a}$ an ideal such that $c' = \operatorname{depth}(\mathfrak{a}, N) = \operatorname{cd}(\mathfrak{a}, N)$. For $0 < n \leq c'$, if $M = \ker(I^n \to I^{n+1})$ is a cosyzygy of N, then M is not finitely generated (otherwise $\operatorname{Gid}_R(M) = d$, contradicting $d = \operatorname{Gid}_R(N) > \operatorname{Gid}_R(M)$). To see why M satisfies the hypotheses of Theorem 5.0.1, first note that since M is an n-th cosyzygy in a minimal injective resolution of N and by [Str90, Corollary 5.2.14], $\operatorname{depth}(N) = \inf\{i | \mu^i_R(\mathfrak{m}, N) \neq 0\} = \operatorname{depth}(M) + n$, hence:

$$\operatorname{depth}(M) = \operatorname{depth}(N) - n = \operatorname{Gid}_R(N) - n = \operatorname{Gid}(M),$$

where the second equality follows again from [CFH06, Theorem II]. Again by [Str90, Corollary 5.2.14], $\operatorname{depth}(\mathfrak{a}, N) = \inf\{i | \mu_R^i(\mathfrak{p}, N) \neq 0, \mathfrak{p} \supseteq \mathfrak{a}\}$, so $\Gamma_{\mathfrak{a}}(I^i) = 0$ for $i < c'$; since $n \leq c'$, we obtain $H_{\mathfrak{a}}^i(M) = H_{\mathfrak{a}}^{i+n}(N) = 0$ for $i \neq c' - n$, and therefore $\operatorname{depth}(\mathfrak{a}, M) = \operatorname{cd}(\mathfrak{a}, M)$.

Proof of Theorem 5.0.1. Recall that by definition of $\operatorname{depth}(\mathfrak{a}, M)$ and $\operatorname{cd}(\mathfrak{a}, M)$, we have that $c = \operatorname{depth}(\mathfrak{a}, M) = \operatorname{cd}(\mathfrak{a}, M)$ if and only if $H_{\mathfrak{a}}^i(M) = 0$ for all $i \neq c$. Also $c \leq t$, since if $c > t$ we would violate Corollary 3.0.3.

Let $M \to I \to U$ be any minimal complete injective resolution (see Construction 2.1.17 for an explicit construction). Apply $\Gamma_{\mathfrak{a}}(-)$ to the map of complexes $I \to U$ to obtain the map of complexes $\Gamma_{\mathfrak{a}}(I) \to \Gamma_{\mathfrak{a}}(U)$ (recall that $\Gamma_{\mathfrak{a}}(U)$ remains exact by Lemma 3.0.1).

Fix $\ell < c = \operatorname{depth}(\mathfrak{a}, M)$. We claim that $\Gamma_{\mathfrak{a}}(I^\ell) = 0$. It will be enough to show that $\mu_R^\ell(\mathfrak{p}, M) = 0$ for all $\mathfrak{p} \supseteq \mathfrak{a}$ (if $\mathfrak{p} \not\supseteq \mathfrak{a}$, then $\Gamma_{\mathfrak{a}}(E(R/\mathfrak{p})) = 0$). So let $\mathfrak{p}$ be any prime containing $\mathfrak{a}$. By [FI03, Proposition 2.10]$\operatorname{depth}_R(\mathfrak{a}, M) = \inf\{\operatorname{depth}_{R_{\mathfrak{q}}} M_{\mathfrak{q}} | \mathfrak{q} \supseteq \mathfrak{a}\}$, so

$$\ell < \operatorname{depth}(\mathfrak{a}, M) \leq \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \inf\{i | \operatorname{Ext}_{R_{\mathfrak{p}}}^i(\kappa(\mathfrak{p}), M_{\mathfrak{p}}) \neq 0\}.$$

Therefore $\mu_R^\ell(\mathfrak{p}, M) = \dim_{\kappa(\mathfrak{p})} \operatorname{Ext}_{R_{\mathfrak{p}}}^\ell(\kappa(\mathfrak{p}), M_{\mathfrak{p}}) = 0$, and so $\Gamma_{\mathfrak{a}}(I^\ell) = 0$.

By minimality of I, $t+1$ is the minimal integer such that $\ker(I^{t+1} \to I^{t+2})$ is reduced Gorenstein injective. To see this, note that [EJ95b, Proposition 2.3] gives that $Z^t(I)$ is Gorenstein injective, and therefore $Z^{t+1}(I)$ is reduced by [EJ00, Theorem 10.1.4] and the proof of [EJ00, Proposition 10.1.8]. Thus for $i \geq t+1$, $I^i \cong U^i$; in particular $Z^{t+1}(I) \cong Z^{t+1}(U)$, and henceforth we identify these modules, setting $N := Z^{t+1}(I) \cong Z^{t+1}(U)$. Note that as N is reduced and Gorenstein injective, so is $\Gamma_{\mathfrak{a}}(N)$. We therefore have the following diagram (using that $\Gamma_{\mathfrak{a}}(I^i) = 0$ for $i < c$ as shown above):

$$\begin{array}{ccccccccccccc}
\cdots \longrightarrow & \Gamma_{\mathfrak{a}}(U^{c-1}) & \longrightarrow & \Gamma_{\mathfrak{a}}(U^{c}) & \longrightarrow & \Gamma_{\mathfrak{a}}(U^{c+1}) & \longrightarrow \cdots \longrightarrow & \Gamma_{\mathfrak{a}}(U^{t}) & \longrightarrow & \Gamma_{\mathfrak{a}}(U^{t+1}) & \longrightarrow \cdots \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow & & \cong\uparrow & \\
\cdots \longrightarrow & 0 & \longrightarrow & \Gamma_{\mathfrak{a}}(I^{c}) & \longrightarrow & \Gamma_{\mathfrak{a}}(I^{c+1}) & \longrightarrow \cdots \longrightarrow & \Gamma_{\mathfrak{a}}(I^{t}) & \longrightarrow & \Gamma_{\mathfrak{a}}(I^{t+1}) & \longrightarrow \cdots
\end{array}$$

Since $\Gamma_{\mathfrak{a}}(N)$ is reduced Gorenstein injective, $\Gamma_{\mathfrak{a}}(U^t) \to \Gamma_{\mathfrak{a}}(N)$ is an injective cover. Also, [EJ00, Theorem 10.1.4] gives $\Gamma_{\mathfrak{a}}(I^t) \to \Gamma_{\mathfrak{a}}(N)$ is an injective precover. Therefore by definition of injective precovers, there exist maps $\Gamma_{\mathfrak{a}}(U^t) \to \Gamma_{\mathfrak{a}}(I^t)$ and $\Gamma_{\mathfrak{a}}(I^t) \to \Gamma_{\mathfrak{a}}(U^t)$ and a commutative diagram

$$\begin{array}{ccccc}
\Gamma_{\mathfrak{a}}(U^t) & \longrightarrow & \Gamma_{\mathfrak{a}}(I^t) & \longrightarrow & \Gamma_{\mathfrak{a}}(U^t) \\
& \searrow & \downarrow & \swarrow & \\
& & \Gamma_{\mathfrak{a}}(N) & &
\end{array}$$

where since $\Gamma_{\mathfrak{a}}(U^t) \to \Gamma_{\mathfrak{a}}(N)$ is an injective cover and the diagram commutes, we must have the composition $\Gamma_{\mathfrak{a}}(U^t) \to \Gamma_{\mathfrak{a}}(I^t) \to \Gamma_{\mathfrak{a}}(U^t)$ is an isomorphism, hence the first horizontal map is an injection and the second is a surjection, such that the composition is isomorphic to the identity on $\Gamma_{\mathfrak{a}}(U^t)$. We have therefore shown that $\Gamma_{\mathfrak{a}}(U^t)$ appears as a direct summand of $\Gamma_{\mathfrak{a}}(I^t)$.

Note that $\Omega^c_{\text{inj}}M \to \Sigma^c(I^{\geq c}) \to \Sigma^c U$ is a minimal complete injective resolution of $\Omega^c_{\text{inj}}M$. Then by definition, $\Gamma^{\text{stab}}_{\mathfrak{a}}(\Omega^c_{\text{inj}}M) = Z^0\Gamma_{\mathfrak{a}}(\Sigma^c U) = Z^c\Gamma_{\mathfrak{a}}(U)$, so we have the following diagram, with exact rows:

$$\begin{array}{ccccccccccccc}
0 \longrightarrow & \Gamma^{\text{stab}}_{\mathfrak{a}}(\Omega^c_{\text{inj}}M) & \longrightarrow & \Gamma_{\mathfrak{a}}(U^{c}) & \longrightarrow & \Gamma_{\mathfrak{a}}(U^{c+1}) & \longrightarrow \cdots \longrightarrow & \Gamma_{\mathfrak{a}}(U^{t}) & \longrightarrow & \Gamma_{\mathfrak{a}}(N) & \longrightarrow 0 \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow & & =\uparrow & \\
0 \longrightarrow & H^c_{\mathfrak{a}}(M) & \longrightarrow & \Gamma_{\mathfrak{a}}(I^{c}) & \longrightarrow & \Gamma_{\mathfrak{a}}(I^{c+1}) & \longrightarrow \cdots \longrightarrow & \Gamma_{\mathfrak{a}}(I^{t}) & \longrightarrow & \Gamma_{\mathfrak{a}}(N) & \longrightarrow 0
\end{array}$$

Totalization induces an exact sequence:

$$0 \to H^c_{\mathfrak{a}}(M) \xrightarrow{\partial^{c-1}} \begin{matrix} \Gamma^{\text{stab}}_{\mathfrak{a}}(\Omega^c_{\text{inj}} M) \\ \oplus \\ \Gamma_{\mathfrak{a}}(I^c) \end{matrix} \xrightarrow{\partial^c} \begin{matrix} \Gamma_{\mathfrak{a}}(U^c) \\ \oplus \\ \Gamma_{\mathfrak{a}}(I^{c+1}) \end{matrix} \xrightarrow{\partial^{c+1}} \cdots \xrightarrow{\partial^t} \begin{matrix} \Gamma_{\mathfrak{a}}(U^t) \\ \oplus \\ \Gamma_{\mathfrak{a}}(N) \end{matrix} \xrightarrow{\partial^{t+1}} \Gamma_{\mathfrak{a}}(N) \to 0, \tag{5.0.3}$$

where ∂^i is defined in the obvious way [EJ00, Proposition 1.4.14]. Note that the complex

$$0 \to \cdots \to 0 \to \Gamma_{\mathfrak{a}}(N) \xrightarrow{\pm \text{id}} \Gamma_{\mathfrak{a}}(N) \to 0$$

appears as a subcomplex of the exact sequence (5.0.3), so we can quotient out by it to obtain another exact sequence. We consider the cases of $c = t$ and $0 \leq c \leq t-1$ separately.

First, suppose $c = t$. After quotienting the exact sequence (5.0.3) out by $0 \to \Gamma_{\mathfrak{a}}(N) \xrightarrow{\pm \text{id}} \Gamma_{\mathfrak{a}}(N) \to 0$, we obtain a short exact sequence

$$0 \to H^t_{\mathfrak{a}}(M) \to \begin{matrix} \Gamma^{\text{stab}}_{\mathfrak{a}}(\Omega^t_{\text{inj}} M) \\ \oplus \\ \Gamma_{\mathfrak{a}}(I^t) \end{matrix} \to \Gamma_{\mathfrak{a}}(U^t) \to 0.$$

Since $\Gamma_{\mathfrak{a}}(-)$ preserves essential injections,

$$\Gamma_{\mathfrak{a}}(I^t) \cong E_R(H^t_{\mathfrak{a}}(M)) \quad \text{and} \quad \Gamma_{\mathfrak{a}}(U^t) \cong E_R(\Gamma^{\text{stab}}_{\mathfrak{a}}(\Omega^t_{\text{inj}} M)),$$

and further, since $\Gamma_{\mathfrak{a}}(I^t) \to \Gamma_{\mathfrak{a}}(U^t)$ is a split surjection, we obtain the desired split short exact sequence when $c = t$:

$$0 \to H^t_{\mathfrak{a}}(M) \to \Gamma^{\mathrm{stab}}_{\mathfrak{a}}(\Omega^t_{\mathrm{inj}} M) \oplus E_R(H^t_{\mathfrak{a}}(M)) \to E_R(\Gamma^{\mathrm{stab}}_{\mathfrak{a}}(\Omega^t_{\mathrm{inj}} M)) \to 0.$$

Next, suppose that $0 \le c \le t-1$. Quotienting out the exact sequence (5.0.3) by $0 \to \Gamma_{\mathfrak{a}}(N) \xrightarrow{\pm \mathrm{id}} \Gamma_{\mathfrak{a}}(N) \to 0$, we obtain the following exact sequence (we abuse notation and use the same names for the maps):

$$0 \to H^c_{\mathfrak{a}}(M) \xrightarrow{\partial^{c-1}} \begin{matrix} \Gamma^{\mathrm{stab}}_{\mathfrak{a}}(\Omega^c_{\mathrm{inj}} M) \\ \oplus \\ \Gamma_{\mathfrak{a}}(I^c) \end{matrix} \xrightarrow{\partial^c} \begin{matrix} \Gamma_{\mathfrak{a}}(U^c) \\ \oplus \\ \Gamma_{\mathfrak{a}}(I^{c+1}) \end{matrix} \xrightarrow{\partial^{c+1}} \cdots \xrightarrow{\partial^{t-1}} \begin{matrix} \Gamma_{\mathfrak{a}}(U^{t-1}) \\ \oplus \\ \Gamma_{\mathfrak{a}}(I^t) \end{matrix} \xrightarrow{\partial^t} \Gamma_{\mathfrak{a}}(U^t) \to 0. \tag{5.0.4}$$

Set $K := \mathrm{coker}(\partial^{c-1})$. If $\mathrm{id}_R M < \infty$, then $\mathrm{Gid}_R M = \mathrm{id}_R M \le d$, hence $U = 0$, so

$$0 \to K \to \Gamma_{\mathfrak{a}}(I^{c+1}) \to \cdots \to \Gamma_{\mathfrak{a}}(I^t) \to 0$$

is a minimal injective resolution (as $\Gamma_{\mathfrak{a}}(-)$ preserves minimal injective resolutions), so $\mathrm{id}_R K = t-c-1$ as desired. Henceforth we assume that $\mathrm{id}_R M = \infty$ (equivalently, $\mathrm{pd}_R M = \infty$).

Since the injective module $\Gamma_{\mathfrak{a}}(U^t)$ is a summand of the injective module $\Gamma_{\mathfrak{a}}(I^t)$, there is an injective module J such that $\Gamma_{\mathfrak{a}}(I^t) \cong \Gamma_{\mathfrak{a}}(U^t) \oplus J$. Set $\pi : \Gamma_{\mathfrak{a}}(I^t) \to J$ as the canonical surjection. This allows us to cancel off the appearance of $0 \to \Gamma_{\mathfrak{a}}(U^t) \xrightarrow{\cong}$

$\Gamma_{\mathfrak{a}}(U^t) \to 0$ in the exact sequence (5.0.4) to obtain an injective resolution for K:

$$0 \to K \xrightarrow{\partial^c} \begin{matrix} \Gamma_{\mathfrak{a}}(U^c) \\ \oplus \\ \Gamma_{\mathfrak{a}}(I^{c+1}) \end{matrix} \xrightarrow{\partial^{c+1}} \cdots \xrightarrow{\partial^{t-2}} \begin{matrix} \Gamma_{\mathfrak{a}}(U^{t-2}) \\ \oplus \\ \Gamma_{\mathfrak{a}}(I^{t-1}) \end{matrix} \xrightarrow{\partial^{t-1}} \begin{matrix} \Gamma_{\mathfrak{a}}(U^{t-1}) \\ \oplus \\ J \end{matrix} \to 0,$$

hence $\operatorname{id}_R K \leq t-1-c$. To show $\operatorname{id}_R K = t-1-c$, it is enough to show that $\operatorname{Ext}_R^{t-c-1}(R/\mathfrak{m}, K) \neq 0$. Apply $\operatorname{Hom}_R(R/\mathfrak{m}, -)$ to the injective resolution of K to obtain:

$$\cdots \longrightarrow \begin{matrix} \operatorname{Hom}_R(R/\mathfrak{m}, \Gamma_{\mathfrak{a}}(U^{t-2})) \\ \oplus \\ \operatorname{Hom}_R(R/\mathfrak{m}, \Gamma_{\mathfrak{a}}(I^{t-1})) \end{matrix} \xrightarrow{(\partial^{t-1})_*} \begin{matrix} \operatorname{Hom}_R(R/\mathfrak{m}, \Gamma_{\mathfrak{a}}(U^{t-1})) \\ \oplus \\ \operatorname{Hom}_R(R/\mathfrak{m}, J) \end{matrix} \longrightarrow 0 \longrightarrow \cdots$$

where if $\Gamma_{\mathfrak{a}}(\partial_U)$ and $\Gamma_{\mathfrak{a}}(\partial_I)$ are the differentials on $\Gamma_{\mathfrak{a}}(U)$ and $\Gamma_{\mathfrak{a}}(I)$, respectively, and if $\Gamma_{\mathfrak{a}}(\rho) : \Gamma_{\mathfrak{a}}(I) \to \Gamma_{\mathfrak{a}}(U)$ is the map induced by the minimal complete injective resolution, then

$$(\partial^{t-1})_* = \begin{pmatrix} (\Gamma_{\mathfrak{a}}(\partial_U^{t-2}))_* & (\Gamma_{\mathfrak{a}}(\rho^{t-1}))_* \\ 0 & (\pi \circ \Gamma_{\mathfrak{a}}(\partial_I^{t-1}))_* \end{pmatrix}.$$

Since $\operatorname{depth}(M) = \operatorname{Gid}_R M = t > t-1$, we obtain $\mu_R^{t-1}(\mathfrak{m}, M) = 0$ by [Str90, Corollary 5.3.14], hence $E(R/\mathfrak{m})$ does not appear in I^{t-1}, and hence also not in $\Gamma_{\mathfrak{a}}(I^{t-1})$. Therefore $\operatorname{Hom}_R(R/\mathfrak{m}, \Gamma_{\mathfrak{a}}(I^{t-1})) = 0$, so $(\Gamma_{\mathfrak{a}}(\rho^{t-1}))_* = 0$. Also, as $\Gamma_{\mathfrak{a}}(I)$ and $\Gamma_{\mathfrak{a}}(U)$ are both minimal complexes, $\operatorname{Hom}_R(R/\mathfrak{m}, -)$ applied to either of their differentials becomes the zero map (see Remark 2.1.6), hence $(\partial^{t-1})_* = 0$.

In order to show that $\operatorname{Ext}_R^{t-c-1}(R/\mathfrak{m}, K) \neq 0$, it is therefore enough to find a nonzero element in $\operatorname{Hom}_R(R/\mathfrak{m}, \Gamma_{\mathfrak{a}}(U^{t-1}))$. Since we are in the case where $\operatorname{pd}_R M =$

∞, we have $E(R/\mathfrak{m})$ appears as a summand of U^{t-1} [AM02, Theorem 10.3], and therefore (since $\mathfrak{a} \subseteq \mathfrak{m}$) appears as a summand of $\Gamma_{\mathfrak{a}}(U^{t-1})$, hence $\operatorname{Hom}_R(R/\mathfrak{m}, \Gamma_{\mathfrak{a}}(U^{t-1})) \neq 0$ [ILL+07, Theorem A.20]. Therefore $\operatorname{Ext}_R^{t-c-1}(R/\mathfrak{m}, K) \neq 0$, and so $\operatorname{id}_R K = t-1-c$. Noting that $\Gamma_{\mathfrak{a}}(I^c) \cong E_R(H^c_{\mathfrak{a}}(M))$, we have the desired short exact sequence when $c \leq t-1$:

$$0 \to H^c_{\mathfrak{a}}(M) \to \Gamma^{\text{stab}}_{\mathfrak{a}}(\Omega^c_{\text{inj}} M) \oplus E_R(H^c_{\mathfrak{a}}(M)) \to K \to 0.$$

□

We highlight a special case of the previous theorem:

Corollary 5.0.5. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring of Krull dimension d. Suppose $M \neq 0$ is a MCM R-module, such that $c = \operatorname{depth}(\mathfrak{a}, M) = \operatorname{cd}(\mathfrak{a}, M)$. Then there exists a short exact sequence*

$$0 \to H^c_{\mathfrak{a}}(M) \to \Gamma^{\text{stab}}_{\mathfrak{a}}(\Omega^c_{\text{inj}} M) \oplus E_R(H^c_{\mathfrak{a}}(M)) \to K \to 0,$$

where $\operatorname{id}_R K < \infty$. Moreover, when $0 \leq c \leq t-1$, we have $\operatorname{id}_R K = t-c-1$ and when $c = t$, the sequence splits and $K \cong E_R(\Gamma^{\text{stab}}_{\mathfrak{a}}(\Omega^t_{\text{inj}} M))$.

Proof. We need only note that since M is finitely generated, $\operatorname{Gid}_R(M) = \operatorname{depth}(R)$ by [CFH06, Theorem II], which in turn coincides with $\operatorname{depth}(M)$ as M is MCM. □

Example 5.0.6. Let $(R, \mathfrak{m})$ be a local Gorenstein ring of finite Krull dimension and $M \neq 0$ a MCM R-module. If $\mathfrak{a}$ is any ideal generated up to radical by a regular sequence, Theorem 5.0.1 applies.

Remark 5.0.7. Let R be Gorenstein of Krull dimension d and N be any R-module. Then $N \simeq 0$ in $\underline{\operatorname{GInj}}(R)$ if and only if $\operatorname{id}_R N < \infty$.

Corollary 5.0.8. *Let R be a Gorenstein local ring of dimension d, $M \neq 0$ a MCM R-module, and $\mathfrak{a} \subset R$ an ideal satisfying $c = \operatorname{depth}(\mathfrak{a}, M) = \operatorname{cd}(\mathfrak{a}, M)$. Then we have an isomorphism in $\underline{\operatorname{GInj}}(R)$,*

$$H^c_{\mathfrak{a}}(M) \simeq \Gamma^{\text{stab}}_{\mathfrak{a}}(\Omega^c_{\text{inj}} M).$$

Proof. Apply Remark 5.0.7 to Theorem 5.0.1. □

This also recovers a result of Zargar and Zakeri in the case of a Gorenstein ring:

Corollary 5.0.9. *[ZZ13] Let R be a Gorenstein local ring of dimension d, and M, $\mathfrak{a}$, and c be as in Theorem 5.0.1. Then*

$$\operatorname{Gid}_R H^c_{\mathfrak{a}}(M) = \operatorname{Gid}_R M - c.$$

Proof. This follows immediately from Theorem 5.0.1. □

Recall that a *MCM approximation* of a finitely generated module N is a short exact sequence $0 \to I \to M \to N \to 0$, where $\operatorname{id}_R I < \infty$ and M is MCM. Often we just refer to M as the MCM approximation of N.

Dually, for an Artinian module N, a short exact sequence of the form $0 \to N \to G \to P \to 0$, where G is Gorenstein injective and $\operatorname{pd}_R P < \infty$ is called a *Gorenstein injective approximation* of N [Kra05, section 7]. Therefore, in light of Theorem 5.0.1, we have:

Corollary 5.0.10. *The short exact sequence given in Theorem 5.0.1 is a Gorenstein injective approximation of $H^c_{\mathfrak{a}}(M)$.*

Chapter 6

Future work on stable local cohomology

We suspect that many of the finiteness properties for classical local cohomology also hold for stable local cohomology, and we would like to understand these ideas better. Additionally, we would like to know if we can learn anything else about finiteness properties of local cohomology from finiteness of stable local cohomology (possibly by Theorem 5.0.1, or similar). Specifically, we would like to further address:

Conjecture 6.0.1. *Let R be a d-dimensional local Gorenstein ring, M a MCM R-module, and $\mathfrak{a}$ an ideal of R. Then*

1. *$\Gamma_{\mathfrak{a}}^{\mathrm{stab}}(M)$ has finitely many associated primes.*
2. *$\Gamma_{\mathfrak{a}}^{\mathrm{stab}}(M)$ has finitely many attached primes.*
3. *The stable Bass numbers of $\Gamma_{\mathfrak{a}}^{\mathrm{stab}}(M)$ are all finite.*

Recall that an *associated* prime of an R-module N is a prime ideal $\mathfrak{p}$ such that there exists $x \in N$ with $\mathfrak{p} = \{r \in R | rx = 0\}$, i.e., $\mathfrak{p}$ is the annihilator of some element of N. A prime ideal $\mathfrak{p}$ is called an *attached* prime of N if every finitely generated ideal contained in $\mathfrak{p}R_{\mathfrak{p}}$ annihilates a nonzero element of $N_{\mathfrak{p}}$; see [Nor82]. Notice that associated primes are attached.

Classically, if M is a holonomic D-module, then $H^i_{\mathfrak{a}}(M)$ is holonomic [Lyu93]. (In particular, R is holonomic as a D-module.) It is natural to also ask:

Question 6.0.2. *Can* $\Gamma^{\mathrm{stab}}_{\mathfrak{a}}(M)$ *be considered as a* D*-module? And, if so, if* M *is a holonomic* D*-module, is* $\Gamma^{\mathrm{stab}}_{\mathfrak{a}}(M)$ *also holonomic?*

Part II

Cosupport via cotorsion-flat resolutions

Introduction to Part II

All rings are assumed to be commutative Noetherian of finite Krull dimension. In this part, we explore applications of *cotorsion-flat resolutions*. We first develop a criterion (Theorem 8.3.4) for a cotorsion-flat resolution to be *minimal*, using the notion of minimality set forth by Avramov and Martsinkovsky [AM02].

One use for such minimal cotorsion-flat resolutions is that they are able to detect the *cosupport* of a module. Cosupport is an invariant developed by Benson, Iyengar, and Krause [BIK12] on the level of R-linear, compactly generated, triangulated categories (a special case was previously investigated by Melkersson and Schenzel [MS95]). We are most interested in the manifestation of cosupport in a commutative Noetherian ring. Recalling that the *(small) support* of a module (see the definition at 9.1.9) can be characterized by the primes appearing in its minimal injective resolution, we show that the cosupport of a module can be characterized by the primes appearing in a minimal cotorsion-flat resolution (which we explain below, see Theorem 9.2.2 for a precise statement).

Theorem A (cf. Theorem 9.2.2)**.** *Let R be a commutative Noetherian ring of finite Krull dimension and M an R-module having a minimal (left or right) cotorsion-flat resolution B. Then $\mathfrak{p} \in \operatorname{cosupp}_R(M)$ if and only if $\mathfrak{p}$ appears in B.*

Both flat modules and cotorsion modules have appropriately minimal cotorsion-flat resolutions, hence this result allows us to compute the cosupport of such modules. In particular, we use work of Enochs [Eno87], where he studies the minimal pure-injective resolution of flat modules, to obtain the following:

Proposition B (Propositions 9.5.2 and 9.5.4). *Assume that R is either a 1-dimensional domain that is not complete local, or that $R = k[x, y]$ for any uncountable field k. Then* $\operatorname{cosupp}_R(R) = \operatorname{Spec}(R)$.

Using the fact that the cosupport of a finitely generated module over a Gorenstein ring depends only on the cosupport of the ring and the support of the module (see Proposition 9.3.2 for details), we provide a generalization of [BIK12, Proposition 4.18]: A complex M with finitely generated cohomology over either of the rings in Proposition B satisfies

$$\operatorname{cosupp}_R(M) = \operatorname{supp}_R(M).$$

For a precise statement, see Corollary 9.5.5 below.

Chapter 7

Preliminaries

Let R be a commutative Noetherian ring. In addition to the basic tools established in Section 1.1 regarding complexes, homotopy equivalences of complexes, and dualizing complexes, we now also define triangulated categories, the derived category of R, limits, cotorsion-flat modules, covers, envelopes, and $\mathcal{F}$-resolutions (for a covering or enveloping class of modules $\mathcal{F}$).

7.1 Triangulated categories

For the axioms defining triangulated categories, refer to [Wei94, Chapter 10] or [Nee01, Chapter 1]. See also [IK06, Section 1].

Definition 7.1.1. An additive category $\mathcal{T}$ equipped with a translation functor $\Sigma : \mathcal{T} \to \mathcal{T}$ and with a distinguished family of *exact triangles* which are subject to Verdier's four axioms listed in [Wei94, Definition 10.2.1] is called a *triangulated category*. A *subcategory* $\mathcal{S}$ of $\mathcal{T}$ is a collection of some of the objects and morphisms of $\mathcal{T}$ such that the morphisms are closed under composition and includes id_X for each object X of $\mathcal{S}$. A subcategory $\mathcal{S}$ of $\mathcal{T}$ is called a *full subcategory* if $\mathrm{Hom}_{\mathcal{S}}(X, X') = \mathrm{Hom}_{\mathcal{T}}(X, X')$ for all $X, X' \in \mathcal{S}$. A *triangulated subcategory* is a full subcategory $\mathcal{S}$ of $\mathcal{T}$ that is closed under isomorphisms, $\mathcal{S} = \Sigma\mathcal{S}$, and if $X \to Y \to Z \to \Sigma X$ is an exact triangle in $\mathcal{T}$ such that X and Y are in $\mathcal{S}$, then Z is in $\mathcal{S}$ as well. Implicitly, all subcategories of a triangulated category are assumed to be full.

Let $\mathcal{T}$ be a triangulated category. A non-empty subcategory $\mathcal{S}$ of $\mathcal{T}$ is *thick* if it is a triangulated subcategory of $\mathcal{T}$ that is closed under direct summands. If $\mathcal{S}$ is also closed under all coproducts allowed in $\mathcal{T}$, then it is *localizing*. For a class of objects $\mathcal{C}$ of $\mathcal{T}$, we use $\mathrm{Thick}(\mathcal{C})$ (resp., $\mathrm{Loc}(\mathcal{C})$) to denote the smallest thick (resp., localizing) subcategory of $\mathcal{T}$ containing $\mathcal{C}$.

Assume that $\mathcal{T}$ admits arbitrary coproducts. An object X of $\mathcal{T}$ is called *compact* if $\mathrm{Hom}_{\mathcal{T}}(X, -)$ commutes with coproducts. The compact objects of $\mathcal{T}$ form a thick subcategory, which is denoted $\mathcal{T}^c$. We say a class of objects $\mathcal{S}$ *generates* $\mathcal{T}$ if $\mathrm{Loc}(\mathcal{S}) = \mathcal{T}$, and that $\mathcal{T}$ is *compactly generated* if there exists a generating set consisting of compact objects.

7.2 Derived category $D(R)$

One of the more important triangulated categories for us will be the derived category $D(R)$ for a ring R. Briefly, $D(R)$ is the category of all complexes of R-modules where we first identify all chain homotopic maps and then localize at the set of quasi-isomorphisms, see [Wei94, Chapter 10]. A *quasi-isomorphism* is a chain map that is an isomorphism on homology. If A, B are two chain complexes, we use $A \sim B$ to mean that A and B are *quasi-isomorphic*, i.e., there exists a zigzag diagram of quasi-isomorphisms between A and B. We also use $\sim$ to denote an isomorphism in $D(R)$.

7.3 Limits

In order to work with cotorsion-flat modules (and complexes of such), we will need to consider the $\mathfrak{p}$-adic completion of R-modules (and of complexes of R-modules), which requires (inverse) limits to define. We will use the notion of a tower of chain complexes in order to define limits for this purpose. This treatment of limits follows that of [AFH16] (see also [AM69, Chapter 10]).

A *tower* of complexes over R is a family $\{f^n : M^n \to M^{n-1}\}_{n\in\mathbb{Z}}$ of chain maps of complexes, such that $M^n = 0$ for $n \ll 0$. To such a tower, one can associate a morphism

$$\xi_M : \prod_{n\in\mathbb{Z}} M^n \longrightarrow \prod_{n\in\mathbb{Z}} M^n$$

$$(m^n)_{n\in\mathbb{Z}} \longmapsto (m^n - f^{n+1}(m^{n+1}))_{n\in\mathbb{Z}}$$

We then define the *limit* of this tower to be $\varprojlim_n M^n = \ker \xi_M$ and the *first derived limit* of this tower to be $\varprojlim_n^{(1)} M^n = \operatorname{coker} \xi_M$. In the case where all maps f^n are surjective, the Mittag-Leffler condition yields that $\lim_n^{(1)} M^n = 0$ (or alternatively, this can be shown directly using more elementary methods as in [AM69, Proposition 10.2]).

Moreover, a *morphism of towers* $\{f^n : M^n \to M^{n-1}\}$ to $\{g^n : L^n \to L^{n-1}\}$ is a family of chain maps $\{h^n : M^n \to L^n\}$ making the appropriate diagrams commute. This induces a map on limits:

$$\varprojlim_n h^n : \varprojlim_n M^n \to \varprojlim_n L^n, \text{ defined by } (m^n) \mapsto (h^n(m^n)).$$

One fact we will need later is the following:

Lemma 7.3.1. *If* $\{f^n : M^n \to M^{n-1}\}$ *and* $\{g^n : L^n \to L^{n-1}\}$ *are two towers of*

R-complexes with f_n, g_n surjective for all n, and $\{h^n : M^n \to L^n\}$ is a morphism of towers such that $h^n : M^n \to L^n$ is a quasi-isomorphism for all n, then $\varprojlim M^n \to \varprojlim L^n$ is a quasi-isomorphism as well.

Proof. The definition of limits above, as well as the fact that the f_n and g_n are surjective for all n, yields a diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \varprojlim_n M^n & \longrightarrow & \prod_{n\in\mathbb{Z}} M^n & \xrightarrow{\xi_M} & \prod_{n\in\mathbb{Z}} M^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \prod h^n} & & \downarrow{\scriptstyle \prod h^n} & & \\
0 & \longrightarrow & \varprojlim_n L^n & \longrightarrow & \prod_{n\in\mathbb{Z}} L^n & \xrightarrow{\xi_L} & \prod_{n\in\mathbb{Z}} L^n & \longrightarrow & 0
\end{array}$$

This diagram induces a morphism between triangles in $D(R)$ (since any short exact sequence of complexes induces a exact triangle in $D(R)$ [Wei94, Example 10.4.9]). As the middle and rightmost vertical maps in the above diagram are quasi-isomorphisms, the Five Lemma for exact triangles [Wei94, Exercise 10.2.2] shows that the map on limits is a quasi-isomorphism as well. □

The data in a tower of complexes above can also be thought of as an *inverse system* as is done in [Mat89, Appendix A]. With this in mind, if M is an R-module and $\mathfrak{a} \subseteq R$ is an ideal, then the limit of the tower

$$\{\pi_n : M/\mathfrak{a}^n M \to M/\mathfrak{a}^{n-1}M\}_{n\geq 1},$$

which is written $\varprojlim M/\mathfrak{a}^n M$, is the $\mathfrak{a}$-*adic completion* of M.

7.4 Cotorsion-flat modules

Recall that an R-module C is called *cotorsion* if $\operatorname{Ext}^1_R(F, C) = 0$ for every flat R-module F. All injective modules, as well as all modules of the form $\operatorname{Hom}_R(M, I)$ for any module M and injective module I, are cotorsion [Eno84, Lemma 2.1]. The class of flat modules and the class of cotorsion modules form what is called a *cotorsion theory*; in particular, if F is any R-module such that $\operatorname{Ext}^1_R(F, C) = 0$ for every cotorsion R-module C, then F is flat [EJ00, Lemma 7.1.4].

A module that is both cotorsion and flat will be referred to as a *cotorsion-flat* module. For an R-module M and index set X, set $M^{(X)} = \bigoplus_X M$. Using the limit as defined in the previous section, we set

$$\widehat{M}^{\mathfrak{p}} = \varprojlim_n (R/\mathfrak{p}^n \otimes_R M) = \varprojlim_n (M/\mathfrak{p}^n M)$$

as the $\mathfrak{p}$-adic completion of M, or when the context is clear, sometimes just $\widehat{M}$. Completions of free modules will play an important role in understanding cotorsion-flat modules later on. Enochs showed [Eno84, Theorem] that cotorsion-flat modules have a unique decomposition indexed by $\operatorname{Spec}(R)$; we sketch a proof below in Theorem 8.2.2.

A useful fact is that a module is cotorsion-flat if and only if it is a direct summand of $\operatorname{Hom}_R(E, E')$ for some injective R-modules E, E' [Eno84, Lemma 2.3]. To spell out a few of the details, first since $\operatorname{Hom}_R(E, E') \otimes_R L \cong \operatorname{Hom}_R(\operatorname{Hom}_R(L, E), E')$ for finitely generated L, it follows that $\operatorname{Hom}_R(E, E')$ is flat. Additionally, for any flat module F, with projective resolution $P \to F$,

$$\operatorname{Ext}^1_R(F, \operatorname{Hom}_R(E, E')) \cong H^1 \operatorname{Hom}_R(P, \operatorname{Hom}_R(E, E')) = H^1 \operatorname{Hom}_R(P \otimes_R E, E') = 0,$$

so $\operatorname{Hom}_R(E, E')$ is also cotorsion. Finally, a direct summand of a flat module is flat, and a direct summand of a cotorsion module is cotorsion. Conversely, if B is a cotorsion-flat module and E' is an *injective cogenerator* for R (e.g., see [Lam99, section 19]), then $B \to \operatorname{Hom}_R(\operatorname{Hom}_R(B, E'), E')$ is a pure submodule (see Definition 8.1.2 below). Note that $\operatorname{Hom}_R(B, E')$ is injective (the usual adjointness does the job here). The fact that B is cotorsion-flat, hence pure-injective, and a pure submodule of $\operatorname{Hom}_R(\operatorname{Hom}_R(B, E'), E')$, makes it a direct summand. To see this, by definition (see Definition 8.1.2 below), there is a surjection $\operatorname{Hom}_R(\operatorname{Hom}_R(\operatorname{Hom}_R(B, E'), E'), B) \to \operatorname{Hom}_R(B, B)$, giving a splitting.

In the case where R is a complete local ring, Matlis duality shows that every finitely generated flat (or projective) R-module M is a cotorsion-flat module. The following result is useful when working with cotorsion-flat modules, a proof of which can be found in [Xu96, Lemma 4.1.5]. In fact, Xu gives an explicit isomorphism, but as we only need the fact that these are abstractly isomorphic R-modules, we give a proof that doesn't require a description of the elements of $\widehat{R_{\mathfrak{p}}^{(X)}}^{\mathfrak{p}}$ as is required in Xu's proof.

Lemma 7.4.1 (Xu)**.** *Let R be a commutative Noetherian ring and $\mathfrak{p} \in \operatorname{Spec}(R)$. Then for any set X, there is an isomorphism:*

$$\operatorname{Hom}_R(E(R/\mathfrak{p}), E(R/\mathfrak{p})^{(X)}) \cong \widehat{R_{\mathfrak{p}}^{(X)}}.$$

Proof. Recall that $E(R/\mathfrak{p}) = \bigcup_{n\geq 1}(0 :_{E(R/\mathfrak{p})} \mathfrak{p}^n)$. Also,

$$\operatorname{Hom}_R(R/\mathfrak{p}^n, E(R/\mathfrak{p})) \cong (0 :_{E(R/\mathfrak{p})} \mathfrak{p}^n),$$

which is finitely generated as an $R_{\mathfrak{p}}$-module for each $n \geq 1$. We have the following:

$$\begin{aligned}
\operatorname{Hom}_R(E(R/\mathfrak{p}), E(R/\mathfrak{p})^{(X)}) &\cong \operatorname{Hom}_R(\varinjlim_n \operatorname{Hom}_R(R/\mathfrak{p}^n, E(R/\mathfrak{p})), E(R/\mathfrak{p})^{(X)}) \\
&\cong \varprojlim_n \operatorname{Hom}_R(\operatorname{Hom}_R(R/\mathfrak{p}^n, E(R/\mathfrak{p})), E(R/\mathfrak{p})^{(X)}) \\
&\cong \varprojlim_n \left(\bigoplus_X \operatorname{Hom}_R(\operatorname{Hom}_R(R/\mathfrak{p}^n, E(R/\mathfrak{p})), E(R/\mathfrak{p})) \right) \\
&\cong \varprojlim_n \left(\bigoplus_X (R/\mathfrak{p}^n \otimes_R \operatorname{Hom}_R(E(R/\mathfrak{p}), E(R/\mathfrak{p}))) \right), \text{ by [Ish65]}, \\
&\cong \varprojlim_n \left(\bigoplus_X (R/\mathfrak{p}^n \otimes_R \varprojlim_t (R/\mathfrak{p}^t \otimes_R R_{\mathfrak{p}})) \right) \\
&\cong \varprojlim_n \left(\bigoplus_X (R/\mathfrak{p}^n \otimes_R R_{\mathfrak{p}}) \right) \\
&\cong \varprojlim_n (R/\mathfrak{p}^n \otimes_R R_{\mathfrak{p}}^{(X)}) \\
&= \widehat{R_{\mathfrak{p}}^{(X)}}^{\mathfrak{p}},
\end{aligned}$$

where the third to last line follows because, for a fixed $n \geq 1$,

$$R/\mathfrak{p}^n \otimes_R \varprojlim_t (R/\mathfrak{p}^t) \cong \varprojlim_t (R/\mathfrak{p}^n \otimes R/\mathfrak{p}^t) \cong R/\mathfrak{p}^n,$$

using that $R/\mathfrak{p}^n$ is finitely generated. □

7.5 Covers, envelopes, and $\mathcal{F}$-resolutions

The material in this section can mostly be found in the book by Enochs and Jenda [EJ00] and applies to any ring R. We start with the definitions of covers and envelopes (that recover the familiar notions of projective covers and injective envelopes/hulls).

Let $\mathcal{F}$ be a class of R-modules closed under isomorphisms. For an R-module M, a morphism $\phi : C \to M$ where $C \in \mathcal{F}$ is called an $\mathcal{F}$*-cover* of M if:

1. For any map $\phi' : C' \to M$ with $C' \in \mathcal{F}$, there exists $f : C' \to C$ such that $\phi \circ f = \phi'$, and
2. If $f : C \to C$ is an endomorphism with $\phi \circ f = \phi$, then f must be an isomorphism.

If $\phi : C \to M$ satisfies condition (1) (but not necessarily condition (2)), we call ϕ an $\mathcal{F}$*-precover*. It is worth remarking that if an $\mathcal{F}$-cover exists, it is unique up to isomorphism. If $\mathcal{F}$ is the class of projective modules, an $\mathcal{F}$-cover is called a projective cover; this can be seen to agree with the usual notion of projective covers (e.g., [Xu96, Theorem 1.2.12]). Also, if the class $\mathcal{F}$ contains the ring R, then $\mathcal{F}$-covers are surjective. We say a class $\mathcal{F}$ is *covering* (resp., *precovering*) if every R-module has an $\mathcal{F}$-cover (resp., an $\mathcal{F}$-precover).

If Flat is the class of flat R-modules and CotFlat is the class of cotorsion-flat R-modules, we also refer to Flat-covers and CotFlat-covers as flat covers and cotorsion-flat covers, respectively (when they exist).

Envelopes and enveloping classes are defined dually. For an R-module M, a morphism $\phi : M \to F$ with $F \in \mathcal{F}$ is an $\mathcal{F}$*-envelope* of M if:

1. For any map $\phi' : M \to F'$ with $F' \in \mathcal{F}$, there exists $f : F \to F'$ such that $f \circ \phi = \phi'$, and
2. If $f : F \to F$ is an endomorphism with $f \circ \phi = \phi$, then f must be an isomorphism.

If $\phi : M \to F$ satisfies (1) but not necessarily (2), it is called an $\mathcal{F}$*-preenvelope*. A class $\mathcal{F}$ is *enveloping* (resp., *preenveloping*) if every R-module has an $\mathcal{F}$-envelope (resp., an $\mathcal{F}$-preenvelope). If an enveloping class contains all injective R-modules, the envelopes will necessarily be injections. In particular, the class of injective modules is enveloping and recovers the usual notion of injective envelope.

If PurInj is the class of pure-injective R-modules (defined below in Definition 8.1.2) and CotFlat is the class of cotorsion-flat R-modules, we also refer to PurInj-envelopes and CotFlat-envelopes as pure-injective envelopes and cotorsion-flat envelopes, respectively (when they exist).

Bican, El Bashir, and Enochs showed that the class of flat modules is covering [BEBE01] (this was shown for a commutative Noetherian ring by Xu [Xu96]). The class of pure-injective modules is enveloping (in a Noetherian ring by Fuchs [Fuc67] and in a locally finitely presented additive category by Herzog [Her03, Theorem 6]). Somewhat surprisingly, the class of projective modules is not a covering class in general; a ring such that the class of projective modules is covering is called a *left perfect* ring (this condition on the ring is equivalent to every flat module being projective, see [Xu96, Theorem 1.2.13]).

Given a class of R-modules $\mathcal{F}$, if this class is either covering or enveloping, we may use it to define $\mathcal{F}$-resolutions. A complex C is said to be $\mathrm{Hom}_R(-, \mathcal{F})$-*exact* if for any $F \in \mathcal{F}$, $\mathrm{Hom}_R(C, F)$ is exact. Dually, we say C is $\mathrm{Hom}_R(\mathcal{F}, -)$-*exact* if for any $F \in \mathcal{F}$, $\mathrm{Hom}_R(F, C)$ is exact.

If $\mathcal{F}$ is an enveloping class, a *right* $\mathcal{F}$-*resolution* of M is a $\mathrm{Hom}_R(-, \mathcal{F})$-exact complex

$$0 \to M \to F^0 \to F^1 \to \cdots$$

with each $F^i \in \mathcal{F}$, constructed so that $M \to F^0$, $\mathrm{coker}(M \to F^0) \to F^1$, and $\mathrm{coker}(F^{i-1} \to F^i) \to F^{i+1}$ for $i \geq 1$ are $\mathcal{F}$-envelopes. Note that this complex need not itself be exact. Dually, if $\mathcal{F}$ is a covering class, a *left* $\mathcal{F}$-*resolution* of M is a

$\mathrm{Hom}_R(\mathcal{F}, -)$ exact complex

$$\cdots \to F_1 \to F_0 \to M \to 0$$

with each $F_i \in \mathcal{F}$, constructed so that $F_0 \to M$, $F_1 \to \ker(F_0 \to M)$, and $F_{i+1} \to \ker(F_i \to F_{i-1})$ for $i \geq 1$ are $\mathcal{F}$-covers.[1] We continue to use the un-decorated term *resolution* to mean an honest resolution in the sense that the augmented sequence is exact.

By the comments above, if we set Flat to be the class of flat R-modules and PurInj to be the class of pure-injective R-modules, then every module has a left Flat-resolution and a right PurInj-resolution.

Remark 7.5.1. Note that for any commutative Noetherian ring R, every R-module has a pure-injective flat envelope [EJ00, Proposition 6.6.6]. By [Her03, Lemma 3], we have that pure-injective modules that are also flat coincide with cotorsion-flat R-modules. Setting CotFlat to be the class of cotorsion-flat R-modules, we have that every R-module has a cotorsion-flat R-envelope, and therefore every module has a right CotFlat-resolution. (Warning: this is not to say that every module has a resolution to the right by cotorsion-flat modules. Keep in mind that a "right CotFlat-resolution" does not need to be exact; instead, it is just $\mathrm{Hom}_R(-, \mathrm{CotFlat})$-exact.)

[1]What we call $\mathcal{F}$-resolutions here are referred to as *minimal* $\mathcal{F}$-resolutions in [EJ00, Chapter 8], but we prefer to reserve the term "minimal" to mean a minimal complex.

Chapter 8

Cotorsion-flat resolutions

Assume throughout this chapter that R is commutative Noetherian of finite Krull dimension. We first give two ways in which cotorsion-flat resolutions arise. After describing a way to decompose cotorsion-flat modules, we use this to describe cotorsion-flat resolutions that are minimal (in the sense given by [AM02]).

Let Flat be the class of flat R-modules and PurInj the class of pure-injective R-modules. For cotorsion modules, we will show that a left Flat-resolution (resolving to the left by taking flat covers) yields a minimal left cotorsion-flat resolution (Fact 8.1.1 below); for flat modules, we will show a right PurInj-resolution (resolving to the right by taking pure-injective envelopes) provides a minimal right cotorsion-flat resolution (see Fact 8.1.3).

First we define cotorsion-flat resolutions:

Definition 8.0.1. For any R-module M, we say a *left cotorsion-flat resolution* of M is a complex B of cotorsion-flat modules with a quasi-isomorphism $B \to M$, with $B_i = 0$ for $i < 0$. A *right cotorsion-flat resolution* of M is a complex B of cotorsion-flat modules with a quasi-isomorphism $M \to B$ such that $B^i = 0$ for $i < 0$.

Remark 8.0.2. As a preliminary caution, we note that such resolutions need not always exist. However, we will show that a left Flat-resolution (which always exists) of a cotorsion module is a left cotorsion-flat resolution and a right PurInj-resolution (which always exists) of a flat module is a right cotorsion-flat resolution. In particular, left cotorsion-flat resolutions exist for cotorsion modules and right cotorsion-flat resolutions exist for flat modules.

8.1 Flat and pure-injective resolutions

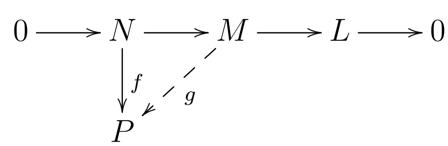
We first show that we can construct *left* cotorsion-flat resolutions of *cotorsion* modules. Set Flat as the class of flat R-modules. Xu showed [Xu96, Theorem 4.3.5] that in a commutative Noetherian ring of finite Krull dimension, every module has a flat cover. Bican, El Bashir, and Enochs showed [BEBE01] that in fact this holds for every associative ring. We can use this to construct a left Flat-resolution, constructed by taking the flat cover, then the flat cover of the kernel of that map, ad infinitum. Since R is flat, we have that in fact flat (pre)covers are surjective. This means that a left Flat-resolution is also a resolution, in the sense that the augmented sequence is exact (stronger than just asserting the sequence is $\mathrm{Hom}_R(\mathrm{Flat}, -)$-exact as in Section 7.5).

The kernel of a flat (pre)cover of a module is cotorsion (by Wakamatsu's Lemma [Xu96, Lemma 2.1.1]). Immediately from a long exact sequence in Ext, we obtain that a flat (pre)cover of a cotorsion module is also cotorsion, so we can see that a left Flat-resolution of a cotorsion module is in fact a left cotorsion-flat resolution. Moreover, a flat cover of a cotorsion module is isomorphic to a cotorsion-flat cover. Additionally, if we have a left cotorsion-flat resolution of an R-module M, Marley and Webb [MW16, Lemma 2.5] show that M is necessarily cotorsion, so we only consider left cotorsion-flat resolutions of cotorsion modules. Explicitly, we have shown:

Fact 8.1.1. *If M is a cotorsion R-module and F is a left* Flat*-resolution, then* $\cdots \xrightarrow{\partial_2^F} F_1 \xrightarrow{\partial_1^F} F_0 \to M \to 0$ *is exact, F is a complex of cotorsion-flat modules, and* $F_i \to \operatorname{coker}(\partial_{i+1}^F)$ *is a cotorsion-flat cover for $i \geq 0$. Moreover, if M is any module having a left cotorsion-flat resolution, then M is cotorsion.*

We can also construct *right* cotorsion-flat resolutions of *flat* modules. In order to do so, we will use pure-injective modules:

Definition 8.1.2 (cf. [Xu96])**.** An exact sequence of R-modules $0 \to N \to M \to L \to 0$ is called *pure* if for every R-module S, the sequence $0 \to S \otimes_R N \to S \otimes_R M \to S \otimes_R L \to 0$ is still exact. In this case, we say that N is a *pure submodule* of M. An R-module P is called *pure-injective* if every diagram



with pure exact top row can be completed to a commutative diagram (i.e., P is pure-injective if for every pure submodule $0 \to N \to M$, we have a surjection $\operatorname{Hom}_R(M, P) \to \operatorname{Hom}_R(N, P)$).

Since pure-injective envelopes exist for all modules [Fuc67] over R, every module has a right PurInj-resolution, constructed by taking the pure-injective envelope, then the pure-injective envelope of the cokernel of that map, and continuing indefinitely. Since PurInj contains all injective R-modules, we can see that a right PurInj-resolution is a resolution in that the augmented sequence is exact.

If F is a flat R-module, the pure-injective envelope of F is a cotorsion-flat module [GJ81] (see also [Eno87, Lemma 1.1 and discussion following]). As the cokernel of this map is again a flat module (by the other version of Wakamatsu's Lemma [Xu96, Lemma 2.1.2]), the PurInj-resolution of a flat module consists of cotorsion-flat modules. The definition of envelopes shows that the pure-injective envelope and the cotorsion-flat envelope of a flat module are isomorphic. Since a flat module injects into its pure-injective envelope (either appealing to the fact that PurInj contains all injective modules or using that Warfield [War69] showed that a flat module N injects into $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$, a pure-injective module, so the pure-injective envelope necessarily maps to this, hence it is an injection), we obtain that the PurInj-resolution of a flat module is an honest resolution in the sense that the augmented sequence is exact. Additionally, if a module has a *bounded* right resolution by flat modules, it is itself flat (which can be shown by analyzing a long exact sequence in Tor), so we primarily consider right cotorsion-flat resolutions of flat modules. Explicitly, this yields:

Fact 8.1.3. *If N is a flat R-module and P is a right* PurInj*-resolution, then* $0 \to N \to P^0 \xrightarrow{\partial_P^0} P^1 \xrightarrow{\partial_P^1} \cdots$ *is exact, P is a complex of cotorsion-flat modules, and* $\ker(\partial_P^i) \to P^i$ *is a cotorsion-flat envelope for $i \geq 0$. Moreover, if N is any module having a bounded right cotorsion-flat resolution, then N is flat.*

8.2 Decomposing cotorsion-flat modules

We will need the following structural lemma when working with cotorsion-flat resolutions.

Lemma 8.2.1. *Let R be a commutative Noetherian ring and $B = \prod_{\mathfrak{q}} T_{\mathfrak{q}}$, where $T_{\mathfrak{q}} = \widehat{R_{\mathfrak{q}}^{(X_{\mathfrak{q}})}}$ for some index sets $X_{\mathfrak{q}}$. Then*

(1) $\widehat{B}^{\mathfrak{p}} \cong \prod_{\mathfrak{p} \subseteq \mathfrak{q}} T_{\mathfrak{q}}$, *and*

(2) $\operatorname{Hom}_R(R_{\mathfrak{p}}, B) \cong \prod_{q \subseteq \mathfrak{p}} T_{\mathfrak{q}}$.

Moreover, if B is a complex of modules of this form, we evidently have an injection of complexes $\operatorname{Hom}_R(R_{\mathfrak{p}}, B) \to B$ and a surjection of complexes $B \to \widehat{B}^{\mathfrak{p}}$, both of which are degreewise split. In particular, the complexes

$$\operatorname{Hom}_R(R_{\mathfrak{p}}, \widehat{B}^{\mathfrak{p}}) \quad \textit{and} \quad \varprojlim_n (R/\mathfrak{p}^n \otimes_R \operatorname{Hom}_R(R_{\mathfrak{p}}, B))$$

can both be identified with the subquotient complex $\cdots \to T^i_{\mathfrak{p}} \to T^{i+1}_{\mathfrak{p}} \to \cdots$ with differential induced from B.

Proof. For (1), consider the following:

$$\begin{aligned}
\widehat{B}^{\mathfrak{p}} &\cong \prod \widehat{T_{\mathfrak{q}}}^{\mathfrak{p}}, \text{ since the direct product commutes with } \varprojlim \text{ and } R/\mathfrak{p}^n \otimes_R -, \\
&= \prod \widehat{\widehat{R_{\mathfrak{q}}^{(X_{\mathfrak{q}})}}^{\mathfrak{q}}}^{\mathfrak{p}} \\
&\cong \prod \widehat{R_{\mathfrak{q}}^{(X_{\mathfrak{q}})}}^{\mathfrak{p}+\mathfrak{q}}, \text{ by [AM69, Chapter 10, Exercise 5]}, \\
&= \prod \varprojlim_n (R/(\mathfrak{p}+\mathfrak{q})^n \otimes_R R_{\mathfrak{q}}^{(X_{\mathfrak{q}})}) \\
&\cong \prod_{\mathfrak{p} \subseteq \mathfrak{q}} T_{\mathfrak{q}},
\end{aligned}$$

where the last isomorphism follows from the following: if $\mathfrak{p} \subseteq \mathfrak{q}$, then $\mathfrak{p}+\mathfrak{q} = \mathfrak{q}$ so $\widehat{R_{\mathfrak{q}}^{(X_{\mathfrak{q}})}}^{\mathfrak{p}+\mathfrak{q}} = T_{\mathfrak{q}}$; if $\mathfrak{p} \not\subseteq \mathfrak{q}$, then $\mathfrak{p}+\mathfrak{q} \not\subseteq \mathfrak{q}$ hence $(\mathfrak{p}+\mathfrak{q})^n \not\subseteq \mathfrak{q}$ as $\mathfrak{q}$ is prime, so $R/(\mathfrak{p}+\mathfrak{q})^n \otimes_R R_{\mathfrak{q}} = 0$, hence in this case $\widehat{R_{\mathfrak{q}}^{(X_{\mathfrak{q}})}}^{\mathfrak{p}+\mathfrak{q}} = 0$.

For (2), we have:

$$\begin{aligned}\mathrm{Hom}_R(R_{\mathfrak{p}}, \prod_{\mathfrak{q}} T_{\mathfrak{q}}) &\cong \prod_{\mathfrak{q}} \mathrm{Hom}_R(R_{\mathfrak{p}}, \mathrm{Hom}_R(E(R/\mathfrak{q}), E(R/\mathfrak{q})^{(X_{\mathfrak{q}})})), \text{ by Lemma 7.4.1,}\\ &\cong \prod_{\mathfrak{q}} \mathrm{Hom}_R(E(R/\mathfrak{q}) \otimes_R R_{\mathfrak{p}}, E(R/\mathfrak{q})^{(X_{\mathfrak{q}})}), \text{ by adjointness,}\\ &\cong \prod_{\mathfrak{q}\subseteq\mathfrak{p}} \mathrm{Hom}_R(E(R/\mathfrak{q}), E(R/\mathfrak{q})^{(X_{\mathfrak{q}})}), \text{ as } E(R/\mathfrak{q}) \text{ is } \mathfrak{q}\text{-local, } \mathfrak{q}\text{-torsion,}\\ &\cong \prod_{\mathfrak{q}\subseteq\mathfrak{p}} T_{\mathfrak{q}}, \text{ again applying Lemma 7.4.1.}\end{aligned}$$

The last remarks follow from the existence of natural maps $R \to R_{\mathfrak{p}}$ and $R \to \widehat{R}^{\mathfrak{p}}$ (see [AM69, Chapters 3 and 10]). □

Using this, we can now sketch a proof of the following fact, which is just a part of [Eno84, Theorem]:

Theorem 8.2.2 (Enochs)**.** *For R commutative and Noetherian, an R-module B is cotorsion-flat if and only if*

$$B \cong \prod_{\mathfrak{p}\in\mathrm{Spec}(R)} \widehat{R_{\mathfrak{p}}^{(X_{\mathfrak{p}})}},$$

where the decomposition is uniquely determined by the dimension of the free modules.

Sketch of proof. Assume B is cotorsion-flat. Then B is a direct summand of $\prod T_{\mathfrak{q}}$ by [Eno84, Lemma 2.3]; also see the discussion in Section 7.4 above. But if $\prod T_{\mathfrak{q}} = G_1 \oplus G_2$, then we get an induced decomposition $T_{\mathfrak{q}} = T_{\mathfrak{q}}^1 \oplus T_{\mathfrak{q}}^2$ for each prime $\mathfrak{q}$ since direct summands of completions of free modules are again completions of free modules (by [Eno84, page 181]), so that $G_1 \cong \prod T_{\mathfrak{q}}^1$ and $G_2 \cong \prod T_{\mathfrak{q}}^2$. (There is a fair amount of work that goes into this implication.)

Conversely, for any $\mathfrak{q}$, $E(R/\mathfrak{q})^{(X_{\mathfrak{q}})}$ is a direct summand of $\oplus_{\mathfrak{p}} E(R/\mathfrak{p})^{(X_{\mathfrak{p}})}$, and so $\mathrm{Hom}_R(E(R/\mathfrak{q}), E(R/\mathfrak{q})^{(X_{\mathfrak{q}})})$ is a direct summand of $\mathrm{Hom}_R(E(R/\mathfrak{q}), \oplus_{\mathfrak{p}} E(R/\mathfrak{p})^{(X_{\mathfrak{p}})})$.

Since $\prod_{\mathfrak{q}} T_{\mathfrak{q}} \cong \prod_{\mathfrak{q}} \mathrm{Hom}_R(E(R/\mathfrak{q}), E(R/\mathfrak{q})^{(X_{\mathfrak{q}})})$ by Lemma 7.4.1, we have that $\prod_{\mathfrak{q}} T_{\mathfrak{q}}$ is isomorphic to a direct summand of

$$\prod_{\mathfrak{q}} \mathrm{Hom}_R(E(R/\mathfrak{q}), \oplus_{\mathfrak{p}} E(R/\mathfrak{p})^{(X_{\mathfrak{p}})}) \cong \mathrm{Hom}_R(\oplus_{\mathfrak{q}} E(R/\mathfrak{q}), \oplus_{\mathfrak{p}} E(R/\mathfrak{p})^{(X_{\mathfrak{p}})}).$$

As R is Noetherian, arbitrary direct sums of injectives are injective. By the remarks earlier (also [Eno84, Lemma 2.1]), direct summands of this module are cotorsion-flat. Hence $\prod_{\mathfrak{q}} T_{\mathfrak{q}}$ is cotorison-flat. □

8.3 Minimal cotorsion-flat resolutions

This section will be devoted to a minimality criterion for cotorsion-flat resolutions. Recall that a complex B is *minimal* if each homotopy equivalence $\gamma : B \to B$ is an isomorphism [AM02] (equivalently, if each map $\gamma : B \to B$ homotopic to id_B is an isomorphism).

Lemma 8.3.1 (cf. Lemma 1.7 of [AM02]). *Let B be a minimal complex of R-modules.*

1. *If A is a contractible subcomplex of B that is degreewise a direct summand, then $A = 0$.*

2. *If A is a contractible quotient complex of B that is degreewise a direct summand, then $A = 0$.*

Proof. The first part is exactly [AM02, Lemma 1.7(3)]. The second part is a dual argument, which we include for completeness. Set $B' = \ker(B \to A)$. The (degreewise split) exact sequence $0 \to B' \xrightarrow{\iota} B \to A \to 0$ induces an exact sequence

$$0 \to \mathrm{Hom}_R(B, B') \xrightarrow{\iota_*} \mathrm{Hom}_R(B, B) \to \mathrm{Hom}_R(B, A) \to 0.$$

We claim that since A is contractible, so is $\mathrm{Hom}_R(B,A)$. Let h be a homotopy between 1_A and 0_A, i.e., $1_A = d_A h + h d_A$. Recall that the differential ∂ of $\mathrm{Hom}_R(B,A)$ is defined as $\partial(f) = d_A f - (-1)^{|f|} f d_B$ for any $f \in \mathrm{Hom}_R(B,A)$. Set $h_* = \mathrm{Hom}_R(B,h)$, which is also a map of cohomological degree -1, defined by $f \mapsto hf$. Then, for any $f \in \mathrm{Hom}_R(B,A)$, consider the following:

$$\begin{aligned}
\partial h_*(f) + h_*\partial(f) &= \partial(hf) + h\partial(f) \\
&= d_A hf - (-1)^{|hf|} hfd_B + hd_A f - (-1)^{|f|} hfd_B \\
&= 1_A(f) + (-1)^{|f|} hfd_B - (-1)^{|f|} hfd_B, \text{ since } |h| = -1, \\
&= 1_A(f).
\end{aligned}$$

Thus A being contractible implies that $\mathrm{Hom}_R(B,A)$ is contractible as well. Hence the map $\mathrm{Hom}_R(B,B') \to \mathrm{Hom}_R(B,B)$ is a quasi-isomorphism; in particular,

$$H^0(\mathrm{Hom}_R(B,B')) \cong H^0(\mathrm{Hom}_R(B,B)),$$

so for any chain map $f : B \to B$, there exists a chain map $g : B \to B'$ such that ιg is homotpic to f. In particular, there exists $g : B \to B'$ such that ιg is homotopic to 1_B. Since B is minimal, ιg is an isomorphism, and therefore ι is surjective by [AM02, Lemma 1.7(1)]. This forces $A = 0$. □

In a local ring $(R,\mathfrak{m})$, a complex P of finitely generated free modules is minimal if and only if $P \otimes R/\mathfrak{m}$ has zero differential [AM02, Proposition 8.1]. A consequence of the following lemma is that the forward implication does not require the finite generation hypothesis.

Lemma 8.3.2. *Let $(R,\mathfrak{m})$ be local. If $d : R^{(X)} \to R^{(Y)}$ is a map of free R-modules,* *for some arbitrary index sets X and Y, that satisfies $d \otimes R/\mathfrak{m} \neq 0$, then there exists a split surjection $\pi : R^{(Y)} \to R$ such that $\pi d : R^{(X)} \to R$ is also split surjective.*

Proof. Set $\overline{d} = d \otimes R/\mathfrak{m}$ and $k = R/\mathfrak{m}$. By assumption, $0 \neq \overline{d} : k^{(X)} \to k^{(Y)}$, so there exists $b \in k^{(X)}$ such that $\overline{d}(b) = c \neq 0$. Extend c to a basis on $k^{(Y)}$, and take $\pi_c : k^{(Y)} \to k$ to be the natural projection onto the cth-component for this new basis. Also let $\pi^X : R^{(X)} \to k^{(X)}$, $\pi^Y : R^{(Y)} \to k^{(Y)}$, and $\pi' : R \to k$ be the canonical surjections. We now have the following commutative diagram, where $\pi : R^{(Y)} \to R$ exists and makes the diagram commute since $R^{(Y)}$ is a projective R-module:

$$\begin{array}{ccccc} R^{(X)} & \xrightarrow{d} & R^{(Y)} & \overset{\pi}{\dashrightarrow} & R \\ \Big\downarrow{\scriptstyle \pi^X} & & \Big\downarrow{\scriptstyle \pi^Y} & & \Big\downarrow{\scriptstyle \pi'} \\ k^{(X)} & \xrightarrow{\overline{d}} & k^{(Y)} & \xrightarrow{\pi_c} & k \end{array}$$

As this commutes, $\pi'\pi d = \pi_c \pi^Y d = \pi_c \overline{d} \pi^X$, and there exists $\widetilde{b} \in R^{(X)}$ such that $\pi^X(\widetilde{b}) = b$. Then $\pi'\pi d(\widetilde{b}) = 1$, so $\pi d(\widetilde{b}) \notin \mathfrak{m}$, hence πd is surjective as desired. Both π and πd are also split as R is projective. $\square$

Under the hypotheses of the lemma, we obtain a commutative diagram with split surjective vertical maps:

$$\begin{array}{ccc} R^{(X)} & \xrightarrow{d} & R^{(Y)} \\ \Big\downarrow{\scriptstyle \pi d} & & \Big\downarrow{\scriptstyle \pi} \\ R & \xrightarrow{=} & R \end{array}$$

Consequently, an application of Lemma 8.3.1 shows that for a local ring $(R, \mathfrak{m})$ and *any* complex of free modules (F, ∂) (not necessarily degreewise finitely generated), if F is minimal then $\partial(F) \subseteq \mathfrak{m}F$.

Lemma 8.3.3. *Let R be a commutative Noetherian ring of finite Krull dimension.*

1. *If M is a cotorsion module and $B \to M$ is a left cotorsion-flat resolution, then B is built from cotorsion-flat precovers.*

2. *If M is a flat module and $M \to B$ is a right cotorsion-flat resolution, then B is built from cotorsion-flat preenvelopes.*

Proof. Let M be cotorsion and $B \to M$ be a left cotorsion-flat resolution. Then the complex

$$\cdots \to B_1 \to B_0 \to M \to 0$$

is an exact complex of cotorsion modules, and therefore (since R has finite Krull dimension) the syzygies are all cotorsion as well [MW16]. Hence for any cotorsion-flat R-module T,

$$\mathrm{Hom}_R(T, B_0) \twoheadrightarrow \mathrm{Hom}_R(T, M)$$

is surjective since $\mathrm{Ext}^1_R(T, \ker(B_0 \to M)) = 0$. This implies that $B_0 \to M$ is a cotorsion-flat precover. Inductively, we see that the entire complex B is built from cotorsion-flat precovers.

If M is flat and $M \to B$ is a right cotorsion-flat resolution, we first note that since the cotorsion-flat resolution

$$0 \to M \to B^0 \to B^1 \to \cdots$$

is exact and $\dim(R) = d < \infty$, its syzygies are flat. This follows because the syzygies have finite flat dimension, and by the Jensen-Raynaud-Gruson theorem (see the discussion in [MW16, Section 1]), must also have finite projective dimension (of at most d) since $\dim(R) < \infty$. In more detail, if Z^i is the i-th syzygy in this complex, then for a cotorsion module C, we have $\mathrm{Ext}^1_R(Z^i, C) \cong \mathrm{Ext}^{d+1}_R(Z^{i+d}, C) = 0$, thus Z^i is flat as C was an arbitrary cotorsion module by [EJ00, Lemma 7.1.4]. Therefore, for any cotorsion-flat R-module T,

$$\operatorname{Hom}_R(B^0, T) \twoheadrightarrow \operatorname{Hom}_R(M, T)$$

is surjective since $\operatorname{Ext}^1_R(\operatorname{coker}(M \to B^0), T) = 0$; in particular, $M \to B^0$ is a cotorsion-flat pre cover. Inductively, we see that at each step, B is built from cotorsion-flat preenvelopes. □

One fact we will use throughout the following proof is that if $\phi : C \to N$ is an $\mathcal{F}$-cover, and $h : N \to L$ is an isomorphism, then $h \circ \phi : B \to L$ is also an $\mathcal{F}$-cover; dually, if $\phi : N \to C$ is an $\mathcal{F}$-envelope and $h : L \to N$ is an isomorphism, then $\phi \circ h : L \to C$ is an $\mathcal{F}$-envelope; we'll usually refer to this just as being a cover/envelope "up to isomorphism," which will be sufficient for our purposes.

To find an appropriate criterion for minimality of cotorsion-flat resolutions, we turn to minimal injective resolutions for inspiration. Recall that an injective resolution $M \to I$ is minimal if and only if for every prime $\mathfrak{p}$, the complex $\operatorname{Hom}_R(R/\mathfrak{p}, I) \otimes_R R_\mathfrak{p}$ has zero differential [ILL+07, Lecture 3, section 3]. The following equivalent conditions give a similar description of minimal (left or right) cotorsion-flat resolutions. (The last condition is equivalent by [AM02, Lemma 1.7(1)] to B being minimal.) We will refer to a complex of cotorsion-flat modules satisfying (1) as *pseudo-minimal*. In the case of a left cotorsion-flat resolution of a cotorsion module, the equivalence of (1) and (2) was pointed out to me by Douglas Dailey [Dai16, Section 4.2].

Theorem 8.3.4. *Let R be a commutative Noetherian ring of finite Krull dimension and M a cotorsion (resp., flat) R-module with left cotorsion-flat resolution $B \to M$ (resp., right cotorsion-flat resolution $M \to B$). Then the following are equivalent:*

1. *For every $\mathfrak{p} \in \operatorname{Spec}(R)$, the complex $\operatorname{Hom}_R(R_\mathfrak{p}, B) \otimes_R R/\mathfrak{p}$ has zero differential;*

2. *Each surjection* $B_i \to \operatorname{coker}(d_{i+1})$ *is a cotorsion-flat cover (resp., each injection* $\ker(d^i) \to B^i$ *is a cotorsion-flat envelope);*

3. *If* $\gamma : B \to B$ *is homotopic to* id_B*, then* γ *is an isomorphism.*

Proof. We begin with the case where $B \to M$ is a left cotorsion-flat resolution of a cotorsion module M as above.

We first show (2) and (3) are equivalent. Suppose $B \to M$ is a resolution formed by taking cotorsion-flat covers, and $\gamma : B \to B$ is a map homotopic to 1_B. Let h be the homotopy such that $\gamma_i - 1_{B_i} = d_{i+1}h_i + h_{i-1}d_i$ for all $i \in \mathbb{Z}$. If $\pi : B_0 \to M$, then

$$\pi\gamma_0 - \pi 1_{B_0} = \pi(d_1 h_0 + 0) = 0 \implies \pi\gamma_0 = \pi 1_{B_0} = 1_M \pi,$$

so γ_0 is an endomorphism of B_0 which commutes with the cotorsion-flat cover $\pi : B_0 \to M$ implying that γ_0 is an isomorphism. For $i \geq 0$, γ_i induces an isomorphism on the $i+1$-st syzygy in B. Assume now that γ_i is an isomorphism. This induces an isomorphism which we call γ_i' of the $i+1$-st syzygy. Inductively, γ_{i+1} is an endomorphism of B_{i+1}. Since $d_{i+1} : B_{i+1} \to \Omega_{i+1}M$ (where $\Omega_{i+1}M \cong \operatorname{coker}(d_{i+2})$) is a cotorsion-flat cover, to show γ_{i+1} is an isomorphism, it is enough to notice that the following diagram commutes (for $i \geq 0$):

$$\begin{array}{ccc}
B_{i+1} & \xrightarrow{d_{i+1}} & \Omega_{i+1}M \\
\downarrow{\scriptstyle \gamma_{i+1}} & & \cong \downarrow{\scriptstyle \gamma_i'} \\
B_{i+1} & \xrightarrow{d_{i+1}} & \Omega_{i+1}M
\end{array}$$

which follows since γ is a map of complexes. Therefore $\gamma : B \to B$ is an isomorphism, so (3) holds.

Conversely, assume (3) holds for B and let $F \to M$ be a left CotFlat-resolution (formed by taking cotorsion-flat covers). First, we obtain a map $\beta : B \to F$ which lifts $\mathrm{id}_M : M \to M$ by definition of cotorsion-flat covers (at each stage, B_i is a cotorsion-flat module mapping to $\mathrm{coker}(F_{i+1} \to F_i)$, hence it factors through F_i by a map β_i). We similarly obtain a map $\alpha : F \to B$ lifting $\mathrm{id}_M : M \to M$ because each $B_i \to \mathrm{coker}(d_{i+1})$ is a cotorsion-flat pre-cover by Lemma 8.3.3. Now $\beta\alpha : F \to F$ is (degree-wise) an endomorphism of a cotorsion-flat cover and is therefore an isomorphism, showing F is a summand of B. Letting B' be the complementary summand of B, we have that B' is a bounded-on-the-right acyclic complex of cotorsion-flat modules, and hence contractible. But since B is minimal, $B' = 0$ by [AM02, Proposition 1.7], and (2) follows.

Assume (1) holds for $B \to M$, and that $F \to M$ is a left CotFlat-resolution of M (built out of cotorsion-flat covers). Again since both $B \to M$ and $F \to M$ are built from cotorsion-flat precovers (the former by Lemma 8.3.3), we obtain maps $\alpha : F \to B$ and $\beta : B \to F$, both lifting $\mathrm{id}_M : M \to M$, and such that $\beta\alpha$ is an isomorphism. We then conclude $B \cong F \oplus B'$ for some bounded-on-the-right acyclic complex of cotorsion-flats B', which is necessarily a contractible complex, i.e., the identity on B' is homotopic to the zero map. Since $\mathrm{Hom}_R(R_{\mathfrak{p}}, B) \otimes_R R/\mathfrak{p}$ has zero differential, so does $\mathrm{Hom}_R(R_{\mathfrak{p}}, B') \otimes_R R/\mathfrak{p}$. Therefore for every $\mathfrak{p} \in \mathrm{Spec}(R)$, the identity equals the zero map on the complex $\mathrm{Hom}_R(R_{\mathfrak{p}}, B') \otimes_R R/\mathfrak{p}$. This is the complex with $\kappa(\mathfrak{p})^{(Y_i^{\mathfrak{p}})}$ in degree i, for an appropriate index set $Y_i^{\mathfrak{p}}$. Consequently, $Y_i^{\mathfrak{p}} = 0$ for all $\mathfrak{p}$ and i, so that $B' = 0$, hence $B \cong F$, and therefore (2) is satisfied.

Finally, we show that (3) implies (1). Assume that B satisfies (3), or equivalently, that B is minimal. Since [Xu96, Theorem 5.2.7] shows that $\mathrm{Hom}_R(R_{\mathfrak{p}}, -)$ preserves cotorsion-flat covers of cotorsion modules, the equivalence of (2) and (3) shows that $\mathrm{Hom}_R(R_{\mathfrak{p}}, B)$ is also a minimal complex, so we reduce the problem to showing that $\widehat{B}^{\mathfrak{m}} \otimes_R R/\mathfrak{m}$ ($\cong B \otimes_R R/\mathfrak{m}$) has zero differential for a local ring $(R, \mathfrak{m})$. By Lemma

8.2.1,

$$\widehat{B}^{\mathfrak{m}} \cong (\cdots \to (T_{\mathfrak{m}})_i \xrightarrow{\widehat{d_i}} (T_{\mathfrak{m}})_{i-1} \to \cdots).$$

Set $\overline{d} = d \otimes R/\mathfrak{m}$. Towards a contradiction, suppose that $\overline{d_i} \neq 0$ for some $i \in \mathbb{Z}$. Note $(T_{\mathfrak{m}})_j \cong \widehat{R^{(X^j_{\mathfrak{m}})}}$ for each j. As $\overline{d_i} \neq 0$, Lemma 8.3.2 yields a split surjective map $\pi : R^{(X^{i-1}_{\mathfrak{m}})} \to R$ such that the composition $R^{(X^i_{\mathfrak{m}})} \to R^{(X^{i-1}_{\mathfrak{m}})} \xrightarrow{\pi} R$ is also surjective (and split). Completion at $\mathfrak{m}$ provides a split surjection $\widehat{\pi} : (T_{\mathfrak{m}})_{i-1} \to \widehat{R}$ such that $\widehat{\pi}\widehat{d_i}$ is split surjective and the following diagram commutes (and is degreewise split):

$$\begin{array}{llllllllll}
\widehat{B}^{\mathfrak{m}} : & \cdots \longrightarrow & (T_{\mathfrak{m}})_{i+1} & \longrightarrow & (T_{\mathfrak{m}})_i & \xrightarrow{\widehat{d_i}} & (T_{\mathfrak{m}})_{i-1} & \longrightarrow & (T_{\mathfrak{m}})_{i-2} & \longrightarrow \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle \widehat{\pi}\widehat{d_i}} & & \downarrow{\scriptstyle \widehat{\pi}} & & \downarrow & \\
A : & \cdots \longrightarrow & 0 & \longrightarrow & \widehat{R} & \xrightarrow{=} & \widehat{R} & \longrightarrow & 0 & \longrightarrow \cdots
\end{array}$$

Set A to be the bottom row; it is clearly a contractible quotient complex. There is a chain map $B \to \widehat{B}^{\mathfrak{m}}$ which is surjective by Lemma 8.2.1 (and degreewise split). We may compose the (degreewise split) surjections $B \to \widehat{B}^{\mathfrak{m}}$ and $\widehat{B}^{\mathfrak{m}} \to A$ to get a (degreewise split) surjective chain map $B \to A$. Thus A is a contractible quotient complex of B, which is degreewise a direct summand, so Lemma 8.3.1 implies $A = 0$, a contradiction. Therefore, the differential of $B \otimes_R R/\mathfrak{m}$ (which is the same as differential of $\widehat{B}^{\mathfrak{m}} \otimes_R R/\mathfrak{m}$) is zero as desired.

Now let us consider the case where $M \to B$ is a right cotorsion-flat resolution of a flat module M. In this case, the equivalence of (2) and (3) as well as the implication (1) implies (2) follow from dual arguments where we use instead the definition of cotorsion-flat envelopes. It is worth noting that the implications (3) implies (2) and (1) implies (2) both require the Jensen-Raynaud-Gruson Theorem, which says that modules of finite flat dimension have finite projective dimension since R has finite Krull dimension, in order for a bounded-on-the-left acyclic complex of cotorsion-flats to be contractible.

The argument for (3) implies (1) in this case requires a small amount of extra care, so we include the proof. If B is minimal, then (3) implies (2) shows that B is built from cotorsion-flat envelopes. By [Eno87, Theorem 4.2], $M/\mathfrak{p}M \to B/\mathfrak{p}B$ is a right cotorsion-flat resolution of the flat $R/\mathfrak{p}$-module $M/\mathfrak{p}M$ that is built from cotorsion-flat envelopes over $R/\mathfrak{p}$, and hence by (2) implies (3), we obtain that $M/\mathfrak{p}M \to B/\mathfrak{p}B$ is a minimal right cotorsion-flat resolution of $M/\mathfrak{p}M$ (over $R/\mathfrak{p}$). Using the ideas of Lemma 8.2.1, we see that for each cotorsion-flat module $B^i \cong \prod_{\mathfrak{q}} T^i_{\mathfrak{q}}$ we have:

$$\operatorname{Hom}_R(R_{\mathfrak{p}}, B^i) \otimes_R R/\mathfrak{p} \cong T^i_{\mathfrak{p}} \otimes_R R/\mathfrak{p} \cong \operatorname{Hom}_R(R_{\mathfrak{p}}, B^i/\mathfrak{p}B^i).$$

Therefore, the following are isomorphisms of complexes:

$$\begin{aligned}
\operatorname{Hom}_R(R_{\mathfrak{p}}, B) \otimes_R R/\mathfrak{p} &\cong \operatorname{Hom}_R(R_{\mathfrak{p}}, B/\mathfrak{p}B) \otimes_R R/\mathfrak{p}, \text{ induced from } B \to B/\mathfrak{p}B,\\
&= \operatorname{Hom}_R(R_{\mathfrak{p}}, \operatorname{Hom}_{R/\mathfrak{p}}(R/\mathfrak{p}, B/\mathfrak{p}B) \otimes_R R/\mathfrak{p}\\
&\cong \operatorname{Hom}_{R/\mathfrak{p}}(\kappa(\mathfrak{p}), B/\mathfrak{p}B) \otimes_R R/\mathfrak{p}, \text{ by standard adjointness.}
\end{aligned}$$

As our goal is to show that $\operatorname{Hom}_R(R_{\mathfrak{p}}, B) \otimes_R R/\mathfrak{p}$ has zero differential, this shows we may assume R is a domain and $\mathfrak{p} = (0)$. By Lemma 8.2.1, we have a (degreewise split) inclusion of complexes $\operatorname{Hom}_R(R_{\mathfrak{p}}, B) \to B$. Since $\mathfrak{p} = (0)$ is minimal, $\mathfrak{p}$ is the only prime that appears in the complex

$$\operatorname{Hom}_R(R_{\mathfrak{p}}, B) = \cdots \to R_{(0)}^{(X^i)} \xrightarrow{d^i} R_{(0)}^{(X^{i+1})} \to \cdots,$$

where for all $j \in \mathbb{Z}$, we have set $X^j = X^j_{(0)}$ and used that $T^j_{\mathfrak{p}} = R_{\mathfrak{p}}^{(X^j_{\mathfrak{p}})}$ since completion at $\mathfrak{p} = (0)$ changes nothing. Assume $d^i \neq 0$ for some i. Then we can find some $u \in R_{(0)}^{(X^i)}$ such that $d^i(u) = v \neq 0$. Set $\iota : R_{(0)} \to R_{(0)}^{(X^i)}$ to be multiplication by u to obtain the following commutative diagram, where $d^i\iota$ is just multiplication by v. Since $v \neq 0$, there is a surjection $\pi : R_{(0)}^{(X^{i+1})} \to R_{(0)}$ such that $\pi d^i \iota$ is an isomorphism; thus ι and $d^i\iota$ are injections. Additionally, these maps are split injections since all the modules are $R_{(0)}$-vector spaces.

$$
\begin{array}{ccccccccccccc}
\operatorname{Hom}_R(R_{\mathfrak{p}}, B) = & & \cdots \longrightarrow & R_{(0)}^{(X^{i-1})} & \longrightarrow & R_{(0)}^{(X^i)} & \xrightarrow{d^i} & R_{(0)}^{(X^{i+1})} & \longrightarrow & R_{(0)}^{(X^{i+2})} & \longrightarrow \cdots \\
\uparrow & & & \uparrow & & \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle d^i\iota} & & \uparrow & \\
A := & & \cdots \longrightarrow & 0 & \longrightarrow & R_{(0)} & = & R_{(0)} & \longrightarrow & 0 & \longrightarrow \cdots
\end{array}
$$

Composing the (degreewise split) injection $A \to \operatorname{Hom}_R(R_{\mathfrak{p}}, B)$ with the (degreewise split) injection $\operatorname{Hom}_R(R_{\mathfrak{p}}, B) \to B$, we obtain a subcomplex A of B that is contractible and degreewise a summand of B. As B is minimal, Lemma 8.3.1 forces $A = 0$, a contradiction. Therefore, $\operatorname{Hom}_R(R_{\mathfrak{p}}, B) \otimes_R R/\mathfrak{p}$ must have zero differential as desired, and so (1) holds.

□

Corollary 8.3.5. *A left* Flat*-resolution of a cotorsion module is a minimal left cotorsion-flat resolution. A right* PurInj*-resolution of a flat module is a minimal right cotorsion-flat resolution.*

The proposition gives some evidence for a positive answer to the following question:

Question 8.3.6. *For any complex B of cotorsion-flat R-modules, is B minimal if and only if the complex* $\operatorname{Hom}_R(R_{\mathfrak{p}}, B) \otimes_R R/\mathfrak{p}$ *has zero differential?*

We also note that the minimal cotorsion-flat resolution (of either a flat module or a cotorsion module) appears as a summand of any other cotorsion-flat resolution:

Lemma 8.3.7. *Let R be a commutative Noetherian ring of finite Krull dimension. If M is a cotorsion module and $B \to M$ is a left cotorsion-flat resolution (resp., if M is a flat module and $M \to B$ is a right cotorsion-flat resolution), then the minimal cotorsion-flat resolution appears as a direct summand of B (in either case).*

Proof. First, suppose $B \to M$ is a left cotorsion-flat resolution of a cotorsion module. By Lemma 8.3.3, we may assume that the resolution is built from cotorsion-flat precovers. The cotorsion-flat cover is a direct summand of the cotorsion-flat precover by [Xu96, Theorem 1.2.7] (or [EJ00, Proposition 5.1.2], more generally). Hence, at each stage, we take B_i' to be the direct summand of B_i that gives rise to a cotorsion-flat cover. Then $B' \to M$ is a left cotorsion-flat resolution built from cotorsion-flat covers such that B' is a direct summand of B, i.e., the minimal left cotorsion-flat resolution is a direct summand of B in this case (by Theorem 8.3.4).

For $M \to B$ a right cotorsion-flat resolution of a flat module, Lemma 8.3.3 shows that B is built from cotorsion-flat preenvelopes. Xu [Xu96, Proposition 1.2.2] shows that the cotorsion-flat envelope is a summand of the cotorsion-flat preenvelope. Taking at each stage $(B^i)'$ to be the cotorsion-flat envelope, we have that $M \to B'$ is a right cotorsion-flat resolution built from cotorsion-flat envelopes such that B' is a summand of B, i.e., the minimal cotorsion-flat resolution is a direct summand of B in this case as well (again applying Theorem 8.3.4). □

Chapter 9

Cosupport

We quickly review the notion of cosupport in a triangulated category. Ultimately, in the context we care about, it is given more simply by the formula (9.1.6) below. As earlier, let $\mathcal{T}$ be a compactly generated R-linear triangulated category and fix a *specialization closed* subset $\mathcal{V} \subseteq \operatorname{Spec}(R)$ (a subset $\mathcal{V} \subseteq \operatorname{Spec}(R)$ such that if $\mathfrak{p} \in \operatorname{Spec}(R)$ and there exists $\mathfrak{q} \in \mathcal{V}$ with $\mathfrak{q} \subseteq \mathfrak{p}$, then $\mathfrak{p} \in \mathcal{V}$ as well). Following [BIK12, Sections 2-4], there exists a localizing functor $L_{\mathcal{V}} : \mathcal{T} \to \mathcal{T}$ whose kernel is the subcategory of $\mathcal{V}$-torsion objects (this is the full subcategory with objects X such that $\operatorname{Hom}_{\mathcal{T}}(C, X)_{\mathfrak{p}} = 0$ for all compact objects C in $\mathcal{T}^c$ and $\mathfrak{p} \in \operatorname{Spec}(R) \backslash \mathcal{V}$). This localization functor induces a colocalization functor, denoted by $\Gamma_{\mathcal{V}}$, which is called the *(derived) local cohomology functor* with respect to $\mathcal{V}$. Then $L_{\mathcal{V}}$ admits a right adjoint if and only if $\Gamma_{\mathcal{V}}$ does; denote their respective right adjoints by $V^{\mathcal{V}}$ and $\Lambda^{\mathcal{V}}$. We call $\Lambda^{\mathcal{V}}$ the *(derived) local homology functor* with respect to $\mathcal{V}$.

Define the following (specialization closed) subsets of $\operatorname{Spec}(R)$:

$$\mathcal{V}(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec}(R) | \mathfrak{a} \subseteq \mathfrak{p}\}, \text{ and}$$

$$\mathcal{Z}(\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Spec}(R) | \mathfrak{q} \not\subseteq \mathfrak{p}\}.$$

Benson, Iyengar, and Krause [BIK12] define the *support* of an object X in $\mathcal{T}$ to be

$$\operatorname{supp}_R X = \{\mathfrak{p} \in \operatorname{Spec}(R) | \Gamma_{\mathcal{V}(\mathfrak{p})} L_{\mathcal{Z}(\mathfrak{p})} X \neq 0\},$$

and define the *cosupport* of an object X in $\mathcal{T}$ to be the set

$$\operatorname{cosupp}_R X = \{\mathfrak{p} \in \operatorname{Spec} R | \Lambda^{\mathcal{V}(\mathfrak{p})} V^{\mathcal{Z}(\mathfrak{p})} X \neq 0\}.$$

Our goal is to understand the manifestation of cosupport in a commutative Noetherian ring. In such a setting, we are able to give a more concrete description of cosupport, which we do next.

9.1 Cosupport in a commutative Noetherian ring

We are interested in cosupport in a commutative Noetherian ring R, where $\mathcal{T} = D(R)$ is the derived category (see Section 7.2). With this in mind, we first analyze $\Lambda^{\mathcal{V}(\mathfrak{p})}$ and $V^{\mathcal{Z}(\mathfrak{p})}$ in the derived category.

In a commutative Noetherian ring, Benson, Iyengar, and Krause show [BIK08, Theorem 9.1] that the derived local cohomology functor $\Gamma_{\mathcal{V}(\mathfrak{p})}$ agrees with the right derived functor of the $\mathfrak{p}$-torsion functor defined in Part I, namely with $\mathbb{R}\Gamma_{\mathfrak{p}}$. Henceforth, we will use $\mathbb{R}\Gamma_{\mathfrak{p}}$ to indicate the derived local cohomology functor $\Gamma_{\mathcal{V}(\mathfrak{p})}$.

We first describe $\Lambda^{\mathcal{V}(\mathfrak{p})}$. By definition, $\Lambda^{\mathcal{V}(\mathfrak{p})}$ is the right adjoint of the derived local cohomology functor $\mathbb{R}\Gamma_{\mathfrak{p}}$. The right adjoint of $\mathbb{R}\Gamma_{\mathfrak{p}}$ is $\mathbb{R}\mathrm{Hom}_R(\mathbb{R}\Gamma_{\mathfrak{p}}(R), -)$ by [Lip02, Section 4]. This also follows from derived Hom-Tensor adjunction [Lip02, 2.2] and the fact that for any complex of R-modules M,

$$\mathbb{R}\Gamma_{\mathfrak{p}}(R) \otimes^{\mathbb{L}}_R M \xrightarrow{\sim} \mathbb{R}\Gamma_{\mathfrak{p}}(M)$$

[Lip02, Corollary 3.3.1]. Therefore it follows that $\Lambda^{\mathcal{V}(\mathfrak{p})}$ and $\mathbb{R}\mathrm{Hom}_R(\mathbb{R}\Gamma_{\mathfrak{p}}(R), -)$ are isomorphic functors on $D(R)$.

For a complex M of R-modules, by identifying $\mathrm{Hom}_R(R/\mathfrak{p}^n, M) \cong \{x \in M | \mathfrak{p}^n x = 0\} \subseteq \Gamma_{\mathfrak{p}}(M)$, we obtain a (filtered) directed system

$$\mathrm{Hom}_R(R/\mathfrak{p}, M) \subseteq \mathrm{Hom}_R(R/\mathfrak{p}^2, M) \subseteq \cdots$$

whose union is $\Gamma_{\mathfrak{p}}(M)$ (cf. [ILL+07, Lecture 7]). Hence we may identify $\Gamma_{\mathfrak{p}}(M) = \varinjlim_n \mathrm{Hom}_R(R/\mathfrak{p}^n, M)$. As the construction of the directed system is functorial, choosing an injective resolution $R \to I$, we have $\Gamma_{\mathfrak{p}}(I) = \varinjlim_n \mathrm{Hom}_R(R/\mathfrak{p}^n, I)$. Therefore $\mathbb{R}\Gamma_{\mathfrak{p}}(R) \cong \varinjlim_n \mathrm{Hom}_R(R/\mathfrak{p}^n, I)$. In conjunction with the previous comment, we obtain

$$\Lambda^{\mathcal{V}(\mathfrak{p})}(-) \cong \mathbb{R}\mathrm{Hom}_R(\varinjlim_n \mathrm{Hom}_R(R/\mathfrak{p}^n, I), -) \tag{9.1.1}$$

are isomorphic functors $D(R) \to D(R)$.

We say a complex F is *semiflat* if $- \otimes_R F$ preserves quasi-isomorphisms and F^i is a flat R-module for all $i \in \mathbb{Z}$. Likewise, we say a complex I is *semiinjective* if $\mathrm{Hom}_R(-, I)$ preserves quasi-isomorphisms and I^i is an injective R-module for all $i \in \mathbb{Z}$. Finally, a complex P is *semiprojective* if $\mathrm{Hom}_R(P, -)$ preserves quasi-isomorphisms and P^i is a projective R-module for all $i \in \mathbb{Z}$. These definitions follow [AFH16]. (Compare these also with *q-flat and q-injective* complexes of [Lip02].) A *semiflat resolution* (resp., *semiinjective resolution* or *semiprojective resolution*) is a semiflat complex F (resp., a semiinjective complex I or semiprojective complex P) along with a quasi-isomorphism $F \to M$ (resp., $M \to I$ or $P \to M$). Semiflat, semiprojective, and semiinjective resolutions exist (see e.g., [AFH16]) for any complex of R-modules.

We will also need the following lifting property (see [AFH16]): If a complex P is semiprojective, M and N are complexes with $\alpha : P \to M$ any map and $\beta : N \to M$ a surjective quasi-isomorphism, then there is a map $\gamma : P \to N$ such that $\alpha = \beta\gamma$.

Lemma 9.1.2. *Suppose F is a semiflat complex and G is a complex such that $G \otimes_R -$ preserves quasi-isomorphisms of degreewise finitely generated complexes. If there is a quasi-isomorphism $F \to G$ and N is any finitely generated R-module, then $N \otimes_R F \to N \otimes_R G$ is a quasi-isomorphism as well.*

Proof. Choose a degreewise finitely generated projective resolution $P \xrightarrow{\sim} N$. We have the following diagram:

$$\begin{array}{ccc} P \otimes_R F & \longrightarrow & P \otimes_R G \\ \downarrow \sim & & \downarrow \sim \\ N \otimes_R F & \longrightarrow & N \otimes_R G \end{array}$$

where the vertical maps are quasi-isomorphisms by hypothesis on F and G. As P is also semiflat, the top map is a quasi-isomorphism as well, hence the result follows. □

Now we have:

Proposition 9.1.3. *Let M be a complex of R-modules with semiprojective resolution $P \to M$. Then*

$$\Lambda^{\mathcal{V}(\mathfrak{p})}(M) \sim \varprojlim_n (R/p^n \otimes_R P),$$

i.e., they are isomorphic in $D(R)$.

Proof. Fix a semiinjective resolution $M \to J$ and an injective resolution $R \to I$. There is a surjective quasi-isomorphism $\mathrm{Hom}_R(I, J) \to J$ and a quasi-isomorphism $P \to J$ (factoring through M). Since P is semiprojective, the lifting property above shows that there exists a map $P \to \mathrm{Hom}_R(I, J)$ commuting with $\mathrm{Hom}_R(I, J) \to J$ and $P \to J$; moreover, since the latter two maps are both quasi-isomorphisms, so is $P \to \mathrm{Hom}_R(I, J)$. Semiprojective complexes are semiflat [AFH16], so P is semiflat.

We claim that $-\otimes_R \mathrm{Hom}_R(I,J)$ preserves quasi-isomorphisms of degreewise finitely generated complexes. For any finitely generated module N, there is an isomorphism

$$N \otimes_R \mathrm{Hom}_R(I,J) \xrightarrow{\cong} \mathrm{Hom}_R(\mathrm{Hom}_R(N,I),J).$$

Thus as $\mathrm{Hom}_R(\mathrm{Hom}_R(-,I),J)$ preserves (all) quasi-isomorphisms (I and J are both semiinjective), $-\otimes_R \mathrm{Hom}_R(I,J)$ preserves quasi-isomorphisms of degreewise finitely generated complexes. Lemma 9.1.2 now gives that $R/\mathfrak{p}^n \otimes_R P \to R/\mathfrak{p}^n \otimes_R \mathrm{Hom}_R(I,J)$ is a quasi-isomorphism for all $n \geq 1$. This is needed for the last quasi-isomorphism in the following string of isomorphisms in $D(R)$:

$$\begin{aligned}
\Lambda^{\mathcal{V}(\mathfrak{p})}(M) &\sim \mathbb{R}\mathrm{Hom}_R(\varinjlim_n \mathrm{Hom}_R(R/\mathfrak{p}^n, I), M), \text{ by 9.1.1,}\\
&= \mathrm{Hom}_R(\varinjlim_n \mathrm{Hom}_R(R/\mathfrak{p}^n, I), J), \text{ see [Lip09, Section 2.4],}\\
&\cong \varprojlim_n \mathrm{Hom}_R(\mathrm{Hom}_R(R/\mathfrak{p}^n, I), J)\\
&\xleftarrow{\cong} \varprojlim_n (R/\mathfrak{p}^n \otimes_R \mathrm{Hom}_R(I,J)), \text{ these are isomorphic degreewise,}\\
&\xleftarrow{\sim} \varprojlim_n (R/\mathfrak{p}^n \otimes_R P), \text{ by Lemma 7.3.1.}
\end{aligned}$$

□

Henceforth, for any object M in $D(R)$, we make the following identification:

$$\Lambda^{\mathcal{V}(\mathfrak{p})}(M) = \varprojlim_n (R/\mathfrak{p}^n \otimes_R P),$$

where $P \to M$ is a semiprojective resolution of M. Looking ahead to using this to compute cosupport, we would like to be able to just take a semiflat resolution of M;

this is possible by the following proposition:

Proposition 9.1.4. *Let M be a complex of R-modules isomorphic in $D(R)$ to a semiflat*[1] *complex F. Then*

$$\Lambda^{\mathcal{V}(\mathfrak{p})}(M) \sim \varprojlim_n (R/\mathfrak{p}^n \otimes_R F).$$

Proof. As $\Lambda^{\mathcal{V}(\mathfrak{p})} : D(R) \to D(R)$ is a functor, $\Lambda^{\mathcal{V}(\mathfrak{p})}(M) \sim \Lambda^{\mathcal{V}(\mathfrak{p})}(F)$. Choose a semiprojective resolution $P \to F$. Then $\Lambda^{\mathcal{V}(\mathfrak{p})}(F) = \varprojlim_n (R/\mathfrak{p}^n \otimes_R P)$. Since F and P are both semiflat, we have $R/\mathfrak{p}^n \otimes_R P \to R/\mathfrak{p}^n \otimes_R F$ is a quasi-isomorphism for all $n \geq 1$ by Lemma 9.1.2. Therefore Lemma 7.3.1 yields

$$\Lambda^{\mathcal{V}(\mathfrak{p})}(M) \sim \Lambda^{\mathcal{V}(\mathfrak{p})}(F) = \varprojlim_n (R/\mathfrak{p}^n \otimes_R P) \xrightarrow{\sim} \varprojlim_n (R/\mathfrak{p}^n \otimes_R F),$$

as desired.

□

If M is a complex of R-modules, this shows that $\Lambda^{\mathcal{V}(\mathfrak{p})}$ can be thought of as the left derived functor of the $\mathfrak{p}$-adic completion functor which assigns M to $\varprojlim M/\mathfrak{p}^n M$, see [GM92, Section 2] and [Lip02, Section 4].

We also have $V^{\mathcal{Z}(\mathfrak{p})}(M) \cong \mathbb{R}\mathrm{Hom}_R(R_\mathfrak{p}, M)$ (see [BIK12, Section 4]). The following lemma is true in a more general setting, namely that $V^{\mathcal{Z}(\mathfrak{p})}\Lambda^{\mathcal{V}(\mathfrak{p})} \cong \Lambda^{\mathcal{V}(\mathfrak{p})}V^{\mathcal{Z}(\mathfrak{p})}$ as functors $D(R) \to D(R)$ (see [BIK12, page 170]), but we include an explicit proof in the setting of a commutative Noetherian ring.

[1]We really only need F to be a complex such that $F \otimes_R -$ preserves quasi-isomorphisms of degreewise finitely generated complexes in order to apply Lemma 9.1.2.

Lemma 9.1.5. *Let M be a complex of R-modules. Then*

$$\mathbb{R}\mathrm{Hom}_R(R_{\mathfrak{p}}, \Lambda^{\mathcal{V}(\mathfrak{p})}(M)) \sim \Lambda^{\mathcal{V}(\mathfrak{p})}\mathbb{R}\,\mathrm{Hom}_R(R_{\mathfrak{p}}, M).$$

Proof. Choose a projective resolution $Q \to R_{\mathfrak{p}}$ and a semiprojective resolution $P \to M$. Observe the following:

$$\begin{aligned}
\mathbb{R}\,\mathrm{Hom}_R(R_{\mathfrak{p}}, \Lambda^{\mathcal{V}(\mathfrak{p})}(M)) &= \mathbb{R}\mathrm{Hom}_R(R_{\mathfrak{p}}, \varprojlim_n (P \otimes_R R/\mathfrak{p}^n)) \\
&= \mathrm{Hom}_R(Q, \varprojlim_n (P \otimes_R R/\mathfrak{p}^n)) \\
&\cong \varprojlim_n \mathrm{Hom}_R(Q, P \otimes_R R/\mathfrak{p}^n) \\
&\cong \varprojlim_n (R/\mathfrak{p}^n \otimes_R \mathrm{Hom}_R(Q, P)).
\end{aligned}$$

For any finitely generated module N, notice that

$$\mathrm{Hom}_R(Q, P \otimes_R N) \cong N \otimes_R \mathrm{Hom}_R(Q, P),$$

hence $\mathrm{Hom}_R(Q, P \otimes_R -)$ preserves quasi-isomorphisms of degreewise finitely generated complexes (since P is semiflat and Q is semiprojective). Choose a semiflat resolution $F \xrightarrow{\sim} \mathrm{Hom}_R(Q, P)$. Lemma 9.1.2 tells us that $R/\mathfrak{p}^n \otimes_R F \xrightarrow{\sim} R/\mathfrak{p}^n \otimes_R \mathrm{Hom}_R(Q, P)$ is a quasi-isomorphism for all $n \geq 1$. Finally, Lemma 7.3.1 allows us to take limits and conclude that

$$\begin{aligned}
\varprojlim_n (R/\mathfrak{p}^n \otimes_R \mathrm{Hom}_R(Q, P)) &\sim \varprojlim_n (R/\mathfrak{p}^n \otimes_R F) \\
&\sim \Lambda^{\mathcal{V}(\mathfrak{p})}\,\mathrm{Hom}_R(Q, P), \text{ by Proposition 9.1.4,} \\
&\sim \Lambda^{\mathcal{V}(\mathfrak{p})}\,\mathbb{R}\mathrm{Hom}_R(R_{\mathfrak{p}}, M).
\end{aligned}$$

□

Therefore we can recast the definition of cosupport in a commutative Noetherian ring as:

$$\operatorname{cosupp}_R(M) = \{\mathfrak{p} \in \operatorname{Spec}(R) | H^* \mathbb{R}\operatorname{Hom}_R(R_\mathfrak{p}, \Lambda^{\mathcal{V}(\mathfrak{p})}(M)) \neq 0\}. \tag{9.1.6}$$

This is the same definition of cosupport as is given in [BIK12].

If M is a cotorsion R-module with a left cotorsion-flat resolution B or M is a flat R-module with a right cotorsion-flat resolution B (as in Section 8.1), then in particular B (and also $\operatorname{Hom}_R(R_\mathfrak{p}, B)$) is a semiflat complex satisfying Proposition 9.1.4, so we can use the complex B to compute cosupport:

$$\operatorname{cosupp}_R(M) = \{\mathfrak{p} \in \operatorname{Spec}(R) | H^* \mathbb{R}\operatorname{Hom}_R(R_\mathfrak{p}, \varprojlim_n B/\mathfrak{p}^n B) \neq 0\} \tag{9.1.7}$$

$$= \{\mathfrak{p} \in \operatorname{Spec}(R) | H^* \varprojlim_n (R/\mathfrak{p}^n \otimes_R \operatorname{Hom}_R(R_\mathfrak{p}, B)) \neq 0\}, \tag{9.1.8}$$

where the second equality follows from Lemma 9.1.5 and the fact that B is a complex of cotorsion modules.

An alternate (and equivalent) definition is given by Christensen and Iyengar in [CI15, Equation 3.1].

One motivation for considering cosupport in a commutative Noetherian ring is its relation to support, which we now briefly review. The *(cohomological or small) support* of a complex M of R-modules, in a commutative Noetherian ring, can be described as follows:

$$\operatorname{supp}_R M = \{\mathfrak{p} \in \operatorname{Spec}(R) | \operatorname{Ext}^*_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, M_\mathfrak{p}) \neq 0\}, \tag{9.1.9}$$

or equivalently, $E(R/\mathfrak{p})$ appears in a minimal injective resolution of M (see [BIK08, Theorem 9.1, Remark 9.2] and [Fox79]). One of our goals is to give a similar description of cosupport in a commutative Noetherian ring.

9.2 Computing cosupport with cotorsion-flat resolutions

For a cotorsion-flat module B, the previous section shows that

$$\operatorname{cosupp}_R B = \{\mathfrak{p} \in \operatorname{Spec}(R) | \operatorname{Ext}^*_R(R_\mathfrak{p}, \widehat{B}^\mathfrak{p}) \neq 0\}.$$

This simpler definition for the cosupport of a cotorsion-flat module partially motivates our use of cotorsion-flat resolutions to understand cosupport better. The following lemma is known to the experts, but we include a proof for completeness.

Lemma 9.2.1. *Let R be a commutative Noetherian ring of finite Krull dimension and F a complex of flat R-modules. If one of the following holds:*

- *$F_i = 0$ for $i \ll 0$, or*
- *$F_i = 0$ for $i \gg 0$ and in addition F_i is cotorsion for $i \in \mathbb{Z}$,*

then $\widehat{F}^\mathfrak{p}$ is acyclic if and only if $F/\mathfrak{p}F$ is acyclic.

Proof. Suppose $\widehat{F}^\mathfrak{p}$ is acyclic. As R has finite Krull dimension, $\widehat{F}^\mathfrak{p}$ is again a complex of flat R-modules [Eno95, Proposition 2.3]. If $\widehat{F}^\mathfrak{p}$ is bounded on the right and acyclic, its syzygies are flat as well (one way to see this is by applying $\operatorname{Hom}_R(-, C)$ for some cotorsion module C). Alternatively, if F is a complex of cotorsion-flats and bounded on the left, then $\widehat{F}^\mathfrak{p}$ is also such a complex. If K^i is the i-th syzygy in this complex, then for a cotorsion module C, we have $\operatorname{Ext}^1_R(K^i, C) \cong \operatorname{Ext}^{d+1}_R(K^{i+d}, C) = 0$, as K^j has finite flat dimension, hence also finite projective dimension by the Jensen-Raynaud-Gruson theorem (see the discussion in [MW16, Section 1]), and thus, as $\dim(R) = d < \infty$, it vanishes. This implies that the syzygies in this case are also flat. Hence applying $- \otimes_R R/\mathfrak{p}$ to either such complex (bounded on the right or bounded on the left and degreewise cotorsion-flat) preserves acyclicity, hence $\widehat{F}^{\mathfrak{p}} \otimes_R R/\mathfrak{p} \cong F/\mathfrak{p}F$ is acyclic as well.

Conversely, assume $F/\mathfrak{p}F$ is acyclic. Consider the exact triangle $\mathfrak{p}F \to F \to F/\mathfrak{p}F \to$ in $D(R)$. As $F/\mathfrak{p}F$ is acyclic, we have $\mathfrak{p}F \to F$ is a quasi-isomorphism. For each $i \geq 1$, applying $\mathfrak{p}^i \otimes_R -$ to the quasi-isomorphism $\mathfrak{p}F \to F$, one obtains $\mathfrak{p}^{i+1}F \to \mathfrak{p}^i F$ is a quasi-isomorphism (this follows since $F/\mathfrak{p}F$ is a complex of flat $R/\mathfrak{p}$-modules, hence $F/\mathfrak{p} \otimes \mathfrak{p}^i/\mathfrak{p}^{i+1} \cong \mathfrak{p}^i F/\mathfrak{p}^{i+1}F$ is acyclic). For each $i \geq 1$, we may compose these quasi-isomorphisms to see that $\mathfrak{p}^i F \to F$ is a quasi-isomorphism for all $i \geq 1$, and therefore $F/\mathfrak{p}^i F$ is acyclic for all $i \geq 1$. Since the system

$$\cdots \to F/\mathfrak{p}^{i+1}F \to F/\mathfrak{p}^i F \to \cdots \to F/\mathfrak{p}F$$

satisfies the Mittag-Leffler condition (all the maps are surjective), there is a short exact sequence [Wei94, Theorem 3.5.8]

$$0 \to \varprojlim_n{}^1 H_{j+1}(F/\mathfrak{p}^n F) \to H_j(\varprojlim_n (F/\mathfrak{p}^n F)) \to \varprojlim_n H_j(F/\mathfrak{p}^n F) \to 0$$

which implies that $\varprojlim_n (F/\mathfrak{p}^n F) = \widehat{F}^{\mathfrak{p}}$ is also acyclic. Notice that we required no boundedness assumption for this implication. $\square$

We are now prepared to prove a result that essentially says that minimal cotorsion-flat resolutions detect cosupport of certain complexes (dual to the fact that minimal injective resolutions detect support). If B is a complex of cotorsion-flat modules, we say B is *pseudo-minimal* if for every $\mathfrak{p} \in \operatorname{Spec}(R)$, the complex $\operatorname{Hom}_R(R_\mathfrak{p}, B) \otimes_R R/\mathfrak{p}$ has zero differential. For left cotorsion-flat resolutions of cotorsion modules and right cotorsion-flat resolutions of flat modules, Theorem 8.3.4 shows that pseudo-minimal is equivalent to minimal.

Theorem 9.2.2. *Let R be a commutative Noetherian ring of finite Krull dimension and M a complex of R-modules that is quasi-isomorphic to a pseudo-minimal complex B of cotorsion-flat modules that is bounded on one side. For each $i \in \mathbb{Z}$, we have $B^i \cong \prod_\mathfrak{q} \widehat{R_\mathfrak{q}^{(X_\mathfrak{q}^i)}}^\mathfrak{q}$ for some (possibly zero or infinite) sets $X_\mathfrak{q}^i$. Then*

$$\mathfrak{p} \in \operatorname{cosupp}_R(M) \iff X_\mathfrak{p}^i \neq \varnothing \text{ for some } i \in \mathbb{Z}.$$

Proof. By definition, $\mathfrak{p} \in \operatorname{cosupp}_R(M)$ if and only if $H^* \mathbb{R}\operatorname{Hom}_R(R_\mathfrak{p}, \Lambda^{\mathcal{V}(\mathfrak{p})} M) \neq 0$. By (9.1.8), we equivalently have $\mathfrak{p} \in \operatorname{cosupp}_R(M)$ if and only if $\varprojlim_n (R/\mathfrak{p}^n \otimes_R \operatorname{Hom}_R(R_\mathfrak{p}, B))$ is not acyclic. By Lemma 9.2.1, this complex is acyclic if and only if $\operatorname{Hom}_R(R_\mathfrak{p}, B) \otimes_R R/\mathfrak{p}$ is acyclic. As B is pseudo-minimal, this latter complex has zero differential, so $\mathfrak{p} \in \operatorname{cosupp}_R(M)$ if and only if $\operatorname{Hom}_R(R_\mathfrak{p}, B) \otimes_R R/\mathfrak{p}$ is not the zero complex. In degree i, the complex $\operatorname{Hom}_R(R_\mathfrak{p}, B) \otimes R/\mathfrak{p}$ is isomorphic to $T_\mathfrak{p}^i \otimes_R R/\mathfrak{p}$ (by Lemma 8.2.1), and

$$T_\mathfrak{p}^i \otimes_R R/\mathfrak{p} = \widehat{R_\mathfrak{p}^{(X_\mathfrak{p}^i)}}^\mathfrak{p} \otimes_R R/\mathfrak{p} \cong R_\mathfrak{p}^{(X_\mathfrak{p}^i)} \otimes_R R/\mathfrak{p} \cong (R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})^{(X_\mathfrak{p}^i)}.$$

Consequently, $\mathfrak{p} \in \operatorname{cosupp}_R(M)$ if and only if $X_\mathfrak{p}^i \neq \varnothing$ for some $i \in \mathbb{Z}$. □

If a complex B satisfies $X_\mathfrak{p}^i \neq \varnothing$ for some $i \in \mathbb{Z}$ as in the theorem, we colloquially say $\mathfrak{p}$ *appears in* B.

Corollary 9.2.3. *Let R be a commutative Noetherian ring of finite Krull dimension. If M is either a cotorsion R-module with left minimal cotorsion-flat resolution B or M is a flat R-module with right minimal cotorsion-flat resolution B, then $\mathfrak{p} \in \operatorname{cosupp}_R(M) \iff \mathfrak{p}$ appears in B.*

Proof. Theorem 8.3.4 shows that such a minimal cotorsion-flat resolution is pseudo-minimal, and so Theorem 9.2.2 applies. □

In Chapter 8 we showed that if M is cotorsion, the left Flat-resolution is a minimal cotorsion-flat resolution; if M is flat, the right PurInj-resolution is a minimal right cotorosion-flat resolution (both of which are bounded on the right).

An immediate consequence is that we are now able to easily construct a module with a given cosupport. Let $W \subseteq \operatorname{Spec}(R)$ be any subset. Then $M := \prod_{\mathfrak{p}\in W} \widehat{R_{\mathfrak{p}}}^{\mathfrak{p}}$ is an R-module with $\operatorname{cosupp}_R(M) = W$.

9.3 Properties of cosupport

Unlike the support of a module, cosupport does not localize well, as noted by [BIK12, Section 4]. However, the cosupport of the *colocalization* of a cotorsion module behaves as we might expect:

Proposition 9.3.1. *Let M be a cotorsion R-module and $\mathfrak{p} \in \operatorname{Spec}(R)$. Then*

$$\operatorname{cosupp}_R \operatorname{Hom}_R(R_{\mathfrak{p}}, M) = \{\mathfrak{q} \in \operatorname{cosupp}_R(M) | \mathfrak{q} \subseteq \mathfrak{p}\}.$$

Proof. Let $B \to M$ be a minimal left cotorsion-flat resolution. Then [Xu96, Theorem 5.2.7] in conjunction with Theorem 8.3.4 shows that $\operatorname{Hom}_R(R_{\mathfrak{p}}, B) \to \operatorname{Hom}_R(R_{\mathfrak{p}}, M)$ is also a minimal left cotorsion-flat resolution. By Theorem 9.2.2 and Lemma 8.2.1, the primes appearing in $\operatorname{Hom}_R(R_{\mathfrak{p}}, B)$ are precisely those in the cosupport of M that are also contained in $\mathfrak{p}$. □

The following is a useful relation between the cosupport and support of finitely generated modules over a Gorenstein ring, which is essentially a consequence of [BIK12, Theorem 9.7] (see also [CI15, Remark 3.4]):

Proposition 9.3.2. *Suppose R is Gorenstein. Let M be a complex of R-modules with finitely generated cohomology. Then*

$$\operatorname{cosupp}_R(M) = \operatorname{cosupp}_R(R) \cap \operatorname{supp}_R(M).$$

Proof. For complexes L and N, there is an equality [CI15, Remark 3.4] (cf. [BIK12, Theorem 9.7]):

$$\operatorname{cosupp}_R \mathbb{R}\operatorname{Hom}_R(L, N) = \operatorname{supp}_R(L) \cap \operatorname{cosupp}_R(N). \tag{9.3.3}$$

Fix a minimal injective resolution of $R \to D$ of R, and recall that D is a dualizing complex for R since R is Gorenstein; in particular, there is a natural quasi-isomorphism

$$R \xrightarrow{\cong} \operatorname{Hom}_R(D, D).$$

Set $P \to M$ to be a semiprojective resolution. By [IK06, Corollary 5.5], P is acyclic if and only if $\operatorname{Hom}_R(P, R)$ is acyclic (since in a Gorenstein ring, acyclic and totally acyclic complexes coincide). This implies that $\operatorname{supp}_R P = \operatorname{supp}_R \operatorname{Hom}_R(P, R)$. Therefore, we have:

$$\begin{aligned}
\operatorname{cosupp}_R M &= \operatorname{cosupp}_R P \\
&= \operatorname{cosupp}_R(P \otimes_R \operatorname{Hom}_R(D, D)), \text{ since } D \text{ is a dualizing complex,} \\
&= \operatorname{cosupp}_R \operatorname{Hom}_R(\operatorname{Hom}_R(P, D), D), \text{ follows from [Ish65, Lemma 1.6],} \\
&= \operatorname{supp}_R \operatorname{Hom}_R(P, D) \cap \operatorname{cosupp}_R D, \text{ by (9.3.3),} \\
&= \operatorname{supp}_R \operatorname{Hom}_R(P, R) \cap \operatorname{cosupp}_R R \\
&= \operatorname{supp}_R P \cap \operatorname{cosupp}_R R, \text{ by the above remark,} \\
&= \operatorname{supp}_R M \cap \operatorname{cosupp}_R R.
\end{aligned}$$

□

9.4 Cosupport of cotorsion modules

We start by computing the cosupport of various cotorsion modules. Refer to Section 7.4 regarding cotorsion modules. Every cotorsion module has a minimal left cotorsion-flat resolution (which is given by taking flat covers; see Section 8.1), which we are able to utilize via Theorem 9.2.2.

We begin with an example:

Example 9.4.1. For a local ring $(R, \mathfrak{m})$, note that $R/\mathfrak{m}$ is cotorsion since $R/\mathfrak{m} \cong \operatorname{Hom}_R(R/\mathfrak{m}, E(R/\mathfrak{m}))$. Moreover, $R/\mathfrak{m}$ has an injective resolution involving only $E(R/\mathfrak{m})$, say $R/\mathfrak{m} \to E$ is the minimal injective resolution. Applying the exact functor $\operatorname{Hom}_R(-, E(R/\mathfrak{m}))$ to this yields:

$$\operatorname{Hom}_R(E, E(R/\mathfrak{m})) \to \operatorname{Hom}_R(R/\mathfrak{m}, E(R/\mathfrak{m})) \cong R/\mathfrak{m},$$

where since $\mathfrak{m}$ is the only prime appearing in E, we obtain:

$$\mathrm{Hom}_R(E, E(R/\mathfrak{m})) = \cdots \to \widehat{R_{\mathfrak{m}}^{(X_{\mathfrak{m}}^1)}} \to \widehat{R_{\mathfrak{m}}^{(X_{\mathfrak{m}}^0)}} \to 0$$

is a left cotorsion-flat resolution of $R/\mathfrak{m}$ with $X_{\mathfrak{p}}^i = 0$ for all $\mathfrak{p} \neq \mathfrak{m}$. We claim $\mathrm{Hom}_R(E, E(R/\mathfrak{m}))$ is minimal: Since E is minimal, we know that $\mathrm{Hom}_R(R/\mathfrak{m}, E)$ has zero differential, and therefore:

$$\begin{aligned}\mathrm{Hom}_R(R_{\mathfrak{m}}, \mathrm{Hom}_R(E, E(R/\mathfrak{m}))) \otimes_R R/\mathfrak{m} &\cong \mathrm{Hom}_R(E, E(R/\mathfrak{m})) \otimes R/\mathfrak{m} \\ &\cong \mathrm{Hom}_R(\mathrm{Hom}_R(R/\mathfrak{m}, E), E(R/\mathfrak{m}))\end{aligned}$$

has zero differential as well, implying that $\mathrm{Hom}_R(E, E(R/\mathfrak{m}))$ is minimal by Theorem 8.3.4. In particular, we have shown the complex $\mathrm{Hom}_R(E, E(R/\mathfrak{m})$ is pseudo-minimal, and so Theorem 9.2.2 yields the desired result:

$$\mathrm{cosupp}_R(R/\mathfrak{m}) = \{\mathfrak{m}\} = \mathrm{supp}_R(R/\mathfrak{m}).$$

As for injective modules, we have:

Proposition 9.4.2. *Let R be a commutative Noetherian ring and $E(R/\mathfrak{p})$ an indecomposable injective R-module. Then*

$$\mathrm{cosupp}_R(E(R/\mathfrak{p})) = \{\mathfrak{q} \in \mathrm{Spec}(R) | \mathfrak{q} \subseteq \mathfrak{p}\}.$$

Proof. Let $R \to I$ be the minimal injective resolution of R. Then we have a quasi-isomorphism

$$\mathrm{Hom}_R(I, E(R/\mathfrak{p})) \to \mathrm{Hom}_R(R, E(R/\mathfrak{p})) \cong E(R/\mathfrak{p}).$$

Since I is minimal, $\operatorname{Hom}_R(R/\mathfrak{q}, I_\mathfrak{q}) \cong \operatorname{Hom}_{R_\mathfrak{q}}(\kappa(\mathfrak{q}), I_\mathfrak{q})$ has zero differential for all $\mathfrak{q} \in \operatorname{Spec}(R)$, and thus

$$\begin{aligned}\operatorname{Hom}_R(R_\mathfrak{q}, \operatorname{Hom}_R(I, E(R/\mathfrak{p}))) \otimes_R R/\mathfrak{q} &\cong \operatorname{Hom}_R(I_\mathfrak{q}, E(R/\mathfrak{p})) \otimes_R R/\mathfrak{q} \\ &\cong \operatorname{Hom}_R(\operatorname{Hom}_R(R/\mathfrak{q}, I_\mathfrak{q}), E(R/\mathfrak{p}))\end{aligned}$$

has zero differential as well. Hence $\operatorname{Hom}_R(I, E(R/\mathfrak{p}))$ is a pseudo-minimal cotorsion-flat resolution (which is also minimal by Theorem 8.3.4). Recalling that

$$\operatorname{Hom}_R(E(R/\mathfrak{q}), E(R/\mathfrak{p})) \cong \begin{cases} \widehat{R_\mathfrak{q}^{(X_\mathfrak{q})}}^{\mathfrak{q}}, & \mathfrak{q} \subseteq \mathfrak{p} \\ 0 & \mathfrak{q} \not\subseteq \mathfrak{p} \end{cases},$$

the result follows from Theorem 9.2.2 (using that for $\mathfrak{q} \subseteq \mathfrak{p}$, $\mathfrak{q}$ appears in I hence there exists $X_\mathfrak{q}^i \neq \varnothing$ for some $i \in \mathbb{Z}$ in $\operatorname{Hom}_R(I, E(R/\mathfrak{p}))$). □

9.5 Cosupport of flat modules, low dimensional rings, and finitely generated modules

Recall that every flat module has a minimal right cotorsion-flat resolution, which is given by taking pure-injective envelopes as discussed in Section 8.1.

Compare the following with [BIK12, Proposition 4.19]:

Proposition 9.5.1. *Suppose R is a complete semi-local ring with maximal ideals $\mathfrak{m}_1, ..., \mathfrak{m}_n$. Then*

$$\operatorname{cosupp}_R(R) = \{\mathfrak{m}_1, ..., \mathfrak{m}_n\}.$$

Proof. In this case, the minimal right cotorsion-flat resolution has one term: $\prod_{i=1}^n \widehat{R_{\mathfrak{m}_i}}^{\mathfrak{m}_i}$, which provides the desired result by Theorem 9.2.2. □

If R is Gorenstein (in addition to being complete semi-local), then Proposition 9.3.2 shows that any complex of R-modules M with H^*M finitely generated satisfies $\operatorname{cosupp}_R M = \{\mathfrak{m}_1, ..., \mathfrak{m}_n\} \cap \operatorname{supp}_R M$.

Proposition 9.5.2. *Suppose R is a 1-dimensional domain that is not a complete local ring. Then*

$$\operatorname{cosupp}_R(R) = \operatorname{Spec}(R).$$

Proof. We use the structure of the right PurInj-resolution of such a ring R, which can be found in [Eno89]. By Theorem 8.3.4, the PurInj-resolution of R is a minimal right cotorsion-flat resolution. Thus the minimal right cotorsion-flat resolution of R in degree 0 is precisely $\prod_{\mathfrak{m}} \widehat{R_{\mathfrak{m}}}$, where the product is over all maximal ideals [War69], cf. also [Eno89]. Only minimal primes may appear in degree 1 by [Eno87, Theorem 2.1], and since R is a domain, the only minimal prime is (0). Hence the minimal right cotorsion-flat resolution of R has the form

$$0 \to \prod_{\mathfrak{m}} \widehat{R_{\mathfrak{m}}} \to \widehat{R_{(0)}^{(X)}} \to 0,$$

for a possibly infinite index set X. If $R \to \prod_{\mathfrak{m}} \widehat{R_{\mathfrak{m}}}$ was an isomorphism, since R is a domain, we would necessarily obtain R is complete local, contrary to hypothesis. Hence the map $R \to \prod_{\mathfrak{m}} \widehat{R_{\mathfrak{m}}}$ is not an isomorphism, so the cardinality of X is at least 1. It only remains to appeal to Theorem 9.2.2 (or Corollary 9.2.3). □

Corollary 9.5.3. *The rings $\mathbb{Z}$ and $k[x]$ (for any field k) both have full cosupport.*

Proposition 9.5.4. *If $R = k[x, y]$ for any uncountable field k, then*

$$\operatorname{cosupp}_R(R) = \operatorname{Spec}(R).$$

Proof. By Theorem 8.3.4, the right PurInj-resolution of R is a minimal right cotorsion-flat resolution. When $R = k[x, y]$ for an uncountable field k, [Eno89, Proposition 2.2] shows that if a prime $\mathfrak{p}$ appears in degree $i+1$ of such a resolution, then there exists a prime q strictly containing p that appears in degree i. Since k is uncountable, [Gru71, Proposition 3.2] yields that $\operatorname{Ext}^2_R(R_{(0)}, R) \neq 0$. Therefore (0) appears in degree 2, hence a height one prime must appear in degree 1. Finally, [Eno89, Remark 3, page 48] says that for a coordinate ring over any field, if a prime $\mathfrak{p}$ appears in degree i of the minimal right cotorsion-flat resolution of R, then every other prime $\mathfrak{q}$ of the same height as $\mathfrak{p}$ also appears in degree i. Theorem 9.2.2 gives the desired result. □

As an immediate consequence of Proposition 9.5.2 and Proposition 9.5.4, we now know that the cosupport of any finitely generated module (or more generally, complex with finitely generated cohomology) over such a ring is equal to its support. This generalizes the case of $R = \mathbb{Z}$ in [BIK12, Proposition 4.18].

Corollary 9.5.5. *Let R be either as in Proposition 9.5.2 or as in Proposition 9.5.4 and M a complex of R-modules with H^*M finitely generated. Then*

$$\operatorname{cosupp}_R(M) = \operatorname{supp}_R(M).$$

Proof. By Proposition 9.3.2, we have $\operatorname{cosupp}_R(M) = \operatorname{cosupp}_R(R) \cap \operatorname{supp}_R(M) = \operatorname{supp}_R(M)$. □

Any complete (semi-)local ring has cosupport equal to the closed set of maximal ideal(s) of the ring (see Proposition 9.5.1), and so complexes with finitely generated cohomology over Gorenstein (semi-)local rings will also have closed cosupport (applying again Proposition 9.3.2). We conjecture that more generally, in a Gorenstein ring R, the cosupport of a complex M of R-modules with H^*M finitely generated is a closed subset of $\operatorname{Spec}(R)$.

9.6 Further questions on cosupport

If the cosupport of one module is contained in the cosupport of another module, can the first module be "built" from the second? More explicitly, we ask:

Question 9.6.1. *For finitely generated R-modules M and N with finite projective dimension, if*

$$\operatorname{cosupp}_R(M) \subseteq \operatorname{cosupp}_R(N),$$

is M in the thick subcategory generated by N?

This question is motivated by [Nee92, Iye06, BIK12], where, in particular, it is shown that over a commutative Noetherian ring R, if P and Q are bounded complexes of finitely generated projective R-modules and $\operatorname{supp}_R P \subseteq \operatorname{supp}_R Q$, then Q "builds" P, i.e., P is in the thick subcategory generated by Q.

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