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Knörrer Periodicity and Bott Periodicity

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KNÖRRER PERIODICITY AND BOTT PERIODICITY

by

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The main goal of this dissertation is to explain a precise sense in which Knörrer periodicity in commutative algebra is a manifestation of Bott periodicity in topological K-theory. In Chapter 2, we motivate this project with a proof of the existence of an 8-periodic version of Knörrer periodicity for hypersurfaces defined over the real numbers. The 2- and 8-periodic versions of Knörrer periodicity for complex and real hypersurfaces, respectively, mirror the 2- and 8-periodic versions of Bott periodicity in KU- and KO-theory. In Chapter 3, we introduce the main tool we need to demonstrate the compatibility between Knörrer periodicity and Bott periodicity: a homomorphism from the Grothendieck group of the homotopy category of matrix factorizations associated to a complex (real) polynomial f into the topological K-theory of its Milnor fiber (positive or negative Milnor fiber). A version of this map first appeared in the setting of complex isolated hypersurface singularities in the paper "An Index Theorem for Modules on a Hypersurface Singularity", by Buchweitz and van Straten. We show that, when f is non-degenerate quadratic (over the real or complex numbers), this map recovers the Atiyah-Bott-Shapiro construction in topology. In Chapter 4, we prove that when f is a complex simple plane curve singularity, this homomorphism is injective.

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Chapter 1

Introduction

Matrix factorizations were introduced by Eisenbud in [Eis80] as a tool for studying the homological behavior of modules over a hypersurface ring; that is, a quotient of a regular ring by a principal ideal generated by a non-unit, non-zero-divisor. Recently, matrix factorizations have begun appearing in a wide variety of contexts, for instance:

- Homological mirror symmetry (e.g. [KKP08], by Katzarkov-Kontsevich-Pantev)
- Knot theory (e.g. [KR04], by Khovanov-Rozansky)
- Singularity theory (e.g. [BVS12], by Buchweitz-van Straten)

The overall goal of this work is to continue the study of an interplay between matrix factorizations and topological K-theory that was begun in the inspiring paper [BVS12].

Let $f \in \mathbb{C}\{x_1, \dots, x_n\}$ be a convergent power series such that

$$R = \mathbb{C}\{x_1, \dots, x_n\}/(f)$$

defines an isolated hypersurface singularity. One of the key insights in [BVS12] is that, by passing to topological information about the hypersurface, the vanishing of the Hochster theta pairing associated to the hypersurface ring R when n is odd can be viewed as a consequence of Bott periodicity in topological K-theory. The main goal of this thesis is to express precisely the manner in which Bott periodicity manifests itself in commutative algebra: it turns out that the answer is *Knörrer periodicity*, a behavior of maximal Cohen-Macaulay modules over certain hypersurface rings discovered by Knörrer ([Knö87] Theorem 3.1).

In Chapter 2, we establish various results concerning differential $\mathbb{Z}/2\mathbb{Z}$-graded categories of matrix factorizations. Most of the results we discuss are well-known; among the new results in this chapter is an 8-periodic version of Knörrer periodicity for isolated hypersurface singularities over the real numbers.

Theorem 1.0.1. *Let $Q := \mathbb{R}[x_1, \dots, x_n]$ and $f \in Q$. Suppose $Q/(f)$ has an isolated singularity at the origin (i.e. $\dim_{\mathbb{R}} \frac{\mathbb{R}[[x_1,\dots,x_n]]}{(\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n})} < \infty$). Set $Q' := \mathbb{R}[u_1, \dots, u_8]$, $q := u_1^2 + \cdots + u_8^2 \in Q'$, and $Q'' := Q \otimes_{\mathbb{R}} Q'$. Then there exists an equivalence of triangulated categories*

$$[\mathrm{MF}(\widehat{Q}, f)] \xrightarrow{\cong} [\mathrm{MF}(\widehat{Q''}, f + q)],$$

where $\widehat{(-)}$ denotes completion at the homogeneous maximal ideal.

We point out that the "period" here is exactly 8; that is, for $1 \leqslant l < 8$, it can happen that

$$[\mathrm{MF}(\mathbb{R}[[x_1, \dots, x_n]], f)] \ncong [\mathrm{MF}(\mathbb{R}[[x_1, \dots, x_n, u_1, \dots, u_l]], f + u_1^2 + \cdots + u_l^2)].$$

Our proof relies heavily on machinery developed by Dyckerhoff and Toën in [Dyc11] and [Toë07]. This result draws a distinction between the maximal Cohen-

Macaulay representation theory of hypersurface rings with ground field $\mathbb{R}$ and those whose ground field is algebraically closed and has characteristic not equal to 2, since the latter exhibit 2-periodic Knörrer periodicity. The maximal Cohen-Macaulay representation theory of hypersurface rings with ground field $\mathbb{R}$ does not seem to be well-studied, and we hope this work motivates further investigation in this direction.

The presence of 2- and 8-periodic versions of Knörrer periodicity over $\mathbb{C}$ and $\mathbb{R}$, respectively, suggests the possibility of a compatibility between Knörrer periodicity and Bott periodicity. Such a compatibility statement is formulated and proved in Chapter 3 (see Theorem 3.4.4):

Theorem 1.0.2. *Let $Q := \mathbb{C}[x_1, \ldots, x_n]$, and suppose f is an element of Q such that $Q/(f)$ has an isolated singularity at the origin (i.e. $\dim_{\mathbb{C}} \frac{\mathbb{C}[[x_1,\ldots,x_n]]}{(\frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_n})} < \infty$) . Then there exists a commutative diagram*

$$\begin{tikzcd}
K_0[\mathrm{MF}(Q,f)] \arrow[rr, "\phi_f"] \arrow[dd, "K"] & & KU^0(B_\epsilon, F_f) \arrow[d, "\beta"] \\
& & KU^0(B_\epsilon, F_f) \otimes KU^0(B_{\epsilon'}, F_{u^2+v^2}) \arrow[d, "\mathrm{ST}_{KU}"] \\
K_0[\mathrm{MF}(Q[u,v], f+u^2+v^2)] \arrow[rr, "\phi_{f+u^2+v^2}"] & & KU^0(B_{\epsilon''}, F_{f+u^2+v^2})
\end{tikzcd}$$

where F_f denotes the Milnor fiber of f, B_ϵ is a closed ball of radius ϵ in $\mathbb{C}^n$, K is induced by the Knörrer functor, β is the Bott periodicity isomorphism, and ST_{KU} is given by the product in relative K-theory followed by the inverse of the map induced by pullback along the Sebastiani-Thom homotopy equivalence.

The Sebastiani-Thom homotopy equivalence to which we refer in Theorem 1.0.2 is discussed in Section 3.1.2.

The key construction in this chapter gives a way of building the horizontal maps above; specifically, given a polynomial f over the complex (real) numbers, we construct a map Φ_f that assigns to a matrix factorization of a complex (real) polynomial f a class in the topological K-theory of the Milnor fiber (positive or negative Milnor fiber) of f; this map first appeared in [BVS12] in the setting of complex isolated hypersurface singularities. We prove that this construction induces a map ϕ_f on the Grothendieck group of the (triangulated) homotopy category of matrix factorizations of f, and we show that it recovers the Atiyah-Bott-Shapiro construction when f is a non-degenerate quadratic (over $\mathbb{R}$ or $\mathbb{C}$). The Atiyah-Bott-Shapiro construction, introduced in Part III of [ABS64], provides the classical link between $\mathbb{Z}/2\mathbb{Z}$-graded modules over Clifford algebras and vector bundles over spheres; the map ϕ_f we discuss in Chapter 3 can be thought of as providing a more general link between algebra and topology.

In Chapter 4, we apply the Atiyah-Bott-Shapiro-type construction ϕ_f from Chapter 3 to matrix factorizations of the ADE singularities, or simple plane curve singularities. The main result of this chapter is:

Theorem 1.0.3. *If $f \in \mathbb{C}[x_1, x_2]$ is an ADE singularity, ϕ_f is injective.*

The proof makes heavy use of key results in [BVS12] and [PV12].

Chapter 2

Knörrer Periodicity over $\mathbb{R}$

In this chapter, we recall some foundational material concerning matrix factorizations in commutative algebra, and we exhibit an 8-periodic version of Knörrer periodicity for matrix factorization categories associated to isolated hypersurface singularities over the real numbers.

2.1 Differential $\mathbb{Z}/2\mathbb{Z}$-graded categories

Let k be a field. We review some facts concerning k-linear differential $\mathbb{Z}/2\mathbb{Z}$-graded categories. In this section, all categories and functors are k-linear.

All of the results in Sections 2.1.1 and 2.1.2 are $\mathbb{Z}/2\mathbb{Z}$-graded variants of results in the setting of differential $\mathbb{Z}$-graded categories appearing in [Toë11]. We refer the reader to Section 5.1 of [Dyc11] for a discussion of how one may reformulate Toën's homotopy theory of dg-categories so that it applies to the $\mathbb{Z}/2\mathbb{Z}$-graded setting.

Henceforth, when we use the term "dg category", we mean "differential $\mathbb{Z}/2\mathbb{Z}$-graded category".

2.1.1 The derived category of a differential $\mathbb{Z}/2\mathbb{Z}$-graded category

Define $C_{\mathbb{Z}/2\mathbb{Z}}(k)$ to be the dg category of $\mathbb{Z}/2\mathbb{Z}$-graded complexes of k-modules. Fix a dg category $\mathcal{T}$.

Definition 2.1.1. The *homotopy category* of $\mathcal{T}$, denoted by $[\mathcal{T}]$, is the category given by the following:

- Objects in $[\mathcal{T}]$ are the same as the objects in $\mathcal{T}$.
- Given two objects X, Y of $[\mathcal{T}]$, the morphisms from X to Y are given by the 0^{th} cohomology vector space $H^0\mathrm{Hom}_{\mathcal{T}}(X,Y)$ of the $\mathbb{Z}/2\mathbb{Z}$-graded complex $\mathrm{Hom}_{\mathcal{T}}(X,Y)$.

Remark 2.1.2. A dg functor $F : \mathcal{S} \to \mathcal{T}$ determines an additive functor $[F] : [\mathcal{S}] \to [\mathcal{T}]$.

We introduce the *derived category* of $\mathcal{T}$:

Definition 2.1.3. A *module* M over $\mathcal{T}$ is a dg functor

$$M : \mathcal{T} \to C_{\mathbb{Z}/2\mathbb{Z}}(k).$$

One may form the dg category $\mathrm{Mod}(\mathcal{T})$ of modules over $\mathcal{T}$ in the evident way.

The dg category $\mathrm{Mod}(\mathcal{T})$ may be equipped with a $C_{\mathbb{Z}/2\mathbb{Z}}(k)$-enriched model structure such that the weak equivalences are given by morphisms

$$F \to F'$$

having the property that the induced maps $F(x) \to F'(x)$ are quasi-isomorphisms of $\mathbb{Z}/2\mathbb{Z}$-graded complexes for all objects $x \in \mathcal{T}$. We refer the reader to Section 3.2 of [Toë11] for details.

Definition 2.1.4. The *derived category* of $\mathcal{T}$, denoted $D(\mathcal{T})$, is the homotopy category $Ho(\mathrm{Mod}(\mathcal{T}))$. That is, there is a functor

$$L_{\mathcal{T}} : \mathrm{Mod}(\mathcal{T}) \to Ho(\mathrm{Mod}(\mathcal{T}))$$

sending weak equivalences to isomorphisms, and the pair $(L_{\mathcal{T}}, Ho(\mathrm{Mod}(\mathcal{T})))$ is universal with respect to this property.

Remark 2.1.5. Note the distinction between the homotopy category $[\mathrm{Mod}(\mathcal{T})]$ of $\mathrm{Mod}(\mathcal{T})$ in the dg sense and the homotopy category $Ho(\mathrm{Mod}(\mathcal{T}))$ of $\mathrm{Mod}(\mathcal{T})$ in the model-theoretic sense. This collision of terminology should cause no confusion in what follows.

2.1.2 Triangulated differential $\mathbb{Z}/2\mathbb{Z}$-graded categories

Denote by $\mathcal{T}^{\mathrm{op}}$ the opposite category of $\mathcal{T}$; that is, the category with the same objects, but with composition $f \circ g$ replaced with $(-1)^{|f||g|} g \circ f$. The Yoneda functor

$$h_{\mathcal{T}} : [\mathcal{T}] \to D(\mathcal{T}^{\mathrm{op}})$$

is given, on objects, by

$$T \mapsto L_{\mathcal{T}}(S \mapsto (\mathrm{Hom}_{\mathcal{T}}(S, T)))$$

and by the evident map on morphisms.

We say an object in $D(\mathcal{T}^{\mathrm{op}})$ is *quasi-representable* if it is in the essential image of $h_{\mathcal{T}}$. An object $M \in D(\mathcal{T}^{\mathrm{op}})$ is *compact* if $\mathrm{Hom}_{D(\mathcal{T}^{\mathrm{op}})}(M, -)$ commutes with coproducts. Quasi-representable objects are compact, but the converse is not true.

Definition 2.1.6. We say $\mathcal{T}$ is *dg-triangulated* if every compact object in $D(\mathcal{T}^{\mathrm{op}})$ is quasi-representable.

Remark 2.1.7. We give an example of a dg category that is not dg-triangulated in Section 2.5.

Remark 2.1.8. Denote by $D(\mathcal{T}^{\mathrm{op}})_c$ the full subcategory of compact objects in $D(\mathcal{T}^{\mathrm{op}})$. If $\mathcal{T}$ is dg-triangulated, $h_{\mathcal{T}} : [\mathcal{T}] \to D(\mathcal{T}^{\mathrm{op}})_c$ is an equivalence. Since $D(\mathcal{T}^{\mathrm{op}})_c$ is triangulated, it follows that, when $\mathcal{T}$ is dg-triangulated, $[\mathcal{T}]$ may be equipped with a canonical triangulated structure.

Every dg category may be embedded in a dg-triangulated category, its *triangulated hull.* To define the triangulated hull, we must introduce the *homotopy category of dg categories.*

Definition 2.1.9. A dg functor $F : \mathcal{S} \to \mathcal{T}$ is a *quasi-equivalence* if $[F] : [\mathcal{S}] \to [\mathcal{T}]$ is an equivalence of categories.

Consider the category $\mathrm{dg}_{\mathbb{Z}/2\mathbb{Z}}k$-cat of k-linear dg categories. There exists a $C_{\mathbb{Z}/2\mathbb{Z}}(k)$-enriched model structure on $\mathrm{dg}_{\mathbb{Z}/2\mathbb{Z}}k$-cat with weak equivalences given by quasi-equivalences. For details, we refer the reader to Section 3.2 of [Toë11].

In particular, there is a category $Ho(\mathrm{dg}_{\mathbb{Z}/2\mathbb{Z}}k\text{-cat})$ and a functor

$$L : \mathrm{dg}_{\mathbb{Z}/2\mathbb{Z}}k\text{-cat} \to Ho(\mathrm{dg}_{\mathbb{Z}/2\mathbb{Z}}k\text{-cat})$$

that maps quasi-equivalences to isomorphisms such that the pair $(L, Ho(\mathrm{dg}_{\mathbb{Z}/2\mathbb{Z}}k\text{-cat}))$ is universal with respect to this property.

Denote by $\mathrm{dg}_{\mathbb{Z}/2\mathbb{Z}}k\text{-cat}^{\mathrm{tr}}$ the full subcategory of $\mathrm{dg}_{\mathbb{Z}/2\mathbb{Z}}k\text{-cat}$ given by dg-triangulated categories.

Proposition 2.1.10 ([Toë11] Prop 4.4.2). *The inclusion functor*

$$i : Ho(\mathrm{dg}_{\mathbb{Z}/2\mathbb{Z}}\text{k-cat}^{\mathrm{tr}}) \to Ho(\mathrm{dg}_{\mathbb{Z}/2\mathbb{Z}}\text{k-cat})$$

admits a left adjoint

$$\mathrm{Perf}(-) : Ho(\mathrm{dg}_{\mathbb{Z}/2\mathbb{Z}}\text{k-cat}) \to Ho(\mathrm{dg}_{\mathbb{Z}/2\mathbb{Z}}\text{k-cat}^{\mathrm{tr}}).$$

Definition 2.1.11. Given a dg category $\mathcal{T}$, we shall call $\mathrm{Perf}(\mathcal{T})$ the *triangulated hull* of $\mathcal{T}$.

It will be useful for us to have an explicit model for the triangulated hull of a dg category. To construct it, we must introduce the notion of a perfect module over a dg category:

Definition 2.1.12. A module $M \in \mathrm{Mod}(\mathcal{T})$ is *perfect* if $L_{\mathcal{T}}(M) \in D(\mathcal{T})$ is a compact object.

Remark 2.1.13. $\mathrm{Perf}(\mathcal{T})$ coincides with the dg subcategory of $\mathrm{Mod}(\mathcal{T})$ consisting of perfect modules, thought of as an object in $Ho(\mathrm{dg}_{\mathbb{Z}/2\mathbb{Z}}k\text{-cat})$.

We may now introduce the notion of a *Morita equivalence* of dg categories:

Definition 2.1.14 ([Toë11] Definition 4.4.4). A morphism $F : \mathcal{S} \to \mathcal{T}$ in $Ho(\mathrm{dg}_{\mathbb{Z}/2\mathbb{Z}}k\text{-cat})$ is called a *Morita equivalence* if $\mathrm{Perf}(F)$ is an isomorphism.

If F is a dg functor such that $L(F)$ is a Morita equivalence, we shall call F a Morita equivalence as well.

Proposition 2.1.15. *A dg functor $F : \mathcal{S} \to \mathcal{T}$ between two dg-triangulated categories is a Morita equivalence if and only if it is a quasi-equivalence.*

Proof. It is immediate that F is a Morita equivalence when F is a quasi-equivalence. By Proposition 2.1.10, if F is a Morita equivalence, $L(F)$ is an isomorphism. By Theorem 1.2.10 in [Hov07], $L(F)$ is an isomorphism if and only if F is a quasi-equivalence. $\square$

2.1.3 Hochschild homology and the Chern character

In this section, we assume $\mathrm{char}(k) = 0$. A k-linear dg category is a generalization of a dg k-algebra; in fact, a dg category with only one object is precisely a dg k-algebra. There exists a notion of *Hochschild homology* for k-linear dg categories that recovers the definition for dg k-algebras; we introduce this notion here, following Section 1.1 of [PV12].

Let $\mathcal{S}$ and $\mathcal{T}$ be dg categories.

Definition 2.1.16. The *tensor product*, $\mathcal{S} \otimes \mathcal{T}$, of $\mathcal{S}$ and $\mathcal{T}$ is the dg category given by the following:

- Objects are pairs (S, T), where $S \in \mathrm{Ob}(\mathcal{S})$ and $T \in \mathrm{Ob}(\mathcal{T})$.
- $\mathrm{Hom}_{\mathcal{S}\otimes\mathcal{T}}((S, T), (S', T')) := \mathrm{Hom}_{\mathcal{S}}(S, S') \otimes_k \mathrm{Hom}_{\mathcal{T}}(T, T')$.

Definition 2.1.17. An $\mathcal{S}$-$\mathcal{T}$ *bimodule* is a module over $(\mathcal{S} \otimes \mathcal{T}^{\mathrm{op}})^{\mathrm{op}} \cong \mathcal{S}^{\mathrm{op}} \otimes \mathcal{T}$.

By Section 6.1 of [Kel94], an $\mathcal{S}$-$\mathcal{T}$ bimodule X determines a functor

$$T_X : \mathrm{Mod}(\mathcal{T}^{\mathrm{op}}) \to \mathrm{Mod}(\mathcal{S}^{\mathrm{op}})$$

in the following way: given an object M of $\mathrm{Mod}(\mathcal{T}^{\mathrm{op}})$, define a functor

$$\mathcal{S}^{\mathrm{op}} \to C_{\mathbb{Z}/2\mathbb{Z}}(k)$$

given by

$$S \mapsto \mathrm{coker}(\bigoplus_{T,T' \in \mathcal{T}} M(T') \otimes_k \mathrm{Hom}_{\mathcal{T}}(T,T') \otimes_k X(S,T) \xrightarrow{\nu} \bigoplus_{T \in \mathcal{T}} M(T) \otimes_k X(S,T)),$$

where $\nu(y,f,x) = (M(f))(y) \otimes x - y \otimes X(\mathrm{id}_S \otimes f)(x)$.

Remark 2.1.18. Suppose S and T have exactly one object; denote the unique object of $\mathcal{T}$ by $Y_{\mathcal{T}}$. In this case, X is a right module over the dg algebra $\mathrm{End}_{\mathcal{T}}(Y_{\mathcal{T}})$, and the functor T_X amounts to the tensor product $- \otimes_{\mathrm{End}_{\mathcal{T}}(Y_{\mathcal{T}})} X$.

There exists a *left derived functor*

$$\mathbb{L}T_X : D(\mathcal{T}^{\mathrm{op}}) \to D(\mathcal{S}^{\mathrm{op}})$$

of T_X; we refer the reader to [Kel94] for details.

Now, fix a dg category $\mathcal{U}$. Let Δ denote the $\mathcal{U} \otimes \mathcal{U}^{\mathrm{op}}$-module given, on objects, by

$$(U,V) \mapsto \mathrm{Hom}_{\mathcal{U}}(V,U).$$

Considering k as a dg category with one object whose endomorphism complex consists of the k-module k concentrated in degree 0, clearly Δ is a k-$(\mathcal{U} \otimes \mathcal{U}^{\mathrm{op}})$ bimodule. Thus, noting that $(\mathcal{U} \otimes \mathcal{U}^{\mathrm{op}})^{\mathrm{op}} \cong \mathcal{U}^{\mathrm{op}} \otimes \mathcal{U}$, we have that Δ determines a functor

$$\mathbb{L}T_{\Delta} : D(\mathcal{U}^{\mathrm{op}} \otimes \mathcal{U}) \to D(k).$$

Δ is also a $\mathcal{U}^{\mathrm{op}} \otimes \mathcal{U}$-module in an evident way; we define the *Hochschild complex* of $\mathcal{U}$ to be the object

$$\mathbb{L}T_{\Delta}(\Delta) \in D(k).$$

The *Hochschild homology* of $\mathcal{U}$, denoted $HH_*(\mathcal{U})$, is the homology of $\mathbb{L}T_{\Delta}(\Delta)$.

As a reality check, let's suppose $\mathcal{U}$ has one object whose endomorphism ring A is concentrated in degree 0. Then Δ is the left $A \otimes_k A^{\mathrm{op}}$-module A. Thus, by Remark 2.1.18, the Hochschild complex of $\mathcal{U}$ is $A \otimes^{\mathbb{L}}_{A \otimes_k A^{\mathrm{op}}} A \in D(k)$; this agrees with the $\mathbb{Z}/2\mathbb{Z}$-folding of the usual Hochschild complex.

We list two properties of Hochschild homology of dg categories that we will make use of:

- Hochschild homology is *Morita invariant*; that is, there is a natural isomorphism

$$HH_*(\mathcal{U}) \xrightarrow{\cong} HH_*(\mathrm{Perf}(\mathcal{U}))$$

  ([Toë11] Section 5.2).

- There is a Künneth isomorphism

$$HH_*(\mathcal{S}) \otimes_k HH_*(\mathcal{T}) \xrightarrow{\cong} HH_*(\mathcal{S} \otimes \mathcal{T})$$

  ([PV12] Proposition 1.1.4).

Now, suppose $\mathcal{U}$ has the following properties:

(1) $\Delta \in \mathrm{Mod}(\mathcal{U} \otimes \mathcal{U}^{\mathrm{op}})$ is perfect

(2) For every pair of objects U, V in $\mathcal{U}$, $\mathrm{Hom}_{\mathcal{U}}(U, V)$ has finite-dimensional cohomology

(3) $D(\mathcal{U})$ admits a compact generator

By Section 1.2 of [PV12], $\mathcal{U}$ is Morita equivalent to a *homologically smooth and proper* dg algebra. Also, when $\mathcal{S}$ and $\mathcal{T}$ are dg categories with the above properties, a dg functor

$$F : \mathrm{Perf}(\mathcal{S}) \to \mathrm{Perf}(\mathcal{T})$$

yields a map

$$F_* : HH_*(\mathcal{S}) \to HH_*(\mathcal{T}).$$

In particular, if U is an object in $\mathcal{U}$, the functor

$$1_U : k \to \mathcal{U}$$

that sends the unique object of k to U yields a map

$$(1_U)_* : k = HH_*(k) \to HH_*(\mathcal{U}).$$

Definition 2.1.19. We define $\mathrm{ch}(U) := (1_U)_*(1) \in HH_*(\mathcal{U})$ to be the *Chern character* of U.

The functor

$$T_\Delta : \mathrm{Mod}(\mathcal{U}^{\mathrm{op}} \otimes \mathcal{U}) \to \mathrm{Mod}(k)$$

restricts to a functor

$$\mathrm{Perf}(\mathcal{U}^{\mathrm{op}} \otimes \mathcal{U}) \to \mathrm{Perf}(k).$$

Combining the map on Hochschild homology induced by this functor with the

Künneth isomorphism, we have a canonical pairing

$$\langle -, - \rangle_{\mathcal{U}} : HH_*(\mathcal{U}^{\mathrm{op}}) \otimes_k HH_*(\mathcal{U}) \to k.$$

On the other hand, one has the *Euler pairing*

$$\chi : \mathrm{Ob}(\mathcal{U}) \times \mathrm{Ob}(\mathcal{U}) \to k$$

given by

$$(U, V) \mapsto \dim_k H^0\mathrm{Hom}_{\mathcal{U}}(U, V) - \dim_k H^1\mathrm{Hom}_{\mathcal{U}}(U, V).$$

The following is an analogue of the Hirzebruch-Riemann-Roch formula:

Theorem 2.1.20 ([PV12] Section 1.2)**.** *Let $\mathcal{U}$ be a k-linear dg category, where k is a field, and assume $\mathcal{U}$ has properties (1) - (3) above. If U and V are objects in $\mathcal{U}$,*

$$\chi(U, V) = \langle \mathrm{ch}(U), \mathrm{ch}(V) \rangle_{\mathcal{U}}.$$

Remark 2.1.21. When $\mathcal{U}$ is a dg category with properties (1) - (3) above, the pairing $\langle -, - \rangle_{\mathcal{U}}$ is non-degenerate. In fact, the map

$$HH_*(\mathcal{U}) \otimes_k HH_*(\mathcal{U}^{\mathrm{op}}) \otimes_k HH_*(\mathcal{U}) \to HH_*(\mathcal{U})$$

given by

$$h \otimes h' \otimes h'' \mapsto \langle h, h' \rangle_{\mathcal{U}^{\mathrm{op}}} \cdot h''$$

sends $h \otimes \mathrm{ch}(\Delta)$ to h for all $h \in HH_*(\mathcal{U})$, where $\mathrm{ch}(\Delta) \in HH_*(\mathrm{Perf}(\mathcal{U}^{\mathrm{op}} \otimes \mathcal{U}))$ is identified with its image under the canonical isomorphisms

$$HH_*(\mathrm{Perf}(\mathcal{U}^{\mathrm{op}} \otimes \mathcal{U})) \cong HH_*(\mathcal{U}^{\mathrm{op}} \otimes \mathcal{U}) \cong HH_*(\mathcal{U}^{\mathrm{op}}) \otimes_k HH_*(\mathcal{U}).$$

2.2 Matrix factorization categories

We provide some background on matrix factorization categories. Fix a commutative algebra Q over a field k and an element f of Q. All categories and functors in this section are assumed to be k-linear.

2.2.1 Definitions and some properties

Definition 2.2.1. The dg category $\mathrm{MF}(Q, f)$ of *matrix factorizations* of f over Q is given by the following:

Objects in $\mathrm{MF}(Q, f)$ are pairs (P, d), where P is a finitely-generated projective $\mathbb{Z}/2\mathbb{Z}$-graded Q-module, and d is an odd-degree endomorphism of P such that $d^2 = f \cdot \mathrm{id}_P$. Henceforth, we will often denote an object (P, d) in $\mathrm{MF}(Q, f)$ by just P.

The morphism complex of a pair of matrix factorizations P, P', which we will denote by $\mathrm{Hom}_{\mathrm{MF}}(P, P')$, is the $\mathbb{Z}/2\mathbb{Z}$-graded module of Q-linear maps from P to P' equipped with the differential ∂ given by

$$\partial(\alpha) = d' \circ \alpha - (-1)^{|\alpha|} \alpha \circ d$$

for homogeneous maps $\alpha : P \to P'$.

It will often be useful to express an object P in $\mathrm{MF}(Q, f)$ in the following way:

$$P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0$$

where P_1, P_0 are the odd and even degree summands of P, and d_1, d_0 are the restrictions of d to P_1 and P_0, respectively.

We now establish several technical results concerning matrix factorization categories that we will need later on.

Define $\mathrm{EMF}(Q, f)$ to be the category with the same objects as $\mathrm{MF}(Q, f)$ and with morphisms given by the degree 0 cycles in $\mathrm{MF}(Q, f)$. When Q is regular with finite Krull dimension and f is a regular element of Q (i.e. f is a non-unit, non-zero-divisor), $\mathrm{EMF}(Q, f)$ is an exact category with the evident family of exact sequences ([Orl03] Section 3.1); the "E" stands for exact.

A degree 0 morphism α in $\mathrm{MF}(Q, f)$ can be represented by a diagram of the following form:

$$\begin{tikzcd} P_1 \arrow[r, "d_1"] \arrow[d, "\alpha_1"'] & P_0 \arrow[r, "d_0"] \arrow[d, "\alpha_0"'] & P_1 \arrow[d, "\alpha_1"] \\ P_1' \arrow[r, "d_1'"] & P_0' \arrow[r, "d_0'"] & P_1' \end{tikzcd}$$

It is straightforward to check that α is a cycle if and only if this diagram commutes. In fact, if $f \in Q$ is a non-zero-divisor, it is easy to see that the left square commutes if and only if the right square commutes.

Remark 2.2.2. If P_1 and P_0 are free and f is non-zero-divisor, P_1 and P_0 must have the same rank.

It will be useful for us to have an alternative characterization for when a morphism in $\mathrm{EMF}(Q, f)$ is a boundary in $\mathrm{MF}(Q, f)$.

Definition 2.2.3. We call a matrix factorization *trivial* if it is a direct sum of matrix factorizations that are isomorphic in $\mathrm{EMF}(Q, f)$ to either

$$E \underset{\mathrm{id}_E}{\overset{f \cdot \mathrm{id}_E}{\rightleftarrows}} E$$

or

$$E \underset{f \cdot \mathrm{id}_E}{\overset{\mathrm{id}_E}{\rightleftarrows}} E.$$

for some finitely generated projective Q-module E.

Proposition 2.2.4. *A morphism $\alpha : P \to P'$ in* $\mathrm{EMF}(Q, f)$ *is a boundary in* $\mathrm{MF}(Q, f)$ *if and only if it factors through a trivial matrix factorization in* $\mathrm{EMF}(Q, f)$*.*

Proof. Suppose α factors through a trivial matrix factorization E. It is easy to see that id_E is a boundary in $\mathrm{MF}(Q, f)$; it follows immediately that α is as well.

Conversely, suppose α is a boundary. Write

$$P = (P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0),\ P' = (P_1' \underset{d_0'}{\overset{d_1'}{\rightleftarrows}} P_0').$$

Since α is a degree 0 cycle, there exist Q-linear maps

$$\alpha_1 : P_1 \to P_1',\ \alpha_0 : P_0 \to P_0'$$

such that $\alpha = \alpha_1 + \alpha_0$ and the following diagram commutes:

$$\begin{tikzcd}
P_1 \arrow[r, "d_1"] \arrow[d, "\alpha_1"'] & P_0 \arrow[r, "d_0"] \arrow[d, "\alpha_0"'] & P_1 \arrow[d, "\alpha_1"] \\
P_1' \arrow[r, "d_1'"] & P_0' \arrow[r, "d_0'"] & P_1'
\end{tikzcd}$$

Choose a Q-linear map

$$h : P \to P'$$

such that $\partial(h) = \alpha$. Since α has degree 0, ∂ evaluated at the degree 0 component of h is 0. Thus, we may as well assume h is homogeneous of degree 1; that is, there are

Q-linear maps

$$h_0 : P_0 \to P_1'$$

$$h_1 : P_1 \to P_0'$$

such that $h = h_0 + h_1$.

Define

$$A : P_1 \oplus P_1' \to P_1 \oplus P_1'$$

$$B : P_1 \oplus P_1' \to P_1 \oplus P_1'$$

to be given by

$$A = \begin{pmatrix} f \cdot \mathrm{id}_{P_1} & 0 \\ 0 & \mathrm{id}_{P_1'} \end{pmatrix}$$

$$B = \begin{pmatrix} \mathrm{id}_{P_1} & 0 \\ 0 & f \cdot \mathrm{id}_{P_1'} \end{pmatrix}$$

Notice that

$$P_1 \oplus P_1' \underset{B}{\overset{A}{\rightleftarrows}} P_1 \oplus P_1'$$

is a trivial matrix factorization, and we have the following commutative diagram:

$$
\begin{tikzcd}
P_1 \arrow[r, "d_1"] \arrow[d, "\begin{pmatrix} \mathrm{id}_{P_1} \\ h_0 \circ d_1 \end{pmatrix}"'] & P_0 \arrow[r, "d_0"] \arrow[d, "\begin{pmatrix} d_0 \\ h_0 \end{pmatrix}"] & P_1 \arrow[d, "\begin{pmatrix} \mathrm{id}_{P_1} \\ h_0 \circ d_1 \end{pmatrix}"] \\
P_1 \oplus P_1' \arrow[r, "A"] \arrow[d, "\begin{pmatrix} d_0' \circ h_1 & \mathrm{id}_{P_1'} \end{pmatrix}"'] & P_1 \oplus P_1' \arrow[r, "B"] \arrow[d, "\begin{pmatrix} h_1 & d_1' \end{pmatrix}"] & P_1 \oplus P_1' \arrow[d, "\begin{pmatrix} d_0' \circ h_1 & \mathrm{id}_{P_1'} \end{pmatrix}"] \\
P_1' \arrow[r, "d_1'"] & P_0' \arrow[r, "d_0'"] & P_1'
\end{tikzcd}
$$

Thus, α factors through a trivial matrix factorization. $\square$

Here is another technical result that will be useful later on:

Proposition 2.2.5. *Let $P = (P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0)$ be a matrix factorization of f over Q. Assume f is a non-zero-divisor. Then the following are equivalent:*

(1) $\operatorname{coker}(d_1)$ *is isomorphic to L/fL for some projective Q-module L.*

(2) There exists a trivial matrix factorization E and a matrix factorization E' that is isomorphic in $\mathrm{EMF}(Q, f)$ *to one of the form*

$$F \underset{f}{\overset{\mathrm{id}_{E'}}{\rightleftarrows}} F$$

such that $P \oplus E'$ is isomorphic to E in $\mathrm{EMF}(Q, f)$.

Before proving the proposition, we establish a general fact about idempotent complete categories.

Definition 2.2.6. We say an additive category $\mathcal{C}$ is *idempotent complete*, or that $\mathcal{C}$ has *split idempotents*, if every idempotent endomorphism $\phi = \phi^2$ of an object X splits; that is, there exists a factorization

$$X \xrightarrow{\pi} Y \xrightarrow{\iota} X$$

of ϕ with $\pi \circ \iota = \mathrm{id}_Y$.

Lemma 2.2.7. *Let $\mathcal{C}$ be an idempotent complete additive category, and let $\mathcal{E}$ be a collection of objects in $\mathcal{C}$ that is*

- *closed under isomorphisms,*
- *closed under finite coproducts, and*
- *closed under taking summands; that is, whenever X is an object in $\mathcal{C}$ such that id_X factors through an object of $\mathcal{E}$, X is an object in $\mathcal{E}$.*

Denote by $\mathcal{L}$ the quotient of $\mathcal{C}$ by those morphisms that factor through an object in $\mathcal{E}$. If X and Y are objects in $\mathcal{C}$, their images in $\mathcal{L}$ are isomorphic if and only if there exist objects E_X, E_Y in $\mathcal{E}$ such that

$$X \oplus E_X \cong Y \oplus E_Y.$$

Proof. Let X and Y be objects in $\mathcal{C}$. Suppose there exist objects E_X, E_Y in $\mathcal{E}$ such that

$$X \oplus E_X \cong Y \oplus E_Y.$$

The quotient functor from $\mathcal{C}$ to $\mathcal{L}$ is additive, and hence preserves finite coproducts. Thus, it suffices to show that objects in $\mathcal{E}$ are mapped to 0 under the quotient functor. This is clear, since, if E is an object in $\mathcal{E}$, id_E factors through an object in $\mathcal{E}$.

Conversely, suppose the images of X and Y in $\mathcal{L}$ are isomorphic. Choose morphisms in $\mathcal{C}$

$$\alpha : X \to Y,\ \beta : Y \to X$$

whose images in $\mathcal{L}$ are mutually inverse. Choose an object E_Y in $\mathcal{E}$ and morphisms

$$\delta : E_Y \to X,\ \epsilon : X \to E_Y$$

in $\mathcal{C}$ such that

$$\delta \circ \epsilon = \beta \circ \alpha - \mathrm{id}_X.$$

We have morphisms

$$\phi := \begin{pmatrix} \alpha \\ \epsilon \end{pmatrix} : X \to Y \oplus E_Y,\ \psi := \begin{pmatrix} \beta & -\delta \end{pmatrix} : Y \oplus E_Y \to X$$

in $\mathcal{C}$. Notice that $\psi \circ \phi = \mathrm{id}_X$. Also, an easy computation shows that

$$\sigma := \mathrm{id}_{Y \oplus E_Y} - \phi \circ \psi$$

is idempotent. Choose an object Z in $\mathcal{C}$ and morphisms

$$\tau : Z \to Y \oplus E_Y,\ \rho : Y \oplus E_Y \to Z$$

in $\mathcal{C}$ such that

$$\tau \circ \rho = \sigma \text{ and } \rho \circ \tau = \mathrm{id}_Z.$$

Given two objects A_1, A_2 in $\mathcal{C}$, we denote by

$$\iota_{A_i} : A_i \to A_1 \oplus A_2,\ \pi_{A_i} : A_1 \oplus A_2 \to A_i$$

the canonical maps associated to the coproduct of A_1 and A_2 in $\mathcal{C}$ (which is also the product of A_1 and A_2 in $\mathcal{C}$). We have mutually inverse morphisms

$$X \oplus Z \xrightarrow{\begin{pmatrix} \alpha & \pi_Y \circ \tau \\ \epsilon & \pi_{E_Y} \circ \tau \end{pmatrix}} Y \oplus E_Y$$

$$Y \oplus E_Y \xrightarrow{\begin{pmatrix} \beta & -\delta \\ \rho \circ \iota_Y & \rho \circ \iota_{E_Y} \end{pmatrix}} X \oplus Z,$$

so it suffices to show that Z is in $\mathcal{E}$. We first show that ϕ descends to an isomorphism in $\mathcal{L}$. Recall that $\psi \circ \phi = \mathrm{id}_X$ in $\mathcal{C}$. Write $\phi \circ \psi - \mathrm{id}_{Y \oplus E_Y}$ as the 2×2 matrix

$$\begin{pmatrix} \alpha \circ \beta - \mathrm{id}_Y & -\alpha \circ \delta \\ \epsilon \circ \beta & -\epsilon \circ \delta - \mathrm{id}_{E_Y} \end{pmatrix}$$

Since each entry of this matrix factors through an object in $\mathcal{E}$, $\phi \circ \psi - \mathrm{id}_{Y \oplus E_Y}$ descends to the zero map in $\mathcal{L}$. This shows that the images of ϕ and ψ in $\mathcal{L}$ are mutually inverse.

Notice that

$$\phi = \begin{pmatrix} \alpha & \pi_Y \circ \tau \\ \epsilon & \pi_{E_Y} \circ \tau \end{pmatrix} \circ \iota_X$$

in $\mathcal{C}$. Thus, ι_X descends to an isomorphism in $\mathcal{L}$. Since $\pi_X \circ \iota_X = \mathrm{id}_X$ in $\mathcal{C}$, it follows that the images of π_X and ι_X in $\mathcal{L}$ are mutually inverse. Choose an object E in $\mathcal{E}$ and morphisms

$$f : E \to X \oplus Z,\ g : X \oplus Z \to E$$

such that

$$\mathrm{id}_{X \oplus Z} - \iota_X \circ \pi_X = f \circ g$$

in $\mathcal{C}$. Observe that

$$\pi_Z \circ f \circ g \circ \iota_Z = \mathrm{id}_Z - \pi_Z \circ \iota_X \circ \pi_X \circ \iota_Z = \mathrm{id}_Z.$$

Thus, id_Z factors through an object in $\mathcal{E}$. $\square$

We are now ready to prove Proposition 2.2.5:

Proof. (2) $\Rightarrow$ (1): Since the cokernel of d_1 is isomorphic to the cokernel of

$$d_1 \oplus \mathrm{id}_{E'} : P_1 \oplus E' \to P_0 \oplus E',$$

we may assume P is trivial. In this case, the result is obvious.

(1) $\Rightarrow$ (2): We have projective resolutions

$$0 \to P_1 \xrightarrow{d_1} P_0 \to \mathrm{coker}(d_1) \to 0$$

$$0 \to L \xrightarrow{f} L \to L/fL \to 0$$

Thus, there exist maps

$$\beta_i : P_i \to L,\ \gamma_i : L \to P_i$$

for $i = 0, 1$ making the following diagrams commute:

$$\begin{tikzcd}
0 \arrow[r] & P_1 \arrow[r, "d_1"] \arrow[d, "\beta_1"'] & P_0 \arrow[r] \arrow[d, "\beta_0"'] & \operatorname{coker} d_1 \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & L \arrow[r, "f"] & L \arrow[r] & L/fL \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "\gamma_1"'] & L \arrow[r] \arrow[d, "\gamma_0"'] & L/fL \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r] & \operatorname{coker} d_1 \arrow[r] & 0
\end{tikzcd}$$

Hence, we have maps

$$h_P : P_0 \to P_1,\ h_L : L \to L$$

such that

$$\gamma_1 \circ \beta_1 - \mathrm{id}_{P_1} = h_P \circ d_1,\ \gamma_0 \circ \beta_0 - \mathrm{id}_{P_0} = d_1 \circ h_P.$$

$$\beta_1 \circ \gamma_1 - \mathrm{id}_L = fh_L,\ \beta_0 \circ \gamma_0 - \mathrm{id}_L = fh_L.$$

We have commutative diagrams

$$\begin{tikzcd}
P_1 \arrow[r, "d_1"] \arrow[d, "h_P \circ d_1"'] & P_0 \arrow[d, "h_p"] & L \arrow[r, "f \cdot \mathrm{id}_L"] \arrow[d, "f \cdot h_L"'] & L \arrow[d, "h_L"] \\
P_1 \arrow[r, "\mathrm{id}_{P_1}"] \arrow[d, "\mathrm{id}_{P_1}"'] & P_1 \arrow[d, "d_1"] & L \arrow[r, "\mathrm{id}_L"] \arrow[d, "\mathrm{id}_L"'] & L \arrow[d, "f \cdot \mathrm{id}_L"] \\
P_1 \arrow[r, "d_1"] & P_0 & L \arrow[r, "f \cdot \mathrm{id}_L"] & L
\end{tikzcd}$$

Denote by $\mathcal{E}$ the collection of matrix factorizations of f over Q isomorphic in

$\mathrm{EMF}(Q,f)$ to a matrix factorization of the form

$$E \underset{f}{\overset{\mathrm{id}_E}{\rightleftarrows}} E.$$

Notice that $\mathrm{EMF}(Q,f)$ is an idempotent complete additive category, and $\mathcal{E}$ is closed under direct sums and direct summands in $\mathrm{EMF}(Q,f)$. Letting $\mathcal{L}$ denote the quotient of $\mathrm{EMF}(Q,f)$ by those morphisms that factor through an object in $\mathcal{E}$, we have that

$$(P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0) \cong (L \underset{\mathrm{id}_L}{\overset{f}{\rightleftarrows}} L)$$

in $\mathcal{L}$. The result now follows from Lemma 2.2.7.

$\square$

2.2.2 Triangulated structure

Suppose Q is regular with finite Krull dimension and f is a regular element of Q. A feature of the homotopy category $[\mathrm{MF}(Q,f)]$ is that it may be equipped with a triangulated structure in the following way ([Orl03] Section 3.1):

The shift functor maps the object

$$P = (P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0)$$

to the object

$$P[1] = (P_0 \underset{-d_1}{\overset{-d_0}{\rightleftarrows}} P_1).$$

That is, shifting a matrix factorization flips the grading on the module and negates the odd-degree endomorphism. On morphisms, the shift functor maps the cycle

$$\begin{array}{ccccc} P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{d_0} & P_1 \\ {\scriptstyle\alpha_1}\downarrow & & {\scriptstyle\alpha_0}\downarrow & & \downarrow{\scriptstyle\alpha_1} \\ P_1' & \xrightarrow{d_1'} & P_0' & \xrightarrow{d_0'} & P_1' \end{array}$$

to the cycle

$$\begin{array}{ccccc} P_0 & \xrightarrow{-d_0} & P_1 & \xrightarrow{-d_1} & P_0 \\ {\scriptstyle\alpha_0}\downarrow & & {\scriptstyle\alpha_1}\downarrow & & \downarrow{\scriptstyle\alpha_0} \\ P_0' & \xrightarrow{-d_0'} & P_1' & \xrightarrow{-d_1'} & P_0' \end{array}$$

Notice that the shift functor applied twice is the identity functor.

Given a morphism $\alpha : (P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0) \to (P_1' \underset{d_0'}{\overset{d_1'}{\rightleftarrows}} P_0')$ in $\mathrm{EMF}(Q, f)$, we define the *mapping cone* of α as follows:

$$\mathrm{cone}(\alpha) = (P_0' \oplus P_1 \underset{\begin{pmatrix} d_1' & \alpha_0 \\ 0 & -d_0 \end{pmatrix}}{\overset{\begin{pmatrix} d_0' & \alpha_1 \\ 0 & -d_1 \end{pmatrix}}{\rightleftarrows}} P_1' \oplus P_0)$$

There are canonical morphisms $P' \to \mathrm{cone}(\alpha)$ and $\mathrm{cone}(\alpha) \to P[1]$ in $\mathrm{EMF}(Q, f)$. Taking the distinguished triangles in $[\mathrm{MF}(Q, f)]$ to be the triangles isomorphic to those of the form

$$P \xrightarrow{\alpha} P' \to \mathrm{cone}(\alpha) \to P[1],$$

$[\mathrm{MF}(Q, f)]$ may be equipped with the structure of a triangulated category.

We define the Grothendieck group

$$K_0[\mathrm{MF}(Q, f)]$$

to be the free abelian group generated by isomorphism classes of $[\mathrm{MF}(Q,f)]$ modulo elements of the form $[P_1] - [P_2] + [P_3]$, where P_1, P_2, and P_3 fit into a distinguished triangle in the following way:

$$P_1 \to P_2 \to P_3 \to P_1[1].$$

Remark 2.2.8. The category $\mathrm{MF}(Q,f)$ is not always dg-triangulated in this setting; a counterexample is given in Section 2.5. When $\mathrm{MF}(Q,f)$ is dg-triangulated, the induced triangulated structure on $[\mathrm{MF}(Q,f)]$ agrees with the triangulated structure just described.

Remark 2.2.9. When Q is a regular local ring and f is a regular element of Q, one has an equivalence of triangulated categories

$$[\mathrm{MF}(Q,f)] \xrightarrow{\cong} \underline{\mathrm{MCM}}(Q/(f)),$$

where $\underline{\mathrm{MCM}}(Q/(f))$ denotes the *stable category* of maximal Cohen-Macaulay (MCM) modules over the ring $Q/(f)$.

The stable category of MCM modules is obtained by taking the quotient of the category of MCM modules over $Q/(f)$ by those morphisms that factor through a projective $Q/(f)$-module. The above equivalence is given, on objects, by

$$(P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0) \mapsto \operatorname{coker}(d_1).$$

Matrix factorizations were first defined by Eisenbud in [Eis80]; this interplay between matrix factorizations and MCM modules over hypersurface rings provided the original motivation for the study of matrix factorization categories.

2.2.3 The Hochster theta pairing

We begin this section with a technical definition:

Definition 2.2.10. If the pair (Q, f) satisfies

- Q is essentially of finite type over k
- Q is equidimensional of dimension n
- The module $\Omega^1_{Q/k}$ of Kähler differentials is locally free of rank n
- The zero locus of $df \in \Omega^1_{Q/k}$ is a 0-dimensional scheme supported on a unique closed point $\mathfrak{m}$ of $\operatorname{Spec}(Q)$ with residue field k and $f \in \mathfrak{m}$

we shall call $Q/(f)$ an *isolated hypersurface singularity*, or IHS. We will sometimes just say f is IHS, if the ambient ring Q is clear.

Remark 2.2.11. Our IHS condition above is precisely condition (B) in Section 3.2 of [Dyc11]. As noted in loc. cit., if $Q/(f)$ and $Q'/(f')$ are IHS, $Q \otimes_k Q'/(f \otimes 1 + 1 \otimes f')$ is as well.

Now, assume $\operatorname{char}(k) = 0$ and that Q is a regular local ring such that $Q/(f)$ is IHS. Set $R := Q/(f)$. One may define a symmetric pairing

$$\theta : K_0[\mathrm{MF}(Q, f)] \times K_0[\mathrm{MF}(Q, f)] \to \mathbb{Z},$$

called the *Hochster theta pairing*, that maps a pair

$$([P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0], [P_1' \underset{d_0'}{\overset{d_1'}{\rightleftarrows}} P_0'])$$

to

$$l(\operatorname{Tor}_2^R(\operatorname{coker}(d_1), \operatorname{coker}(d_1'))) - l(\operatorname{Tor}_1^R(\operatorname{coker}(d_1), \operatorname{coker}(d_1'))),$$

where l denotes length as an R-module; our assumption that the singular locus of R is dimension 0 guarantees that these lengths are finite. The pairing θ was introduced in [Hoc81]; for more detailed discussions related to this pairing, we refer the reader to [BVS12], [MPSW11], [PV12], and [Wal14b].

The Euler pairing χ from Section 2.1.3 applied to the dg category $\mathrm{MF}(Q, f)$ can be thought of as a pairing on the homotopy category, since, for matrix factorizations $P, P' \in \mathrm{MF}(Q, f)$,

$$\chi(P, P') = \dim_k H^0\mathrm{Hom}_{\mathrm{MF}}(P, P') - \dim_k H^1\mathrm{Hom}_{\mathrm{MF}}(P, P')$$

$$= \dim_k H^0\mathrm{Hom}_{\mathrm{MF}}(P, P') - \dim_k H^0\mathrm{Hom}_{\mathrm{MF}}(P, P'[1]).$$

It is straightforward to check that χ induces a pairing on $K_0[\mathrm{MF}(Q, f)]$.

Write

$$P = (P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0),\ P' = (P'_1 \underset{d'_0}{\overset{d'_1}{\rightleftarrows}} P'_0).$$

By Section 3 of [Wal14a], the Euler pairing χ applied to (P, P') corresponds, via the equivalence

$$[\mathrm{MF}(Q, f)] \xrightarrow{\text{coker}} \underline{\mathrm{MCM}}(Q/(f)),$$

to the pairing

$$(\mathrm{coker}(d_1), \mathrm{coker}(d'_1)) \mapsto l(\mathrm{Ext}^2_R(\mathrm{coker}(d_1), \mathrm{coker}(d'_1))) - l(\mathrm{Ext}^1_R(\mathrm{coker}(d_1), \mathrm{coker}(d'_1))).$$

As above, these lengths must be finite because of our assumption on the singular locus of R. By Remark 3.2 of [BVS12], it follows that

$$\chi(P, P') = \theta(\mathrm{coker}(d_1)^*, \mathrm{coker}(d'_1)),$$

where $(-)^*$ denotes the R-linear dual $\mathrm{Hom}_R(-, R)$. In particular, since MCM modules over R are reflexive,

$$\chi(-, [P]) : K_0[\mathrm{MF}(Q, f)] \to \mathbb{Z}$$

is the zero map if and only if

$$\theta(-, \mathrm{coker}(d_1)) : K_0[\mathrm{MF}(Q, f)] \to \mathbb{Z}$$

is the zero map.

2.2.4 Stabilization

Assume now that Q is a regular local ring of Krull dimension n, and suppose f is a regular element of Q. Denote by $\mathrm{D}^{\mathrm{b}}(Q/(f))$ the bounded derived category of $Q/(f)$. We will say an object C in $\mathrm{D}^{\mathrm{b}}(Q/(f))$ is *perfect* if it is isomorphic, in $\mathrm{D}^{\mathrm{b}}(Q/(f))$, to a complex of finitely generated projective $Q/(f)$-modules; set $\mathrm{D}^{\mathrm{b}}_{\mathrm{perf}}(Q/(f))$ to be the full subcategory of $\mathrm{D}^{\mathrm{b}}(Q/(f))$ given by perfect complexes. It turns out that $\mathrm{D}^{\mathrm{b}}_{\mathrm{perf}}(Q/(f))$ is a thick subcategory of $\mathrm{D}^{\mathrm{b}}(Q/(f))$; define $\underline{\mathrm{D}^{\mathrm{b}}}(Q/(f))$ to be the Verdier quotient of $\mathrm{D}^{\mathrm{b}}(Q/(f))$ by $\mathrm{D}^{\mathrm{b}}_{\mathrm{perf}}(Q/(f))$. In [Buc86], Buchweitz defines this quotient to be the *stabilized derived category* of $Q/(f)$.

By [Buc86], the functor

$$\underline{\mathrm{MCM}}(Q/(f)) \to \underline{\mathrm{D}^{\mathrm{b}}}(Q/(f))$$

that sends an MCM module M to the complex with M concentrated in degree 0 is a triangulated equivalence. Hence, composing with the equivalence in Remark 2.2.9, one has an equivalence

$$[\mathrm{MF}(Q, f)] \to \underline{\mathrm{D}^{\mathrm{b}}}(Q/(f))$$

Following [Dyc11], given an object C in $\underline{\mathrm{D}}^{\mathrm{b}}(Q/(f))$, we denote by C^{stab} the isomorphism class in $[\mathrm{MF}(Q,f)]$ corresponding to C under the above equivalence ("stab" stands for "stabilization").

In particular, thinking of the residue field k of $Q/(f)$ as a complex concentrated in degree 0, we may associate to k an isomorphism class k^{stab} in $[\mathrm{MF}(Q,f)]$. We now construct an object E_f in $\mathrm{MF}(Q,f)$ that represents k^{stab}; this construction appears in [Dyc11]. Choose a regular system of parameters $x_1,\dots,x_n$ for Q, and consider the Koszul complex

$$(\bigoplus_{i=0}^{n} \textstyle\bigwedge^i Q^n, s_0)$$

as a $\mathbb{Z}/2\mathbb{Z}$-graded complex of free Q-modules with even (odd) degree piece given by the direct sum of the even (odd) exterior powers of Q^n. Here, s_0 denotes the $\mathbb{Z}/2\mathbb{Z}$-folding of the Koszul differential associated to $x_1,\dots,x_n$. Choose an expression of $f \in Q$ of the form

$$f = g_1x_1 + \cdots + g_nx_n.$$

Fix a basis $e_1,\dots,e_n$ of Q^n, and set s_1 to be the odd-degree endomorphism of $\bigoplus_{i=0}^{n} \bigwedge^i Q^n$ given by exterior multiplication on the left by $g_1e_1 + \cdots + g_ne_n$. Set

$$E_f := (\bigoplus_{i=0}^{n} \textstyle\bigwedge^i Q^n, s_0 + s_1).$$

It is easy to check that E_f is a matrix factorization of f. By Corollary 2.7 in [Dyc11], E_f represents k^{stab} in $[\mathrm{MF}(Q,f)]$. In particular, E_f does not depend on the choice of regular system of parameters $x_1,\dots,x_n$ or coefficients $g_1,\dots,g_n$ up to homotopy equivalence.

We will be interested in the dga $\mathrm{End}_{\mathrm{MF}}(E_f)$. $\mathrm{End}_{\mathrm{MF}}(E_f)$ may be expressed, in terms of generators and relations, as the $\mathbb{Z}/2\mathbb{Z}$-graded Q-algebra

$$Q\langle \lambda_1, \dots, \lambda_n, c_1, \dots, c_n \rangle / ([\lambda_i, \lambda_j], [c_i, c_j], [\lambda_i, c_j] - \delta_{ij})$$

equipped with the differential ∂ determined by $\partial(\lambda_i) = x_i$ and $\partial(c_i) = g_i$. Here, the λ_i and c_i are non-commuting variables of odd degree, $[-,-]$ denotes the $\mathbb{Z}/2\mathbb{Z}$-graded commutator, and δ_{ij} is the Kronecker delta. An isomorphism from this algebra to $\mathrm{End}_{\mathrm{MF}}(E_f)$ is given by

$$\lambda_i \mapsto \text{left multiplication by } e_i$$

$$c_i \mapsto \text{contraction by } e_i$$

where, by contraction by e_i, we mean the map that sends a basis element

$$e_{i_1} \cdots e_{i_r}$$

to 0 if $i \notin \{i_1, \dots, i_r\}$ and to $(-1)^{r-1} e_{i_1} \cdots \widehat{e_i} \cdots e_{i_r}$ otherwise.

We set

$$A_{(Q,f)} := \mathrm{End}_{\mathrm{MF}}(E_f),$$

and we emphasize that $A_{(Q,f)}$ does not depend on the choice of regular system of parameters $x_1, \dots, x_n$ or coefficients $g_1, \dots, g_n$ up to quasi-isomorphism.

2.3 The tensor product of matrix factorizations

Suppose Q and Q' are commutative algebras over a field k. Given objects P and P' in $\mathrm{MF}(Q, f)$, $\mathrm{MF}(Q', f')$, one can form their *tensor product* over k:

$$P \otimes_{\mathrm{MF}} P' := ((P_1 \otimes_k P_0') \oplus (P_0 \otimes_k P_1') \underset{\begin{pmatrix} d_0 \otimes \mathrm{id}_{P_0'} & -\mathrm{id}_{P_1} \otimes d_1' \\ \mathrm{id}_{P_0} \otimes d_0' & d_1 \otimes \mathrm{id}_{P_1'} \end{pmatrix}}{\overset{\begin{pmatrix} d_1 \otimes \mathrm{id}_{P_0'} & \mathrm{id}_{P_0} \otimes d_1' \\ -\mathrm{id}_{P_1} \otimes d_0' & d_0 \otimes \mathrm{id}_{P_1'} \end{pmatrix}}{\rightleftarrows}} (P_0 \otimes_k P_0') \oplus (P_1 \otimes_k P_1')).$$

This construction first appeared in [Yos98]; it can be thought of as a $\mathbb{Z}/2\mathbb{Z}$-graded analogue of the tensor product of complexes. It is straightforward to check that $P \otimes_{\mathrm{MF}} P'$ is an object in $\mathrm{MF}(Q \otimes_k Q', f \otimes 1 + 1 \otimes f')$.

In fact, setting $f \oplus f' := f \otimes 1 + 1 \otimes f' \in Q \otimes_k Q'$, and noting that there is a canonical map

$$\mathrm{Hom}_{\mathrm{MF}}(P, L) \otimes_k \mathrm{Hom}_{\mathrm{MF}}(P', L') \to \mathrm{Hom}_{\mathrm{MF}}(P \otimes_{\mathrm{MF}} P', L \otimes_{\mathrm{MF}} L'),$$

we have the following:

Proposition 2.3.1. *There is a dg functor*

$$\mathrm{ST}_{\mathrm{MF}} : \mathrm{MF}(Q, f) \otimes_k \mathrm{MF}(Q', f') \to \mathrm{MF}(Q \otimes_k Q', f \oplus f')$$

that sends an object (P, P') to $P \otimes_{\mathrm{MF}} P'$.

Further,

Proposition 2.3.2. $\mathrm{ST}_{\mathrm{MF}}$ *induces a pairing*

$$K_0[\mathrm{MF}(Q,f)] \otimes K_0[\mathrm{MF}(Q',f')] \to K_0[\mathrm{MF}(Q \otimes_k Q', f \oplus f')].$$

Proof. Suppose P is a contractible matrix factorization in $\mathrm{MF}(Q,f)$. Choose a contracting homotopy (h_0,h_1). Then, if P' is any matrix factorization in $\mathrm{MF}(Q',f')$, the maps

$$(P_0 \otimes P_0') \oplus (P_1 \otimes P_1') \xrightarrow{\begin{pmatrix} h_0 \otimes \mathrm{id}_{P_0'} & 0 \\ 0 & h_1 \otimes \mathrm{id}_{P_1'} \end{pmatrix}} (P_1 \otimes P_0') \oplus (P_0 \otimes P_1')$$

$$(P_1 \otimes P_0') \oplus (P_0 \otimes P_1') \xrightarrow{\begin{pmatrix} h_1 \otimes \mathrm{id}_{P_0'} & 0 \\ 0 & h_0 \otimes \mathrm{id}_{P_1'} \end{pmatrix}} (P_0 \otimes P_0') \oplus (P_1 \otimes P_1')$$

yield a contracting homotopy of $P \otimes_{\mathrm{MF}} P'$.

Suppose $\alpha : L \to L'$ is a morphism in $\mathrm{EMF}(Q,f)$. One easily checks that, if B is a matrix factorization in $\mathrm{MF}(Q',f')$,

$$\mathrm{cone}(\alpha) \otimes_{\mathrm{MF}} B = \mathrm{cone}(\alpha \otimes \mathrm{id}_B).$$

Thus,

$$[(L \oplus \mathrm{cone}(\alpha)) \otimes_{\mathrm{MF}} B] - [L' \otimes_{\mathrm{MF}} B] = [L \otimes_{\mathrm{MF}} B] + [\mathrm{cone}(\alpha \otimes \mathrm{id}_B)] - [L' \otimes_{\mathrm{MF}} B] = 0.$$

If $\alpha : P \to P'$ is an isomorphism in $[\mathrm{MF}(Q,f)]$, then $\mathrm{cone}(\alpha)$ is contractible.

Thus, for all matrix factorizations B of f' over Q',

$$[P \otimes_{\mathrm{MF}} B] = [P' \otimes_{\mathrm{MF}} B].$$

It follows that the pairing respects isomorphism in the homotopy category. Since every distinguished triangle is isomorphic to one of the form

$$P \xrightarrow{\alpha} P' \to \mathrm{cone}(\alpha) \to P[1],$$

and we have shown that the pairing preserves triangles of this form, this finishes the proof. $\square$

Remark 2.3.3. The "ST" in the name $\mathrm{ST}_{\mathrm{MF}}$ stands for "Sebastiani-Thom", since this tensor product operation is related to the Sebastiani-Thom homotopy equivalence discussed in Section 3.1.2. A precise sense in which the tensor product of matrix factorizations is related to the Sebastiani-Thom homotopy equivalence is illustrated by the proof of Proposition 3.4.1 below; see Remark 3.4.3 for further details.

Now, suppose $Q/(f)$ and $Q'/(f')$ are IHS (see Definition 2.2.10 above). Henceforth, we will denote by $\widehat{Q}$ the $\mathfrak{m}$-adic completion of $Q_{\mathfrak{m}}$, where $\mathfrak{m}$ is as in the definition of IHS.

Set $Q'' := Q \otimes_k Q'$, and define

$$\phi : \widehat{Q} \otimes_k \widehat{Q'} \to \widehat{Q''}$$

to be the canonical ring homomorphism. ϕ induces a dg functor

$$\mathrm{MF}(\phi) : \mathrm{MF}(\widehat{Q} \otimes_k \widehat{Q'}, f \oplus f') \to \mathrm{MF}(\widehat{Q''}, f \oplus f').$$

Set $\widehat{\mathrm{ST}}_{\mathrm{MF}} := \mathrm{MF}(\phi) \circ \mathrm{ST}_{\mathrm{MF}}$.

Proposition 2.3.4. *If $Q/(f)$ and $Q'/(f')$ are* IHS*,*

$$\widehat{\mathrm{ST}}_{\mathrm{MF}} : \mathrm{MF}(\widehat{Q}, f) \otimes_k \mathrm{MF}(\widehat{Q'}, f') \to \mathrm{MF}(\widehat{Q''}, f \oplus f')$$

is a Morita equivalence.

Remark 2.3.5. This proposition is really just a straightforward application of several results in [Dyc11].

Proof. Suppose $Q_{\mathfrak{m}}$ and $Q'_{\mathfrak{m}'}$ have Krull dimensions n and m, respectively. $Q_{\mathfrak{m}}$ and $Q'_{\mathfrak{m}'}$ are regular local rings; choose regular systems of parameters $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ in $Q_{\mathfrak{m}}$ and $Q'_{\mathfrak{m}'}$, and choose expressions

$$f = g_1 x_1 + \cdots + g_n x_n$$

$$f' = h_1 y_1 + \cdots + h_m y_m$$

of f and f'. Use these expressions to construct the dga's $A_{(Q_{\mathfrak{m}}, f)}$ and $A_{(Q'_{\mathfrak{m}'}, f')}$.

Note that $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ form regular systems of parameters in $\widehat{Q}$ and $\widehat{Q'}$ as well, so we may use these expressions to construct $A_{(\widehat{Q_{\mathfrak{m}}}, f)}$ and $A_{(\widehat{Q'_{\mathfrak{m}'}}, f')}$. Also, $x_1 \otimes 1, \ldots, x_n \otimes 1, 1 \otimes y_1, \ldots, 1 \otimes y_m$ is a regular system of parameters in $Q''_{\mathfrak{m}''}$, where $\mathfrak{m}'' := \mathfrak{m} \otimes 1 + 1 \otimes \mathfrak{m}'$, so we may use the expression

$$f \oplus f' = g_1 x_1 \otimes 1 + \cdots + g_n x_n \otimes 1 + 1 \otimes h_1 y_1 + \cdots + 1 \otimes h_m y_m$$

to construct $A_{(Q''_{\mathfrak{m}''}, f \oplus f')}$ and $A_{(\widehat{Q''}, f \oplus f')}$.

By Section 6.1 of [Dyc11], we have a quasi-isomorphism

$$F : A_{(Q_{\mathfrak{m}},f)} \otimes_k A_{(Q'_{\mathfrak{m}'},f')} \xrightarrow{\cong} A_{(Q''_{\mathfrak{m}''},f\oplus f')}.$$

We also have a canonical map

$$G : A_{(\widehat{Q},f)} \otimes_k A_{(\widehat{Q'},f')} \to A_{(\widehat{Q''},f\oplus f')}.$$

By the proof of Theorem 5.7 in [Dyc11], the inclusions

$$A_{(Q_{\mathfrak{m}},f)} \hookrightarrow A_{(\widehat{Q},f)}$$

$$A_{(Q'_{\mathfrak{m}'},f')} \hookrightarrow A_{(\widehat{Q'},f')}$$

$$A_{(Q''_{\mathfrak{m}''},f\oplus f')} \hookrightarrow A_{(\widehat{Q''},f\oplus f')}$$

are all quasi-isomorphisms.

By Exercise 4.4.11 in [Toë11], it follows that the induced map

$$\operatorname{Perf}(A_{(Q_{\mathfrak{m}},f)} \otimes_k A_{(Q'_{\mathfrak{m}'},f')}) \to \operatorname{Perf}(A_{(\widehat{Q},f)} \otimes_k A_{(\widehat{Q'},f')})$$

is an isomorphism in $Ho(\mathrm{dg}_{\mathbb{Z}/2\mathbb{Z}}k\text{-cat})$.

It is clear that we have the following commutative square in $Ho(\mathrm{dg}_{\mathbb{Z}/2\mathbb{Z}}k\text{-cat})$:

$$\begin{tikzcd}
\operatorname{Perf}(A_{(Q_{\mathfrak{m}},f)} \otimes_k A_{(Q'_{\mathfrak{m}'},f')}) \arrow[r, "\cong"] \arrow[d, "\cong"] & \operatorname{Perf}(A_{(\widehat{Q},f)} \otimes_k A_{(\widehat{Q'},f')}) \arrow[d, "\operatorname{Perf}(L(G))"] \\
\operatorname{Perf}(A_{(Q''_{\mathfrak{m}''},f\oplus f')}) \arrow[r, "\cong"] & \operatorname{Perf}(A_{(\widehat{Q''},f\oplus f')})
\end{tikzcd}$$

It follows that $\mathrm{Perf}(L(G))$ is an isomorphism in $Ho(\mathrm{dg}_{\mathbb{Z}/2\mathbb{Z}}k\text{-cat})$.

One may think of a dga as a dg category with a single object. Adopting this point of view, we have inclusion functors

$$i : A_{(\widehat{Q},f)} \hookrightarrow \mathrm{MF}(\widehat{Q}, f)$$

$$j : A_{(\widehat{Q'},f')} \hookrightarrow \mathrm{MF}(\widehat{Q'}, f')$$

$$l : A_{(\widehat{Q''},f\oplus f')} \hookrightarrow \mathrm{MF}(\widehat{Q''}, f \oplus f')$$

Combining Theorem 5.2 and Lemma 5.6 in [Dyc11], we conclude that i, j, and l are Morita equivalences. In particular, applying Exercise 4.4.11 in [Toë11] again, we have that

$$\mathrm{Perf}(L(i) \otimes L(j)) : \mathrm{Perf}(A_{(\widehat{Q},f)} \otimes_k A_{(\widehat{Q'},f')}) \to \mathrm{Perf}(\mathrm{MF}(\widehat{Q}, f) \otimes_k \mathrm{MF}(\widehat{Q'}, f'))$$

is an isomorphism in $Ho(\mathrm{dg}_{\mathbb{Z}/2\mathbb{Z}}k\text{-cat})$.

Finally, observe the following commutative diagram in $Ho(\mathrm{dg}_{\mathbb{Z}/2\mathbb{Z}}k\text{-cat})$:

$$\begin{tikzcd}
\mathrm{Perf}(A_{(\widehat{Q},f)} \otimes_k A_{(\widehat{Q'},f')}) \arrow[r, "\mathrm{Perf}(L(i) \otimes L(j))"] \arrow[d, "\mathrm{Perf}(L(G))"] & \mathrm{Perf}(\mathrm{MF}(\widehat{Q}, f) \otimes_k \mathrm{MF}(\widehat{Q'}, f')) \arrow[d, "\mathrm{Perf}(L(\widehat{\mathrm{TS}}_{\mathrm{MF}}))"] \\
\mathrm{Perf}(A_{(\widehat{Q''},f\oplus f')}) \arrow[r, "\mathrm{Perf}(L(l))"] & \mathrm{Perf}(\mathrm{MF}(\widehat{Q''}, f \oplus f'))
\end{tikzcd}$$

Since the left-most vertical map and both horizontal maps are isomorphisms in $Ho(\mathrm{dg}_{\mathbb{Z}/2\mathbb{Z}}k\text{-cat})$, $\mathrm{Perf}(L(\widehat{\mathrm{TS}}_{\mathrm{MF}}))$ is as well.

□

Remark 2.3.6. Under the assumptions of Proposition 2.3.4, the functor

$$\mathrm{MF}(Q, f) \otimes_k \mathrm{MF}(Q', f') \to \mathrm{MF}(Q'', f \oplus f')$$

given by tensor product of matrix factorizations is also a Morita equivalence.

Here is a proof: by Theorem 5.2 in [Dyc11], the inclusion functors

$$A_{(Q_{\mathfrak{m}}, f)} \hookrightarrow \mathrm{MF}(Q_{\mathfrak{m}}, f)$$

$$A_{(Q'_{\mathfrak{m}'}, f')} \hookrightarrow \mathrm{MF}(Q'_{\mathfrak{m}'}, f')$$

are Morita equivalences.

By arguments similar to those in the proof Proposition 2.3.4, one has a commutative square in $Ho(\mathrm{dg}_{\mathbb{Z}/2\mathbb{Z}} k\text{-cat})$:

$$\begin{tikzcd}
\mathrm{Perf}(A_{(Q_{\mathfrak{m}}, f)} \otimes_k A_{(Q'_{\mathfrak{m}'}, f')}) \arrow[r, "\cong"] \arrow[d, "\cong"] & \mathrm{Perf}(\mathrm{MF}(Q_{\mathfrak{m}}, f) \otimes_k \mathrm{MF}(Q'_{\mathfrak{m}'}, f')) \arrow[d] \\
\mathrm{Perf}(A_{(Q''_{\mathfrak{m}''}, f \oplus f')}) \arrow[r, "\cong"] & \mathrm{Perf}(\mathrm{MF}(Q''_{\mathfrak{m}''}, f \oplus f'))
\end{tikzcd}$$

It follows that the dg functor

$$\mathrm{MF}(Q_{\mathfrak{m}}, f) \otimes_k \mathrm{MF}(Q'_{\mathfrak{m}'}, f') \to \mathrm{MF}(Q''_{\mathfrak{m}''}, f \oplus f')$$

given by tensor product of matrix factorizations is a Morita equivalence. Finally, consider the square

$$\begin{tikzcd}
\mathrm{MF}(Q,f) \otimes_k \mathrm{MF}(Q',f') \arrow[r] \arrow[d] & \mathrm{MF}(Q'', f \oplus f') \arrow[d] \\
\mathrm{MF}(Q_{\mathfrak{m}},f) \otimes_k \mathrm{MF}(Q'_{\mathfrak{m}'},f') \arrow[r] & \mathrm{MF}(Q''_{\mathfrak{m}''}, f \oplus f')
\end{tikzcd}$$

where the vertical maps are induced by localization. By Theorems 4.11 and 5.2 in [Dyc11] and an application of Exercise 4.4.11 in [Toë11], the vertical maps are Morita equivalences; hence, the top map is a Morita equivalence.

2.4 Clifford algebras

Fix a field k such that $\mathrm{char}(k) \neq 2$ and a finite-dimensional vector space V over k. Let $q : V \to k$ be a quadratic form.

The *Clifford algebra*, $\mathrm{Cliff}_k(q)$, of q over k is defined to be the quotient

$$\mathcal{T}(V)/(v \otimes v - q(v)),$$

where $\mathcal{T}(V)$ denotes the tensor algebra of V over k.

$\mathrm{Cliff}_k(q)$ is a $\mathbb{Z}/2\mathbb{Z}$-graded k-algebra; let $\mathrm{mod}_{\mathbb{Z}/2\mathbb{Z}}(\mathrm{Cliff}_k(q))$ denote the category of finitely generated $\mathbb{Z}/2\mathbb{Z}$-graded left modules over $\mathrm{Cliff}_k(q)$. Henceforth, when we refer to a module over a Clifford algebra, we will always mean it to be a left module.

Assume q is non-degenerate, and choose a basis $\{e_1, \ldots, e_n\}$ of V with respect to which q is *diagonal*; that is,

$$q = a_1 x_1^2 + \cdots + a_n x_n^2 \in S^2(V^*)$$

where the x_i comprise the dual basis corresponding to the e_i, and the a_i are nonzero elements of k. Denote by Q the localization of $S(V^*)$ at the ideal $(x_1, \ldots, x_n)$.

We quote the following theorem from Section 14 of [Yos90]; it is due to Buchweitz-Eisenbud-Herzog ([BEH87]).

Theorem 2.4.1. $[\mathrm{MF}(\widehat{Q}, q)]$ *and* $\mathrm{mod}_{\mathbb{Z}/2\mathbb{Z}}(\mathrm{Cliff}_k(q))$ *are equivalent* k*-linear categories.*

It will be useful for us to exhibit a bijection between the isomorphism classes of objects of these two categories; this bijection is described in Section 14 of [Yos90]:

Given an isomorphism class $[P]$, where P is an object in $[\mathrm{MF}(\widehat{Q}, q)]$, we may choose an object

$$\widetilde{P} = (\widetilde{P_1} \underset{d_0}{\overset{d_1}{\rightleftarrows}} \widetilde{P_0})$$

in $[\mathrm{MF}(\widehat{Q}, q)]$ such that

(a) $\widetilde{P} \cong P$ in $[\mathrm{MF}(\widehat{Q}, q)]$, and

(b) there exist choices of bases of $\widetilde{P_0}$ and $\widetilde{P_1}$ as free modules over $\widehat{Q}$ so that d_1 and d_0 may be expressed by matrices A and B with entries in $S^1(V^*) = V^*$.

That such an object $\widetilde{P}$ exists is a theorem due to Buchweitz-Eisenbud-Herzog in [BEH87].

Recall that $\widetilde{P_0}$ and $\widetilde{P_1}$ have the same rank as free $\widehat{Q}$-modules. Set m to be this rank. Let W be a k-vector space of dimension m equipped with a basis. Set U_0 and U_1 to be copies of W.

Given $x \in V$ and a matrix D with entries in V^*, define

$$\mathrm{ev}_x(D)$$

to be the matrix over k given by evaluating the entries of D at x.

Define an action of $\mathcal{T}(V)$ on $U_1 \oplus U_0$ by

$$x \cdot u_1 = \mathrm{ev}_x(A) \cdot u_1 \in U_0$$

for $x \in V$ and $u_1 \in U_1$, and

$$x \cdot u_0 = \mathrm{ev}_x(B) \cdot u_0 \in U_1$$

for $x \in V$ and $u_0 \in U_0$. Notice that $(v \otimes v)u = q(v)u$ for all $u \in U_1 \oplus U_0$ and $v \in V$. It follows that $U_1 \oplus U_0$ is a finitely generated $\mathrm{Cliff}_k(V)$-module. It turns out that $U_1 \oplus U_0$ does not depend on the choice of $\widetilde{P}$ up to isomorphism of $\mathrm{Cliff}_k(V)$-modules, so that we may set $\Delta_q([P])$ to be the isomorphism class of the $\mathrm{Cliff}_k(V)$-module $U_1 \oplus U_0$.

Going the other direction, let $[M]$ denote the isomorphism class of a finitely generated $\mathbb{Z}/2\mathbb{Z}$-graded $\mathrm{Cliff}_k(q)$-module $M = M_1 \oplus M_0$. Let $v \in V$. Multiplication by v determines k-linear maps

$$\phi(v) : M_1 \to M_0$$

$$\psi(v) : M_0 \to M_1$$

Since q is non-degenerate, we may choose $w \in V$ such that $q(w) \neq 0$. It follows that $\phi(w)$ and $\psi(w)$ are isomorphisms; in particular, M_1 and M_0 have the same rank m as k-vector spaces.

Choosing bases of M_0 and M_1, we may think of the maps $\phi(v)$ and $\psi(v)$ as $m \times m$ matrices with entries in k.

Define

$$\phi_{ij} : V \to k$$

to be the k-linear map assigning an element v of V to the (i, j) entry of $\phi(v)$. Define

ψ_{ij} similarly.

The maps ϕ_{ij}, ψ_{ij} are elements of V^*, so they may be written as linear combinations of $x_1, \ldots, x_n \in S^1(V^*)$.

Define $\Theta_q(M)$ to be the isomorphism class of the matrix factorization

$$\widehat{Q}^m \underset{\psi}{\overset{\phi}{\rightleftarrows}} \widehat{Q}^m$$

where ϕ is the square matrix with entries ϕ_{ij}, and ψ is defined similarly. It is elementary to check that the assignments Δ_q and Θ_q are inverses on isomorphism classes.

Remark 2.4.2. Note that the inclusion

$$k[x_1, \ldots, x_n] \hookrightarrow \widehat{Q}$$

induces an equivalence

$$[\mathrm{MF}(k[x_1, \ldots, x_n], q_n)] \xrightarrow{\cong} [\mathrm{MF}(\widehat{Q}, q_n)].$$

To see this, we first recall that, as noted above, every matrix factorization of q_n over $\widehat{Q}$ is isomorphic in $[\mathrm{MF}(\widehat{Q}, q_n)]$ to one with (linear) polynomial entries (Proposition 14.3, [Yos90]); hence, the functor is essentially surjective.

Also, one has a commutative diagram

$$\begin{tikzcd}
{[\mathrm{MF}(Q, q_n)]} \arrow[r, "\cong"] \arrow[d] & \underline{\mathrm{MCM}}(Q/(q_n)) \arrow[d] \\
{[\mathrm{MF}(\widehat{Q}, q_n)]} \arrow[r, "\cong"] & \underline{\mathrm{MCM}}(\widehat{Q}/(q_n))
\end{tikzcd}$$

The morphism sets in $\underline{\mathrm{MCM}}(Q/(q_n))$ are Artinian modules, and hence complete. Thus, the functor on the right is fully faithful, and so the functor on the left is as well.

It now follows easily from Corollary 4.11, Theorem 5.2, and Theorem 5.7 in [Dyc11] that the functor

$$[\mathrm{MF}(k[x_1, \dots, x_n], q_n)] \to [\mathrm{MF}(\widehat{Q}, q_n)]$$

is fully faithful.

Remark 2.4.3. Suppose $q' : V' \to k$ is another non-degenerate quadratic form. Choose a basis of V' with respect to which q' is diagonal, and let $x_1, \dots, x_m$ denote the basis of $(V')^*$ corresponding to this choice of basis. As above, we may think of q' as an element of $S^2((V')^*)$. Set Q' to be the localization of $S((V')^*)$ at the ideal $(x_1, \dots, x_m)$.

It is well-known that the $\mathbb{Z}/2\mathbb{Z}$-graded tensor product of $\mathrm{Cliff}_k(q)$ and $\mathrm{Cliff}_k(q')$ is canonically isomorphic to $\mathrm{Cliff}_k(q + q')$. Further, by Remark 1.3 in [Yos98], the $\mathbb{Z}/2\mathbb{Z}$-graded tensor product of Clifford modules is compatible, via this canonical isomorphism and the equivalence in Theorem 2.4.1, with the tensor product $\mathrm{ST}_{\mathrm{MF}}$ in Proposition 2.3.1. That is, one has a commutative diagram

$$\begin{tikzcd}
{[\mathrm{Ob}(\mathrm{mod}_{\mathbb{Z}/2\mathbb{Z}}(\mathrm{Cliff}_k(q)))] \times [\mathrm{Ob}(\mathrm{mod}_{\mathbb{Z}/2\mathbb{Z}}(\mathrm{Cliff}_k(q)))]} \arrow[r] \arrow[d, "\Theta_q \times \Theta_{q'}"] & {[\mathrm{Ob}(\mathrm{mod}_{\mathbb{Z}/2\mathbb{Z}}(\mathrm{Cliff}_k(q+q')))]} \arrow[d, "\Theta_{q+q'}"] \\
{[\mathrm{Ob}([\mathrm{MF}(Q,q)])] \times [\mathrm{Ob}([\mathrm{MF}(Q',q')])]} \arrow[r, "\mathrm{ST}_{\mathrm{MF}}"] & {[\mathrm{Ob}([\mathrm{MF}(Q \otimes_k Q', q+q')])]}
\end{tikzcd}$$

where $[\mathrm{Ob}(\mathcal{C})]$ denotes the collection of isomorphism classes of objects in a category $\mathcal{C}$.

Let C be a rank 1 free $\mathbb{Z}/2\mathbb{Z}$-graded $\mathrm{Cliff}_k(q)$-module. If $\dim(V) = 1$ and $q = x^2$, it is easy to see that $\Theta_q([C]) = k^{\mathrm{stab}}$, where k^{stab} is as defined in Section 2.2.4.

By the discussion in Section 6.1 of [Dyc11], the tensor product of a stabilization of the residue field of $k[x]/(x^2)$ with itself is a stabilization of the residue field of $k[x_1, x_2]_{(x_1,x_2)}/(x_1^2 + x_2^2)$. Thus, by Remark 2.4.3, we have:

Proposition 2.4.4. *If $a_i = 1$ for $1 \leqslant i \leqslant n$, $\Theta_q([C]) = k^{\text{stab}}$.*

Corollary 2.4.5. *If k is algebraically closed, $\Theta_q(C) = k^{\text{stab}}$.*

2.5 An example: $f = y^2 - x^2(x+1) \in \mathbb{C}[x,y]_{(x,y)}$.

We now consider the category of matrix factorizations of $f = y^2 - x^2(x+1)$ over the ring $Q = \mathbb{C}[x,y]_{(x,y)}$. Our main goal in this section is to show that $[\mathrm{MF}(Q,f)]$ is not idempotent complete.

Let $\widehat{Q}$ denote the (x,y)-adic completion of Q. We first prove:

Proposition 2.5.1. $K_0[\mathrm{MF}(Q,f)] \ncong K_0[\mathrm{MF}(\widehat{Q},f)]$.

Proof. We will show that $K_0[\mathrm{MF}(Q,f)]$ is a torsion group, while $K_0[\mathrm{MF}(\widehat{Q},f)]$ is not.

Set $R := Q/(f)$, let $\mathrm{MCM}(R)$ denote the category of maximal Cohen-Macaulay modules over R, and let $\mathrm{mod}(R)$ denote the category of finitely generated modules over R.

The inclusion of exact categories

$$\mathrm{MCM}(R) \hookrightarrow \mathrm{mod}(R).$$

induces an isomorphism on Grothendieck groups. By the bottom of page 7 of [Dyc11], $\underline{\mathrm{MCM}}(R)$ is the stable category of the Frobenius exact category $\mathrm{MCM}(R)$; hence, by

Section 7.4 of [Kra07], one has a well-defined map

$$K_0(\mathrm{MCM}(R)) \to K_0(\underline{\mathrm{MCM}}(R))$$

given by $[M] \mapsto [M]$.

Thus, one has a surjection

$$\Phi : G_0(R) \to K_0(\underline{\mathrm{MCM}}(R)).$$

Since $f = y^2 - x^2(x+1)$ is irreducible over $\mathbb{C}[x, y]$, R has exactly two prime ideals: (0) and (x, y). It follows that $G_0(R)$ is generated by $[R]$ and $[R/(x, y)]$.

Choose a nonzero element $r \in R$. One has an exact sequence

$$0 \to R \xrightarrow{\cdot r} R \to R/(r) \to 0,$$

and $R/(r)$ is a finite length R-module. Thus, the class $[R/(x, y)] \in G_0(R)$ is torsion. Since $\Phi([R]) = 0$, it follows that $K_0(\underline{\mathrm{MCM}}(R))$ is torsion. Hence, by Remark 2.2.9, $K_0[\mathrm{MF}(Q, f)]$ is a torsion group.

$x+1 \in \widehat{Q}$ has a square root $z \in \widehat{Q}$; this follows, for instance, from Hensel's Lemma. Thus, there is an isomorphism

$$\widehat{Q}/(y^2 - x^2) \to \widehat{Q}/(f)$$

given by $x \mapsto zx$ and $y \mapsto y$.

Finally, note that $K_0[\mathrm{MF}(\widehat{Q}, y^2 - x^2)]$ is not a torsion group. One way to see this is that the Hochster theta pairing (see Section 2.2.3)

$$\theta : K_0[\mathrm{MF}(\widehat{Q}, y^2 - x^2)] \times K_0[\mathrm{MF}(\widehat{Q}, y^2 - x^2)] \to \mathbb{Z}$$

is non-zero (Examples 1.1 in [BVS12]). $\square$

Remark 2.5.2. In fact, $K_0[\mathrm{MF}(\widehat{Q}, y^2 - x^2)] \cong \mathbb{Z}$. To see this, we need only show that $K_0[\mathrm{MF}(\widehat{Q}, y^2 - x^2)]$ is cyclic, since we demonstrated above that it is not torsion.

The only primes of $\widehat{Q}/(y^2 - x^2)$ are $(x, y), (x)$, and (y). Hence, $G_0(\widehat{Q}/(y^2 - x^2))$ is generated by $[\widehat{Q}/(x, y)], [\widehat{Q}/(x)]$, and $[\widehat{Q}/(y)]$. By the reasoning in the proof above, one has a surjection

$$G_0(\widehat{Q}/(y^2 - x^2)) \to K_0[\mathrm{MF}(\widehat{Q}, y^2 - x^2)].$$

Since there is an exact sequence

$$0 \to \widehat{Q}/(y) \xrightarrow{\cdot x} \widehat{Q}/(y) \to \widehat{Q}/(x, y) \to 0,$$

$[\widehat{Q}/(x, y)] = 0$ in $G_0(\widehat{Q}/(y^2 - x^2))$. It is also easy to see that the image of $[\widehat{Q}/(x)]$ in $K_0[\mathrm{MF}(\widehat{Q}, y^2 - x^2)]$ is the negative of the image of $[\widehat{Q}/(y)]$. Thus, $K_0[\mathrm{MF}(\widehat{Q}, y^2 - x^2)]$ is cyclic.

The following is now a straightforward consequence of Proposition 2.5 and several results in [Dyc11]:

Proposition 2.5.3. *The triangulated category* $[\mathrm{MF}(Q, f)]$ *is not idempotent complete.*

Proof. $Q/(f)$ is an isolated hypersurface singularity in the sense of Definition 2.2.10; thus, by Theorem 5.2 and Theorem 5.7 in [Dyc11], $[\mathrm{MF}(\widehat{Q}, f)]$ is the idempotent completion of $[\mathrm{MF}(Q,f)]$. By Prop 2.5, this means $[\mathrm{MF}(Q,f)]$ cannot be idempotent complete. $\square$

Since homotopy categories of dg-triangulated categories are idempotent complete, we immediately obtain:

Corollary 2.5.4. $\mathrm{MF}(Q,f)$ *is not dg-triangulated.*

It is an illuminating exercise to produce an idempotent morphism in $[\mathrm{MF}(Q,f)]$ that does not split; we conclude this section by doing so.

As discussed in the proof of Proposition 2.5, there is a ring isomorphism

$$\widehat{Q}/(x^2-y^2) \xrightarrow{\cong} \widehat{Q}/(f).$$

We construct a model for the stabilization of the residue field of $\widehat{Q}/(x^2-y^2)$, as in Section 2.2.4.

Let F be a rank 2 free $\widehat{Q}$-module. Choose a basis e_1, e_2 of F, so that one has a basis $1, e_1, e_2, e_1e_2$ of $\bigoplus_{i=0}^2 \bigwedge^i F$.

As in Section 2.2.4, we may use the expression

$$x^2 - y^2 = x \cdot x + (-y) \cdot y$$

to build the matrix factorization

$$E_{x^2-y^2} = \Big(\textstyle\bigwedge^0 F \oplus \bigwedge^2 F \underset{\begin{pmatrix} x & y \\ y & x \end{pmatrix}}{\overset{\begin{pmatrix} x & -y \\ -y & x \end{pmatrix}}{\rightleftarrows}} \bigwedge^1 F\Big).$$

Again following Section 2.2.4, express $\mathrm{End}_{\mathrm{MF}}(E_{x^2-y^2})$ in terms of generators and relations as the dg $\widehat{Q}$-algebra

$$\widehat{Q}\langle\lambda_1, \lambda_2, c_1, c_2\rangle/([\lambda_i, \lambda_j], [c_i, c_j], [\lambda_i, c_j] - \delta_{ij})$$

with differential ∂ determined by $\partial(\lambda_1) = x = \partial(c_1)$, $\partial(\lambda_2) = y$, and $\partial(c_2) = -y$.

As observed in Section 5.5 of [Dyc11], the cycles $z_1 = c_1 - \lambda_1$ and $z_2 = c_2 + \lambda_2$ generate $H^0\mathrm{End}_{\mathrm{MF}}(E_{x^2-y^2})$ as a $\mathbb{C}$-algebra (in fact, $H^0\mathrm{End}_{\mathrm{MF}}(E_{x^2-y^2})$ is isomorphic to the Clifford algebra $\mathrm{Cliff}_{\mathbb{C}}(y^2 - x^2)$).

Notice that the element $\frac{1+z_1z_2}{2}$ of the algebra $H^0\mathrm{End}_{\mathrm{MF}}(E_{x^2-y^2})$ is idempotent.

Since $Q/(f)$ is IHS, the functor

$$[\mathrm{MF}(Q, f)] \to [\mathrm{MF}(\widehat{Q}, f)]$$

induced by the inclusion $Q \hookrightarrow \widehat{Q}$ is fully faithful. Letting E_f and $E_{\widehat{f}}$ denote the stabilizations of the residue fields of $Q/(f)$ and $\widehat{Q}/(f)$, we may trace through the isomorphisms

$$H^0\mathrm{End}_{\mathrm{MF}}(E_f) \xrightarrow{\cong} H^0\mathrm{End}_{\mathrm{MF}}(E_{\widehat{f}}) \xrightarrow{\cong} H^0\mathrm{End}_{\mathrm{MF}}(E_{x^2-y^2})$$

to obtain an idempotent z of $[\mathrm{MF}(Q, f)]$. We now demonstrate that z does not split.

The morphism $\frac{1+z_1z_2}{2}$, thought of as an element of $\mathrm{End}_{\mathrm{MF}}(E_{x^2-y^2})$, may be expressed by the diagram

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\bigwedge^0 F \oplus \bigwedge^2 F \arrow[r, "\begin{pmatrix} x & -y \\ -y & x \end{pmatrix}"] \arrow[d, "P"] & \bigwedge^1 F \arrow[r, "\begin{pmatrix} x & y \\ y & x \end{pmatrix}"] \arrow[d, "P"] & \bigwedge^0 F \oplus \bigwedge^2 F \arrow[d, "P"] \\
\bigwedge^0 F \oplus \bigwedge^2 F \arrow[r, "\begin{pmatrix} x & -y \\ -y & x \end{pmatrix}"] & \bigwedge^1 F \arrow[r, "\begin{pmatrix} x & y \\ y & x \end{pmatrix}"] & \bigwedge^0 F \oplus \bigwedge^2 F
\end{tikzcd}$$

where P is the matrix $\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

Since idempotents split in $\mathrm{MF}(\widehat{Q}, x^2 - y^2)$, the kernel and cokernel of $\frac{1+z_1z_2}{2}$ determine objects in $\mathrm{MF}(\widehat{Q}, x^2 - y^2)$. It is easy to see that $\ker(\frac{1+z_1z_2}{2})$ is isomorphic to

$$\widehat{Q} \underset{x+y}{\overset{x-y}{\rightleftarrows}} \widehat{Q}.$$

As established in Remark 2.5.2, $[E_f] = 0$ and $[\widehat{Q} \underset{x+y}{\overset{x-y}{\rightleftarrows}} \widehat{Q}] \neq 0$ in $K_0[\mathrm{MF}(\widehat{Q}, x^2 - y^2)]$; this implies that $[\mathrm{coker}(\frac{1+z_1z_2}{2})] = -[\widehat{Q} \underset{x+y}{\overset{x-y}{\rightleftarrows}} \widehat{Q}] \neq 0$.

Now, suppose z splits. Choose an object Y of $[\mathrm{MF}(Q, f)]$ such that there is a factorization

$$E_f \xrightarrow{\pi} Y \xrightarrow{\iota} E_f$$

of z, where $\pi \circ \iota = \mathrm{id}_Y$. Applying the composition

$$\Theta : [\mathrm{MF}(Q, f)] \to [\mathrm{MF}(\widehat{Q}, f)] \xrightarrow{\cong} [\mathrm{MF}(\widehat{Q}, x^2 - y^2)]$$

yields a splitting of the idempotent $\frac{1+z_1z_2}{2}$; this means $\Theta(Y) \cong \mathrm{coker}(\frac{1+z_1z_2}{2})$ in $[\mathrm{MF}(\widehat{Q}, x^2 - y^2)]$.

Since the class $[\mathrm{coker}(\frac{1+z_1z_2}{2})]$ is nonzero in $K_0[\mathrm{MF}(\widehat{Q}, x^2 - y^2)] \cong \mathbb{Z}$, this implies that the map

$$K_0[\mathrm{MF}(Q, f)] \to K_0[\mathrm{MF}(\widehat{Q}, x^2 - y^2)]$$

induced by Θ is nonzero. But this is impossible, since $K_0[\mathrm{MF}(Q, f)]$ is torsion. Thus, z does not split.

2.6 Periodicity

The following phenomenon, discovered by Knörrer in [Knö87], is known as *Knörrer periodicity*:

Theorem 2.6.1. *Suppose k is an algebraically closed field and* $\mathrm{char}(k) \neq 2$. *Let $q = u^2 + v^2 \in k[[u, v]]$. If $f \in (x_1, \dots, x_n) \smallsetminus \{0\} \subseteq k[[x_1, \dots, x_n]]$, there is a triangulated equivalence*

$$K : [\mathrm{MF}(k[[x_1, \dots, x_n]], f)] \to [\mathrm{MF}(k[[x_1, \dots, x_n, u, v]], f + q)].$$

Remark 2.6.2. Set X to be the matrix factorization

$$k[[u, v]] \underset{u-iv}{\overset{u+iv}{\rightleftarrows}} k[[u, v]]$$

of $u^2 + v^2$ over $k[[u, v]]$. K may be given by

$$P \mapsto P \otimes_{\mathrm{MF}} X$$

on objects and

$$\alpha \mapsto \alpha \otimes \mathrm{id}_X$$

on morphisms.

A version of Knörrer periodicity for isolated hypersurface singularities may be deduced from the following proposition:

Proposition 2.6.3. *Suppose Q and Q' are algebras over a field k. Let $f \in Q$ and $f' \in Q'$, and suppose $Q/(f)$ and $Q'/(f')$ are* IHS*. Set $Q'' := Q \otimes_k Q'$. If there exists an object X in $\mathrm{MF}(Q', f')$ such that*

(a) X is a compact generator of $[\mathrm{MF}(\widehat{Q'}, f')]$, and

(b) the inclusion $k \hookrightarrow \mathrm{End}_{\mathrm{MF}(\widehat{Q'}, f')}(X)$ is a quasi-isomorphism

then the dg functor

$$K_X : \mathrm{MF}(\widehat{Q}, f) \to \mathrm{MF}(\widehat{Q''}, f \oplus f')$$

given by

$$P \mapsto P \otimes_{\mathrm{MF}} X$$

on objects and

$$\alpha \mapsto \alpha \otimes \mathrm{id}_X$$

on morphisms is a quasi-equivalence.

Proof. By Sections 4 and 5 of [Dyc11], the inclusion

$$\mathrm{End}_{\mathrm{MF}}(X) \hookrightarrow \mathrm{MF}(\widehat{Q'}, f').$$

is a Morita equivalence. Applying Exercise 4.4.11 in [Toë11], we have a chain of Morita equivalences

$$\mathrm{MF}(\widehat{Q}, f) \otimes_k k \hookrightarrow \mathrm{MF}(\widehat{Q}, f) \otimes_k \mathrm{End}_{\mathrm{MF}}(X) \hookrightarrow \mathrm{MF}(\widehat{Q}, f) \otimes_k \mathrm{MF}(\widehat{Q'}, f').$$

Composing with $\widehat{\mathrm{ST}}_{\mathrm{MF}}$, Proposition 2.3.4 yields a Morita equivalence

$$\mathrm{MF}(\widehat{Q}, f) \to \mathrm{MF}(\widehat{Q''}, f \oplus f').$$

This composition is clearly the functor K_X. Since both $\mathrm{MF}(\widehat{Q}, f)$ and $\mathrm{MF}(\widehat{Q''}, f \oplus f')$ are dg-triangulated by Lemma 5.6 in [Dyc11], an application of Proposition 2.1.15 finishes the proof. $\square$

To deduce a version of Knörrer periodicity for isolated hypersurface singularities, assume k to be an algebraically closed field such that $\mathrm{char}(k) \neq 2$, set $Q' = k[u, v]$ and $f' = u^2 + v^2$, and take X to be the matrix factorization

$$k[u, v] \underset{u-iv}{\overset{u+iv}{\rightleftarrows}} k[u, v].$$

This is the approach taken in Section 5.3 of [Dyc11].

We point out that k is not assumed to be algebraically closed in Proposition 2.6.3, and no assumptions on the characteristic of k are made, either. In particular, we may use Proposition 2.6.3 to prove an 8-periodic version of Knörrer periodicity over $\mathbb{R}$ (this result implies Theorem 1.0.1 from the introduction):

Theorem 2.6.4. *Suppose Q is an $\mathbb{R}$-algebra. Let $f \in Q$, and suppose $Q/(f)$ is* IHS. *Set $Q' := \mathbb{R}[u_1, \dots, u_8]$, $q := u_1^2 + \cdots + u_8^2 \in Q'$, and $Q'' := Q \otimes_{\mathbb{R}} Q'$. Then there exists a matrix factorization X of q over Q' such that the dg functor*

$$\mathrm{MF}(\widehat{Q}, f) \to \mathrm{MF}(\widehat{Q''}, f + q)$$

given by

$$P \mapsto P \otimes_{\mathrm{MF}} X$$

on objects and

$$\alpha \mapsto \alpha \otimes \mathrm{id}_X$$

on morphisms is a quasi-equivalence.

Proof. We equip the matrix algebra $\mathrm{Mat}_{16}(\mathbb{R})$ of 16×16 of matrices over $\mathbb{R}$ with a $\mathbb{Z}/2\mathbb{Z}$-grading in the following way: $A = (a_{ij})$ is homogeneous of even degree if $a_{ij} = 0$ whenever $i + j$ is odd, and A is homogeneous of odd degree if $a_{ij} = 0$ whenever $i + j$ is even. Then, by Proposition V.4.2 in [Lam05],

$$\mathrm{Cliff}_k(q) \cong \mathrm{Mat}_{16}(\mathbb{R})$$

as $\mathbb{Z}/2\mathbb{Z}$-graded algebras. In particular,

$$[\mathrm{MF}(\widehat{Q'}, q)] \cong \mathrm{mod}_{\mathbb{Z}/2\mathbb{Z}}(\mathrm{Mat}_{16}(\mathbb{R}))$$

by Theorem 2.4.1.

Let $M \in \mathrm{mod}_{\mathbb{Z}/2\mathbb{Z}}(\mathrm{Mat}_{16}(\mathbb{R}))$ be the module of matrices with nonzero entries only in the first column, and let X be a matrix factorization corresponding to M under the equivalence of categories in Theorem 2.4.1. Since

$$\mathrm{End}_{\mathrm{Mat}_{16}(\mathbb{R})}(M) \cong \mathbb{R}$$

as $\mathbb{Z}/2\mathbb{Z}$-graded algebras, where $\mathbb{R}$ is concentrated in even degree, we have

$$H^0(\mathrm{End}_{\mathrm{MF}}(X)) \cong \mathbb{R}.$$

Thus, the inclusion

$$\mathbb{R} \hookrightarrow \mathrm{End}_{\mathrm{MF}}(X)$$

is a quasi-isomorphism. Also, by Proposition 2.4.4,

$$(X \oplus X[1])^{\oplus 8} \cong \mathbb{R}^{\mathrm{stab}}$$

in $\mathrm{MF}(\widehat{Q'}, q)$; it follows from Sections 4 and 5 of [Dyc11] that X is a compact generator of $\mathrm{MF}(\widehat{Q'}, q)$. Now apply Proposition 2.6.3. $\square$

Remark 2.6.5. Explicitly, one may take X to be the matrix factorization

$$\widehat{Q'}^{\oplus 8} \underset{B}{\overset{A}{\rightleftarrows}} \widehat{Q'}^{\oplus 8},$$

where

$$A = \begin{pmatrix} u_1 & u_2 & -u_3 & u_4 & u_5 & -u_6 & -u_7 & -u_8 \\ -u_2 & u_1 & -u_4 & -u_3 & u_6 & u_5 & -u_8 & u_7 \\ u_3 & u_4 & u_1 & -u_2 & u_7 & u_8 & u_5 & -u_6 \\ -u_4 & u_3 & u_2 & u_1 & u_8 & -u_7 & u_6 & u_5 \\ u_5 & u_6 & u_7 & u_8 & u_1 & -u_2 & u_3 & -u_4 \\ -u_6 & u_5 & u_8 & -u_7 & u_2 & u_1 & u_4 & u_3 \\ -u_7 & -u_8 & u_5 & u_6 & -u_3 & -u_4 & u_1 & u_2 \\ -u_8 & u_7 & -u_6 & u_5 & u_4 & -u_3 & -u_2 & u_1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} u_1 & -u_2 & u_3 & -u_4 & u_5 & u_6 & u_7 & u_8 \\ u_2 & u_1 & u_4 & u_3 & -u_6 & u_5 & u_8 & -u_7 \\ -u_3 & -u_4 & u_1 & u_2 & -u_7 & -u_8 & u_5 & u_6 \\ u_4 & -u_3 & -u_2 & u_1 & -u_8 & u_7 & -u_6 & u_5 \\ -u_5 & u_6 & u_7 & u_8 & u_1 & u_2 & -u_3 & u_4 \\ -u_6 & -u_5 & u_8 & -u_7 & -u_2 & u_1 & -u_4 & -u_3 \\ -u_7 & -u_8 & -u_5 & u_6 & u_3 & u_4 & u_1 & -u_2 \\ -u_8 & u_7 & -u_6 & -u_5 & -u_4 & u_3 & u_2 & u_1 \end{pmatrix}$$

This can be verified by computing $\Theta_{u_1^2+\cdots+u_8^2}(M)$ using the formula provided in Section 2.4.

Remark 2.6.6. Theorem 2.6.4 implies the existence of a Knörrer-type periodicity for matrix factorizations over $\mathbb{R}$ of period at most 8. We point out that the period is exactly 8, since the Brauer-Wall group of $\mathbb{R}$ is the cyclic group $\mathbb{Z}/8\mathbb{Z}$ generated by the class of $\mathrm{Cliff}_{\mathbb{R}}(x^2)$ (see [Yos90] Remark 14.9).

It is natural to ask whether one may use Proposition 2.6.3 to exhibit additional periodic behaviors of matrix factorization categories. We conclude this section with some remarks in this direction.

The existence of an object X as in the setup of 2.6.3 implies that the dga $A_{(Q',f')}$ is formal, since, by Theorem 5.1 in [Dyc11], $A_{(Q',f')}$ is quasi-isomorphic, in this setting, to the endomorphism dga of a $\mathbb{Z}/2\mathbb{Z}$-graded complex of k-vector spaces. On the other hand, by Theorem 5.9 in [Dyc11], $A_{(Q',f')}$ can only be formal when f' has no terms of degree higher than 2. It follows that, when $\mathrm{char}(k) \neq 2$, we may use the Buchweitz-Eisenbud-Herzog equivalence (Theorem 2.4.1) to reduce the problem of determining whether $\mathrm{MF}(Q', f')$ is Morita equivalent to k to studying the image of the group homomorphism

$$WG(k) \to BW(k)$$

where $WG(k)$ is the Witt-Grothendieck ring of k, thought of as an additive group, and $BW(k)$ is the Brauer-Wall group of k. When $\mathrm{char}(k) = 2$, less is known about the structure of matrix factorization categories over non-degenerate quadratics. We leave for future work the problem of finding sufficient conditions for such a matrix factorization category to be Morita equivalent to its ground field.

Chapter 3

Matrix Factorizations and the K-theory of the Milnor Fiber

We have demonstrated that matrix factorization categories associated to isolated hypersurface singularities over $\mathbb{C}$ and $\mathbb{R}$ exhibit 2- and 8-periodic versions of Knörrer periodicity, respectively. This pattern resembles Bott periodicity in topological K-theory; the goal of this chapter is to explain this resemblance.

We give a rough sketch of our approach. The classical link between the periodicity of Clifford algebras up to $\mathbb{Z}/2\mathbb{Z}$-graded Morita equivalence and Bott periodicity in topological K-theory is the *Atiyah-Bott-Shapiro construction*, which first appeared in Part III of [ABS64]. Loosely speaking, the Atiyah-Bott-Shapiro construction is a way of mapping a finitely generated $\mathbb{Z}/2\mathbb{Z}$-graded module over a real or complex Clifford algebra to a class in the K-theory of a sphere.

Composing the Buchweitz-Eisenbud-Herzog equivalence (Theorem 2.4.1) with the Atiyah-Bott-Shapiro construction, we have a way of assigning a class in the topological K-theory of a sphere to a matrix factorization of a non-degenerate quadratic form over

$\mathbb{R}$ or $\mathbb{C}$:

$$\text{mf's of real/complex quadratics} \xrightarrow{\text{ABS} \circ \text{BEH}} \text{K-theory of spheres}$$

The idea is to lift this composition; that is, we wish to associate a space X_f to a real or complex polynomial f and construct a map from matrix factorizations of f to the topological K-theory of X_f so that the diagram

$$\begin{tikzcd}
\text{mf's of real/complex quadratics} \arrow[r, "\text{ABS} \circ \text{BEH}"] \arrow[d, hook] & \text{K-theory of spheres} \arrow[d, hook] \\
\text{mf's of real/complex polynomials} \arrow[r, dotted] & \text{K-theory of spaces of the form } X_f
\end{tikzcd}$$

commutes.

It turns out that the right choice of X_f is the *Milnor fiber* (resp. *positive or negative Milnor fiber*) associated to the complex (resp. real) polynomial.

We begin this chapter with discussions of known results concerning the Milnor fiber and relative topological K-theory. Then, using the work of Atiyah-Bott-Shapiro in [ABS64] as a guide, we will complete the above diagram, and we will use the bottom arrow to explain a precise sense in which Knörrer periodicity and Bott periodicity are compatible phenomena.

3.1 The real and complex Milnor fibers

Let $f \in \mathbb{C}[x_0, \ldots, x_n]$, and suppose $f(0) = 0$. We begin this section by describing the construction of the *Milnor fiber* associated to f, following the exposition in Section 1 of [BVS12]. We then discuss various properties of the Milnor fiber that we will make use of later on.

3.1.1 Construction of the Milnor fibration and some properties of the Milnor fiber

For $\epsilon > 0$, define B_ϵ to be the closed ball centered at the origin of radius ϵ in $\mathbb{C}^{n+1}$, and for $\delta > 0$, set D^*_δ to be the punctured disk centered at the origin in $\mathbb{C}$ of radius δ.

Choose $\epsilon > 0$ so that, for $0 < \epsilon' \leqslant \epsilon$, $\partial B_{\epsilon'}$ intersects $f^{-1}(0)$ transversely. Upon choosing such an ϵ, choose $\delta \in (0, \epsilon)$ such that $f^{-1}(t)$ intersects ∂B_ϵ transversely for all $t \in D^*_\delta$. Then the map

$$\psi : B_\epsilon \cap f^{-1}(D^*_\delta) \to D^*_\delta$$

given by $\psi(x) = f(x)$ is a locally trivial fibration.

The map ψ depends, of course, on our choices of ϵ and δ. However, if ϵ', δ' is another pair of positive numbers satisfying the above conditions, the fibration associated to these choices is equivalent to the one above (see Definition 1.5 in Chapter 2 of [Dim92] for a description of what it means for two fibrations to be equivalent). We are thus justified in calling ψ the *Milnor fibration* associated to f.

Remark 3.1.1. The Milnor fibration was originally introduced in [Mil68]. The above construction is not the same as the construction of the Milnor fibration in [Mil68], but the two constructions yield equivalent fibrations; this is a result due to Lê in [Lê76].

We will call the fiber of this fibration the *Milnor fiber* of f and denote it by F_f. F_f is independent of our choices of ϵ and δ up to homeomorphism, so we suppress these choices in the notation. However, these choices will be significant at various points later on.

Definition 3.1.2. Let k be a field. A polynomial $f \in k[x_0, \dots, x_n]$ is called *quasi-homogeneous* of degree d if there exist positive integers $w_0, \dots, w_n$ such that f is a homogeneous element of degree d in the $\mathbb{Z}$-graded ring $k[x_0, \dots, x_n]$, where each variable x_i has degree w_i.

Remark 3.1.3. Suppose f is quasi-homogeneous, and let $t > 0$. Define

$$h : \mathbb{C}[x_0, \dots, x_n] \to \mathbb{C}[x_0, \dots, x_n]$$

to be the ring automorphism given by

$$x_i \mapsto \frac{x_i}{t^{w_i/d}}.$$

Then, if the Milnor fiber of f may be taken to be a fiber of f over t, the Milnor fiber of $h(f)$ may be taken to be a fiber over 1; hence one may often assume without loss that the Milnor fiber associated to a quasi-homogeneous polynomial is a fiber over 1.

If $\mathbb{C}[x_0, \dots, x_n]_{(x_0, \dots, x_n)}/(f)$ is IHS (see Definition 2.2.10), set

$$\mu := \dim_{\mathbb{C}} \frac{\mathbb{C}[x_0, \dots, x_n]_{(x_0, \dots, x_n)}}{(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n})} < \infty,$$

the *Milnor number* of f.

Theorem 3.1.4 (Milnor, 1968)**.** *If* $\mathbb{C}[x_0, \dots, x_n]_{(x_0, \dots, x_n)}/(f)$ *is* IHS*,* F_f *is homotopy equivalent to a wedge sum of* μ *copies of* S^n*.*

Remark 3.1.5. Since ψ restricts to a fibration over a circle, F_f comes equipped with a *monodromy homeomorphism*

$$h_f : F_f \xrightarrow{\cong} F_f.$$

3.1.2 The Sebastiani-Thom homotopy equivalence

We recall the definition of the *join* of two topological spaces:

Definition 3.1.6. Let X and Y be compact Hausdorff spaces. The *join* of X and Y, denoted $X * Y$, is the quotient of $X \times Y \times I$ by the relations

$$(x_1, y, 0) \sim (x_2, y, 0)$$

$$(x, y_1, 1) \sim (x, y_2, 1)$$

equipped with the quotient topology.

Remark 3.1.7. The *cone* CX over a compact Hausdorff space X can be expressed explicitly as the quotient of

$$X \times [0, 1]$$

by the relation

$$(x_1, 0) \sim (x_2, 0)$$

for all $x_1, x_2 \in X$.

When X and Y are compact Hausdorff,

$$X * Y \cong (CX \times Y) \cup (X \times CY) \subseteq CX \times CY;$$

here, we identify X and Y with the subsets $X \times \{1\}$ and $Y \times \{1\}$ of CX and CY, respectively.

By [Bro06] 5.7.4, one has an explicit homeomorphism

$$CX \times CY \to C(X * Y)$$

given by

$$(x,t,y,t') \mapsto ((x,y,\frac{t}{2t'}),t'), \text{ if } t' \geqslant t,\ t' \neq 0$$

$$(x,t,y,t') \mapsto ((x,y,1-\frac{t'}{2t}),t), \text{ if } t \geqslant t',\ t \neq 0$$

$$(x,0,y,0) \mapsto ((x,y,0),0),$$

and this map restricts to a homeomorphism

$$(CX \times Y) \cup (X \times CY) \to X * Y.$$

Example 3.1.8. Let X be a compact Hausdorff space.

- $X * \text{point} \cong CX$.
- $X * S^0 \cong SX$, the *suspension* of X.

Remark 3.1.9. When forming the join of spaces X and Y that are not necessarily compact Hausdorff, the set $X * Y$ is typically equipped with the weakest topology such that the coordinate projections from $X \times Y \times I$ to $X * Y$ are continuous. When X and Y are compact Hausdorff, this topology coincides with the quotient topology ([Ehl92] Section 3.2).

Now, suppose $f \in \mathbb{C}[x_0, \dots, x_n]$, $f' \in \mathbb{C}[y_0, \dots, y_m]$, and $f(0) = 0 = f'(0)$. Let $f \oplus f'$ denote the sum of f and f' thought of as an element of $\mathbb{C}[x_0, \dots, x_n, y_0, \dots, y_m]$.

We have the following classical result relating the Milnor fibers of f, f', and $f \oplus f'$, due to Sebastiani-Thom:

Theorem 3.1.10 ([ST71])**.** *There is a homotopy equivalence*

$$\mathrm{ST} : F_f * F_{f'} \to F_{f \oplus f'}$$

that is compatible with monodromy; that is, the square

$$\begin{array}{ccc} F_f * F_{f'} & \xrightarrow{\text{ST}} & F_{f\oplus f'} \\ {\scriptstyle h_f * h_{f'}}\downarrow & & {\scriptstyle h_{f\oplus f'}}\downarrow \\ F_f * F_{f'} & \xrightarrow{\text{ST}} & F_{f\oplus f'} \end{array}$$

commutes up to homotopy.

We refer the reader to Section 2.7 of [AGZV12], §3 of Chapter 3 in [Dim92], and [Oka73] for discussions related to Theorem 3.1.10.

Suppose $\mathbb{C}[x_0,\ldots,x_n]_{(x_0,\ldots,x_n)}/(f)$, $\mathbb{C}[y_0,\ldots,y_m]_{(y_0,\ldots,y_m)}/(f')$ are IHS (see Definition 2.2.10). We now exhibit an explicit map realizing the homotopy equivalence in Theorem 3.1.10 in this setting, following Section 2.7 of [AGZV12].

Choose real numbers ϵ'', δ'', such that the map

$$B_{\epsilon''} \cap (f \oplus f')^{-1}(D^*_{\delta''}) \to D^*_{\delta''}$$

given by $x \mapsto (f \oplus f')(x)$ is a locally trivial fibration, as above.

Similarly, choose ϵ, δ and ϵ', δ', as well as $t'' \in D^*_{\delta''}$, so that the analogous maps

$$B_\epsilon \cap f^{-1}(D^*_\delta) \to D^*_\delta$$

$$B_{\epsilon'} \cap (f')^{-1}(D^*_{\delta'}) \to D^*_{\delta'}$$

are locally trivial fibrations, and also so that

(a) ϵ, ϵ' are sufficiently small so that $B_\epsilon \times B_{\epsilon'} \subseteq B_{\epsilon''}$.

(b) $|t''| < \min\{\delta, \delta'\}$.

Set F_f, $F_{f'}$, and $F_{f\oplus f'}$ to be the Milnor fibers of f, f', and $f \oplus f'$ over t''.

The goal is to construct an injective homotopy equivalence

$$CF_f \times F_{f'} \cup F_f \times CF_{f'} \to F_{f \oplus f'}.$$

Applying Lemma 2.10 in [AGZV12], choose an injection

$$H : CF_f \to B_\epsilon$$

such that

- $H(x, 1) = x \in F_f \subseteq B_\epsilon$,
- $H(-, s) : F_f \to B_\epsilon$ maps into the Milnor fiber $B_\epsilon \cap f^{-1}(st'')$ for $s \in (0, 1)$, and
- $H(x, 0) = 0$ for all $x \in F_f$

Example 3.1.11. If f is quasi-homogeneous of degree d with weights $w_0, \ldots, w_d$, such a map H may be given by

$$(x, s) \mapsto (s^{\frac{w_0}{d}} x_0, \ldots, s^{\frac{w_n}{d}} x_n).$$

Notice that our isolated singularity assumption is not necessary in this example.

Choose H' similarly for the Milnor fiber $F_{f'}$.

By the discussion on pages 54-55 of [AGZV12] and Remark 3.1.7, there is an injective homotopy equivalence

$$(\mathrm{im}(H) \times F_{f'} \cup F_f \times \mathrm{im}(H')) \to F_{f \oplus f'}$$

given by

$$(H(x,s), H'(y,s')) \mapsto (H(x, \frac{1+s-s'}{2}), H'(y, \frac{1-s+s'}{2})).$$

Composing, one has an injective homotopy equivalence

$$g : CF_f \times F_{f'} \cup F_f \times CF_{f'} \to F_{f\oplus f'},$$

as desired. The map obtained by composing g with the inverse of the homeomorphism from $CF_f \times F_{f'} \cup F_f \times CF_{f'}$ to $F_f * F_{f'}$ in Remark 3.1.7 enjoys the same properties as the map ST in Theorem 2.3.4.

Remark 3.1.12. The homotopy equivalence

$$(\mathrm{im}(H) \times F_{f'} \cup F_f \times \mathrm{im}(H')) \to F_{f\oplus f'}$$

above extends to an injection of pairs

$$G : (\mathrm{im}(H) \times \mathrm{im}(H'), \mathrm{im}(H) \times F_{f'} \cup F_f \times \mathrm{im}(H')) \to (B_{\epsilon''}, F_{f\oplus f'})$$

that maps a point $(H(x,s), H'(y,s'))$ to

$$(H(x, \frac{s}{2}), H'(y, \frac{2s'-s}{2}), \text{ if } s \leqslant s',\ s' \neq 0$$

$$(H(x, \frac{2s-s'}{2}), H'(y, \frac{s'}{2}), \text{ if } s' \leqslant s,\ s \neq 0$$

$$0, \text{ if } s = 0 = s'.$$

The image of $\mathrm{im}(H) \times \mathrm{im}(H')$ under this injection is homeomorphic to $CF_{f\oplus f'}$ in an evident way.

3.1.3 An analogue of the Milnor fibration for polynomials over $\mathbb{R}$

Now, suppose $f \in \mathbb{R}[x_0, \ldots, x_n]$ and $f(0) = 0$. One may construct a topological locally trivial fibration

$$\psi : B_\epsilon \cap f^{-1}((-\delta, 0) \cup (0, \delta)) \to (-\delta, 0) \cup (0, \delta)$$

for some $\epsilon > 0$ and δ such that $0 < \delta << \epsilon$ in the same way as above, where B_ϵ is now the closed ball of radius ϵ centered at the origin in $\mathbb{R}^{n+1}$.

But now, fibers over $(-\delta, 0)$ and $(0, \delta)$ need not be homeomorphic. For instance, if $f = x_0^2 + \cdots + x_n^2$, the positive fibers of ψ are homeomorphic to S^n, while the negative fibers are empty.

We denote by F_f^+ and F_f^- the positive and negative Milnor fibers of f. The topology of the real Milnor fibers is more complicated than that of the complex Milnor fiber. However, there is a version of Theorem 3.1.10 for real Milnor fibers:

Theorem 3.1.13 ([DP92] Remark 11)**.** *Suppose*

$$f \in \mathbb{R}[x_0, \ldots, x_n],\ g \in \mathbb{R}[y_0, \ldots, y_m]$$

are quasi-homogeneous, and $f(0) = 0 = g(0)$*.*

If F_f^+ *and* F_g^+ *are nonempty, there is a homotopy equivalence*

$$\mathrm{ST} : F_f^+ * F_g^+ \to F_{f\oplus g}^+.$$

Remark 3.1.14. Since $F_{-f}^+ = F_f^-$, we have a similar result for negative Milnor fibers.

3.2 Relative topological K-theory and the Euler characteristic

We introduce some facts concerning relative topological K-theory that we will need along the way. All results in this section are essentially due to Atiyah-Bott-Shapiro in [ABS64], though we modify their exposition at various points to fit our purposes.

Let X be a compact topological space, and let Y be a closed subspace of X such that there exists a homotopy equivalence of pairs between (X, Y) and a finite CW pair; that is, a pair (X', Y') where X' is a finite CW complex and Y' is a subcomplex of X'. We construct a category $\mathcal{C}_1(X, Y)$ from (X, Y) in the following way:

• Objects of $\mathcal{C}_1(X, Y)$ are pairs of real vector bundles V_1, V_0 over X equipped with isomorphisms

$$V_1|_Y \xrightarrow{\sigma} V_0|_Y.$$

Denote objects of $\mathcal{C}_1(X, Y)$ by $(V_1, V_0; \sigma)$.

• Morphisms in $\mathcal{C}_1(X, Y)$ are pairs of morphisms of vector bundles over X

$$\alpha_1 : V_1 \to V_1',\ \alpha_0 : V_0 \to V_0'$$

such that the following diagram of maps of vector bundles over Y commutes:

$$\begin{tikzcd} V_1|_Y \arrow[r, "\sigma"] \arrow[d, "\alpha_1|_Y"'] & V_0|_Y \arrow[d, "\alpha_0|_Y"'] \\ V_1'|_Y \arrow[r, "\sigma'"] & V_0'|_Y \end{tikzcd}$$

We write morphisms in $\mathcal{C}_1(X, Y)$ as ordered pairs (α_1, α_0).

Remark 3.2.1. The reason for the subscript in the notation $\mathcal{C}_1(X, Y)$ is that, for any $n \geqslant 1$, one may build a category $\mathcal{C}_n(X, Y)$ with objects given by ordered $n+1$-tuples of vector spaces on X whose restrictions to Y fit into an exact sequence (cf. [ABS64] §8).

Remark 3.2.2. We will work with real vector bundles throughout this section; however, there is an analogous version of every result in this section for complex vector bundles.

Proposition 3.2.3. *A map $g : (X_1, Y_1) \to (X_2, Y_2)$ of pairs of spaces as above induces a functor*

$$g^* : \mathcal{C}_1(X_2, Y_2) \to \mathcal{C}_1(X_1, Y_1).$$

Proof. On objects,

$$g^*((V_1, V_0; \sigma)) = (g^*(V_1), g^*(V_0); (g|_{Y_1})^*(\sigma)).$$

If $\alpha = (\alpha_1, \alpha_0) : (V_1, V_0; \sigma) \to (V_1', V_0'; \sigma')$ is a morphism,

$$g^*(\alpha) = (g^*(\alpha_1), g^*(\alpha_0)).$$

The diagram

$$\begin{array}{ccc} g^*(V_1)|_{Y_1} & \xrightarrow{(g|_{Y_1})^*(\sigma)} & g^*(V_0)|_{Y_1} \\ {\scriptstyle g^*(\alpha_1)|_{Y_1}}\downarrow & & {\scriptstyle g^*(\alpha_0)|_{Y_1}}\downarrow \\ g^*(V_1')|_{Y_1} & \xrightarrow{(g|_{Y_1})^*(\sigma')} & g^*(V_0')|_{Y_1} \end{array}$$

commutes, since pullback of vector bundles respects composition. □

Given objects $V = (V_1, V_0; \sigma)$ and $V' = (V_1', V_0'; \sigma')$ in $\mathcal{C}_1(X, Y)$, define an object

$$V \oplus V' := (V_1 \oplus V_1', V_0 \oplus V_0'; \sigma \oplus \sigma').$$

We have evident canonical morphisms

$$\iota_V := V \to V \oplus V'$$

$$\iota_{V'} := V' \to V \oplus V'$$

Proposition 3.2.4. *Let $V = (V_1, V_0; \sigma)$ and $V' = (V_1', V_0'; \sigma')$ be objects in $\mathcal{C}_1(X,Y)$. Then*

$$(V \oplus V', \iota_V, \iota_{V'})$$

is the coproduct of V and V' in $\mathcal{C}_1(X,Y)$.

Proof. This follows easily from the fact that

$$(V_1 \oplus V_1', (\iota_V)_1, (\iota_{V'})_1)$$

and

$$(V_0 \oplus V_0', (\iota_V)_0, (\iota_{V'})_0)$$

are the coproducts in the category of vector bundles over X of V_1, V_1' and V_0, V_0'. $\square$

Proposition 3.2.5. *$\mathcal{C}_1(X,Y)$ is an additive category.*

Proof. It is well-known that the category of vector bundles over any topological space is additive ([Kar08] Theorem I.6.1).

Given morphisms

$$(\alpha_1, \alpha_0), (\beta_1, \beta_0) : (V_1, V_0; \sigma) \to (V_1', V_0'; \sigma'),$$

define

$$\alpha + \beta := (\alpha_1 + \beta_1, \alpha_0 + \beta_0).$$

It is easy to check that $\mathrm{Hom}_{\mathcal{C}(X,Y)}((V_1, V_0; \sigma), (V_1', V_0'; \sigma'))$, equipped with this operation, is an abelian group and that composition in $\mathcal{C}_1(X, Y)$ is $\mathbb{Z}$-bilinear.

Finally, apply Proposition 3.2.4 to conclude that $\mathcal{C}_1(X, Y)$ admits finite coproducts. $\square$

Remark 3.2.6. A morphism in (α_1, α_0) in $\mathcal{C}_1(X, Y)$ is an isomorphism (resp. monomorphism, epimorphism) if and only if α_1 and α_0 are isomorphisms (resp. monomorphisms, epimorphisms) of vector bundles over X.

We shall call an object of $\mathcal{C}_1(X, Y)$ *elementary* if it is isomorphic to an object of the form $(V, V; \mathrm{id}_{V|_Y})$. Notice that the direct sum of two elementary objects in $\mathcal{C}_1(X, Y)$ is again elementary.

There is a useful alternative definition of an elementary object:

Lemma 3.2.7. *Let $(V_1, V_0; \sigma)$ be an object in $\mathcal{C}_1(X, Y)$. The following are equivalent:*

(1) σ can be extended to an isomorphism $\widetilde{\sigma} : V_1 \to V_0$.

(2) $(V_1, V_0; \sigma)$ is elementary.

Proof. Suppose $(V_1, V_0; \sigma)$ is elementary. Then we have a commutative square on Y:

$$\begin{array}{ccc} V_1|_Y & \xrightarrow{\sigma} & V_0|_Y \\ {\scriptstyle \alpha_1|_Y}\downarrow & & \downarrow{\scriptstyle \alpha_0|_Y} \\ V|_Y & \xrightarrow{\mathrm{id}_{V|_Y}} & V|_Y \end{array}$$

where V is a vector bundle over X, and

$$\alpha_1 : V_1 \to V,\ \alpha_0 : V \to V_0$$

are isomorphisms of vector bundles over X. Observe that σ lifts to $\alpha_0^{-1} \circ \alpha_1$.

Conversely, suppose σ can be extended to an isomorphism $\widetilde{\sigma} : V_1 \to V_0$. Then we have a commutative square of maps of vector bundles on X:

$$\begin{array}{ccc} V_1 & \xrightarrow{\widetilde{\sigma}} & V_0 \\ {\scriptstyle \mathrm{id}_{V_1}}\downarrow & & \downarrow{\scriptstyle \widetilde{\sigma}^{-1}} \\ V_1 & \xrightarrow{\mathrm{id}_{V_1}} & V_1 \end{array}$$

□

If V and V' are objects in $\mathcal{C}_1(X, Y)$, we will say $V \sim V'$ if and only if there exist elementary objects E, E' such that

$$V \oplus E \cong V' \oplus E'.$$

The relation $\sim$ is an equivalence relation. Define $L_1(X, Y)$ to be the commutative monoid of equivalence classes under $\sim$ with operation given by $\oplus$.

Remark 3.2.8. Let (X_1, Y_1), (X_2, Y_2) be pairs as above, and let $g : (X_1, Y_1) \to (X_2, Y_2)$ be a map of pairs. Then the functor

$$g^* : \mathcal{C}_1(X_2, Y_2) \to \mathcal{C}_1(X_1, Y_1)$$

applied to an elementary object is again elementary. Hence, g^* induces a map of monoids

$$\overline{g^*} : L_1(X_2, Y_2) \to L_1(X_1, Y_1).$$

One may similarly define monoids $L_n(X, Y)$ involving longer sequences of bundles; see [ABS64] Definition 7.1 for details. Denote elements of $L_n(X, Y)$ by

$$[V_n, \ldots, V_0; \sigma_n, \ldots, \sigma_1].$$

We point out that there is an inclusion map

$$j_n : L_1(X, Y) \to L_n(X, Y)$$

given by

$$[V_1, V_0; \sigma] \mapsto [0, \dots, 0, V_1, V_0; 0, \dots, 0, \sigma],$$

and, by Proposition 7.4 in [ABS64], j_n is an isomorphism for all n.

The main reason we are interested in the monoid $L_1(X, Y)$ is the following result due to Atiyah-Bott-Shapiro:

Proposition 3.2.9 ([ABS64] 9.1)**.** *There exists a unique natural homomorphism*

$$\chi : L_1(X, Y) \to KO^0(X, Y)$$

which, when $Y = \emptyset$, is given by

$$\chi(E) = [V_0] - [V_1].$$

Moreover, χ is an isomorphism.

In particular, $L_1(X, Y)$ is an abelian group. Atiyah-Bott-Shapiro call the map χ an *Euler characteristic.*

Let (X, Y), (X', Y') be pairs as above. We conclude this section by exhibiting a product map

$$L_1(X, Y) \otimes L_1(X', Y') \to L_1(X \times X', X \times Y' \cup Y \times X')$$

that agrees, via χ, with the usual product on relative K-theory.

Let $V = (V_1, V_0; \sigma) \in \mathrm{Ob}(\mathcal{C}_1(X,Y))$ and $V' = (V_1', V_0'; \sigma') \in \mathrm{Ob}(\mathcal{C}_1(X',Y'))$. By Proposition 10.1 in [ABS64], we may lift σ, σ' to maps $\widetilde{\sigma}, \widetilde{\sigma}'$ of bundles over X and X', respectively.

Thinking of

$$0 \to V_1 \xrightarrow{\widetilde{\sigma}} V_0 \to 0$$

$$0 \to V_1' \xrightarrow{\widetilde{\sigma}'} V_0' \to 0$$

as complexes of bundles with V_1, V_1' in degree 1 and V_0, V_0' in degree 0, we may take their tensor product

$$V \otimes V' = 0 \to V_1 \otimes V_1' \xrightarrow{\tau_2} (V_1 \otimes V_0') \oplus (V_0 \otimes V_1') \xrightarrow{\tau_1} V_0 \otimes V_0' \to 0,$$

where

$$\tau_1 = \begin{pmatrix} \widetilde{\sigma} \otimes \mathrm{id}_{V_0'} & \mathrm{id}_{V_0} \otimes \widetilde{\sigma}' \end{pmatrix}$$

$$\tau_2 = \begin{pmatrix} -\mathrm{id}_{V_1} \otimes \widetilde{\sigma}' \\ \widetilde{\sigma} \otimes \mathrm{id}_{V_1'} \end{pmatrix}$$

The result is a complex of vector bundles over $X \times X'$ that is exact upon restriction to $X \times Y' \cup Y \times X'$.

Choose a splitting π of $\tau_2|_{X \times Y' \cup Y \times X'}$. Then,

$$[(V_1 \otimes V_0') \oplus (V_0 \otimes V_1'), (V_0 \otimes V_0') \oplus (V_1 \otimes V_1'); \begin{pmatrix} \tau_1|_{X \times Y' \cup Y \times X'} \\ \pi \end{pmatrix}]$$

is an element of $L_1(X \times X', X \times Y' \cup Y \times X')$.

Now, the pairing

$$L_1(X,Y) \otimes L_1(X',Y') \to L_1(X \times X', X \times Y' \cup Y \times X')$$

described in Proposition 10.4 of [ABS64] is given by sending a simple tensor

$$[V_1, V_0; \sigma] \otimes [V_1', V_0'; \sigma']$$

to

$$j_2^{-1}([V_1 \otimes V_1', (V_1 \otimes V_0') \oplus (V_0 \otimes V_1'), V_0 \otimes V_0'; \tau_2|_{X \times Y' \cup Y \times X'}, \tau_1|_{X \times Y' \cup Y \times X'}]);$$

this follows from the proof of Proposition 10.4.

Thus, in order to show that the assignment

$$\mathrm{Ob}(\mathcal{C}_1(X,Y)) \times \mathrm{Ob}(\mathcal{C}_1(X',Y')) \to L_1(X \times X', X \times Y' \cup Y \times X')$$

given by

$$(V, V') \mapsto [(V_1 \otimes V_0') \oplus (V_0 \otimes V_1'), (V_0 \otimes V_0') \oplus (V_1 \otimes V_1'); \begin{pmatrix} \tau_1|_{X \times Y' \cup Y \times X'} \\ \pi \end{pmatrix}]$$

determines

(a) a well-defined pairing on $\mathrm{Ob}(\mathcal{C}_1(X,Y)) \times \mathrm{Ob}(\mathcal{C}_1(X',Y'))$ up to our choices of liftings $\widetilde{\sigma}$, $\widetilde{\sigma'}$ and splitting π, and

(b) a pairing

$$L_1(X,Y) \otimes L_1(X',Y') \to L_1(X \times X', X \times Y' \cup Y \times X')$$

that coincides with the pairing in Proposition 10.4 of [ABS64],

we need only prove:

Lemma 3.2.10. *Let (X,Y) be a pair as above, and let $[V_2, V_1, V_0; \sigma_2, \sigma_1] \in L_2(X,Y)$. If π is a splitting of σ_2,*

$$j_2([V_1, V_0 \oplus V_2; \begin{pmatrix} \sigma_1 \\ \pi \end{pmatrix}]) = [V_2, V_1, V_0; \sigma_2, \sigma_1].$$

Proof. First, suppose $\dim(V_1) > \dim(V_2) + \dim(X)$. Apply Lemma 7.2 in [ABS64] to construct a monomorphism

$$h : V_2 \to V_1$$

that extends σ_2. By the proof of Lemma 7.3 in [ABS64],

$$j_2([\operatorname{coker}(h), V_0; \overline{\sigma_1}]) = [V_2, V_1, V_0; \sigma_2, \sigma_1],$$

and so

$$j_2([\operatorname{coker}(h) \oplus V_2, V_0 \oplus V_2; A]) = [V_2, V_1, V_0; \sigma_2, \sigma_1],$$

where

$$A = \begin{pmatrix} \overline{\sigma_1} & 0 \\ 0 & \mathrm{id}_{V_2}|_Y \end{pmatrix}.$$

Hence, it suffices to show

$$[\operatorname{coker}(h) \oplus V_2, V_0 \oplus V_2; A] = [V_1, V_0 \oplus V_2; \begin{pmatrix} \sigma_1 \\ \pi \end{pmatrix}]$$

Choose a splitting s of h, and let

$$p : V_1 \to \operatorname{coker}(h)$$

denote the canonical surjection. Then we have an isomorphism

$$\begin{pmatrix} p \\ s \end{pmatrix} : V_1 \to \operatorname{coker}(h) \oplus V_2.$$

Since $s|_Y$ is a splitting of σ_2, we also have an isomorphism

$$\begin{pmatrix} \sigma_1 \\ s|_Y \end{pmatrix} : V_1|_Y \to V_0|_Y \oplus V_2|_Y.$$

We have a commutative square

$$\begin{tikzcd}
V_1|_Y \arrow[r, "\begin{pmatrix} \sigma_1 \\ s|_Y \end{pmatrix}"] \arrow[d, "\begin{pmatrix} p|_Y \\ s|_Y \end{pmatrix}"] & V_0|_Y \oplus V_2|_Y \arrow[d, "\mathrm{id}_{V_0|_Y \oplus V_2|_Y}"] \\
\operatorname{coker}(h)|_Y \oplus V_2|_Y \arrow[r, "A"] & V_0|_Y \oplus V_2|_Y
\end{tikzcd}$$

Thus,

$$[\mathrm{coker}(h) \oplus V_2, V_0 \oplus V_2; A] = [V_1, V_0 \oplus V_2; \begin{pmatrix} \sigma_1 \\ s|_Y \end{pmatrix}].$$

Notice that we have an object

$$[V_1 \times I, (V_0 \oplus V_2) \times I; t \begin{pmatrix} \sigma_1 \\ s|_Y \end{pmatrix} + (1-t) \begin{pmatrix} \sigma_1 \\ \pi \end{pmatrix}]$$

in $\mathcal{C}_1(X \times I, Y \times I)$ whose restrictions to $X \times \{0\}$ and $X \times \{1\}$ are $[V_1, V_0 \oplus V_2; \begin{pmatrix} \sigma_1 \\ \pi \end{pmatrix}]$ and $[V_1, V_0 \oplus V_2; \begin{pmatrix} \sigma_1 \\ s|_Y \end{pmatrix}]$, respectively. It now follows from Proposition 9.2 in [ABS64] that

$$[V_1, V_0 \oplus V_2; \begin{pmatrix} \sigma_1 \\ s|_Y \end{pmatrix}] = [V_1, V_0 \oplus V_2; \begin{pmatrix} \sigma_1 \\ \pi \end{pmatrix}].$$

This finishes the case where $\dim(V_1) > \dim(V_2) + \dim(X)$.

For the general case, choose a bundle E such that

$$\dim(E) + \dim(V_1) > \dim(V_2) + \dim(X).$$

Define

$$U := [V_2, V_1 \oplus E, V_0 \oplus E; \begin{pmatrix} \sigma_2 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_1 & 0 \\ 0 & \mathrm{id}_{V_1}|_Y \end{pmatrix}],$$

$$U' := [V_1 \oplus E, V_0 \oplus E \oplus V_2; \begin{pmatrix} \sigma_1 & 0 \\ 0 & \mathrm{id}_{V_1}|_Y \\ \pi & 0 \end{pmatrix}]$$

Notice that

$$[V_2, V_1, V_0; \sigma_2, \sigma_1] = U,$$

and

$$[V_1, V_0 \oplus V_2; \begin{pmatrix} \sigma_1 \\ \pi \end{pmatrix}] = U',$$

so that it suffices to show that $j(U') = U$. Since $\begin{pmatrix} \pi & 0 \end{pmatrix}$ is a splitting of $\begin{pmatrix} \sigma_2 \\ 0 \end{pmatrix}$, this follows from the case we have already considered. □

Let $[V], [V']$ denote the classes of V, V' in $L_1(X,Y), L_1(X',Y')$, and define

$$[V] \otimes_{L_1} [V'] := [(V_1 \otimes V_0') \oplus (V_0 \otimes V_1'), (V_0 \otimes V_0') \oplus (V_1 \otimes V_1'); \begin{pmatrix} \tau_1|_{X \times X', X \times Y' \cup Y \times X'} \\ \pi \end{pmatrix}]$$

Remark 3.2.11. By Proposition 10.4 in [ABS64] and the above remarks,

$$\chi([V]) \otimes \chi([V']) = \chi([V] \otimes_{L_1} [V']).$$

3.3 A generalized Atiyah-Bott-Shapiro construction

We now recall the classical Atiyah-Bott-Shapiro construction ([ABS64] Part III). Following Atiyah-Bott-Shapiro, we work with real Clifford algebras and KO-theory, and we point out that one may perform a similar construction involving complex Clifford algebras and KU-theory.

Define

$$q_n := -x_1^2 - \cdots - x_n^2 \in \mathbb{R}[x_1, \dots, x_n]$$

for all $n \geqslant 1$, and set $C_n := \mathrm{Cliff}_{\mathbb{R}}(q_n)$. We also set $C_0 := \mathbb{R}$.

Let $M(C_n)$ denote the free abelian group generated by isomorphism classes of finitely-generated, indecomposable $\mathbb{Z}/2\mathbb{Z}$-graded modules over C_n. When we say a $\mathbb{Z}/2\mathbb{Z}$-graded module is indecomposable, we mean that if the module is written as a direct sum of two $\mathbb{Z}/2\mathbb{Z}$-graded modules, then one of the two summands must be 0.

There are evident injective maps

$$i_n : C_n \to C_{n+1}$$

for all $n \geqslant 0$; these injections induce homomorphisms

$$i_n^* : M(C_{n+1}) \to M(C_n)$$

via restriction of scalars. Set

$$A_n := M(C_n)/i_n^*(M(C_{n+1})).$$

Define D^n to be the closed disk of radius 1 in $\mathbb{R}^n$. An important special case of the classical Atiyah-Bott-Shapiro construction is the group isomorphism

$$\alpha_n : A_n \xrightarrow{\cong} L_1(D^n, \partial D^n)$$

that appears in [ABS64] Theorem 11.5.

α_n is defined as follows: let $M = M_1 \oplus M_0$ be a finitely generated $\mathbb{Z}/2\mathbb{Z}$-graded C_n-module. We use the $\mathbb{R}$-vector spaces M_1 and M_0 to construct real vector bundles over D^n:

$$V_1 := D^n \times M_1$$

$$V_0 := D^n \times M_0$$

and we define a map

$$\sigma : V_1 \to V_0$$

given by $(x, m) \mapsto (x, x \cdot m)$, where $\cdot$ denotes the action of C_n on M. Here, we are thinking of $D^n \subseteq \mathbb{R}^n$ as a subset of C_n. Notice that σ restricts to an isomorphism of bundles over ∂D^n. Thus, we have constructed an element $[V_1, V_0; \sigma] \in L_1(D^n, \partial D^n)$.

Define

$$\alpha_n([M]) = [V_1, V_0; \sigma].$$

We refer the reader to [ABS64] for verification that the mapping

$$[M] \mapsto [V_1, V_0; \sigma]$$

is well-defined on the quotient A_n and determines an isomorphism.

Now, let $f \in (x_1, \ldots, x_n) \subseteq Q := \mathbb{R}[x_1, \ldots, x_n]$. Choose real numbers ϵ, δ, and t such that $\epsilon > 0$, $0 < \delta << \epsilon$, and $t \in (-\delta, 0)$ in such a way that we may construct a negative Milnor fiber F_f^- associated to f as in Section 3.1.3.

Denote by B_ϵ the closed ball of radius ϵ in $\mathbb{R}^n$ centered at the origin. We now construct a map

$$\mathrm{Ob}(\mathrm{MF}(Q, f)) \to L_1(B_\epsilon, F_f^-)$$

that

(a) recovers the Atiyah-Bott-Shapiro construction via the Buchweitz-Eisenbud-Herzog equivalence (Theorem 2.4.1) when $f = q_n$, and

(b) descends to a group homomorphism

$$K_0[\mathrm{MF}(Q, f)] \to L_1(B_\epsilon, F_f^-).$$

We emphasize that a similar construction involving complex polynomials and their Milnor fibers may be performed *mutatis mutandis.* One may also perform the following construction using the positive Milnor fiber F_f^+ of f.

Let

$$P = (P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0)$$

be a matrix factorization of f over Q. Denote by $C(B_\epsilon)$ the ring of $\mathbb{R}$-valued continuous functions on B_ϵ.

Applying extension of scalars along the inclusion

$$Q \hookrightarrow C(B_\epsilon),$$

we obtain a map

$$P_1 \otimes_Q C(B_\epsilon) \xrightarrow{d_1 \otimes \mathrm{id}} P_0 \otimes_Q C(B_\epsilon)$$

of finitely generated projective $C(B_\epsilon)$-modules.

The category of real vector bundles over B_ϵ is equivalent to the category of finitely generated projective $C(B_\epsilon)$-modules; on objects, the equivalence sends a bundle to its space of sections. Let

$$V_1 \xrightarrow{d_1} V_0$$

be a map of real vector bundles over B_ϵ corresponding to the above map $d_1 \otimes \mathrm{id}$ under this equivalence.

Notice that $d_1|_{F_f^-}$ is an isomorphism; its inverse is the restriction to F_f^- of the map $d_0 : V_0 \to V_1$ determined by

$$P_0 \otimes_Q C(B_\epsilon) \xrightarrow{\frac{1}{t}(d_0 \otimes \mathrm{id})} P_1 \otimes_Q C(B_\epsilon).$$

Define $\Phi_f(P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0) = (V_1, V_0; d_1|_{F_f^-}) \in \mathrm{Ob}(\mathcal{C}_1(B_\epsilon, F_f^-))$.

Remark 3.3.1. The map analogous to Φ_f in the setting of polynomials over $\mathbb{C}$ and KU-theory appears in [BVS12]; we discuss this in detail in the proof of Proposition 4.3.2.

A morphism in $\mathrm{EMF}(Q, f)$ determines a morphism in $\mathcal{C}_1(B_\epsilon, F_f^-)$ in an obvious way (see Section 2.2.1 for the definition of the category $\mathrm{EMF}(Q, f)$). Hence, we have shown:

Proposition 3.3.2. *There is an additive functor*

$$\Phi_f : \mathrm{EMF}(Q, f) \to \mathcal{C}_1(B_\epsilon, F_f^-)$$

given, on objects, by

$$(P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0) \mapsto [V_1, V_0; d_1|_{F_f^-}].$$

In particular, we have a map

$$\mathrm{Ob}(\mathrm{MF}(Q, f)) \to L_1(B_\epsilon, F_f^-).$$

Suppose $f = q_n$. Then ϵ can be chosen to be 1 in the construction of the negative Milnor fiber F_f^-, and the fiber can be chosen to be exactly $S^{n-1} \subseteq \mathbb{R}^n$.

Let $[\mathrm{Ob}([\mathrm{MF}(Q, f)])]$ denote the set of isomorphism classes of objects in $[\mathrm{MF}(Q, f)]$. It is easy to check that one has a commutative triangle

$$\begin{tikzcd}[\mathrm{Ob}([\mathrm{MF}(Q,f)])] \arrow[rr, "\Phi_f"] \arrow[d, "\mathrm{BEH}"] & & L_1(B_1, F_f^-) \\ [\mathrm{Ob}(\mathrm{mod}_{\mathbb{Z}/2\mathbb{Z}}(\mathrm{Cliff}_{\mathbb{R}}(q_n)))] \arrow[urr, "\mathrm{ABS}"] & & \end{tikzcd}$$

where BEH denotes the bijection induced by the Buchweitz-Eisenbud-Herzog equivalence (discussed in detail in Section 2.4), and ABS denotes the Atiyah-Bott-Shapiro construction. Hence, our construction recovers the Atiyah-Bott-Shapiro construction via the Buchweitz-Eisenbud-Herzog equivalence when $f = q_n$.

Our next goal is to show that Φ_f induces a map on K-theory:

Proposition 3.3.3. Φ_f *induces a group homomorphism*

$$\phi_f : K_0[\mathrm{MF}(Q, f)] \to L_1(B_\epsilon, F_f^-).$$

We will adopt the following notational conventions for the purposes of this proof:

(1) A pair (ϵ, t) is a *good pair* if $\epsilon > 0$, $t < 0$, and the map

$$\psi : B_\epsilon \cap f^{-1}((-\delta, 0) \cup (0, \delta)) \to (-\delta, 0) \cup (0, \delta)$$

from Section 3.1 is a locally trivial fibration for some $\delta > 0$ such that

$$0 < |t| < \delta << \epsilon.$$

(2) If (ϵ, t) is a good pair, we denote the negative Milnor fiber $B_\epsilon \cap f^{-1}(t)$ by F_t^-.

We will need the following technical lemma:

Lemma 3.3.4. *Let $(\epsilon_1, t_1), (\epsilon_2, t_2)$ be good pairs. Then there is an isomorphism*

$$g : L_1(B_{\epsilon_1}, F^-_{t_1}) \xrightarrow{\cong} L_1(B_{\epsilon_2}, F^-_{t_2})$$

yielding a commutative triangle

$$\begin{tikzcd}
L_1(B_{\epsilon_1}, F^-_{t_1}) \arrow[r, "g"] & L_1(B_{\epsilon_2}, F^-_{t_2}) \\
\mathrm{Ob}(\mathrm{MF}(Q, f)) \arrow[u, "\Phi_f"] \arrow[ur, "\Phi_f"] &
\end{tikzcd}$$

Proof. The case where $t_1 = t_2$ is immediate, so we may assume $t_1 \neq t_2$. First, suppose $\epsilon_1 = \epsilon_2$. Without loss, assume $t_2 < t_1$.

Set $F^-_{[t_2,t_1]} := f^{-1}([t_2, t_1])$. Since the inclusions

$$F^-_{t_1} \hookrightarrow F^-_{[t_2,t_1]}$$

$$F^-_{t_2} \hookrightarrow F^-_{[t_2,t_1]}$$

are homotopy equivalences, the pullback maps

$$L_1(B_{\epsilon_1}, F_{[t_2,t_1]}) \to L_1(B_{\epsilon_1}, F_{t_1})$$

$$L_1(B_{\epsilon_1}, F_{[t_2,t_1]}) \to L_1(B_{\epsilon_1}, F_{t_2})$$

are isomorphisms.

We have commuting triangles

$$\begin{tikzcd} L_1(B_{\epsilon_1}, F_{[t_2,t_1]}) \arrow[r, "\cong"] & L_1(B_{\epsilon_1}, F_{t_i}^-) \\ \mathrm{Ob}(\mathrm{MF}(Q,f)) \arrow[u, "\Phi_f"] \arrow[ur, "\Phi_f"] & \end{tikzcd}$$

for $i = 1, 2$. It follows that the result holds when $\epsilon_1 = \epsilon_2$.

For the general case, assume, without loss, that $|t_2| < |t_1|$. Then (ϵ_1, t_2) is also a good pair. By the cases we've already considered, the result holds for the pairs (ϵ_1, t_1) and (ϵ_1, t_2), and also for the pairs (ϵ_1, t_2) and (ϵ_2, t_2). Hence, the result holds for the pairs (ϵ_1, t_1), (ϵ_2, t_2). $\square$

We now prove Proposition 3.3.3:

Proof. It is not hard to see that $\Phi_f(P \oplus P') = \Phi_f(P) \oplus \Phi_f(P')$; we need only show that ϕ_f is well-defined. First, suppose $P \cong 0$ in $[\mathrm{MF}(Q,f)]$. Then id_P is a boundary in $\mathrm{MF}(Q,f)$, and so id_P factors through a trivial matrix factorization, by Proposition 2.2.4.

Write

$$P = (P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0).$$

Since P is a summand of a trivial matrix factorization, $\mathrm{coker}(d_1)$ is a projective $Q/(f)$ module. Choose $g \in Q$ such that $g(0) \neq 0$ and $\mathrm{coker}(d_1)_g$ is free over $Q_g/(f)$, and choose $\epsilon' \in (0, \epsilon)$ such that $B_{\epsilon'} \cap g^{-1}(0) = \emptyset$.

The inclusion

$$Q \hookrightarrow Q_g$$

induces a functor

$$\mathrm{MF}(Q,f) \to \mathrm{MF}(Q_g, f).$$

Choose t' such that (ϵ', t') is a good pair. Applying Lemma 3.3.4, we have a commutative diagram

$$\begin{tikzcd}
L_1(B_\epsilon, F_t^-) \arrow[r, "\cong"] & L_1(B_{\epsilon'}, F_{t'}^-) \\
\mathrm{Ob}(\mathrm{MF}(Q,f)) \arrow[u, "\Phi_f"] \arrow[ur, "\Phi_f"] \arrow[r] & \mathrm{Ob}(\mathrm{MF}(Q_g,f)) \arrow[u, "\Phi_f"']
\end{tikzcd}$$

By Proposition 2.2.5, the image of P in $\mathrm{Ob}(\mathrm{MF}(Q_g, f))$ maps to 0 via Φ_f. Hence, the map $\Phi_f : \mathrm{Ob}(\mathrm{MF}(Q, f)) \to L_1(B_\epsilon, F_t^-)$ sends P to 0, as required.

We now show that, if $\alpha : P \to P'$ is a morphism in $\mathrm{EMF}(Q, f)$, $\Phi_f(P) \oplus \Phi_f(\mathrm{cone}(\alpha))$ and $\Phi_f(P')$ represent the same class in $L_1(B_\epsilon, F_t^-)$.

We start by showing $\Phi_f(P[1]) = -\Phi_f(P)$ in $L_1(B_\epsilon, F_t^-)$. Write $\Phi_f(P) = (V_1, V_0; d_1|_{F_f^-})$, so that $\Phi_f(P[1]) = (V_0, V_1; -d_0|_{F_f^-})$. Since $\mathrm{cone}(\mathrm{id}_P)$ is contractible, the class represented by

$$\Phi_f(\mathrm{cone}(\mathrm{id}_P)) = (V_0 \oplus V_1, V_1 \oplus V_0; \begin{pmatrix} d_0|_{F_f^-} & \mathrm{id} \\ 0 & -d_1|_{F_f^-} \end{pmatrix})$$

in $L_1(B_\epsilon, F_t^-)$ is 0.

The object

$$(V_0 \oplus V_1, V_1 \oplus V_0; \begin{pmatrix} d_0|_{F_f^-} & t \cdot \mathrm{id} \\ 0 & -d_1|_{F_f^-} \end{pmatrix})$$

of $\mathcal{C}_1(B_\epsilon \times I, F_t^- \times I)$ restricts to $\Phi_f(\mathrm{cone}(\mathrm{id}_P))$ at $t = 1$ and $\Phi_f((P \oplus P[1])[1])$ at $t = 0$. Since $(P \oplus P[1])[1] \cong P \oplus P[1]$, we may use Proposition 9.2 in [ABS64] to conclude that $\Phi_f(P[1]) = -\Phi_f(P)$ in $L_1(B_\epsilon, F_t^-)$.

Now, we have

$$\Phi_f(\mathrm{cone}(\alpha)) = (V_0 \oplus V_1, V_1 \oplus V_0; \begin{pmatrix} d_0|_{F_f^-} & \alpha_1 \\ 0 & -d_1'|_{F_f^-} \end{pmatrix}).$$

Using Proposition 9.2 in [ABS64] in the same manner as above, we may conclude that $\Phi_f(\mathrm{cone}(\alpha))$ and $\Phi_f(P') \oplus \Phi_f(P[1])$ represent the same class in $L_1(B_\epsilon, F_t^-)$.

Finally, suppose $\alpha : P \cong P'$ is an isomorphism in $[\mathrm{MF}(Q, f)]$. Then $\mathrm{cone}(\alpha)$ is contractible, and so the results we just established imply that $\Phi_f(P) = \Phi_f(P')$. Thus, Φ_f preserves isomorphisms in $[\mathrm{MF}(Q, f)]$. Since every distinguished triangle in $[\mathrm{MF}(Q, f)]$ is isomorphic to one of the form

$$P \xrightarrow{\alpha} P' \to \mathrm{cone}(\alpha) \to P[1],$$

and we have shown that Φ_f preserves such triangles, we are done. □

We now use our construction ϕ_f to exhibit a compatibility between Knörrer periodicity (Theorem 2.6.1) and Bott periodicity; we study the map ϕ_f more closely in the case where f is an ADE singularity in Chapter 4.

3.4 Knörrer periodicity and Bott periodicity

In this section, we work with polynomials and vector bundles over $\mathbb{C}$. The author fully expects results analogous to those in this section to hold for polynomials and vector bundles over $\mathbb{R}$; we leave the details for future work.

Set

$$Q := \mathbb{C}[x_1, \dots, x_n],\ Q' := \mathbb{C}[y_1, \dots, y_m]$$

and let

$$f \in (x_0, \dots, x_n) \subseteq Q,\ f' \in (y_0, \dots, y_m) \subseteq Q'$$

be such that $Q_{(x_1,\dots,x_n)}/(f), Q'_{(y_1,\dots,y_m)}/(f')$ are IHS.

We now construct the Milnor fibers of f and f'. Choose real numbers ϵ'', δ'', such that the map

$$B_{\epsilon''} \cap (f \oplus f')^{-1}(D^*_{\delta''}) \to D^*_{\delta''}$$

given by $x \mapsto (f \oplus f')(x)$ is a locally trivial fibration.

Similarly, choose ϵ, δ and ϵ', δ', as well as $t'' \in D^*_{\delta''}$, so that the analogous maps

$$B_{\epsilon} \cap f^{-1}(D^*_{\delta}) \to D^*_{\delta}$$

$$B_{\epsilon'} \cap (f')^{-1}(D^*_{\delta'}) \to D^*_{\delta'}$$

are locally trivial fibrations, and also so that

(a) ϵ, ϵ' are sufficiently small so that $B_{\epsilon} \times B_{\epsilon'} \subseteq B_{\epsilon''}$.

(b) $|t''| < \min\{\delta, \delta'\}$.

Set F_f, $F_{f'}$, and $F_{f \oplus f'}$ to be the Milnor fibers of f, f', and $f \oplus f'$ over t''.

Recall from Proposition 2.3.2 that we have a map

$$K_0[\mathrm{MF}(Q, f)] \otimes K_0[\mathrm{MF}(Q', f')] \to K_0[\mathrm{MF}(Q \otimes_{\mathbb{C}} Q', f \oplus f')]$$

given by

$$[P] \otimes [P'] \mapsto [P \otimes_{\mathrm{MF}} P'].$$

The following proposition is the key technical result in this section.

Proposition 3.4.1. *There exists a map*

$$\mathrm{ST}_{L_1} : L_1(B_\epsilon, F_f) \otimes L_1(B_{\epsilon'}, F_{f'}) \to L_1(B_{\epsilon''}, F_{f\oplus f'})$$

such that, given matrix factorizations P and P' of f and f', respectively,

$$\mathrm{ST}_{L_1}(\phi_f([P]) \otimes \phi_{f'}([P'])) = \phi_{f\oplus f'}([P \otimes_{\mathrm{MF}} P']).$$

Proof. Write

$$P = (P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0),\ P' = (P_1' \underset{d_0'}{\overset{d_1'}{\rightleftarrows}} P_0')$$

and

$$\Phi_f(P) = [V_1, V_0; d_1|_{F_f}],\ \Phi_{f'}(P') = [V_1', V_0'; d_1'|_{F_{f'}}].$$

We note that

$$\phi_{f\oplus f'}([P \otimes_{\mathrm{MF}} P']) = [(V_1 \otimes V_0') \oplus (V_0 \otimes V_1'), (V_0 \otimes V_0') \oplus (V_1 \otimes V_1'); A],$$

where A is the restriction of the matrix

$$\begin{pmatrix} d_1 \otimes \mathrm{id} & \mathrm{id} \otimes d_1' \\ -\mathrm{id} \otimes d_0' & d_0 \otimes \mathrm{id} \end{pmatrix}$$

to $F_{f\oplus f'}$.

As in Section 3.1.2, choose an injection

$$H : CF_f \to B_\epsilon$$

such that

- $H(x, 1) = x \in F_f \subseteq B_\epsilon$,
- $H(-, s) : F_f \to B_\epsilon$ maps into the Milnor fiber $B_\epsilon \cap f^{-1}(st'')$ for $s \in (0, 1)$, and
- $H(x, 0) = 0$ for all $x \in F_f$

Choose $H' : CF_{f'} \to B_{\epsilon'}$ similarly.

Clearly $\operatorname{im}(H)$ is contractible, since it is homeomorphic to CF_f (and of course the same is true for $\operatorname{im}(H')$). It follows that the inclusions of pairs

$$g : (\operatorname{im}(H), F_f) \hookrightarrow (B_\epsilon, F_f)$$

$$g' : (\operatorname{im}(H'), F_{f'}) \hookrightarrow (B_{\epsilon'}, F_{f'})$$

induce isomorphisms on L_1 upon pullback; this is immediate from the long exact sequence in topological K-theory and the naturality of the Euler characteristic from Section 3.2 with respect to maps of pairs.

Recall from Section 3.2 that we have a map

$$L_1(\operatorname{im}(H), F_f) \otimes L_1(\operatorname{im}(H'), F_{f'}) \to L_1(\operatorname{im}(H) \times \operatorname{im}(H'), \operatorname{im}(H) \times F_{f'} \cup F_f \times \operatorname{im}(H'))$$

denoted by

$$[V] \otimes [V'] \mapsto [V] \otimes_{L_1} [V'].$$

Define

$$\mathrm{ST}_{L_1} : L_1(B_\epsilon, F_f) \otimes L_1(B_{\epsilon'}, F_{f'}) \to L_1(B_{\epsilon''}, F_{f \oplus f'})$$

to be given by

$$[V] \otimes [V'] \mapsto (G^*)^{-1}(g^*([V]) \otimes_{L_1} (g')^*([V'])),$$

where G is the homotopy equivalence of pairs in Remark 3.1.12. We now compute

$$g^*(\phi_f(P)) \otimes_{L_1} (g')^*(\phi_f(P')) = g^*([V_1, V_0; d|_{F_f}]) \otimes_{L_1} (g')^*([V_1', V_0'; d'|_{F_{f'}}])$$

explicitly.

There are obvious liftings of $d_1|_{F_f}$ and $d_1'|_{F_{f'}}$ to maps of bundles over $\mathrm{im}(H)$ and $\mathrm{im}(H')$, namely $d_1|_{\mathrm{im}(H)}$ and $d_1'|_{\mathrm{im}(H')}$. A splitting of the restriction of

$$\begin{pmatrix} -\mathrm{id} \otimes d_1'|_{\mathrm{im}(H')} \\ d_1|_{\mathrm{im}(H)} \otimes \mathrm{id} \end{pmatrix}$$

to $\mathrm{im}(H) \times F_{f'} \cup F_f \times \mathrm{im}(H')$ is given, on the fiber over $(H(x,s), H'(y,s'))$, by

$$\frac{1}{f(H(x,s)) + f'(H'(y,s'))} \begin{pmatrix} -\mathrm{id} \otimes d_0'|_{\mathrm{im}(H')} & d_0|_{\mathrm{im}(H)} \otimes \mathrm{id} \end{pmatrix}$$

(notice that $f(H(x,s)) + f'(H'(y,s')) = (s+s')t'' \neq 0$ when $(H(x,s), H'(y,s')) \in \mathrm{im}(H) \times F_{f'} \cup F_f \times \mathrm{im}(H')$, since either s or s' is equal to 1).

Thus, by the discussion at the end of Section 3.2, the product

$$g^*([V_1, V_0; d|_{F_f}]) \otimes_{L_1} (g')^*([V_1', V_0'; d'|_{F_{f'}}])$$

is equal to

$$[(V_1|_{\mathrm{im}(H)} \otimes V_0'|_{\mathrm{im}(H')}) \oplus (V_0|_{\mathrm{im}(H)} \otimes V_1'|_{\mathrm{im}(H')}), (V_0|_{\mathrm{im}(H)} \otimes V_0'|_{\mathrm{im}(H')}) \oplus (V_1|_{\mathrm{im}(H)} \otimes V_1'|_{\mathrm{im}(H')}); B],$$

where B is given, on the fiber over $(H(x,s), H'(y,s')) \in \mathrm{im}(H) \times F_{f'} \cup F_f \times \mathrm{im}(H')$, by the matrix

$$\begin{pmatrix} d_1|_{\mathrm{im}(H)} \otimes \mathrm{id} & \mathrm{id} \otimes d_1'|_{\mathrm{im}(H')} \\ \frac{1}{f(H(x,s))+f'(H'(y,s'))}(-\mathrm{id} \otimes d_0'|_{\mathrm{im}(H')}) & \frac{1}{f(H(x,s))+f'(H'(y,s'))}(d_0|_{\mathrm{im}(H)} \otimes \mathrm{id}) \end{pmatrix}$$

restricted to $\mathrm{im}(H) \times F_{f'} \cup F_f \times \mathrm{im}(H')$.

We wish to show that, upon applying $(G^*)^{-1}$ to this class, one obtains

$$[(V_1 \otimes V_0') \oplus (V_0 \otimes V_1'), (V_0 \otimes V_0') \oplus (V_1 \otimes V_1'); C],$$

where C is the restriction of the matrix

$$\begin{pmatrix} d_1 \otimes \mathrm{id} & \mathrm{id} \otimes d_1' \\ \frac{1}{t''}(-\mathrm{id} \otimes d_0') & \frac{1}{t''}(d_0 \otimes \mathrm{id}) \end{pmatrix}$$

to $F_{f \oplus f'}$. This will finish the proof, since the class

$$[(V_1 \otimes V_0') \oplus (V_0 \otimes V_1'), (V_0 \otimes V_0') \oplus (V_1 \otimes V_1'); C]$$

is clearly equal to

$$[(V_1 \otimes V_0') \oplus (V_0 \otimes V_1'), (V_0 \otimes V_0') \oplus (V_1 \otimes V_1'); A],$$

where A is as above.

Observe that we have an object

$$[((V_1 \otimes V_0') \oplus (V_0 \otimes V_1')) \times I, ((V_0 \otimes V_0') \oplus (V_1 \otimes V_1')) \times I; D]$$

in $\mathcal{C}_1(\mathrm{im}(H) \times \mathrm{im}(H') \times I, (\mathrm{im}(H) \times F_{f'} \cup F_f \times \mathrm{im}(H')) \times I)$, where D is given, on the fiber over

$$(H(x,s), H'(y,s'), T) \in (\mathrm{im}(H) \times F_{f'} \cup F_f \times \mathrm{im}(H')) \times I,$$

by the matrix

$$\begin{pmatrix} d_1 \otimes \mathrm{id} & \mathrm{id} \otimes d_1' \\ \frac{1}{f(a(T))+f'(b(T))}(-\mathrm{id} \otimes d_0') & \frac{1}{f(a(T))+f'(b(T))}(d_0 \otimes \mathrm{id}) \end{pmatrix}.$$

Here, f, f', and the entries of d_1, d_1', d_0, d_0' are evaluated at the point

$$(a(T), b(T)) := (H(x, \frac{T(1-s'-s)+2s}{2}), H'(y, \frac{T(1-s'-s)+2s'}{2})).$$

Notice that $f(a(T)) + f'(b(T)) \neq 0$ for all

$$(H(x,s), H'(y,s'), T) \in (\mathrm{im}(H) \times F_{f'} \cup F_f \times \mathrm{im}(H')) \times I,$$

so this matrix is indeed an isomorphism on every fiber over $(\mathrm{im}(H) \times F_{f'} \cup F_f \times \mathrm{im}(H')) \times I$.

Restricting to $T = 0$, one obtains the object

$$((V_1|_{\mathrm{im}(H)} \otimes V_0'|_{\mathrm{im}(H')}) \oplus (V_0|_{\mathrm{im}(H)} \otimes V_1'|_{\mathrm{im}(H')}), (V_0|_{\mathrm{im}(H)} \otimes V_0'|_{\mathrm{im}(H')}) \oplus (V_1|_{\mathrm{im}(H)} \otimes V_1'|_{\mathrm{im}(H')}); B).$$

Restricting to $T = 1$ and applying G^{*-1}, one obtains

$$((V_1 \otimes V_0') \oplus (V_0 \otimes V_1'), (V_0 \otimes V_0') \oplus (V_1 \otimes V_1'); C).$$

Now apply Proposition 9.2 in [ABS64]. $\square$

Remark 3.4.2. It follows easily from the naturality of the Euler characteristic χ from Section 3.2 and Remark 3.2.11 that ST_{L_1} induces a map

$$\mathrm{ST}_{KU} : KU^0(B_\epsilon, F_f) \otimes KU^0(B_{\epsilon'}, F_{f'}) \to KU^0(B_{\epsilon''}, F_{f \oplus f'}).$$

Remark 3.4.3. We point out that the group homomorphism ST_{L_1} in Proposition 3.4.1 is given by the composition of the tensor product in topological K-theory with a specific formulation of the the Sebastiani-Thom homotopy equivalence. Hence, Proposition 3.4.1 yields a precise sense in which the tensor product of matrix factorizations is related to the Sebastiani-Thom homotopy equivalence (cf. Remark 2.3.3).

Let us now consider the case where $Q' = \mathbb{C}[u,v]$ and $f = u^2 + v^2$. Note that $K_0[\mathrm{MF}(\mathbb{C}[u,v], u^2+v^2)] \cong \mathbb{Z}$ (Remark 2.5.2 and Remark 2.4.2); it is generated by the class

$$X = [\mathbb{C}[u,v] \underset{u-iv}{\overset{u+iv}{\rightleftarrows}} \mathbb{C}[u,v]].$$

Also, by Theorem 3.1.4, $F_{u^2+v^2}$ is homotopy equivalent to S^1, and so $L_1(B_{\epsilon'}, F_{u^2+v^2})$ is isomorphic to $\mathbb{Z}$. This group is generated by $\phi_{u^2+v^2}(X)$; a way to see this is to apply Theorem 11.5 in [ABS64] and observe the compatibility between the Atiyah-Bott-Shapiro construction and the map ϕ_f via the Buchweitz-Eisenbud-Herzog functor, as discussed in the previous section. Thus, $\phi_{u^2+v^2}(X)$ is a Bott element in the group $L_1(B_{\epsilon'}, F_{u^2+v^2}) \cong \widetilde{KU}^0(S^2)$; we shall denote by β the map

$$KU^0(B_\epsilon, F_f) \to KU^0(B_\epsilon, F_f) \otimes KU^0(B_{\epsilon'}, F_{u^2+v^2})$$

given by $(\chi \otimes \chi) \circ (- \otimes \phi_{u^2+v^2}(X)) \circ \chi^{-1}$. β is the Bott periodicity isomorphism.

Since Knörrer periodicity is induced by tensoring with the matrix factorization

$\mathbb{C}[u,v] \underset{u-iv}{\overset{u+iv}{\leftrightarrows}} \mathbb{C}[u,v]$, we will denote by K the map

$$K_0[\mathrm{MF}(Q,f)] \to K_0[\mathrm{MF}(Q[u,v], f \oplus u^2+v^2)]$$

given by $- \otimes_{\mathrm{MF}} X$.

The following result gives a precise sense in which Bott periodicity and Knörrer periodicity are compatible; it follows immediately from Proposition 3.4.1:

Theorem 3.4.4. *Let $f \in (x_1,\ldots,x_n) \subseteq \mathbb{C}[x_1,\ldots,x_n]$, and suppose the hypersurface $\mathbb{C}[x_1,\ldots,x_n]_{(x_1,\ldots,x_n)}/(f)$ is* IHS *(see Definition 2.2.10). Then the diagram*

$$\begin{tikzcd}
K_0[\mathrm{MF}(Q,f)] \arrow[rr, "\chi \circ \phi_f"] \arrow[dd, "K"] & & KU^0(B_\epsilon, F_f) \arrow[d, "\beta"] \\
& & KU^0(B_\epsilon, F_f) \otimes KU^0(B_{\epsilon'}, F_{u^2+v^2}) \arrow[d, "\mathrm{ST}_{KU}"] \\
K_0[\mathrm{MF}(Q[u,v], f \oplus u^2+v^2)] \arrow[rr, "\chi \circ \phi_{f \oplus u^2+v^2}"] & & KU^0(B_{\epsilon''}, F_{f \oplus u^2+v^2})
\end{tikzcd}$$

commutes.

Chapter 4

Examples: the ADE singularities

In Section 3.3, we constructed a map ϕ_f from the Grothendieck group of the homotopy category of matrix factorizations associated to a complex (real) polynomial f into the topological K-theory of its Milnor fiber (positive or negative Milnor fiber). We established that, when f is a non-degenerate quadratic, this map recovers the Atiyah-Bott-Shapiro construction.

In this chapter, we examine some properties of the map ϕ_f when $f \in \mathbb{C}[x_1, \ldots, x_n]$ is an ADE singularity.

4.1 Maximal Cohen-Macaulay modules over the ADE singularities

Let $f \in (x_1, \ldots, x_n) \subseteq \mathbb{C}[x_1, \ldots, x_n]$, and assume the hypersurface

$$\mathbb{C}[x_1, \ldots, x_n]_{(x_1, \ldots, x_n)}/(f)$$

is IHS (Definition 2.2.10).

Choose $\epsilon, \delta > 0$ so that the map

$$B_\epsilon \cap f^{-1}(D_\delta^*) \to D_\delta^*$$

given by $x \mapsto f(x)$ is a locally trivial fibration, as in Section 3.1. Let F_f denote the fiber of this fibration, the Milnor fiber of f.

Recall that F_f is homotopy equivalent to a wedge sum of μ copies of S^{n-1}, where μ is the Milnor number of f (Theorem 3.1.4).

Suppose n is odd. Then

$$KU^0(B_\epsilon, F_f) \cong \widetilde{KU}^0(\Sigma F_f) \cong \bigoplus_\mu \widetilde{KU}^0(S^n) = 0$$

Thus, in this case, the map

$$\phi_f : K_0[\mathrm{MF}(\mathbb{C}[x_1, \ldots, x_n], f)] \to L_1(B_\epsilon, F_f)$$

is the zero map.

For more interesting examples, we look to the *ADE singularities*, or *simple plane curve singularities*:

$$\begin{aligned}
A_k &= x_1^{k+1} + x_2^2, & k \geqslant 1 \\
D_k &= x_1^{k-1} + x_1 x_2^2, & k \geqslant 4 \\
E_6 &= x_1^3 + x_2^4 \\
E_7 &= x_1^3 + x_1 x_2^3 \\
E_8 &= x_1^3 + x_2^5
\end{aligned}$$

It turns out that, if f is a simple plane curve singularity and $n \geqslant 2$, the ring

$$R = \mathbb{C}[[x_1, \dots, x_n]]/(f + x_3^2 + \cdots + x_n^2)$$

has finite MCM type; that is, R has only finitely many indecomposable MCM modules up to isomorphism (this follows from results in [Yos90] Chapters 9 and 11 along with Knörrer periodicity). By a theorem of Buchweitz-Greuel-Schreyer in 1987, the converse is also true:

Theorem 4.1.1 ([BGS87])**.** *If $R = \mathbb{C}[[x_1, \dots, x_n]]/(f)$ has finite MCM type and $n \geqslant 2$, $R \cong \mathbb{C}[[x_1, \dots, x_n]]/(g + x_3^2 + \cdots + x_n^2)$, where $g \in \mathbb{C}[[x_1, x_n]]$ is an ADE singularity.*

Remark 4.1.2. A more general result is stated in Theorem 9.8 of [LW12].

In particular, when $f \in \mathbb{C}[[x, y]]$ is an ADE singularity, $K_0[\mathrm{MF}(\mathbb{C}[[x, y]], f)]$ is a finitely generated abelian group; this makes the ADE singularities a convenient source of examples for studying the properties of the map

$$\phi_f : K_0[\mathrm{MF}(\mathbb{C}[x, y], f)] \to L_1(B_\epsilon, F_f).$$

The results we mentioned above involve ADE singularities thought of as elements of power series rings, not polynomial rings. But, for the purposes of studying homotopy categories of matrix factorizations, this makes no difference:

Proposition 4.1.3. *If $f \in \mathbb{C}[x, y]$ is an ADE singularity, the functor*

$$i : [\mathrm{MF}(\mathbb{C}[x, y], f)] \to [\mathrm{MF}(\mathbb{C}[[x, y]], f)]$$

induced by inclusion is an equivalence.

Proof. Every matrix factorization of f over $\mathbb{C}[[x, y]]$ can be expressed, up to isomorphism in $[\mathrm{MF}(\mathbb{C}[[x, y]], f)]$, as one involving a pair of matrices with polynomial entries ([Yos90] Chapter 9). Thus, i is essentially surjective. One can argue that i is fully faithful in the same manner as in Remark 2.4.2. □

Before going further, we return to a discussion of formalities involving Hochschild homology of dg categories, this time applied to matrix factorization categories.

4.2 Hochschild homology of matrix factorization categories

Set $Q := \mathbb{C}[x_1, \ldots, x_n]$, and let $f \in (x_1, \ldots, x_n) \subseteq Q$ so that $Q_{(x_1,\ldots,x_n)}/(f)$ is IHS. Assume n is even.

Let $\Omega^1_{Q/\mathbb{C}}$ denote the module of Kähler differentials of Q over $\mathbb{C}$. We consider the exterior algebra

$$\bigwedge \Omega^1_{Q/\mathbb{C}},$$

as a $\mathbb{Z}/2\mathbb{Z}$-graded complex of Q-modules with odd (even) degree piece given by the direct sum of the odd (even) exterior powers, equipped with differential given by left exterior multiplication by df.

A computation due to Dyckerhoff in Section 6 of [Dyc11] yields a canonical isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded complexes of $\mathbb{C}$-vector spaces between the above complex and Hochschild complex of the dg category $\mathrm{MF}(Q, f)$, and hence a canonical isomorphism

$$HH_*(\mathrm{MF}(Q, f)) \xrightarrow{\cong} \Omega^n_{Q/\mathbb{C}}/(df \wedge \Omega^n_{Q/\mathbb{C}}) \cong J_f \otimes_Q \Omega^n_{Q/\mathbb{C}},$$

where J_f is the algebra

$$\frac{\mathbb{C}[x_1, \dots, x_n]}{(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})}$$

thought of as a Q-module. In particular, $HH_*(\mathrm{MF}(Q, f))$ is concentrated in even degree.

Let $P = (P_1 \underset{B}{\overset{A}{\rightleftarrows}} P_0)$ be a matrix factorization of f over Q. Choose bases of P_1, P_0, so that we may view A and B as matrices with entries in Q.

By Example 2.30 in [Wal14a], upon applying the above isomorphism

$$HH_*(\mathrm{MF}(Q, f)) \xrightarrow{\cong} \Omega^n_{Q/\mathbb{C}}/(df \wedge \Omega^n_{Q/\mathbb{C}}),$$

the Chern character

$$\mathrm{ch} : \mathrm{Ob}(\mathrm{MF}(Q, f)) \to \Omega^n_{Q/\mathbb{C}}/(df \wedge \Omega^n_{Q/\mathbb{C}})$$

is given by

$$(P_1 \underset{B}{\overset{A}{\rightleftarrows}} P_0) \mapsto \frac{2}{n!}(-1)^{\binom{n}{2}} \operatorname{tr}(\underbrace{dAdB \cdots dAdB}_{\frac{n}{2} \text{ factors of } dAdB}) dx_1 \wedge \cdots \wedge dx_n$$

where dA and dB denote the matrices resulting from applying $d : Q \to \Omega^1_{Q/\mathbb{C}}$ to the entries of A and B.

By Corollary 5.12 in [Yu15], the Chern character map descends, in this setting, to a map

$$\mathrm{ch} : K_0[\mathrm{MF}(Q, f)] \to \Omega^n_{Q/\mathbb{C}}/(df \wedge \Omega^n_{Q/\mathbb{C}}).$$

Example 4.2.1. It will be useful for us to have a formula for the Chern character of

$$\Delta \in \mathrm{Perf}(\mathrm{MF}(Q, f)^{\mathrm{op}} \otimes \mathrm{MF}(Q, f)),$$

where Δ is as in Section 2.1.3.

Applying [PV12] (2.14) and Remark 2.3.6, we have natural isomorphisms

$$HH_*(\mathrm{Perf}(\mathrm{MF}(Q,f)^{\mathrm{op}} \otimes \mathrm{MF}(Q,f))) \cong HH_*(\mathrm{MF}(Q,-f) \otimes \mathrm{MF}(Q,f)).$$

$$\cong HH_*(\mathrm{MF}(Q \otimes_{\mathbb{C}} Q, -f \oplus f)).$$

For $1 \leqslant j \leqslant n$ and $g \in Q$, set $\Delta_j(g) \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ to be the polynomial

$$\frac{g(x_1, \ldots, x_{j-1}, y_j, y_{j+1}, \ldots, y_n) - g(x_1, \ldots, x_{j-1}, x_j, y_{j+1}, \ldots, y_n)}{y_j - x_j}.$$

Let C denote the $n \times n$ matrix over $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ with $C_{ij} = \Delta_j(\frac{\partial f}{\partial x_i})$.

By Proposition 4.1.1 in [PV12], $\mathrm{ch}(\Delta)$ corresponds, via the above isomorphisms, to the class

$$(-1)^{\binom{n}{2}} \cdot \det(C) \cdot dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n \in \Omega^n_{Q/\mathbb{C}}/(df \wedge \Omega^n_{Q/\mathbb{C}})^{\otimes 2}.$$

We now wish to use our formula for the Chern character map, along with Theorem 2.1.20 and several results of [BVS12], to examine the map

$$\phi_f : K_0[\mathrm{MF}(Q,f)] \to KU^0(B_\epsilon, F_f)$$

when f is an ADE singularity.

4.3 An application of the Hirzebruch-Riemann-Roch formula for differential $\mathbb{Z}/2\mathbb{Z}$-graded categories

Let $f = x_1^3 + x_1x_2^2 \in Q = \mathbb{C}[x_1, x_2]$, the D_4 singularity. By results of Chapters 9 and 13 in [Yos90], $K_0[\mathrm{MF}(Q, f)]$ is generated as an abelian group by the classes

$$[Q \underset{x_1^2+x_2^2}{\overset{x_1}{\rightleftarrows}} Q],\ [Q \underset{x_1-ix_2}{\overset{x_1(x_1+ix_2)}{\rightleftarrows}} Q].$$

The Chern characters of these classes are

$$-2x_2dx_1dx_2, (3ix_1 - x_2)dx_1dx_2 \in \Omega^2_{Q/\mathbb{C}}/(df \wedge \Omega^2_{Q/\mathbb{C}})$$

These classes are linearly independent over $\mathbb{C}$. This implies that

(a) $K_0[\mathrm{MF}(Q, f)]$ is a rank 2 free abelian group generated by the two classes above, and

(b) the Chern character map $\mathrm{ch} : K_0[\mathrm{MF}(Q, f)] \to \Omega^2_{Q/\mathbb{C}}/(df \wedge \Omega^2_{Q/\mathbb{C}})$ is injective.

Remark 4.3.1. Using the same argument, it is straightforward to check that, if f is any ADE singularity, $K_0[\mathrm{MF}(Q, f)]$ is free abelian and the Chern character map

$$\mathrm{ch} : K_0[\mathrm{MF}(Q, f)] \to \Omega^2_{Q/\mathbb{C}}/(df \wedge \Omega^2_{Q/\mathbb{C}}).$$

is injective.

We are now prepared to prove:

Proposition 4.3.2. *If* $f = x_1^3 + x_1 x_2^2 \in Q = \mathbb{C}[x_1, x_2]$, *the homomorphism*

$$\phi_f : K_0[\mathrm{MF}(Q, f)] \to L_1(B_\epsilon, F_f) \cong \mathbb{Z}^{\oplus 4}$$

is injective.

Proof. Suppose $\phi_f([P]) = 0$. As alluded to in Remark 3.3.1, the map

$$\Phi_f : \mathrm{Ob}(\mathrm{MF}(Q, f)) \to L_1(B_\epsilon, F_f)$$

in this setting agrees with a map discussed in [BVS12]. More specifically,

$$\Phi_f(E = (E_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} E_0)) = \alpha(\mathrm{coker}(d_1))|_{F_f}$$

for all matrix factorizations E of f, where α is as described on page 252 of [BVS12].

It follows from Proposition 4.1 and Theorem 4.2 in [BVS12], as well as the discussion in Section 2.2.3, that

$$\chi(-, [P]) : K_0[\mathrm{MF}(Q, f)] \to \mathbb{Z}$$

is the zero map. Thus, by the Hirzebruch-Riemann-Roch formula in Theorem 2.1.20,

$$\langle \mathrm{ch}((-)^\vee), \mathrm{ch}([P]) \rangle_{\mathrm{MF}(Q,f)} : K_0[\mathrm{MF}(Q, f)] \to \mathbb{C}$$

is the zero map, where $E^\vee$ is the matrix factorization of $-f$ corresponding to the object $E \in \mathrm{MF}(Q, f)^{\mathrm{op}}$ under the equivalence in [PV12] (2.14). By the discussion on page 11 of [PV12], it follows that

$$\langle \mathrm{ch}([P]), \mathrm{ch}([(-)^\vee]) \rangle_{\mathrm{MF}(Q,-f)} : K_0[\mathrm{MF}(Q,f)] \to \mathbb{C}$$

is the zero map.

Clearly $\Omega^2_{Q/\mathbb{C}}/(d(-f) \wedge \Omega^2_{Q/\mathbb{C}}) = \Omega^2_{Q/\mathbb{C}}/(df \wedge \Omega^2_{Q/\mathbb{C}})$. Also, the images of the Chern character maps

$$\mathrm{ch} : K_0[\mathrm{MF}(Q,-f)] \to \Omega^2_{Q/\mathbb{C}}/(df \wedge \Omega^2_{Q/\mathbb{C}})$$

$$\mathrm{ch} : K_0[\mathrm{MF}(Q,f)] \to \Omega^2_{Q/\mathbb{C}}/(df \wedge \Omega^2_{Q/\mathbb{C}})$$

are identical, since one has an isomorphism

$$K_0[\mathrm{MF}(Q,f)] \xrightarrow{\cong} K_0[\mathrm{MF}(Q,-f)]$$

given by

$$[P_1 \underset{d_0}{\overset{d_1}{\rightleftarrows}} P_0] \mapsto [P_1 \underset{d_0}{\overset{-d_1}{\rightleftarrows}} P_0].$$

By the computations above, the $\mathbb{C}$-span of the image of the Chern character map in $\Omega^2_{Q/\mathbb{C}}/(df \wedge \Omega^2_{Q/\mathbb{C}})$ is precisely the elements of the form $l dx_1 dx_2$, where $l \in Q$ is a homogeneous linear form. Thus, $\langle \mathrm{ch}([P]), l dx_1 dx_2 \rangle_{\mathrm{MF}(Q,-f)} = 0$ for all homogeneous linear forms $l \in Q$.

Let $\Delta \in \mathrm{Ob}(\mathrm{MF}(\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n], -f \oplus f))$ denote the matrix factorization described in Example 4.2.1. An easy computation yields

$$\mathrm{ch}(\Delta) = (-6x_1^2 - 6x_1 y_1 + 2x_2 y_2 + 2y_2^2) \cdot dx_1 \wedge dx_2 \wedge dy_1 \wedge dy_2.$$

By Remark 2.1.21, the map

$$\Omega^2_{Q/\mathbb{C}}/(df \wedge \Omega^2_{Q/\mathbb{C}})^{\otimes 3} \to \Omega^2_{Q/\mathbb{C}}/(df \wedge \Omega^2_{Q/\mathbb{C}})$$

given by

$$h \otimes h' \otimes h'' \mapsto \langle h, h' \rangle_{\mathrm{MF}(Q,-f)} \cdot h''$$

maps $\mathrm{ch}([P]) \otimes \mathrm{ch}(\Delta)$ to $\mathrm{ch}([P])$. That is, denoting $\langle -, - \rangle_{\mathrm{MF}(Q,-f)}$ by just $\langle -, - \rangle$, we have

$$-6\langle \mathrm{ch}([P]), 1 \rangle \cdot x_1^2 - 6\langle \mathrm{ch}([P]), y_1 \rangle \cdot x_1 + 2\langle \mathrm{ch}([P]), y_2 \rangle \cdot x_2 + 2\langle \mathrm{ch}([P]), y_2^2 \rangle \cdot 1$$

$$= -6\langle \mathrm{ch}([P]), 1 \rangle \cdot x_1^2 + 2\langle \mathrm{ch}([P]), y_2^2 \rangle \cdot 1 = \mathrm{ch}([P]).$$

Since x_1^2 and 1 are not homogeneous linear forms, the only way this equality can hold is if $\mathrm{ch}([P]) = 0$. Since

$$\mathrm{ch} : K_0[\mathrm{MF}(Q, f)] \to \Omega^2_{Q/\mathbb{C}}/(df \wedge \Omega^2_{Q/\mathbb{C}})$$

is injective, it follows that $[P] = 0$. □

Remark 4.3.3. The only properties of the polynomial D_4 that we used in the proof were

(1) $\mathbb{C}[x_1, x_2]_{(x_1,x_2)}/D_4$ is IHS.

(2) The map

$$K_0[\mathrm{MF}(\mathbb{C}[x_1, x_2], D_4)] \to K_0[\mathrm{MF}(\mathbb{C}[[x_1, x_2]], D_4)]$$

induced by inclusion is an isomorphism, and

(3) If

$$\langle \mathrm{ch}([P]), \mathrm{ch}([(-)^\vee]) \rangle_{\mathrm{MF}(\mathbb{C}[x_1,x_2],-D_4)} : K_0[\mathrm{MF}(\mathbb{C}[x_1,x_2],D_4)] \to \mathbb{C}$$

is the zero map, then $[P] = 0$.

The ADE singularities clearly have property (1), and we showed in Proposition 4.1.3 that they have property (2). An easy (but tedious) series of computations shows that the ADE singularities satisfy property (3) as well; one can show this for each ADE singularity using exactly the same argument that we used in the D_4 example above. Hence, if $f \in \mathbb{C}[x_1, x_2]$ is an ADE singularity, ϕ_f is injective (this is Theorem 1.0.3 in the introduction). In fact, more is true:

Theorem 4.3.4. *If $f \in \mathbb{C}[x_1, x_2]$ is an ADE singularity and $n \geqslant 0$ is even, $\phi_{f \oplus (x_3^2 + \cdots + x_n^2)}$ is injective.*

Proof. Since ϕ_f is injective, this follows immediately from Theorem 3.4.4. $\square$

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