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Antichains and Diameters of Set Systems

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ANTICHAINS AND DIAMETERS OF SET SYSTEMS

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In this thesis, we present a number of results, mostly concerning set systems that are antichains and/or have bounded diameter. Chapter 1 gives a more detailed outline of the thesis. In Chapter 2, we give a new short proof of Kleitman's theorem concerning the maximal size of a set system with bounded diameter. In Chapter 3, we turn our attention to antichains with bounded diameter. Šileikis conjectured that an antichain of diameter D has size at most $\binom{n}{\lfloor D/2 \rfloor}$. We present several partial results towards the conjecture.

In 2014, Leader and Long gave asymptotic bounds on the size of a set system where $|A \setminus B| \neq 1$ and more generally, when $|A \setminus B| \neq k$. In Chapter 4, we present streamlined versions of their proofs, with slightly better bounds.

The final chapter presents a proof for the following poset analog of an elementary graph theory problem: every poset with $|R|$ relations contains a height two subposet with at least $|R|/2$ relations.

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Chapter 1

Introduction

Let X be a set. A *set system* $\mathcal{F}$ (often referred to as a *family* when the context is clear) is just a subset of 2^X. Oftentimes, X is taken to be finite and, perhaps most often, $X = [n] = \{1, 2, \ldots, n\}$. Set systems have been studied extensively since the early 1900's.

One of the first and most well-known results is due to Sperner. A *Sperner system* is a set system where no element is contained in another. In the language of posets, this is called an *antichain*. We will use $\mathcal{A}$ to denote this special type of set system. In 1928 (see [16]), Sperner bounded the size of such families:

Theorem 1.1. *Let $\mathcal{A}$ be a Sperner system on $X = [n]$. Then*

$$|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

A *chain*, as one might expect, is the opposite of an antichain. It is a sequence $A_1, A_2, \ldots, A_k \subseteq \mathcal{F}$ such that $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_k$. One of the proofs of Sperner's theorem involves counting maximal chains. Because of its importance in Chapter 4, we present this proof below.

Proof of Theorem 1.1. We double count pairs of the form $(A, \mathcal{C})$ where a $A \in \mathcal{A}$ and $\mathcal{C}$ is a maximal chain in $2^{[n]}$ containing A. On the one hand, each member A can be extended to a maximal chain in one of $|A|! \cdot (n - |A|)!$ ways. So the number of pairs is exactly

$$\sum_{A \in \mathcal{A}} |A|! \cdot (n - |A|)!$$

On the other hand, each maximal chain contains at most 1 member of $\mathcal{A}$, lest the antichain condition be violated. So the number of pairs is at most

$$\sum_{\text{maximal } \mathcal{C}} 1 = n!$$

Dividing, we conclude

$$\sum_{A \in \mathcal{A}} \binom{n}{|A|}^{-1} \leq 1 \tag{1.1}$$

Finally, since $\binom{n}{|A|}$ is maximized when $|A| = \lfloor \frac{n}{2} \rfloor$, we have that

$$|\mathcal{A}| \cdot \binom{n}{\lfloor \frac{n}{2} \rfloor}^{-1} = \sum_{A \in \mathcal{A}} \binom{n}{\lfloor \frac{n}{2} \rfloor}^{-1} \leq 1$$

Multiplying appropriately yields the result. $\square$

Technically, we have just derived the famous LYM inequality (1.1), and used it to prove Sperner's Theorem. This stronger inequality is due to Lubell, Yamamoto, and Meshalkin (see [11], [17], and [13], respectively).

Since Sperner's theorem, a plethora of other families have been studied. Some of the most notable are *intersecting families* and, more generally, *t-intersecting families*. A family $\mathcal{F} \subseteq 2^{[n]}$ is called *intersecting* (respectively, *t-intersecting*) if for all $A, B \in \mathcal{F}$, $|A \cap B| \geq 1$ (respectively, $|A \cap B| \geq t$). Additionally, a family is called *k-uniform* if $|A| = k$ for all $A \in \mathcal{F}$. In 1961 (see [5]), Erdős, Ko, and Rado bounded the size of k-uniform intersecting families.

Theorem 1.2. *Let $n \geq 2k$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ be a k-uniform intersecting family. Then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1}$$

Their original proof was one of the first to use standard left-compression, a technique we will discuss in greater detail in Chapters 2 and 3.

Erdős, Ko, and Rado also made progress on the t-intersecting case, but it wasn't until 1997 that Ahlswede and Khachatrian completely classified all k-uniform t-intersecting families with their Complete Intersection Theorem (see [2]). The proof of their result also uses standard left-compression.

In 1964, Katona bounded the size of a non-uniform t-intersecting family (see [8], Theorem 4). He found that when $n + t = 2k$ is even, the largest such a family could be is $\sum_{i=k}^{n} \binom{n}{i}$, with the unique optimal family being all sets of size at least k. In the case when $n + t = 2k - 1$ is odd, he found the optimal size to increase by $\binom{n-1}{k-1}$, and the optimal family, now unique only up to isomorphism, also included all sets of size $k - 1$ avoiding a fixed point.

In the even case, Katona's extremal family can be simply described as the collection of all sets at distance at most $n-k$ away from $[n]$. Given any two sets $A, B \subseteq [n]$, the *(Hamming) distance* is $\text{dist}(A, B) = |A \Delta B|$ where Δ is the *symmetric difference* symbol. Given a set system $\mathcal{F} \subseteq 2^{[n]}$, the *diameter of $\mathcal{F}$* is

$$\text{diam}(\mathcal{F}) = \max_{A,B \in \mathcal{F}} \text{dist}(A, B)$$

In 1966, Kleitman proved that families of diameter $n - t$ had the same maximal size as Katona's extremal t-intersecting families (see [9]). Furthermore, Kleitman's proof yields a structure similar to Katona's extremal families. When $n - t = 2d$, take all sets of size at most d, that is, the ball of radius d about $\emptyset$. In the $n - t = 2d + 1$

case, we need only add all sets of size $d+1$ that *contain* a fixed point, g for the $\binom{n-1}{d}$ term. It's worth noting that this structure is only unique up to isomorphism, as taking the symmetric difference with any set $A \subseteq [n]$ yields a family of the same diameter. Kleitman's original proof uses standard left-compression and deletion (another form of compression). In Chapter 2, we present a new proof of Kleitman's theorem using generalized compressions.

In 1968, Milner took Katona's setup and added an antichain condition. He found that the largest layer of the cube in Katona's extremal example, $\binom{[n]}{k}$ where $k = \lceil \frac{n+t}{2} \rceil$, is a maximal t-intersecting antichain (regardless of the parity of $n+t$). It was conjectured in [12] that adding an antichain condition to Kleitman's bounded diameter theorem would allow one to make the analogous conclusion to Milner's t-intersecting theorem, but in the bounded diameter case.

Conjecture 1.3. *Let $\mathcal{A} \subseteq 2^{[n]}$ be an antichain with* $\operatorname{diam}(\mathcal{A}) \leq D$ *and let* $d = \lfloor \frac{D}{2} \rfloor$. *Then*

$$|\mathcal{A}| \leq \binom{n}{d}$$

While we were not able to prove the conjecture in full generality, in Chapter 3, we present some partial results in that direction.

Next, we turn our attention to a related property. Consider the following alternate definition of *Sperner family*: a set system $\mathcal{A} \subseteq 2^{[n]}$ is called a *Sperner family* if for all $A, B \in \mathcal{A}$, $|A \setminus B| \neq 0$. This condition can be generalized by instead insisting for all $A, B \in \mathcal{A}$, $|A \setminus B| \neq k$ for some $k \in \mathbb{N}$. Leader and Long gave asymptotic bounds on the size of such a family, focusing first on the case $k = 1$. In Chapter 4, we present streamlined versions of their proofs that are easier to follow and that slightly sharpen their results.

Finally, in Chapter 5, we present some work done in the area of poset partitions.

An elementary induction argument shows that every graph $G = (V, E)$ with $|E|$ edges has a bipartite subgraph with at least $|E|/2$ edges. We prove an analogous result for general posets: every poset $P = (V, R)$ with $|R|$ relations has a height two subposet with with at least $|R|/2$ relations. While we later discovered that this result was already proven by Patel [15] for general height subposets, our main height two theorem is stronger.

Chapter 2

A Short Proof of Kleitman's Theorem

In 1966, Kleitman proved the following isodiametric theorem originally conjectured by Erdős in [5]:

Theorem 2.1 (Kleitman, 1966 [9]). *Let $\mathcal{F} \subseteq 2^{[n]}$ have* $\operatorname{diam}(\mathcal{F}) \leq D$. *Then*

$$|\mathcal{F}| \leq \begin{cases} \sum_{j=0}^{d} \binom{n}{j} & \text{if } D = 2d \\ \binom{n-1}{d} + \sum_{j=0}^{d} \binom{n}{j} & \text{if } D = 2d+1 \end{cases}.$$

Furthermore, equality is attained by the family $\binom{[n]}{\leq d}$ in the first case and $\binom{[n]}{\leq d} \cup \left(\binom{[n-1]}{d} \vee \{\{n\}\}\right)$ in the second.

The notation used in the last line is non-standard, so we define it below. For two set systems, $\mathcal{F}, \mathcal{G} \subseteq 2^{[n]}$, we define the $\vee$ and $\wedge$ operators:

$$\mathcal{F} \vee \mathcal{G} = \{A \cup B : A \in \mathcal{F}, B \in \mathcal{G}\}$$

$$\mathcal{F} \wedge \mathcal{G} = \{A \cap B : A \in \mathcal{F}, B \in \mathcal{G}\}$$

Finally, if two sets $A, B \subseteq [n]$ are disjoint, we shall write $A \mathbin{\dot\cup} B$ instead of $A \cup B$.

It is important to recognize that the diameter condition does not guarantee uniqueness, even up to reordering of $[n]$. Let $C \subseteq [n]$ and consider the following function

$f_C : 2^{[n]} \to 2^{[n]}$ defined by $f_C(A) = C\Delta A$. First of all, we claim f_C is one-to-one. That is, if $f_C(A) = C\Delta A = C\Delta B = f_C(B)$, necessarily $A = B$.

Let $x \in A = (A\backslash C) \cup (C \cap A)$. If $x \in A\backslash C \subseteq C\Delta A = C\Delta B$, then since $x \notin C$, $x \in B\backslash C \subseteq B$. If, on the other hand, $x \in C \cap A$, then $x \notin C\Delta A = C\Delta B$. But since $x \in C$, we must have $x \in C \cap B \subseteq B$. Therefore, $A \subseteq B$ and a symmetric argument shows $A = B$.

If we define $f_C(\mathcal{A}) = \{f_C(A) : A \in \mathcal{A}\}$, then by the argument above, we may conclude $|\mathcal{A}| = |f_C(\mathcal{A})|$ for all $A \subseteq [n]$. Next, we claim f_C is distance-preserving and therefore diameter-preserving. Another elementary set-theoretic argument shows that for all $A, B, C \subseteq [n]$, $A\Delta B = (C\Delta A)\Delta(C\Delta B)$ and hence

$$\begin{aligned}\operatorname{dist}(f_C(A), f_C(B)) =& \operatorname{dist}(C\Delta A, C\Delta B) = |(C\Delta A)\Delta(C\Delta B)| \\ =& |A\Delta B| = \operatorname{dist}(A, B).\end{aligned}$$

We've just shown that we can "move" the extremal family described by Kleitman's theorem to the corresponding ball centered at the point $C \subseteq [n]$ by applying f_C to the family.

Kleitman's original proof is rather long and involved. In 1991, Bollobás and Leader used generalized compressions to present a short proof of the Kruskal-Katona Theorem (see [4]). Here we use similar techniques to prove Kleitman's Theorem.

Definition 2.2. Let $X, Y \subseteq [n]$ be disjoint and $A \subseteq [n]$. The *generalized* $Y \to X$ *compression* of A is

$$A_{Y\to X} = \begin{cases} (A\backslash Y) \cup X & \text{if } Y \subseteq A, X \cap A = \emptyset \\ A & \text{otherwise} \end{cases}.$$

We also define the *generalized* $Y \to X$ *compression* of $\mathcal{F}$ as

$$\mathcal{F}_{Y\to X} = \{A_{Y\to X} : A \in \mathcal{F}\} \cup \{A : A, A_{Y\to X} \in \mathcal{F}\}.$$

We say that $\mathcal{F}$ is $Y \to X$ *compressed* if $\mathcal{F}_{Y\to X} = \mathcal{F}$.

It is important to note that the definition of $\mathcal{F}_{Y\to X}$ guarantees $|\mathcal{F}_{Y\to X}| = |\mathcal{F}|$. In fact, we often think of compression as a bijection from $\mathcal{F}$ to $\mathcal{F}_{Y\to X}$; the image of $A \in \mathcal{F}$, typically denoted A', is defined to be $A_{Y\to X}$ unless $A_{Y\to X} \in \mathcal{F}$, in which case $A' = A$.

Definition 2.3. Let $\mathcal{F} \subseteq 2^{[n]}$ and $A \in \mathcal{F}$. We say that A *moves* if $A_{Y\to X} \neq A$ and $A_{Y\to X} \notin \mathcal{F}$ (i.e., $A' \neq A$). Otherwise, we say A is *fixed* (i.e., $A' = A$).

Compression-style proofs all have the same general formula. There is some property of the family (e.g. the diameter or the size of the shadow) that we wish to show is monotone increasing or decreasing under the "right" conditions. The following example demonstrates that even applying $Y \to X$ compressions where $X < Y$ in the colexicographic order can increase the diameter.

Example 2.4. Consider $\mathcal{A} = \{12, 13, 14, 15\}$, a family of diameter 2. Applying the compression $\{14\} \to \{23\}$ yields $\mathcal{A}_{\{14\}\to\{23\}} = \{12, 13, 23, 15\}$, a family of diameter 4.

The crux of the matter is determining when a given compression does not increase the diameter of the family. We start with a very simple lemma.

Lemma 2.5. *Given $\mathcal{F} \subseteq 2^{[n]}$, $A, B \in \mathcal{F}$, and disjoint $X, Y \subseteq [n]$, if A and B are either both moved or both fixed by the $Y \to X$ compression, then the distance between them is preserved.*

For convenience, we let $[n]_0 = [n] \setminus (X \cupdot Y)$, $A_0 = A \cap [n]_0$, $B_0 = B \cap [n]_0$, and $\Delta_0 = |A_0 \Delta B_0|$.

Proof. If both sets are moved by the compression, then $Y \subseteq A, B$ and $X \cap (A \cup B) = \emptyset$ and the images of A, B are $A' = A_{Y \to X}$ and $B' = B_{Y \to X}$, respectively. Then

$$\begin{aligned}\operatorname{dist}(A', B') &= \operatorname{dist}(A_{Y \to X}, B_{Y \to X}) = \operatorname{dist}(A_0 \cupdot X, B_0 \cupdot X) \\ &= \operatorname{dist}(A_0, B_0) = \operatorname{dist}(A_0 \cupdot Y, B_0 \cupdot Y) = \operatorname{dist}(A, B)\end{aligned}$$

Obviously, if both sets are fixed by the compression, then so is the distance between them. $\square$

The previous lemma establishes that we can only have $\operatorname{diam}(\mathcal{F}_{Y \to X}) > \operatorname{diam}(\mathcal{F})$ if there exists a pair of sets, $A, B \in \mathcal{F}$, one of which is moved by the $Y \to X$ compression while the other is fixed. Without loss of generality, we will assume A moves while B is fixed. We set out to show $\operatorname{dist}(A', B') = \operatorname{dist}(A_{Y \to X}, B) \leq D$. Next, we present two lemmas containing sufficient conditions on X and Y for $\operatorname{diam}(\mathcal{F}_{Y \to X}) \leq \operatorname{diam}(\mathcal{F})$.

Lemma 2.6. *Let $Y \subseteq [n]$ and let $\mathcal{F} \subseteq 2^{[n]}$. Suppose for all $y \in Y$, $\mathcal{F}$ is $(Y \setminus y) \to \emptyset$ compressed. Then* $\operatorname{diam}(\mathcal{F}_{Y \to \emptyset}) \leq \operatorname{diam}(\mathcal{F})$.

Proof. We assume $\operatorname{diam}(\mathcal{F}_{Y \to \emptyset}) > \operatorname{diam}(\mathcal{F})$ so that by the discussion above, there exists a pair of sets $A, B \in \mathcal{F}$ with A moved and B fixed by $Y \to X$ compression. This implies that $A = A_0 \cupdot Y$ and $A_{Y \to \emptyset} = A_0 \notin \mathcal{F}$. Let $B = B_0 \cupdot Y_B$ where $Y_B = B \cap Y$. Since B is fixed under the compression, then either

1. B was eligible to move (i.e. $Y_B = Y$), but $B_{Y \to \emptyset} = B_0 \in \mathcal{F}$ or

2. B was ineligible to move (i.e. $Y_B \subsetneq Y$).

In the first case,

$$\begin{aligned}\operatorname{dist}(A_{Y\to\emptyset}, B) &= \operatorname{dist}(A_0, B_0 \dot\cup Y) = \Delta_0 + |Y| \\ &= \operatorname{dist}(A_0 \dot\cup Y, B_0) = \operatorname{dist}(A, B_0) \leq D,\end{aligned}$$

since $B_0 \in \mathcal{F}$, as desired.

In this diagram and the diagrams to follow, undecorated lines represent intersections between two sets, the weights of which represent the size of the intersection with the "$+\Delta_0$" term suppressed. Single arrows indicate the act of compression while a double arrow from one set to another represents the existence of one guaranteeing the existence of another.

$$\begin{tikzcd} A_0 \dot\cup Y \arrow[r, dash, "|Y|"] \arrow[d, "Y\to\emptyset"'] & B_0 \\ A_0 \arrow[r, dash, "|Y|"'] & B_0 \dot\cup Y \arrow[u, Rightarrow, "Y\to\emptyset"'] \end{tikzcd}$$

Figure 2.1: The $Y_B = Y$ case.

In the second case, since $Y_B \subsetneq Y$, select any $y_0 \in Y \backslash Y_B$. Then by assumption, $\mathcal{F}$ is $(Y\backslash y_0) \to \emptyset$ compressed; i.e. $A_{(Y\backslash y_0)\to\emptyset} = A \dot\cup \{y_0\} \in \mathcal{F}$. Then, since $y_0 \notin Y_B$,

$$\begin{aligned}\operatorname{dist}(A_{Y\to\emptyset}, B) &= \operatorname{dist}(A_0, B_0 \dot\cup Y_B) = \Delta_0 + |Y_B| < \Delta_0 + |Y_B| + 1 \\ &= \operatorname{dist}(A_0 \dot\cup \{y_0\}, B_0 \dot\cup Y_B) = \operatorname{dist}(A_0 \dot\cup \{y_0\}, B) \leq D,\end{aligned}$$

since $A_0 \dot\cup \{y_0\} \in \mathcal{F}$, as desired.

$$
\begin{tikzcd}
A_0 \dot\cup Y \arrow[r, Rightarrow, "Y\setminus y_0 \to \emptyset"] \arrow[d, "Y\to\emptyset"'] & A_0 \dot\cup y_0 \arrow[d, dash, "|Y_B|+1"] \\
A_0 \arrow[r, dash, "|Y_B|"'] & B_0 \dot\cup Y_B
\end{tikzcd}
$$

Figure 2.2: The $Y_B \subsetneq Y$ Case.

$\square$

Lemma 2.7. *Let $X, Y \subseteq [n]$ be disjoint and nonempty with $|X| \leq |Y|$ and let $\mathcal{F} \subseteq 2^{[n]}$. Suppose*

(i) For all $y \in Y$ there exists $x \in X$ s.t. $\mathcal{F}$ is $(Y\backslash y) \to (X\backslash x)$ compressed and

(ii) For all $X' \subsetneq X$, $\mathcal{F}$ is $Y \to X'$ compressed.

Then $\operatorname{diam}(\mathcal{F}_{Y\to X}) \leq \operatorname{diam}(\mathcal{F})$.

Proof. As in the previous lemma, we assume $\operatorname{diam}(\mathcal{F}_{Y\to X}) > \operatorname{diam}(\mathcal{F})$ so that there exists $A, B \in \mathcal{F}$ with A moved and B fixed by $Y \to X$ compression. This implies that $A = A_0 \dot\cup Y$ and $A_{Y\to X} = A_0 \dot\cup X \notin \mathcal{F}$. Let $B = B_0 \dot\cup X_B \dot\cup Y_B$ where $X_B = B \cap X$ and $Y_B = B \cap Y$. If B is fixed by the compression, then either

1. B was eligible to move (i.e., $X_B = \emptyset$, $Y_B = Y$, and $B = B_0 \dot\cup Y$), but $B_{Y\to X} = B_0 \cup X \in \mathcal{F}$ already or

2. B was ineligible to move (i.e. $X_B \neq \emptyset$ or $Y_B \subsetneq Y$).

In the first case,

$$
\begin{aligned}
\operatorname{dist}(A_{Y\to X}, B) &= \operatorname{dist}(A_0 \dot\cup X, B_0 \dot\cup Y) = \Delta_0 + |X| + |Y| \\
&= \operatorname{dist}(A_0 \dot\cup Y, B_0 \dot\cup X) = \operatorname{dist}(A, B_{Y\to X}) \leq D
\end{aligned}
$$

since $A, B' \in \mathcal{F}$, as desired.

$$\begin{tikzcd}
A_0 \uplus Y \arrow[r, dash, "|X|+|Y|"] \arrow[d, "Y\to X"'] & B_0 \uplus X \\
A_0 \uplus X \arrow[r, dash, "|X|+|Y|"'] & B_0 \uplus Y \arrow[u, Rightarrow]
\end{tikzcd}$$

Figure 2.3: The $X_B = \emptyset, Y_B = Y$, and $B = B_0 \uplus Y$ case.

As for the second case, let's first assume $Y_B \subsetneq Y$. Select any $y_0 \in Y \backslash Y_B$. By (i), there exists $x_0 \in X$ such that $\mathcal{F}$ is $(Y\backslash y_0) \to (X\backslash x_0)$ compressed, i.e. $A_{Y\backslash y_0 \to X\backslash x_0} = A_0 \uplus (X\backslash x_0) \uplus y_0 \in \mathcal{F}$.

Since $y_0 \notin Y_B$, but possibly $x_0 \in X_B$

$$\begin{aligned}
\operatorname{dist}(A_{Y\to X}, B) &= \operatorname{dist}(A_0 \uplus X, B_0 \uplus X_B \uplus Y_B) = \Delta_0 + |X| - |X_B| + |Y_B| \\
&= \Delta_0 + (|X| - |X_B| - 1) + (|Y_B| + 1) \\
&\leq \operatorname{dist}\left(A_0 \uplus (X\backslash x_0) \uplus y_0, B_0 \uplus X_B \uplus Y_B\right) = \operatorname{dist}(A_{Y\backslash y_0 \to X\backslash x_0}, B) \leq D,
\end{aligned}$$

as desired.

$$\begin{tikzcd}
A_0 \uplus Y \arrow[r, Rightarrow, "Y\backslash y_0 \to X\backslash x_0"] \arrow[d, "Y\to X"'] & A_0 \uplus (X\backslash x_0) \uplus y_0 \arrow[d, dash, "\geq (|X|-|X_B|-1)+(|Y_B|+1)"] \\
A_0 \uplus X \arrow[r, dash, "|X|-|X_B|+|Y_B|"'] & B_0 \uplus X_B \uplus Y_B
\end{tikzcd}$$

Figure 2.4: The $Y_B \subsetneq Y$ case.

We may now assume $Y_B = Y$, but $X_B \neq \emptyset$. Setting $X' = X \backslash X_B \neq X$, by (ii) we know that $\mathcal{F}$ is $Y \to X'$ compressed, i.e. $B_{Y \to X'} = B_0 \dot\cup X \in \mathcal{F}$. Therefore

$$\begin{aligned}
\operatorname{dist}(A_{Y\to X}, B) &= \operatorname{dist}(A_0 \dot\cup X, B_0 \dot\cup X_B \dot\cup Y) = \Delta_0 + (|X| - |X_B|) + |Y| \\
&< \Delta_0 + |X| + |Y| = \operatorname{dist}(A_0 \dot\cup Y, B_0 \dot\cup X) \\
&= \operatorname{dist}(A, B_{Y\to X'}) \leq D,
\end{aligned}$$

as desired.

$$\begin{tikzcd}
A_0 \dot\cup Y \arrow[r, dash, "|X|+|Y|"] \arrow[d, "Y\to X"'] & B_0 \dot\cup X \\
A_0 \dot\cup X \arrow[r, dash, "|X|-|X_B|+|Y|"'] & B_0 \dot\cup X_B \dot\cup Y \arrow[u, Rightarrow, "Y\to(X\setminus X_B)"']
\end{tikzcd}$$

Figure 2.5: The $Y_B = Y$ and $X_B \neq \emptyset$ case.

$\square$

The theorem we are about to prove states that among all families with m members, the family consisting of the first m elements of the *weight lexicographic order* achieves the minimum possible diameter.

Definition 2.8. The *weight lexicographic order* on $2^{[n]}$ is defined by $A <_{\text{wl}} B \iff |A| < |B|$ or if $|A| = |B|$ and $A <_{\text{lex}} B$.

Let $I_{\text{wl}}(m)$ denote the first m elements in the weight lexicographic order on $2^{[n]}$.

Lemma 2.9. *Let* $\Gamma = \left\{(X,Y) \in 2^{[n]} \times 2^{[n]} : X \cap Y = \emptyset, X <_{wl} Y\right\}$. *A family* $\mathcal{F} \subseteq 2^{[n]}$ *is an initial segment of the weight lexicographic order if and only if it is* $Y \to X$ *compressed for all pairs* $(X,Y) \in \Gamma$.

Proof. ($\Leftarrow$) Let $A \in \mathcal{F}$ and select any $B <_{\mathrm{wl}} A$. We claim $B \in \mathcal{F}$ as well. Clearly, $(B \backslash A, A \backslash B) \in \Gamma$ and so by assumption, $\mathcal{F}$ is $A \backslash B \to B \backslash A$ compressed. That is, $A_{A\backslash B \to B\backslash A} = B \in \mathcal{F}$, as desired.

($\Rightarrow$) Now suppose $\mathcal{F}$ is an initial segment of the weight lexicographic order, but some compression is effective. That is, $A_{Y \to X} = B \notin \mathcal{F}$. This implies $A >_{\mathrm{wl}} B$, contradicting the fact that $B \notin \mathcal{F}$. ☐

Definition 2.10. Given a family $\mathcal{F} \subseteq 2^{[n]}$ and disjoint $X, Y \subseteq [n]$, the family $\mathcal{F}_{Y \to X}$ is called a *legal compression* if it satisfies the hypothesis of Lemmas 2.6 or 2.7.

Definition 2.11. Let $\mathcal{R}(\mathcal{F})$ be the set of families, $\mathcal{G}$, *reachable from* $\mathcal{F}$ via a finite sequence of legal compressions. That is, there exists $(\mathcal{F}_i, X_i, Y_i)$ for $i = 0$ to N such that $\mathcal{F}_0 = \mathcal{F}$, $\mathcal{F}_N = \mathcal{G}$, $X_i, Y_i \subset [n]$ are disjoint and $\mathcal{F}_{i+1} = (\mathcal{F}_i)_{Y_{i+1} \to X_{i+1}}$ is a legal compression.

For our proof of Kleitman's Theorem, we introduce a non-standard notion of "weight."

Definition 2.12. For $\mathcal{F} \subseteq 2^{[n]}$, let $I(\mathcal{F})$ denote the set of corresponding indices of the members of $\mathcal{F}$ in the weight lexicographic order. The *weight* of $\mathcal{F}$ is defined to be

$$\mathrm{wt}(\mathcal{F}) = \sum_{i \in I(\mathcal{F})} i$$

We are now in position to prove the following theorem:

Theorem 2.13. *Let $\mathcal{F} \subseteq 2^{[n]}$. Then*

$$\mathrm{diam}\,(\mathcal{F}) \geq \mathrm{diam}\,(I_{wl}\,(|\mathcal{F}|))\,.$$

Proof. Let $\tilde{\mathcal{F}} \in \mathcal{R}(\mathcal{F})$ have $\mathrm{wt}\big(\tilde{\mathcal{F}}\big)$ minimal and let $m = |\mathcal{F}| = |\tilde{\mathcal{F}}|$. We claim $\tilde{\mathcal{F}} = I_{\mathrm{wl}}(m)$. If not, $\tilde{\mathcal{F}}$ is missing one of the first m members of the weight lexicographic order. Let Λ be the collection of pairs (A, B) such that $A \notin \tilde{\mathcal{F}}$, $B \in \tilde{\mathcal{F}}$, $A <_{\mathrm{wl}} B$ and $|A\backslash B|$ minimal. Select $(A_0, B_0) \in \Lambda$ with $|B_0\backslash A_0|$ minimal as well. We claim $(B_0\backslash A_0) \to (A_0\backslash B_0)$ is a legal compression. Let $Y = B_0\backslash A_0$ and $X = A_0\backslash B_0$.

If $A_0 \subseteq B_0$, then $X = \emptyset$. We claim for all $y_0 \in Y$, that $\tilde{\mathcal{F}}$ is $\big(Y\backslash\{y_0\}\big) \to \emptyset$ compressed. If not, then there exists $B_1 \in \tilde{\mathcal{F}}$ and $A_1 \notin \tilde{\mathcal{F}}$ such that $(B_1)_{Y\backslash y_0 \to \emptyset} = A_1$. However, $A_1 \subseteq B_1$, so $A_1 <_{\mathrm{wl}} B_1$, $|A_1\backslash B_1| = 0 = |A_0\backslash B_0|$ and $|B_1\backslash A_1| = |B_0\backslash A_0| - 1$, a contradiction.

If on the other hand, $X \neq \emptyset$, we claim (i) for all $y \in Y$, there exists $x \in X$, such that $\tilde{\mathcal{F}}$ is $\big((Y\backslash y) \to (X\backslash x)\big)$-compressed and (ii) for all $X' \subsetneq X$, $\tilde{\mathcal{F}}$ is $\big(Y \to X'\big)$-compressed.

If (i) is false, then there exists $y_0 \in Y$ such that $\tilde{\mathcal{F}}$ is *not* $\big((Y\backslash y_0) \to (X\backslash x)\big)$-compressed for all $x \in X$. If $x_0 = \max(X)$, then this implies there exists $A_2 \notin \tilde{\mathcal{F}}$ and $B_2 \in \tilde{\mathcal{F}}$ such that $(B_2)_{Y\backslash y_0 \to X\backslash x_0} = A_2$. We claim $(A_2, B_2) \in \Lambda$. If $|B_0\backslash A_0| > |A_0\backslash B_0|$, then $|B_2\backslash A_2| > |A_2\backslash B_2|$ and clearly $A_2 <_{\mathrm{wl}} B_2$. If, on the other hand, $|B_0\backslash A_0| = |A_0\backslash B_0|$, then $|B_2\backslash A_2| = |A_2\backslash B_2|$. Here the choice of x_0 is very important; it guarantees that

$$\min\big(Y\backslash y_0\big) \geq \min(Y)) > \min(X) > \min\big(X\backslash x_0\big)$$

and hence $A_2 <_{\mathrm{wl}} B_2$. In either case we have $|A_2\backslash B_2| = |A_0\backslash B_0| - 1$ and yet (A_2, B_2) is otherwise eligible to be in Λ, a contradiction. So (i) must hold.

If (ii) is false, then there exists $X' \subsetneq X$ such that $\tilde{\mathcal{F}}$ is *not* $\big(Y \to X'\big)$-compressed. That is, there exists $A_3 \notin \tilde{\mathcal{F}}$ and $B_3 \in \tilde{\mathcal{F}}$ such that $(B_3)_{Y \to X'} = A_3$. We claim $(A_3, B_3) \in \Lambda$. This follows easily from the fact that $|Y| \geq |X|$ and hence $|B_3\backslash A_3| =$

$|Y| > |X'| = |A_3 \backslash B_3|$ and so $|B_3| > |A_3$ and hence $A_3 <_{\mathrm{wl}} B_3$. As before, this implies that (A_3, B_3) should have been included in Λ, but $|A_3 \backslash B_3| < |A_0 \backslash B_0|$, a contradiction.

In both cases, $Y \to X$ is a legal compression, so $\tilde{\mathcal{F}}_{Y \to X} \in \mathcal{R}(\mathcal{F})$. However, $\mathrm{wt}\big(\tilde{\mathcal{F}}_{Y \to X}\big) < \mathrm{wt}\big(\tilde{\mathcal{F}}\big)$, a contradiction. We conclude that $\tilde{\mathcal{F}} = I_{\mathrm{wl}}(m)$. $\square$

Finally, we derive Kleitman's Theorem as a corollary to our theorem.

Corollary 2.14. *Let* $\mathcal{F} \subseteq 2^{[n]}$ *have* $\operatorname{diam}(\mathcal{F}) \leq D$. *Then*

$$|\mathcal{F}| \leq \begin{cases} \sum_{j=0}^{d} \binom{n}{j} & \text{if } D = 2d \\ \binom{n-1}{d} + \sum_{j=0}^{d} \binom{n}{j} & \text{if } D = 2d+1 \end{cases}.$$

Furthermore, equality is attained by the family $\binom{[n]}{\leq d}$ *in the first case and* $\binom{[n]}{\leq d} \cup \left(\{\{1\}\} \vee \binom{[2,n]}{d}\right)$ *in the second.*

Proof. Let $\mathcal{F}$ be a family of diameter D. By the previous theorem, we may assume $\mathcal{F}$ is an initial segment of the weight lexicographic order. First suppose $D = 2d$ is even, but $|\mathcal{F}| > \sum_{j=0}^{d} \binom{n}{j}$. Here we are guaranteed $A = [1, d+1] \in \mathcal{F}$ as well as every set of size d. In particular, the set $B = [d+2, 2d+1] \in \mathcal{F}$ (recall, $n > D = 2d$). However, $\mathrm{dist}(A, B) = 2d+1 > D$, a contradiction.

If, on the other hand, $D = 2d+1$ and $|\mathcal{F}| > \binom{n-1}{d} + \sum_{j=0}^{d} \binom{n}{j}$, then we are guaranteed $A = [2, d+2] \in \mathcal{F}$. We are also guaranteed every set of size $d+1$ that contains 1. In particular, the set $B = \{1, d+2, d+3, \ldots, 2d+2\} \in \mathcal{F}$ (recall, $n > D = 2d+1$). However, $\mathrm{dist}(A, B) = 2d+2 > D$, another contradiction. $\square$

Chapter 3

Bounded Diameter Antichains

3.1 Background

Definition 3.1. A family of sets $\mathcal{F} \subseteq 2^{[n]}$ is called *t-intersecting* if for all $A, B \in \mathcal{F}$, $|A \cap B| \geq t$.

Observe,

$$|A \cap B| = |A| + |B| - |A \cup B| \geq |A| + |B| - n$$

So if we want $\mathcal{F}$ to be t-intersecting, requiring $|A|+|B| \geq n+t$ is certainly sufficient. To establish a lower bound on the maximum size of such a family, simply insist that $|A| \geq \frac{n+t}{2}$ for all $A \in \mathcal{F}$.

In 1964, Katona proved that this is essentially the best possible.

Theorem 3.2 (Katona (1964) [8])**.** *If $\mathcal{F} \subseteq 2^{[n]}$ is t-intersecting,*

$$|\mathcal{F}| \leq \begin{cases} \sum_{j=k}^{n} \binom{n}{j} & \text{if } t+n = 2k \\ \binom{n-1}{k-1} + \sum_{j=k}^{n} \binom{n}{j} & \text{if } t+n = 2k-1 \end{cases}$$

with equality attained by $\binom{[n]}{\geq k}$ and $\binom{[n]}{\geq k} \cup \binom{[n-1]}{k-1}$, respectively.

For $i, j \in [n]$, $i \neq j$, and $A \subseteq [n]$ we define the *standard $(j \to i)$-compression* of

A, $A_{j\to i}$ as

$$A_{j\to i} = \begin{cases} (A\setminus\{j\}) \cup \{i\} & \text{if } j \in A, i \notin A \\ A & \text{otherwise} \end{cases}$$

Note, this is just a special case of the more general $Y \to X$ compression we defined in Chapter 2. Observe, $\{A_{j\to i} : A \in \mathcal{F}\}$ need not have the same size as $\mathcal{F}$. If A is *eligible* to move, i.e., if $j \in A$, but $i \notin A$, but $A\setminus\{j\}\cup\{i\} \in \mathcal{F}$, then $A_{j\to i} = (A\setminus\{j\}\cup\{i\})_{j\to i}$. For this reason, we define

$$\mathcal{F}_{j\to i} = \{A_{j\to i} : A \in \mathcal{F}\} \cup \{A : A, A_{j\to i} \in \mathcal{F}\}$$

ensuring $|\mathcal{F}_{j\to i}| = |\mathcal{F}|$. Put simply, $\mathcal{F}_{j\to i}$ is the family obtained from $\mathcal{F}$ by replacing j with i whenever possible. We say that $\mathcal{F}$ is *left-compressed* if $\mathcal{F} = \mathcal{F}_{j\to i}$ for all $1 \leq i < j \leq n$.

In [5], Erdős, Ko, and Rado showed that if $\mathcal{F}$ is t-intersecting, then so is $\mathcal{F}_{j\to i}$. This incredibly useful fact allows one to assume that $\mathcal{F}$ is left-compressed.

Proposition 3.3 (Erdős, Ko, Rado (1961) [5]). *If $\mathcal{F}$ is a t-intersecting family, then there is a left-compressed t-intersecting family, $\mathcal{F}'$, so that $|\mathcal{F}| = |\mathcal{F}'|$.*

We now present a proof of Katona's Theorem due to its relevance to our work:

Proof of Theorem 3.2. We perform induction on t and n. For the case $t = 1$, the maximum size of such a family is clearly 2^{n-1}, obtained by taking all sets containing a fixed point (only one of $\{A, A^c\}$ may be contained in $\mathcal{F}$). So assume $t > 1$. When $n \leq t$, the result is trivial, so assume $n > t$. By the previous proposition, we may also assume $\mathcal{F}$ is left-compressed. Let

$$\mathcal{F}_0 = \{A \in \mathcal{F} : 1 \notin A\} \qquad \text{and} \qquad \mathcal{F}_1 = \{A \in \mathcal{F} : 1 \in \mathcal{A}\}.$$

Furthermore, let $\mathcal{F}_1' = \{A \backslash \{1\} : A \in \mathcal{F}_1\}$ and observe $\mathcal{F}_1'$ is a $(t-1)$-intersecting family on $n-1$ elements. If we assume $n+t$ is even (the odd case is similar) by induction

$$|\mathcal{F}_1'| \leq \sum_{j=\frac{t+n-2}{2}}^{n-1} \binom{n-1}{j}.$$

Claim: Since $\mathcal{F}$ is left-compressed, $\mathcal{F}_0$ is a $(t+1)$-intersecting family on $n-1$ elements.

Proof of Claim: Let $A, B \in \mathcal{F}_0$ and select $j \in A \cap B$. Since $\mathcal{F}$ is left-compressed, $A_{j\to 1} \in \mathcal{F}$. Since neither A nor B contains 1, replacing j with 1 in A strictly decreases the size of the intersection. But since $\mathcal{F}$ is t-intersecting,

$$|A \cap B| > |A_{j\to 1} \cap B| \geq t.$$

That is, $|A \cap B| \geq t+1$, as desired.

Now again, by induction,

$$|\mathcal{F}_0| \leq \sum_{j=\frac{t+n}{2}}^{n-1} \binom{n-1}{j}.$$

Then, using Pascal's Identity,

$$\begin{aligned}
|\mathcal{F}| = |\mathcal{F}_0| + |\mathcal{F}_1'| &\leq \sum_{j=\frac{t+n}{2}}^{n-1} \binom{n-1}{j} + \sum_{j=\frac{t+n}{2}-1}^{n-1} \binom{n-1}{j} \\
&= \sum_{j=\frac{t+n}{2}}^{n} \binom{n-1}{j} + \sum_{j=\frac{t+n}{2}}^{n} \binom{n-1}{j-1} \\
&= \sum_{j=\frac{t+n}{2}}^{n} \binom{n}{j},
\end{aligned}$$

as desired. $\square$

Observe that $\mathcal{F}$ is t-intersecting if and only if $\mathcal{F}^c = \{A^c : A \in \mathcal{F}\}$ has $|A^c \cup B^c| \leq n - t$ for all $A^c, B^c \in \mathcal{F}^c$:

$$|A^c \cup B^c| = |(A \cap B)^c| = n - |A \cap B| \leq n - t.$$

if and only if $|A \cap B| \geq t$ for all $A, B \in \mathcal{F}$. This yields the following corollary:

Corollary 3.4. *If $\mathcal{F} \subseteq 2^{[n]}$, $|A \cup B| \leq n - t$ for all $A, B \in \mathcal{F}$, then*

$$|\mathcal{F}| \leq \begin{cases} \sum_{j=0}^{k} \binom{n}{j}, & \text{if } n - t = 2k \\ \sum_{j=0}^{k} \binom{n}{j} + \binom{n-1}{k}, & \text{if } n - t = 2k+1 \end{cases}$$

with equality attained by $\binom{[n]}{\leq k} = \binom{[n]}{\leq \frac{n-t}{2}}$ *and* $\binom{[n]}{\leq k} \cup \left[\binom{[n-1]}{k} \vee \{n\}\right] = \binom{[n]}{\leq \frac{n-t-1}{2}} \cup \left[\binom{[n-1]}{\frac{n-t-1}{2}} \vee \{n\}\right]$.

Definition 3.5. Given a family $\mathcal{F} \subseteq 2^{[n]}$, the *diameter* of $\mathcal{F}$ is

$$\operatorname{diam}(\mathcal{F}) = \max\{|A \Delta B| : A, B \in \mathcal{F}\}.$$

Kleitman was able to arrive at the same conclusion as Katona after replacing the "t-intersecting" condition with the condition "$\mathcal{F}$ has diameter at most $n - t$."

Theorem 3.6 (Kleitman (1966) [9])**.** *If $\mathcal{F} \subseteq 2^{[n]}$ has $\operatorname{diam}(\mathcal{F}) \leq n - t$, then*

$$|\mathcal{F}| \leq \begin{cases} \sum_{j=0}^{k} \binom{n}{j}, & \text{if } n - t = 2k \\ \binom{n-1}{k} + \sum_{j=0}^{k} \binom{n}{j}, & \text{if } n - t = 2k+1 \end{cases}$$

with equality attained by $\binom{[n]}{\leq k} = \binom{[n]}{\leq \frac{n-t}{2}}$ *and* $\binom{[n]}{\leq k} \cup \left[\binom{[n-1]}{k} \vee \{n\}\right] = \binom{[n]}{\leq \frac{n-t-1}{2}} \cup \left[\binom{[n-1]}{\frac{n-t-1}{2}} \vee \{n\}\right]$.

Kleitman's original proof used standard $(j \to i)$ compressions and $(j \to \emptyset)$ compressions (otherwise known as *deletions*). For $A \in \mathcal{F}$,

$$A_{j\to\emptyset} = \begin{cases} A \setminus \{j\} & \text{if } j \in A \\ A & \text{otherwise} \end{cases}$$

As with standard $(j \to i)$ compression, we define the analogous $(j \to \emptyset)$ compression of $\mathcal{F}$,

$$\mathcal{F}_{j\to\emptyset} = \{A_{j\to\emptyset} : A \in \mathcal{F}\} \cup \{A : A, A_{j\to\emptyset} \in \mathcal{F}\}$$

which ensures $|\mathcal{F}_{j\to\emptyset}| = |\mathcal{F}|$. We say $\mathcal{F}$ is *down-compressed* if $\mathcal{F}_{j\to\emptyset} = \mathcal{F}$ for all $j \in [n]$.

Bollobás was able to combine elements of Kleitman's original proof and (the corollary to) Katona's Theorem to give another nice proof of Kleitman's Theorem. He makes use of the following lemma, which is a special case of our Theorem 2.13

Lemma 3.7 (Kleitman)**.** *If* $\operatorname{diam}(\mathcal{F}) \leq D$*, then there is a down-compressed family* $\mathcal{F}'$ *such that* $\operatorname{diam}(\mathcal{F}) \leq D$ *and* $|\mathcal{F}'| = |\mathcal{F}|$*.*

Now we present Bollobás' proof of Kleitman's Theorem.

Proof of Theorem 3.6. Let $\mathcal{F}$ be a family with $\operatorname{diam}(\mathcal{F}) \leq D$. By Lemma 3.7, we may assume $\mathcal{F}$ is down-compressed, that is $\mathcal{F}_{j\to\emptyset} = \mathcal{F}$ for all $1 \leq j \leq n$. Hence, $\mathcal{F}$ is a down-set.

Next, we'll bound $|A \cup B|$ in $\mathcal{F}_0$. Set $A' = A \backslash B \subseteq A$. Then $A' \in \mathcal{F}_0$ and

$$|A \cup B| = |A' \Delta B| \leq D.$$

Applying Corollary 3.4, we obtain the desired result. $\square$

3.2 The Main Conjecture

We now return to the main problem of this chapter: describing bounded diameter antichains. First, we discuss the inspiration for Šileikis' conjecture.

We can turn the extremal t-intersecting *families* that Katona found into t-intersecting *antichains* by simply keeping only the largest level sets. As it happens, this is best possible:

Theorem 3.8 (Milner (1968) [14])**.** *If $\mathcal{A} \subseteq 2^{[n]}$ is a t-intersecting antichain, then*

$$|\mathcal{A}| \leq \binom{n}{k}$$

where $k = \left\lceil \frac{n+t}{2} \right\rceil$ and equality attained only by $\binom{[n]}{k}$ or (in the odd case only)

$$\left[\binom{[n]}{k} \setminus \left(\binom{[n-t]}{k-t} \vee [n-t+1, n]\right)\right] \cup \left(\binom{[n-t]}{k-t-1} \vee [n-t+1, n]\right).$$

We present an outline of Milner's proof, as we will apply a similar strategy later on. If $|A| = k$ for all $A \in \mathcal{A}$, then there is nothing to prove. As it happens, this is the only instance where the bound is tight. If $|A| \leq k$ for all $A \in \mathcal{A}$ and at least one $A' \in \mathcal{A}$ has $|A'| < k$, then Milner used up-compression and Katona's theorem on shadows of intersecting families (see below). Finally, if $|A| > k$ for some $A \in \mathcal{A}$, Milner used down-compression on sets of size greater than k to reduce to the second case.

Before we state the conjecture, we define upper and lower shadows, and state Katona's theorem on shadows of intersecting families; both of which we use many times over.

Definition 3.9. Let $l < k < m$. For a k-uniform family $\mathcal{A} \subseteq \binom{[n]}{k}$, *upper m-shadow* and *lower l-shadow* are defined, respectively, as follows:

$$\partial^m(\mathcal{A}) = \left\{B \in \binom{[n]}{m} : \exists A \in \mathcal{A} \text{ s.t. } B \supset A\right\}$$
$$\partial_l(\mathcal{A}) = \left\{B \in \binom{[n]}{l} : \exists A \in \mathcal{A} \text{ s.t. } B \subset A\right\}$$

As a word of warning, equivalent definitions exist in the literature where the super- and subscripts denote the number of levels above or below level k, rather than specifying the actual level numbers as we have.

Theorem 3.10 (Katona [8]). *Let $1 \leq l, t \leq k \leq l + t$ and let $\mathcal{A}$ be a k-uniform, t-intersecting family. Then*

$$|\partial_l(\mathcal{A})| \geq \frac{\binom{2k-t}{l}}{\binom{2k-t}{k}}|\mathcal{A}| = \frac{(k)_{(k-l)}}{(2k-t-l)_{(k-l)}}|\mathcal{A}|$$

In 2015, Markström published a collection of open problems and conjectures, including this one due to Šileikis:

Conjecture 3.11 (Šileikis [12]). *Let $\mathcal{A} \subseteq 2^{[n]}$ be an antichain with* $\operatorname{diam}(\mathcal{A}) = D < n$ *and set $d = \left\lfloor \frac{D}{2} \right\rfloor$. Then*

$$|\mathcal{A}| \leq \binom{n}{d}$$

Note, if $D \geq n$, then Sperner's Theorem (1.1) applies and

$$|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$$

The problem is that the standard compression operations preserve neither property we care about. Standard down compression, for instance, fails to preserve the antichain condition. Consider $\{ij\}$ and $\{jk\}$. If we were to apply $C_{i \to \emptyset}$ to the two sets, we would be left with $\{j\}$ and $\{jk\}$, which is clearly not an antichain.

Naturally, one might try a *modified* down compression operation which only compresses when the antichain condition does not break. This, too, fails. Take $\mathcal{A} = \{A, B, C\}$ where $A = \{123\}$, $B = \{145\}$, and $C = \{245\}$. Then $\text{dist}(A, B) = 4$, $\text{dist}(A, C) = 4$, and $\text{dist}(A, C) = 2$. However, if we apply the modified version of $C_{1\to\emptyset}$, the resulting family is $\mathcal{A}' = \{23, 145, 245\}$ and $\text{dist}(A', B') = 5$, violating the diameter condition. As another example, let $\mathcal{A} = \{123, 124, 134, 234\}$, a family of diameter 2. Ideally, we should be able to compress this family to a set of singletons, but every $i \to \emptyset$ compressions violates the antichain condition. Similarly, every $ij \to k$ compression increases the diameter. Perhaps there is a less natural order to apply these modified compressions that would allow this approach to go through, but it is not clear what that order might be.

It's worth noting that in Bollobás' proof of Kleitman's theorem (see [3]), the last step hinges on the compressed family being a downset. It allowed him to use the union-version of Katona's theorem to bound $\text{dist}(A, B)$. However antichains can never be downsets, rendering this approach fruitless.

3.3 General Techniques

First, we make a simple observation about structure of a bounded diameter antichain.

Proposition 3.12. *An antichain $\mathcal{A} \subseteq 2^{[n]}$ of diameter D must be contained in $D-1$ adjacent levels of the cube.*

Proof. First, recall

$$|A\Delta B| = |A\backslash B| + |B\backslash A|.$$

Second, since $\mathcal{A}$ is an antichain, $|A \backslash B| > 0$ and $|B \backslash A| > 0$. Thus

$$|B \backslash A| < |A \backslash B| + |B \backslash A| \leq D$$

and similarly, $|A \backslash B| < D$. So the list of possible maximal pairs of $(|A \backslash B|, |B \backslash A|)$ is

$$(1, D-1), (2, D-2), \ldots, (D-1, 1).$$

If we interpret these pairs as instructions on how to get from A to B on the cube, they read "go down $|A \backslash B|$ levels and then back up by $|B \backslash A|$ levels, for a net movement of $||A \backslash B| - |B \backslash A||$ levels." So the list of possible "net level movements" is $\{0, 1, 2, \ldots, D-2\}$. That is, the elements of $\mathcal{A}$ occupy a maximum of $D-1$ levels. $\square$

In a later section, we will impose a condition on the maximum number of levels that a bounded diameter antichain is allowed to occupy. Our first partial result takes care of the case that the family lies on the first $d+1$ levels of the cube.

Proposition 3.13. *Let $\mathcal{A}$ be an antichain of diameter D and let $d = \lfloor D/2 \rfloor$. If $\mathcal{A} \subseteq \binom{[n]}{\leq d}$, then $|\mathcal{A}| \leq \binom{n}{d}$.*

Here there is no diameter condition, and we may simply use upper shadows to push the family up, one level at a time. To guarantee that this process does not decrease the size of the family, we will appeal to the well-known Local LYM inequality (see [3]).

Theorem 3.14 (Local LYM)**.** *Let $1 \leq k \leq n$ and let $\mathcal{A} \in \binom{[n]}{k}$. Then*

$$\frac{|\mathcal{A}|}{\binom{n}{k}} \leq \frac{\left|\partial^{k+1}(\mathcal{A})\right|}{\binom{n}{k+1}}$$

Proof. Each k-set has $n-k$ possible extensions. Each $k+1$ set is the extension of at most $k+1$ members of $\mathcal{A}$. So

$$\begin{aligned} |\mathcal{A}|(n-k) &\leq \left|\partial^{k+1}(\mathcal{A})\right|(k+1) \\ |\mathcal{A}| &\leq \frac{\left|\partial^{k+1}(\mathcal{A})\right|}{\frac{n-k}{k+1}} \\ \frac{|\mathcal{A}|}{\binom{n}{k}} &\leq \frac{\left|\partial^{k+1}(\mathcal{A})\right|}{\frac{n-k}{k+1}\binom{n}{k}} = \frac{\left|\partial^{k+1}(\mathcal{A})\right|}{\binom{n}{k+1}}. \end{aligned} \tag{3.1}$$

□

Before we prove Proposition 3.13, we need a bit of notation that will be used throughout the remainder of the thesis. Let $\mathcal{A}^{(k)}$ denote the portion of $\mathcal{A}$ contained on the k-th level. That is,

$$\mathcal{A}^{(k)} = \mathcal{A} \cap \binom{[n]}{k}.$$

Proof of Proposition 3.13. Recursively replace the lowest level of the family with its upper shadow. The antichain condition guarantees there are no collisions with existing elements and since $n \geq k + (k+j) \geq 2k+1$, (3.1) guarantees that

$$\begin{aligned} |\mathcal{A}| &= \left|\mathcal{A}^{(k)}\right| + \left|\mathcal{A}^{(k+1)}\right| + \cdots + \left|\mathcal{A}^{(k+j)}\right| \\ &\leq \left|\partial^{k+1}\left(\mathcal{A}^{(k)}\right)\right| + \left|\mathcal{A}^{(k+1)}\right| + \cdots + \left|\mathcal{A}^{(k+j)}\right|. \end{aligned}$$

Add $\partial^{k+1}\left(\mathcal{A}^{(k)}\right)$ to $\mathcal{A}^{(k+1)}$ to form a new family, $\mathcal{A}'$ occupying one fewer level. Repeat j times until the resulting family lies on a single level $k+j \leq d$. Then

$$|\mathcal{A}| \leq \cdots \leq \binom{n}{k+j} \leq \binom{n}{d},$$

as desired. □

We can improve that slightly by employing Katona's theorem concerning shadows of intersecting families (Theorem 3.10).

Proposition 3.15. *Let $\mathcal{A}$ be an antichain of diameter D and let $d = \lfloor D/2 \rfloor$. If $\mathcal{A} \subseteq \binom{[n]}{\leq d+1}$, then $|\mathcal{A}| \leq \binom{n}{d}$.*

Proof. Applying Theorem 3.10 to $\mathcal{A}^{(d+1)}$ with $t = k - d$ and $l = d$, we have

$$\left|\partial_d \left(\mathcal{A}^{(d+1)}\right)\right| \geq \left|\mathcal{A}^{(d+1)}\right| \cdot \frac{(d+1)_1}{(d+1)_1} = \left|\mathcal{A}^{(d+1)}\right|.$$

So we replace $\mathcal{A}^{(d+1)}$ with its lower shadow. Again, the antichain condition prevents any collision with members of $\mathcal{A}^{(\leq d)}$. Applying Proposition 3.13, we have

$$|\mathcal{A}| = \left|\mathcal{A}^{(\leq d)}\right| + \left|\mathcal{A}^{(d+1)}\right| \leq \left|\mathcal{A}^{(\leq d)}\right| + \left|\partial_d\left(\mathcal{A}^{(d+1)}\right)\right| \leq \binom{n}{d}$$

as desired. □

3.4 The Single Layer Case

One of the main approaches we tried was imposing strong conditions on the number of levels that a bounded diameter antichain was allowed to occupy. If we restrict the antichain to a single level, k, then we can appeal to the Ahlswede and Khachatrian's Complete Intersection Theorem (see [2]) to achieve the desired bound. For fixed $t \leq k \leq n$, and $0 \leq i \leq \frac{n-t}{2}$ they define

$$\mathcal{F}_i = \left\{F \in \binom{[n]}{k} : |F \cap [t+2i]| \geq t + i\right\}$$

Clearly, this class of families is always t-intersecting. They proved that these *are* the optimal k-uniform t-intersecting families.

Theorem 3.16 (The Complete Intersection Theorem [2]). *Let $1 \le t \le k \le n$ and $\mathcal{A} \subseteq \binom{n}{k}$ be t-intersecting.*

1. *With $(k-t+1)\left(2+\frac{t-1}{r+1}\right) < n < (k-t+1)\left(2+\frac{t-1}{r}\right)$, for some $r \in \mathbb{N}_0$, we have*

$$\max|\mathcal{A}| = |\mathcal{F}_r|$$

and $\mathcal{F}_r$ is – up to permutations – the unique optimum. By convention, $\frac{t-1}{r} = \infty$ for $r = 0$.

2. *With $(k-t+1)\left(2+\frac{t-1}{r+1}\right) = n$ for $r \in \mathbb{N}_0$, we have*

$$\max|\mathcal{A}| = |\mathcal{F}_r| = |\mathcal{F}_{r+1}|$$

and an optimal system equals – up to permutations – either $\mathcal{F}_r$ or $\mathcal{F}_{r+1}$.

We are less concerned with the structure of these extremal families and more with their size. We show that bounded diameter antichains (on a single level), when considered as t-intersecting families for the appropriate value of t, do not exceed the conjectured bound.

Theorem 3.17. *Let $\mathcal{A} \subseteq \binom{[n]}{k}$ be an antichain with $\operatorname{diam}(\mathcal{A}) = D < n$ and let $d = \lfloor D/2 \rfloor$. Then*

$$|\mathcal{A}| \le \binom{n}{d}.$$

Proof. Since $D < N$, $d \le \lfloor n/2 \rfloor$ and by Proposition 3.15, $k \ge d+2$. For $A, B \in \mathcal{A}$, since $\operatorname{dist}(A, B) \le D$, we have

$$D \ge |A| + |B| - 2|A \cap B| = 2k - 2|A \cap B|$$

and hence

$$|A \cap B| \geq k - \frac{D}{2} \geq k - d.$$

That is, $\mathcal{A}$ is a k-uniform $(k-d)$-intersecting family and we may employ the Complete Intersection Theorem (CIT). Set $t = k - d$ so that $d = k - t$.

We make use of the following basic identity:

$$\binom{n}{k} = \sum_{j=0}^{\min\{l,k\}} \binom{l}{j}\binom{n-l}{k-j} \tag{3.2}$$

Given r as in the statement of the CIT, we know

$$|\mathcal{A}| \leq \sum_{j_1=t+r}^{\min\{t+2r,k\}} \binom{t+2r}{j_1}\binom{n-t-2r}{k-j_1}$$

Replacing k with d and letting $l = t + 2r$, equation (3.2) says

$$\binom{n}{d} = \sum_{j_2=0}^{\min\{t+2r,d\}} \binom{t+2r}{j_2}\binom{n-t-2r}{d-j_2}.$$

We claim every time $k - j_1 = d - j_2$, (i.e. $j_2 = j_1 + d - k = j_1 - t$)

$$\binom{t+2r}{j_1} \leq \binom{t+2r}{j_2} = \binom{t+2r}{j_1-t}.$$

Since $j_1 \geq t + r$,

$$2t + 2r - j_1 \leq t + r \leq j_1$$

and hence

$$(2t+2r-j_1)_{(t)} \le (j_1)_{(t)}$$
$$\frac{(2t+2r-j_1)!}{(t+2r-j_1)!} \le \frac{(j_1)!}{(j_1-t)!}$$

$$\frac{(t+2r)!}{(t+2r-j_1)!\,(j_1)!} = \binom{t+2r}{j_1} \le \binom{t+2r}{j_1-t} = \frac{(t+2r)!}{(2t+2r-j_1)!\,(j_1-t)!}.$$

We've just shown

$$|\mathcal{A}| \le \sum_{j_1=t+r}^{\min\{t+2r,\}} \binom{t+2r}{j_1}\binom{n-t-2r}{k-j_1} = \sum_{j_2=r}^{\min\{2r,d\}} \binom{t+2r}{j_2}\binom{n-t-2r}{d-j_2}.$$

In particular, if $r=0$, we have

$$|\mathcal{A}| \le \binom{n-k+d}{d} < \binom{n}{d}$$

since $k>d$ and for $r \ge 1$,

$$|\mathcal{A}| \le \binom{n}{d} - \sum_{j_2=0}^{r-1} \binom{t+2r}{j_2}\binom{n-t-2r}{d-j_2} \le \binom{n}{d} - \binom{n-t-2r}{d}.$$

$\square$

3.5 The Two Level Case

For this section, we assume $\mathcal{A} \subseteq \binom{[n]}{k,k+1}$ is an antichain of diameter D. Assuming both levels are nonempty implies $D \ge 3$. We've already dealt with the case that $k+1 \le d+1$, so we may assume $k \ge d+1$. As before, $n \ge 2k+1$. First, we present a very specific result that requires n to be odd and $\mathcal{A}$ to occupy the two middle levels of the cube.

Proposition 3.18. *Let D be given, $d = \lfloor \frac{D}{2} \rfloor$, $k = d+1$, $n = 2k+1 = 2d+3$ and let $\mathcal{A} \subseteq \binom{[n]}{k,k+1}$ be an antichain with* $\operatorname{diam}(\mathcal{A}) = D$*. Then*

$$|\mathcal{A}| \leq \binom{n}{d}$$

Proof. Here, we must assume that $\mathcal{A}$ is maximal: every member in $\binom{[n]}{k,k+1} \backslash \mathcal{A}$ cannot be added to $\mathcal{A}$ without breaking the antichain or diameter condition. We double-count pairs of the form (A, B) where $A \in \mathcal{A}$ and $B \in \binom{[n]}{k,k+1} \backslash \mathcal{A}$. If A is a k-set, then A^c is a forbidden $k+1$ set, as are all $k+1$ sets that contain A. Furthermore, all $\binom{k+1}{k}$ k-sets contained in A^c are disjoint from A, so A forbids exactly $2k+3 = n+2$ sets in $\binom{[n]}{k,k+1}$. Similarly, if A is a $(k+1)$-set, all $\binom{k+1}{k}$ k-sets contained in A are forbidden, as is A^c any $(k+1)$-set containing A^c, as diameter between such a set and A is $D+2$. So again, A forbids exactly $n+2$ sets in $\binom{[n]}{k,k+1}$ and hence the number of such pairs is exactly:

$$(n+2)\,|\mathcal{A}|\,.$$

Now since $n = 2k+1$, $\binom{n}{k} = \binom{n}{k+1}$ and so the number of forbidden sets is exactly $2\binom{n}{k} - |\mathcal{A}|$. Given a set $B \in \binom{[n]}{k,k+1} \backslash \mathcal{A}$, let $f(B)$ be the number of sets in $\mathcal{A}$ forbidding B. Note, if $B^c \in \mathcal{A}$, $f(B) = 1$, since the sets forbidding B are exactly the sets forbidden by B^c, and hence must not be in $\mathcal{A}$. If $B^c \notin \mathcal{A}$, we claim at most half of the remaining $n+1$ forbidding B are contained in $\mathcal{A}$. If $|B| = k$, we pair them up in the following way: $(B + \{x\}, B^c - x)$ and if $|B| = k+1$, we pair them up as $(B - \{x\}, B^c + \{x\})$. Either way, the pairs are disjoint and in this case $f(B) \leq \frac{n+1}{2}$.

Thus the total number of pairs is

$$\sum_{B\in\binom{[n]}{k,k+1}} f(B) \leq |\mathcal{A}|\cdot 1 + \frac{n+1}{2}\left(2\binom{n}{k} - 2|\mathcal{A}|\right) = (n+1)\binom{n}{k} - n|\mathcal{A}|$$

and hence

$$\begin{aligned}(2n+2)|\mathcal{A}| &\leq (n+1)\binom{n}{k}\\ |\mathcal{A}| &\leq \frac{1}{2}\binom{n}{k}.\end{aligned}$$

Now, recalling that $k = d+1$ and since $d \geq 2$, we have

$$n - d = d + 3 < 2(d+1)$$

and hence

$$\frac{1}{2}\binom{n}{k} = \frac{1}{2}\frac{n!}{(d+1)!\,(n-d-1)!} = \frac{n-d}{2(d+1)}\binom{n}{d} < \binom{n}{d}.$$

This completes the case $k = d+1$ and $n = 2k+1$. $\square$

We now relax the conditions on n and k, but specify values for D.

3.5.1 The Two Level Case when $D = 3$

Proposition 3.19. *If $\mathcal{A}$ is an antichain of diameter $D = 3$, then*

$$|\mathcal{A}| \leq \binom{n}{\lfloor 3/2\rfloor} = n$$

Note the condition that $\mathcal{A}$ is contained on two adjacent levels is actually implied by the diameter in this case (see Proposition 3.12). First, we need a pair of complementary definitions.

Definition 3.20. A *sunflower* $\mathcal{F}$ with *core* S is a set system on $[n]$ where every pairwise intersection is S. That is,

$$\mathcal{F} = S \vee \{P_1, P_2, \ldots, P_m\}$$

for some m where the P_i are pairwise disjoint, and all are disjoint from S. The P_i are referred to as the *petals* of $\mathcal{F}$.

By taking the complements of the petals on $[n] \setminus S$, we can form *co-petals*, $\{\overline{P_i} = ([n] \setminus S) \setminus P_i : P_i \text{ a pedal of } \mathcal{F}\}$. Together with the core, this forms a *co-sunflower*,

$$\overline{\mathcal{F}} = S \vee \{\overline{P_1}, \overline{P_2}, \ldots, \overline{P_m}\}$$

Note that it is easy to determine if a given set system is a sunflower; simply take the intersection of every member to form the candidate for S, then check if the would-be leaves are, in fact, disjoint. Co-sunflower's, on the other hand, require one to take complements of the would-be co-petals to see if the corresponding petals are disjoint. Sunflowers have been studied quite extensively; see, for example, Chapter 6 of [7].

The proof of Proposition 3.19 will rely heavily on the following structural lemma, which states that every k-uniform, $(k-1)$-intersecting family is either a sunflower or a co-sunflower.

Lemma 3.21. *Let $\mathcal{A} \subseteq \binom{[n]}{k}$ be $(k-1)$-intersecting with $|\mathcal{A}| = m \geq 2$. Then $\mathcal{A}$ is isomorphic to one of the following:*

1. *A sunflower with a core S of size $(k-1)$ and petals isomorphic to $\binom{[m]}{1}$.*

2. *A co-sunflower with a core S of size $(k+1-m)$ and co-petals isomorphic to* $\binom{[m]}{m-1}$.

Note, this doesn't require $\mathcal{A}$ to be maximal (in which case, we could appeal to the Complete Intersection Theorem). Both objects have a common intersection, or core, S. We'll use knowledge of the core's size to determine the exact shape of $\mathcal{A}$:

1. If $|S| = 0$, then either $k = 1$, an easily understood case, or $|\mathcal{A}| = k+1$ and $\mathcal{A} \cong \binom{[k+1]}{k}$.

2. If $|S| = k-1$, then we're either in cases (1) or $|\mathcal{A}| = 2$ and in this second case, (1) and (2) actually coincide.

3. If $|S| = k$, then obviously $|\mathcal{A}| = 1$.

We can also use knowledge of $|\mathcal{A}|$ to deduce the size of the core and hence the shape of $\mathcal{A}$: If $|\mathcal{A}| > k+1$, then $k+1-|\mathcal{A}| < 0$ and hence $\mathcal{A}$ must be a sunflower.

Proof of Lemma 3.21. Induction on $m = |\mathcal{A}|$. If $m = 2$, then, up to isomorphism, there is only one possible option for $\mathcal{A}$: $[1, k-1] \vee \{k, k+1\}$. In this case, both (1) and (2) describe $\mathcal{A}$.

For $m = 3$, we'll think of building $\mathcal{A}$ one set at time. Given $\{A_1, A_2\}$, we can manipulate A_2 into A_3 by deleting an element and adding an element. Suppose $S = [k-1]$, $A_1 = S \cup \{x\}$, and $A_2 = S \cup \{y\}$.

- If we delete y, everything is still $(k-1)$-intersecting, so we must introduce a new element, say z. In this case, $A_1 = S \cup \{x\}$, $A_2 = S \cup \{y\}$, and $A_3 = S \cup \{z\}$. That is $\{A_1, A_2, A_3\}$ is a sunflower with three pedals of size 1.

- If we delete an element in S, say i, then we have to add x in order to properly intersect A_1. This reduces the common intersection by one: $S' = [k-1] \setminus i$. Now

$A_1 = S' \cup \{i, x\}$, $A_2 = S' \cup \{i, y\}$, and $A_3 = S' \cup \{x, y\}$. That is, $\{A_1, A_2, A_2\} \cong \binom{[3]}{2} \vee S'$.

Once the first choice has been made, there is no turning back: we must continue as we did in the first step. Assume the lemma holds for $3 < m < |\mathcal{A}|$. Delete some member of $\mathcal{A}$. By induction, the remaining family, $\mathcal{A}'$ is in one of the two cases.

Case 1: $\mathcal{A}' = S \vee \{\{x_1\}, \{x_2\}, \ldots, \{x_m\}\}$ where S is a $(k-1)$-set and the x_i are distinct.

In this case, we can obtain the missing member by deleting some element of $S \cup \{x_m\}$ and adding something different. If we delete something in S, then we have to add back $x_1, x_2, \ldots, x_{m-1}$ in order to remain $(k-1)$-intersecting with the other members of $\mathcal{A}$. But since $m - 1 > 2$, this is impossible. Hence, we must delete x_m and add a new element, x_{m+1} and $\mathcal{A} = \mathcal{A}' \cup \{S \cup \{x_{m+1}\}\}$, as desired.

Case 2: $\mathcal{A}' = S \vee \{\{T_1\}, \{T_2\}, \ldots, \{T_m\}\}$ where the $T_i = [m] \setminus \{i\}$ are members of $\binom{[m]}{m-1}$ and $|S| = k + 1 - m$.

In this case, the T_i are $(m-2)$-intersecting. Again, we can obtain the missing member of $\mathcal{A}$ by deleting an element from $S \cup T_m$ and adding something different. If we delete something in T_m, say element $i < m$ then we need to add something to intersect T_j for $j \neq i, m$ in $m - 2$ places. But there's nothing we could add (except i or m) that would make this happen. Hence, we must delete an element, x, from S. Note, $S \cup T_m \setminus \{x\}$ is $(k-2)$-intersecting with everything but $S \cup T_m$, and every other set contains m, so we must add $\{m\}$ to form the new set. Thus $S' = S \setminus \{x\}$, $T_i' = T_i \cup \{x\}$ for $1 \leq i \leq m$ and $T_{m+1}' = [m]$. That is,

$$\mathcal{A} \cong S' \vee \{T_1', T_2', \ldots, T_m', T_{m+1}'\}$$

where the $T_i = [m+1] \setminus i$. $\square$

It's worth noting that, in the second case, each set we added reduced the core size by one, while in the second case, the core remained fixed. We are now ready to prove the proposition.

Proof of Proposition 3.19. By Proposition 3.12, $\mathcal{A}$ must be contained in 2 adjacent levels of the cube, i and $i+1$.

Claim: We may assume $2 \leq i \leq n-2$.

Proof of Claim: If $i = 0$, then $\mathcal{A} \subseteq \binom{[n]}{1}$ and hence $|\mathcal{A}| \leq n$. If $i = n-1$, then either $\mathcal{A}_{i+1} = \mathcal{A}_n = [n]$ and hence $|\mathcal{A}_i| = 0$, $|\mathcal{A}_{i+1}| = 1$, or $|\mathcal{A}_{i+1}| = 0$ and so $|\mathcal{A}_i| = |\mathcal{A}| \leq n$. If $i = 1$, then the key lemma implies $\mathcal{A}_2$ is either a sunflower with a singleton core, or $\binom{[3]}{2}$. The antichain condition implies that the points in $\mathcal{A}_1$ must be disjoint from $\mathcal{A}_2$. In the first case, the sunflower occupies $|\mathcal{A}_2| + 1$ points, leaving $n - |\mathcal{A}_2| - 1$ for $\mathcal{A}_1$. Thus $|\mathcal{A}| \leq |\mathcal{A}_2| + (n - |\mathcal{A}| - 1) = n - 1$. In the second case, $\mathcal{A}_2$ occupies 3 points, leaving $n-3$ for $\mathcal{A}_1$. Thus $|\mathcal{A}| \leq 3 + (n-3) = n$.

From here, we proceed with induction on n. The cases when $n = 1, 2$ or 3 are trivial and handled by the claim. So assume $n > 3$ and let S be the common intersection of everything in $\mathcal{A}$. If $|S| > 0$, then deleting this common intersection preserves the antichain and diameter conditions. By induction, $|\mathcal{A}| = |\mathcal{A}'| \leq n - |S| < n$, as desired.

So assume $|S| = 0$ and instead consider $|S_i|$ and $|S_{i+1}|$, the common intersections of $\mathcal{A}_i$ and $\mathcal{A}_{i+1}$, respectively.

Case 1: If $|S_i| = 0$, then by the lemma, $\mathcal{A}_i \cong \binom{[i+1]}{i}$. Since $i \geq 2$, $|\mathcal{A}_i| \geq 3$. Let $A_1 = [i+1] \setminus \{1\}$, $A_2 = [i+1] \setminus \{2\}$, and let $B \in \mathcal{A}_{i+1}$. Since B must intersect every set in $i-1$ places, it follows that $B = [i+1] \setminus \{1, a\} \cup \{x, y\}$ for some a, x, y. In order for B to properly intersect A_2, we must have $x = 1$. But then $B = [i+1] \setminus \{a\} \cup \{y\}$. and in particular, $B \supset A_a = [i+1] \setminus \{a\}$, a contradiction. So no such $B \in \mathcal{A}_{i+1}$

exists and we conclude $|\mathcal{A}| = |\mathcal{A}_i| = i + 1 \leq n - 1$.

Case 2: If $|S_{i+1}| \neq 0$ and $|S_i| \neq 0$, yet $|S| = 0$. We have $S_{i+1} \cap S_i = \emptyset$. Take $A \in \mathcal{A}_i$ and $B \in \mathcal{A}_{i+1}$ so that $A = S_i \dot\cup T_i$ and $B = S_{i+1} \dot\cup T_{i+1}$. Then

$$i - 1 = |A \cap B| = |T_i \cap T_{i+1}| \subseteq T_i, T_{i+1}.$$

That is, both $|T_i|, |T_{i+1}| \geq i - 1$ and hence $|S_i| \leq 1$ and $|S_{i+1}| \leq 2$. By the lemma, $\mathcal{A}_i \cong \binom{[i]}{i-1} \vee \{x\}$ and $\mathcal{A}_{i+1} \cong \binom{[i+1]}{i} \vee \{x\}$ or $\binom{[i]}{i-1} \vee \{x, y\}$. Take an element B from $\mathcal{A}_{i+1}$. If B contains $\{x\}$, then we need to show $B \backslash \{x\}$ intersects $\binom{[i]}{i-1}$ in $i-2$ places. But by an identical argument as above, this is impossible. So B must not contain $\{x\}$ but must instead intersect every set in $\binom{[i]}{i-1}$ in $i-1$ places. To intersect $A_1 = [i] \backslash \{1\}$, B must contain $A_a = [i] \backslash \{a\}$ for some a, a contradiction. So no such B exists and in this final case, $|\mathcal{A}_{i+1}| = 0$. But this implies $S = S_i \neq 0$, so this case is impossible.

Case 3: If $|S_{i+1}| = 0$, then by the lemma, $\mathcal{A}_{i+1} \cong \binom{[i+2]}{i+1}$. Since $i \geq 2$, $|\mathcal{A}_{i+1}| \geq 4$ (although we only need ≥ 3). Let $A_1 = [i+2] \backslash \{1\}$, $A_2 = [i+2] \backslash \{2\}$ and let $B \in \mathcal{A}_i$. Since B must intersect every set in $i-1$ places, it follows that $B = [i+2] \backslash \{1, a, b\} \cup \{x\}$ for same a, b, x. In order for B to properly intersect A_2, we must have $x = 1$. Thus $B = [i+2] \backslash \{a, b\}$. However, this implies $B \subset A_a = [i+2] \backslash \{a\}$. Hence, so such $B \in \mathcal{A}_i$ exists and we conclude $|\mathcal{A}| = |\mathcal{A}_{i+1}| = i + 2 \leq n$. $\square$

3.5.2 The Two Level Case with $D = 4$

Proposition 3.22. *If $\mathcal{A} \subseteq \binom{[n]}{k,k+1}$ is an antichain of diameter $D = 4$, then*

$$|\mathcal{A}| \leq \binom{n}{\lfloor 4/2 \rfloor} = \binom{n}{2}$$

Proof. Here we use induction on k. The cases $k \in \{1, 2\}$ have already been dealt with. So we may assume $k \geq 3$. If there is a point in common to every member of $\mathcal{A}$, simply delete the point from every set to obtain $\mathcal{A}' \subseteq \binom{[n-1]}{k-1,k}$. For $A, B \in \mathcal{A}$, removing a point in $A \cap B$ does not change $\operatorname{dist}(A, B)$, so $\operatorname{diam}(\mathcal{A}') = 4$ as well. Applying the inductive hypothesis, we conclude $|\mathcal{A}| \leq \binom{n-1}{2}$.

If, on the other hand, there exists a point common to every member of $\mathcal{A}^{(k+1)}$, simply delete the point from every set in $\mathcal{A}^{(k+1)}$ to obtain $\mathcal{B}^{(k)}$ and a new family $\mathcal{A}' = \mathcal{A}^{(k)} \cup \mathcal{B}^{(k)} \subseteq \binom{[n]}{k}$. The sets in $\mathcal{B}^{(k)}$ will now be $(k-2)$-intersecting, and since they originally cross-$(k-1)$-intersected the sets in $\mathcal{A}^{(k)}$, they will now be cross-$(k-2)$-intersect them. Furthermore, the antichain condition guarantees that $\mathcal{A}^{(k)} \cap \mathcal{B}^{(k)} = \emptyset$. Therefore, $|\mathcal{A}'| = |\mathcal{A}|$, $\mathcal{A}'$ is $(k-2)$-intersecting and hence $\operatorname{diam}(\mathcal{A}') = 4$, so we may apply Theorem 3.17 to achieve the desired bound.

Similarly, if there exists $j \in [n]$ that is *missing* from all of the k-sets, we may *add* it and reduce to the single-level case. Finally, if there exists a point common to every member of $\mathcal{A}^{(k)}$, say x, then since some member $A \in \mathcal{A}^{(k+1)}$ avoids x, we may uniquely specify each of $\mathcal{A}^{(k)}$ by the pair of elements in A it avoids. That is, $\left|\mathcal{A}^{(k)}\right| \leq \binom{k+1}{2}$. Now we apply the single-level-case to $\mathcal{A}^{(k+1)}$.

Case 1: If $n \geq 3\,(2 + (k-2)) = 3k$, then

$$\left|\mathcal{A}^{(k+1)}\right| \leq \binom{n-k+1}{2} = \frac{(n-(k-1))\,(n-k)}{2} = \frac{n^2 - (2k-1)\,n + k\,(k-1)}{2}$$

and hence

$$\begin{aligned} |\mathcal{A}| &\leq \frac{n^2 - (2k-1)\,n + 2k^2}{2} = \binom{n}{2} - \frac{(2k-2)\,n - 2k^2}{2} \\ &= \binom{n}{2} - (k-1)\,n + k^2. \end{aligned}$$

Put another way, $\binom{[n]}{2}$ can be counted in the following way: take all pairs in $[1, k+1]$, all pairs in $[k,n]$, but then subtract the pair $\{k,k+1\}$ which was counted twice and add pairs containing one element each from $[1,k-1]$ and $[k+2,n]$. That is,

$$\binom{n}{2} = \binom{k+1}{2} + \binom{n-k+1}{2} - 1 + (k-1)(n-k-1)$$

and hence

$$|\mathcal{A}| \leq \binom{n}{2} + 1 - (k-1)(n-k-1),$$

the quantity from above.

Case 2: If $3k \geq n \geq 3\left(2 + \frac{k-2}{2}\right)$, then

$$\left|\mathcal{A}^{(k+1)}\right| \leq (k+1)(n-k-1) + 1 = kn - k^2 - 2k + n.$$

So

$$\begin{aligned} |\mathcal{A}| &\leq \frac{2kn - k^2 - 3k + 2n}{2} \leq \frac{(n-1)\,n - k^2 - 3k + 6k}{2} \\ &= \binom{n}{2} + \frac{3k - k^2}{2} \leq \binom{n}{2} \end{aligned}$$

since $k \geq 3$

Case 3: If $k+4 \leq n \leq 3\left(2 + \frac{k-2}{2}\right)$, then $\left|\mathcal{A}^{(k+1)}\right| \leq \binom{k+3}{2}$. Therefore

$$\begin{aligned} |\mathcal{A}| &\leq \binom{k+1}{2} + \binom{k+3}{2} = \frac{2k^2 + 6k + 6}{2} = k^2 + 3k + 3 \\ &\leq 2k^2 + k = \binom{2k+1}{2} \leq \binom{n}{2} \end{aligned}$$

since $k \geq 3$. We may now assume that $\bigcap \mathcal{A}^{(k)} = \bigcap \mathcal{A}^{(k+1)} = \emptyset$ and $\bigcup \mathcal{A}^{(k)} = [n]$.

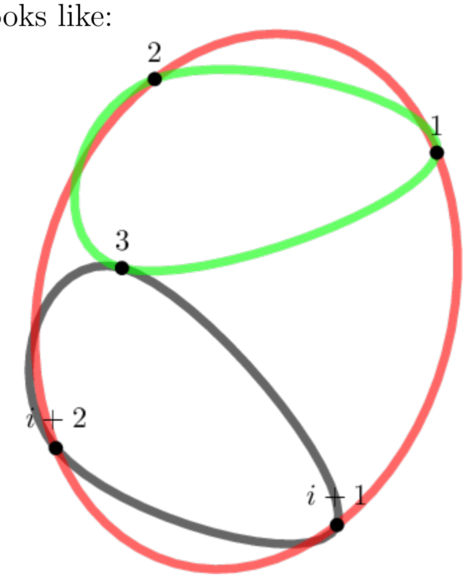
Claim: We must have that $\mathcal{A}^{(k)}$ is actually $(k-1)$-intersecting instead and hence, by the Complete Non-Trivial Intersection Theorem ([1]), (with $n > (t+1)(k-t+1) =$

$2k$ and $k = k < 2t + 1 = 2k - 1$)

$$\left|\mathcal{A}^{(k)}\right| \leq |\mathcal{F}_1| = \binom{[k+1]}{k} = k + 1.$$

Since $k + 1 < \binom{k+1}{2}$ for $k \geq 3$, by the work done above, we are done.

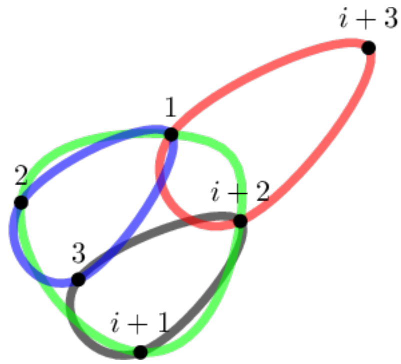
Proof of Claim: Rather than referring to elements in the intersection of two members of $\mathcal{A}$, we'll refer to which elements a given member omits from another (as well as the elements that it adds). We assume that $\mathcal{A}^{(k)}$ is $(k-2)$-intersecting, but not $(k-1)$-intersecting. That is, there exists two sets, $A, B \in \mathcal{A}^{(k)}$ such that $|A \cap B| = k - 2$. Without loss of generality, $A = [1, k]$ and $B = [3, k + 2]$ so $A \cap B = [3, k]$. By hypothesis, there exists some $C \in \mathcal{A}^{(k+1)}$ that avoids some point in $[3, k]$, without loss of generality, $\{3\}$. Since a $(k + 1)$-set may only omit a single point in an k-set, we must have $C = [1, k + 2]\backslash\{3\}$. Note that $A \cap B \cap C = [4, k]$.[1] The rest of the picture looks like:



Now, also by hypothesis, there exists a k-set D avoiding $\{3\}$. Since $A \cap B = [3, k]$, D may avoid at most one other point from $[3, k]$. However, it must also avoid two points in C (of which we have omitted only one so far), so it must also omit one of

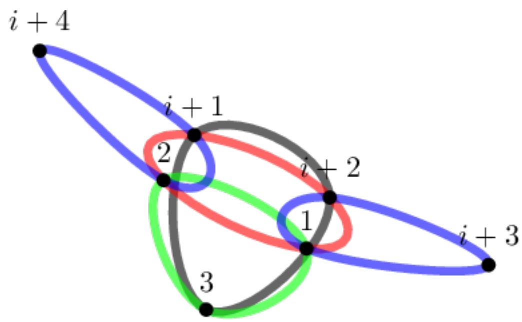
[1]When $k = 3$, we have $A \cap B \cap C = \emptyset$.

$\{1, 2, k+1, k+2\}$, all of which are in A or B. So C can only omit $\{3\}$ from $[3, k]$. It must also omit two from $\{1, 2, k+1, k+2\}$. If it omits $\{1, 2\}$, then $|A \Delta D| \geq 5$ and if it omits $\{k+1, k+2\}$, then $|B \Delta D| \geq 5$. So it must omit exactly one each from $\{1, 2\}$ and $\{k+1, k+2\}$. Let $D' = D \backslash [4, k]$ so without loss of generality $D' \supset \{1, k+2\}$. Next, D must add an unused point, without loss of generality $k+3$ in order to be an k-set. We still have $A \cap B \cap C \cap D = [4, k]$ and the rest of the picture looks like:

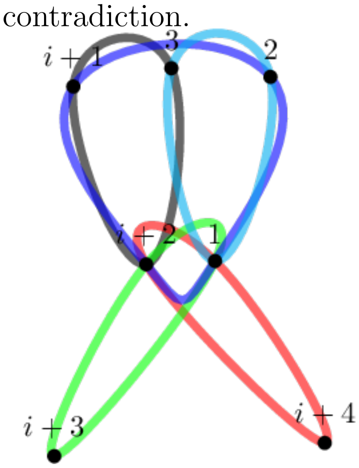


Now since $3 \leq k$, $k+4 \leq 2k+1 \leq n$, so there exists $E \in \mathcal{A}^{(k)}$ containing the element $k+4$. If E omits two points within $[4, k]$, then must contain $\{1, 2, 3, k+1, k+2, k+3\}$ and hence $|E| = 1 + (k-5) + 6 = k+2$, which is impossible. Similarly, if E omits a single point from $[4, k]$, then E must omit a single point from $\{1, 2, k+1, k+2\}$ and at most one from $\{1, 2, 3\}, \{3, k+1, k+2\}, \{1, k+2, k+3\}$. No matter what is chosen to omit from the triple, we will need at least four elements of $\{1, 2, 3, k+1, k+2, k+3\}$. Thus $|E| \geq 1 + (k-4) + 4 = k+1$, a contradiction. Hence $E \supset [4, k] \cup \{k+4\}$ and, setting $E' = E \backslash [4, k]$ we know E' must *contain* one element each from $\{1, 2\}$ and $\{k+1, k+2\}$.

Case 1: If $E' = \{2, k+1, k+4\}$, then $|D \Delta E| = 6$, a contradiction.

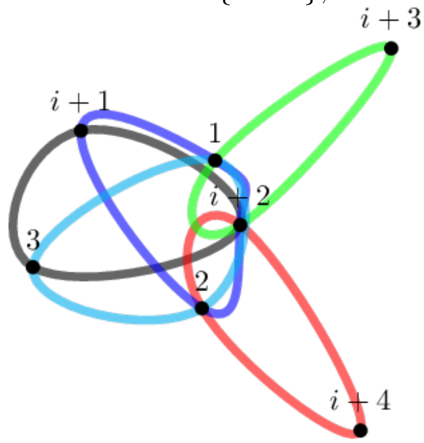


Case 2: If $E' = \{1, k+2, k+4\}$, we consider adding another $F \in \mathcal{A}^{(k+1)}$. It must omit at least one point from C. If that point is in $[4, k]$, it must contain $\{1, 2, 3, k+1, k+2, k+3, k+4\}$ since it may not omit any other points from A, B, D, E. Hence $|F| \geq (k-2)+7 = k+5$, a contradiction. So $F \supset [4, k]$ and we set $F' = F \backslash [4, k]$ (note, $|F'| = 4$). F' must *contain* exactly two points from each of $\{1, 2, 3\}, \{3, k+1, k+2\}, \{1, k+2, k+3\}, \{1, k+2, k+4\}$ and at least two from $\{1, 2, k+1, k+2\}$. If we choose to must omit $\{1\}$, then $F' \supset \{2, 3, k+2, k+3, k+4\}$, contradicting the size of F'. If we choose to omit $\{3\}$, then $F' = \{1, 2, k+1, k+2\} = C'$. Thus $F' \supset \{1, 3\}$. If we chose to omit $\{k+2\}$, then $F' \supset \{1, 3, k+1, k+3, k+4\}$, again contradicting the size of F'. Thus $F' \supset \{1, 3, k+2\}$. From here, the fourth and final member of F' makes no difference, as we've just shown every member of $\mathcal{A}^{(k+1)}$ contains $\{1, k+2\}$, a contradiction.



Case 3: If $E' = \{2, k+2, k+4\}$, we again consider adding another $F \in \mathcal{A}^{(k+1)}$. If

it omits a point from $[4,k]$, then $|F| \geq k+5$ as above. So $F \supset [4,k]$ and we set $F' = F \backslash [4,k]$. Similarly, F' must *contain* exactly two points from each of $\{1,2,3\}, \{3,k+1,k+2\}, \{1,k+2,k+3\}, \{2,k+2,k+4\}$ and at least two from $\{1,2,k+1,k+2\}$. If we choose to omit $\{1\}$, then $F' = \{2,3,k+2,k+3\}$. If we choose to omit $\{2\}$, then $F' = \{1,3,k+2,k+4\}$. If we choose to omit $\{3\}$, then $F' = \{1,2,k+1,k+2\} = C'$. Since these are the only possible 4-sets we could form, it's now clear every member of $\mathcal{A}^{(k+1)}$ contains $\{k+2\}$, another contradiction.



Case 4: If $E = [4,k] \cup \{1,k+1,k+4\}$, we use the symmetry of the picture at the end of the previous step and Case 3 to arrive at the same conclusion. $\square$

We would, of course, like to weaken the diameter condition and obtain a result for all antichains contained on two adjacent levels of the cube. One approach might involve proving a "multi-level" analogue of the Complete Intersection Theorem. Another that we considered involved "rigging" the family so that the antichain condition was negligible. Consider an antichain $\mathcal{A} \subseteq \binom{[n]}{k,k+1}$ where every member of $\mathcal{A}^{(k)}$ contains a fixed point, say n, which every member of $\mathcal{A}^{(k+1)}$ avoids. Deleting this point from every member $\mathcal{A}^{(k)}$ yields a new family, $\mathcal{A}' \subseteq \binom{[n-1]}{k-1,k+1}$, that *need not be an antichain* and whose maximum distance between layers has decreased. It would be interesting to know if these families achieve the desired bound and whether or not all optimal two-layer families are of this form.

Chapter 4

An Improvement Upon a Theorem of Leader and Long

In this chapter, we discuss two existing results regarding families with a forbidden (non-symmetric) set difference. For this entire chapter, we fix $k \in \mathbb{N}$ and let $\mathcal{F} \subseteq 2^{[n]}$ have $|A \setminus B| \neq k$ for all $A, B \in \mathcal{F}$. When $k = 0$, this is equivalent to the antichain condition, so the next natural step is to consider the case $k = 1$. First we present a simple lemma which we will use later.

Lemma 4.1. *If $\mathcal{F} \subseteq \binom{[n]}{k}$ has $|A \backslash B| \neq 1$ for all $A, B \in \mathcal{F}$, then*

$$|\mathcal{F}| \leq \frac{1}{k}\binom{n}{k-1} = \frac{1}{n-k+1}\binom{n}{k}.$$

Proof. Let $\mathcal{P} = \{(A, S) : A \in \mathcal{F}, S \in \partial_{k-1}(A)\}$ and count $|\mathcal{P}|$ in two ways. For a fixed $A \in \mathcal{F}$, there is precisely k elements in $\partial_{k-1}(A)$, so

$$|\mathcal{P}| = k \cdot |\mathcal{F}|.$$

On the other hand, for a fixed $S \in \binom{n}{k-1}$, we claim at most one $A \in \mathcal{F}$ contains S. If not, say $A \neq B$ both contain S, then $|A \backslash B| = 1$, a contradiction. So

$$|\mathcal{P}| \leq 1 \cdot \binom{n}{k-1},$$

proving the result. $\square$

In the non-uniform $k = 1$ case, Leader and Long proved the following asymptotic bound.

Theorem 4.2 (Leader-Long [10]). *Suppose* $\mathcal{F} \subseteq 2^{[n]}$ *has* $|A \backslash B| \neq 1$ *for* $A, B \in \mathcal{F}$. *Then*

$$|\mathcal{F}| \leq \frac{2 + o(1)}{n} \binom{n}{\lfloor n/2 \rfloor}.$$

A disclaimer on the proof that follows. This is not a"new" proof of their theorem, so much as it is a streamlining of their original proof. Many of the elements are the same or similar, but we feel it conveys the result more clearly.

Proof of Theorem 4.2. Remove, from $\mathcal{F}$, all sets of size $< \frac{n}{2} - n^\alpha$ and $> \frac{n}{2} + n^\alpha$ where $\frac{1}{2} < \alpha < 1$. Let $A \in 2^{[n]}$ be chosen uniformly at random. By Chernoff's Inequality,

$$\mathbb{P}\left(A \in \binom{[n]}{< \frac{n}{2} - n^\alpha}\right) < \exp\left(-\frac{(n^\alpha)^2}{2\left(\frac{n}{2}\right)}\right) = \exp\left(-n^{2\alpha - 1}\right).$$

Assuming $n^\alpha < \frac{n}{2}$, we need $\alpha < 1 - \log_n 2 < 1$ and in order for the right hand side to vanish, we need $2\alpha - 1 > 0$. Thus we only require $\frac{1}{2} < \alpha < 1$ (the original authors take $\alpha = \frac{2}{3}$ for convenience).

At the asymptotic level, the parity of n does not matter. So for the remainder of the proof of this theorem and the theorem that follows, we shall assume omit floor and ceiling functions.

Noting

$$\begin{aligned}\binom{n}{n/2} &\sim \frac{\sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n}{2\pi \cdot \frac{n}{2} \left(\frac{n}{2e}\right)^{\frac{n}{2} \cdot 2}} \\ &= \frac{\sqrt{2}}{\sqrt{\pi n}} 2^n\end{aligned}$$

we have

$$\frac{1}{n}\binom{n}{n/2} \sim \sqrt{\frac{2}{\pi}} n^{-3/2} \cdot 2^n.$$

Thus, provided $1 > \alpha > \frac{1}{2}$, we've deleted at most

$$\begin{aligned} 2^n \cdot 2\exp\left(-n^{2\alpha-1}\right) &= 2^n o\left(\sqrt{\frac{2}{\pi}} n^{-3/2}\right) \\ &= o\left(\frac{1}{n}\binom{n}{n/2}\right) \end{aligned}$$

elements.

Let $\mathcal{B}$ be the leftover sets. We need only show

$$|\mathcal{B}| \leq \frac{(2+o(1))}{n}\binom{n}{n/2}.$$

Next, we consider all truncated maximal chains running from level $\frac{n}{2} - n^\alpha$ to level $\frac{n}{2} + n^\alpha$. We group these chains into "classes" in the following way: $\mathcal{C} \sim \mathcal{C}'$ if and only if $C_{i+1} \backslash C_i = C'_{i+1} \backslash C'_i$ for all i. That is, the sequences of elements added to the base elements of $\mathcal{C}$ and $\mathcal{C}'$, respectively, are identical. Note, there are $\binom{n}{2n^\alpha} \cdot (2n^\alpha)! = n_{(2n^\alpha)}$ such sequences and consequently $n_{(2n^\alpha)}$ equivalence classes. First, we choose $2n^\alpha$ elements to be in the sequence and order them in one of $(2n^\alpha)!$ ways. Since every chain in a given class is uniquely determined by its base element, we see that there are $\binom{n-2n^\alpha}{\frac{n}{2}-n^\alpha}$ chains in a class.

Choose a chain $\mathcal{C}$ uniformly at random from among all maximal truncated chains and let Γ denote its chain class. We say $\mathcal{C}$ *hits* $\mathcal{B}$ if $\mathcal{B} \cap \mathcal{C} \neq \emptyset$. Let X_Γ be the number of chains in Γ that hit $\mathcal{B}$. Since all the chain classes have the same size, each chain in Γ had an equal chance of being selected. Therefore

$$X_\Gamma = \binom{n-2n^\alpha}{\frac{n}{2}-n^\alpha}\mathbb{P}\left(\mathcal{C} \text{ hits } \mathcal{B}|\mathcal{C}\in\Gamma\right)$$

and consequently

$$\mathbb{E}X_\Gamma = \binom{n-2n^\alpha}{\frac{n}{2}-n^\alpha}\mathbb{P}\left(\mathcal{C} \text{ hits } \mathcal{B}\right).$$

However,

$$\begin{aligned} X_\mathcal{C} &= \mathbf{1}\left(\mathcal{C} \text{ hits } \mathcal{B}\right) \\ &= \sum_i \mathbf{1}\left(\mathcal{C} \text{ hits } \mathcal{B} \text{ at level } i \text{ and not before}\right) \end{aligned}$$

yielding

$$\begin{aligned} \mathbb{E}X_\mathcal{C} &= \mathbb{P}\left(\mathcal{C} \text{ hits } \mathcal{B}\right) \\ &= \sum_i \mathbf{1}\left(\mathcal{C} \text{ hits } \mathcal{B} \text{ at level } i \text{ and not before}\right). \end{aligned}$$

Therefore,

$$\mathbb{E}X_\Gamma = \binom{n-2n^\alpha}{\frac{n}{2}-n^\alpha}\sum_i \mathbf{1}\left(\mathcal{C} \text{ hits } \mathcal{B} \text{ at level } i \text{ and not before}\right)$$

On the other hand, suppose $\mathcal{C}$ and $\mathcal{C}'$ hit $\mathcal{B}$ with $\mathcal{C}\sim\mathcal{C}'$. Thus there exists $C_i\in\mathcal{C}$ and $C'_j\in\mathcal{C}'$ with $C_i, C'_j\in\mathcal{B}$. Without loss of generality, $i<j$, so

$$1\neq \left|C_i\backslash C'_j\right| = \left|C_{\frac{n}{2}-n^\alpha}\backslash C'_{\frac{n}{2}-n^\alpha}\right|.$$

Thus, we can bound the number of chains in a class that hit $\mathcal{B}$ by bounding the number of elements A, B in layer $\frac{n}{2} - n^\alpha$ with $|A \backslash B| \neq 1$. By the Lemma above,

$$\mathbb{E}X_\Gamma \leq \frac{1}{\frac{n}{2} - n^\alpha} \binom{n - 2n^\alpha}{\frac{n}{2} - n^\alpha - 1} = \frac{1}{\frac{n}{2} - n^\alpha + 1} \binom{n - 2n^\alpha}{\frac{n}{2} - n^\alpha}.$$

Thus far, we have shown

$$\binom{n - 2n^\alpha}{\frac{n}{2} - n^\alpha} \sum_i \mathbb{P}\left(\mathcal{C}0 \text{ hits at level } i \text{ and not before}\right) = \mathbb{E}X_\Gamma \leq \frac{1}{\frac{n}{2} - n^\alpha + 1} \binom{n - 2n^\alpha}{\frac{n}{2} - n^\alpha}$$

and hence

$$\sum_i \mathbb{P}\left(\mathcal{C} \text{ hits at level } i \text{ and not before}\right) \leq \frac{1}{\frac{n}{2} - n^\alpha + 1}.$$

Let $C_i = \mathcal{C} \cap \binom{[n]}{i}$, i.e. the member of the chain on level i. Since the events $\mathbb{P}(C_i = B)$ are mutually exclusive for all $B \in \mathcal{B}$, and zero for all B with $|B| \neq i$, we have

$$\begin{aligned}\sum_i \mathbb{P}\left(\mathcal{C} \text{ hits at level } i \text{ and not before}\right) &= \sum_i \mathbb{P}\left(C_i \in \mathcal{B}, C_{i'} \notin \mathcal{B} \text{ for } i' < i\right) \\ &= \sum_i \sum_{B \in \mathcal{B}^{(i)}} \mathbb{P}\left(C_i = B, C_{i'} \notin \mathcal{B} \text{ for } i' < i\right).\end{aligned}$$

Thus we've shown

$$\sum_i \sum_{B \in \mathcal{B}^{(i)}} \mathbb{P}\left(C_i = B, C_{i'} \notin \mathcal{B} \text{ for } i' < i\right) \leq \frac{1}{\frac{n}{2} - n^\alpha + 1}.$$

Major Claim:

$$\mathbb{P}\left(C_i = B, C_{i'} \notin \mathcal{B} \text{ for } i' < i\right) = (1 - o(1))\mathbb{P}(C_i = B).$$

For then

$$\begin{aligned}
\frac{1}{\frac{n}{2}-n^{\alpha}+1} &\geq \sum_{i}\sum_{B\in\mathcal{B}^{(i)}}\mathbb{P}\left(C_i=B, C_{i'}\notin\mathcal{B}\text{ for } i'<i\right)\\
&= (1-o(1))\sum_{i}\sum_{B\in\mathcal{B}^{(i)}}\mathbb{P}\left(C_i=B\right)\\
&= (1-o(1))\sum_{i}\sum_{B\in\mathcal{B}^{(i)}}\binom{n}{i}^{-1}\\
&= (1-o(1))\sum_{i}\left|\mathcal{B}^{(i)}\right|\binom{n}{i}^{-1}\\
&\geq (1-o(1))\sum_{i}\left|\mathcal{B}^{(i)}\right|\binom{n}{n/2}^{-1}\\
&= (1-o(1))\left|\mathcal{B}\right|\binom{n}{n/2}^{-1}
\end{aligned}$$

and hence

$$\begin{aligned}
|\mathcal{B}| &\leq \frac{(1+o(1))}{\frac{n}{2}-n^{\alpha}+1}\binom{n}{n/2}\\
&= \frac{(1+o(1))}{\frac{n}{2}}\binom{n}{n/2}\\
&= \frac{(2+o(1))}{n}\binom{n}{n/2}.
\end{aligned}$$

Proof of Claim: First note, by successive conditioning:

$$\begin{aligned}
\mathbb{P}\left(C_i=\mathcal{B}, C_{i'}\notin\mathcal{B}\text{ for } i'<i\right) &= \mathbb{P}\left(C_i=B\right)\mathbb{P}\left(C_{i-1}\notin\mathcal{B}\mid C_i=B\right)\\
&\quad\cdots\mathbb{P}\left(C_1\notin\mathcal{B}\mid C_2,\ldots C_{i-1}\notin\mathcal{B}, C_i=B\right).
\end{aligned}$$

By the Law of Total Probability, however,

$$\mathbb{P}\left(C_{i'} \notin \mathcal{B} \mid C_{i'+1} \notin \mathcal{B}, \ldots, C_{i-1} \notin \mathcal{B}, C_i = B\right) = \sum_{\mathbf{F}} \mathbb{P}\left(C_{i'} \notin \mathcal{B} \mid \mathbf{F}\right) \cdot \mathbb{P}\left(\mathbf{F}\right)$$

where $\mathbf{F} = \{C_{i'+1} = F_{i'+1}, \ldots, C_{i-1} = F_{i-1}, C_i = B\}$, each $F_j \notin \mathcal{B}$, and the sum ranges over all possible $\mathbf{F}$. In particular, $\sum_{\mathbf{F}} \mathbb{P}\left(\mathbf{F}\right) = 1$. Now since the event $C_{i'} \notin \mathcal{B}$ only depends on the $C_{i'+1}$, we are dealing with a Markov process, hence

$$\mathbb{P}\left(C_{i'} \notin \mathcal{B} \mid \mathbf{F}\right) = \mathbb{P}\left(C'_i \notin \mathcal{B} \mid C_{i'+1} = F_{i'+1}\right).$$

For $i' + 1 > \frac{n}{2} - n^\alpha$, $C_{i'}$ is equally likely to be any of $C_{i'+1}$'s i'-subsets. But since at most one $C_{i'+1} \backslash \{d\} \in \mathcal{B}$, we have

$$\mathbb{P}\left(C_{i'} \in \mathcal{B} \mid C_{i'+1} = F_{i'+1}\right) = \frac{1}{i'+1}$$

and hence for $i' + 1 > \frac{n}{2} - n^\alpha$

$$\mathbb{P}\left(C_{i'} \notin \mathcal{B} \mid C_{i'+1} = F_{i'+1}\right) = 1 - \frac{1}{i'+1} > 1 - \frac{1}{\frac{n}{2} - n^\alpha}.$$

Therefore, for $i' + 1 > \frac{n}{2} - n^\alpha$

$$\begin{aligned} &\mathbb{P}\left(C_{i'} \notin \mathcal{B} \mid C_{i'+1} \notin \mathcal{B}, \ldots, C_{i-1} \notin \mathcal{B}, C_i = B\right) \\ &\geq \sum_{\mathbf{F}} \left(1 - \frac{1}{\frac{n}{2} - n^\alpha}\right) \cdot \mathbb{P}\left(\mathbf{F}\right) = \left(1 - \frac{1}{\frac{n}{2} - n^\alpha}\right). \end{aligned}$$

Now for $i'+1 \leq \frac{n}{2}-n^{\alpha}$, $\mathbb{P}\left(C_{i'} \notin \mathcal{B} \mid C_{i'+1} \notin \mathcal{B}, \ldots, C_{i-1} \notin \mathcal{B}, C_i = B\right) = 1$. Therefore,

$$\begin{aligned}
\mathbb{P}\left(C_i = \mathcal{B}, C_{i'} \notin \mathcal{B} \text{ for } i' < i\right) &\geq \left(1 - \frac{1}{\frac{n}{2} - n^{\alpha}}\right)^{(i-1)-(n/2-n^{\alpha}-1)} \cdot 1^{\frac{n}{2}-n^{\alpha-1}} \cdot \mathbb{P}\left(C_i = B\right) \\
&\geq \left(1 - \frac{1}{\frac{n}{2} - n^{\alpha}}\right)^{2n^{\alpha}} \mathbb{P}\left(C_i = B\right) \\
&= \left(1 - o\left(1\right)\right)\mathbb{P}\left(C_i = B\right)
\end{aligned}$$

as desired. $\square$

Leader and Long also proved an asymptotic bound for the general k case.

Theorem 4.3 (Leader-Long [10]). *Let $k \in \mathbb{N}$. Suppose $\mathcal{F} \subseteq 2^{[n]}$ has $|A \backslash B| \neq k$ for $A, B \in \mathcal{F}$. Then*

$$|\mathcal{F}| \leq \frac{C_k}{n^k}\binom{n}{n/2}$$

where C_k is a constant depending only on k.

Their proof makes use of a theorem due to Frankl and Füredi which we will use as well.

Theorem 4.4 (Frankl-Füredi [6]). *Let $0 \leq k < r$ and suppose $\mathcal{A} \subseteq \binom{[n]}{r}$ with $|A \cap B| \neq k$ for all $A, B \in \mathcal{A}$. Then $|\mathcal{A}| \leq d_r n^{\max(k, r-k-1)}$ where d_r is a constant depending only on r.*

Note, the upper bound technique will not work exactly as before: given a family $\mathcal{F} \subseteq \binom{n}{m}$ with $|A \backslash B| \neq k$, we can count pairs $\left\{(A, S) : A \in \mathcal{F}, S \in \binom{A}{m-k}\right\}$ to get

$$|\mathcal{F}| \cdot \binom{m}{m-k}$$

on one hand. However, for a fixed set S, there could be many sets A containing S. Equivalently, there could be many U such that $S \cup U \in \mathcal{F}$. However the collection of such U form an intersecting family: if not $|(S \cup U_1) \setminus (S \cup U_2)| = |U_1 \backslash U_2| = k$. So by Erdős-Ko-Rado, there are at most $\binom{n-m+k-1}{k-1}$ such U. Hence

$$|\mathcal{F}| \leq \frac{\binom{n-m+k-1}{k-1} \cdot \binom{n}{m-k}}{\binom{m}{m-k}} = \frac{k}{n-m+k}\binom{n}{m} \cdot \leq \frac{k}{n-m+k}\binom{n}{n/2},$$

which isn't good enough.

Furthermore, in the $k = 1$ case, we considered skipless chains. We were able to do this since, given any chain element at level i, we were equally likely to be any subset on level $i-1$. However, the $|A \backslash B| \neq 1$ condition only allowed for at most one subset to be in $\mathcal{B}$.

For $k \neq 1$, deleting 1 element doesn't guarantee anything; every subset on level $i-1$ could potentially be in $\mathcal{B}$. So we instead consider chains that skip k levels at a time.

The disclaimer from the previous theorem applies here as well.

Proof of Theorem 4.3. Remove the same sets as before. By Chernoff, we've removed at most $o\left(\frac{1}{n^k}\binom{n}{n/2}\right)$. We now partition $\mathcal{B}$ into k classes: for each $l \in [0, \ldots, k-1]$, let $\mathcal{B}_l = \{B \in \mathcal{B} : |B| \equiv l \mod k\}$. We'll show for each l,

$$|\mathcal{B}_l| \leq \frac{C_k}{n^k}\binom{n}{n/2}.$$

where C_k is a constant depending only on k. It suffices to prove the result for $\mathcal{B}_0$ and assume that $\frac{n}{2} - n^\alpha$ is multiple of k. Our partial chains will start at level $\frac{n}{2} - n^\alpha$, and subsequently add k elements at a time until it reaches (close to) layer $\frac{n}{2} + n^\alpha$. In fact, they will add somewhere between $2n^\alpha - k + 1$ and $2n^\alpha$. Pick such a partial chain $\mathcal{C}$ uniformly at random. Since the number elements not in the chain depends on n, we let J be the $n - 2n^\alpha \leq |J| \leq n - 2n^\alpha + k - 1$ things *not* added at some point in the chain. Let

$$\mathcal{D} = \left\{ S \in \binom{J}{\frac{n}{2} - n^\alpha} : \mathcal{C}_S \cap \mathcal{B}_0 \neq \emptyset \right\},$$

i.e., the set of bases of chains hitting $\mathcal{B}_0$ in the same class as $\mathcal{C}$.

Instead of going k layers below the first layer in $\mathcal{B}_0$, we go down $2k$. For any $T \in \binom{J}{\frac{n}{2}-n^\alpha-2k}$, let $D_T = \left\{ U \in \binom{J\setminus T}{2k} : U \cup T \in \mathcal{D} \right\}$. We now count pairs (S,T). For each $S \in \mathcal{D}$, there is exactly $\binom{\frac{n}{2}-n^\alpha}{2k}$ subsets U of size $2k$ we could delete and obtain a T. On the other hand, for a fixed $T \in \binom{J}{\frac{n}{2}-n^\alpha-2k}$, there is exactly $|\mathcal{D}_T|$ U we could add and obtain a member of $\mathcal{D}$. So

$$\sum_{T \in \binom{J}{\frac{n}{2}-n^\alpha-2k}} |\mathcal{D}_T| = |\mathcal{D}| \cdot \binom{\frac{n}{2} - n^\alpha}{2k}.$$

So by the averaging principle, there exists a T' with

$$|\mathcal{D}_{T'}| \geq \frac{|\mathcal{D}| \cdot \binom{\frac{n}{2}-n^\alpha}{2k}}{\binom{|J|}{|T|}}.$$

On the other hand, for any T, there exists $U_1, U_2 \in \mathcal{D}_T \subseteq \binom{J\setminus T}{2k}$ such that $|U_1 \cap U_2| = k$, then $U_1 \cup T, U_2 \cup T \in \mathcal{D}$. So there exists $i_1 \leq i_2$ s.t.

$$\begin{aligned} |C_{U_1 \cup T, i_1} \setminus C_{U_2 \cup T, i_2}| &= |(U_1 \cup T) \setminus (U_2 \cup T)| = |U_1 \setminus U_2| \\ &= |U_1| - |U_1 \cap U_2| = 2k - k = k, \end{aligned}$$

a contradiction. Hence, for all $U_1, U_2 \in \mathcal{D}_T$, $|U_1 \cap U_2| \neq k$ for every T. By Theorem 4.4, there exists some constant d_{2k}, relying only on k, s.t.

$$|\mathcal{D}_T| \leq d_{2k} \left(|J \setminus T| \right)^{\max(k,k-1)}.$$

So

$$\frac{|\mathcal{D}| \cdot \binom{\frac{n}{2}-n^\alpha}{2k}}{\binom{|J|}{|T'|}} \leq |\mathcal{D}_{T'}| \leq d_{2k} \left(|J \backslash T|\right)^k$$

and hence

$$\begin{aligned}
|\mathcal{D}| &\leq d_{2k} \left(|J \backslash T|\right)^k \cdot \frac{|J|!}{|T|! \, |J \backslash T|!} \cdot \frac{(2k)! \left(\frac{n}{2} - n^\alpha - 2k\right)!}{\left(\frac{n}{2} - n^\alpha\right)!} \\
&\leq d_{2k} \left(|J \backslash T|\right)^k \cdot \frac{|J|!}{|J \backslash T|!} \cdot \frac{(2k)!}{\left(\frac{n}{2} - n^\alpha\right)!} \cdot \frac{\left(|J| - \frac{n}{2} + n^\alpha\right)!}{\left(|J| - \frac{n}{2} + n^\alpha\right)!} \\
&= d_{2k} \left(|J \backslash T|\right)^k \cdot \binom{|J|}{\frac{n}{2} - n^\alpha} \cdot \frac{(2k)! \cdot \left(|J| - \frac{n}{2} + n^\alpha\right)!}{|J \backslash T|!} \\
&= d_{2k} \left(|J \backslash T|\right)^k \cdot \binom{|J|}{\frac{n}{2} - n^\alpha} \cdot \binom{|J \backslash T|}{2k}^{-1} \\
&\leq d_{2k} \left(|J \backslash T|\right)^k \cdot \binom{|J|}{\frac{n}{2} - n^\alpha} \cdot \frac{(2k)^{2k}}{\left(|J \backslash T|\right)^{2k}} \\
&\leq \frac{d_{2k} \cdot (2k)^{2k}}{\left(\frac{n}{2} - n^\alpha + 2k\right)^k} \cdot \binom{|J|}{\frac{n}{2} - n^\alpha}
\end{aligned}$$

As before

$$\binom{|J|}{\frac{n}{2} - n^\alpha} \sum_i \mathbb{P}\left(\mathcal{C}' \text{ hits first at } i\right) = \mathbb{E}X = \mathbb{E}\,|\mathcal{D}| \leq \frac{d_{2k} \cdot (2k)^{2k}}{\left(\frac{n}{2} - n^\alpha + 2k\right)^k} \cdot \binom{|J|}{\frac{n}{2} - n^\alpha}$$

and so

$$\sum_i \mathbb{P}\left(\mathcal{C}' \text{ hits first at } i\right) \leq \frac{d_{2k} \cdot (2k)^{2k}}{\left(\frac{n}{2} - n^\alpha + 2k\right)^k}.$$

Again,

$$\sum_i \mathbb{P}\left(\mathcal{C} \text{ hits first at } i\right) = \sum_i \sum_{B \in \mathcal{B}_0^{(ik)}} \mathbb{P}\left(C_i = B, C_{i'} \notin \mathcal{B}_0 \text{ for } i' < i\right)$$

and we claim

$$\mathbb{P}\left(C_i = B, C_{i'} \notin \mathcal{B}_0 \text{ for } i' < i\right) = \left(1 - o\left(1\right)\right) \mathbb{P}\left(C_i = B\right)$$

so that

$$\begin{aligned} \frac{d_{2k} \cdot \left(2k\right)^{2k}}{\left(\frac{n}{2} - n^{\alpha} + 2k\right)^{k}} &\geq \left(1 - o\left(1\right)\right) \sum_i \sum_{B \in \mathcal{B}_0^{(ik)}} \mathbb{P}\left(C_i = B\right) \\ &\geq \left(1 - o\left(1\right)\right) \left|\mathcal{B}_0\right| \binom{n}{n/2}^{-1} \end{aligned}$$

and hence

$$\left|\mathcal{B}_0\right| \leq \frac{d_{2k} \cdot \left(2k\right)^{2k} \left(1 + o\left(1\right)\right)}{\left(\frac{n}{2} - n^{\alpha} + 2k\right)^{k}} \binom{n}{n/2}.$$

Multiplying by k, we have

$$\left|\mathcal{B}\right| \leq \frac{k \cdot d_{2k} \cdot \left(2k\right)^{2k} \left(1 + o\left(1\right)\right)}{\left(\frac{n}{2} - n^{\alpha} + 2k\right)^{k}} \binom{n}{n/2}.$$

Proof of Claim: For our chain $\mathcal{C}$, we write C_i for the member of $\mathcal{C}$ on level ik. By successive conditioning

$$\begin{aligned} \mathbb{P}\left(C_i = \mathcal{B}_0, C_{i'} \notin \mathcal{B}_0 \text{ for } i' < i\right) &= \mathbb{P}\left(C_i = B\right) \mathbb{P}\left(C_{i-1} \notin \mathcal{B}_0 \mid C_i = B\right) \\ &\qquad \cdots \mathbb{P}\left(C_1 \notin \mathcal{B}_0 \mid C_2, \ldots C_{i-1} \notin \mathcal{B}_0, C_i = B\right) \end{aligned}$$

and by the Law of Total Probability

$$\begin{aligned}
&\mathbb{P}\left(C_{i'} \notin \mathcal{B}_0 \mid C_{i'+1} \notin \mathcal{B}_0, \ldots, C_{i-1} \notin \mathcal{B}_0, C_i = B\right) \\
&= \sum_{\mathbf{F}} \mathbb{P}\left(C_{i'} \notin \mathcal{B}_0 \mid \mathbf{F}\right) \cdot \mathbb{P}\left(\mathbf{F}\right) \\
&= \sum_{\mathbf{F}} \mathbb{P}\left(C_i' \notin \mathcal{B}_0 \mid C_{i'+1} = F_{i'+1}\right) \cdot \mathbb{P}\left(\mathbf{F}\right)
\end{aligned}$$

where $\mathbf{F} = \{C_{i'+1} = F_{i'+1}, \ldots, C_{i-1} = F_{i-1}, C_i = B\}$, each $F_j \notin \mathcal{B}_0$, and the sum ranges over all possible $\mathbf{F}$.

Fixing $C_{i'+1}$, $C_{i'}$ is equally likely to be any of $C_{i'+1}$'s $i'k$ subsets. Equivalently, the k elements we remove from $C_{i'+1}$ to get to $C_{i'}$ are chosen uniformly at random from $C_{i'+1}$. Hence, there are $\binom{k(i'+1)}{k}$ possible choices for $C_{i'}$. How many of them can be in $\mathcal{B}_0$? If E_1, E_2 are two subsets of $C_{i'+1}$ of size k and $E_1 \cap E_2 = \emptyset$, then

$$|(C_{i'+1} \backslash E_1) \setminus (C_{i'+1} \backslash E_2)| = |E_1 \backslash E_2| = k.$$

Thus the set of possible E s.t. $C_{i'+1} \backslash E$ hit $\mathcal{B}_0$ form an intersecting family. By Erdős-Ko-Rado, there are at most

$$\binom{k\,(i'+1)-1}{k-1} = \frac{k}{k\,(i'+1)} \binom{k\,(i'+1)}{k}.$$

Hence,

$$\mathbb{P}\left(C_{i'} \in \mathcal{B}_0 \mid C_{i'+1} = F_{i'+1}\right) \leq \frac{\frac{k}{k(i'+1)}\binom{k(i'+1)}{k}}{\binom{k(i'+1)}{k}} = \frac{1}{i'+1}.$$

So

$$\mathbb{P}\left(C_{i'} \notin \mathcal{B}_0 \mid C_{i'+1} \notin \mathcal{B}_0, \ldots, C_{i-1} \notin \mathcal{B}_0, C_i = B\right)$$
$$\geq \sum_{\mathbf{F}} \left(1 - \frac{1}{i'+1}\right) \mathbb{P}\left(\mathbf{F}\right) = \left(1 - \frac{1}{i'+1}\right).$$

Here, for $(i'+1)\, k > \frac{n}{2} - n^{\alpha}$,

$$\mathbb{P}\left(C_{i'} \notin \mathcal{B}_0 \mid C_{i'+1} \notin \mathcal{B}_0, \ldots, C_{i-1} \notin \mathcal{B}_0, C_i = B\right) > 1 - \frac{k}{\frac{n}{2} - n^{\alpha}}$$

and for $(i'+1)\, k \leq \frac{n}{2} - n^{\alpha}$

$$\mathbb{P}\left(C_{i'} \notin \mathcal{B}_0 \mid C_{i'+1} \notin \mathcal{B}_0, \ldots, C_{i-1} \notin \mathcal{B}_0, C_i = B\right) = 1.$$

Thus

$$\begin{aligned} \mathbb{P}\left(C_i = \mathcal{B}_0, C_{i'} \notin \mathcal{B}_0 \text{ for } i' < i\right) &\geq \left(1 - \frac{k}{\frac{n}{2} - n^{\alpha}}\right)^{2n^{\alpha}} \mathbb{P}\left(C_i = B\right) \\ &= \left(1 - o\left(1\right)\right) \mathbb{P}\left(C_i = B\right), \end{aligned}$$

as desired. $\square$

Chapter 5

Subposets of a Specific Height

Recall, the classic result from graph theory:

Proposition 5.1. *Every graph $G = (V, E)$ contains a bipartite subgraph with $\geq \frac{|E|}{2}$ edges.*

We present the proof as a basis for our results:

Proof of Proposition 5.1. Pick a bipartite subgraph that preserves a maximal number of relations, $|E_B|$. If every vertex x has at least half its original degree, i.e. $d_B(x) \geq d(x)/2$, then

$$2|E_B| = \sum_x d_B(x) \geq \sum_x d(x)/2 = |E|$$

and $|E_B| \geq |E|/2$. So assume that some x has $d_B(x) < \frac{d(x)}{2}$. This implies that x has more edges within its partition than across. Simply moving x to the other partition will cause the number of preserved edges to increase by at least one, contradicting maximality. Thus no such x exists and $|E_B| \geq |E|/2$. $\square$

Partially ordered sets can be thought of as directed graphs whose edges have the transitive property: if $(A, B) \in E$ and $(B, C) \in E$, then $(A, C) \in E$ as well. To avoid confusion, we will write a poset P as an ordered pair (V, R) where V is the set of elements in the poset and R is the set of *relations*. Recall, the *height* of a poset is length of the longest chain. The analogous result for posets is as follows:

Theorem 5.2. *Every poset $P = (V, R)$ contains a height two subposet with $\geq \frac{R}{2}$ relations.*

Here's where things get complicated: one cannot simply partition the elements of a poset at random. Consider putting every other element of a chain in the upper and lower partitions. The relations that are "preserved" force relations within the partitions via transitivity. This contradicts the height two requirement.

If the poset is already height two, for instance, it makes no matter how it is partitioned. In this case, we could have relations going both ways across the partitions. In cases where the height is at least three, this will almost surely induce relations via transitivity. It is therefore sufficient to partition the poset into an upset and a downset (which we will refer to as U and D), which ensures that all relations across are going the same direction. We call such a partition a (U, D)-partition.

Given any element x in a poset $\mathcal{P}$, define the *down* and *up degrees* of x:

$$d(x) = |\{y \in \mathcal{P} \mid x > y\}|$$
$$u(x) = |\{y \in \mathcal{P} \mid x < y\}|.$$

Given a (U, D)-partition of $\mathcal{P}$, we define also the *across* and *within degrees* of x. For $x \in U$

$$a(x) = |\{y \in D \mid x > y\}|$$
$$w(x) = |\{y \in U \mid x > y \text{ or } x < y\}|$$

and for $y \in D$,

$$a(y) = |\{x \in U : y < x\}|$$
$$w(y) = |\{x \in D : y < x \text{ or } y > x\}|.$$

Additionally, once the poset has been partitioned, we partition the set of relations, $R = A \uplus W$ where A is the collection of relations that occur between elements of different partitions of the ground set and W is the collection of relations between elements of the same partition of the ground set. We car further partition W based on which partition the relation occurs: $W = W_D \uplus W_U$.

Obviously, for any (U, D)-decomposition, the total number of relations across the partition is

$$|A| = \sum_{x \in U} a(x) = \sum_{y \in D} a(y)$$

and by counting the within degrees on each partition, we get twice the number of relations within. That is,

$$2|W| = \sum_{x \in U} w(x) + \sum_{y \in D} w(y)$$
$$= 2|W_U| + 2|W_D|.$$

Additionally,

$$|W| = \sum_{x \in U} u(x) + \sum_{y \in D} d(x)$$

so

$$|W_U| = \frac{1}{2}\sum_{x\in U} w(x) = \sum_{x\in U} u(x)$$
$$|W_D| = \frac{1}{2}\sum_{y\in D} w(y) = \sum_{y\in U} d(y)$$

We now introduce a method for generating a good (U, D)-partition.

Algorithm 5.3 (Down-Up)**.** *Define*

$$P = \{x \in \mathcal{P} \mid d(x) - u(x) > 0\}$$
$$N = \{x \in \mathcal{P} \mid d(x) - u(x) < 0\}$$
$$Z = \{x \in \mathcal{P} \mid d(x) = u(x)\}$$

Let $U = P$ *and* $D = Z \cup N$.

Remark 5.4. The choice to place every member of Z in D was arbitrary, as moving any $z \in Z$ across the partition preserves the number of relations across.

To show that this is, in fact, a (U, D)-partition, we need a lemma.

Lemma 5.5. *If* $x > y$*, then*

$$u(y) \geq u(x) + 1$$
$$d(x) \geq d(y) + 1$$

and hence

$$d(x) - u(x) \geq d(y) - u(y) + 2.$$

Proof. Let $x > y$. For every $z > x$, $z > y$ by transitivity. Taking into account x itself yields the first inequality. Similarly, for every $z < y$, $z < x$ again by transitivity. Taking into account y itself yields the second inequality. Subtraction yields the final inequality. $\square$

Now to show that DOWN-UP yields an (U, D)-partition. Let $y \in U = P$, i.e. $d(y) - u(y) > 0$. If $x > y$, then by the lemma $d(x) - u(x) \geq d(y) - u(y) + 2 > 2$. Hence, $x \in U$ as well. Showing D is a downset is similar. Note, we can move any member of Z to U and not affect the number of relations across, so we only put Z in D by convention.

It turns out that this particular (U, D)-partition is the best possible:

Proposition 5.6. DOWN-UP *yields a height two subposet with the maximal number of relations.*

Proof. Note, for *any* (U, D)-partition, the number of relations within the partitions (i.e., the number of relations deleted) is always

$$|W| = \sum_{x \in U} u(x) + \sum_{x \in D} d(x).$$

The number of total relations is also

$$|R| = \sum_{x} u(x) = \sum_{x} d(x)$$

Thus, the number of relations across (i.e, the number of relations preserved) is always

$$\begin{aligned} |A| = |R| - |W| &= \sum_{x \in U} d(x) + \sum_{x \in D} d(x) - \sum_{x \in U} u(x) - \sum_{x \in D} d(x) \\ &= \sum_{x \in U} d(x) - u(x). \end{aligned}$$

This last sum is maximized when every x with $d(x) - u(x) > 0$ is in U and every x with $d(x) - u(x) < 0$ is in D, i.e., when we have partitioned the poset according to Down-Up. □

We now present an easy lower bound:

Proposition 5.7. Down-Up *yields a height two subposet with more than* $\frac{1}{3}$ *of the original relations.*

Proof. We make the following claim: for every $x \in P$, $a(x) \geq u(x) + 1$ and for every $y \in N \cup Z$, $a(y) \geq d(y)$. This implies

$$|A| = \sum_{x \in P} a(x) \geq \sum_{x \in P} (u(x) + 1)$$

and

$$|A| = \sum_{y \in D = N \cup Z} a(y) \geq \sum_{y \in N \cup Z} d(y).$$

Hence

$$\begin{aligned} 2|A| &\geq \sum_{x \in P} u(x) + \sum_{y \in N \cup Z} d(y) + |P| \\ &= |W| + |P| \\ &= |R| - |A| + |P| \end{aligned}$$

So

$$\begin{aligned} 3|A| &\geq |R| + |P| \\ |A| &\geq \frac{|R|}{3} + \frac{|P|}{3} \\ &\geq \frac{|R|+1}{3} > \frac{|R|}{3}. \end{aligned}$$

To prove the claim, let $x \in P$ be minimal. This implies $d(x) = a(x)$, but since $d(x) \geq u(x) + 1$ in order for x to be in U, $a(x) \geq u(x) + 1$. Now consider $x \in P$ that is not minimal. Simply select a minimal $y \in P$ with $x > y$. Then

$$\begin{aligned} u(y) &\geq u(x) + 1 \\ a(x) &\geq a(y) = d(y) \end{aligned}$$

and thus

$$a(x) \geq a(y) = d(y) \geq u(y) + 1 \geq u(x) + 2.$$

Hence, for non-minimal elements $x \in P$, $a(x) \geq u(x) + 2$. For $y \in N \cup Z$, the argument is similar with the following caveat: maximal elements y have $a(y) \geq d(y)$, not $d(y) + 1$ (since y could be in Z). □

An algebraic proof improving the constant to $\frac{1}{2}$ has eluded us. However, there is a straightforward pairing algorithm that easily yields the result and leads to a few stronger results.

Pairing Process: Fix a linear extension of $\mathcal{P}$ that preserves the order of P, Z, and N (i.e., find linear extensions of all three, and glue them together). Order the elements of $\mathcal{P}$ according to the extension, but don't actually add the missing relations. We have already established that every $x \in P$ has $a(x) \geq u(x)+1$ and every $y \in N \cup Z$

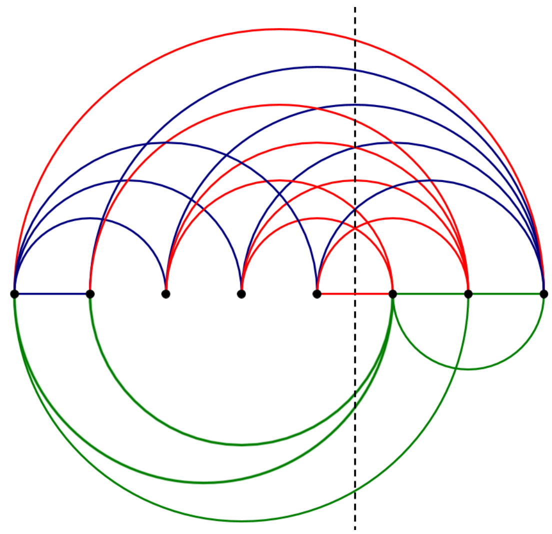


Figure 5.1: A pairing of a $(1, 4, 1, 1, 1)$-chain. Unpaired relations are colored in red. By taking $P_0 = P$ to be the entire upset, we see that $5 = a \geq c = 3$. Notice every element of P has one unpaired relation and $a - c + 1 = 5 - 3 + 1 = 3$ elements in P have two unpaired relations.

has $a(y) \geq d(y)$. So for each element $x \in P$, pair the $u(x)$ relations up with the last $u(x)$ relations across. Similarly, for $y \in N \cup Z$, pair the $d(y)$ relations down with the first $d(y)$ relations across. See Figure 5.1.

Claim: This process is well-defined.

Clearly, each relation within gets paired exactly once. It is less clear, however, that each relation across gets paired at most once. Suppose there is some $x > y$ where $x \in P$ and $y \in N \cup Z$ that is paired to two different relations within (one of x's up relations and one of y's down relations). This implies that y is in the last $u(x)$ down neighbors of x across. By transitivity, each of y's down-neighbors (which occur to the right of y) must also be in first $u(x)$ down neighbors of x across. Thus $d(y) + 1 \leq u(x)$. On the other hand, x must be in the first $d(y)$ up neighbors of y. By transitivity, each of x's up-neighbors (which occur to the left of x) must also be in the first $d(y)$ up neighbors of y across. Thus, $u(x) + 1 \leq d(y)$ and hence

$$u(x) + 1 \leq d(y) \leq u(x) - 1,$$

a contradiction. Thus this operation is well-defined; every relation within is paired to a unique relation across. In fact, it leaves behind $|A| - |W| \geq 0$ unpaired relations across and we obtain the original conjecture as an immediate consequence:

Corollary 5.8. Down-Up *yields a* (U, D)*-partition with* $|A| \geq |W|$*. Equivalently,* Down-Up *yields a height two subposet with at least* $\frac{1}{2}$ *of the original relations.*

Proof. To see the equivalence, observe $|A| \geq |W| = |R| - |A|$ and so $2|A| \geq |R|$ and $|A| \geq |R|/2$. □

With a little more work, we see that Down-Up does even better:

Proposition 5.9. Down-Up *yields a* (U, D)*-partition with* $|A| - |W| \geq \max\{|P|, |N|\}$.

Proof. For each $x \in P$, let $k_x = a(x) - u(x)$ and let y_x be the k_x-th neighbor across for x. Such a neighbor exists since for every $x \in P$, $a(x) \geq u(x) + 1$, and so $k_x \geq 1$. Note the relation $x > y_x$ is not paired to an up relation of x. It's possible, however, that it is paired with a down relation of some $y \in N \cup Z$.

Claim: For every element $x \in P$, the relation to y_x is *always* unpaired.

Proof of Claim: If $x > y_x$ is the k_x-th relation across, it is not paired to one of the $u(x)$ up-relations of x. We need to show it's not paired with one of the $d(y)$ down-relations of y. First, $d(y) \leq u(x)$, otherwise y is not the k_x-th relation (by transitivity). But since y pairs its down-relations with its first $d(y) \leq u(x)$ up-relations, it will never pair with the relation to x itself (as all of x's up-neighbors occur to the left of x). Thus, every $x \in P'$ always has the its k_x-th relation unpaired.

Hence, after we pair the $|W|$ relations within with $|W|$ relations across, of the $|A|-|W|$ relations across left over, there's at least one for every element in P. Therefore,

$$|A| - |W| \geq |P|.$$

If we invert the poset (i.e., reverse the direction of every relation), we get a (U, D)-partition of $(N, Z \cup P)$. Reapplying the pairing process and the same argument as above, we see that $|A| - |W| \geq |N|$ and hence $|A| - |W| \geq \max\{|P|, |N|\}$. □

It turns out that containing a downset with certain properties guarantees even more unpaired relations.

Theorem 5.10. *Let $\mathcal{P}$ be a poset and $\mathcal{P}_0$ a downset in $\mathcal{P}$ with*

$$P_0 = \{x \in \mathcal{P}_0 \mid d(x) - u(x) \geq 2\}$$
$$N_0 = \{x \in \mathcal{P}_0 \mid d(x) - u(x) \leq 0\}.$$

If $a = |P_0|$ and $c = |N_0|$, then applying Down-Up *to $\mathcal{P}$ yields a (U, D)-partition $(P, N \cup Z)$ with*

$$|A| - |W| \geq \max\{|P|, |P| + a - c + 1\}.$$

Proof. If $a \leq c - 1$, then we are done by the previous result. So assume $a \geq c$ (as in Figure 5.1). Let $x \in P_0 \subseteq P$. If x is minimal in P, then x has all $a(x) = d(x) \geq u(x) + 2$ relations across. If x is not minimal, then select x' that is minimal in P with $x' < x$.[1] Then $a(x) \geq a(x') \geq u(x') + 1 \geq u(x) + 2$. Thus every $x \in P_0$ has $a(x) \geq u(x) + 2$ and hence $k_x - 1 \geq 1$. Let y'_x be the $(k_x - 1)$-st neighbor across for x. Note also, the relation $x > y'_x$ is not paired to an up relation of x. So when we are

[1]Such an element might have $d(x') - u(x') = 1$, and therefore may not be in P_0.

pairing x's up relations with relations across, there is always 2 relations unpaired for each $x \in P_0 \subseteq P'$ *thus far*. We already established in the previous result, the k_x-th relation is always unpaired. We now show that enough of $(k_x - 1)$st relations remain unpaired as well.

Claim: There exists $\geq a - c + 1$ elements $x \in P_0 \subseteq P$ that also have the relation to y'_x unpaired.

Proof of Claim: Let $x_1, x_2, \ldots, x_m$ be the elements of P_0 that have their relation to y'_{x_i} *paired.* We will show that $m \leq c - 1$ so that the number of elements in P_0 that have their $(k_x - 1)$-st relation *unpaired* is at least $a - c + 1$.[2] First note that it is impossible for last element of N_0 (in the ordering) to be the $(k_x - 1)$ down-neighbor of *any* $x \in P_0$. Thus, the number of possible y_x is therefore $\leq |N_0| - 1 = c - 1$. Next, we show the map

$$P_0 \to N_0$$
$$x \mapsto y'_x$$

is injective when restricted to $x_1, \ldots, x_m$. Suppose, to the contrary, that $x_i \mapsto y'$ and $x_j \mapsto y'$. If $d(y') \leq u(x_i)$ or $d(y') \leq u(x_j)$, then y' doesn't pair a down-relation with its relation to x_i or x_j. So we must have $d(y') = u(x_i) + 1 = u(x_j) + 1$. But since $d(y') = u(x_i) + 1$, x_i is the last up neighbor that y' pairs an edge with. Similarly, x_j is the last up-neighbor that y' pairs an edge with so $x_i = x_j$. Thus each $x_1, \ldots, x_m$ map to a unique y'_{x_i}, of which there are $c - 1$ available. The remaining $\geq a - c + 1$ members of P_0 have an unpaired relation to their corresponding y'_x. That is, they each have ≥ 2 unpaired relations.

In summary, we started out with a poset that has $|A|$ relations across. We then

[2]Recall, P_0 is part of a down set; all of its across relations go to N_0.

paired $|W|$ relations across with the $|W|$ relations within, leaving behind $|A| - |W|$ unpaired relations across. Once that was done, everything in P still had at least one more relation across and at least $a-c+1$ of the things in P_0 had ≥ 2 relations across. Thus

$$|A| - |W| \geq |P| + a - c + 1,$$

as desired. □

Example 5.11. The bound $|A| - |W| \geq \max\{|P|, |N|\}$ is tight.

- Even chains have $|A| = \frac{n}{2} \cdot \frac{n}{2}$ relations across out of $|R| = \frac{n(n-1)}{2}$. Then

$$|A| - |W| = 2|A| - |R| = \frac{n^2}{2} - \frac{n^2 - n}{2} = \frac{n}{2} = |P| = |N|.$$

- Odd chains have $|A| = \frac{n-1}{2} \cdot \frac{n+1}{2} = \frac{n^2-1}{4}$ relations across out of $|R| = \frac{n(n-1)}{2}$. Then

$$|A| - |W| = 2|A| - |R| = \frac{n^2 - 1}{2} - \frac{n^2 - n}{2} = \frac{n-1}{2} = |P| = |N|$$

  (since $|Z| = 1$).

- $(1, n-2, 1)$-chains have $|A| = n - 1$ relations across out of $|R| = 2n - 3$. Then

$$|A| - |W| = 2|A| - |R| = (2n-2) - (2n-3) = 1 = |P| = |N|$$

  (since $|Z| = n - 2$).

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