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On a Family of Generalized Wiener Spaces and Applications

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ON A FAMILY OF GENERALIZED WIENER SPACES AND APPLICATIONS

by

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ON A FAMILY OF GENERALIZED WIENER SPACES AND APPLICATIONS

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We investigate the structure and properties of a variety of generalized Wiener spaces. Our main focus is on Wiener-type measures on spaces of continuous functions; our generalizations include an extension to multiple parameters, and a method of adjusting the distribution and covariance structure of the measure on the underlying function space.

In the second chapter, we consider single-parameter function spaces and extend a fundamental integration formula of Paley, Wiener, and Zygmund for an important class of functionals on this space. In the third chapter, we discuss measures on very general function spaces and introduce the specific example of a generalized Wiener space of several parameters; this will be the setting for the fourth chapter, where we extend some interesting results of Cameron and Storvick. In the final chapter, we apply the work of the preceding chapters to the question of reflection principles for single-parameter and multiple-parameter Gaussian stochastic processes.

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Whatever is of quality in this dissertation is built upon the foundation, earnest efforts, and good offices of those who preceded and assisted me; any failures or shortcomings are mine alone. I lay this work at the feet of God, the Father, Son, and Holy Spirit, to whom be all glory and honor unto the ages of ages. Amen.

Contents

Chapter 1

Background and Introduction

1.1 Background

For finite-dimensional spaces, Lebesgue measure serves as a canonical example; it has the important (and intuitive) properties of scale and translation invariance. However, it is well-known that there can be no reasonable translation invariant measure on an infinite-dimensional space. The problem of defining a somewhat reasonable measure on such a space was first solved by Norbert Wiener in [58]. Wiener constructed his measure on the space $C[0,1]$ of continuous functions on the unit interval. Elements of the support of this measure satisfy the conditions to be sample paths of an ordinary Brownian motion; thus Wiener's work made mathematically rigorous the model proposed by Albert Einstein (thus foreshadowing the significant applications to problems in mathematical physics which were to follow).

A standard Brownian motion is a model originally intended to describe the apparently random motion of a particle moving in a liquid. For a Brownian motion, the particle trajectories must satisfy several conditions, the first being continuity. In addition, the position at time t must be normally distributed with mean 0 and variance t, and this distribution must have independent increments. Viewing the path of a particle as a stochastic process and using these properties, one can show that the transition probabilities for a discretization of the process using the partition $0 = t_0 < t_1 < \ldots < t_n = 1$ have density functions

$$\frac{1}{\sqrt{2\pi(t_{j+1} - t_j)}} \exp\left(-\frac{(u_{j+1} - u_j)^2}{2(t_{j+1} - t_j)}\right),$$

as per Einstein's proposed model.

Wiener's task was to start with sets defined in terms of finite discretizations of this type, and from them to obtain a countably additive set function on the Borel algebra of subsets of $C[0, T]$. This is quite involved, and is all the more impressive when one recalls that the Lebesgue Theory was really quite new at the time, and that much of the modern machinery of functional analysis was unavailable to Wiener.

From the basic facts discussed above we can easily obtain the ability to integrate "tame" functionals of the form $F(x) = f(x(t_1), \ldots, x(t_n))$, where $f : \mathbb{R}^n \to \mathbb{C}$ is Lebesque measurable against Wiener's measure $\mathfrak{w}$ using the formula

$$\int_{C[0,1]} F(x)\mathfrak{w}(dx) = \left(\prod_{j=1}^{n} (2\pi(t_j - t_{j-1}))\right)^{-\frac{1}{2}} \int_{\mathbb{R}^n} f(u_1, \ldots, u_n) \exp\left(-\frac{1}{2}\sum_{j=1}^{n} \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}}\right) \mathrm{d}u_n \cdots \mathrm{d}u_1. \quad (1.1)$$

A considerable amount of work was accomplished with little more than this formula and very clever estimation and limiting arguments, and the additional theorem from [42] of Paley, Wiener, and Zygmund stating that for $\{h_1, \ldots, h_n\}$ an orthonormal collection (in the L^2-sense) of functions of bounded variation on $[0, 1]$ and for $F : C[0, 1] \to \mathbb{C}$ defined by

$F(x) = f\left(\int_0^1 h_1 \mathrm{d}x, \dots, \int_0^1 h_n \mathrm{d}x\right)$, the following holds:

$$\int_{C[0,1]} F(x)\mathfrak{w}(dx) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(u_1, \dots, u_n) \exp\left(-\sum_{j=1}^n \frac{u_j^2}{2}\right) \mathrm{d}u_n \cdots \mathrm{d}u_1. \tag{1.2}$$

Perhaps the best-known contributers to the field following Paley, Wiener, and Zygmund were R.H. Cameron and W.T. Martin and their disciples. Of great significance was the Cameron-Martin translation theorem, which drew on the underlying structure hinted at by (1.2). This translation theorem showed that while the Wiener measure was not translation invariant, it was quasi-invariant with respect to translation by a certain subset of $C[0,1]$. Now called the Cameron-Martin space, this is the collection of absolutely continuous functions with L^2 derivatives, (*i.e.* the Sobolev space $H_0^2[0,1]$). If $T : C[0,1] \to C[0,1]$ is defined by $x \mapsto x + x_0$, where x_0 is in the Cameron-Martin Space and x_0' is of bounded variation, then the translation theorem asserts that the Radon-Nicodym derivative of the measure $\gamma \circ T$ with respect to the untranslated measure γ is given by

$$\frac{d(\gamma \circ T)}{d\gamma}(x) = \exp\left(-\frac{1}{2}\left|\left|x_0'\right|\right|_2^2 - \int_0^1 x_0'(t)\mathrm{d}x(t)\right). \tag{1.3}$$

The condition that x_0' be of bounded variation can be relaxed by substituting a stochastic integral for the Riemann-Stieltjes integral in (1.3). This translation theorem has many significant implications. As a connection to the theory of probability, we note also that this is a special case of the Girsanov Theorem.

Considerable development and generalization was accomplished by many of Cameron's students and associates; of particular note for my research have been Donsker, Baxter, Varberg, Kuelbs, Yeh, C. Park, and Skoug. Two forms of generalization are of particular interest to us; the first is in considering a wider variety of transition probability densities and the second is in allowing for parameter sets other than the interval $[0,1]$.

Speaking quite generally, a measure γ on a Banach space E is Gaussian if for every $x^* \in E^*$, the "push forward" of γ by x^* is a Gaussian measure on $\mathbb{R}$; that is, the image measure $\gamma \circ (x^*)^{-1}$ is either a Dirac mass δ_a or has density

$$\phi(u) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(u-a)^2}{2\sigma^2}\right)$$

for some $a \in \mathbb{R}$ and $\sigma > 0$. Our discussion will be primarily concerned with Gaussian measures on function spaces, including the Lebesgue spaces $L^p(S : \nu)$, where ν is a Borel measure on the compact metric space S and $1 \leq p < \infty$, and particularly with the "canonical" space $C(S)$ of continuous functions from S to $\mathbb{R}$, equipped with usual supremum norm. A considerable amount of work has been done to describe the behavior and structure of Gaussian measures on very general spaces. A compendium of many of these results is found in Bogachev's informative text [5].

In 1968, Leonard Gross simultaneously codified and extended the relationship of the Cameron-Martin space, the larger Banach space containing it, and the corresponding Gaussian measure γ in [28], christening the construction an *Abstract Wiener Space* (AWS). This development opened a floodgate of new work in the area, and additional significant results were obtained by many researchers; some commonly seen names include Dudley, Feldman, LeCam, Kallianpur, and Kuo. A good representation of where the subject stood in 1975 can be found in [36] and in [2]. A more recent retrospective is given by Strook in [53].

Gross' work on Abstract Wiener Spaces was partially inspired by the extension of the Cameron-Martin translation theorem obtained by his advisor Segal, and drew on the notion of a cylindrical measure (for a very general treatment see [49]). Briefly, the basic idea of the AWS recognizes that the structure of a Gaussian measure is captured by its Cameron-Martin space; in some sense, the larger Banach space is merely needed to "fill in" what is lacking in a Hilbert space in order to allow the canonical centered cylindrical Gaussian measure having

Fourier transform

$$\hat{\gamma}(f) = \exp\left(-\frac{1}{2}\,||f||^2\right) \tag{1.4}$$

to extend to a countably additive measure with nontrivial support. This is probably the most studied case of what is generally termed the radonification problem: when does a cylindrical set measure on a Banach space extend to a countably additive Radon measure? For interesting and general work on the radonification problem, one might consult Fernique, Fernholz, Linde, Pietsch, Kwapién, etc. Van Neerven has a very nice survey article [57] that discusses many of these endeavors.

1.2 Overview

A natural starting point is to begin with a thorough investigation of the spaces $C_{a,b}[0,T]$. Here we begin with the "classical" formulation used extensively by Chang, Chung, Choi, Ryu, Skoug, Yeh, in [10, 11, 12, 65], and by others.

Briefly, these spaces are quite natural extensions of Wiener's original construction, with a structure determined by the $\mathbb{R}$-valued functions a and b defined on $[0,T]$, for which the transition probabilities are of the form

$$\frac{1}{\sqrt{2\pi[b(t_{j+1}) - b(t_j)]}} \exp\left(-\frac{[u_{j+1} - a(t_{j+1}) - u_j + a(t_j)]^2}{2[b(t_{j+1}) - b(t_j)]}\right). \tag{1.5}$$

Generally, the function b has been taken to be continuous and increasing and the function a taken continuous. The choice of a and b has important consequences for the properties of the measure thus obtained. Most formulations (*e.g.* [11, 13, 64]) have further stipulated that b be continuously differentiable with b' bounded away from 0 and a absolutely continuous with L^2 derivative.

Chapter 1 is focused on understanding these single-parameter spaces, with the aim of understanding the effect of the choice of a and b on the measure. One result, which appeared in [48], is an extension of (1.2) to the generalized Wiener space setting with less restrictive conditions on the choice of functions $\{h_1, \dots, h_n\}$.

One of the original motivations for the general direction of this research was to formulate a suitable generalization of the spaces $C_{a,b}[0,T]$, both in terms of the number of parameters and in terms of the available choices of a and b. Moreover, our intent is to formulate this in a manner that preserves the form of basic tools and techniques wherever possible and clearly demonstrates any limitations that the more general setting might necessitate.

If we take $Q = \prod_{j=1}^{d}[0, T_j]$ as our parameter space and functions a and b defined on Q with a certain amount of regularity, we can obtain suitable Gaussian measures on $C(Q)$ that directly extend the single-parameter construction. In this setting, one can obtain appropriate formulations of (1.2) and (1.3), and then it is fairly easy to extend a considerable amount of work from the single-parameter setting to the multiple parameter setting.

In considering the previous ideas, questions arose about the abstract structure underlying the AWS construction. While there has been considerable work done with extremely general spaces elsewhere (*e.g.* locally convex spaces, Banach spaces), here we trade generality in exchange for a more tractable collection of problems whose solutions can be expressed in a concrete fashion, and have focused on Banach spaces of real-valued functions.

Classical Wiener space theory usually begins by choosing a particular Banach space and a desired covariance structure for the measure, and then finding some suitable Hilbert space contained in the Banach space that will serve as the Cameron-Martin space. Meanwhile, abstract Wiener space theory typically starts with a pre-chosen Hilbert space and then finds a suitable Banach space in which the Hilbert space can be embedded and function as the Cameron-Martin space for a non-trivial Gaussian measure. In some sense, one is left with a "chicken and egg" problem, for in the first situation the necessary Hilbert space may not be desirable, while in the latter the resulting Banach space may not be desirable. One aim of this work is to strike a reasonable balance between these approaches.

By way of example, taking S to be a compact metric space, ν a Borel measure on S, and $C(S)$ as our function space, a basic structure of interest is shown in the following diagram.

$$\begin{tikzcd}
C(S)^* \arrow[dd, "TT^*"] \arrow[dr, "T^*"] & \\
& L^2(S;\nu) \arrow[dl, "T"] \\
C(S) &
\end{tikzcd}$$

In this example, the measure γ on $C(S)$ is centered with Fourier transform (or characteristic function)

$$\hat{\gamma}(\mu) = \exp\left(-\frac{1}{2}\langle TT^*\mu, \mu\rangle\right) = \exp\left(-\frac{1}{2}\left(T^*\mu, T^*\mu\right)_{L^2(S;\nu)}\right).$$

The structure and properties of the maps T and T^* will ultimately determine the properties of the measure obtained on $C(S)$ having TT^* for its covariance operator. In Chapter 3, we demonstrate a method for constructing these maps in such a manner that the resulting measure γ on $C(S)$ satisfies given desired properties.

Given these families of Gaussian measures, a question of interest is to determine which general properties are common to all and which are dependent upon the particular choices of a, b, and parameter space. One such question involves the idea of reflection; it is well-known that an ordinary Brownian motion process exhibits this property.

In the $C_{a,b}[0,T]$ spaces, if the point evaluation process $x(t)$ is not centered (*i.e.* is not mean 0, which is equivalent to the condition that a is not the zero function) then it is not hard to see that it should not satisfy the reflection principle. It will however satisfy a reflection principle *about its mean*; that is, the process $X(t,x) = x(t) - a(t)$ will have the appropriate distribution.

In the multiple-parameter case, the question becomes less clear. To begin, what is meant by "reflection" in this setting? In Chapter 5, we will investigate several possible notions that could serve as the analogue to the reflection principle. The answers to these questions about reflection can then be applied to barrier-crossing problems for the appropriate associated stochastic processes.

1.3 Questions for Future Work

Much of our original motivation described above still remains to be accomplished. This is to continue the program of generalizing and extending the body of work already completed for $C_{a,b}[0,T]$ to the more general function spaces we have obtained. Which constructions generalize easily and which are more difficult? Which properties are essentially independent of parameter dimension, drift, and covariance structure, and which must be investigated on a case by case basis? As noted above, the reflection principle is a good example of a such a property.

Note also that the choice of $Q = \prod_{j=1}^{d}[0,T_j]$ is a fairly limiting assumption, though it has the advantage of imposing a very clear structure. It may be interesting to investigate the effect of replacing Q with a finite-dimensional manifold $\mathcal{M}$. How does the choice of smooth structure on $\mathcal{M}$ affect the measure γ on $C(\mathcal{M})$?

Along the same lines, what is the changed when constructing γ on the space $C(\mathcal{M}, \mathcal{N})$, where both $\mathcal{M}$ and $\mathcal{N}$ are manifolds? The case where $\mathcal{M} = [0,1]$ with a suitably adjusted Wiener measure has been the subject of a considerable amount of work; for instance, see [17, 18, 19, 27, 29].

As an extension of these ideas, it is possible to obtain or construct a suitable notion of the space of cádlag functions (also commonly called Skorohod Space) over d-parameters, and if possible determine necessary and sufficient conditions to assure that $\mathrm{supp}(\gamma) = D$ for

Gaussian measures γ of the sort considered above? It seems that this should be a natural space of which such measures should live, as the reproducing kernel $K(s,t)$ is well-defined in a pointwise fashion (and hence all elements of the Cameron-Martin space for γ). Thus it seems reasonable that elements in $\mathrm{supp}(\gamma)$ should behave somewhat reasonably. These spaces would occupy a sort of middle ground between Lebesgue spaces and spaces of continuous functions.

Chapter 2

Single Parameter Spaces

2.1 An Introduction to the Function Space $C_{a,b}[0,T]$

Let $C_0[0,T]$ denote a one-parameter Wiener space; that is the space of real-valued continuous functions $x(t)$ on $[0,T]$ with $x(0) = 0$. Let $\mathcal{M}$ denote the class of all Wiener-measurable subsets of $C_0[0,T]$ and let $\mathfrak{w}$ denote Wiener measure. Then $(C_0[0,T], \mathcal{M}, \mathfrak{w})$ is a complete measure space and we denote the Wiener integral of a Wiener integrable functional F by

$$\int_{C_0[0,T]} F(x)\mathfrak{w}(dx).$$

The Wiener process found in [8, 30, 31, 44, 45, 63] is stationary in time and is free of drift. We shall concern ourselves with a more general class of stochastic processes which may be non-stationary in time and subject to drift.

A generalized Brownian motion process is a real-valued stochastic process $X(t)$ on a probability space $(\Omega, \mathcal{A}, P)$ with parameter space $[0,T]$ if $X(0,x) = 0$ almost surely on Ω and for $0 = t_0 < t_1 < \ldots < t_n \leq T$, the density function for the random vector $(X(t_1,x), \ldots, X(t_n,x))$ is given by

$$\left(\prod_{j=1}^{n} 2\pi(b(t_j) - b(t_{j-1}))\right)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\sum_{j=1}^{n}\frac{((u_j - a(t_j)) - (u_{j-1} - a(t_{j-1})))^2}{b(t_j) - b(t_{j-1})}\right), \tag{2.1}$$

where $a(t)$ is a continuous function on $[0,T]$ satisfying $a(0) = 0$ and $b(t)$ is a continuous, strictly increasing function on $[0,T]$ satisfying $b(0) = 0$. In Chapter 3 of [66], Yeh shows that the generalized Brownian motion process determined by a and b is a Gaussian process with mean function $a(t)$ and covariance function $r(s,t) = \min\{b(s), b(t)\}$.

For the present we will adopt the formulation of [11], though we will later see that the associated requrements can be relaxed considerably while retaining a workable theory. Let a and b be functions defined on $[0,T]$ with $a' \in L^2[0,T]$ and b' continuous, positive, and bounded away from 0 on $[0,T]$. Observe that a and b are absolutely continuous and b is strictly increasing on $[0,T]$, and so one can define a generalized Brownian motion as above. We desire to take $\Omega = C_0[0,T]$ and find a measure $\mathfrak{m}$ on $C_0[0,T]$ with respect to which the coordinate evaluation map $\delta_t : C_0[0,T] \to \mathbb{R}$ by $x \mapsto x(t)$ is the generalized Brownian motion process determined by a and b.

For $n = 1, 2, \ldots$, let $0 = t_0 < t_1 < \cdots < t_n \leq T$. Take the collection of finite-dimensional distributions on $\mathbb{R}^n$ given by the density function in (2.1), which we will express as

$$W_n(\mathbf{t};\mathbf{u}) = \left(\prod_{i=1}^{n} 2\pi\Delta_i b(t)\right)^{-\frac{1}{2}} \exp\left(-\sum_{i=1}^{n}\frac{(\Delta_i(\mathbf{u} - a(t)))^2}{2\Delta_i b(t)}\right), \tag{2.2}$$

where $\Delta_i b(t) = b(t_i) - b(t_{i-1})$ and $\Delta_i(\mathbf{u} - a(t)) = u_i - a(t_i) - u_{i-1} + a(t_{i-1})$. We will refer to $W_n(\mathbf{t};\mathbf{u})$ as the generalized Wiener kernel.

Let $C_0[0,T]$ be the space of continuous functions on $[0,T]$ for which $x(0) = 0$, equipped with the supremum norm $||x|| = \sup_{0\leq t\leq T}|x(t)|$. We want a measure $\mathfrak{m}$ on $C_0[0,T]$ that has finite-dimensional distributions $\{\mathfrak{W}(\mathbf{t}) : \mathbf{t} \in R^n;\ n = 1, 2, \ldots\}$, each having density

$W_n(\mathbf{t};\mathbf{u})$; that is for $0 = t_0 < t_1 < \cdots < t_n \leq T$ and Lebesgue measurable $E \subset \mathbb{R}^n$, if $\tilde{E} = \{x \in C_0[0,T] : (x(t_1), \ldots x(t_n)) \in E\}$, then

$$\mathfrak{m}\left(\tilde{E}\right) = \int_{\tilde{E}} d\mathfrak{W}(\mathbf{t}) = \int_E W_n(\mathbf{t};\mathbf{u})\mathrm{d}\mathbf{u}. \tag{2.3}$$

For $0 = t_0 < t_1 < \cdots < t_n \leq T$ and $\mathbf{d}$ and $\mathbf{e}$ in $\mathbb{R}^n$, define $I(\mathbf{t};\mathbf{d},\mathbf{e})) = \{x \in \mathbb{C}_0[0,T] : d_i < x(t_i) \leq e_i$ for $i = 1, 2, \ldots, n\}$. We will refer to these sets as cylinder sets or intervals. Put

$$\mathfrak{W}(\mathbf{t})(I(\mathbf{t};\mathbf{d},\mathbf{e})) = \int_{I(\mathbf{t};\mathbf{d},\mathbf{e})} d\mathfrak{W}(\mathbf{t}) = \int_{\prod_{j=1}^n [d_j,e_j]} W_n(\mathbf{t};\mathbf{u})\mathrm{d}\mathbf{u}. \tag{2.4}$$

From this beginning, each distribution $\mathfrak{W}(\mathbf{t}_0)$ can be extended to a probability measure on the sigma algebra generated by the collection $\{I(\mathbf{t}_0;\mathbf{d},\mathbf{e})\}$ by the usual Carathéodory extension process.

We will need the following proposition. It is stated with its proof in Chapter 3 of [33].

Proposition 1 (Chapman-Kolmogorov Equation)**.** *Let r, s, and t be real numbers with $r < s < t$ and let $\lambda > 0$. Then*

$$\int_{\mathbb{R}} \left(\frac{\lambda}{2\pi(t-s)}\right)^{\frac{1}{2}} \exp\left(-\frac{\lambda(w-v)^2}{2(t-s)}\right) \left(\frac{\lambda}{2\pi(s-r)}\right)^{\frac{1}{2}} \exp\left(-\frac{\lambda(v-u)^2}{2(s-r)}\right) \mathrm{d}v$$
$$= \left(\frac{\lambda}{2\pi(t-r)}\right)^{\frac{1}{2}} \exp\left(-\frac{\lambda(w-u)^2}{2(t-r)}\right). \tag{2.5}$$

Coupled with careful bookkeeping, the Chapman-Kolmogorov Equation can be used to demonstrate that the family of distributions $\{\mathfrak{W}(\mathbf{t}) : \mathbf{t} \in R^n;\ n = 1, 2, \ldots\}$ is consistent; that is if $\mathbf{t} = (t_1, \ldots, t_n)$ and $\mathbf{t}' = (t_1, \ldots, t_{j-1}, t', t_j, \ldots, t_n)$ and $A \subseteq \mathbb{R}^n$ is measurable with respect to the distribution $\mathfrak{W}(\mathbf{t})$, then

$$\mathfrak{W}(\mathbf{t})(A) = \int_A d\mathfrak{W}(\mathbf{t}) = \int_{A\times\mathbb{R}} d\mathfrak{W}(\mathbf{t}') = \mathfrak{W}(\mathbf{t}')(A \times \mathbb{R}). \tag{2.6}$$

By the celebrated theorem of Kolmogorov (see [35, 46]), the consistency of this family of distributions implies that there is a Radon probability measure on $\mathbb{R}^{[0,T]}$ that has the desired finite-dimensional distributions $\{\mathfrak{W}(\mathbf{t})\}$. It only remains to show that the space $C_0[0,T]$ supports this measure.

There are several approaches to this. One approach is to use an adaptation of the Kolmogorov-Čentsov Theorem (see [35]) to demonstrate that the sample paths for the stochastic process induced by the measure just obtained are almost surely continuous. This is in some sense unsatisfying, because the space $\mathbb{R}^{[0,T]}$ on which the measure is defined is not $C_0[0,T]$ as desired.

In fact, it is well-known that $C_0[0,T]$ is not even an element of the Borel class for $\mathbb{R}^{[0,T]}$. By a theorem of Doob [16], one could show that the outer measure of $C_0[0,T]$ is 1, and then the desired measure can be obtained by simply using the outer measure and intersecting $C_0[0,T]$ with the Borel class of $\mathbb{R}^{[0,T]}$. This is the approach that Yeh takes in constructing the ordinary Wiener measure in [66].

Another approach is an adaptation of that taken by Nelson in [38] and [39] and discussed in [24]. Nelson's solution is to adapt Kolmogorov's theorem to obtain a measure on $(\mathbb{R}^*)^{[0,T]}$, where $\mathbb{R}^*$ is the one-point compactification of $\mathbb{R}$. Then $C_0[0,T]$ is a Borel set for this space, as

$$C_0[0,T] \cup \{x_\infty\} = \bigcap_{n=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{\substack{0 \le s,t \le n \\ |s-t|<1/k}} \left\{ x : |x(s) - x(t)| \le \tfrac{1}{j} \right\},$$

where x_∞ is the function that always takes the value ∞ in $\mathbb{R}^*$. It then suffices to demonstrate that the measure of the complement of this set is 0, which is facilitated by the fact that the measure is Radon (and hence inner regular), which allows the estimation of the measure of the sets

$$\bigcup_{\substack{s,t\in[0,T]\\ |s-t|<1/k}} \left\{x : |x(s) - x(t)| > \tfrac{1}{j}\right\}$$

from within by compact sets. This in turn depends on the estimate

$$\begin{aligned}\sup_{t-s\leq\delta} \left(\frac{2}{\pi(b(t)-b(s))}\right)^{\frac{1}{2}} \int_{\varepsilon}^{\infty} \exp\left(-\frac{[u-(a(t)-a(s)]^2}{2(b(t)-b(s))}\right) \mathrm{d}u \\ \leq \frac{2\sqrt{2(b(t)-b(s))}}{[\varepsilon-(a(t)-a(s))]\sqrt{\pi}} \exp\left(-\frac{[\varepsilon-(a(t)-a(s))]^2}{2(b(t)-b(s))}\right) \quad (2.7)\end{aligned}$$

for $0 \leq s < t$ and for δ sufficiently small that $a(t) - a(s) < \varepsilon$. With these adjustments, the existence of the desired measure is obtained in generally the same fashion as found in Section 10.5 of [24].

Of course, one could also do as Wiener did in [58] and build the desired measure from scratch by building a set function with the desired finite-dimensional distributions and then proving countable additivity. Whichever method is used, a probability measure $\mathfrak{m}$ is obtained on the Borel class of $C_0[0,T]$ having finite-dimensional distributions given by $\{\mathfrak{W}(\mathbf{t})\}$. This measure can then be completed in the usual manner. The resulting function space, which we denote by $C_{a,b}[0,T]$, is considered by Yeh in [65] and by Yeh and Hudson in [67] and was investigated extensively by Chang and Chung in [12] and Chang and Skoug in [13].

Observe that the functions a and b induce a Lebesgue-Stieltjes measure $\nu_{a,b}$ on $[0,T]$ by

$$\nu_{a,b}(E) = \int_E \mathrm{d}(b(t) + |a|(t)) = \int_E \mathrm{d}b(t) + \int_E \mathrm{d}\,|a|(t) = \nu_b(E) + \nu_{|a|}(E),$$

for Lebesgue measurable $E \subseteq [0,T]$, where $|a|$ denotes the total variation of a. This leads to the following definition.

Definition 1. Define the space $L^2_{a,b}[0,T]$ to be the space of functions on $[0,T]$ that are square integrable with respect to the measure induced by a and b. That is,

$$L^2_{a,b}[0,T] = \{f : [0,T] \to \mathbb{R} : \int_0^T f^2(t)\mathrm{d}(b(t) + |a|(t)) < \infty\}.$$

The space $L^2_{a,b}[0,T]$ is in fact a Hilbert space (as our notation suggests), and has the obvious inner product,

$$(f,g)_{a,b} = \int_0^T f(t)g(t)\mathrm{d}(b(t) + |a|(t)).$$

As $b' > 0$, the measure ν_b is mutually absolutely continuous with Lebesgue measure. Note that the measure $\nu_{|a|}$ is absolutely continuous with respect to Lebesgue measure, but that the converse need not hold. Thus $\nu_{a,b}$ is absolutely continuous with respect to Lebesgue measure and $L^2_{a,b}[0,T] \subseteq L^2[0,T]$ in general, with $L^2_{0,b}[0,T] = L^2[0,T]$ in the case where a is the zero function.

We briefly discuss some important facts and theorems that will prove useful. The first is the analog to the 'Wiener Integration Formula' for ordinary Wiener space $C_0[0,T]$. This is a well-known and often used theorem. It follows quite naturally from equation (2.2) and from the definition of the function space integral; see [33] for the proof in the case of $C_0[0,T]$ and make the obvious adaptations.

Theorem 1 (Tame Functionals)**.** *Let $0 = t_0 < t_1 < \cdots < t_n \leq T$ and let $F(x) = f(x(t_1),\ldots,x(t_n))$, where $f : \mathbb{R}^n \to \mathbb{C}$. Then F is $\mathfrak{m}$-measurable if and only if f is Lebesgue measurable and*

$$\int_{C_{a,b}[0,T]} F(x)\mathfrak{m}(dx) \overset{*}{=} \int_{\mathbb{R}^n} f(\mathbf{u})W_n(\mathbf{t};\mathbf{u})\mathrm{d}\mathbf{u}, \tag{2.8}$$

where $\overset{*}{=}$ is strong in the sense that if one side exists, then the other side exists with equality.

The following useful facts can be proven using Theorem 1. For $t \in [0,T]$, put $X(t,x) =$

$x(t)$, and then by (2.8), the first and second moments of $X(t, x)$ are

$$\mathbb{E}[X(t,x)] = \mathbb{E}[x(t)] = \int_{C_{a,b}[0,T]} x(t)\mathfrak{m}(dx) = a(t), \tag{2.9}$$

and

$$\mathbb{E}[X(t,x)^2] = \mathbb{E}[x(t)^2] = \int_{C_{a,b}[0,T]} x^2(t)\mathfrak{m}(dx) = b(t) + a^2(t). \tag{2.10}$$

It is also straightforward to compute the covariance of $X(t_1, x)$ and $X(t_2, x)$; for then

$$\mathbb{E}[X(t_1,x)X(t_2,x)] = \int_{C_{a,b}[0,T]} x(t_1)x(t_2)\mathfrak{m}(dx) = \min\{b(t_1), b(t_2)\} + a(t_1)a(t_2). \tag{2.11}$$

The next theorem is analogous to the celebrated translation theorem of Cameron and Martin found in [7] and has been used extensively. The statement we give here is from [13] and a proof can be found in [12].

Theorem 2 (Translation Theorem)**.** *Let a and b be defined as above. Let $z \in L^2_{a,b}[0,T]$ and let $x_0(t) = \int_0^t z(s)\mathrm{d}b(s)$. If F is an integrable function on $C_{a,b}[0,T]$, then*

$$\mathbb{E}[F(x + x_0)] = \exp\left(-\frac{1}{2}\int_0^T z^2(s)\mathrm{d}b(s) - \int_0^T z(s)\mathrm{d}a(s)\right)\mathbb{E}[F(x)\exp\left(\langle z, x\rangle\right)]. \tag{2.12}$$

The translation theorem shows that it is possible to compute the Radon-Nikodym derivative of the measure $\mathfrak{m} \circ T_{x_0}$ induced by a translation $x \mapsto x + x_0$ provided that the translating function x_0 'behaves nicely'. In the classical Wiener space $C_0[0,T]$, absolutely continuous functions with derivatives in $L^2[0,T]$ are the class of functions for which the translation theorem holds. This collection of functions is commonly known as the Cameron-Martin space and is known to be dense and of zero measure in $C_0[0,T]$. In our setting, we observe that the existence of the integral $\int_0^T x_0'(t)\mathrm{d}a(t)$ is necessary in order to use the translation theorem. In view of the possibly strict containment of $L^2_{a,b}[0,T]$ in $L^2[0,T]$, this may decrease the available pool of functions by which one can translate, depending on the behavior of the function a. This leads us to make some further observations.

In the original statement of the translation theorem in [7] and in certain other statements since, the hypothesis on the translating function has been given as "let x_0 be a given function in $C_0[0,T]$ with a first derivative x_0' of bounded variation on $0 \leq t \leq T$." We note that this hypothesis can be understood in several ways.

First, one could take this statement to mean that x_0' is derived from x_0 in the purely classical sense and exists on all of $[0,T]$. As Darboux's Theorem prohibits derivatives obtained in this manner from having jump discontinuities, this would essentially force x_0 to have a continuous derivative on $[0,T]$. While we can certainly translate by such functions, this is a very strong condition that needlessly reduces the available pool of translators. It seems clear that this is not the understanding that was originally intended.

A second view that one could take is that x_0 is only required to have a classical derivative almost everywhere on $[0,T]$ and that on the exceptional set the derivative is assigned appropriate values to assure that it is of bounded variation. While this initially seems to be a viable option, it is not tenable. To see why, let x_0 be the classic Cantor function on $[0,1]$. Note that x_0 is continuous with $x_0(0) = 0$ and that $x_0'(t)$ exists and is equal to zero almost everywhere on $[0,1]$. Define $x_0'(t)$ in any appropriate manner on the exceptional set and then we attempt to apply the translation theorem. By inspection, the translation theorem shows that $\mathfrak{m}(B_1(x_0)) = \mathfrak{m}(B_1(0))$. Take $x_k(t) = 2kx_0(t)$ for $k = 1, 2, \ldots$, and we obtain a sequence of functions satisfying $||x_k - x_m||_\infty \geq 2$ whenever $k \neq m$. Moreover, $\mathfrak{m}(B_1(x_k)) = \mathfrak{m}(B_1(0))$ for each k. As the collection $\{B_1(x_k)\}$ is disjoint, we obtain the obvious contradiction

$$1 = \mathfrak{m}(C_0[0,1]) \geq \sum_{k=1}^{\infty} \mathfrak{m}(B_1(x_k)) \sum_{k=1}^{\infty} \mathfrak{m}(B_1(0)) = \infty.$$

Lastly, one can understand the hypothesis in the translation theorem to mean that x_0 is absolutely continuous in some appropriate sense (for instance, in the classical case, having the form $x_0(t) = \int_0^t x_0'(s)\, ds$ for some function x_0' of bounded variation on $[0,T]$). This view is prevalent in most of the more recent statements of the translation theorem. Moreover, by the appropriate limiting processes, one can show that this requirement can be relaxed further, and that it is sufficient for x_0' to be square-integrable in an appropriate sense for the space $C_{a,b}[0,T]$ under consideration. This is the understanding that captures the essential property of allowable translations.

Finally, we note that the hypothesis that $z \in L^2_{a,b}[0,T]$ is actually stronger than necessary for (2.12) to hold. It is sufficient that the function z be merely integrable with respect to the measure $\nu_{|a|}$ instead of square-integrable. We will defer a demonstration of this fact until a later discussion of the translation theorem in a more general setting.

In the remainder of this chapter we have two main goals. The first is to develop machinery and techniques that we will use in subsequent chapters. The second is to establish useful tools for integrating certain functionals on $C_{a,b}[0,T]$.

It is often desirable to consider functionals of the form $F(x) = f(\langle\theta, x\rangle)$, where $\langle\theta, x\rangle$ is the Paley-Wiener-Zygmund stochastic integral of the function $\theta \in L^2[0,T]$. Functionals of this type naturally appear in the Cameron-Martin translation theorem and in virtually all of the integral transform theory for functionals on Wiener space. The first integration formula of this type for functionals on $C_0[0,T]$ was established by Paley, Wiener, and Zygmund in [42] and can be found in [41]. A shorter, modern proof of this theorem was given by Yeh in [66]. Applications of this theorem abound in the literature; for example see [30, 31, 44, 45, 63]. In [23], Robert Ewan obtained a generalization of this result. In Section 2.2 we define the PWZ integral and demonstrate some of its important properties. In Section 2.3 we obtain a generalization of the Paley-Wiener type integration formula for the function space $C_{a,b}[0,T]$.

2.2 Stochastic Integrals on $C_{a,b}[0,T]$

For our purposes, we broadly classify stochastic integrals into three types. These are time integrals, Paley-Wiener-Zygmund integrals (often called simply Paley-Wiener or Wiener integrals), and Itô-type integrals. Our results in this section are concerned with the first two types. We will use fairly classical methods in our work in this section. In later chapters similar notions will be explored using somewhat different techniques.

We begin with a brief discussion of our first object of interest, the time integral (or parameter integral). Simply put, this is the Reimann integral of a function of the continuous random variable $X(t,x) = x(t)$ with respect to the parameter t; thus the time integral of $X(t,x)$ is a random variable $Y(x)$ that satisfies

$$Y(x) = \int_0^T F(t, X(t,x))\mathrm{d}t = \int_0^T F(t, x(t))\mathrm{d}t, \tag{2.13}$$

where $F(t, x(t))$ is a functional on $[0,T] \times C_{a,b}[0,T]$ that is Riemann integrable on $[0,T]$.

The study of the Feynman integral provides ready examples of the utility of the time integral. The Feynman-Kac formula represents an important step in the process of providing a rigorous definition of the Feynman integral. A detailed explanation of this formula can be found in [33]. The Feynman-Kac functional is given by

$$F(x) = \exp\left(-\int_0^t V(s, X(s,x))\mathrm{d}s\right), \tag{2.14}$$

where $X(t,x)$ is a standard Brownian motion process and $V : [0,T] \times C_0[0,T] \to \mathbb{C}$. Some simple examples of time integrals include

$$F_{\alpha,\beta}(x) = \int_0^T \theta(t, \alpha x(t) + \beta)\mathrm{d}t, \tag{2.15}$$

where $\alpha, \beta \in \mathbb{R}$ and $\theta : [0, T] \times \mathbb{R} \to \mathbb{R}$ is bounded and continuous, and also

$$G(x) = \left(\int_0^T x(t)\mathrm{d}t \right)^2, \tag{2.16}$$

$$H(x) = \int_0^T x^2(t)\mathrm{d}t. \tag{2.17}$$

Calculations involving the expectation of the first two examples are important in the study of the Feynman integral. For more examples of functionals involving time integrals see [9, 32, 33].

The Paley-Wiener-Zygmund (PWZ) integral is a simple type of stochastic integral that proves extremely useful in the study of the space $C_{a,b}[0, T]$. Given any complete orthonormal set of functions $\{\phi_j\}_{j=1}^{\infty}$ of bounded variation on $[0, T]$ and any function $f \in L^2_{a,b}[0, T]$, we can write $f = \lim_{n\to\infty} f_n$, where $f_n(t) = \sum_{j=1}^{n} (f, \phi_j)_{a,b}\phi_j(t)$. As each ϕ_j is of bounded variation, the Reimann-Stieltjes integral $\int_0^T \phi_j(t)\mathrm{d}x(t)$ exists for every $x \in C_{a,b}[0, T]$, and thus for $f \in L^2_{a,b}[0, T]$, the integral

$$\int_0^T f_n(t)\mathrm{d}x(t) = \int_0^T \sum_{j=1}^{n} (f, \phi_j)_{a,b}\phi_j(t)\mathrm{d}x(t) = \sum_{j=1}^{n} (f, \phi_j)_{a,b} \int_0^T \phi_j(t)\mathrm{d}x(t)$$

is well-defined for each n and for every $x \in C_{a,b}[0, T]$.

Definition 2. Let $\{\phi_j\}_{j=1}^{\infty}$ be a complete orthonormal set of functions of bounded variation in $L^2_{a,b}[0, T]$ and let $f_n(t) = \sum_{j=1}^{n} (f, \phi_j)_{a,b}\phi_j(t)$. For $f \in L^2_{a,b}[0, T]$, we define the Paley-Wiener-Zygmund stochastic integral by the formula

$$\langle f, x \rangle = \lim_{n\to\infty} \int_0^T f_n(t)\mathrm{d}x(t)$$

for all $x \in C_{a,b}[0, T]$ for which this limit exists.

We will presently demonstrate the following important properties of the PWZ integral.

1. $\langle f, x\rangle$ exists for almost every $x \in C_{a,b}[0,T]$.

2. The PWZ integral is essentially independent of the choice of the orthonormal set used to define it. Any complete orthonormal set of functions of bounded variation can be used to define $\langle f, x\rangle$.

3. $\langle f, x\rangle$ has the usual linearity properties.

4. If a function f is of bounded variation on $[0,T]$, then $\langle f, x\rangle$ is equal to the Riemann-Stieltjes integral $\int_0^T f(t)\mathrm{d}x(t)$ for almost every $x \in C_{a,b}[0,T]$.

Before proving these four facts, we will first establish some basic properties of the PWZ integral as a random variable. Yeh establishes similar properties for a standard Brownian Motion process in Chapter 5 of [66]; our development is quite similar. Some adjustments are required due to the fact that our generalized Brownian Motion process is dependent on the functions a and b. The following lemma is essentially due to Yeh, with a few necessary changes.

Lemma 1. *Let $S[0,T]$ be the collection of simple functions of the form $\varphi(t) = \sum_{k=1}^{n} c_k \chi_{I_k}(t)$, where $I_k = [t_k, t_{k+1}] \subseteq [0,T]$. Then*

1. $\mathbb{E}[\int_0^T \varphi(t)\mathrm{d}x(t)] = \int_0^T \varphi(t)\mathrm{d}a(t)$.

2. $\mathbb{E}[\Big(\int_0^T \varphi(t)\mathrm{d}x(t)\Big)^2] = \int_0^T \varphi^2(t)\mathrm{d}b(t) + \Big(\int_0^T \varphi(t)\mathrm{d}a(t)\Big)^2$

3. $\int_0^T \varphi(t)\mathrm{d}x(t)$ *is a normally distributed random variable with mean* $\int_0^T \varphi(t)\mathrm{d}a(t)$ *and variance* $\int_0^T \varphi^2(t)\mathrm{d}b(t)$.

Proof. As $S[0,T] \subseteq \mathrm{BV}[0,T]$, the Riemann-Stieltjes integrals appearing in the lemma all exist. Using (2.9) we see that

$$\int_{C_{a,b}[0,T]} \int_0^T \varphi(t)\mathrm{d}x(t)\mathfrak{m}(dx) = \int_{C_{a,b}[0,T]} \int_0^T \sum_{k=1}^n c_k \chi_{I_k}(t)\mathrm{d}x(t)\mathfrak{m}(dx)$$
$$= \sum_{k=1}^n c_k \int_{C_{a,b}[0,T]} [x(t_{k+1}) - x(t_k)]\mathfrak{m}(dx) = \sum_{j=1}^n c_k[a(t_{k+1}) - a(t_k)] = \int_0^T \varphi(t)\mathrm{d}a(t).$$

This gives the first conclusion.

For the second conclusion, we use equation (2.11) to compute

$$\int_{C_{a,b}[0,T]} \left(\sum_{k=1}^n c_k[x(t_{k+1}) - x(t_k)]\right)^2 \mathfrak{m}(dx)$$
$$= \sum_{j=1}^n \sum_{k=1}^n c_j c_k \int_{C_{a,b}[0,T]} [x(t_{j+1}) - x(t_j))(x(t_{k+1}) - x(t_k)]\mathfrak{m}(dx)$$
$$= \sum_{j=1}^n \sum_{k=1}^n c_j c_k \int_{C_{a,b}[0,T]} [x(t_{j+1})x(t_{k+1}) - x(t_{j+1})x(t_k) - x(t_j)x(t_{k+1}) + x(t_j)x(t_k)]\,\mathfrak{m}(dx)$$
$$= \sum_{j=1}^n \sum_{k=1}^n c_j c_k \big[\min\{b(t_{j+1}), b(t_{k+1})\} + a(t_{j+1})a(t_{k+1}) - \min\{b(t_j), b(t_{k+1})\}$$
$$- a(t_j)a(t_{k+1}) - \min\{b(t_{j+1}), b(t_k)\} - a(t_{j+1})a(t_k) + \min\{b(t_j), b(t_k)\} + a(t_j)a(t_k)\big]$$
$$= \sum_{\substack{j=1\\ j\neq k}}^n \sum_{k=1}^n c_j c_k \big[\min\{b(t_{j+1}), b(t_{k+1})\} - \min\{b(t_j), b(t_{k+1})\}$$
$$- \min\{b(t_{j+1}), b(t_k)\} + \min\{b(t_j), b(t_k)\}\big]$$
$$+ \sum_{j=1}^n c_j^2\big[b(t_{j+1}) - b(t_j)\big] + \sum_{j=1}^n \sum_{k=1}^n c_j c_k \big[a(t_{j+1}) - a(t_j)\big]\big[a(t_{k+1}) - a(t_k)\big]$$
$$= \int_0^T \varphi^2(t)\mathrm{d}b(t) + \left(\int_0^T \varphi(t)\mathrm{d}a(t)\right)^2.$$

Finally, observe that the collection of random variables $\{x(t), t \in [0,T]\}$ is a Gaussian system by Theorem 17.1 of [66], and therefore the linear combination

$$\int_0^T \varphi(t)\mathrm{d}x(t) = \sum_{k=1}^{n} c_k[x(t_{k+1}) - x(t_k)]$$

is normally distributed by Theorem 16.4 of [66]. From the first two conclusions we can compute the mean and variance for $\int_0^T \varphi(t)\mathrm{d}x(t)$; this completes the proof. □

We remark that the Riemann-Stieltjes integral is linear; whence we have the useful fact that

$$\int_0^T [r\varphi(t) + s\psi(t)]\,\mathrm{d}x(t) = r\int_0^T \varphi(t)\mathrm{d}x(t) + s\int_0^T \psi(t)\mathrm{d}x(t)$$

for φ and ψ in $S[0,T]$ and $r, s \in \mathbb{R}$. We now 'extend' this use of the Riemann-Stieltjes integral as a functional on $C_{a,b}[0,T]$ to include a much wider pool of functions.

Definition 3. For $f \in L^2_{a,b}[0,T]$, let $\{\varphi_n\}_{n=1}^{\infty}$ be a sequence of simple functions in $L^2_{a,b}[0,T]$ satisfying $f = \lim_{n\to\infty} \varphi_n$. Define the pointwise limit

$$\mathfrak{I}f(x) = \lim_{n\to\infty} \int_0^T \varphi_n(t)\mathrm{d}x(t) \tag{2.18}$$

for all $x \in C_{a,b}[0,T]$ for which this limit exists.

If $f \in L^2_{a,b}[0,T]$ then there is a sequence of simple functions $\varphi_n(t) = \sum_{k=1}^{n} c_k\chi_{I_k}(t)$ for which $|\varphi_n| \le |f|$ on $[0,T]$ and $\lim_{n\to\infty} ||f - \varphi_n||_{a,b} = 0$. In addition, $\int_0^T \varphi_n(t)\mathrm{d}x(t)$ is an element of $L^2(C_{a,b}[0,T], \mathfrak{m})$ for each n by part (b) of Lemma 1. Moreover, $\varphi_n - \varphi_m$ is an element of $S[0,T]$. Applying Lemma 1 and using Jensen's Inequality (cf. [24]) shows that

$$\begin{aligned}\left|\left|\int_0^T \varphi_n(t)\mathrm{d}x(t) - \int_0^T \varphi_m(t)\mathrm{d}x(t)\right|\right|^2_{L^2(\mathfrak{m})} &= \int_{C_{a,b}[0,T]} \left(\int_0^T [\varphi_n - \varphi_m](t)\mathrm{d}x(t)\right)^2 \mathfrak{m}(dx) \\ &= \int_0^T [\varphi_n - \varphi_m]^2(t)\mathrm{d}b(t) \\ &\quad + \left(\int_0^T [\varphi_n - \varphi_m](t)\mathrm{d}a(t)\right)^2 \\ &\leq \int_0^T [\varphi_n - \varphi_m]^2(t)\mathrm{d}b(t) \\ &\quad + \nu_{|a|}\left([0,T]\right) \int_0^T [\varphi_n - \varphi_m]^2(t)\mathrm{d}\left|a\right|(t) \\ &\leq \max\left\{1, \nu_{|a|}\left([0,T]\right)\right\} \left|\left|\varphi_n - \varphi_m\right|\right|^2_{a,b}.\end{aligned}$$

From this, we conclude that the sequence $\{\int_0^T \varphi_n(t)\mathrm{d}x(t)\}_{n=1}^\infty$ is Cauchy in $L^2(C_{a,b}[0,T], \mathfrak{m})$.

Notice that $\mathfrak{I}f$ exists for a.e. $x \in C_{a,b}[0,T]$. Moreover, if $\lim_{n\to\infty} \varphi_n = f = \lim_{n\to\infty} \psi_n$ for sequences $\{\varphi_n\}_{n=1}^\infty$ and $\{\psi_n\}_{n=1}^\infty$ in $S[0,T]$ and $\mathfrak{I}f = \lim_{n\to\infty} \int_0^T \varphi_n(t)\mathrm{d}x(t)$, then again by Lemma 1 and Jensen's Inequality we have

$$\left|\left|\int_0^T \varphi_n(t)\mathrm{d}x(t) - \int_0^T \psi_n(t)\mathrm{d}x(t)\right|\right|^2_{L^2(\mathfrak{m})} \leq \max\left\{1, \nu_{|a|,0}\left([0,T]\right)\right\} \left|\left|s_n - s_m\right|\right|^2_{a,b},$$

and therefore

$$\mathfrak{I}f(x) = \lim_{n\to\infty} \int_0^T \varphi_n(t)\mathrm{d}x(t) = \lim_{n\to\infty} \int_0^T \psi_n(t)\mathrm{d}x(t)$$

in $L^2(C_{a,b}[0,T], \mathfrak{m})$. We conclude that the definition of $\mathfrak{I}f$ is essentially independent of the choice of sequence from $S[0,T]$ that is used to define it.

In some ways it is easier to work with $\mathfrak{I}f$ than with $\langle f, x\rangle$. Fortunately, they are actually the same entity for almost every $x \in C_{a,b}[0,T]$, as we will show. We first establish the desired properties for $\mathfrak{I}f$ and then show that Definition 3 is equivalent to Definition 2, and hence $\mathfrak{I}f(x) = \langle f, x\rangle$. To this end, we state the following lemma.

Lemma 2. *If $f \in L^2_{a,b}[0,T]$, then*

1. *$\mathfrak{I}(f)$ is a normally distributed random variable.*

2. *$\mathbb{E}[\mathfrak{I}(f)] = \int_0^T f(t)\mathrm{d}a(t)$.*

3. *$\mathbb{E}[\mathfrak{I}(f)^2] = \int_0^T f^2(t)\mathrm{d}b(t) + \left(\int_0^T f(t)\mathrm{d}a(t)\right)^2$*

4. *If r and s are in $\mathbb{R}$ and f and g are in $L^2_{a,b}[0,T]$, then $\mathfrak{I}(rf + sg) = r\mathfrak{I}(f) + s\mathfrak{I}(g)$ for a.e. $x \in C_{a,b}[0,T]$.*

Proof. The integrals on the right hand sides of items 2 and 3 are well-defined Lebesgue-Stieltjes integrals, because $f \in L^2_{a,b}[0,T]$. Take $\{\varphi_n\}_{n=1}^{\infty}$ to be a sequence of simple functions with $|\varphi_n| \leq |f|$ on $[0,T]$ and $\lim_{n\to\infty} \varphi_n = f$ in $L^2_{a,b}[0,T]$. Then

$$\left|\int_0^T \varphi_n(t)\mathrm{d}a(t) - \int_0^T f(t)\mathrm{d}a(t)\right| \leq \int_0^T |\varphi_n(t) - f(t)| \ d\,|a|\,(t) \leq ||\varphi_n - f||_{a,b}\,,$$

and therefore $\lim_{n\to\infty} \int_0^T \varphi_n(t)\mathrm{d}a(t) = \int_0^T f(t)\mathrm{d}a(t)$. Moreover, $\varphi^2 \leq f^2$ and $\varphi^2 \to f^2$ pointwise for almost every $t \in \ [0,T]$, and so by dominated convergence

$$\lim_{n\to\infty} \int_0^T \varphi_n^2(t)\mathrm{d}b(t) = \int_0^T f^2(t)\mathrm{d}b(t).$$

Finally, $\int_0^T \varphi_n(t)\mathrm{d}x(t)$ is real-valued and thus $\left|\exp\left(iu\int_0^T \varphi_n(t)\mathrm{d}x(t)\right)\right| = 1$ for $u \in \mathbb{R}$.

Using these facts, we can compute the characteristic function for $\mathfrak{I}f$ and find that

$$
\begin{aligned}
\mathbb{E}[\exp(iu\mathfrak{I}f)] &= \int_{C_{a,b}[0,T]} \exp\left(iu \lim_{n\to\infty}\int_0^T \varphi_n \mathrm{d}x\right)\mathfrak{m}(dx)\\
&= \lim_{n\to\infty}\int_{C_{a,b}[0,T]} \exp\left(iu \int_0^T \varphi_n \mathrm{d}x\right)\mathfrak{m}(dx)\\
&= \lim_{n\to\infty}\mathbb{E}[\exp\left(iu\mathfrak{I}\varphi_n\right)]\\
&= \lim_{n\to\infty}\exp\left(-\frac{u^2}{2}\int_0^T \varphi_n^2(t)\mathrm{d}b(t) + iu\int_0^T \varphi_n(t)\mathrm{d}a(t)\right)\\
&= \exp\left(-\frac{u^2}{2}\int_0^T f^2(t)\mathrm{d}b(t) + iu\int_0^T f(t)\mathrm{d}a(t)\right).
\end{aligned}
$$

This is the (unique) characteristic function of a normally distributed random variable with expected value $\int_0^T f(t)\mathrm{d}a(t)$ and variance $\int_0^T f^2(t)\mathrm{d}b(t)$. This proves items 1, 2, and 3.

Finally, we take sequences $\{\varphi_n\}_{n=1}^\infty$ and $\{\psi_n\}_{n=1}^\infty$ of simple functions with $\lim_{n\to\infty}\varphi_n = f$ and $\lim_{n\to\infty}\psi_n = g$ in $L^2_{a,b}[0,T]$. Then for real r and s each function $r\varphi_n + s\psi_n$ is also a simple function with $\lim_{n\to\infty} r\varphi_n + s\psi_n = rf + sg$ in $L^2_{a,b}[0,T]$, and therefore

$$
r\mathfrak{I}f + s\mathfrak{I}g = r\lim_{n\to\infty}\int_0^T \varphi_n(t)\mathrm{d}x(t) + s\lim_{n\to\infty}\int_0^T \psi_n(t)\mathrm{d}x(t) = \mathfrak{I}(rf+sg)
$$

for almost every $x \in C_{a,b}[0,T]$. □

Lemma 3. *If $f \in \mathrm{BV}[0,T]$, then $\mathfrak{I}f(x) = \int_0^T f(t)\mathrm{d}x(t)$ for a.e. $x \in C_{a,b}[0,T]$.*

Proof. The proof of this lemma is precisely the same proof that Yeh uses for Theorem 22.5 of page 322 in [66], except that we reference our Definition 3 for $\mathfrak{I}f$ and our Lemma 2 from above at the appropriate points. □

Lemma 4. *If $f \in L^2_{a,b}[0,T]$, then $\mathfrak{I}f(x) = \langle f, x\rangle$ for a.e $x \in C_{a,b}[0,T]$.*

Proof. Note that $f = \lim_{n\to\infty}\varphi_n = \lim_{n\to\infty} f_n$ for sequences $\{f_n\}$ and $\{\varphi_n\}$ as before, and that each φ_n and each f_n is an element of $\mathrm{BV}[0,T]$. Therefore $\varphi_n - f_n \in \mathrm{BV}[0,T]$. Then by

Lemma 2, Lemma 3, and Jensen's Inequality,

$$\begin{aligned}\left|\left|\int_0^T \varphi_n(t)\mathrm{d}x(t) - \int_0^T f_n(t)\mathrm{d}x(t)\right|\right|^2_{L^2(\mathfrak{m})} &= \int_0^T [\varphi_n - f_n]^2(t)\mathrm{d}b(t) \\ &\quad + \left(\int_0^T [\varphi_n - f_n](t)\mathrm{d}a(t)\right)^2 \\ &\leq ||\varphi_n - f_n||^2_{a,b} + \left(\int_0^T |\varphi_n - f_n|(t)\mathrm{d}\,|a|\,(t)\right)^2 \\ &\leq \int_0^T (\varphi_n - f_n)^2(t)\mathrm{d}b(t) \\ &\quad + \nu_{|a|}\left([0,T]\right)\int_0^T [\varphi_n - f_n]^2(t)\mathrm{d}\,|a|\,(t) \\ &\leq \max\left\{1, \nu_{|a|}\left([0,T]\right)\right\} ||\varphi_n - f_n||^2_{a,b}\,.\end{aligned}$$

Then note that

$$\begin{aligned}||\mathfrak{I}f - \langle f, x\rangle||_{L^2(\mathfrak{m})} \leq &\left|\left|\int_0^T \varphi_n(t)\mathrm{d}x(t) - \mathfrak{I}f\right|\right|_{L^2(\mathfrak{m})} + \left|\left|\langle f, x\rangle - \int_0^T f_n(t)\mathrm{d}x(t)\right|\right|_{L^2(\mathfrak{m})} \\ &+ \left|\left|\int_0^T \varphi_n(t)\mathrm{d}x(t) - \int_0^T f_n(t)\mathrm{d}x(t)\right|\right|_{L^2(\mathfrak{m})},\end{aligned}$$

and thus we conclude that $\mathfrak{I}f(\cdot) = \langle f, \cdot\rangle$ in $\text{Ł}^2(C_{a,b}[0,T])$, whence $\mathfrak{I}f(x) = \langle f, x\rangle$ for almost every $x \in C_{a,b}[0,T]$. □

We are now free to dispense with $\mathfrak{I}f$, after ascribing its desired properties to $\langle f, x\rangle$. The following theorem follows immediately from Lemma 2 and Lemma 4.

Theorem 3. *If $f \in L^2_{a,b}[0,T]$, then*

1. *$\langle f, x\rangle$ is a normally distributed random variable.*

2. *$\mathbb{E}[\langle f, x\rangle] = \int_0^T f(t)\mathrm{d}a(t)$.*

3. *$\mathbb{E}[\langle f, x\rangle^2] = \int_0^T f^2(t)\mathrm{d}b(t) + \left(\int_0^T f(t)\mathrm{d}a(t)\right)^2$*

4. For r and s in $\mathbb{R}$ and f and g in $L^2_{a,b}[0,T]$, $\langle rf+sg, x\rangle = r\langle f,x\rangle + s\langle g,x\rangle$ for a.e. $x \in C_{a,b}[0,T]$.

The following corollary gives one way of characterizing the relationship between the space of functions $L^2_{a,b}[0,T]$ and the collection of PWZ integrals obtained from it.

Corollary 1. *The map $\Phi : L^2_{a,b}[0,T] \to L^2(C_{a,b}[0,T])$ by $\Phi(f) = \langle f,x\rangle$ is an injective and bounded linear transformation. Its image $\Phi(L^2_{a,b}[0,T])$ is a closed subspace of $L^2(C_{a,b}[0,T])$. Thus Φ is a linear homeomorphism of Hilbert spaces.*

Proof. Φ is linear by (d). Now, $0 = ||\langle f,x\rangle||_2 = E[\langle f,x\rangle^2] \geq \int_0^T f^2(t)\mathrm{d}b(t) \geq 0$ if and only if $f = 0$ in $L^2_{a,b}[0,T]$ by (c); thus Φ is injective. Also by (c),

$$\begin{aligned}||\langle f,x\rangle||^2_{L^2(\mathfrak{m})} &= \int_0^T f^2(t)\mathrm{d}b(t) + \left(\int_0^T f(t)\mathrm{d}a(t)\right)^2 \\ &\leq \int_0^T f^2(t)\mathrm{d}b(t) + \nu_{|a|,0}\left([0,T]\right)\int_0^T f^2(t)\mathrm{d}\left|a\right|(t) \leq \max\{1, \nu_{|a|,0}\left([0,T]\right)\}\left|\left|f\right|\right|^2_{a,b}, \quad (2.19)\end{aligned}$$

and so Φ is bounded with $||\Phi|| \leq \max\{1, \nu_{|a|,0}\left([0,T]\right)\}$. As Φ is bounded and $L^2_{a,b}[0,T]$ is closed in itself, $\Phi(L^2_{a,b}[0,T])$ is closed in $L^2(C_{a,b}[0,T])$. Since $\Phi : L^2_{a,b}[0,T] \to \Phi(L^2_{a,b}[0,T])$ is a bounded bijective linear transformation, the Open Mapping Theorem guarantees that Φ^{-1} is also bounded. □

In [66], Yeh demonstrates that the analogous mapping from $L^2[0,T]$ into $L^2(C_0[0,T])$ is an isometry (often referred to as the Itô Isometry) and is then able to exploit this fact to obtain further results. As one should expect, the situation is more complicated for more general function spaces. We consider a simple example that illustrates the difficulty. Let $C_{a,b}[0,T]$ be a generalized Wiener space with parameter space $[0,T]$ and having drift function

$a(t) = 2t$ and consider the functions f_1 and f_2 given by

$$f_1(t) = \begin{cases} 1 & \text{if } 0 \le t \le \frac{1}{2}, \\ -1 & \text{if } \frac{1}{2} < t \le 1, \end{cases}$$

and $f_2(t) = 1$. From (c) of Theorem 3 it is easy to see that $||f_1||_{a,b} = ||f_2||_{a,b} = b(1) + 2$. However, a simple calculation also shows that

$$\int_0^1 f_2^2(t)\mathrm{d}b(t) + \left(\int_0^1 f_2(t)\mathrm{d}a(t)\right)^2 = b(1) + (2)^2,$$

and that

$$\int_0^1 f_1^2(t)\mathrm{d}b(t) + \left(\int_0^1 f_1(t)\mathrm{d}a(t)\right)^2 = b(1).$$

Thus $||\Phi(f_2)||_2^2 > ||f_2||_{a,b} = ||f_1||_{a,b} > ||\Phi(f_1)||_2^2$ and it is difficult to say very much about the exact value of $||\Phi||$ except that Φ is certainly not an isometry.

With suitable adjustments, one might still carry out a program of orthogonal expansions for a general Brownian Motion process similar to that detailed in Section 23 of [66], albeit with some difficulty. As one might expect, the very possibility of carrying out such a program may be largely dependent upon the nature of the functions a and b. In Section 2.3 we will show that for f and g in $L^2_{a,b}[0,T]$,

$$(\Phi(f), \Phi(g))_{L^2(\mathfrak{m})} = \int_0^T f(t)g(t)\mathrm{d}b(t) + \int_0^T f(t)\mathrm{d}a(t) \int_0^T g(t)\mathrm{d}a(t), \tag{2.20}$$

and then taking the generalized Wiener space of our last example, we note that $f_3(t) = \chi_{[0,\frac{1}{2}]}(t)$ and $f_4(t) = \chi_{[\frac{1}{2},1]}(t)$ are orthogonal in $L^2_{a,b}[0,T]$, but that $\Phi(f_3)$ and $\Phi(f_4)$ are not orthogonal in $L^2(C_{a,b}[0,T])$ by equation (2.20). Thus one cannot necessarily build orthogonal sets in $L^2(C_{a,b}[0,T])$ from orthogonal sets in $L^2_{a,b}[0,T]$, which raises doubts about whether one could obtain many of the results of Section 23 of [66]. Of course, if the function a is identically zero it is not hard to see that then the map Φ is in fact an isomorphism of Hilbert spaces and one need only make minimal adjustments for the behavior of b in order to carry out Yeh's program. The problem here is that $L^2_{a,b}[0,T]$ is in some sense the "wrong" space to be working with. In Chapter 3, we will have something to say about what the "correct" space is.

We briefly return our attention to time integrals to observe an interesting relationship between PWZ integrals and certain time integrals. If $F(t, X(t,x)) = rX(t,x) + s$ with $r, s \in \mathbb{R}$, we can use integration by parts, and taking our sample paths $x \in C_{a,b}[0,T]$ we obtain

$$\begin{aligned} Y(x) = \int_0^T F(t, X(t,x))\mathrm{d}t &= \int_0^T (rx(t) + s)\mathrm{d}t \\ &= r\left(Tx(T) - \int_0^T t\mathrm{d}x(t)\right) + sT \\ &= sT + r\langle T - t, x\rangle. \end{aligned} \tag{2.21}$$

We conclude that time integrals of this type can be written in terms of PWZ integrals. In the examples of time integrals found in equations (2.15) and (2.16), $G(x)$ is clearly of this type and $F_{\alpha,\beta}(x)$ can also be handled in this manner in the case where $\theta(\cdot,\cdot)$ is a linear function of its second argument. In section 2.3 we will obtain integration formulas for a wide class of functionals involving PWZ integrals, which will consequently yield formulas for functionals of time integrals.

2.3 Paley-Wiener-Zygmund Theorem

In this section, we continue to assume that the function a is absolutely continuous with derivative $a' \in L^2[0,T]$ and that b is continuously differentiable with b' positive and bounded away from 0.

Functionals that involve Paley-Wiener-Zygmund (PWZ) stochastic integrals are quite common. A very important method for evaluating function space integrals of these functionals is the following formula which is stated without proof on page 2929 of [11] by Chang, Choi and Skoug.

> Let $\{\alpha_1, \ldots, \alpha_n\}$ be an orthonormal set in $L^2_{a,b}[0,T]$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be Lebesgue measurable and let $A_j = \int_0^T \alpha_j(s)\mathrm{d}a(s)$ and $B_j = \int_0^T \alpha_j^2(s)\mathrm{d}b(s)$ for $j = 1, \ldots, n$. If $F(x) = f(\langle \alpha_1, x\rangle, \ldots, \langle \alpha_n, x\rangle)$, then
>
> $$\begin{aligned}\int_{C_{a,b}[0,T]} F(x)d\mu(x) &\equiv \int_{C_{a,b}[0,T]} f(\langle \alpha_1, x\rangle, \ldots, \langle \alpha_n, x\rangle)\ d\mu(x)\\ &= \left(\prod_{j=1}^n 2\pi B_j\right)^{-1/2} \int_{\mathbb{R}^n} f(u_1, \ldots, u_n)\ \exp\left(-\frac{1}{2}\sum_{j=1}^n \frac{(u_j - A_j)^2}{B_j}\right) \mathrm{d}u_1 \cdots \mathrm{d}u_n,\end{aligned} \tag{2.22}$$
>
> in the sense that if one side exists then the other exists with equality.

Observe that this formula is a generalization to the space $C_{a,b}[0,T]$ of a theorem of Paley and Wiener for ordinary Wiener space, which can be found in chapter 7 of [66]. We will establish a more general theorem that subsumes the formula above as a special case. Our theorem has considerably relaxed conditions on the choice of functions $\{\alpha_1, \ldots, \alpha_n\}$. This proof is in part motivated by the proof of a similar theorem for ordinary Wiener space found in Robert Ewan's thesis [23], completed at UNL under the direction of Skoug in 1973.

The first theorem of this section is the most general of our Paley-Wiener type theorems.

Theorem 4. *Let* $\{\theta_1, \ldots, \theta_n\}$ *be a collection of functions in* $L^2_{a,b}[0,T]$*. Also, let* $M = (\mathrm{Cov}[\langle\theta_i, x\rangle, \langle\theta_j, x\rangle])_{i,j=1}^n$ *be the covariance matrix for the collection* $\{\langle\theta_i, x\rangle\}$*. Further suppose that* M *is nonsingular and let* $M^{-1} = (\hat{m}_{i,j})_{i,j=1}^n$*. Put* $A_i = \mathbb{E}[\langle\theta_i, x\rangle]$ *and let* $f : \mathbb{R}^n \to \mathbb{C}$ *be Lebesgue measurable and put* $F(x) = f(\langle\alpha_1, x\rangle, \ldots, \langle\alpha_n, x\rangle)$*. Then*

$$\int_{C_{a,b}[0,T]} F(x)\mathfrak{m}(dx) = \int_{C_{a,b}[0,T]} f(\langle\theta_1, x\rangle, \ldots, \langle\theta_n, x\rangle)\mathfrak{m}(dx)$$
$$\stackrel{*}{=} (2\pi)^{-\frac{n}{2}} |M|^{-\frac{1}{2}} \int_{\mathbb{R}^n} f(\mathbf{u}) \; \exp\left(-\frac{1}{2}\sum_{j=1}^n\sum_{i=1}^n \hat{m}_{i,j}(u_i - A_i)(u_j - A_j)\right) \mathrm{d}\mathbf{u}. \quad (2.23)$$

Proof. Let $\{\theta_1, \ldots \theta_n\}$ be as given in the theorem. By Theorem 3 we see that $(\langle\theta_j, x\rangle - A_j) \stackrel{d}{\sim} \mathrm{N}(0, \sqrt{B_j})$ for $j = 1, \ldots, n$. If $M = (\mathrm{Cov}[\langle\theta_i, x\rangle, \langle\theta_j, x\rangle])_{i,j=1}^n$ is nonsingular, then $|M| > 0$, and so $|M^{-1}| = |M|^{-1} \in \mathbb{R}$. Let $\{Y_i\}_{i=1}^n$ be a collection of random variables with $Y_i \stackrel{d}{\sim} N(\overline{y}_i, \sigma_i)$. Then from page 164 of [59] we see that the density function of the n-variate normal distribution for $(Y_1, \ldots, Y_n)$ is given by

$$\phi(\mathbf{y}) = \frac{\sqrt{|M^{-1}|}}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2}\langle\mathbf{y} - \mathbf{w}, M^{-1}(\mathbf{y} - \mathbf{w})\rangle\right), \quad (2.24)$$

where $\mathbf{y} = (y_1, \ldots, y_n)$, $\mathbf{w} = (\overline{y}_1, \ldots, \overline{y}_n)$, and $\langle\cdot,\cdot\rangle$ is the Euclidean inner product. Note that we can write

$$\langle\mathbf{y} - \mathbf{w}, M^{-1}(\mathbf{y} - \mathbf{w})\rangle = \sum_{j=1}^n\sum_{i=1}^n \hat{m}_{i,j}(y_i - \overline{y}_i)(y_j - \overline{y}_j),$$

where $M^{-1} = (\hat{m}_{i,j})_{i,j=1}^n$.

Now, from page 41 of [54] we observe that for a collection of random variables $\{y_1, \ldots, y_n\}$ in a probability space $(\Omega, \mathcal{A}, P)$ with joint probability density function ϕ we have

$$\mathbb{E}[f(y_1, \ldots, y_n)] = \int_\Omega f(y_1, \ldots, y_n) \; dP(\mathbf{y}) = \int_{\mathbb{R}^n} f(\mathbf{u})\phi(\mathbf{u})\mathrm{d}\mathbf{u}. \quad (2.25)$$

Theorem 4 follows from equations (2.24) and (2.25). $\square$

The next Lemma gives a useful means for computing the covariance matrix M appearing in Theorem 4.

Lemma 5. *If θ_1 and θ_2 are in $L^2_{a,b}[0,T]$, then* $\mathrm{Cov}[\langle\theta_1,x\rangle,\langle\theta_2,x\rangle] = \int_0^T \theta_1(s)\theta_2(s)\mathrm{d}b(s)$.

Proof. Note that if $\theta_1 = 0$ in $L^2_{a,b}[0,T]$, then $\theta_1(t) = 0$ for a.e. $t \in [0,T]$. Therefore $\langle\theta_1,x\rangle = 0$ and hence

$$\mathrm{Cov}[\langle\theta_1,x\rangle,\langle\theta_2,x\rangle] = E[\langle\theta_1,x\rangle\langle\theta_2,x\rangle] - E[\langle\theta_1,x\rangle]E[\langle\theta_2,x\rangle] = E[0] - E[0]E[\langle\theta_2,x\rangle] = 0 \tag{2.26}$$

in the case where either θ_1 or θ_2 is 0 in $L^2_{a,b}[0,T]$. Take θ_i nonzero in $L^2_{a,b}[0,T]$ and put $A_i = \int_0^T \theta_i(s)\mathrm{d}a(s)$ and $B_i = \int_0^T \theta_i^2(s)\mathrm{d}b(s)$ for $i = 1,2$. Observe that b induces a positive measure on $[0,T]$ and hence $B_i > 0$. Apply Theorem 4 to the singleton set $\{\theta_i\}$ to obtain

$$\mathbb{E}[\langle\theta_i,x\rangle^2] = \int_{C_{a,b}[0,T]} \langle\theta_i,x\rangle^2\mathfrak{m}(dx) = \frac{1}{\sqrt{2\pi B_i}}\int_{\mathbb{R}} u^2\ \exp\left(-\frac{(u-A_i)^2}{2B_i}\right)\mathrm{d}u,$$

and then a routine calculation yields the equality

$$\mathbb{E}[\langle\theta_i,x\rangle^2] = B_i + A_i^2. \tag{2.27}$$

Similar computations performed with the function $\theta_1+\theta_2$ show that

$$\begin{aligned}\mathbb{E}[\langle\theta_1+\theta_2,x\rangle^2] &= \int_0^T (\theta_1(s)+\theta_2(s))^2\mathrm{d}b(s) + \left(\int_0^T \theta_1(s)+\theta_2(s)\mathrm{d}a(s)\right)^2 \\ &= \left(\int_0^T \theta_1(s)\mathrm{d}a(s)\right)^2 + 2\int_0^T \theta_1(s)\mathrm{d}a(s)\int_0^T \theta_2(s)\mathrm{d}a(s) + \left(\int_0^T \theta_2(s)\mathrm{d}a(s)\right)^2 \\ &\qquad + \int_0^T \theta_1^2(s)\mathrm{d}b(s) + 2\int_0^T \theta_1(s)\theta_2(s)\mathrm{d}b(s) + \int_0^T \theta_2^2(s)\mathrm{d}b(s),\end{aligned}$$

which establishes that

$$\mathbb{E}[\langle \theta_1 + \theta_2, x\rangle^2] = A_1^2 + 2A_1A_2 + A_2^2 + B_1 + B_2 + 2\int_0^T \theta_1(s)\theta_2(s)\mathrm{d}b(s). \tag{2.28}$$

Using equation (2.27), we also obtain the equality

$$\begin{aligned}\mathbb{E}[\langle \theta_1 + \theta_2, x\rangle^2] &= \int_{C_{a,b}[0,T]} \langle \theta_1 + \theta_2, x\rangle\langle \theta_1 + \theta_2, x\rangle \mathfrak{m}(dx) \\ &= \int_{C_{a,b}[0,T]} \langle \theta_1, x\rangle^2 \mathfrak{m}(dx) + 2\int_{C_{a,b}[0,T]} \langle \theta_1, x\rangle\langle \theta_2, x\rangle \mathfrak{m}(dx) + \int_{C_{a,b}[0,T]} \langle \theta_2, x\rangle^2 \mathfrak{m}(dx) \\ &= B_1 + A_1^2 + B_2 + A_2^2 + 2E[\langle \theta_1, x\rangle\langle \theta_2, x\rangle]. \end{aligned} \tag{2.29}$$

Finally, combining equations (2.28) and (2.29) yields that

$$\mathbb{E}[\langle \theta_1, x\rangle\langle \theta_2, x\rangle] = \int_0^T \theta_1(s)\theta_2(s)\mathrm{d}b(s) + A_1A_2, \tag{2.30}$$

from which we deduce that

$$\mathrm{Cov}[\langle \theta_1, x\rangle, \langle \theta_2, x\rangle] = E[\langle \theta_1, x\rangle\langle \theta_2, x\rangle] - E[\langle \theta_1, x\rangle]E[\langle \theta_2, x\rangle] = \int_0^T \theta_1(s)\theta_2(s)\mathrm{d}b(s), \tag{2.31}$$

which also agrees with (2.26) in the case where either function is zero. □

Lemma 6. *If $\{\theta_1, \ldots, \theta_n\}$ is a collection of linearly independent functions in $L^2_{a,b}[0,T]$, then the random variables $\{\langle \theta_i, x\rangle\}$ are linearly independent.*

Proof. Suppose that $0 = \sum_{j=1}^n c_j\langle \theta_j, x\rangle$ for almost every $x \in C_{a,b}[0,T]$ with $c_j \neq 0$ for some j. Then

$$0 = \int_{C_{a,b}[0,T]} \left| \sum_{j=1}^n c_j \langle \theta_j, x\rangle \right|^2 \mathfrak{m}(dx) = \int_{C_{a,b}[0,T]} \left| \langle \sum_{j=1}^n c_j\theta_j(t), x\rangle \right|^2 \mathfrak{m}(dx). \tag{2.32}$$

Take $\varphi(t) = \sum_{j=1}^{n} c_j\theta_j(t)$ and then put $A = E[\langle \sum_{j=1}^{n} \theta_j, x\rangle] = E[\langle \varphi, x\rangle]$ and $B = \mathrm{Var}[\langle \sum_{j=1}^{n} \theta_j, x\rangle] = \mathrm{Var}[\langle \varphi, x\rangle]$, observing that $B \geq 0$ by Lemma 3.

If $B > 0$, then we apply Theorem 4 to the singleton set $\{\varphi\}$ to obtain that

$$\int_{C_{a,b}[0,T]} |\langle \varphi, x\rangle|^2 \ d\mu(x) = \frac{1}{\sqrt{2\pi B}} \int_{\mathbb{R}} u^2 \ \exp\left(-\frac{(u-A)^2}{2B}\right) \mathrm{d}u. \tag{2.33}$$

From equations (2.32) and (2.33) a routine computation shows that

$$0 = \int_{C_{a,b}[0,T]} \left| \sum_{j=1}^{n} c_j \langle \theta_j, x\rangle \right|^2 \mathfrak{m}(dx) = \int_{C_{a,b}[0,T]} |\langle \varphi, x\rangle|^2 \, \mathfrak{m}(dx) = B + A^2, \tag{2.34}$$

from which we quickly deduce that we must have $B = 0$, which is a contradiction.

If $B = 0$ then we have $0 = \mathrm{Var}[\langle \varphi, x\rangle] = \int_0^T \varphi^2(s)\mathrm{d}b(s)$ by Lemma 3. Therefore $\sum_{j=1}^{n} c_j\theta_j(s) = \varphi(s) = 0$ for a.e. $s \in [0,T]$ because b induces a positive measure on $[0,T]$. Then the functions $\{\theta_1, \ldots, \theta_n\}$ cannot be linearly independent. □

Lemma 6 allows us to obtain our next Paley-Wiener type theorem by taking the collection of functions $\{\theta_1, \ldots, \theta_n\}$ to be linearly independent.

Theorem 5. *Let $\{\theta_1, \ldots, \theta_n\}$ be a linearly independent set in $L^2_{a,b}[0,T]$. Then the covariance matrix $M = (\mathrm{Cov}[\langle \theta_i, x\rangle, \langle \theta_j, x\rangle])_{i,j=1}^{n}$ is nonsingular and the result of Theorem 4 holds.*

Proof. By Lemma 6 we see that $\{\langle \theta_1, x\rangle, \ldots, \langle \theta_n, x\rangle\}$ are linearly independent. Then by Theorem 3.5.1 of [59], the covariance matrix M is positive-definite (hence nonsingular). Now apply Theorem 4. □

In the special case where the functions $\{\theta_1, \ldots, \theta_n\}$ are orthonormal, we obtain as a corollary the very useful formula on page 2929 of [11] which was discussed at the beginning of this section.

Corollary 2. *Let* $\{\theta_1, \ldots, \theta_n\}$ *be an orthonormal set in* $L^2_{a,b}[0,T]$*, let* $f : \mathbb{R}^n \to \mathbb{R}$ *be Lebesgue measurable, and let* $A_j = \int_0^T \theta_j(s)\mathrm{d}a(s)$ *and* $B_j = \int_0^T \theta_j^2(s)\mathrm{d}b(s)$*. If* $F(x) = f(\langle\theta_1, x\rangle, \ldots, \langle\theta_n, x\rangle)$*, then*

$$\int_{C_{a,b}[0,T]} F(x)\mu(dx) = \int_{C_{a,b}[0,T]} f(\langle\theta_1, x\rangle, \ldots, \langle\theta_n, x\rangle)\mu(dx)$$
$$\overset{*}{=} \left(\prod_{j=1}^n 2\pi B_j\right)^{-1/2} \int_{\mathbb{R}^n} f(u_1, \ldots, u_n)\ \exp\left(-\frac{1}{2}\sum_{j=1}^n \frac{(u_j - A_j)^2}{B_j}\right)\ du_1 \cdots du_n. \quad (2.35)$$

Proof. In the case that a collection of functions $\{\theta_1, \ldots, \theta_n\}$ is orthonormal in $L^2_{a,b}[0,T]$, we can compute the covariance matrix $M = (\mathrm{Cov}[\langle\theta_i, x\rangle, \langle\theta_j, x\rangle])_{i,j=1}^n$ quite easily.

Recall that θ_i and θ_j are members of an orthonormal set in $L^2_{a,b}[0,T]$. Using Lemma 5 we determine that

$$m_{i,j} = \mathrm{Cov}[\langle\theta_i, x\rangle, \langle\theta_j, x\rangle] = \int_0^T \theta_i(s)\theta_j(s)\mathrm{d}b(s) = \begin{cases} B_j & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Thus we can compute $|M| = \prod_{j=1}^n B_j$, while $|M^{-1}| = \prod_{j=1}^n B_j^{-1}$ and $M^{-1} = (\hat{m}_{i,j})_{i,j=1}^n$. Applying Theorem 5 with the set $\{\theta_1, \ldots, \theta_n\}$ immediately yields the result. $\square$

We now consider some applications and examples of the use of these theorems. Begin by making the observation that for functionals of the form

$$F(x) = f(x(t_1), \ldots, x(t_n)) \quad (2.36)$$

where $0 = t_0 < t_1 < \cdots < t_n \leq T$, one can evaluate the integral $\int_{C_{a,b}[0,T]} F(x)\mathfrak{m}(dx)$ by at least three different methods. The first of these methods is to use formula (2.8). A second option is to use formula (2.23) with $\theta_j(t) = \chi_{[0,t_j]}(t)$ for $j = 1, \ldots, n$ and noting that

$$x(t_j) = \int_0^{t_j} \mathrm{d}x(t) = \int_0^T \chi_{[0,t_j]}(t)\mathrm{d}x(t) = \langle \theta_j, x \rangle. \tag{2.37}$$

It is easy to see that the collection $\{\theta_1, \ldots, \theta_n\}$ is linearly independent in $L^2_{a,b}[0,T]$, and thus the collection of random variables $\{\langle \theta_1, x\rangle, \ldots, \langle \theta_n, x\rangle\}$ has nonsingular covariance matrix $M = (m_{i,j})_{i,j=1}^n = (\min\{b(t_i), b(t_j)\})_{i,j=1}^n$.

Finally, a third method for evaluating function the space integral for the functional in (2.36) is to again take $\theta_j(t) = \chi_{[0,t_j]}(t)$ for $j = 1, \ldots, n$ and use the Gram-Schmidt process on the set $\{\theta_1, \ldots, \theta_n\}$ to obtain an orthonormal set in $L^2_{a,b}[0,T]$ and then apply (2.35) after an appropriate change of variables.

A combination of integration by parts and equation (2.37) can be used to find integration formulas for certain other functionals in terms of Lebesgue integrals. In Section 2.2 we introduced the time integral and noted that certain time integrals could be written in terms of PWZ integrals. Under favorable conditions, this notion can be extended. For example, observe that if $\phi(t)$ is of bounded variation on $[0,T]$ we can write

$$\int_0^T x(t)\mathrm{d}\phi(t) = \int_0^T (\phi(T) - \phi(t))\, \mathrm{d}x(t) = \langle \phi(T) - \phi(t), x \rangle. \tag{2.38}$$

Therefore, if $f(u_1, \ldots, u_n)$ is Lebesgue measurable, and if the collection $\{\phi_1, \ldots, \phi_n\}$ is linearly independent in $L^2_{a,b}[0,T]$ with no ϕ_j constant on $[0,T]$, and if we define a functional F on $C_{a,b}[0,T]$ by

$$F(x) = f\left(\int_0^T x(t)\mathrm{d}\phi_1(t), \ldots, \int_0^T x(t)\mathrm{d}\phi_n(t) \right), \tag{2.39}$$

then we can express the function space integral of F in terms of a Lebesgue integral by using (2.38) and Theorem 5.

We close this section by working two examples. The first highlights the method for using the general Paley-Wiener type theorem. For most choices of functions in $L^2_{a,b}[0,T]$, the computations involved in using the theorem can become quite complicated. In order to simplify the computations and concentrate on the techniques used, we assume in this first example that a is the zero function. Let $\{\theta_1, \theta_2\}$ be a linearly independent set in $L^2_{a,b}[0,T]$ and define the functional

$$F(x) = f(\langle\theta_1, x\rangle, \langle\theta_2, x\rangle) = \exp\left(-\frac{1}{2}\left(\langle\theta_1, x\rangle^2 + \langle\theta_2, x\rangle^2\right)\right). \tag{2.40}$$

As $a(t) \equiv 0$, we see that $A_i = \int_0^T \theta_i(t)\mathrm{d}a(t) = 0$ for $i = 1, 2$. The covariance matrix for the random variables $\{\langle\theta_1, x\rangle, \langle\theta_2, x\rangle\}$ is

$$M = \begin{pmatrix} B_1 & K \\ K & B_2 \end{pmatrix} = \begin{pmatrix} \int_0^T \theta_1^2(t)\mathrm{d}b(t) & \int_0^T \theta_1(t)\theta_2(t)\mathrm{d}b(t) \\ \int_0^T \theta_1(t)\theta_2(t)\mathrm{d}b(t) & \int_0^T \theta_2^2(t)\mathrm{d}b(t) \end{pmatrix}. \tag{2.41}$$

Put $D = \det M = B_1B_2 - K^2$, and one can compute

$$M^{-1} = (\det M)^{-1}\mathrm{adj}(M) = \frac{1}{D}\begin{pmatrix} B_2 & -K \\ -K & B_1 \end{pmatrix}, \tag{2.42}$$

so that we have $\hat{m}_{1,1} = \frac{B_2}{D}$, $\hat{m}_{2,2} = \frac{B_1}{D}$, and $\hat{m}_{2,1} = \hat{m}_{1,2} = \frac{-K}{D}$. Now using Theorem 5 we can write

$$
\begin{aligned}
&\int_{C_{a,b}[0,T]} F(x)\mathfrak{m}(dx) \\
&= \frac{1}{2\pi\sqrt{D}} \int_{\mathbb{R}^2} \exp\left(-\frac{1}{2}\left(u_1^2+u_2^2\right)\right) \exp\left(-\frac{1}{2D}\left(B_2u_1^2+B_1u_2^2-2Ku_1u_2\right)\right) \mathrm{d}u_2\mathrm{d}u_1 \\
&= \frac{1}{2\pi\sqrt{D}} \int_{\mathbb{R}^2} \exp\left(-\frac{1}{2D}\left((B_2+D)u_1^2+(B_1+D)u_2^2-2Ku_1u_2\right)\right) \mathrm{d}u_2\mathrm{d}u_1 \\
&= \frac{1}{2\pi\sqrt{D}} \int_{\mathbb{R}} \exp\left(\left(\frac{K^2-(B_1+D)(B_2+D)}{2D(B_1+D)}u_1^2\right)\right) \\
&\qquad\qquad \int_{\mathbb{R}} \exp\left(-\frac{B_1+D}{2D}\left(u_2-\frac{Ku_1}{B_1+D}\right)^2\right) \mathrm{d}u_2\mathrm{d}u_1 \\
&= \frac{1}{\sqrt{B_1+D}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left(-\frac{1}{2D}\left(\frac{-K^2+(B_1+D)(B_2+D)}{B_1+D}u_1^2\right)\right) \mathrm{d}u_1 \\
&= \frac{1}{\sqrt{B_1+D}} \left(\frac{-K^2+(B_2+D)(B_1+D)}{D(B_1+D)}\right)^{-\frac{1}{2}} \\
&= \left(\frac{D}{D^2-K^2+DB_1+DB_2+B_1B_2}\right)^{\frac{1}{2}} \\
&= \left(\frac{D}{D^2+DB_1+DB_2+D}\right)^{\frac{1}{2}} \\
&= \left(\frac{1}{1+B_1+B_2+B_1B_2-K^2}\right)^{\frac{1}{2}},
\end{aligned}
$$

where the third equality follows from completing the square.

In our second example we no longer require that $a(\cdot)=0$, and we impose the stronger hypothesis that $\{\theta_1,\ldots,\theta_n\}$ be an orthogonal set in $L^2_{a,b}[0,T]$. Define the functional

$$
F(x) = f(\langle\theta_1,x\rangle,\ldots,\langle\theta_n,x\rangle) = \exp\left(-\sum_{j=1}^{n}\langle\theta_j,x\rangle^2\right). \tag{2.43}
$$

Then by applying Theorem 5 with the method of the previous example and exploiting the fact that the functions θ_j are orthogonal (and thus M is diagonal), we find that

$$\begin{aligned}\int_{C_{a,b}[0,T]} F(x)\mathfrak{m}(dx) &= \int_{C_{a,b}[0,T]} \exp\left(-\sum_{j=1}^{n}\langle\theta_j, x\rangle^2\right)\mathfrak{m}(dx)\\ &= \left(\prod_{j=1}^{n}\frac{1}{\sqrt{1+2B_j}}\right)\exp\left(-\sum_{j=1}^{n}\frac{A_j^2}{1+2B_j}\right),\end{aligned}$$

where $A_j = \int_0^T \theta_j(t)\mathrm{d}a(t)$ and $B_j = \int_0^T \theta_j^2(t)\mathrm{d}b(t)$.

Chapter 3

General Spaces

3.1 Cylinder sets and cylindrical Gaussian measures

In this chapter, all functions are understood to be $\mathbb{R}$-valued unless otherwise specified. The symbol $\langle \cdot, \cdot \rangle$ will generally denote a duality pairing, while $(\cdot, \cdot)$ will be reserved for inner products. Let $\mathcal{B}(X)$ denote the Borel σ-algebra of a topological space X. We begin by stating some important definitions and theorems. See [5, 36, 56] for a more thorough treatment. For consistency, most of our terminology and statements of definitions and theorems are substantially borrowed from [5].

A cylinder set in a locally convex space X is a set of the form

$$C(x_1^*, \ldots, x_n^*) = \{x \in X : (\langle x_1^*, x \rangle, \ldots, \langle x_n^*, x \rangle) \in C'\} \tag{3.1}$$

for some Borel set $C' \subseteq \mathbb{R}^n$. Let $\mathcal{E}(X)$ denote the σ-algebra generated by the collection of cylindrical subsets of X. Note that every $x^* \in X^*$ is measurable on $\mathcal{E}(X)$ by definition.

We remark that $\mathcal{E}(X)$ is always contained in the Borel σ-algebra $\mathcal{B}(X)$ but the two do not generally coincide. However, if B is a separable Banach space (hence Fréchet), it follows that the Borel σ-algebra $\mathcal{B}(B) = \mathcal{E}(B)$. For a proof of this fact, see [55].

Definition 4. A measure γ on a locally convex space X is called Gaussian if $\gamma \circ (x^*)^{-1}$ is a Gaussian measure on $\mathbb{R}$; that is, $\gamma \circ (x^*)^{-1}$ is either a point mass or has density

$$\frac{d(\gamma \circ (x^*)^{-1})}{d\mathrm{m}}(u) = \frac{1}{\sqrt{2\pi b}} \exp\left(-\frac{(u-a)^2}{2b}\right)$$

for some real a and some $b > 0$. Equivalently, one can show that γ is Gaussian if $\gamma \circ P^{-1}$ is a Gaussian measure on $\mathcal{B}(\mathbb{R}^n)$ for every continuous linear map $P : B \to \mathbb{R}^n$.

The following theorem characterizes Gaussian measures on locally convex spaces. It is essentially taken from Chapter 2 of [5].

Theorem 6. *A measure γ on a locally convex space X is Gaussian if and only if its Fourier transform has the form*

$$\hat{\gamma}(x^*) = \exp\left(iL(x^*) - \frac{1}{2}Q(x^*, x^*)\right), \tag{3.2}$$

where L is a linear function on X^* and $Q(\cdot,\cdot)$ is symmetric and bilinear on X^* such that the form $Q(x^*, x^*) \geq 0$ for all x^*.

The maps L and Q capture the structure of the measure γ by determining where and how it is distributed on X. The next definition gives us two useful tools for analyzing the measure γ.

Definition 5. For a locally convex space X, the covariance operator R_γ for a measure γ on $\mathcal{E}(X)$ with $X^* \subseteq L^2(\gamma)$ is defined by the formula

$$R_\gamma x^*(y^*) = \int_X \left(\langle x^*, x\rangle - m(x^*)\right)\left(\langle y^*, x\rangle - m(y^*)\right)\gamma(dx), \tag{3.3}$$

where

$$m(x^*) = \int_X \langle x^*, x\rangle \gamma(dx) \tag{3.4}$$

is the mean of the functional x^* with respect to γ.

We say that a Gaussian measure γ on X is centered if $m(x^*) = 0$ for all $x^* \in X^*$ and observe that a measure γ with Fourier transform given by (3.2) is centered if and only if $L = 0$. From (3.2) and the fact that the measure γ is Gaussian it is not hard to see that $m(x^*) = L(x^*)$ and that $R_\gamma x^*(y^*) = Q(x^*, y^*)$ for $x^*, y^* \in X^*$. We will consider the bilinear map $Q(\cdot,\cdot)$ to be defined by $Q(x^*, y^*) = \langle Qx^*, y^*\rangle$ whenever there is an appropriate operator $Q : X^* \to X$. This is always possible if X is a separable Banach space.

To see why this is so, note that our notation $R_\gamma x^*(y^*)$ is suggestive of the fact that for each x^*, the entity $R_\gamma x^*$ is itself a map on X^*. This is in fact the case, and for a locally convex space X, each $R_\gamma x^*$ will be an element of $(X^*)'$, the algebraic (not necessarily topological) dual of X^*. Thus we can define $Q : B^* \to B$ by $\langle Qx^*, y^*\rangle = R_\gamma x^*(y^*) = Q(x^*, y^*)$. In light of this discussion, we will often take $L = \langle a, \cdot\rangle$ for some $a \in B$ and $Q(\cdot,\cdot) = \langle Q\cdot,\cdot\rangle$ and write (3.2) as

$$\hat{\gamma}(x^*) = \exp\left(i\langle a, x^*\rangle - \frac{1}{2}\langle Qx^*, x^*\rangle\right).$$

We now introduce a more general related notion.

Definition 6. A cylindrical Gaussian measure on a Banach space B is a non-negative finitely additive function γ on the algebra (note: not necessarily a σ-algebra) of cylinder sets of B such that $\gamma \circ P^{-1}$ is countably additive and Gaussian on $\mathcal{B}(\mathbb{R}^n)$ for any continuous linear projection $P : B \to \mathbb{R}^n$. We note that the Fourier transform of a cylindrical Gaussian measure γ is the function $\hat{\gamma} : B^* \to \mathbb{C}$ defined by

$$\hat{\gamma}(x^*) = \int_{\mathbb{R}} \exp(it)\gamma \circ (x^*)^{-1}(dt). \tag{3.5}$$

For a Hilbert space H, the canonical cylindrical Gaussian measure is understood to be the set function γ_0 on $\mathcal{E}(H)$ whose Fourier transform is

$$\hat{\gamma}_0(h) = \exp\left(-\frac{1}{2}\,||h||^2\right) = \exp\left(-\frac{1}{2}\,(h,h)\right). \tag{3.6}$$

It is well-known that the canonical cylindrical Gaussian measure is not countably additive on H unless the identity operator on H is trace class (*i.e.* if H is finite dimensional).

Example 1. Suppose that γ_0 is actually a measure on a separable, infinite dimensional Hilbert space H, let $\{h : ||h|| \le r\}$ be a ball of any radius r in H and let P_n be an increasing sequence of orthogonal projections converging to the identity operator in the strong operator sense. From (3.6) we determine that

$$\hat{\gamma}_0(e_n) = \exp\left(-\frac{1}{2}\,||e_n||^2\right) = \exp\left(-\frac{1}{2}\right)$$

for each basis element $e_n \in H$. Then $\{(e_n, \cdot) : n = 1, 2, \ldots\}$ is a collection of mean 0 Gaussian random variables for which the covariance of $(e_i, \cdot)$ and $(e_j, \cdot)$ is the Kronecker delta $\delta_{i,j}$, whence they are independent. For every n, we have

$$\begin{aligned}
\gamma_0\,\{h : ||h|| \le r\} &\le \gamma_0\,\{h : ||P_n h|| \le r\} \\
&= \gamma_0\left\{h : \left|\left|\sum_{j=1}^{n}(h, e_j)e_j\right|\right| \le r\right\} \\
&= \int_{\{\sum_{j=1}^{n} u_j^2 \le r\}}\!\!\cdots\int (2\pi)^{-\frac{n}{2}} \exp\left(-\frac{1}{2}\sum_{j=1}^{n} u_j^2\right) \mathrm{d}u_1 \cdots \mathrm{d}u_n \\
&\le \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{r}{2}} \mathrm{m}(\mathrm{Ball}(0; r)) \\
&= \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}+1\right)} e^{-\frac{r}{2}}
\end{aligned}$$

Observe that as $n \to \infty$, the last expression converges to 0, whence it must be the case that $\gamma_0 \{h : ||h|| \leq r\} = 0$ for every $r \geq 0$. This forces $\gamma_0(H) = 0$, which yields a contradiction because then

$$0 = \gamma_0(H) \geq \gamma_0 \{h : (h, e_1) > 0\} = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{\frac{-u^2}{2}} \mathrm{d}u = \frac{1}{2}.$$

Alternatively, one can use the Cameron-Martin translation theorem (which we will discuss later), to show that $\gamma_0(B)$ must be 0 or ∞ for every open ball B in H. In either case, γ_0 cannot be a measure on H. However, the canonical cylindrical Gaussian measure will extend to a countably additive measure on a suitable larger space, as we will see.

Definition 7. Let H be a separable Hilbert space with canonical cylindrical Gaussian measure γ_0. A seminorm q on H is said to be measurable (in the sense of Gross) if for all $\varepsilon > 0$ there is a finite-dimensional orthogonal projection P_ε such that

$$\gamma_0 \{h \in H : q(Ph) > \varepsilon\} < \varepsilon \tag{3.7}$$

for every finite dimensional orthogonal projection P satisfying $P \perp P_\varepsilon$.

We remark that sometimes the definition of a measurable seminorm is also stated in terms of a sequence of projections P_n converging to the identity operator in the strong operator topology; this amounts to the same condition as shown above and is often more useful in practice. In Example 1, it is just such a sequence that led to the undoing of γ_0 as a measure. Very loosely speaking, one should think of (3.7) as indicating that according to a weighting provided by the seminorm q, the "mass" of H (with respect to the cylindrical measure) is in some manner concentrated on certain of its finite-dimensional subspaces.

Observe that for a continuous linear embedding of Banach spaces $E \hookrightarrow F$, the norm $||\cdot||_F$ can be taken as a seminorm on E. This fact is important in the next definition and theorem, which are essential to all that follows. See Chapter 3 of [5] or [28] for a proof of the theorem.

Definition 8. Let H be a separable Hilbert space and let B be a separable Banach space such that $i : H \hookrightarrow B$ is a continuous linear embedding with dense range. The triple (i, H, B) is called an abstract Wiener space if the composition of the norm of B with i is a measurable seminorm on H.

Theorem 7. *If (i, H, B) is an abstract Wiener space, then the canonical cylindrical Gaussian measure γ_0 on H extends to a countably additive measure γ on B. In addition, $i(H)$ coincides with the Cameron-Martin space of this measure.*

The embedding $H \hookrightarrow B$ in the previous theorem is said to radonify the cylindrical measure γ_0 on H. The best intuition for this situation is that the Hilbert space H was "too small" to support the cylindrical Gaussian measure γ_0 as a measure, but that by expanding to the larger space B with its somewhat weaker norm, we can achieve the countable additivity lacking in the cylindrical measure γ_0. Two other good pictures to have in mind are of H forming a skeleton and B the flesh surrounding it, or H acting as a chicken wire form and B acting as a paper-mâché covering of the wire.

3.2 Centered Gaussian measures on Lebesgue spaces

This section contains neither new nor (in and of themselves) deep results. Its intent is to motivate and frame the subsequent discussion. We are generally interested in constructing Gaussian measures on spaces of functions, and we start by considering such measures on certain Lebesgue spaces.

A necessary and sufficient condition for the existence of a centered Gaussian measure on $L^p(S; \nu)$ is offered by Vakhaniya, Tarieladze, and Chobanyan in [55], in the following theorem (statement adapted from Theorem 3.11.15 from [5]).

Theorem 8. *Let ν be a positive measure on S and for $1 \leq p < \infty$ and $q = p(1-p)^{-1}$ let $L^p(S;\nu)$ be separable. If γ is a centered Gaussian measure on $L^p(S;\nu)$, then the covariance operator $R_\gamma : L^q(S;\nu) \to L^p(S;\nu)$ is given by*

$$R_\gamma f(s) = \int_S K(s,t) f(t) \nu(dt), \tag{3.8}$$

where K is a symmetric nonnegative definite measurable function on S^2 satisfying

$$\int_S K(t,t)^{\frac{p}{2}} \nu(dt) < \infty. \tag{3.9}$$

Conversely, for any symmetric nonnegative definite measurable function K satisfying (3.9)*, the operator defined by* (3.8) *coincides with the covariance operator of some centered Gaussian measure on $L^p(S;\nu)$.*

We will consider a slightly more restrictive situation in which we can formulate a slightly more specific result. Let (S,ν) be a σ-finite measure space. For measurable $f : S^2 \to \mathbb{C}$ and $1 \leq p < \infty$, we will write

$$f_p(s) = \left(\int_S |f(s,t)|^p \, \nu(dt) \right)^{\frac{1}{p}},$$
$$f_p(t) = \left(\int_S |f(s,t)|^p \, \nu(ds) \right)^{\frac{1}{p}}.$$

Note that $0 \leq f_p(s), f_p(t) \leq \infty$.

Lemma 7. *If $f : S^2 \to \mathbb{C}$ is measurable then the following are equivalent.*

1. $f \in L^p(S^2; \nu \times \nu)$,

2. $f_p(s) \in L^p(S;\nu)$,

3. $f_p(t) \in L^p(S;\nu)$.

Proof. That the first assertion implies the other two (up to a ν null set) is a conclusion of Fubini's theorem. If $f_p(t) \in L^p(S;\nu)$ we simply compute

$$||f||^p_{L^p(\nu\times\nu)} = \int_S \int_S |f(s,t)|^p \, \nu(ds)\nu(dt) = ||f_p(t)||^p_{L^p(S;\nu)} < \infty;$$

in the other case,the computation is identical. □

In particular, we note that $f \in L^2(S^2;\nu\times\nu)$ if and only if $f_2(s) \in L^2(S;\nu)$ and $f_2(t) \in L^2(S;\nu)$. We are now ready to prove our first theorem, which is in some sense merely a corollary of Theorem 8. It does, however, serve as a sort of motivation for what will follow in the subsequent sections of this chapter by demonstrating a construction of which we will make frequent use.

Theorem 9. *Let ν be a positive σ-finite measure on S and let $1 \le p < \infty$. Given $k : S^2 \to \mathbb{R}$ with $k_2(t) \in L^p(S;\nu)$ and $k_p(u) \in L^2(S;\nu)$, let $T : L^2(S;\nu) \to L^p(S;\nu)$ by $Tf(t) = \int_S k(u,t)f(u)\nu(du)$. Then the operator TT^* is the covariance operator for a centered Gaussian measure on $L^p(S;\nu)$.*

Proof. Take T as defined in the theorem and $f \in L^2(S;\nu)$; then

$$\begin{aligned}
\int_S |Tf(t)|^p \, \nu(dt) &\le \int_S \left(\int_S |k(u,t)| \, |f(u)| \, \nu(du) \right)^p \nu(dt) \\
&\le \int_S \left(\int_S |k(u,t)|^2 \, \nu(du) \right)^{\frac{p}{2}} \left(\int_S |f(u)|^2 \, \nu(du) \right)^{\frac{p}{2}} \nu(dt) \\
&= ||f||^p_{L^2(S;\nu)} \int_S |k_2(t)|^p \, \nu(du) \\
&< \infty,
\end{aligned}$$

so that T is a well-defined integral operator. Now, let $T' : L^q(S;\nu) \to L^2(S;\nu)$ by $T'g(u) = \int_S k(u,t)g(t)\nu(dt)$ and then for $g \in L^q$ we see that

$$\begin{aligned}\int_S \left(\int_S k(u,t)g(t)\nu(dt)\right)^2 \nu(du) &\leq \int_S \left(\int_S |k(u,t)|^p \nu(dt)\right)^{\frac{2}{p}} ||g||^2_{L^q(S;\nu)} \nu(du)\\ &= ||g||^2_{L^q(S;\nu)} \int_S |k_p(u)|^2 \nu(du)\\ &< \infty\end{aligned}$$

whence T' is also well-defined.

Now we show that in fact $T' = T^*$. To this end, let $f \in L^2(S;\nu)$ and $g \in L^q(S;\nu)$ and observe that

$$\begin{aligned}\langle Tf, g\rangle &= \int_S Tf(t)g(t)\nu(dt)\\ &= \int_S \int_S k(u,t)f(u)\nu(du)g(t)\nu(dt)\\ &= \int_S f(u) \int_S k(u,t)g(t)\nu(dt)\nu(du)\\ &= (f, T'g),\end{aligned}$$

where the exchange of integrals is justified by the fact that $k_2(t) \in L^p(S;\nu)$, and hence

$$\int_S |g(t)| \int_S |k(u,t)f(u)|\, \nu(du)\nu(dt) \leq ||f||_{L^2(S;\nu)} \int_S |g(t)|\, |k_2(t)|\, \nu(dt) < \infty.$$

Now by Theorem 8, it suffices to show that

$$\int_S K(t,t)^{\frac{p}{2}} \nu(dt) = \int_S \left(\int_S |k(u,t)|^2 \nu(du)\right)^{\frac{p}{2}} \nu(dt) = \int_S |k_2(t)|^p \nu(dt) < \infty,$$

completing the proof. $\square$

Thus for $1 \leq p < \infty$ we can construct centered Gaussian measures on $L^p(S;\nu)$. The situation for $p = \infty$ is more delicate, both because $L^\infty(S;\nu)$ is not separable and because the dual of $L^\infty(S;\nu)$ is not very friendly. (We will later see how to construct some satisfactory Gaussian measures on $L^\infty(S;\nu)$ should we be so inclined.) Of course, the case $p = 2$ is of particular interest to us.

Corollary 3. *Let ν be a positive σ-finite measure on S. Given $k : S^2 \to \mathbb{R}$ with $k \in L^2(S^2; \nu \times \nu)$, put $Tf(t) = \int_S k(u,t) f(u) \nu(du)$. Then the operator TT^* is the covariance operator for a centered Gaussian measure on $L^2(S;\nu)$.*

Proof. By Lemma 7, we have $k_2(t) \in L^2(S;\nu)$. Note that $p = q = 2$, and then $K(s,t) = \int_S k(u,s)k(u,t)\nu(du)$ satisfies equation (3.9), as

$$\int_S K(t,t)\nu(dt) = \int_S \int_S |k(u,t)|^2 \nu(du)\nu(dt) = \int_S |k_2(t)|^2 \nu(dt) < \infty.$$

In addition, $TT^* f(t) = \int_S K(s,t) f(s) \nu(ds)$, and the conclusion follows by Theorem 8. $\square$

We also note that one could adapt such a construction for $L^p(S;\nu)$ to include measures that are not centered. For example, take some $a \in L^2(S)$ and T as above and then $(\cdot, a)$ is a linear map on $L^2(S;\nu)$ and there is a measure γ on $L^2(S;\nu)$ with Fourier transform

$$\hat{\gamma}(f) = \exp\left(i\,(f,a) - \frac{1}{2}\,(TT^* f, f) \right).$$

The result is a Gaussian measure on $L^2(S;\nu)$ that is centered around the element a and has covariance operator TT^*.

To conclude this section, we briefly consider a situation arising if we choose a purely singular finite measure ν with finite support on S.

Corollary 4. *Let $k : S^2 \to \mathbb{R}$ and let ν be a positive finite measure on S. If $D = \mathrm{supp}(\nu)$ is a finite subset of S, then the measure γ as constructed in Theorem 9 is the probability measure for a centered $|D|$-variate normal distribution having covariance matrix with entries $c_{i,j} = a_k k(t_k, t_i) k(t_k, t_j)$ for some collection $\{a_k\}$ of positive numbers.*

Proof. If $D = \{t_1, t_2, \ldots, T_{|D|}\}$ is finite, then $\nu = \sum_{k=1}^{|D|} a_k \delta_{t_k}$ and $L^2(S;\nu) \cong \mathbb{R}^{|D|}$. □

3.3 General construction and properties

We draw attention to the fact that in the previous section, the covariance operator is being factored through the Hilbert space $L^2(S;\nu)$, as shown below.

$$\begin{tikzcd} L^2(S;\nu)^* \arrow[rr, "TT^*"] \arrow[dr, "T^*"'] & & L^2(S;\nu) \\ & L^2(S;\nu) \arrow[ur, "T"'] & \end{tikzcd}$$

This notion of factoring the covariance operator through a suitably chosen Hilbert space will inform our further discussion. Recall also that $k \in L^2(S^2; \nu \times \nu)$ is a necessary and sufficient condition for an integral operator with kernel k to be Hilbert-Schmidt on $L^2(S;\nu)$, so it is immediately clear that TT^* will be trace-class, *i.e.* it has a sequence of orthonormal eigenvectors with eigenvalues (α_j) satisfying $\sum_{j=1}^{\infty} |\alpha_j| < \infty$. This is exactly the condition that is required for the covariance operator of a Gaussian measure on a Hilbert space (recall the requirement for the canonical cylindrical Gaussian measure on H to be a measure), as the following theorem from [5] shows.

Theorem 10. *Let γ be a Gaussian measure on a separable Hilbert space H. Then there exist $a \in H$ and a symmetric non-negative trace-class operator K such that the Fourier transform* *of the measure γ is*

$$\hat{\gamma}(h) = \exp\left(i(a,h) - \frac{1}{2}(Kh,h)\right).$$

Conversely, for every pair (a, K) as above, the function $\hat{\gamma}$ above is the Fourier transform of a Gaussian measure on H with mean a and covariance operator K.

In this section we discuss our general method of constructing measures. We begin by introducing the Cameron-Martin space H_γ for a Gaussian measure γ on a separable Banach space B.

Definition 9. For a Gaussian measure γ on a separable Banach space B, we denote by B^*_γ the closure of the set $\{x^* - m(x^*) : x^* \in B^*\}$ of affine maps in $L^2(B;\gamma)$. Thus B^*_γ is a Hilbert space with inner product

$$(f,g)_{L^2(\gamma)} = \int_B f(x)g(x)\gamma(dx).$$

Let $\tau : B^* \to B^*_\gamma$ by $\tau x^* = x^* - m(x^*)$, so that $B^*_\gamma = \mathrm{cl}_{L^2(\gamma)}(\tau(B^*))$.

From Definition 5, we have $m(x^*) = \int_B \langle x^*, x\rangle \gamma(dx)$ and

$$R_\gamma x^*(y^*) = \int_B \left(\langle x^*, x\rangle - m(x^*)\right)\left(\langle y^*, x\rangle - m(y^*)\right)\gamma(dx), \tag{3.10}$$

and from the discussion following Definition 4 we know that $m(x^*)$ and $R_\gamma x^*$ are elements of B. We wish to extend the operator R_γ to B^*_γ. From (3.10) we see that

$$R_\gamma x^*(y^*) = (\tau x^*, \tau y^*)_{L^2(\gamma)}. \tag{3.11}$$

We also observe that

$$\int_B \tau x^*(x)\gamma(dx) = \int_B [\langle x^*, x\rangle - m(x^*)]\,\gamma(dx) = \int_B \langle x^*, x\rangle\gamma(dx) - m(x^*)\int_B \gamma(dx) = 0,$$

and hence if we extend τ to B^*_γ in the obvious fashion we will have $m(\tau x^*) = 0$, so that $\tau(\tau x^*) = \tau x^* - m(\tau x^*) = \tau x^*$ for each $x \in B^*$. Now, from (3.11) it is clear that we can extend R_γ to B^*_γ, with

$$R_\gamma g(x^*) = (g, \tau x^*)_{L^2(\gamma)}$$

for $g \in B^*_\gamma$.

For any separable Banach space B, $R_\gamma x^*$ will necessarily be an element of B^{**}, and in fact can be taken as an element of $B \subseteq B^{**}$ under the natural embedding; in other words, both $m(\cdot) \in B$ and $R_\gamma g \in B$ for each $g \in B^*_\gamma$. This results from the facts that γ is Radon and B is complete and locally convex so that $m(\cdot)$ and $R_\gamma g(\cdot)$ are continuous in the Mackey topology on X^* (see Section 3.2 of [5] for a proof).

Note that $m(g) = \int_B g(x)\gamma(dx) = 0$ for every $g \in B^*_\gamma$. To see this, observe that $||g_n - g||_{L^2(\gamma)} \to 0$ with $g_n = x^*_n - m(x^*_n)$ for some sequence (x^*_n) in B^*. It is clear that $m(g_n) = 0$ for every n. Then

$$\left|\int_B g(x)\gamma(dx)\right| = \left|\int_B g_n(x) - g(x)\gamma(dx)\right| \leq \int_B |g_n(x) - g(x)|\,\gamma(dx) \leq ||g_n - g||_{L^2(\gamma)}\,\gamma(B),$$

whence $m(g) = 0$.

From here forward, we take R to be the operator R_γ with domain restricted to B^*, while by R_γ we will mean the operator having domain B^*_γ. That is, R factors as $R = R_\gamma\tau$, as shown below.

$$\begin{tikzcd} B^* \arrow[rr, "R"] \arrow[dr, "\tau"'] & & B \\ & B^*_\gamma \arrow[ur, "R_\gamma"'] & \end{tikzcd} \tag{3.12}$$

From this we see that

$$\langle Rx^*, y^*\rangle = R_\gamma \tau x^*(y^*) = R_\gamma x^*(y^*) = (\tau x^*, \tau y^*)_{L^2(\gamma)} \tag{3.13}$$

for all $x^*, y^* \in B^*$.

Lemma 8. *Let γ be a Gaussian measure on a separable Banach space B and put $\tau : B^* \to B^*_\gamma$ by $\tau x^* = x^* - m(x^*)$. Then:*

1. *τ is linear and (weak*, weak) continuous.*

2. *R_γ is (weak*, weak) continuous.*

3. *$R^*_\gamma = \tau$.*

Proof. The linearity of τ follows from the fact that B^* is a linear space and the linearity of the integral $\int_B \langle x^*, x\rangle \gamma(dx) = m(x^*)$. Now let $x^*_n \to 0$ in B^* and let h be any element of B^*_γ. Then $(\tau x^*_n, h)_{L^2(\gamma)} = R_\gamma h(x^*_n)$, and then the fact that $R_\gamma h \in B$ and the magic of linearity establish that τ is (weak*, weak) continuous. The (weak*, weak) continuity of R_γ follows by the same argument and (3.11). Finally, let $x^* \in B^*$ and let $g \in B^*_\gamma = \mathrm{cl}_{L^2(\gamma)}(\tau(B^*))$ and then for $h = R_\gamma g$ we have $\langle h, x^*\rangle = R_\gamma g(x^*) = (g, \tau x^*)_{L^2(\gamma)}$. $\square$

It is known from a theorem of Kallianpur from [34] that every Gaussian measure γ on a Banach space B has an associated Cameron-Martin space H_γ; moreover, if (i, H_γ, B) is an abstract Wiener space, we always have

$$\begin{tikzcd} B^* \arrow[rr, "ii^*"] \arrow[dr, "i^*"'] & & B \\ & H_\gamma \arrow[ur, "i"'] & \end{tikzcd} \tag{3.14}$$

where i is the embedding $H_\gamma \hookrightarrow B$. Thus one can always factor the covariance operator of a Gaussian measure through some Hilbert space. It is also a fact that the covariance operator of any cylindrical Gaussian measure on B will also have a Cameron-Martin space and will factor as in (3.14). However, there is no guarantee that the cylindrical measure γ will be radonified by the inclusion $H_\gamma \hookrightarrow B$. This radonification question has been the subject of a considerable amount of inquiry; for a very good history and survey see [57].

The following lemma from Chapter 3 of [5] characterizes the Cameron-Martin space in terms of B^*_γ and R_γ.

Lemma 9. *An element h of B belongs to the Cameron-Martin space H_γ of γ if and only if there is some $g \in B^*_\gamma$ with $h = R_\gamma g$. In this case $||h||_{H_\gamma} = ||g||_{L^2(\gamma)}$.*

We can also consider the Cameron-Martin space as a Hilbert space associated with a particular inner product. Let R be a positive, symmetric operator from B^* to B. For x^* and y^* in B^*, define the form $(Rx^*, Ry^*) = \langle Rx^*, y^* \rangle$. It is easy to check that this form is symmetric and bilinear and that $0 \leq (Rx^*, Rx^*) = \langle Rx^*, x^* \rangle$. Now, to see that this semi-inner product is positive definite, observe that $|(Rx^*, Ry^*)|^2 \leq (Rx^*, Rx^*)(Ry^*, Ry^*)$ by Cauchy-Schwarz, and hence

$$0 \leq |\langle Rx^*, y^* \rangle|^2 = |(Rx^*, Ry^*)| \leq (Rx^*, Rx^*)(Ry^*, Ry^*) = 0$$

for every $y^* \in B^*$ whenever $(Rx^*, Rx^*) = 0$. The Cameron-Martin space H can be thought of as the completion of $R(B^*)$ in the Hilbert norm induced by an inner product of this type.

Understanding the Cameron-Martin space and how it "sits" in the larger Banach space reveals a great deal about a Gaussian measure γ. The next theorems are from Chapter 3 of [5]; the statements have been specified to our setting.

Theorem 11. *Let γ be a Radon Gaussian measure on a locally convex space X which is continuously and linearly embedded into a locally convex space Y. Then the Cameron-Martin space is independent of whether the measure γ is considered on X or Y.*

Theorem 12. *Let γ be a Radon Gaussian measure on a locally convex space X with mean $a_\gamma \in X$. Then the topological support of γ coincides with the affine subspace $a_\gamma + \mathrm{cl}_X(H_\gamma)$. In particular, the support of γ is separable.*

Recall that we are interested in constructing covariance operators of the form $R = TT^*$ for some Hilbert-Schmidt operator $T : H \to B$ (such as the integral operator we were using). In fact, it is necessary for the operator T to be Hilbert-Schmidt for this construction to work, as we will presently show. Recall that for a normed space X, a collection $F \subseteq X^*$ is said to separate the points of X if for every pair of distinct elements x and y there is some $x^* \in F$ such that $\langle x^*, x\rangle \neq \langle x^*, y\rangle$.

Lemma 10. *Let X be a normed space and let D be a collection of elements of X^*. Then D separates the points of X if and only if $\bigcap_{x^* \in D} \ker(x^*) = (0)$.*

Proof. Suppose that $(0) \neq \bigcap_{x^* \in D} \ker(x^*) = \{x : \langle x^*, x\rangle \text{ for all } x^* \in D\}$. Say that y is a nonzero element. Then $\langle x^*, y\rangle = \langle x^*, 0\rangle = 0$ for each $x^* \in D$ and D does not separate points of X.

Now suppose that D does not separate points of B. Then there is some pair of distinct elements x_0 and y_0 such that $\langle x^*, x_0\rangle = \langle x^*, y_0\rangle$, and hence $0 = \langle x^*, x_0 - y_0\rangle$ for every $x^* \in D$. But then there is a nonzero element $x_0 - y_0$ in $\bigcap_{x^* \in D} \ker(x^*)$. $\square$

Lemma 11. *Let X be a normed space. Then for any sequence (x_j^*) that separates points of X, there is a subsequence (x_k^*) which also separates points of X and satisfies*

$$\bigcap_{\substack{j=1 \\ j\neq n}}^{\infty} \ker(x_k^*) \neq (0)$$

for each $n = 1, 2, \ldots$.

Proof. Let (x_j^*) separate points of X. If for some n we have

$$\bigcap_{\substack{j=1 \\ j\neq n}}^{\infty} \ker(x_k^*) = (0), \tag{3.15}$$

then the subsequence $x_1^*, \ldots, x_{n-1}^*, x_{n+1}^*, \ldots$ will also separate points of X by Lemma 10. Consider each x_n^* in turn and discard those for which (3.15) is true to obtain the desired sequence. □

Lemma 12. *If B is a separable Banach space and H is a separable Hilbert space, then there is a continuous linear embedding $\iota : B \hookrightarrow H$. Moreover, ι can be constructed so that it has dense range in H.*

Proof. Let $D = \{x_j^* : j = 1, 2, \ldots\} \subseteq \mathrm{Ball}(B^*)$ separate points of B. Take an orthonormal basis (e_j) of H and an ℓ^2 sequence (α_j) of nonzero terms. Define $i : B \to H$ by $\iota(x) = \sum_{j=1}^{\infty} \alpha_j \langle x_j^*, x\rangle e_j$.

Note that $\left|\langle x_j^*, x\rangle\right| \leq ||x||_B$ for any $x \in B$, and then

$$||\iota(x)||_H = \sum_{j=1}^{\infty} \alpha_j^2 \left|\langle x_j^*, x\rangle\right|^2 \leq ||x||_B^2 \sum_{j=1}^{\infty} \alpha_j^2 < \infty, \tag{3.16}$$

so that i is well-defined. It is easy to see that i is linear. The map is injective due to the fact that D separates points of B, and each α_j is nonzero, so that at least one $\alpha_j \langle x_j^*, x\rangle \neq 0$

for each $x \neq 0$. By (3.16) we also see that $||\iota(x)||_H \leq C\,||x||_B$ for some $C > 0$, whence ι is continuous.

To ensure that ι has dense range, use Lemma 11 choose the collection $D \subseteq B^*$ so that

$$M_k = \bigcap_{\substack{j=1 \\ j \neq k}}^{\infty} \ker(x_j^*) \neq (0)$$

for each $k = 1, 2, \ldots$. Note that this ensures that each $M_k \nsubseteq \ker(x_k^*)$, for otherwise D could not separate points of B by Lemma 10. Now, let any $h = \sum_{j=1}^{\infty}(h, e_j)e_j$ in H and any $\varepsilon > 0$ be given. Take N sufficiently large that $\sum_{j=N+1}^{\infty} |(h, e_j)|^2 < \varepsilon$. We will obtain an $x \in B$ for which $||\iota(x) - h|| < \varepsilon$.

To do this, put $a_j = \frac{1}{\alpha_j}(h, e_j)$. Now, take $x_1 \in M_1$ so that $\langle x_1^*, x_1 \rangle = a_1$, which is possible by the linearity of x_1^* and the fact that $M_1 \nsubseteq \ker(x_1^*)$. Do the same for $j = 2, \ldots, N$. Now put $x = \sum_{j=1}^{N} x_j$ and observe that $\langle x_k^*, x \rangle = \sum_{j=1}^{N} \langle x_k^*, x_j \rangle = \langle x_k^*, x_k \rangle = a_k$ for $k = 1, 2, \ldots, N$ and $\langle x_k^*, x \rangle = 0$ for $k \geq N + 1$ because each $x_j \in M_j \subseteq \ker(x_k^*)$. Then

$$||\iota(x) - h||_H = \sum_{j=1}^{\infty} \left|\alpha_j \langle x_j^*, x \rangle - (h, e_j)\right|^2 = \sum_{j=N+1}^{\infty} |(h, e_j)|^2 < \varepsilon,$$

whence ι has dense range in H. □

Lemma 13. *Let H be a separable Hilbert space, B be a separable Banach space, let $\iota : B \hookrightarrow H$ be a continuous linear embedding with dense range, and let $T : H \to B$ be continuous.*

1. *If* $\ker(T) = (0)$, *then* $\operatorname{ran}(T^*\iota^*)$ *is dense.*

2. *If* $\ker(T^*) = (0)$, *then* $\ker(\iota^*) = \ker(T^*\iota^*) = \operatorname{ran}(iT)^{\perp}$.

3. *ι is (weak, weak) continuous and ι^* is (weak, weak*) continuous.*

4. *If ι^* is injective, then $\iota^*\iota : B \hookrightarrow B^*$ is a (weak, weak*) continuous linear embedding.*

Proof. By the duality relationships, we have the following diagrams.

$$\begin{tikzcd}
& B^* \arrow[dr, "T^*"] \arrow[dd, "TT^*"] & \\
H \arrow[ur, "\iota^*"] & & H \arrow[dl, "T"] \\
& B \arrow[ul, "\iota"] &
\end{tikzcd}$$

$$\begin{tikzcd}
& B^* \arrow[dr, "T^*"] & \\
H \arrow[ur, "\iota^*"] \arrow[rr, bend left, "T^*i^*"] & & H \arrow[dl, "T"] \arrow[ll, bend left, "iT"] \\
& B \arrow[ul, "\iota"] &
\end{tikzcd}$$

The first and second statements result by "chasing the arrows" of the diagrams and noting the fact that for a bounded linear operator A on a Hilbert space H, $\ker(A) = \operatorname{ran}(A^*)^{\perp}$.

To see that the third statement holds, let $x_n \to x$ weakly in B and $h_n \to h$ weakly in H. Then $(\iota x_n, g) = \langle x_n \iota^* g \rangle \to \langle x, \iota^* g \rangle = (\iota x, g)$ for all $g \in H$ and $\langle \iota^* h_n, y \rangle = (h_n, \iota y) \to (h, \iota y) = \langle \iota^* h, y \rangle$ for all $y \in B$. The fourth statement follows directly from the third. $\square$

Theorem 13. *Let H be a separable Hilbert space and B be a separable Banach space with $T : H \to B$ an operator such that $R = TT^*$ is the covariance operator of a Gaussian measure γ on B. Then T must be a Hilbert-Schmidt operator.*

Proof. For convenience, let γ be a centered measure and suppose that T is as given. By Lemma 12, we can embed $B \hookrightarrow H$ densely. From Theorem 11 we note that we can take γ as a Gaussian measure on H with the same Cameron-Martin space. In addition, we have

$\iota^* : H \to B^*$. We now have the situation shown in the following diagram.

$$\begin{tikzcd} & B^* \arrow[dr, "T^*"] & \\ H \arrow[ur, "\iota^*"] & & H \arrow[dl, "T"] \\ & B \arrow[ul, "\iota"] & \end{tikzcd} \tag{3.17}$$

The measure γ on B has Fourier transform

$$\hat{\gamma}(x^*) = \exp\left(-\frac{1}{2}\left(T^*x^*, T^*x^*\right)_H\right).$$

Taken on $H = \iota(B)$, we see that $\iota^* h \in B^*$ when $h \in H$, and

$$\hat{\gamma}(h) = \exp\left(-\frac{1}{2}\left(T^*\iota^*h, T^*\iota^*h\right)_H\right) = \exp\left(-\frac{1}{2}\left(\iota TT^*\iota^*h, h\right)_H\right),$$

because $(\iota T)^* = T^*\iota^*$, and thus γ is a Gaussian measure on H with covariance operator $\iota TT^*\iota^* = \iota R \iota^*$, which must be trace-class by Theorem 10.

Now we show that T must be Hilbert-Schmidt. Because ι is a dense injective embedding, it will suffice to show that ιT is Hilbert-Schmidt, as then an orthonormal basis of eigenvectors for T can be recovered. To do this, we note that $\iota TT^*\iota$ is trace-class, positive, and symmetric. Thus we have the polar decomposition

$$\iota TT^*\iota^* = U\left|\iota TT^*\iota^*\right| = \left(U\sqrt{|\iota TT^*\iota^*|}\right)\sqrt{|\iota TT^*\iota^*|},$$

where U is a partial isometry on the range of $|\iota TT^*\iota^*| = \sqrt{(\iota TT^*\iota^*)(\iota TT^*\iota^*)^*}$ and $\sqrt{|\iota TT^*\iota^*|}$ is Hilbert-Schmidt. Because $\iota TT^*\iota^*$ is positive, $|\iota TT^*\iota^*| = \iota TT^*\iota^*$, and thus we see that $\sqrt{\iota TT^*\iota^*}$ is a Hilbert-Schmidt operator. But $\sqrt{\iota TT^*\iota} = \sqrt{(\iota T)(\iota T)^*} = |\iota T|$, whence $|\iota T|$ and

ιT are also Hilbert-Schmidt operators. $\square$

The following theorems yield additional interesting information about these measures. We will say that a Gaussian measure γ on a Banach space B has mean $a_\gamma \in B$ if each $x^* \in B^*$ has mean $\langle x^*, a_\gamma \rangle$.

Theorem 14. *Let γ be a Radon Gaussian measure on a Banach space B, having mean a_γ. Then* $\mathrm{supp}(\gamma)$ *is a subspace of B if and only if $a_\gamma \in \mathrm{cl}_B(H_\gamma)$.*

Proof. Note that $\mathrm{supp}(\gamma) = a_\gamma + \mathrm{cl}_B(H_\gamma)$ by Theorem 12. H_γ is a linear subspace (not necessarily closed) of B, whence $\mathrm{cl}_B(H_\gamma)$ is a closed linear subspace. Now, if $\mathrm{supp}(\gamma)$ is a subspace, then $a_\gamma = -h$ for some $h \in \mathrm{cl}_B(H_\gamma)$. Conversely, if $a_\gamma \in \mathrm{cl}_B(H_\gamma)$, then so is $-a_\gamma$, whence $0 \in \mathrm{supp}(\gamma)$. For any $x \in \mathrm{supp}(\gamma)$, note that $x = a_\gamma + x_0$ with $x_0 \in \mathrm{cl}_B(H_\gamma)$. For any $t \in \mathbb{R}$ we will have $tx = ta_\gamma + tx_0 = a_\gamma + (tx_0 - (1-t)a_\gamma) \in \mathrm{supp}(\gamma)$. Similarly, for x and y in $\mathrm{supp}(\gamma)$, $x + y = a_\gamma + x_0 + a_\gamma + y_0 = a_\gamma + (x_0 + y_0 + a_\gamma) \in \mathrm{supp}(\gamma)$. $\square$

Theorem 15. *Let H be a separable Hilbert space and B be a separable Banach space with $T : H \to B$ a Hilbert-Schmidt operator such that $R = TT^*$ is the covariance operator for a centered Gaussian measure γ on B. Then:*

1. *$T(H) = H_\gamma$, the Cameron-Martin space for γ, and $T : \mathrm{cl}_H(T^*(B^*)) \hookrightarrow H_\gamma$ is an isometry.*

2. $\mathrm{supp}(\gamma) \subseteq \bigcap_{\ker T^*} \ker(x^*)$.

Proof. As γ is a centered measure, $m(x^*) = 0$ for all $x^* \in B^*$ and τ is merely the identity operator on B^*. Thus B^*_γ is just the closure of B^* in $L^2(\gamma)$ and the operator $R_\gamma = R = TT^*$.

$$\begin{tikzcd} B^* \arrow[rr, "R=R_\gamma=TT^*"] \arrow[dr, "T^*"'] & & B \\ & H \arrow[ur, "T"'] & \end{tikzcd} \tag{3.18}$$

Thus by Lemma 9 the Cameron-Martin space H_γ is the completion of $R(B^*)$ under the norm satisfying

$$||Rx^*||^2_{H_\gamma} = (Rx^*, Rx^*)_{H_\gamma} = \langle Rx^*, x^*\rangle = (T^*x^*, T^*x^*)_H = ||T^*x^*||^2_H \tag{3.19}$$

for every x^*. Every Cauchy sequence (Rx^*_n) in B has a corresponding sequence $(T^*x^*_n)$ in H for which

$$y = \lim_{n\to\infty} Rx^*_n = \lim_{n\to\infty} TT^*x^*_n = T\left(\lim_{n\to\infty} T^*x^*_n\right) = Th,$$

for some $h \in H$, and hence we must have $H_\gamma = T(H)$. From (3.19) we see that the restriction of T is the desired isometry from $\mathrm{cl}_H(T^*(B^*))$ to H_γ.

Next, note that $\ker(T^*)$ always contains the zero functional, whose kernel is obviously all of B, and thus the intersection over $\ker T^*$ is never empty. Observe that $0 = \langle Th, x^*\rangle = (h, T^*x^*)_H$ for all $h \in H$ if and only if $T^*x^* = 0$ in H. Thus $T(H) \subseteq \ker(x^*)$ if and only if $x^* \in \ker(T^*)$, and hence $H_\gamma = T(H) \subseteq \bigcap_{\ker T^*} \ker(x^*)$. From this, we conclude that

$$\mathrm{supp}(\gamma) = \mathrm{cl}_B(T(H)) \subseteq \bigcap_{\ker T^*} \ker(x^*) \tag{3.20}$$

as desired. □

We are interested in the question of how to identify the Cameron-Martin space of a measure that is not centered, and to do so without relying on the definition of H_γ in terms of R_γ and B^*_γ because we would need to first identify B^*_γ, which may be difficult. We would like to do it using the raw information about the operators T and T^* and the Hilbert space H. We have two parallel factorizations for the covariance, so we can hope to play them off against each other. The following diagram shows the situation.

$$\begin{tikzcd} & B^* \arrow[dl, "T^*"'] \arrow[dd, "R"] \arrow[dr, "\tau"] & \\ H \arrow[dr, "T"'] & & B^*_\gamma \arrow[dl, "R_\gamma"] \\ & B & \end{tikzcd}$$

In Theorem 15 we were able to avoid having to deal with reconciling TT^* and $R_\gamma \tau$ because in the centered case τ is the identity and R_γ agrees with R, whence the right-hand side of the diagram collapses, leaving us free to work with $R = R_\gamma$ and only one factorization to obtain the desired result. In general, we need to work a little harder.

Theorem 16. *Let H be a separable Hilbert space and B be a separable Banach space. Let $T : H \to B$ a Hilbert-Schmidt operator for which $R = TT^*$ is the covariance of a Gaussian measure γ on B. Then there is a subspace $H_0 \subseteq H$ and an isometric embedding $B^*_\gamma \hookrightarrow H_0 \subseteq \mathrm{cl}_H(T^*(B^*))$.*

Proof. Let (e_n) be a basis of B^*_γ with each $e_n = \tau x^*_n$ for some sequence (x^*_n) in B^* and define $\phi : \tau(B^*) \to H$ by $\phi(\tau x^*) = T^* x^*$. Then note that $\phi(e_n) = T^* x^*_n$ for each x^*_n. In addition, $(T^* x^*_n, T^* x^*_m)_H = \langle TT^* x^*_n, x^*_m \rangle = R_\gamma x^*_n(x^*_m) = (\tau x^*_n, \tau x^*_m)_{L^2(\gamma)} = \delta_{m,n}$, and thus $(T^* x^*_n)$ is an orthonormal set in H. Put $H_0 = \mathrm{cl}_H \mathrm{span}\left(\{T^* x^*_n\}\right)$. Extend ϕ by linearity and continuity. $\square$

The following diagram illustrates the conclusion of the previous theorem.

$$\begin{tikzcd}
 & & B^* \arrow[dll, "T^*"'] \arrow[dr, "\tau"] & \\
H \arrow[drr, "T"'] & \supseteq H_0 & & B^*_\gamma \arrow[ll, "\cong", "\phi"'] \arrow[dl, "R_\gamma"] \\
 & & B &
\end{tikzcd} \tag{3.21}$$

Corollary 5. *If H_0, B, T, and γ are as in the previous theorem, then the restriction $T : H_0 \to H_\gamma$ is an isometry.*

Proof. Recall that $H_\gamma = R_\gamma(B^*_\gamma)$ with R_γ an isometry by Lemma 9 and then from (3.21) we see that $T = R_\gamma \phi^{-1}$ is an isometry. $\square$

3.4 Measures on $C(S)$

We will now restrict our setting as we examine Gaussian measures on spaces of continuous functions. We take (S, ϱ) to be a compact metric space and let $C(S)$ denote the space of continuous real-valued functions on S and identify $C(S)^*$ with the space of regular Borel (Radon) measures on S by the Riesz representation. Note that $C(S)^*$ is a Banach space under the total variation norm $||\cdot||_{Var}$. Recall that the weak* topology on $C(S)^*$ is the weak topology generated by $C(S)$ under the natural embedding $C(S) \hookrightarrow C(S)^{**}$.

While the space $C(S)$ has the usual supremum norm $||\cdot||_S$ (we reserve $||\cdot||_\infty$ for the essential supremum), we will ultimately desire more regularity on the part of certain functions. For $\alpha > 0$, let $C^\alpha(S)$ denote the usual space of Hölder continuous functions on S. As S is compact, we can define a norm on C^α by

$$||f||_\alpha = ||f||_S + |f|_\alpha = ||f||_S + \sup_{s \neq t} \frac{|f(s) - f(t))|}{\varrho(s,t)^\alpha}. \tag{3.22}$$

In addition, we will need to quantify the complexity of the parameter space S in a particular fashion. By $H(S,\varepsilon)$, we will denote the ε-entropy of S; that is $H(S,\varepsilon) = \log(N(S,\varepsilon))$, where $N(S,\varepsilon)$ counts the minimum number of balls of diameter at most ε needed to cover S, which is guaranteed to exist by the compactness of S. This notion of metric entropy is originally borrowed from Kolmogorov; it figures prominently in the work of Dudley (see [20, 21]).

Recall that a reproducing kernel Hilbert space is a Hilbert space of functions for which pointwise evaluation is a continuous linear functional. That is, if X is a set and H is a Hilbert space of functions on X, then H is a reproducing kernel Hilbert space if there is some function K on X^2 such that $K_x \in H$ and $f(x) = (f, K_x)_H$ for every $x \in X$, where $K_x(y) = K(x,y)$. In this event the function K is the reproducing kernel for H and one often uses the notation $H(K)$ for the space. See [1] for a wealth of details about reproducing kernel Hilbert spaces.

We are now in a position to state the following theorem from [2].

Theorem 17. *Let (S,ϱ) be a compact metric space, and K a reproducing kernel on S with reproducing kernel Hilbert space $\mathcal{H}(K)$. Suppose also that there are fixed $C, \alpha > 0$ such that:*

1. *$K(s,s) + K(t,t) - 2K(s,t) \leq C\ \varrho(s,t)^{2\alpha}$ for all $s,t \in S$,*

2. *$\sum_{n=1}^{\infty} 2^{-n\alpha} H(S, 2^{-n})^{1/2} < \infty$.*

Then $\mathcal{H}(K) \subset C(S)$ and the inclusion is radonifying.

Using this theorem as a base, we can proceed as in Section 3.2 to construct a wide variety of Gaussian measures on $C(S)$. In Section 3.2 we saw the function

$$K(s,t) = \int_S \int_S k(u,v)\delta_s(dv)k(u,t)\nu(du) = \int_S k(u,s)k(u,t)\nu(du), \tag{3.23}$$

which we will refer to as the covariance function for our measure. Notice that $K(s,t) = \langle R\delta_s, \delta_t \rangle = (T^*\delta_s, T^*\delta_t)_{L^2(S;\nu)}$, where T^* and R are as described following Definition 9.

Theorem 18. *Let (S, ϱ) be a compact metric space and $k : S^2 \to \mathbb{R}$ be a bounded Borel function. Let ν be a positive regular Borel measure on S for which $\int_S |k(u,s) - k(u,t)|^2 \nu(du) \leq C\varrho(s,t)^{2\alpha}$ for some $C > 0$ and $\alpha > 0$ and suppose that (S, ϱ) satisfies the metric entropy condition of the previous theorem for this α. If an operator T is defined on $L^2(S;\nu)$ by $Tf(t) = \int_S k(u,t)f(u)\nu(du)$, then TT^* is the covariance operator of a centered Gaussian measure γ on $C(S)$.*

Proof. To prove Theorem 18, we will use Theorem 17 by showing that $\mathcal{H}(K) = T(L^2(S;\nu))$ is a suitable reproducing kernel Hilbert space and showing that the canonical cylindrical Gaussian measure on this space is radonified by the inclusion $T(L^2(S;\nu) \hookrightarrow C(S)$.

Put $T_1\mu(u) = \int_S k(u,t)\mu(dt)$ for each $\mu \in C(S)^*$. For every $f \in L^2(S;\nu)$ and every $\mu \in C(S)^*$ we see that

$$\int_S \int_S |k(u,t)f(u)|\, \nu(du)\, |\mu|\,(dt) \leq ||f||_{L^2(S;\nu)}\, ||k||_\infty\, |\mu|\,(S)$$

by Hölder's inequality, and the hypotheses on k ensure that $|k(u,t)f(u)| \in L^1(S^2; \nu \times |\mu|)$. Thus $T_1 = T^*$, because

$$\begin{aligned}
\langle Tf, \mu \rangle &= \int_S Tf(t)\mu(dt) \\
&= \int_S \int_S k(u,t)f(u)\nu(du)\mu(dt) \\
&= \int_S f(u) \int_S k(u,t)\mu(dt)\nu(du) \\
&= (f, T_1\mu)_{L^2(S;\nu)}
\end{aligned}$$

for every $\mu \in C(S)^*$.

Now note that $K_t(s) = K(s,t) = \langle TT^*\delta_s, \delta_t \rangle$ is a reproducing kernel on S for the space $H = T(L^2(S;\nu))$ with inner product $(Tf, Tg)_H = (f,g)_{L^2(S;\nu)}$, for certainly

$$||K_t||_H^2 = \int_S |k(s,t)|^2 \, \nu(ds) < \infty$$

for each $t \in S$, and also

$$Tf(t) = \int_S k(u,t) f(u) \nu(du) = (k_t, f)_{L^2(\nu)} = (K_t, Tf)_H$$

for each element $Tf \in H$ and each $t \in S$.

It suffices to show that our reproducing kernel K_t satisfies the remaining condition of the previous theorem. This is equivalent to the condition that H is continuously embedded into $C^\alpha(S)$, for then $|h(s) - h(t)| \leq C \, ||h||_H \, \varrho(s,t)^\alpha$ for each $h \in H$, and then taking $f = K_t - K_s$ we have

$$\begin{aligned}
K(t,t) + K(s,s) - 2K(s,t) &= ||K_t - K_s||_H^2 \\
&= |(K_t - K_s, K_t)_H - (K_t - K_s, K_s)_H| \\
&= |f(s) - f(t)| \\
&\leq C^2 \varrho(s,t)^{2\alpha},
\end{aligned}$$

as the previous theorem requires.

To demonstrate the necessary embedding, note that

$$\begin{aligned}
||Tf||_S &= \sup_{t \in S} \left| \int_S k(u,t) f(u) \nu(du) \right| \\
&\leq ||k||_\infty \sup_{t \in S} \int_S |f(u)| \, \nu(du) \\
&\leq ||f||_{L^2(\nu)} \, ||k||_\infty \, \nu(S)^{1/2},
\end{aligned}$$

and also that

$$\begin{aligned}|Tf(t)-Tf(s)| &= \left|\int_S (k(u,t)-k(u,s))f(u)\nu(du)\right| \\ &\leq \int_S |k(u,t)-k(u,s)|\,|f(u)|\,\nu(du) \\ &\leq ||f||_{L^2(S;\nu)}\left(\int_S |k(u,t)-k(u,s)|^2\,\nu(du)\right)^{\frac{1}{2}} \\ &\leq ||f||_{L^2(S;\nu)}\,C\varrho(s,t)^\alpha,\end{aligned}$$

by the hypotheses on the function k. This, in combination with (3.22) and the linearity of T assures the continuity of the embedding $H \hookrightarrow C^\alpha(S)$, completing the proof. □

We remark on something of interest; in the theorem above, a cylindrical Gaussian measure was radonified, but as there were multiple Hilbert spaces in view, a pertinent question arises. Which cylindrical measure? Was it the canonical Gaussian cylindrical measure on $L^2(S;\nu)$ or the one on its image H under T? The answer will be manifest in the following corollary.

Corollary 6. *The Fourier transform of the measure γ obtained in the previous theorem is given by*

$$\hat{\gamma}(x^*) = \exp\left(-\frac{1}{2}\langle TT^*x^*, x^*\rangle\right) = \exp\left(-\frac{1}{2}\left(T^*x^*, T^*x^*\right)_{L^2(S;\nu)}\right). \tag{3.24}$$

Proof. The canonical cylindrical Gaussian measure γ_0 on $T(L^2(S,\nu))$ has the Fourier transform

$$\hat{\gamma_0}(Tf) = \exp\left(-\frac{1}{2}\left(Tf, Tf\right)_H\right) = \exp\left(-\frac{1}{2}\left(f, f\right)_{L^2(S;\nu)}\right)$$

Notice that $T^* : C(S)^* \to L^2(S,\nu)$ and $TT^* : C(S)^* \to H$. As γ extends γ_0,

$$\begin{aligned}\hat{\gamma}(x^*) &= \hat{\gamma}_0(TT^*x^*)\\ &= \exp\left(-\frac{1}{2}\left(TT^*x^*, TT^*x^*\right)_H\right)\\ &= \exp\left(-\frac{1}{2}\left(T^*x^*, T^*x^*\right)_{L^2(S;\nu)}\right),\end{aligned}$$

for $x^* \in C(S)^*$, and thus (3.24) holds. □

Speaking in the terms of Theorem 17 it was the canonical cylindrical Gaussian measure on the reproducing kernel space $H = T(L^2(S;\nu))$, with reproducing kernel

$$K_t(s) = \langle TT^*\delta_s, \delta_t\rangle = (T^*\delta_t, T^*\delta_s)_{L^2(S;\nu)}$$

and norm

$$||Tf||_H = (Tf, Tf)_H = (f, f)_{L^2(S;\nu)},$$

that was radonified by the embedding. However, another view that we might have is one where the compact operator T on $L^2(S;\nu)$ carrying the canonical cylindrical Gaussian measure on *that* space to a particularly nice subspace H which in turn embeds nicely into $C(S)$. This is the perspective of [56, 57]; under this view, the operator $T : H \to B$ is said to be a radonifying operator.

The following corollary now follows quite easily. It is a somewhat obvious consequence of Theorem 18 and the definition of the n-variate Gaussian distribution, but we state it for future reference.

Corollary 7. *Let $P : C(S) \to \mathbb{R}^n$ be continuous and linear and let E be a Borel set in R^n. Write $\langle x_j^*, \cdot\rangle$ for the jth component of Px and let C be the matrix with entries $c_{i,j} =$*

$\langle TT^*x_i^*, x_j^*\rangle = (T^*x_i^*, T^*x_j^*)_{L^2(S;\nu)}$. *If C is nondegenerate, then*

$$\gamma\{x : Px \in E\} = \left(|\det(C)|\,(2\pi)^k\right)^{-\frac{1}{2}} \int_E \exp\left(-\frac{1}{2}C^{-1}\mathbf{u}\cdot\mathbf{u}\right) \mathrm{d}\mathbf{u}, \tag{3.25}$$

Proof. If P is linear and continuous, then its component functions must be as given. Now, $\hat{\gamma}(x^*) = \exp\left(-\frac{1}{2}\,||T^*x^*||_{L^2(S;\nu)}\right)$ by Corollary 6, whence each x_j^* is a normal random variable with mean 0 and variance $\left|\left|T^*x_j^*\right|\right|_{L^2(S;\nu)}$. From this it is easy to compute the covariance matrix C for the collection $\{x_j^*\}$, and then the corollary follows by the definition of the density function for the n-variate normal distribution (cf. chapter 2 of [59]). □

Observe that $\langle a, \cdot\rangle$ is a continuous linear map on $C(S)^*$ for any $a \in C(S)$. The following corollary then follows immediately from Theorem 18 and from Theorem 6, taking $L(x) = \langle a, x\rangle$. It gives us a non-centered Gaussian measure on $C(S)$.

Corollary 8. *Let (S, ϱ), k, and ν be as in Theorem 18 and let $a \in C(S)$. Then there is a Radon Gaussian measure on $C(S)$ with Fourier transform*

$$\hat{\gamma}(x^*) = \exp\left(i\langle a, x^*\rangle - \frac{1}{2}\langle TT^*x^*, x^*\rangle\right) = \exp\left(i\langle x^*, a\rangle - \frac{1}{2}\left(T^*x^*, T^*x^*\right)_{L^2(S;\nu)}\right). \tag{3.26}$$

Observe that under the method of construction in Theorem 18 and Corollary 8, the structure of the Cameron-Martin space and the support of the measure γ are determined by the choice of measure ν and kernel function $k : S^2 \to R$, because these determine the operators T and T^* on $L^2(S;\nu)$ and $C(S)^*$.

A particularly important subset of $C(S)^*$ is the collection of point evaluation functionals $\{\delta_t : t \in S\}$. We pause to state and prove a useful theorem, which is surely already very well-known. Its proof is quite enjoyable, so we include it here.

Theorem 19. *If S is compact and $\mu \in C(S)^*$, then μ is the weak* limit of finite linear combinations of point evaluation functionals. Moreover, if D is a dense subset of the compact metric space (S, ϱ), it is sufficient to consider the collection of point evaluation functionals for points in D.*

Proof. Note that the unit ball of $C(S)^*$ is convex and nonempty, and is weak* compact by Alaoglu's theorem. Then by the Krein-Milman theorem, the unit ball of $C(S)^*$ coincides with the weak*-closed convex hull of its extreme points. It is not hard to show that the set $\{\pm\delta_t : t \in S\}$ comprises the extreme points of the unit ball of $C(S)^*$; see section V.8 of [15]. Thus, every element of the unit ball of $C(S)^*$ is the weak* limit of a sequence whose elements are of the form $\sum_{j=1}^{n} c_j \delta_{t_j}$, where $\sum c_j = 1$. If μ is not in the unit ball of $C(S)^*$, then we can normalize μ to obtain $\mu' = \frac{\mu}{||\mu||_{Var}}$, which is. Then μ is the weak* limit of a sequence having coefficients $||\mu||_{Var}\, c_j$.

Let D be a dense subset of S. Note that for each point $t \in S$, there is a sequence $(t_n) \subseteq D$ with $t_n \to t$. Observe that

$$\langle \delta_{t_n}, x \rangle = x(t_n) \to x(t) = \langle \delta_t, x \rangle \tag{3.27}$$

for each $x \in C(S)$, and thus each $\delta_t \in C(S)^*$ is the weak* limit of (δ_{t_n}).

Now, let μ be the weak* limit of $\sum_{j=1}^{n} a_j \delta_{t_j}$, and let each δ_{t_j} be the weak* limit of $\delta_{s_{k,j}}$ for a sequence $(s_{k,j})$ in D. Then for each $x \in C(S)$, we have

$$\left| \langle \mu - \sum_{j=1}^{n} a_j \delta_{s_{k,j}}, x \rangle \right| \leq \left| \langle \mu - \sum_{j=1}^{n} a_j \delta_{t_j}, x \rangle \right| + \sum_{j=1}^{n} a_j \left| \langle \delta_{t_j} - \delta_{s_{k,j}}, x \rangle \right| . \tag{3.28}$$

Then for $\varepsilon > 0$, we can obtain n and k sufficiently large that (3.28) is less than ε, and thus μ is the weak* limit of finite linear combinations of $\{\delta_t : t \in D\}$. $\square$

Another useful observation obtains from the proof of the preceding theorem. It is an obvious consequence of (3.27) and the compactness of S.

Corollary 9. *Let (S, ϱ) be a compact metric space. Then the map $S \to C(S)^*$ by $t \mapsto \delta_t$ is (ϱ, weak*) continuous and the set $\{\delta_t : t \in S\}$ is weak* compact in $C(S)^*$.*

The next lemma investigates the properties of certain maps from $C(S)^*$ to Hilbert space.

Lemma 14. *Let H be a real separable Hilbert space and $\Phi : C(S)^* \to H$ be linear and (weak*, weak) continuous.*

1. *For each $f \in H$, if we define $F : S \to \mathbb{R}$ by $F(t) = (\Phi(\delta_t), f)_H$, then F is continuous.*
2. *The set $\{\Phi(\delta_t) : t \in S\}$ is weakly compact (and hence bounded) in H.*
3. *If D is a countable dense subset of S then the collection $\{\Phi(\delta_t) : t \in D\}$ is a complete set in $H_0 = \mathrm{cl}_H \Phi(C(S)^*)$.*

Proof. The first and second statements follow from the compactness of S, Corollary 9, and the (weak*, weak) continuity of Φ. That weak compactness implies boundedness follows by Banach-Steinhaus.

For the third statement, take any $f \in H_0$ and suppose that $(f, \Phi(\delta_t)) = 0$ for every $t \in D$. Recall that every $\mu \in C(S)^*$ is the weak* limit of a sequence of finite linear combinations of point evaluation functionals at points in D by Theorem 19. By the (weak*, weak) continuity of Φ, we see that every $\Phi(\mu)$ is a weak limit of $\Phi\left(\sum c_k \delta_{t_k}\right)$. For the functional $(f, \cdot)$, we have

$$(f, \Phi(\mu)) = \lim_{j\to\infty} \left(f, \sum_{k=1}^{j} c_k \Phi(\delta_{t_k}) \right) = \lim_{j\to\infty} \sum_{k=1}^{j} c_k \left(f, \Phi(\delta_{t_k})\right) = 0.$$

From this, we determine that $(f, \cdot) = 0$ on the dense set $\Phi(C(S)^*)$, whence $(f, \cdot)$ is the zero functional on H_0. Then we see that $f = 0$ in H_0. $\square$

The following theorem is an appropriate formulation of the classic Cameron-Martin translation theorem from [5], which guarantees quasi-invariance of γ under a specific type of transformation, namely by translations by elements of the Cameron-Martin space.

Theorem 20 (Cameron-Martin Theorem)**.** *Let γ be a Gaussian measure on a locally convex space X with Cameron-Martin space H_γ, let $h \in H_\gamma$ with $h = R_\gamma g$ for some $g \in X_\gamma^*$, and let $T : X \to X$ by $x \mapsto x + h$. Then γ and $\gamma_h = \gamma \circ T$ are equivalent with Radon-Nicodym derivative*

$$\frac{d\gamma_h}{d\gamma} = \exp\left(g(x) - \frac{1}{2}\,||h||_{H_\gamma}^2\right) = \exp\left(g(x) - \frac{1}{2}\,||g||_{L^2(\gamma)}^2\right). \tag{3.29}$$

We can make use of the translation theorem to prove the following very useful result.

Theorem 21. *Let γ be a Gaussian measure on a locally convex space X. If $\xi \in \mathbb{C}$ and $g \in X_\gamma^*$ then*

$$\int_X \exp\left(\xi g(x)\right)\gamma(dx) = \exp\left(\frac{\xi^2}{2}\,||g||_{L^2(\gamma)}^2\right). \tag{3.30}$$

Proof. We first show that (3.30) holds for real values of ξ. This is easy, for given such ξ put $h = R_\gamma(-\xi g)$ for the proffered $g \in X_\gamma^*$ and then use Theorem 20 to obtain that

$$1 = \int_X \gamma(dx) = \int_{X+h} \gamma(dx) = \exp\left(-\frac{1}{2}\,||\xi g||_{L^2(\gamma)}^2\right)\int_X \exp\left(\xi g(x)\right)\gamma(dx). \tag{3.31}$$

Now, let Z be any closed contour in $\mathbb{C}$ with a piecewise smooth parameterization $z : [0,1] \to \mathbb{C}$ for which $|z'(t)|$ is bounded (certainly this includes all triangular contours). Let

$M = \sup_{[0,1]} |z(t)|$ and $N = \sup_{[0,1]} |z'(t)|$ and observe that

$$\begin{aligned}
\int_0^1 \int_X |\exp(z(t)g(x))\, z'(t)|\, \gamma(dx)\mathrm{d}t &= \int_X \int_0^1 |\exp(z(t)g(x))\, z'(t)|\, \mathrm{d}t\gamma(dx) \\
&\leq N \int_X \int_0^1 2\exp(g(x)\mathrm{Re}(z(t)))\, \mathrm{d}t\gamma(dx) \\
&\leq N \int_X \int_0^1 2\exp([\mathrm{sign}(g(x))M]g(x))\, \mathrm{d}t\gamma(dx) \\
&= 2N \int_X \exp([\mathrm{sign}(g(x))M]g(x))\, \gamma(dx) \\
&= 2N \exp\left(\frac{M^2}{2}\, ||g||^2_{L^2(\gamma)}\right) \\
&< \infty.
\end{aligned}$$

Note that $f(z) = \exp(zg(x))$ is an entire function and put $F(z) = \int_X \exp(zg(x))\, \gamma(dx)$. Using the previous computation as justification to exchange the integrals, we see that

$$\int_Z F(z)\mathrm{d}z = \int_0^1 \int_X \exp(z(t)g(x))\, \gamma(dx)z'(t)\mathrm{d}t = \int_X \int_Z \exp(zg(x))\, \mathrm{d}z\gamma(dx) = 0,$$

so that F is entire by Morera's theorem. By (3.31), $F(z)$ agrees with the entire function $\exp\left(\frac{z^2}{2}\, ||g||^2_{L^2(\gamma)}\right)$ for all real z, whence it must agree on all of $\mathbb{C}$, completing the proof. $\square$

Corollary 10. *Let $\{g_1, \ldots, g_n\}$ be a linearly independent set in $C(S)^*_\gamma$ and let $F(x) = f(g_1(x), \ldots, g_n(x))$ for some function $f : \mathbb{R}^n \to \mathbb{C}$. If f is Lebesgue measurable then F is measurable, and*

$$\int_{C(S)} F(x)\gamma(dx) \stackrel{*}{=} \frac{\det(M^{-1})}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(\mathbf{u}) \exp\left(-\frac{1}{2}M\mathbf{u}\cdot\mathbf{u}\right) \mathrm{d}\mathbf{u}, \tag{3.32}$$

where M is an $n \times n$ matrix with entries $m_{i,j} = (g_i, g_j)_{L^2(\gamma)}$ and $\stackrel{*}{=}$ indicates that if one side of the equality exists then so does the other.

Proof. Take $\xi = i$ in (3.30) to see that the characteristic function of the random variable g is $\exp\left(-\frac{1}{2}\left|\left|g\right|\right|_L^2(\gamma)\right)$, and thus each g has a mean zero Gaussian distribution and the covariance of g and h is $(g,h)_{L^2(\gamma)}$. Then the linear independence of the set $\{g_1, \ldots g_n\}$ in $C(S)^*_\gamma$ ensures that the covariance matrix M will be non-degenerate by Theorem 3.5.1 of [59]. Then (3.32) follows by the definition of the n-variate normal distribution (cf. [59]). □

It is also well-known (again, see [5]) that for a Gaussian measure γ on a locally convex space X, the Cameron-Martin space H_γ coincides with the set $\{h \in X : \gamma_h \sim \gamma\}$. Intuitively speaking, it is the collection of elements along which translation does not move a set off of the support of γ. Translation by an element of X not in H_γ results in a measure that is mutually singular with γ.

The question remains: can we identify the Cameron-Martin space? While we made some progress in Section 3.3 we still do not have an explicit answer to this question. By Theorem 16 we can think of $C(S)^*_\gamma$ as a centered copy of H_0 of $L^2(S;\nu)$. Then $H_0 \cong C(S)^*_\gamma \cong H_\gamma$. To work with H_0 directly, we will need some sort of "canonical" representation for the action of an element $f \in L^2(S;\nu)$ on $C(S)$, which in turn requires more information about the structure of the operator T. In the next section we will generate a wide class of examples demonstrating how this works out.

We conclude this section with an aside that makes good on our promise in Section 3.2 to demonstrate a method for obtaining Gaussian measures on $L^\infty(S;\nu)$. There are some interesting difficulties and questions that arise from this which we do not wish to address in detail here, but we will offer some brief remarks.

Proposition 2. *Let a centered Gaussian measure γ be constructed as in Theorem 18. Then γ can be taken as a measure on $L^\infty(S;\nu)$.*

Proof. We can use Theorem 18 to build a centered measure γ on $C(S)$. Note that the embedding $C(S) \hookrightarrow L^\infty(S;\nu)$ is continuous for any choice of $\nu \in C(S)^*$ because uniform convergence implies convergence in $L^\infty(S;\nu)$. Now by Theorem 11 we can consider γ as a measure on $L^\infty(S;\nu)$ with the same Cameron-Martin space as on $C(S)$. □

In considering the previous proposition, there are a few caveats that bear mention. First, note that the relationship between γ and $L^\infty(S;\nu)$ is quite artificial. Recall that $\mathrm{supp}(\gamma) = \mathrm{cl}_{C(S)}(H_\gamma)$ by Theorem 12, whence H_γ cannot be dense in $L^\infty(S;\nu)$, because the former space is separable and the latter is not. This leads us to conclude that the support of γ must be some closed proper subspace of $L^\infty(S;\nu)$, and thus in some sense we have "tacked on" a "large" set of measure 0.

In addition, we must ask the uncomfortable question about how elements $x^* \in L^\infty(S;\nu)^*$ behave with respect to this measure, as this space of functionals does not generally agree with $C(S)^*$, which is what we built our covariance operator on in the first place. As our present focus lies elsewhere, we only pause briefly to offer an example illustrating the problem.

Example 2. Let $S = [0,1]$ and let $\nu = \mathrm{m}$ be Lebesgue measure. Then it is well-known that no $\delta_t \notin L^\infty[0,1]^*$, and that there are certainly elements $f^* \in L^\infty[0,1]^*$ that are not in $C(S)^*$. It is not the absence of the evaluation functionals δ_t that concerns us; rather, it is the question of what happens for $f^* \notin C(S)^*$. Recall that k is a bounded Borel function on S^2, and then taking $k(u,t) = k_u(t)$ we see that $\langle k_u, f^*\rangle$ is a perfectly good bounded function in $L^2[0,1]$. Then

$$TT^*f^*(t) = \int_S \langle k_u, f^*\rangle k(u,t)\mathrm{d}u. \tag{3.33}$$

The issue is that we must have $\langle TT^*f^*, g^*\rangle = (T^*f^*, T^*g^*)\, L^2[0,1]$, and showing this directly involves exchanging the order of integration in (3.33). However, it is not at all clear that this can be done, as spaces of finitely additive measures are strange and frightening places where Fubini type-theorems are very hard to come by. Exactly what happens in this case is an interesting question that we will leave unanswered here.

3.5 Measurability

In all of these constructions, there is necessarily the question of measurability. Recall that the Gaussian measures we have been considering arise as the radonification of a cylindrical measure on the σ-algebra $\mathcal{E}(B)$. Thus the finite-dimensional distributions of γ are specifically constructed so that whenever $E \subseteq \mathbb{R}^n$ is Borel and P is a finite dimensional projection onto $\mathbb{R}^n$ it must follow that $P^{-1}(E)$ is measurable by definition. By the standard measure-theoretic arguments this then carries through to tame functionals defined on B in terms of finite-dimensional projections and measurable functions on $\mathbb{R}^n$ of the form $F(x) = f(P(x))$. The question of converse measurability is less clear. Must it be the case that the measurability of F will guarantee the measurability of $f \circ P$?

The first known successful attempt at addressing this question was due to Fulton Koehler at the University of Minnesota, who had heard of the problem in the case of the classical Wiener measure $\mathfrak{w}$, as posed by Robert Cameron. This result provided for the converse measurability of tame functionals with respect to the ordinary Wiener measure. In the end, Koehler's result remained unpublished, but known to Cameron and his associates.

In [51], David Skoug extended Koehler's result to address the question of measurability not only with respect to the one parameter Wiener measure, but also for the two parameter Yeh-Wiener measure. One of his motivations for this was a desire to streamline the statement of the theorems in [8].

Here the question remained for some time, until Skoug's colleague Gerald Johnson was visiting the university of Erlängen and mentioned the problem in a seminar there. A far more general converse measurability result was subsequently proven by Siegfried Graf. One can also find a general converse measurability result by Chang and Ryu in [10].

By $\mathcal{M}(B)$ we will denote the usual completion of $\mathcal{B}(B)$. We will also abuse notation by taking γ to be the completion of the corresponding Gaussian measure on $\mathcal{B}(B)$. The statement of the following theorem is adapted from [33]; the proof is the same as presented there. This result is very much a consequence of the fact that γ is a finite measure on a separable Banach space, and is hence a regular measure. The proof is originally by Graf.

Theorem 22. *Let $C = \left\{x : (\langle x_1^*, x\rangle, \ldots, \langle x_n^*, x\rangle) \in \tilde{C}\right\}$ be a cylinder set in a separable Banach space B and let γ be a Radon Gaussian measure on $\mathcal{M}(B)$. Then C is measurable if and only if $\tilde{C}$ is Lebesgue measurable in $\mathbb{R}^n$.*

Of course, one could consider the question of scale invariance of sets in $\mathcal{M}(B)$. As important as this idea is, we will not engage it here.

3.6 Bounded Variation and Absolute Continuity

In the next section, we will consider in greater detail a class of function spaces and measures sharing a particular basic structure. First, we will need several results from [3] and will have to reframe these results to our setting. Take $Q = \prod_{j=1}^{d}[0, T_j]$ and let $\leq$ be the usual partial order on Q such that $s \leq t$ if and only if each $s_j \leq t_j$. By a rectangle (or interval) in Q we refer to the set $[s,t] = \{u \in Q : s \leq u \leq t\}$. We say that a rectangle $R = [s,t]$ is degenerate if $s_i = t_i$ for some i. In general, for sets A and B in $\mathbb{R}^d$ let $A \triangle B$ denote the symmetric difference of A and B.

Let $\alpha = (\alpha_1, \ldots, \alpha_d)$ be a multi-index with $\alpha_j \in \{0,1\}$ and let $|\alpha| = \sum_{j=1}^{d} \alpha_j$ as usual. The collection of multi-indices $\{\alpha\}$ forms a bounded, graded lattice having $(1, \ldots, 1)$ as its top, $(0, \ldots, 0)$ as its bottom, and rank function $|\alpha|$. The order is $\alpha \leq \beta$ if and only if $\alpha_i \leq \beta_i$, and the meet ($\wedge$) and join ($\vee$) of this lattice are determined by:

$$\alpha \wedge \beta = (\min(\alpha_1, \beta_1), \ldots, \min(\alpha_d, \beta_d)),$$

$$\alpha \vee \beta = (\max(\alpha_1, \beta_1), \ldots, \max(\alpha_d, \beta_d)).$$

For a rectangle $R = [s, t]$, let $\mathrm{ext}(R) = \{\mathbf{r} \in [s, t] : r_i = s_i \text{ or } r_i = t_i \text{ for each } i\}$ be the extreme points (corners) of R. For $r \in \mathrm{ext}(R)$, let $\#r$ be the number of indices for which $r_i = s_i$. Note that the multi-index α can be used to order these extreme points with the order in the lattice of multi-indices directly corresponding to the application of the partial order on $\mathbb{R}^d$ to $\mathrm{ext}(R)$. In terms of the multi-index, if $\mathrm{ext}(R) = \{r_\alpha\}$ we then have $\#r_\alpha = |\alpha|$. For a rectangle $R \subseteq \mathbb{R}^d$, put

$$\Delta_R f = \sum_{r \in \mathrm{ext}(R)} (-1)^{\#r} f(r), \tag{3.34}$$

and note that this expression yields an alternating sum of point evaluations of f at the corners of R. For each α, we let $\underline{\alpha} t = (\alpha_1 t_1, \ldots, \alpha_d t_d)$ and $\overline{\alpha} t = (\alpha_1 t_1 + (1-\alpha_1)T_1, \ldots, \alpha_d t_d + (1-\alpha_d)T_d)$, and then $Q_\alpha = \{\underline{\alpha} t : t \in Q\}$ is the lower α-face of Q and $Q^\alpha = \{\overline{\alpha} t : t \in Q\}$ is the upper α-face of Q.

Definition 10. Let P be a finite partition of Q into non-degenerate intervals R_j and let $\mathcal{P}$ be the collection of all such partitions. A function $f : Q \to \mathbb{C}$ is said to be of bounded variation on Q in the sense of Vitali if the quantity

$$V(f; Q) = \sup_{P \in \mathcal{P}} \left\{ \sum_{R_j \in P} \left| \Delta_{R_j} f \right| \right\} \leq K \tag{3.35}$$

for some $K \geq 0$.

We define the variation of f on Q in the sense of Hardy-Krause to be the quantity

$$\mathrm{Var}(f; Q) = \sum_{|\alpha|=0}^{d} V(f; Q_\alpha), \tag{3.36}$$

where $V(f; Q_\alpha) = |f(0)|$ when $|\alpha| = 0$ and is defined as in (3.35) otherwise. We will denote the collection of functions of bounded variation on Q in the sense of Hardy-Krause by $BV(Q) = \{f : Q \to \mathbb{C} : \mathrm{Var}(f; Q) < \infty\}$.

In short, for f to be of bounded variation in the sense of Hardy-Krause simply requires that each restriction of f to a lower α-face (or to each upper α-face) has bounded variation in the sense of Vitali. Note that if $d = 1$, these reduce to the usual definition of bounded variation. For a more detailed development in the cases $d = 1, 2$, see [3, 14, 37, 52], which our definitions follow and extend. Adjusting for the more cumbersome computations resulting from the fact that $d > 2$ and using the same methods as in [3] one can obtain the following theorem.

Theorem 23. *For $f \in BV(Q)$, put $||f||_{BV} = \mathrm{Var}(f; Q)$. Then $||\cdot||_{BV}$ is a norm, and under pointwise products the space $BV(Q)$ is a unital Banach algebra.*

Definition 11. Let f and g be defined on a rectangle $I \subseteq \mathbb{R}^d$, let $P = \{R_j\}$ be a finite partition of I into non-degenerate rectangles, and let $c_j \in R_j$ for each j. Put

$$S(f; g, P) = \sum_{j=1}^{n} f(c_j)\Delta_{R_j} g. \tag{3.37}$$

We say that f is Riemann-Stieltjes integrable with respect to g on I if there is some number J such that given any $\varepsilon > 0$ there is a $\delta > 0$ so that $|S(f; g, P) - J| < \varepsilon$ whenever $|P| < \delta$. In this case, J is the Riemann-Stieltjes integral of f with respect to g on I.

From [62], we take the following two results after adjusting to our notation. The first gives the existence of the Riemann-Stieltjes integral.

Theorem 24. *If g is continuous and f is of bounded variation (in the sense of Hardy-Krause) on a closed rectangle $I \subseteq \mathbb{R}^d$, then the Riemann-Stieltjes integral $\int_I g\mathrm{d}f$ exists.*

The next theorem shows that the Riemann-Stieltjes integral satisfies an integration by parts formula. We will need a little notation to state the theorem. For a rectangle $I =$

$\prod_{j=1}^{d}[a_j, b_j]$ and $0 \leq k \leq d$, let $\mathfrak{I}_k$ denote the collection of (possibly degenerate) subrectangles for which $n-k$ coordinates are fixed at either a_j or b_j and the remaining k coordinates are free in their corresponding intervals. Note that there are $\binom{d}{k} 2^{d-k}$ such subrectangles in each $\mathfrak{I}_k$. For a given subrectangle $I_m \in \mathfrak{I}$ we will take $\# I_m$ to be the number of coordinates of that subrectangle that are fixed at the corresponding b_j

Theorem 25. *Suppose that g is continuous and f is of bounded variation (in the sense of Hardy-Krause) on a closed rectangle $I \subseteq \mathbb{R}^d$. Then $\int_I f \mathrm{d}g$ exists, and*

$$\int_I f(u)\mathrm{d}g(u) = \sum_{k=0}^{d} \sum_{I_m \in \mathfrak{I}_k} (-1)^{d-(\# I_m)} \int_{I_m} g(u)\mathrm{d}f(u). \tag{3.38}$$

We will say that two sets A and B in $\mathbb{R}^d$ are essentially disjoint if $\mathrm{m}(A \cap B) = 0$. In addition, we will take the measure m_α on Q_α to be the image measure of Lebesgue measure on $\mathbb{R}^{|\alpha|}$ under the natural bijection $\mathbb{R}^{|\alpha|} \to Q_\alpha$. We next introduce a notion of absolute continuity on Q that is intimately connected with that of bounded variation in the sense of Hardy-Krause. See [3, 52] for further information.

Definition 12. For each α, let $P_\alpha = \{R_{\alpha,j}\}$ be a finite collection of essentially disjoint intervals in Q_α. A function $f : Q \to \mathbb{C}$ is said to be absolutely continuous on Q if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\sum_{R_{\alpha,j} \in P_\alpha} \left|\Delta_{R_{\alpha,j}} f\right| < \varepsilon \tag{3.39}$$

whenever each P_α satisfies

$$\sum_{R_{\alpha,j} \in P_\alpha} \mathrm{m}_\alpha \left(R_{\alpha,j}\right) < \delta.$$

Unpacking this definition, we see that f is absolutely continuous in this sense whenever f is absolutely continuous on Q with respect to Lebesgue measure, and f restricted to each lower α-face is also absolutely continuous as a function of the remaining free parameters.

Let $AC(Q)$ denote the collection of absolutely continuous functions on Q. Again, we can obtain the following result by the same methods as used in [3]; it characterizes $AC(Q)$ in several very convenient ways.

Theorem 26. *Let $f : Q \to \mathbb{C}$. Then the following are equivalent:*

1. *$f \in AC(Q)$,*

2. *for $|\alpha| = 1, \ldots d$, there exist $F_\alpha \in L^1(Q_\alpha)$ such that*

$$f(t) = \sum_{|\alpha|=1}^{d} \int_{Q_\alpha} \chi_{[0,\underline{\alpha}t]}(u) F_\alpha(u) \mathrm{m}_\alpha(du) \tag{3.40}$$

3. *f belongs to the closure in $BV(Q)$ of the polynomials in d-variables.*

We will use the following lemma in the next section.

Lemma 15. *For each $x \in C(Q)$, the map $BV(Q) \to \mathbb{R}$ given by $f \mapsto \int_Q f(u)\mathrm{d}x(u)$ is linear and continuous.*

Proof. By the linearity of the Riemann-Stieltjes integral, it is clear that each such map is linear. Let $f \in BV(Q)$. Then, by integration by parts,

$$\begin{aligned}
\left|\int_Q f_n(u)\mathrm{d}x(u)\right| &= \left|\sum_{k=0}^{d}\sum_{Q_m\in\mathfrak{Q}_k}(-1)^{d-(\#Q_m)}\int_{I_m} g(u)\mathrm{d}f(u)\right| \\
&\leq \sum_{k=0}^{d}\binom{d}{k}2^{d-k}\int_{Q_m}|g(u)|\,\mathrm{d}\,|f|\,(u) \\
&\leq \left(\sum_{k=0}^{d}\binom{d}{k}2^{d-k}\right)||g||_S\int_{Q_m}\mathrm{d}\,|f|\,(u) \\
&\leq C\,||g||_S\,\mathrm{Var}(f;Q) \\
&= C\,||g||_S\,||f||_{BV}\,,
\end{aligned}$$

where $C > 0$ can be appropriately chosen, and thus the map is continuous. □

3.7 Measures on the space $C_0(Q)$

The next theorem is our first aim of this section; it shows how to build a wide class of "generalized Wiener measures" on the space $C(Q)$. Throughout our discussion, we will fix an element $a \in C(Q)$ satisfying $a(t) = 0$ whenever $t_j = 0$ for some $j = 1, \ldots, d$; this function will serve as the mean for our family of measures. We begin by dispensing with the technical condition on ε-entropy of Theorem 17 for these spaces.

Proposition 3. *For every $\alpha > 0$, $Q = \prod_{j=1}^{d}[0, T_j]$ satisfies the metric entropy condition $\sum_{n=1}^{\infty} 2^{-n\alpha}H(Q, 2^{-n}) < \infty$.*

Proof. We use the ℓ^∞ norm $||\mathbf{u} - \mathbf{v}||_\infty = \max_{1\leq j\leq d}|u_j - v_j|$ on $\mathbb{R}^d$. Put $T = \max_{1\leq j\leq d}\{T_j\}$. Now, there is an $n_0 \in \mathbb{N}$ sufficiently large that for each $n \geq n_0$ there is some $M_n \in \mathbb{N}$ so that $T \leq 2^{-n}M_n \leq 2T$; from this we see that $M_n \leq 2^{n+1}T$. Now note that it takes at most M_n^d balls of size 2^{-n} (in the ℓ^∞ sense) to cover Q.

Thus $N(Q,2^{-n}) \leq M_n^d \leq (2^{n+1}T)^d$, and hence

$$H(Q,2^{-n}) = \left(\log N(Q,2^{-n})\right)^{\frac{1}{2}} \leq \left(\log(2^{n+1}T)^d\right)^{\frac{1}{2}} \leq C\sqrt{n+3},$$

where $C = (d\max(\log 2, \log T))^{\frac{1}{2}}$. Now observe that

$$\sum_{n=n_0}^{\infty} 2^{-n\alpha} H(Q,2^{-n}) \leq C \sum_{n=n_0}^{\infty} 2^{-n\alpha}\sqrt{n+3},$$

which converges by the ratio test for any $\alpha > 0$. □

Theorem 27. *Let ν be absolutely continuous on Q with non-negative Radon-Nicodym derivative $\frac{d\nu}{\mathrm{dm}} \in L^p(Q)$ for some $1 < p \leq \infty$ and let $k(u,t) = \chi_{[0,t]}(u)$ be a Volterra kernel. For $f \in L^2(Q;\nu)$, put $Tf(t) = \int_Q k(u,t)f(u)\nu(du)$. Then there is a Radon Gaussian measure γ on $C(Q)$ with Fourier transform*

$$\hat{\gamma}(x^*) = \exp\left(i\langle a, x^*\rangle - \frac{1}{2}\langle TT^*x^*, x^*\rangle\right). \tag{3.41}$$

Proof. Note that $\langle a, \cdot\rangle$ is a perfectly good continuous linear map on $C(Q)$, and the result is that $m(x^*) = \langle a, x^*\rangle$ for each $x^* \in C(Q)^*$. We need to demonstrate that our intended covariance operator TT^* does what it should. By Theorem 18, it suffices to show that

$$\int_Q |k(u,s) - k(u,t)|^2 \nu(du) \leq C\,|s-t|^{2\alpha} \tag{3.42}$$

for some $\alpha > 0$ and $C > 0$, because Proposition 3 assures that the metric entropy hypothesis of Theorem 18 is automatically met.

If $1 < p < \infty$, take $0 < \alpha < \frac{p-1}{2p}$ and then

$$\begin{aligned}
\frac{1}{|s-t|^{2\alpha}} \int_Q |k(u,s) - k(u,t)|^2 \nu(du) &= \frac{1}{|s-t|^{2\alpha}} \int_Q \left|\chi_{[0,s]}(u) - \chi_{[0,t]}(u)\right|^2 \nu(du) \\
&= \frac{1}{|s-t|^{2\alpha}} \int_Q \chi_{[0,s]\triangle[0,t]}(u) \frac{d\nu}{d\mathrm{m}} \mathrm{m}(du) \\
&\le \frac{1}{|s-t|^{2\alpha}} \left|\left|\frac{d\nu}{d\mathrm{m}}\right|\right|_p \mathrm{m}\left([0,s]\triangle[0,t]\right)^{\frac{p-1}{p}} \\
&\le C \left|\left|\frac{d\nu}{d\mathrm{m}}\right|\right|_p |s-t|^{\frac{p-1}{p}-2\alpha} \\
&\le C \left|\left|\frac{d\nu}{d\mathrm{m}}\right|\right|_p .
\end{aligned}$$

In the case that $p = \infty$, take $0 < \alpha < \frac{1}{2}$, and then by a similar computation

$$\begin{aligned}
\frac{1}{|s-t|^{2\alpha}} \int_Q |k(u,s) - k(u,t)|^2 \nu(du) &\le \frac{1}{|s-t|^{2\alpha}} \left|\left|\frac{d\nu}{d\mathrm{m}}\right|\right|_\infty \mathrm{m}\left([0,s]\triangle[0,t]\right) \\
&\le C \left|\left|\frac{d\nu}{d\mathrm{m}}\right|\right|_\infty |s-t|^{1-2\alpha} \\
&\le C \left|\left|\frac{d\nu}{d\mathrm{m}}\right|\right|_\infty .
\end{aligned}$$

In either case, (3.42) is satisfied. □

Corollary 11. *Let γ be as in Theorem 27. Then*

1. $\mathrm{supp}(\gamma) \subseteq C_0(Q) = \{x : x(t) = 0$ *whenever* $t_j = 0$ *for some* $j = 1, \ldots, d\}$,
2. $H_\gamma \subseteq AC(Q)$.

Proof. Let $t \in Q$ with $t_j = 0$ for some j and observe that $T^*\delta_t(u) = \chi_{[0,t]}(u)$, so that

$$||T^*\delta_t||_{L^2(Q;\nu)} = \int_Q \chi_{[0,t]}(u)\nu(du) = \int_Q \chi_{[0,t]}(u) \frac{d\nu}{d\mathrm{m}}(u)\mathrm{m}(du) = 0,$$

because $\mathrm{m}[0,t]=0$ for such t. Thus $\delta_t \in \ker(T^*)$ whenever some $t_j=0$ and then by Theorem 15, $\mathrm{supp}(\gamma)\subseteq \ker(\delta_t)$, whence $x(t)=\langle \delta_t, x\rangle = 0$. If γ has mean $a\in C(Q)$ as in Theorem 12, then $x=a+x_0$ for some $x_0\in \mathrm{cl}_{C(Q)}(H_\gamma)$, whence $x(t)=a(t)+x_0(t)=0$.

As $H_\gamma \subseteq T(L^2(Q;\nu))$, each $h\in H_\gamma$ is $h=Tf$ for some $f\in L^2(Q;\nu)$, and thus

$$h(t)=\int_Q \chi_{[0,t]}(u)f(u)\nu(du)=\int_Q \chi_{[0,t]}(u)\left(f(u)\frac{d\nu}{d\mathrm{m}}\right)\mathrm{m}(du).$$

Note that $f\frac{d\nu}{d\mathrm{m}}\in L^1(Q)$ and that we can take each $F_\alpha=0$ on the lower α-faces of Q, which shows that $h\in AC(Q)$ by Theorem 26. □

We will now take some time to develop a useful means of representing linear functionals on $C_0(Q)$. To accomplish this, we will continue to view of T^* as acting on $C(Q)^*$, but will consider it to have a different codomain.

Lemma 16. *Let T^* be as in Theorem 27. Then T^* can be also considered as a bounded linear map from $C(Q)^*$ to $BV(Q)$.*

Proof. Note that T^* is linear by definition. For $x^*\in C(Q)^*$ let x^* be represented by the measure μ and put $T^*x^*=T^*\mu$ and then $T^*x^*(u)=\int_Q \chi_{[0,t]}(u)\mu(dt)=\mu\{t : u\le t\}$. For each α and any partition $P_\alpha=\{R_{\alpha,j}\}$ of Q_α by rectangles we have a uniform bound

$$\sum_{R_{\alpha,j}\in P_\alpha}\left|\Delta_{R_{\alpha,j}}T^*x^*\right|=\sum_{R_{\alpha,j}\in P_\alpha}\left|\Delta_{R_{\alpha,j}}\mu\{t : u\le t\}\right|=\sum_{R_{\alpha,j}\in P_\alpha}\left|\mu(R_{\alpha,j})\right|\le |\mu|\,(Q),$$

whence we can easily obtain the coarse (but sufficient) estimate

$$||T^*x^*||_{BV}=\sum_{|\alpha|=0}^{d} V(T^*x^*;Q_\alpha)\le 2^d\,||\mu||_{Var}=2^d\,||x^*||_{Var}\,,$$

and thus T^* is bounded. □

Theorem 28. *Let T^* be as in Lemma 16 and let $x \in C_0(Q)$. Then*

$$\langle x, x^* \rangle = \int_Q T^* x^*(u) \mathrm{d}x(u) \tag{3.43}$$

for each $x^* \in C(Q)^*$.

Proof. Begin with the case that $x^* = \delta_t$. Observe that $T^*\delta_t(u) = \int_Q \chi_{[0,s]}(u)\delta_t(ds) = \chi_{[0,t]}(u)$, and then we can compute that

$$\int_Q T^*\delta_t(u)\mathrm{d}x(u) = \int_Q \chi_{[0,t]}(u)\mathrm{d}x(u) = x(t) = \langle x, \delta_t \rangle,$$

because of the fact that $x(t) = 0$ whenever $t_j = 0$ for some $j = 1, 2, \ldots, d$. The linearity of the Riemann-Stieltjes integral then assures that

$$\langle x, \sum_{j=1}^{n} c_j \delta_{t_j} \rangle = \int_Q \sum_{j=1}^{n} c_j \delta_{t_j}(u)\mathrm{d}x(u)$$

for any finite linear combination of point evaluation functionals.

Now, let x^* be any element of $C(Q)^*$ and recall by Theorem 19 that x^* is the weak* limit of finite linear combinations of point evaluations, say $x^* = \lim_{n\to\infty} x_n^*$. Note the composition map $x^* \mapsto T^*x^* \mapsto \int_Q T^*x^*(u)\mathrm{d}x(u)$ is continuous by Lemmas 15 and 16, whence

$$\langle x, x^* \rangle = \lim_{n\to\infty} \langle x, x_n^* \rangle = \lim_{n\to\infty} \int_Q T^*x_n^*(u)\mathrm{d}x(u) = \int_Q T^*x^*(u)\mathrm{d}x(u),$$

as desired. □

Corollary 12. *If $f \in BV(Q)$, then there is some $f^* \in C(Q)^*$ so that $\int_Q f(u)\mathrm{d}x(u) = \langle f^*, x \rangle$ for every $x \in C_0(Q)$, and in this case we have $T^*f^* = f$.*

Proof. Let $f \in BV(Q)$. Then the map $C(Q) \to \mathbb{R}$ given by $x \mapsto \int_Q f(u)\mathrm{d}x(u)$ is certainly linear; moreover, the proof of Lemma 15 shows that

$$\left|\int_Q f(u)\mathrm{d}x(u)\right| \leq C\,||f||_{BV}\,||x||_S\,,$$

whence there is a bounded linear functional $f^* \in C(Q)^*$ so that $\langle f^*, x\rangle = \int_Q f(u)\mathrm{d}x(u)$, as desired. Then by Theorem 28 we see that $T^*f^* = f$. $\square$

Lemma 17. *For each $f \in BV(Q)$,*

1. *the Riemann-Stieltjes integral $\int_Q f\mathrm{d}x$ exists for every $x \in C(Q)$ and for every $x \in C_0(Q)$ satisfies $\int_Q f\mathrm{d}x = \langle f^*, x\rangle$ for some $f^* \in C(Q)^*$ for which $T^*f^* = f$,*
2. *$\int_{C_0(Q)} \left(\int_Q f\mathrm{d}x\right)\gamma(dx) = \int_Q f\mathrm{d}a$,*
3. *$\int_{C_0(Q)} \left(\int_Q f\mathrm{d}x\right)^2 \gamma(dx) = ||f||^2_{L^2(Q;\nu)} + \left(\int_Q f\mathrm{d}a\right)^2$.*

Proof. The first statement follows directly from Theorem 28 and Corollary 12. The second two statements follow immediately from the first and the fact that

$$\hat{\gamma}(f^*) = \exp\left(i\langle f^*, a\rangle - \frac{1}{2}\,||T^*f^*||^2_{L^2(Q;\nu)}\right) = \exp\left(i\int_Q f\mathrm{d}a - \frac{1}{2}\,||f||^2_{L^2(Q;\nu)}\right),$$

so that f^* is a normal random variable with mean $\int_Q f\mathrm{d}a$ and variance $||f||^2_{L^2(Q;\nu)}$. $\square$

Lemma 18. *Let ν be a positive Borel measure on Q and fix some $x_0 \in C(Q)$. For $f, g \in BV(Q)$, let*

$$[f, g]_{x_0} = (f, g)_{L^2(Q;\nu)} + \int_Q f\mathrm{d}x_0 \int_Q g\mathrm{d}x_0.$$

Then $[\cdot,\cdot]_{x_0}$ is a semi-inner product, and the semi-norm associated with $[\cdot,\cdot]$ dominates the $L^2(Q;\nu)$-norm.

Proof. By the linearity of the map $f \mapsto \int_Q f\mathrm{d}x_0$, it is easy to see that $[\cdot,\cdot]_{x_0}$ is symmetric and bilinear. In addition, $[f, f]_{x_0} = ||f||^2_{L^2(Q;\nu)} + (\int_Q f\mathrm{d}x_0)^2 \geq 0$, and if $f = 0$ in $BV(Q)$, we must have $||f||_{L^2(Q;\nu)} = 0$ and $\int_Q f \mathrm{d}x_0 = 0$. Thus $[\cdot,\cdot]_{x_0}$ is a semi-inner product on $BV(Q)$. The domination of $||\cdot||_{L^2(Q;\nu)}$ by the associated seminorm is clear. $\square$

In light of this, $[f,g]_a$ is a semi-inner product on $BV(Q)$. Define the quotient space $BV(Q)_a = BV(Q)/\{f : [f,f]_a = 0\}$, and then we can take $[\cdot,\cdot]_a$ as an inner product on $BV(Q)_a$. Observe that $f = g$ in $BV(Q)_a$ if and only if $f = g$ in $L^2(Q;\nu)$ and $\int_Q f \mathrm{d}a = \int_Q g \mathrm{d}a$ for every choice of representative for the corresponding equivalence classes of f and g. Now, define H to be the completion of $BV(Q)_a$ in the norm associated with $[\cdot,\cdot]_a$, which we will denote by $||\cdot||_H$. We will use $(\cdot,\cdot)_a$ to denote this inner product.

Lemma 19. *If H is as above, then:*

1. $H \subseteq \mathrm{cl}_{L^2(Q;\nu)} T^*(C(Q)^*)$,
2. *for each* $x^* \in C(Q)^*$,

$$||x^*||^2_{L^2(\gamma)} = ||T^*x^*||^2_H = ||T^*x^*||^2_{L^2(Q;\nu)} + \langle x^*, a\rangle^2 = ||\tau x^*||^2_{L^2(\gamma)} + \langle x^*, a\rangle^2. \tag{3.44}$$

Proof. Let $(f_n) \subseteq BV(Q)$ with $\lim_{n\to\infty} f_n = f$ in H. Note that by Corollary 12 each f_n satisfies $f_n = T^* f_n^*$ for some $f_n^* \in C(Q)^*$. Then the fact that $||\cdot||_H$ dominates $||\cdot||_{L^2(Q;\nu)}$ forces the desired containment.

Now let $x^* \in C(Q)^*$. Then x^* is Gaussian with mean $\langle x^*, a\rangle$ and variance $||T^*x^*||^2_{L^2(Q;\nu)}$, whence $||x^*||^2_{L^2(\gamma)} = ||T^*x^*||^2_{L^2(Q;\nu)} + \langle x^*, a\rangle^2$. The second conclusion is then immediate from the fact that $||T^*x^*||^2_H = ||T^*x^*||^2_{L^2(Q;\nu)} + \left(\int_Q T^*x^* \mathrm{d}a\right)^2$ by Theorem 28 and the definition of $||\cdot||_H$, and the fact that $||\tau x^*||_{L^2(\gamma)} = ||T^*x^*||_{L^2(Q;\nu)}$ from Theorem 16. $\square$

We pause to note that we are really thinking about the map T^* in several different ways:

$$\begin{tikzcd}
C(Q)^* \arrow[r, "Lem.16"'] \arrow[rr, bend left, "T^*"] & BV(Q) \arrow[r] \arrow[d, "quotient"] & L^2(Q;\nu) \\
& BV(Q)_a \arrow[r] & H \arrow[u, "Lem.19"']
\end{tikzcd}$$

Thus we are frequently abusing notation and taking T^* to be the map $C(Q)^* \to BV(Q)$ as in Lemma 16, as a map $C(Q)^* \to H$ in Lemma 19, or (by including the composition with an appropriate quotient or inclusion) as a map $C(Q)^* \to L^2(Q;\nu)$, as in Theorem 27.

Definition 13. Let $\{\phi_j\}$ be a complete orthonormal set of functions of bounded variation in H. For $f \in H$, put

$$I_n f(x) = \sum_{j=1}^{n} (f, \phi_j)_a \int_Q \phi_j(u) \mathrm{d}x(u),$$

Define the Paley-Wiener-Zygmund (PWZ) integral to be $If(x) = \lim_{n\to\infty} I_n f(x)$ for all $x \in C(Q)$ for which this limit exists.

Theorem 29.

1. *For each $f \in BV(Q)$, the PWZ integral $If(x)$ exists and agrees with the Riemann-Stieltjes integral $\int_Q f(u)\mathrm{d}x(u)$ for* a.e. $x \in C_0(Q)$*,*

2. *the PWZ integral is essentially independent (up to $L^2(\gamma)$ equivalence) of the choice of orthonormal basis in Definition 13,*

3. $\int_{C_0(Q)} If(x)\gamma(dx) = \int_Q f(u)\mathrm{d}a(u),$

4. $\int_{C_0(Q)} (If(x))^2 \gamma(dx) = ||f||^2_{L^2(Q;\nu)} + \left(\int_Q f(u)\mathrm{d}a(u)\right)^2.$

5. *If f and g are in $BV(Q)$, then the covariance of If and Ig is $(f,g)_{L^2(Q;\nu)}$.*

Proof. To prove the first two statements, let $\{e_j\}$ be an orthonormal basis of functions in $BV(Q)$ for H and let $f \in BV(Q)$. Take the PWZ integrals $I_n f(x)$ as in Definition 13, note that $\sum_{j=1}^n (f, e_j)_a e_j \in BV(Q)_a$, and recall that the Riemann-Stieltjes integral $\int_Q f(u)\mathrm{d}x(u)$ exists for every $x \in C(Q)$. Now we observe that

$$\begin{aligned}\int_{C_0(Q)} \left| I_n f(x) - \int_Q f \mathrm{d}x \right|^2 \gamma(dx) &= \int_{C_0(Q)} \left| \sum_{j=1}^n (f, e_j)_a \int_Q e_j \mathrm{d}x - \int_Q f \mathrm{d}x \right|^2 \gamma(dx) \\ &= \int_{C_0(Q)} \left| \sum_{j=1}^n (f, e_j)_a \langle e_j^*, x\rangle - \langle f^*, x\rangle \right|^2 \gamma(dx), \qquad (3.45)\end{aligned}$$

with Corollary 12 yielding the appropriate functionals e_j^* and f^* in $C(Q)^*$. Now the integrand of (3.45) can be written as

$$\langle \sum_{j=1}^n (f, e_j)_a e_j^*, x\rangle^2 - 2\sum_{j=1}^n (f, e_j)_a \langle e_j^*, x\rangle \langle f^*, x\rangle + \langle f^*, x\rangle^2,$$

and then integrating over $C_0(Q)$ with respect to γ, Lemma 19 shows that (3.45) is equal to

$$\left|\left| T^* \left(\sum_{j=1}^n (f, e_j)_a e_j^* \right) \right|\right|_H^2 - 2\sum_{j=1}^n (f, e_j)_a \left[\left(T^* e_j^*, T^* f^* \right)_{L^2(Q;\nu)} + \langle e_j^*, a\rangle \langle f^*, a\rangle \right] + ||T^* f^*||_H \, ,$$

By the linearity of T^*, Corollary 12, and the fact that $(f, g)_a = (f, g)_{L^2(Q;\nu)} + \int_Q f \mathrm{d}a \int_Q g \mathrm{d}a$, this is equal to

$$\left(\left|\left| \sum_{j=1}^n (f, e_j)_a e_j \right|\right|_H^2 - 2\sum_{j=1}^n (f, e_j)_a^2 + ||f||_H^2 \right),$$

whence $\lim_{n\to\infty} I_n f(x) = \int_Q f(u)\mathrm{d}x(u)$ in $L^2(\gamma)$, and hence for a.e. $x \in C_0(Q)$.

Let (f_n) and (g_n) be sequences in $BV(Q)$ converging to f in $L^2(Q;\nu)$. Then we have

$$\begin{aligned}\lim_{n\to\infty}\int_{C_0(Q)}\left|\int_Q f_n \mathrm{d}x - \int_Q g_n \mathrm{d}x\right|^2 \gamma(dx) &= \lim_{n\to\infty}\int_{C_0(Q)} \left(\langle f^*, x\rangle - \langle g^*, x\rangle\right)^2 \gamma(dx)\\ &= \lim_{n\to\infty} ||f_n||_H^2 - 2\left(f_n, g_n\right)_a + ||g_n||_H^2\\ &= \lim_{n\to\infty} ||f_n - g_n||_H^2\,,\end{aligned}$$

and thus $\lim_{n\to\infty} If_n(x) = \lim_{n\to\infty} Ig_n(x)$ in $L^2(\gamma)$, and hence for a.e. $x \in C_0(Q)$ and thus $If(x)$ is essentially independent of the choice of approximating sequence in $BV(Q)$.

Now, taking f^* in $C(Q)^*$ corresponding to $f \in BV(Q)$ by Corollary 12, we note that $\tau f^* = \langle f^*, \cdot\rangle - \langle f^*, a\rangle \in C(Q)^*_\gamma$, and then we can apply Theorem 21 with $\xi = i$ to see that the characteristic function of the random variable τf^* is given by

$$\exp\left(-\frac{1}{2}\,||T^* f^*||^2_{L^2(Q;\nu)}\right) = \exp\left(-\frac{1}{2}\,||f||^2_{L^2(Q;\nu)}\right),$$

and then from the fact that $If(x) = \int_Q f(u)\mathrm{d}x(u) = \langle f^*, x\rangle$ for a.e. $x \in C_0(Q)$ we quickly see that $If(\cdot)$ is a Gaussian random variable with mean $\int_Q f(u)\mathrm{d}a(u) = \langle f^*, a\rangle$ and variance $||f||^2_{L^2(Q;\nu)}$, whence the next two conclusions follow.

Finally, let $f = T^*f^*$ and $g = T^*g^*$ for some f^* and g^* by Corollary 12. Then $If(x) - \int_Q f\mathrm{d}a = \langle f^*, x\rangle - \langle f^*, a\rangle$ and $Ig(x) - \int_Q g\mathrm{d}a = \langle g^*, x\rangle - \langle g^*, a\rangle$, whence the covariance of If and Ig is

$$\begin{aligned}\left(If(\cdot) - \int_Q f\mathrm{d}a, Ig(\cdot) - \int_Q g\mathrm{d}a\right)_{L^2(\gamma)} &= \left(\langle f^*, \cdot\rangle - \langle f^*, a\rangle, \langle g^*, \cdot\rangle - \langle g^*, a\rangle\right)_{L^2(\gamma)}\\ &= \left(\tau f^*, \tau g^*\right)_{L^2(\gamma)}\\ &= \left(T^* f^*, T^* g^*\right)_{L^2(Q;\nu)}\\ &= \left(f, g\right)_{L^2(Q;\nu)}\,,\end{aligned}$$

and the final conclusion holds. □

As a consequence of Theorem 29, we also have the following corollary.

Corollary 13. *If $f \in BV(Q)$, then $\int_Q f\mathrm{d}x \in L^2(\gamma)$ with*

$$\left|\left|\int_Q f\mathrm{d}x\right|\right|_{L^2(\gamma)} = ||If(\cdot)||_{L^2(\gamma)} = ||f||_H \,. \tag{3.46}$$

Lemma 20. *Let f and g be elements of H. Then:*

1. *There is a number $L \in \mathbb{R}$ such that $\lim_{n\to\infty} \int_Q f_n\mathrm{d}a = L$ whenever (f_n) is a sequence in $BV(Q)$ with $f_n \to f$ in H. In fact, $L = If(a)$.*

2. *$(f,g)_a = (f,g)_{L^2(Q;\nu)} + If(a)Ig(a)$ and $||f||_H^2 = ||f||^2_{L^2(Q;\nu)} + (If(a))^2$.*

Proof. If (f_n) in $BV(Q)$ is convergent in H, then $\left(||f_n||_{L^2(Q;\nu)}\right)$ and $\left(\int_Q f_n\mathrm{d}a\right)$ must each be convergent. Put $L = \lim_{n\to\infty}\int_Q f_n\mathrm{d}a$. Note that if (g_n) is another sequence in $BV(Q)$ converging to f in H, then $||f_n - g_n||_{L^2(Q;\nu)} + \left(\int_Q (f_n - g_n)\,\mathrm{d}a\right)^2 = ||f_n - g_n||_H \to 0$ as $n \to \infty$, whence $\int_Q g_n\mathrm{d}a \to L$. Let (e_j) be an orthonormal basis in $BV(Q)$ for H and then $f_n = \sum_{j=1}^n (f, e_j)_a\, e_j$ is in $BV(Q)$; moreover, we see that

$$If(a) = \lim_{n\to\infty}\sum_{j=1}^n (f,e_j)_a \int_Q e_j\mathrm{d}a = \lim_{n\to\infty}\int_Q \sum_{j=1}^n (f,e_j)_a e_j\mathrm{d}a = \lim_{n\to\infty}\int_Q f_n\mathrm{d}a = L.$$

The second conclusion follows from the first and the fact that

$$(f,g)_a = \lim_{n\to\infty}(f_n,g_n)_a = \lim_{n\to\infty}\left[(f_n,g_n)_{L^2(Q;\nu)} + \int_Q f_n\mathrm{d}a \int_Q g_n\mathrm{d}a\right],$$

where (f_n) and (g_n) are any sequences in $BV(Q)$ converging to f and g, respectively. □

Theorem 30.

1. *If* $f \in H$*, then the PWZ integral* $If(x)$ *exists for a.e.* $x \in C_0(Q)$ *and is essentially independent of the choice of orthonormal basis in Definition 13.*

2. *If* $f \in H$*, then* If *is a Gaussian random variable with mean* $If(a)$ *and variance* $||f||^2_{L^2(Q;\nu)}$

3. *If* f *and* g *are in* H*, then the covariance of the random variables* If *and* Ig *is* $(f,g)_{L^2(Q;\nu)}$.

Proof. Let (e_j) be an orthonormal basis in $BV(Q)$ for H and then $f_n = \sum_{j=1}^{n} (f, e_j)_a e_j$ defines a sequence (f_n) in $BV(Q)$; note that $||f_n - f_m||_H \to 0$ as $m, n \to \infty$; *i.e.* (f_n) is Cauchy, because it is convergent. Then

$$||If_n - If_m||_{L^2(\gamma)} = ||I(f_n - f_m)||_{L^2(\gamma)} = ||f_n - f_m||_H$$

by Corollary 13, and hence (If_n) is Cauchy in $L^2(\gamma)$. Let $F(x)$ be its limit, and then

$$F(x) = \lim_{n\to\infty} If_n(x) = \lim_{n\to\infty} \int_Q \sum_{j=1}^{n} (f, e_j)_a e_j \mathrm{d}x = \lim_{n\to\infty} \sum_{j=1}^{n} (f, e_j)_a \int_Q e_j \mathrm{d}x = \lim_{n\to\infty} I_n f(x)$$

in $L^2(\gamma)$. Thus $F = \lim_{n\to\infty} I_n f = If$ in $L^2(\gamma)$, and so If exists for a.e. x. If (h_j) is another orthonormal basis in $BV(Q)$, then by Corollary 12 and Lemma 19 we see that

$$\begin{aligned}\left\|\sum_{j=1}^{n}(f,e_j)_a\int_Q e_j\mathrm{d}a-\sum_{j=1}^{n}(f,h_j)_a\int_Q h_j\mathrm{d}a\right\|_{L^2(\gamma)} &= \left\|\sum_{j=1}^{n}\left[(f,e_j)_a\langle e_j^*,x\rangle-(f,h_j)_a\langle h_j^*,x\rangle\right]\right\|_{L^2(\gamma)}\\ &= \left\|\langle\sum_{j=1}^{n}\left[(f,e_j)_a e_j^*-(f,h_j)_a h_j^*\right],x\rangle\right\|_{L^2(\gamma)}\\ &= \left\|T^*\left(\sum_{j=1}^{n}\left[(f,e_j)_a e_j^*-(f,h_j)_a h_j^*\right]\right)\right\|_H\\ &= \left\|\sum_{j=1}^{n}(f,e_j)_a e_j-\sum_{j=1}^{n}(f,h_j)_a h_j\right\|_H,\end{aligned}$$

and then we must have $\lim_{n\to\infty} I_n f(x)=\lim_{n\to\infty} I_n g(x)$.

Again using the sequence (f_n) as above, note that $If_n(x)=I_nf(x)$ for a.e. x by the linearity of the Reimann-Stieltjes integral. Now, by Theorem 29 we see that each If_n is Gaussian with mean $\int_Q f_n\mathrm{d}a$ and variance $||f_n||^2_{L^2(Q;\nu)}$, and $\lim_{n\to\infty} If_n(x)=If(x)$ for a.e. x. Thus If_n has characteristic function

$$\int_{C_0(Q)}\exp\left(iIf_n(x)\right)\gamma(dx)=\exp\left(-\frac{1}{2}\,||f_n||^2_{L^2(Q;\nu)}+iIf_n(a)\right),$$

and then the fact that $|\exp\left(iIf_n(x)\right)|\le 1$ allows the use of dominated convergence to show that If is Gaussian with the desired mean and variance.

Now we can compute the covariance. Note that

$$\int_{C_0(Q)}\left(If(x)-If(a)\right)\left(Ig(x)-Ig(a)\right)\gamma(dx)=\int_{C_0(Q)}If(x)Ig(x)\gamma(dx)-If(a)Ig(a),$$

so that it suffices to show that $(If,Ig)^2_{L^2(\gamma)}=(f,g)_{L^2(Q;\nu)}+If(a)Ig(a)$. Once more we enlist our orthonormal basis (e_j) and construct (f_n) and (g_n) as before, noting that $If=\lim_{n\to\infty} I_nf=If_n$ and $Ig=\lim_{n\to\infty} I_ng=Ig_n$ in $L^2(\gamma)$ and also that $f_n\to f$ and $g_n\to g$

in H. As each f_n and g_n are in $BV(Q)$, we see that

$$|(f_n, g_n)_a - (f, g)_a| \leq ||g_n||_H \, ||f_n - f||_H + ||f||_H \, ||g_n - g||_H \,,$$

so that $(f_n, g_n)_a \to (f, g)$ as $n \to \infty$. In the same way, $(I_n f, I_n g)_{L^2(\gamma)} \to (If, Ig)_{L^2(\gamma)}$. With this and Corollary 13, we see that

$$(If, Ig)_{L^2(\gamma)} = \lim_{n\to\infty} (I_n f, I_n g)_{L^2(\gamma)} = \lim_{n\to\infty} (f_n, g_n)_H = (f, g)_H,$$

and then the desired result follows from the fact that $(f, g)_H = (f, g)_{L^2(Q;\nu)} + If(a)Ig(a)$. $\square$

We can now state another version of the classic Cameron-Martin translation theorem.

Theorem 31 (Translation Theorem)**.** *Let $f \in H$ and take any $x_0 \in C_0(Q)$ satisfying $x_0(t) = Tf(t) = \int_Q \chi_{[0,t]}(u) f(u) \nu(du)$. If $q : C_0(Q) \to C_0(Q)$ by $q(x) = x + x_0$, then $\gamma \circ q$ is absolutely continuous with respect to γ, and*

$$\frac{d\gamma \circ q}{d\gamma}(x) = \exp\left(-\frac{1}{2} ||f||^2_{L^2(Q;\nu)} + If(x) - If(a) \right). \tag{3.47}$$

Proof. Taking an orthonormal basis (e_j) in $BV(Q)$ for H, we have the $L^2(\gamma)$ limit

$$\begin{aligned} If(x) - If(a) &= \lim_{n\to\infty} \left[\sum_{j=1}^{n} (f, e_j)_a \int_Q e_j \mathrm{d}x - \sum_{j=1}^{n} (f, e_j)_a \int_Q e_j \mathrm{d}a \right] \\ &= \lim_{n\to\infty} \left[\langle \sum_{j=1}^{n} (f, e_j)_a e_j^*, x \rangle - \langle \sum_{j=1}^{n} (f, e_j)_a e_j^*, a \rangle \right], \end{aligned}$$

where e_j^* is taken as in Corollary 12. Noting that

$$\int_{C_0(Q)} \langle \sum_{j=1}^{n} (f, e_j)_a e_j^*, x \rangle \gamma(dx) = \langle \sum_{j=1}^{n} (f, e_j)_a e_j^*, a \rangle$$

we see that $If(\cdot) - If(a)$ must be an element of $C(Q)^*_\gamma$. Now observe that

$$\begin{aligned}
R_\gamma(If(\cdot) - If(a))(\delta_t) &= \int_{C_0(Q)} (If(x) - If(a))(\langle \delta_t, x\rangle - a(t))\gamma(dx) \\
&= \int_{C_0(Q)} If(x)\langle \delta_t, x\rangle \gamma(dx) - a(t)If(a) \\
&= (f, T^*\delta_t)_{L^2(Q;\nu)} + a(t)If(a) - a(t)If(a) \\
&= \int_Q \chi_{[0,t]}(u) f(u)\nu(du) \\
&= x_0(t),
\end{aligned}$$

where R_γ is as in Lemma 9.

Then the hypotheses of Theorem 20 are satisfied, and

$$\begin{aligned}
\frac{d\gamma \circ q}{d\gamma}(x) &= \exp\left(If(x) - If(a) - \frac{1}{2} \left|\left| If(\cdot) - If(a) \right|\right|^2_{L^2(\gamma)} \right) \\
&= \exp\left(If(x) - If(a) - \frac{1}{2} \left|\left| f \right|\right|^2_{L^2(Q;\nu)} \right),
\end{aligned}$$

as desired. $\square$

The following theorem is the multiple-parameter version of Theorem 4. It is the basic integration formula for cylindrical functions.

Theorem 32 (PWZ Theorem)**.** *Let $\{\phi_1, \ldots, \phi_n\}$ be a linearly independent set in H_0 and let $f : \mathbb{R}^n \to \mathbb{C}$. Then $F(x) = f(I\phi_1(x), \ldots, I\phi_n(x))$ is measurable if and only if f is Lebesgue measurable, and in this case*

$$\int_{C(Q)} F(x)\gamma(dx) \overset{*}{=} \frac{\det(M^{-1})}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(\mathbf{u}) \exp\left(-\frac{1}{2} M(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) \right) d\mathbf{u},$$

where M is the $n \times n$ matrix with entries $m_{i,j} = (\phi_i, \phi_j)_{L^2(Q;\nu)}$, $\mathbf{v} = (I\phi_1(a), \ldots, I\phi_n(a))$, *and* $\stackrel{*}{=}$ *indicates that if one side of the equality exists then so does the other.*

Proof. The proof follows directly from Theorem 30, the definition of the multivariate normal distribution, and the same observations as in the proof of Corollary 10. □

Corollary 14. *Let* $\{\phi_1, \ldots, \phi_n\}$ *be an orthonormal set in* H_0 *and let* $f : \mathbb{R}^n \to \mathbb{C}$*. Then* $F(x) = f(I\phi_1(x), \ldots, I\phi_n(x))$ *is measurable if and only if* f *is Lebesgue measurable, and in this case*

$$\int_{C(Q)} F(x)\gamma(dx) \stackrel{*}{=} (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(\mathbf{u}) \exp\left(-\sum_{j=1}^{n} \frac{(u_j - I\phi_j(a))^2}{2} \right) \mathrm{d}\mathbf{u}, \tag{3.48}$$

where $\stackrel{*}{=}$ *indicates that if one side exists, the other also exists with equality.*

Proof. The fact that $\{\phi_1, \ldots, \phi_n\}$ is orthonormal ensures that the covariance matrix C is the identity matrix. Then apply Theorem 32. □

3.8 Examples

In this section, we collect and comment on several examples, most of which are well-known in other contexts. In these examples we take m to denote Lebesgue measure. If $k(u,t)$ is the characteristic function of the set $\{(u,t) : u \leq t\}$, then k is a Volterra kernel and for suitable choices of ν absolutely continuous with respect to m, some well-known examples result.

Example 3. If we take $Q = [0,1]$ and $\nu = \mathrm{m}$ and $a = 0$ on $[0,1]$, it is not hard to see that

$$K(s,t) = \langle R\delta_s, \delta_t \rangle = \int_{[0,1]} \chi_{[0,s]}(u)\chi_{[0,t]}(u)\mathrm{m}(du) = \mathrm{m}([0,s] \cap [0,t]) = \min(s,t),$$

and we have obtained the classical Wiener space with Cameron-Martin space

$$H_\gamma = \left\{ x : x(t) = \int_0^t x'(u)\mathrm{d}u \text{ for some } x' \in L^2[0,1] \right\}.$$

Note that this is the space of absolutely continuous functions on $[0,1]$ with derivative in $L^2[0,1]$ that vanish at 0 (*i.e.* the Sobolev space $H_0^1[0,1]$), as expected.

Example 4. Taking $k(u,t) = \chi_{[t,T]}(u)$ in the previous example, we obtain a similar function space, with its elements a.e. "tied down" at T, as in [50].

Example 5. In a similar fashion to the previous example, if we take $Q = [0,1]^2$ with Lebesgue measure and $a = 0$ on Q, the result is the classical two-parameter Wiener (Yeh-Wiener) space with

$$K(\mathbf{s},\mathbf{t}) = \int_{[0,1]^2} \chi_{[\mathbf{0},\mathbf{s}]}(\mathbf{u})\chi_{[\mathbf{0},\mathbf{t}]}(\mathbf{u})\mathrm{m}(d\mathbf{u}) = \mathrm{m}([\mathbf{0},\mathbf{s}] \cap [\mathbf{0},\mathbf{t}]) = \min(s_1,t_1)\min(s_2,t_2).$$

In this case, the Cameron-Martin space is

$$H_\gamma = \left\{ x : x(t_1,t_2) = \int_0^{t_1}\int_0^{t_2} x'(u_1,u_2)\mathrm{d}u_1\mathrm{d}u_2 \text{ for some } x' \in L^2[0,1]^2 \right\},$$

which is the collection of absolutely continuous functions on $[0,1]^2$ for which $\frac{\partial^2 x}{\partial t_1 \partial t_2} \in L^2[0,1]^2$ and for which $x(t_1,t_2) = 0$ whenever either $t_1 = 0$ or $t_2 = 0$.

Example 6. Again take Lebesgue measure on $[0,1]$ and put $k(u,t) = \chi_{[0,t]}(u) - \mathrm{m}([0,t]) =$

$\chi_{[0,t]}(u) - t$. Now we have

$$\begin{aligned} K(s,t) &= \int_0^1 \big(\chi_{[0,s]}(u) - s\big)\big(\chi_{[0,t]}(u) - t\big)\,\mathrm{m}(du) \\ &= \int_0^1 \big[\chi_{[0,s]\cap[0,t]}(u) - t\chi_{[0,s]}(u) - s\chi_{[0,t]}(u) + st\big]\,\mathrm{m}(du) \\ &= \mathrm{m}([0,s]\cap[0,t]) - st \\ &= \min(s,t) - st, \end{aligned}$$

and we have obtained a Gaussian measure on $C[0,1]$ for which the stochastic process $X_t = \langle \delta_t, x\rangle$ corresponds to the well-known Brownian bridge. Note that

$$H_\gamma = \left\{ x : x(t) = \int_0^1 \big(\chi_{[0,t]}(u) - \mathrm{m}([0,t])\big)\, x'(u)\mathrm{m}(du) \text{ for some } x' \in L^2[0,1] \right\}.$$

Note that

$$\int_0^1 \big(\chi_{[0,t]}(u) - \mathrm{m}([0,t])\big)\, x'(u)\mathrm{m}(du) = \int_0^t x'(u)\mathrm{m}(du) - t\int_0^1 x'(u)\mathrm{m}(du)$$

so that the Cameron-Martin space for this measure is the space of absolutely continuous functions on $[0,1]$ with derivative in $L^2[0,1]$ and for which $x(0) = x(1) = 0$.

Example 7. Let $b(t) = \int_0^t b'(u)\mathrm{d}u$ for some nonnegative $b' \in L^p[0,1]$ for $p > 1$. Take ν to be the Lebesgue-Stieltjes measure with respect to b; that is $\nu(E) = \int_E \mathrm{d}b(u) = \int_E b'(u)\mathrm{d}u$ for measurable sets E. Then

$$K(s,t) = \int_0^1 \chi_{[0,s]}(u)\chi_{[0,t]}(u)\mathrm{d}b(u) = b(\min(s,t)),$$

is the covariance function for a Gaussian measure on $C[0,1]$ and the corresponding Cameron-

Martin space is

$$H_\gamma = \left\{ x : x(t) = \int_0^t x'(u)\mathrm{d}b(u) \text{ for some } x' \in L_b^2[0,1] \right\}.$$

We denote the resulting measure space by $(C_{0,b}[0,1], \gamma)$.

Example 8. Taking the previous two examples in combination, we consider the interval $[0, L]$ and let $b(t) = \int_0^t b'(u)\mathrm{d}u$ for some continuous nonnegative b'. Put $k(u,t) = \sqrt{b(L)}\chi_{[0,t]}(u) - \frac{b(t)}{\sqrt{b(L)}}$, and the resulting covariance function is

$$\begin{aligned}
K(s,t) &= \int_0^L \left(\sqrt{b(L)}\chi_{[0,s]}(u) - \frac{b(s)}{\sqrt{b(L)}}) \right) \left(\sqrt{b(L)}\chi_{[0,t]}(u) - \frac{b(t)}{\sqrt{b(L)}} \right) \mathrm{d}b(u) \\
&= \int_0^L \left[b(L)\chi_{[0,s]\cap[0,t]}(u) - b(s)\chi_{[0,t]}(u) - b(t)\chi_{[0,s]}(u) + \frac{b(s)b(t)}{b(L)} \right] b'(u)\mathrm{d}u \\
&= b(L)b(\min(s,t)) - b(s)b(t) \\
&= \begin{cases} b(s)(b(L) - b(t)) & \text{if } s \leq t, \\ b(t)(b(L) - b(s)) & \text{if } t \leq s. \end{cases}
\end{aligned}$$

In this way we obtain a measure with respect to which the stochastic process $X_t = \langle \delta_t, x \rangle$ is a Brownian bridge whose covariance structure is governed by the Lebesgue-Stieltjes measure $\mathrm{d}b$. Note that Example 7 corresponds to the choices $L = 1$ and $b(t) = t$.

We remark that the Cameron-Martin space for the measure we obtain in this case is

$$H_\gamma = \left\{ x : x(t) = \int_0^L \left(\sqrt{b(L)}\chi_{[0,t]}(u) - \frac{b(t)}{\sqrt{b(L)}} \right) x'(u)\mathrm{d}b(u) \text{ for some } x' \in L_b^2[0,L] \right\}.$$

Noting that

$$x(t) = \sqrt{b(L)} \int_0^t x'(u)b'(u)\mathrm{d}u - \frac{b(t)}{\sqrt{b(L)}} \int_0^L x'(u)b'(u)\mathrm{d}u$$

for each $x \in H_\gamma$ and that $b(0) = 0$, we again see that $x(0) = x(L) = 0$, as expected.

Example 9. With $d = 1$ and functions a and b chosen such that $a' \in L^2[0,T]$ and $b \in C^1[0,T]$ with b' positive and bounded away from 0 we obtain the measure spaces $(C_{a,b}[0,T], \gamma)$ of Chapter 2. In this case, also recall that we have $L^2_b[0,1] = L^2[0,1]$ by these stricter conditions on b', as then we have both $||b'||_\infty < \infty$ and $\left|\left|\frac{1}{b'}\right|\right|_\infty < \infty$. We remark that the translation theorem from [12, 13] requires translations by elements of the form $x_0(t) = \int_0^t f(u)\mathrm{d}b(u)$ for f satisfying $\int_0^1 |f|^2 \, \mathrm{d}b < \infty$ and $\int_0^1 |f|^2 \, \mathrm{d}a < \infty$; however, by our Theorem 31, one can relax the second of these requirements to $\int_0^1 |f| \, \mathrm{d}\, |a| < \infty$.

Example 10. This nonexample exhibits a measure that is not of the type discussed in Section 8. Let $\mathfrak{c} : [0,1] \to [0,1]$ be the Cantor function. Note that $\mathfrak{c}$ is monotone and that $\mathfrak{c}' = 0 \quad a.e.$ on $[0,1]$. As $\mathfrak{c}$ is of bounded variation, we can define a measure ν on $[0,1]$ by $\nu(E) = \int_E \mathrm{d}\mathfrak{c}(u)$; moreover, we will certainly have $\nu \in C[0,1]^*$. Take $k(u,t)$ to be the Volterra kernel.

As before, we can build a covariance function $K(s,t) = \mathfrak{c}(\min(s,t))$. However, under our method of construction we have no guarantee that the support of the resulting measure γ must be contained in the space of continuous functions $C[0,1]$, because the necessary Hölder-type condition $\nu([0,s]\triangle[0,t]) = \int_0^1 |k(u,t) - k(u,s)|^2 \, \nu(du) < C\, |s-t|^\alpha$ might not hold. There is certainly an $\mathbb{R}$-valued stochastic process with this covariance, but we don't know whether its sample paths are continuous.

Example 11. Our final example is the family of spaces $C_{a,b}(Q)$, in the case were $d = 2$. Let $Q = [0,S] \times [0,T]$. Now, take $b \in AC(Q)$ with $b(s,t) = 0$ whenever some $s = 0$ or $t = 0$; then we have

$$b(s,t) = \int_0^s \int_0^t \frac{\partial^2 b}{\partial s \partial t} \mathrm{d}v\mathrm{d}u = \int_Q \chi_{[0,(s,t)]}(u,v)\nu(du,dv) = (T^*\delta_{(s,t)}, T^*\delta_{(s,t)})$$

for some suitable choice of ν, *i.e.* if ν is absolutely continuous with respect to Lebesgue measure with nonnegative Radon-Nicodym derivative $\frac{\partial^2 b}{\partial s \partial t} \in L^2(Q)$. In the same way, take

$$a(s,t) = \int_0^s \int_0^t \frac{\partial^2 a}{\partial s \partial t}(u,v)\mathrm{d}v\mathrm{d}u,$$

with $\frac{\partial^2 a}{\partial s \partial t} \in L^2(Q)$. With these and the Volterra kernel $k((u,v),(s,t)) = \chi_{[0,s]\times[0,t]}(u,v)$, we can obtain a measure in the same manner as Section 3.7. In this case we have the convenient representations

$$If(a) = \int_0^S \int_0^T f(u,v)\frac{\partial^2 a}{\partial s \partial t}(u,v)\mathrm{d}v\mathrm{d}u$$

and

$$||f||_{L^2(Q;\nu)} = \int_0^S \int_0^T |f(u,v)|^2 \frac{\partial^2 b}{\partial s \partial t}(u,v)\mathrm{d}v\mathrm{d}u,$$

provided the former exists. In this case we will refer to the measure γ as being subordinate to the functions a and b. Variations on this example will be the setting for the next chapter.

Chapter 4

Integration Over Paths

In this chapter, we will take $\mathfrak{m}$ to be the generalized Yeh-Wiener measure on $C_0(Q)$ subordinate to the functions a and b, as in Example 11 of the previous chapter. We will denote the resulting measure space by $C_{a,b}(Q)$.

For $0 < s_1 < \cdots < s_m \le S$ and $0 < t_1 < \cdots < t_n \le T$, the distribution of a finite-dimensional projection onto $\mathbb{R}^{mn}$ with component projections given by $\{\delta_{s_i,t_j)} : 1 \le i \le m, 1 \le j \le n\}$ (the generalized Yeh-Wiener kernel) is given by

$$W_{m,n}(\mathbf{u},\mathbf{s},\mathbf{t}) = \left(\prod_{i=1}^{m}\prod_{j=1}^{n} 2\pi\Delta_i\Delta_j b(s,t)\right)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{(\Delta_i\Delta_j(u-a(s,t)))^2}{\Delta_i\Delta_j b(s,t)}\right), \quad (4.1)$$

where $\Delta_i\Delta_j u = u_{i,j} - u_{i-1,j} - u_{i,j-1} + u_{i-1,j-1}$ and $u_{i,0} = u_{0,j} = 0$ for all $i, j \ge 0$.

Theorem 33 (Tame Functionals). *Let* $0 < s_1 < \cdots < s_m \le S$ *and* $0 < t_1 < \cdots < t_n \le T$ *and let* $f : \mathbb{R}^{mn} \to \mathbb{C}$ *and* $F : C_{a,b}(Q) \to \mathbb{C}$ *be defined by* $F(x) = f(x(s_1,t_1),\ldots,x(s_m,t_n))$. *Then* F *is measurable if and only if* f *is Lebesgue measurable, and in this case,*

$$\int_{C_{a,b}[Q]} F(x)\mathfrak{m}(dx) \overset{*}{=} \int_{\mathbb{R}^{mn}} f(u_{1,1},\ldots u_{m,n}))W_{m,n}(\mathbf{u},\mathbf{s},\mathbf{t})d\mathbf{u}, \quad (4.2)$$

where the equality (*) is in the sense that if one of the integrals exists then the other exists with equality.

Proof. Let $\phi_{i,j} = \chi_{[s_{i-1},s_i]\times[t_{j-1},t_j]}(u,v)$. An easy calculation shows that

$$x(s_k,t_l) = \sum_{\substack{0\le i\le k\\0\le j\le l}} \Delta_i\Delta_j x(s,t).$$

for any (s_k,t_l). It is also not hard to see that

$$\Delta_i\Delta_j x(s,t) = I\phi_{i,j}(x) = \int_Q \chi_{[s_{i-1},s_i]\times[t_{j-1},t_j]}(u,v)\mathrm{d}x(u,v).$$

Thus we have

$$F(x) = f(x(s_1,t_1),\ldots,x(s_m,t_n)) = f\left(I\phi_{1,1}(x),\ldots,\sum_{\substack{0\le i\le k\\0\le j\le l}} I\phi_{i,j}(x),\ldots,\sum_{\substack{0\le i\le m\\0\le j\le n}} I\phi_{i,j}(x)\right).$$

As $\phi_{i,j}\in BV(Q)$, we use Lemma 17 to compute

$$\int_{C_{a,b}(Q)} I\phi_{i,j}(x)\mu(dx) = I\phi_{i,j}(a) = \Delta_i\Delta_j a(s,t),$$

and also observe that

$$\begin{aligned}\int_{C_{a,b}(Q)} \left(I\phi_{i,j}(x) - I\phi_{i,j}(a)\right)\left(I\phi_{l,m}(x) - I\phi_{l,m}(a)\right)\mathfrak{m}(dx)\\ &= \int_Q \phi_{i,j}(u,v)\phi_{l,m}(u,v)\mathrm{d}b(u,v)\\ &= \begin{cases}\Delta_i\Delta_j b(s,t) & \text{if } i=l, j=m\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

From this, we see that the covariance matrix M for the collection $\{\phi_{i,j}\}$ is a diagonal matrix whose nonzero entries are $\Delta_i\Delta_j b(s,t)$. Now we can apply Theorem 32 to complete the proof. $\square$

We will be concerned with integrating functionals defined in terms of certain paths in Q. We will confine our discussion to paths $\phi : [0,S] \to Q$ for which $\phi(\tau) = (\phi_1(\tau), \phi_2(\tau))$ satisfies the condition that its component functions ϕ_1 and ϕ_2 are each piecewise continuously differentiable. We will say that ϕ is an increasing path in Q if $\phi' \cdot \phi' > 0$ on $[0,S]$ and $\phi(\tau_1) \le \phi(\tau_2)$ whenever $\tau_1 \le \tau_2$.

4.1 Preliminaries

Our first theorem in this section establishes a special case of the tame functionals theorem in the case that one defines the functional in terms of a sequence of points lying on an increasing path.

Theorem 34. *Let* $0 = s_0 < s_1 \le \cdots \le s_n \le S$ *and* $0 = t_0 < t_1 \le \cdots \le t_n \le T$ *and let* $f : \mathbb{R}^n \to \mathbb{C}$ *be Lebesgue measurable. If* $F : C_{a,b}(Q) \to \mathbb{C}$ *is defined by* $F(x) = f(x(s_1,t_i)), x(s_2,t_2), \ldots, x(x_n,t_n))$, *then* F *is* μ*-measurable and*

$$\int_{C_{a,b}(Q)} F(x)\mathfrak{m}(dx) \overset{*}{=} \left(\prod_{j=1}^{n} 2\pi(b(s_j,t_j) - b(s_{j-1},t_{j-1})\right)^{-\frac{1}{2}} \int_{\mathbb{R}^n} f(u_1,\ldots,u_n) \exp\left(-\frac{1}{2}\sum_{j=1}^{n} \frac{(u_j - a(s_j,t_j) - u_{j-1} + a(s_{j-1},t_{j-1}))^2}{b(s_j,t_j) - b(s_{j-1},t_{j-1})}\right) \mathrm{d}u_1 \cdots \mathrm{d}u_n, \tag{4.3}$$

where the equality ($\overset{*}{=}$) *is in the sense that if one of the integrals exists then the other exists with equality.*

Proof. The proof is by induction on n. The theorem is clearly true for $n = 1$, for by (4.2),

$$\int_{C_{a,b}(Q)} f(x(s_1,t_1))\mathfrak{m}(dx) = \frac{1}{\sqrt{2\pi b(s_1,t_1)}} \int_{\mathbb{R}} f(u_1)\exp\left(-\frac{1}{2}\sum_{j=1}^{1}\frac{(u_j - a(s_j,t_j))^2}{b(s_j,t_j) - b(s_{j-1},t_{j-1})}\right) \mathrm{d}u_1,$$

and thus (4.3) holds.

Assume that the theorem holds for $n = k \geq 1$. Then for $n = k+1$, we have

$$\begin{aligned} &\int_{C_{a,b}(Q)} f(x(s_1,t_1),\ldots,x(s_n,t_n))\mu(dx) \\ &\qquad = \left(\prod_{i=1}^{k+1}\prod_{j=1}^{k+1} 2\pi\Delta_i\Delta_j b(s,t)\right)^{-\frac{1}{2}} \int_{\mathbb{R}^{(k+1)^2}} f(u_{1,1},\ldots,u_{k+1,k+1}) \\ &\qquad\qquad \exp\left(-\frac{1}{2}\sum_{i=1}^{k+1}\sum_{j=1}^{k+1}\frac{(\Delta_i\Delta_j(u - a(s,t)))^2}{\Delta_i\Delta_j b(s,t)}\right)\mathrm{d}u_1\cdots\mathrm{d}u_n. \end{aligned} \tag{4.4}$$

Notice that for $j = 1,\ldots,k$ the variables $u_{k+1,j}$ and $u_{j,k+1}$ appear in (4.4) only in the kernel as the functional $F(x)$ does not depend on the values of x at these points. Also observe that $b(s_{k+1},t_1) - b(s_k,t_1) = \Delta_{k+1}\Delta_1 b(s,t)$, $b(s_{k+1},t_2) - b(s_k,t_2) = \Delta_{k+1}\Delta_2 b(s,t) + \Delta_{k+1}\Delta_1 b(s,t)$, and eventually $b(s_{k+1},t_k) - b(s_k,t_k) = \Delta_{k+1}\Delta_k b(s,t) + \cdots + \Delta_{k+1}\Delta_1 b(s,t)$, and also that

$$\begin{aligned} u_{k+1,1} - a(s_{k+1},t_1) - u_{k,1} + a(s_k,t_1) &= \Delta_{k+1}\Delta_1(u - a(s,t)), \\ u_{k+1,2} - a(s_{k+1},t_2) - u_{k,2} + a(s_k,t_2) &= \Delta_{k+1}\Delta_2(u - a(s,t)) + \Delta_{k+1}\Delta_1(u - a(s,t)), \\ &\vdots \\ u_{k+1,k} - a(s_{k+1},t_k) - u_{k,k} + a(s_k,t_k) &= \Delta_{k+1}\Delta_k(u - a(s,t)) + cldots + \Delta_{k+1}\Delta_1(u - a(s,t)). \end{aligned}$$

Applying Proposition 1 (the Chapman-Kolmogorov equation) $2k-2$ times to the right side of (4.4) will yield

$$\left(\prod_{i=1}^{k}\prod_{j=1}^{k} 2\pi\Delta_i\Delta_j b(s,t)\right)^{-\frac{1}{2}} \int\limits_{\mathbb{R}^{k^2+1}} f(u_{1,1},\ldots,u_{k,k}) \exp\left(-\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k}\frac{(\Delta_i\Delta_j(u-a(s,t)))^2}{\Delta_i\Delta_j b(s,t)}\right)$$
$$\frac{1}{\sqrt{2\pi\Delta_{k+1}\Delta_{k+1}b(s,t)}}\frac{1}{\sqrt{2\pi(b(s_k,t_{k+1})-b(s_k,t_k)))}}\frac{1}{\sqrt{2\pi(b(s_k,t_{k+1})-b(s_k,t_k))}} \tag{4.5}$$
$$\int\limits_{\mathbb{R}^2} \exp\left(-\frac{(\Delta_{k+1}\Delta_{k+1}(u-a(a,t)))^2}{2\Delta_{k+1}\Delta_{k+1}b(s,t)}\right)\exp\left(-\frac{(u_{k+1,k}-a(s_{k+1},t_k)-u_{k,k}+a(s_k,t_k))^2}{b(s_{k+1},t_k)-b(s_k,t_k)}\right)$$
$$\exp\left(-\frac{(u_{k,k+1}-a(s_k,t_{k+1})-u_{k,k}+a(s_k,t_k))^2}{b(s_k,t_{k+1})-b(s_k,t_k)}\right)\mathrm{d}u_{k+1,k}\mathrm{d}u_{k,k+1}\mathrm{d}u_{k+1,k+1}\mathrm{d}u_{k,k}\ldots\mathrm{d}u_{1,1}$$

Now notice that

$$\begin{aligned}
&\Delta_{k+1}\Delta_{k+1}(\mathbf{u}-a(s,t))\\
&=u_{k+1,k+1}-a(s_{k+1},t_{k+1})-u_{k+1,k}+a(s_{k+1},t_k)-u_{k,k+1}+a(s_k,t_{k+1})+u_{k,k}-a(s_k,t_k)\\
&=[(u_{k+1,k+1}-a(s_{k+1},t_{k+1}))-(u_{k,k}-a(s_k,t_k))]\\
&-[(u_{k,k+1}-a(s_k,t_{k+1}))-(u_{k,k}-a(s_k,t_k))]-[(u_{k+1,k}-a(s_{k+1},t_k))-(u_{k,k}-a(s_k,t_k))],
\end{aligned}$$

and also that

$$\begin{aligned}
\Delta_{k+1}\Delta_{k+1}b(s,t)&=b(s_{k+1},t_{k+1})-b(s_k,t_{k+1})-b(s_{k+1},t_k)+b(s_k,t_k)\\
&=[b(s_{k+1},t_{k+1})-b(s_k,t_k)]-[b(s_k,t_{k+1})-b(s_k,t_k)]-[b(s_{k+1},t_k)-b(s_k,t_k)],
\end{aligned}$$

and apply the Chapman-Kolmogorov equation twice to the inner double integral in (4.5),

$$\frac{1}{\sqrt{2\pi\Delta_{k+1}\Delta_{k+1}b(s,t)}}\frac{1}{\sqrt{2\pi(b(s_k,t_{k+1})-b(s_k,t_k))}}\frac{1}{\sqrt{2\pi(b(s_k,t_{k+1})-b(s_k,t_k))}}$$
$$\int_{\mathbb{R}^2}\exp\left(-\frac{(\Delta_{k+1}\Delta_{k+1}(u-a(a,t)))^2}{2\Delta_{k+1}\Delta_{k+1}b(s,t)}\right)\exp\left(-\frac{(u_{k+1,k}-a(s_{k+1},t_k)-u_{k,k}+a(s_k,t_k))^2}{b(s_{k+1},t_k)-b(s_k,t_k)}\right)$$
$$\exp\left(-\frac{(u_{k,k+1}-a(s_k,t_{k+1})-u_{k,k}+a(s_k,t_k))^2}{b(s_k,t_{k+1})-b(s_k,t_k)}\right)\mathrm{d}u_{k+1,k}\mathrm{d}u_{k,k+1},$$

which yields

$$\left(\frac{1}{2\pi(b(s_{k+1},t_{k+1})-b(s_k,t_k))}\right)^{\frac{1}{2}}\exp\left(-\frac{(u_{k+1,k+1}-a(s_{k+1},t_{k+1})-u_{k,k}+a(s_k,t_k))^2}{2(b(s_{k+1},t_{k+1})-b(s_k,t_k))}\right)$$

Thus (4.4) becomes

$$\int_{C_{a,b}(Q)} f(x(s_1,t_1),\ldots,x(s_n,t_n))\mathfrak{m}(dx)$$
$$=\left(\prod_{i=1}^{k}\prod_{j=1}^{k}2\pi\Delta_i\Delta_j b(s,t)\right)^{-\frac{1}{2}}\left(\frac{1}{2\pi(b(s_{k+1},t_{k+1})-b(s_k,t_k))}\right)^{\frac{1}{2}}\int_{\mathbb{R}^{k^2}}$$
$$\left\{\int_{\mathbb{R}} f(u_{1,1},\ldots,u_{k+1,k+1})\exp\left(-\frac{(u_{k+1,k+1}-a(s_{k+1},t_{k+1})-u_{k,k}+a(s_k,t_k))^2}{2(b(s_{k+1},t_{k+1})-b(s_k,t_k))}\right)\mathrm{d}u_{k+1,k+1}\right\}$$
$$\exp\left(-\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k}\frac{(\Delta_i\Delta_j(u-a(s,t)))^2}{\Delta_i\Delta_j b(s,t)}\right)\prod_{\substack{i=1\\j=1}}^{k}\mathrm{d}u_{i,j}. \quad (4.6)$$

Define a function $g:\mathbb{R}^{k^2}\to\mathbb{C}$ so that $g(u_{1,1},\ldots,u_{k,k})$ is equal to

$$\int_{\mathbb{R}} f(u_{1,1},\ldots,u_{k+1,k+1})\exp\left(-\frac{(u_{k+1,k+1}-a(s_{k+1},t_{k+1})-u_{k,k}+a(s_k,t_k))^2}{2(b(s_{k+1},t_{k+1})-b(s_k,t_k))}\right)\mathrm{d}u_{k+1,k+1} \quad (4.7)$$

and define a tame functional $G(x) : C_{a,b}(Q) \to \mathbb{R}$ by

$$G(x) = g(x(s_1,t_1),\ldots,x(s_k,t_k)). \tag{4.8}$$

Combine (4.6) and (4.8) to obtain

$$\begin{aligned}
&\int_{C_{a,b}(Q)} f(x(s_1,t_1),\ldots,x(s_n,t_n))\mathfrak{m}(dx)\\
&\quad= \left(\prod_{i=1}^{k}\prod_{j=1}^{k} 2\pi\Delta_i\Delta_j b(s,t)\right)^{-\frac{1}{2}} \int_{\mathbb{R}^{k^2}} g(u_{1,1},\ldots,u_{k,k})\\
&\qquad \exp\left(-\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k}\frac{(\Delta_i\Delta_j(u-a(s,t)))^2}{\Delta_i\Delta_j b(s,t)}\right)\prod_{\substack{i=1\\j=1}}^{k} \mathrm{d}u_{i,j}\\
&\quad= \int_{C_{a,b}(Q)} G(x)\mathfrak{m}(dx).
\end{aligned} \tag{4.9}$$

We now apply the induction hypothesis to the functional G. Put $u_{k+1,k+1} = u_{k+1}$ and $u_{k,k} = u_k$ in equation (4.7) and then using (4.9) we can obtain

$$\begin{aligned}
&\int_{C_{a,b}(Q)} f(x(s_1,t_1),\ldots,x(s_n,t_n))\mathfrak{m}(dx)\\
&\quad= \left(\prod_{j=1}^{k} 2\pi(b(s_j,t_j)-b(s_{j-1},t_{j-1}))\right)^{-\frac{1}{2}} \int_{\mathbb{R}^k} g(u_1,\ldots,u_k)\\
&\qquad \exp\left(-\frac{1}{2}\sum_{j=1}^{k}\frac{(u_j-a(s_j,t_j)-u_{j-1}+a(s_{j-1},t_{j-1}))^2}{b(s_j,t_j)-b(s_{j-1},t_{j-1})}\right)\mathrm{d}u_k\cdots\mathrm{d}u_1 \qquad (4.10)\\
&= \left(\prod_{j=1}^{k+1} 2\pi(b(s_j,t_j)-b(s_{j-1},t_{j-1}))\right)^{-\frac{1}{2}} \int_{\mathbb{R}^{k+1}} f(u_1,\ldots,u_{k+1})\\
&\qquad \exp\left(-\frac{1}{2}\sum_{j=1}^{k+1}\frac{(u_j-a(s_j,t_j)-u_{j-1}+a(s_{j-1},t_{j-1}))^2}{b(s_j,t_j)-b(s_{j-1},t_{j-1})}\right)\mathrm{d}u_{k+1}\cdots\mathrm{d}u_1,
\end{aligned}$$

and so for $n = k+1$, equation (4.3) holds by induction. $\square$

As usual, the order of assumptions in the previous theorem, where the Lebesgue measurability of f is assumed and the μ-measurability of F is a conclusion, is not actually necessary. Per Section 3.5, the hypothesis of measurability can be either that F is μ-measurable on $C_{a,b}(Q)$ or that f is Lebesgue measurable, and the measurability of one of these will imply the measurability of the other.

4.2 One-line Theorems

Now we are prepared to investigate formulas for the integration of functionals depending only on the values of x on certain well-behaved paths in Q. The following theorem allows reduction of certain integrals over $C_{a,b}(Q)$ to integrals over an appropriately chosen space $C_{\tilde{a},\tilde{b}}[0,S]$.

Theorem 35. *Let $\phi : [0,S] \to Q$ be an increasing path. Let $a_\phi(\tau) = a(\phi(\tau)) - a(\phi(0))$ and $b_\phi(\tau) = b(\phi(\tau)) - b(\phi(0))$, and let $\mathfrak{m}_\phi$ be the Gaussian measure on $C_0[0,S]$ subordinate to a_ϕ and b_ϕ. If $F(x) = f(x(\phi(\cdot)))$ is a measurable functional on $C_{a,b}(Q)$, then*

$$\int_{C_{a,b}(Q)} F(x)\mathfrak{m}(dx) \stackrel{*}{=} \int_{C_{a_\phi,b_\phi}[0,S]} f(w)\mathfrak{m}_\phi(dw), \tag{4.11}$$

where the equality ($\stackrel{*}{=}$) is in the sense that if one of the integrals exists then the other exists with equality.

Proof. Let $0 = \tau_0 < \tau_1 < \cdots < \tau_j < \cdots < \tau_n \leq S$ and let $I = \{x \in C_{a,b}(Q) : \alpha_j < x(\phi(\tau_j)) < \beta_j\}$and $J = \{w \in C_{a_\phi,b_\phi}[0,S] : \alpha_j < w(\tau_j) < \beta_j\}$. Note that by the conditions on γ we have

$\phi_1(0) \leq \phi_1(\tau_1) \leq \cdots \leq \phi_1(\tau_n)$ and $\phi_2(0) \leq \phi_2(\tau_1) \leq \cdots \leq \phi_2(\tau_n)$. Then by Theorem 34,

$$\begin{aligned}
\mathfrak{m}(I) &= \int_{C_{a,b}(Q)} \chi_I(x)(dx) \\
&= \left(\prod_{j=1}^{n} 2\pi(b(\phi(\tau_j)) - b(\phi(\tau_{j-1})))\right)^{-\frac{1}{2}} \\
&\qquad \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\sum_{j=1}^{n} \frac{(u_j - a(\phi(\tau_j)) - u_{j-1} + a(\phi(\tau_{j-1})))^2}{b(\phi(\tau_j)) - b(\phi(\tau_{j-1}))}\right) \mathrm{d}u_n \cdots \mathrm{d}u_1 \\
&= \left(\prod_{j=1}^{n} 2\pi(b_\phi(\tau_j) - b_\phi(\tau_{j-1}))\right)^{-\frac{1}{2}} \\
&\qquad \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\sum_{j=1}^{n} \frac{(u_j - a_\phi(\tau_j) - u_{j-1} + a_\phi(\tau_{j-1}))^2}{b_\phi(\tau_j) - b_\phi(\tau_{j-1})}\right) \mathrm{d}u_n \cdots \mathrm{d}u_1 \\
&= \int_{C_{a_\phi,b_\phi}[0,S]} \chi_J(y)\mathfrak{m}_\phi(dy) \\
&= \mathfrak{m}_\phi(J).
\end{aligned}$$

Thus the theorem is true for characteristic functions of sets of the form $\{x \in C_{a,b}(Q) : \alpha_j < x(\phi(\tau_j)) < \beta_j\}$. The general theorem follows by taking the function f to successively be the characteristic function of a Borel set, and then a simple function. Then by monotone convergence the theorem holds for positive functions, and hence for general functions by taking positive and negative and real and imaginary parts. $\square$

As a corollary to Theorem 35 we have the following one-line theorem of Cameron and Storvick from [8].

Corollary 15. *Let $0 < \beta \leq T$ and let $f(\cdot)$ be a real or complex valued functional defined on $C_0[0,S]$ such that $f(\sqrt{\beta}w)$ is a Wiener measurable functional on $C_0[0,S]$. Then $f(x(\cdot,\beta))$*

is a Yeh-Wiener measurable functional of x on $C_0(Q)$ and

$$\int_{C_0(Q)} f(x(\cdot,\beta))(dx) \overset{*}{=} \int_{C_0[0,S]} f(\sqrt{\beta}\ w)\mathfrak{w}(dw), \tag{4.12}$$

where the equality ($\overset{*}{=}$) is in the sense that if one of the integrals exists then the other exists with equality.

Proof. We take $\phi : [0, S] \to Q$ to be $\gamma(\tau) = (\tau, \beta)$ and note that $a(s, t) = 0$ and $b(s, t) = st$. Apply Theorem 35 to any tame functional $F(x) = f(x(s_1, \beta), \ldots, x(s_n, \beta))$ to obtain

$$\begin{aligned}
&\int_{C_0(Q)} F(x)\mathfrak{m}(dx) \\
&= \left(\prod_{j=1}^{n} 2\pi(\beta s_j - \beta s_{j-1})\right)^{-\frac{1}{2}} \int_{\mathbb{R}^n} f(u_1, \ldots, u_n) \exp\left(-\frac{1}{2}\sum_{j=1}^{n} \frac{(u_j - u_{j-1})^2}{\beta s_j - \beta s_{j-1}}\right) \mathrm{d}\mathbf{u} \\
&= \left(\prod_{j=1}^{n} 2\pi(s_j - s_{j-1})\right)^{-\frac{1}{2}} \int_{\mathbb{R}^n} f(\sqrt{\beta}\ w_1, \ldots, \sqrt{\beta}\ w_n) \exp\left(-\frac{1}{2}\sum_{j=1}^{n} \frac{(w_j - w_{j-1})^2}{s_j - s_{j-1}}\right) \mathrm{d}\mathbf{w} \\
&= \int_{C_0[0,S]} f(\sqrt{\beta}\ y(s_1), \ldots, \sqrt{\beta}\ y(s_n))\mathfrak{w}(dw).
\end{aligned}$$

The theorem holds in the general case by the same argument used to finish the proof of Theorem 35. □

4.3 n-line Theorem

Using Theorem 35, we can extend the n-line theorem of Cameron and Storvick from [8].

Theorem 36. *Let $0 < \beta_1 < \cdots < \beta_n \le T$ and let $F(x) = f(x(\cdot, \beta_1), \ldots, x(\cdot, \beta_n))$ be μ-measurable. Put $a_1(s) = a(s, \beta_1)$ and $a_k(s) = a(s, \beta_k) - a(s, \beta_{k-1})$ and put $b_1(s) = b(s, \beta_1)$ and $b_k(s) = b(s, \beta_k) - b(s, \beta_{k-1})$ for $k = 2, \ldots, n$. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ be Gaussian measures on*

$C_0[0,S]$, each subordinate to the corresponding a_k and b_k. Then

$$\begin{aligned}
&\int_{C_{a,b}(Q)} F(x)\mathfrak{m}(dx) \qquad (4.13)\\
&\quad \stackrel{*}{=} \int_{C_{a_n,b_n}[0,S]} \cdots \int_{C_{a_1,b_1}[0,S]} f(y_1, y_1+y_2, \cdots, y_1+y_2+\cdots+y_n)\mathfrak{m}_1(dy_1)\cdots\mathfrak{m}_n(dy_n),
\end{aligned}$$

where the equality ($\stackrel{*}{=}$) is in the sense that if one of the integrals exists then the other exists with equality.

Proof. Let $0 = s_0 < s_1 < \ldots < s_m \leq S$ and $t_k = \beta_k$ for $k = 1, \ldots, n$ and let $p_{j,k} < q_{j,k}$ for all $j = 1, \ldots m$ and $k = 1, \ldots, n$. Define

$$I_j = \{x \in C_{a,b}(Q) : p_{j,k} < x(s_j, \beta_k) \leq q_{j,k} \text{ for } k = 1, \ldots, n\}, \tag{4.14}$$

$$E_j = \{(u_{j,1}, \ldots, u_{j,n}) \in \mathbb{R}^n : p_{j,k} < u_{j,k} \leq q_{j,k} \text{ for } k = 1, \ldots, n\}, \tag{4.15}$$

$$J_j = \{(y_1, \ldots, y_n) \in \times_{k=1}^n C_{a_k,b_k}[0,S] : p_{j,k} < \sum_{l=1}^{k} y_l(s_j) \leq q_{j,k} \text{ for } k = 1, \ldots, n\}. \tag{4.16}$$

Observe that the measurability of E_j in $\mathbb{R}^n$ assures the measurability of I_j and J_j in their respective spaces. Moreover, we note that for a cylinder set $I(p_{1,1}, \ldots, p_{m,n}, q_{1,1}, \ldots, q_{m,n}) \subseteq C_{a,b}(Q)$ determined solely by the values of $x(\cdot,\cdot)$ at the points (s_j, β_k) for $j = 1, \ldots, m$ and $k = 1, \ldots, n$, we have

$$\begin{aligned}
&I(p_{1,1}, \ldots, p_{m,n}, q_{1,1}, \ldots, q_{m,n})\\
&\quad = \{x \in C_{a,b}(Q) : p_{j,k} < x(s_j, \beta_k) \leq q_{j,k} \text{ for } j = 1, \ldots, m; k = 1, \ldots, n\}\\
&\quad = \bigcap_{j=1}^{m} I_j.
\end{aligned}$$

We first consider the case where

$$F(x) = \chi_I(x) = \prod_{j=1}^{m} \chi_{I_j}(x) = \prod_{j=1}^{m} \chi_{E_j}(x(\cdot,\beta_1),\ldots,x(\cdot,\beta_n)).$$

By Theorem 33,

$$\begin{aligned}\int_{C_{a,b}(Q)} F(x)\mathfrak{m}(dx) &= \int_{C_{a,b}(Q)} \prod_{j=1}^{m} \chi_{E_j}(x(s_j,\beta_1),\ldots,x(s_j,\beta_n))\mathfrak{m}(dx)\\ &= \left(\prod_{k=1}^{n}\prod_{j=1}^{m} 2\pi\Delta_k\Delta_j b(s,\beta)\right)^{-\frac{1}{2}} \int_{\mathbb{R}^{mn}} \prod_{j=1}^{m} \chi_{E_j}(u_{j,1},\ldots,u_{j,n}) \qquad (4.17)\\ &\qquad \prod_{k=1}^{n} \exp\left(-\frac{1}{2}\sum_{j=1}^{m} \frac{(\Delta_k\Delta_j(u-a(s,t)))^2}{\Delta_k\Delta_j b(s,t)}\right) \mathrm{d}u_{1,1}\cdots \mathrm{d}u_{m,n}.\end{aligned}$$

Note that

$$\begin{aligned}&\Delta_k\Delta_j(u-a(s,t))\\ &= u_{j,k} - u_{j,k-1} - a(s_j,\beta_k) + a(s_j,\beta_{k-1}) - u_{j-1,k} + u_{j-1,k-1} + a(s_{j-1},\beta_k) - a(s_{j-1},\beta_{k-1})\\ &= [u_{j,k} - u_{j,k-1}] - [a(s_j,\beta_k) - a(s_j,\beta_{k-1})] - [u_{j-1,k} - u_{j-1,k-1}] \qquad (4.18)\\ &\qquad + [a(s_{j-1},\beta_k) - a(s_{j-1},\beta_{k-1})],\end{aligned}$$

and also that

$$\begin{aligned}\Delta_k\Delta_j b(s,t) &= b(s_j,\beta_k) - b(s_{j-1},\beta_k) - b(s_j,\beta_{k-1}) + b(s_{j-1},\beta_{k-1})\\ &= [b(s_j,\beta_k) - b(s_j,\beta_{k-1})] - [b(s_j,\beta_k) - b(s_{j-1},\beta_{k-1})]. \qquad (4.19)\end{aligned}$$

Take $b_1(\cdot) = b(\cdot,\beta_1)$, $b_k(\cdot) = b(\cdot,\beta_k) - b(\cdot,\beta_{k-1})$, $a_1(\cdot) = a(\cdot,\beta_1)$, and $a(\cdot) = b(\cdot,\beta_k) -$

$b(\cdot, \beta_{k-1})$ for $k = 2, \ldots, n$ as in the statement of the theorem. Put

$$v_{j,k} = u_{j,k} - u_{j,k-1}, \tag{4.20}$$

and note that $\mathrm{d}v_{j,k} = \mathrm{d}u_{j,k}$ under this change of variables, and that

$$u_{j,k} = v_{j,k} + u_{j,k-1} = v_{j,k} + v_{j,k-1} + \cdots + v_{j,1} \tag{4.21}$$

for $1 \leq k \leq n$. Substitute (4.18), (4.19), (4.20), and (4.21) in (4.17) to obtain

$$\begin{aligned}
&\prod_{k=1}^{n} \left(\prod_{j=1}^{m} 2\pi \Delta_j b_k(s) \right)^{-\frac{1}{2}} \int_{\mathbb{R}^{mn}} \prod_{j=1}^{m} \chi_{E_j}(v_{j,1}, v_{j,1} + v_{j,2}, \ldots, v_{j,1} + \cdots + v_{j,n}) \\
&\qquad \prod_{k=1}^{n} \exp\left(-\frac{1}{2} \sum_{j=1}^{m} \frac{(\Delta_j(v_j - a_j(s)) - v_{j-1} + a_{j-1}(s))^2}{\Delta_j b_k(s)} \right) \mathrm{d}v_{1,1} \cdots \mathrm{d}v_{m,n} \\
&= \int_{C_{a_1,b_1}[0,S]} \cdots \int_{C_{a_n,b_n}[0,S]} \prod_{j=1}^{m} \chi_{E_j}(y_1(s_j), \ldots, y_1(s_j) + \cdots + y_n(s_j)) \mathfrak{m}_n(dy_n) \cdots \mathfrak{m}_1(dy_1) \\
&\quad = \int_{C_{a_1,b_1}[0,S]} \cdots \int_{C_{a_n,b_n}[0,S]} \prod_{j=1}^{m} \chi_{E_j}(y_1(\cdot), \ldots, y_1(\cdot) + \cdots + y_n(\cdot)) \mathfrak{m}_n(dy_n) \cdots \mathfrak{m}_1(dy_1).
\end{aligned}$$

Thus the theorem is true for characteristic functions of cylinder sets dependent only on the value of $x(\cdot, \cdot)$ at the points $\{(s_j, \beta_k)$ for $j = 1, \ldots, m;\ k = 1, \ldots, n\}$. In the usual manner we can prove the theorem for characteristic functions of measurable sets depending only on the values of $x(\cdot, \beta_k)$ for $k = 1, \ldots n$. The proof is then completed in the same fashion as the proof of Theorem 35. $\square$

4.4 Applications and Examples

For our first example, we demonstrate the use of Theorem 35. Let $Q = [0, S]^2$, $a(s,t) = st$, $b(s,t) = s^2t^2$, and $F(x) = \exp\left(\int_0^S x(s,s)\mathrm{d}s\right)$. Note that $\phi : [0, S] \to Q$ defined by $\phi(s) = (s, s)$ is increasing. Then

$$\int_{C_{a,b}(Q)} F(x)\mathfrak{m}(dx) = \int_{C_{a_1,b_1}[0,S]} \exp\left(\int_0^S y(s)\mathrm{d}s\right) \mathfrak{m}_\phi(dy), \tag{4.22}$$

where $a_1(s) = a(\phi(s)) - a(\phi(0)) = a(s,s) - a(0,0) = s^2$ and $b_1(s) = b(\phi(s)) - b(\phi(0)) = s^4$. Integrating by parts we obtain that

$$\int_0^S y(s)\mathrm{d}s = Sy(S) - \int_0^S s\mathrm{d}y(s) = \langle S, y\rangle - \langle s, y\rangle = \langle S - s, y\rangle,$$

for $\mathfrak{m}$ a.e. $y \in C_{a_\phi,b_\phi}[0, S]$, where in this case we will let $\langle f, y\rangle$ denote the Paley-Wiener-Zygmund integral of the function $f \in L^2_{a_\phi,b_\phi}[0, S]$. It is easy to compute the values $A = \int_0^S (S - s)\mathrm{d}a_\phi(s) = \frac{1}{3}S^3$ and $B = \int_0^S (S - s)^2\mathrm{d}b_\phi(s) = \frac{1}{15}S^6$. We now make use of Theorem 32 to compute the right-hand side of (4.22); thus

$$\begin{aligned}
&\int_{C_{a_1,b_1}[0,S]} \exp\left(\int_0^S y(s)\mathrm{d}s\right) \mathfrak{m}_\phi(dy) \\
&\quad= \frac{1}{\sqrt{2\pi B}} \int_{-\infty}^{\infty} \exp(u) \exp\left(-\frac{(u-A)^2}{2B}\right) \mathrm{d}u \\
&\quad= \frac{1}{\sqrt{2\pi B}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2B}[u^2 - 2Au - 2Bu + A^2]\right) \mathrm{d}u \\
&\quad= \exp\left(-\frac{A^2}{2B}\right) \exp\left(\frac{(A+B)^2}{2B}\right) \int_{-\infty}^{\infty} \exp\left(-\frac{[u-(A+B)]^2}{2B}\right) \mathrm{d}u \\
&\quad= \exp\left(\frac{2AB + B^2}{2B}\right) \\
&\quad= \exp\left(\frac{1}{3}S^3 + \frac{1}{30}S^6\right).
\end{aligned}$$

Next follows an example of the use of Theorem 36. Let $a(s,t) = b(s,t) = st$ on $[0,S] \times [0,2T]$ and put $F(x) = \int_0^S x(s,T)x(s,2T)\mathrm{d}s$. We compute the value of $\int_{C_{a,b}(Q)} F(x)\mu(dx)$ using the theorem. We have $a_1(s) = b_1(s) = sT$ and $a_2(s) = b_2(s) = 2sT - sT = sT$, and thus

$$\begin{aligned}
\int_{C_{a,b}(Q)} F(x)\mathfrak{m}(dx) &= \int_{C_{a_2,b_2}[0,S]} \int_{C_{a_1,b_1}[0,S]} \int_0^S y_1(s)(y_1(s) + y_2(s))\mathrm{d}s\mathfrak{m}_1(dy_1)\mathfrak{m}_2(dy_2) \\
&= \int_0^S \int_{C_{a_2,b_2}[0,S]} \int_{C_{a_1,b_1}[0,S]} \left(y_1^2(s) + y_1(s)y_2(s)\right) \mathfrak{m}_1(dy_1)\mathfrak{m}_2(dy_2)\mathrm{d}s \\
&= \int_0^S \int_{C_{a_2,b_2}[0,S]} \left(sT + s^2T^2 + sTy_2(s)\right) \mathfrak{m}_2(dy_2)\mathrm{d}s \\
&= \int_0^S \left(sT + s^2T^2 + s^2T^2\right) \mathrm{d}s \\
&= \frac{1}{2}S^2T + \frac{2}{3}S^3T^2,
\end{aligned}$$

where Fubini's theorem can be used to justify the change in order of integration. In this example, we can easily complete a similar computation without using Theorem 36 and verify our result, for

$$\begin{aligned}
\int_{C_{a,b}(Q)} F(x)\mathfrak{m}(dx) &= \int_{C_{a,b}(Q)} \int_0^S x(s,T)x(s,2T)\mathrm{d}s\mathfrak{m}(dx) \\
&= \int_0^S \int_{C_{a,b}(Q)} x(s,T)x(s,2T)\mathfrak{m}(dx)\mathrm{d}s \\
&= \int_0^S \left(sT + 2s^2T^2\right) \mathrm{d}s \\
&= \frac{1}{2}S^2T + \frac{2}{3}S^3T^2.
\end{aligned}$$

Chapter 5

Reflection Principles

5.1 Introduction

Let $C_0[0,T]$ denote the single parameter Wiener space; this is the space of $\mathbb{R}$-valued continuous functions on $[0,T]$ with $x(0) = 0$. Let $\mathcal{M}$ denote the class of Wiener measurable subsets of $C_0[0,T]$ and let $\mathfrak{w}$ denote Wiener measure. Then $(C_0[0,T], \mathcal{M}, \mathfrak{w})$ is a complete measure space and we denote the Wiener integral of a Wiener-integrable functional F by $\int_{C_0[0,T]} F(x)\mathfrak{w}(dx)$. Note that the point evaluation functional $\langle \delta_t, x\rangle = x(t)$ defines a stochastic process with parameter t having mean

$$\mathbb{E}[x(t)] = \int_{C_0[0,T]} x(t)\mathfrak{w}(dx) = 0$$

and covariance

$$\mathbb{E}[x(s)x(t)] = \int_{C_0[0,T]} x(s)x(t)\mathfrak{w}(dx) = \min(s,t);$$

observe that this is the standard Brownian motion process.

It is well-known that the Wiener space $C_0[0,T]$ exhibits a reflection principle about its mean; that is, for all $c \geq 0$,

$$\mathfrak{w}\left\{x \in C_0[0,T] : \sup_{[0,T]} x(t) \geq c\right\} = 2\mathfrak{w}\left\{x \in C_0[0,T] : x(T) \geq c\right\}. \tag{5.1}$$

Proofs and discussions of this result can be found in [25, 35, 66] and elsewhere; a particularly good explanation is given in [4].

In Section 2, we show that the generalized function space $C_{a,b}[0,T]$ also exhibits a reflection principle about its mean function $a(t)$; that is, for $c \geq 0$,

$$\mathfrak{m}\left\{x \in C_{a,b}[0,T] : \sup_{[0,T]}[x(t) - a(t)] \geq c\right\} = 2\mathfrak{m}\left\{x \in C_{a,b}[0,T] : [x(T) - a(T)] \geq c\right\}. \tag{5.2}$$

For $Q = [0,S] \times [0,T]$, let $C_2(Q)$ denote the two parameter Wiener space (see [61]); this is the space of all $\mathbb{R}$-valued continuous functions on Q satisfying $x(s,0) = x(0,t) = 0$ for all $(s,t) \in Q$. In Sections 3 and 4, we consider several ways in which one might formulate the notion of a reflection principle and then discuss whether any of these formulations actually holds on this space.

5.2 A reflection principle for the general function space $C_{a,b}[0,T]$

We turn our attention to the generalized Wiener space. We follow the same formulation as [11]. Let a and b be functions defined on $[0,T]$ with $a' \in L^2[0,T]$ and b' continuous, positive, and bounded away from 0 on $[0,T]$. Observe that a and b are absolutely continuous and b is strictly increasing on $[0,T]$, and so one can define a generalized Brownian motion as in Chapter 3 of [66]. We take $\mathfrak{m}$ to be the Gaussian measure on $C_0[0,T]$ with finite-dimensional distributions having density

$$\left(\prod_{j=1}^{n} 2\pi[b(t_j) - b(t_{j-1})]\right)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\sum_{j=1}^{n} \frac{([u_j - a(t_j)] - [u_{j-1} - a(t_{j-1})])^2}{b(t_j) - b(t_{j-1})}\right).$$

Observe that with respect to $\mathfrak{m}$ the coordinate evaluation map $\langle \delta_t, x\rangle = x(t)$ is the generalized Brownian motion process determined by a and b, having mean

$$\mathbb{E}[x(t)] = \int_{C_{a,b}[0,T]} x(t)\mathfrak{m}(dx) = a(t)$$

and covariance function

$$r(s,t) = \int_{C_{a,b}[0,T]} [x(s) - a(s)][x(t) - a(t)]\mathfrak{m}(dx) = \min\{b(s), b(t)\}.$$

For more information about these function spaces, consult [11, 12, 13].

We will also make use of the following lemma from Chapter 3 of [66].

Lemma 21. *Let $\{X_j : j = 1, \ldots, n\}$ be an independent set of of symmetrically distributed random variables on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$, let $S_0 = 0$, and let $S_j = X_1 + \cdots + X_j$ for $j = 1, \ldots, n$. Then for every $\varepsilon > 0$,*

$$2\mathbb{P}[S_n \geq c] \geq \mathbb{P}\left[\max_{1\leq j\leq n} S_j \geq c\right] \geq 2\mathbb{P}\left[S_n \geq c + 2\varepsilon\right] - 2\sum_{j=1}^{n} \mathbb{P}[X_j \geq \varepsilon]. \tag{5.3}$$

We are now ready to establish equation (5.2) above, demonstrating that the generalized Wiener space $C_{a,b}[0,T]$ satisfies a reflection principle about its mean function $a(t)$. Our proof uses ideas from Chapter 3 of [66] as well as from unpublished lecture notes of R.H. Cameron.

Theorem 37. *For all $c \geq 0$,*

$$\mathfrak{m}\left\{x : \sup_{[0,T]}[x(t) - a(t)] \geq c\right\} = 2\mathfrak{m}\left\{x : [x(T) - a(T)] \geq c\right\}. \tag{5.4}$$

Proof. Let $D \subseteq [0,T]$ be countable and dense, containing 0 and T, and let $P_n = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ denote a nested sequence of partitions of $[0,T]$ with each $t_j \in D$ and $||P_n|| \to 0$ as $n \to \infty$. Note that the process $X_t = x(t)$ is continuous and separable, and thus for all $c > 0$ and $\varepsilon > 0$,

$$\begin{aligned}
\mathfrak{m}\left\{x : \sup_{[0,T]}[x(t) - a(t)] \geq c\right\} &= \mathfrak{m}\left\{x : \sup_{D}[x(t) - a(t)] \geq c\right\} \\
&\leq \mathfrak{m}\left(\bigcup_{n=1}^{\infty}\left\{x : \max_{1\leq k\leq n+1}[x(t_k) - a(t_k)] \geq c - \varepsilon\right\}\right) \\
&= \lim_{n\to\infty}\mathfrak{m}\left\{x : \max_{1\leq k\leq n+1}[x(t_k) - a(t_k)] \geq c - \varepsilon\right\} \\
&\leq 2\mathfrak{m}\left\{x : [x(T) - a(T)] \geq c - \varepsilon\right\},
\end{aligned}$$

where the last inequality is due to Lemma 21. Taking the limit as $\varepsilon \to 0$ yields

$$\mathfrak{m}\left\{x : \sup_{[0,T]}[x(t) - a(t)] \geq c\right\} \leq 2\mathfrak{m}\left\{x : x(T) - a(T) \geq c\right\}.$$

For the other inequality, we specify partitions $P_n = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ with $t_k = \frac{kT}{n}$. Then for any $c > 0$ and $\varepsilon > 0$ we again use Lemma 21 to obtain

$$\begin{aligned}
\mathfrak{m}\left\{x : \sup_{[0,T]}[x(t) - a(t)] \geq c\right\} &\geq \mathfrak{m}\left\{x : \max_{1\leq k\leq n}[x(t_k) - a(t_k)] \geq c\right\} \\
&\geq 2\mathfrak{m}\left\{x : [x(T) - a(T)] \geq c + 2\varepsilon\right\} \qquad (5.5) \\
&\quad - 2\sum_{k=1}^{n}\mathfrak{m}\left\{x : [x(t_k) - a(t_k) - x(t_{k-1}) + a(t_{k-1})] \geq \varepsilon\right\}.
\end{aligned}$$

We estimate

$$\frac{1}{\sqrt{2\pi s}}\int_{\varepsilon}^{\infty}\exp\left(-\frac{u^2}{2s}\right)\mathrm{d}u \leq \frac{1}{\sqrt{2\pi s}}\int_{\varepsilon}^{\infty}\exp\left(-\frac{\varepsilon u}{2s}\right)\mathrm{d}u \tag{5.6}$$
$$= \frac{\sqrt{2s}}{\varepsilon\sqrt{\pi}}\exp\left(-\frac{\varepsilon^2}{2s}\right),$$

and (noting that b' is positive and bounded) that

$$\begin{aligned}\max_{1\leq k\leq n}[b(t_k)-b(t_{k-1})] &= \max_{1\leq k\leq n}\int_{t_{k-1}}^{t_k} b'(s)\mathrm{d}s\\ &\leq \max_{1\leq k\leq n}\left([t_k-t_{k-1}]\int_{t_{k-1}}^{t_k}(b'(s))^2\,\mathrm{d}s\right)^{\frac{1}{2}} \qquad (5.7)\\ &\leq \frac{\|b'\|_2\sqrt{T}}{\sqrt{n}}.\end{aligned}$$

Recall that $x(t_k)-a(t_k)-x(t_{k-1})+a(t_{k-1})$ is distributed normally with mean 0 and variance $b(t_k)-b(t_{k-1})$, and then using our estimates in (5.6) and (5.7), we find that

$$\begin{aligned}&\lim_{n\to\infty}\sum_{k=1}^{n}\mathfrak{m}\left\{x:[x(t_k)-a(t_k)-x(t_{k-1})+a(t_{k-1})]\geq\varepsilon\right\}\\ &\quad=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{2\pi[b(t_k)-b(t_{k-1})]}}\int_{\varepsilon}^{\infty}\exp\left(-\frac{u^2}{b(t_k)-b(t_{k-1})}\right)\mathrm{d}u\\ &\quad\leq\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sqrt{2[b(t_k)-b(t_{k-1})]}}{\varepsilon\sqrt{\pi}}\exp\left(-\frac{\varepsilon^2}{2[b(t_k)-b(t_{k-1})]}\right)\\ &\quad\leq\lim_{n\to\infty}n\left[\frac{\sqrt{2\,\|b'\|_2\sqrt{T}}}{\varepsilon\sqrt{\pi n}}\exp\left(-\frac{\varepsilon^2}{2\,\|b'\|_2\sqrt{T}}\right)\right]\\ &\quad\leq\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sqrt{2n\,\|b'\|_2\sqrt{T}}}{\varepsilon\sqrt{\pi}}\exp\left(-\frac{\sqrt{n}\varepsilon^2}{2\,\|b'\|_2\sqrt{T}}\right).\end{aligned}$$

Now use this estimate and let $n\to\infty$ and then $\varepsilon\to 0$ in (5.5). $\square$

The previous theorem has several useful corollaries.

Corollary 16. *Let E be a Lebesgue measurable subset of $\mathbb{R}$. Then*

$$\mathfrak{m}\left\{x : \sup_{[0,T]}[x(t)-a(t)] \in E\right\} = 2\mathfrak{m}\left\{x : [x(T)-a(T)] \in E \cap [0,\infty)\right\} \tag{5.8}$$

and

$$\int_{C_{a,b}[0,T]} \chi_E\left(\sup_{[0,T]}[x(t)-a(t)]\right)\mathfrak{m}(dx) = \frac{2}{\sqrt{2\pi b(T)}}\int_0^\infty \chi_E(u)\exp\left(-\frac{u^2}{2b(T)}\right)\mathrm{d}u. \tag{5.9}$$

Proof. The proof is a standard exercise in measure theory. Begin with E an open interval and the result follows easily. The case for an open set E follows by decomposing E into a countable union of disjoint intervals. From this, demonstrate that (5.8) holds for G_δ and then null sets. Finally use this to demonstrate the conclusion for Lebesgue measurable sets. Then (5.9) follows immediately from (5.8). □

Corollary 17. *Let $f : \mathbb{R} \to \mathbb{C}$ be Lebesgue measurable with $f = 0$ on $(-\infty, 0)$ and put $F(x) = f\left(\sup_{[0,T]}[x(t)-a(t)]\right)$. Then F is $\mathfrak{m}$-measurable and*

$$\begin{aligned}\int_{C_{a,b}[0,T]} F(x)\mathfrak{m}(dx) &= \int_{C_{a,b}[0,T]} f\left(\sup_{[0,T]}[x(t)-a(t)]\right)\mathfrak{m}(dx)\\ &= 2\int_{C_{a,b}[0,T]} f\left(x(T)-a(T)\right)\mathfrak{m}(dx)\\ &= \frac{2}{\sqrt{2\pi b(T)}}\int_0^\infty f(u)\exp\left(-\frac{u^2}{2b(T)}\right)\mathrm{d}u.\end{aligned} \tag{5.10}$$

Proof. Begin with the case where $f(u) = \chi_E(u)$ for a measurable set E and apply the previous corollary to show the desired conclusion. Then successively consider the cases where f is a measurable simple function, then a nonnegative function; finally take positive and negative and real and imaginary parts of f. □

Choosing the mean function $a(t)$ to be the zero function, we immediately recover a direct extension of the reflection principle for ordinary Wiener space, as expected.

Corollary 18. *Let* $\mathfrak{m}$ *be a generalized Wiener measure on* $C_{a,b}[0,T]$ *with* $a(t) = 0$ *on* $[0,T]$. *Then for all* $c \geq 0$ *and* $t_0 \in (0,T]$,

$$\mathfrak{m}\left\{x : \sup_{[0,t_0]} x(t) \geq c\right\} = 2\mathfrak{m}\left\{x : x(t_0) \geq c\right\}. \tag{5.11}$$

We have some additional useful corollaries, which can be used to yield error estimates when approximating function space integrals using interpolation by tame functionals. For examples of the use of these types of results, see [6, 60]. As before, let $||\cdot||_S$ denote the usual supremum norm.

Corollary 19. *If* f *is Lebesgue measurable and nonnegative on* $[0,\infty)$, *then* $f\left(||x-a||_S\right)$ *is* $\mathfrak{m}$*-measurable and*

$$\int_{C_{a,b}[0,T]} f\left(||x-a||_S\right)\mathfrak{m}(dx) \leq \frac{4}{\sqrt{2\pi b(T)}}\int_0^\infty f(u)\exp\left(-\frac{u^2}{2b(T)}\right)\mathrm{d}u. \tag{5.12}$$

Proof. Partition $C_{a,b}[0,T]$ into

$$A = \left\{x : \sup_{[0,T]}[x(t)-a(t)] \geq \sup_{[0,T]}[a(t)-x(t)]\right\},$$

and

$$B = \left\{x : \sup_{[0,T]}[a(t)-x(t)] > \sup_{[0,T]}[x(t)-a(t)]\right\}.$$

Then we find that

$$\begin{aligned}
\int_{C_{a,b}[0,T]} f\left(||x-a||_S\right)\mathfrak{m}(dx) &= \int_A f\left(||x-a||_S\right)\mathfrak{m}(dx) + \int_B f\left(||x-a||_S\right)\mathfrak{m}(dx)\\
&= \int_A f\left(\sup_{[0,T]}[x(t)-a(t)]\right)\mathfrak{m}(dx)\\
&\qquad + \int_B f\left(\sup_{[0,T]}[a(t)-x(t)]\right)\mathfrak{m}(dx)\\
&\leq \int_{C_{a,b}[0,T]} f\left(\sup_{[0,T]}[x(t)-a(t)]\right)\mathfrak{m}(dx)\\
&\qquad + \int_{C_{a,b}[0,T]} f\left(\sup_{[0,T]}[a(t)-x(t)]\right)\mathfrak{m}(dx)\\
&= \frac{4}{\sqrt{2\pi b(T)}}\int_0^\infty f(u)\exp\left(-\frac{u^2}{2b(T)}\right)\mathrm{d}u,
\end{aligned}$$

where the last equality follows from Corollary 17, the positivity of f, and the symmetry of the centered normal distribution. □

Corollary 20. *Let f be Lebesgue measurable and monotonically increasing on $[0,\infty)$. Then*

$$\frac{2}{\sqrt{2\pi b(T)}}\int_0^\infty f(u)\exp\left(-\frac{u^2}{2b(T)}\right)\mathrm{d}u \leq \int_{C_{a,b}[0,T]} f(||x-a||_S)\mathfrak{m}(dx) \tag{5.13}$$

whenever both sides are defined. Moreover, if f is Lebesgue measurable and monotonically decreasing the reverse inequality holds whenever both sides are defined.

Proof. Note that

$$||x-a||_S = \max\left\{\sup_{[0,T]}[x(t)-a(t)], \sup_{[0,T]}[a(t)-x(t)]\right\},$$

Using this fact, the monotonicity of f, and Corollary 17, we have

$$\frac{2}{\sqrt{2\pi b(T)}}\int_0^\infty f(u)\exp\left(-\frac{u^2}{2b(T)}\right)\mathrm{d}u = \int_{C_{a,b}[0,T]} f\left(\sup_{[0,T]}[x(t)-a(t)]\right)\mathfrak{m}(dx)$$
$$\leq \int_{C_{a,b}[0,T]} f\left(||x-a||_S\right)\mathfrak{m}(dx),$$

as desired. For decreasing f the inequality clearly reverses. □

The final corollary follows immediately from the previous two.

Corollary 21. *If f is Lebesgue measurable, nonnegative, and monotonically increasing on $[0,\infty)$, then there exists some M satisfying $2 \leq M \leq 4$ such that*

$$\int_{C_{a,b}[0,T]} f(||x-a||_S)\mathfrak{m}(dx) = \frac{M}{\sqrt{2\pi b(T)}}\int_0^\infty f(u)\exp\left(-\frac{u^2}{2b(T)}\right)\mathrm{d}u. \tag{5.14}$$

5.3 Reflection principles for two parameter Wiener space

Let $Q = [0,S]\times[0,T]$ and $\partial Q = \{(s,t) \in Q : s = 0, S \text{ or } t = 0, T\}$ be the boundary of Q, and let $C_2(Q)$ denote the space of continuous $\mathbb{R}$-valued functions defined on Q for which $x(0,t) = x(s,0) = 0$. In [61, 62], Yeh constructed a Gaussian measure $\mathfrak{m}_y$ on $C_2(Q)$ with respect to which the point evaluation functional $\delta_{(s,t)}$ defines a stochastic process with parameter $(s,t) \in Q$ having mean

$$\mathbb{E}[x(s,t)] = \int_{C_2(Q)} x(s,t)\mathfrak{m}_y(dx) = 0$$

and covariance

$$\mathbb{E}[x(s,t)x(u,v)] = \int_{C_2(Q)} x(s,t)x(u,v)\mathfrak{m}_y(dx) = \min(s,u)\min(t,v).$$

Recall that for functions of one variable, the classical notion of a function of bounded variation is unambiguously defined and very well understood. However, when considering functions of two (or more) variables, there are many possible definitions for the concept of bounded variation. See [3, 14, 52] for several such definitions and a considerable amount of discussion.

In the same way, for multiple parameter Wiener spaces, one can formulate the idea of a reflection principle in a variety of manners. In this section, we will consider several such formulations and determine whether the space $C_2(Q)$ satisfies each of them.

For ordinary single parameter Wiener space, we note that $x(0) = 0$ for $x \in C_0[0,T]$; therefore considering again the single parameter reflection principle, we see that

$$\begin{aligned}\mathfrak{m}\left\{x : \sup_{[0,T]} x(t) \geq c\right\} &= 2\mathfrak{m}\left\{x : x(T) \geq c\right\} \\ &= 2\mathfrak{m}\left\{x : x(T) \geq c; x(0) \geq c\right\} + 2\mathfrak{m}\left\{x : x(T) \geq c; x(0) < c\right\} \\ &= 2\mathfrak{m}\left\{x : \max_{\{0,T\}} x(t) \geq c\right\}\end{aligned}$$

for $c \geq 0$. From this, we might consider the reflection principle to be a means of expressing either of the following:

1. a relationship between the behavior of the supremum of the process $x(t)$ on the interval to the behavior of the process at the endpoint T of the interval, or

2. a relationship between the behavior of the supremum of the process $x(t)$ on the interval to the behavior of the process on the boundary $\{0,T\}$ of the interval.

Thus we immediately have two candidate formulations for a reflection principle in the two parameter setting; we can ask the following corresponding questions:

1. Is there a constant $k_1 \geq 0$ so that for every $c \geq 0$,

$$\mathfrak{m}_y \left\{ x : \sup_Q x(s,t) \geq c \right\} = k_1 \; \mathfrak{m}_y \left\{ x : x(S,T) \geq c \right\}? \tag{5.15}$$

2. Is there a constant $k_2 \geq 0$ so that for every $c \geq 0$,

$$\mathfrak{m}_y \left\{ x : \sup_Q x(s,t) \geq c \right\} = k_2 \; \mathfrak{m}_y \left\{ x : \sup_{\partial Q} x(s,t) \geq c \right\}? \tag{5.16}$$

In fact, the answer to both questions is negative, as we will demonstrate below.

Now, we wish to compare $\mathfrak{m}_y \left\{ x : \sup_Q x(s,t) \geq c \right\}$ with either $\mathfrak{m}_y \left\{ x : x(S,T) \geq c \right\}$ or $\mathfrak{m}_y \left\{ x : \sup_{\partial Q} x(s,t) \geq c \right\}$; therefore we define

$$\gamma_1(c) = \frac{\mathfrak{m}_y \left\{ x : \sup_Q x(s,t) \geq c \right\}}{\mathfrak{m}_y \left\{ x : x(S,T) \geq c \right\}} \tag{5.17}$$

and

$$\gamma_2(c) = \frac{\mathfrak{m}_y \left\{ x : \sup_Q x(s,t) \geq c \right\}}{\mathfrak{m}_y \left\{ x : \sup_{\partial Q} x(s,t) \geq c \right\}}. \tag{5.18}$$

Observe that both γ_1 and γ_2 are continuous on $[0, \infty)$; moreover it is easy to see that $\gamma_1(0) = 2$ and $\gamma_2(0) = 1$. Also, in [68], Zimmerman shows that $\gamma_1(c) \leq 4$ for all $c \geq 0$.

First, we show that the first question has a negative answer. For each $c \geq 0$, we consider the following sets:

$$A_c = \left\{ x : \sup_{[0,S]} x(s,T) \geq c \right\}, \tag{5.19}$$

$$B_c = \left\{ x : \sup_Q x(s,t) \geq c; \sup_{[0,S]} x(s,T) < c \right\}, \tag{5.20}$$

and

$$D_c = \left\{ x : \sup_Q x(s,t) < c \right\}. \tag{5.21}$$

It is clear that A_c, B_c, and D_c are disjoint and that

$$C_2(Q) = A_c \cup B_c \cup D_c, \tag{5.22}$$

and putting $f(c) = \mathfrak{m}_y(A_c)$, $g(c) = \mathfrak{m}_y(B_c)$, and $h(c) = \mathfrak{m}_y(D_c)$, we observe that

$$1 = f(c) + g(c) + h(c). \tag{5.23}$$

We will make use of the following theorem of Cameron and Storvick from [8].

Theorem 38. *Let F be a functional defined on $C_0[0,S]$ such that $F(\sqrt{T}w)$ is a Wiener measurable functional of w on $C_0[0,S]$. Then $F(x(\cdot,T))$ is a Yeh-Wiener measurable functional of x on $C_2(Q)$ and*

$$\int_{C_2(Q)} F(x(\cdot,T))\mathfrak{m}_y(dx) = \int_{C_0[0,S]} F(\sqrt{T}w)\mathfrak{w}(dw), \tag{5.24}$$

where the existence of either integral implies the existence of the other with equality.

As shown by Skoug in [51], the hypothesis of measurability in the previous theorem can be assumed either for $F(x)$ on $C_2(Q)$ or for $F(\sqrt{T}w)$ on $C_0[0,S]$, and the measurability of one will imply the measurability of the other.

Using this theorem, we demonstrate that, as one would reasonably expect, the space $C_2(Q)$ exhibits a reflection principle when restricted to any horizontal or vertical line in Q.

Proposition 4. *For $c \geq 0$,*

$$\begin{aligned} f(c) &= \mathfrak{m}_y \left\{ x : \sup_{[0,S]} x(s,T) \geq c \right\} \\ &= 2\ \mathfrak{m}_y \left\{ x : x(S,T) \geq c \right\} \\ &= \frac{2}{\sqrt{2\pi ST}} \int_c^\infty \exp\left(-\frac{u^2}{2ST} \right) \mathrm{d}u. \end{aligned} \tag{5.25}$$

Proof. Using (5.1) and (5.24) above, a computation and an easy change of variable show that

$$\begin{aligned} f(c) &= \int_{C_2(Q)} \chi_{[c,\infty)} \left(\sup_{[0,S]} x(s,T) \right) \mathfrak{m}_y(dx) \\ &= \int_{C_0[0,S]} \chi_{[c,\infty)} \left(\sup_{[0,S]} \sqrt{T} w(s) \right) \mathfrak{w}(dw) \\ &= \frac{2}{\sqrt{2\pi S}} \int_0^\infty \chi_{[c,\infty)} \left(\sqrt{T} u \right) \exp\left(-\frac{u^2}{2S} \right) \mathrm{d}u \\ &= \frac{2}{\sqrt{2\pi ST}} \int_0^\infty \chi_{[c,\infty)}(u) \exp\left(-\frac{u^2}{2ST} \right) \mathrm{d}u \\ &= \frac{2}{\sqrt{2\pi ST}} \int_c^\infty \exp\left(-\frac{u^2}{2ST} \right) \mathrm{d}u \\ &= 2\mathfrak{m}_y \left\{ x : x(S,T) \geq c \right\}, \end{aligned}$$

for each $c \geq 0$. □

The next lemma follows readily from (5.21), (5.22), (5.23), and (5.25).

Lemma 22.

1. *The function $f(c)$ is smooth, strictly decreasing, and concave upward on $[0,\infty)$. Furthermore, $f(0) = 1$, $\lim_{c\to\infty} f(c) = 0$, and f has a fixed point in $(0,1)$.*

2. *The function* $h(c) = \mathfrak{m}_y(D_c)$ *is continuous and strictly increasing on* $[0, \infty)$, *with* $h(0) = 0$, $\lim_{c\to\infty} h(c) = 1$, *and* $h(c) = 1 - c$ *for some* c *in* $(0, 1)$.

3. *The function* $g(c) = \mathfrak{m}_y(B_c)$ *is continuous on* $[0, \infty)$, *with* $g(0) = 0$, $\lim_{c\to\infty} g(c) = 0$ *and* $f(c) + g(c) = c$ *for some* c *in* $(0, 1)$.

Now, using (5.17), (5.21), (5.23), and (5.25), it follows that

$$\begin{aligned}
\gamma_1(c) &= \frac{2\mathfrak{m}_y\left\{x : \sup_Q x(s,t) \geq c\right\}}{\mathfrak{m}_y(A_c)} \\
&= \frac{2(1 - h(c))}{f(c)} \\
&= \frac{2(f(c) + g(c))}{f(c)} \\
&= 2 + \frac{2g(c)}{f(c)} \\
&\geq 2
\end{aligned} \tag{5.26}$$

for each $c \geq 0$. Moreover, $2 = \gamma_1(0) \leq \gamma_1(c) \leq 4$ by of Zimmerman's result; this and (5.26) imply that

$$0 = g(0) \leq g(c) \leq f(c) \leq f(0) = 1 \tag{5.27}$$

on $[0, \infty)$. Now, $2 = \gamma_1(0) = 2 + \frac{2g(c)}{f(c)}$, so that if γ_1 is to be equal to a constant k_1 it must be the case that g is identically zero on $[0, \infty)$.

We now show that this cannot be true. For $c > 0$, put

$$\phi(s,t) = \begin{cases} \frac{8c}{ST} & \text{if } 0 \leq s \leq \frac{S}{2}, 0 \leq t \leq \frac{T}{2} \\ -\frac{8c}{ST} & \text{if } \frac{S}{2} < s \leq S, 0 \leq t \leq \frac{T}{2} \\ -\frac{8c}{ST} & \text{if } \frac{T}{2} < t \leq T, 0 \leq s \leq \frac{S}{2} \\ \frac{8c}{ST} & \text{if } \frac{S}{2} < s \leq S, \frac{T}{2} < t \leq T \end{cases}$$

and let

$$x_0(s,t) = \int_0^s \int_0^t \phi(u,v)\mathrm{d}v\mathrm{d}u. \tag{5.28}$$

Note that $x_0\left(\frac{S}{2}, \frac{T}{2}\right) = 2c$ and $x_0 = 0$ on ∂Q. Denote by $\mathrm{B}(x_0; \frac{c}{2})$ the ball of radius $\frac{c}{2}$ around x_0 and observe that this ball is contained in the set B_c, whence

$$\mathfrak{m}_y\left(\mathrm{B}(x_0; \tfrac{c}{2})\right) \leq \mathfrak{m}_y(B_c) = g(c). \tag{5.29}$$

As ϕ is of bounded variation in the sense of Hardy-Krause (see [3] for explanation), we apply the Cameron-Martin theorem for $C_2(Q)$ as found in [62] to see that

$$\begin{aligned}
\mathfrak{m}_y\left(\mathrm{B}(x_0; \tfrac{c}{2})\right) &= \int_{\mathrm{B}(x_0, \frac{c}{2})} \mathrm{m}_y(dx) \\
&= \int_{\mathrm{B}(0, \frac{c}{2})} \exp\left(-\frac{1}{2}\,||\phi||_2^2 + \int_Q \phi(s,t)\mathrm{d}x(s,t)\right) \mathrm{m}_y(dx) \\
&\geq \exp\left(-\frac{32c^2}{ST}\right) \int_{\mathrm{B}(0, \frac{c}{2})} \exp\left(-\frac{36c^2}{ST}\right) \mathrm{m}_y(dx) \\
&\geq \exp\left(-\frac{68c^2}{ST}\right) \mathrm{m}_y\left(\mathrm{B}(0, \tfrac{c}{2})\right) \\
&> 0,
\end{aligned} \tag{5.30}$$

where we have used the fact that the stochastic integral $\int_Q \phi(s,t)\mathrm{d}x(s,t)$ is equal $\mathfrak{m}_y$-a.e. to the ordinary Riemann-Stieltjes integral, whence we can integrate to obtain

$$\int_Q \phi(s,t)dx(s,t) = \frac{8c}{ST}\left[x(S,T) + 4x(\tfrac{S}{2}, \tfrac{T}{2}) - 2x(S, \tfrac{T}{2}) - 2x(\tfrac{S}{2}, T)\right],$$

which we can easily bound from below on $\mathrm{B}(0, \frac{c}{2})$. Thus by (5.29) we see that $g(c) > 0$ whenever $c > 0$, and thus (5.15) cannot hold for any constant k_1.

Now we show that the second question must also have a negative answer. In a similar fashion as above, put

$$A'_c = \left\{ x : \sup_{\partial Q} x(s,t) \geq c \right\}, \tag{5.31}$$

$$B'_c = \left\{ x : \sup_{Q} x(s,t) \geq c; \sup_{\partial Q} x(s,t) < c \right\} \tag{5.32}$$

and

$$D'_c = \left\{ x : \sup_{Q} x(s,t) < c \right\}. \tag{5.33}$$

As above, $C_2(Q)$ is the disjoint union of these sets. We let $F(c) = \mathfrak{m}_y(A'_c)$, $G(c) = \mathfrak{m}_y(B'_c)$, and $H(c) = \mathfrak{m}_y(D'_c)$, so that

$$1 = F(c) + G(c) + H(c). \tag{5.34}$$

From this, we can write

$$\begin{aligned} \gamma_2(c) &= \frac{\mathfrak{m}_y \left\{ x : \sup_Q x(s,t) \geq c \right\}}{\mathfrak{m}_y \left\{ x : \sup_{\partial Q} x(s,t) \geq c \right\}} \\ &= \frac{\mathfrak{m}_y \left\{ x : \sup_{\partial Q} x(s,t) \geq c \right\} + \mathfrak{m}_y \left\{ x : \sup_Q x(s,t) \geq c; \sup_{\partial Q} x(s,t) < c \right\}}{\mathfrak{m}_y \left\{ x : \sup_{\partial Q} x(s,t) \geq c \right\}} \\ &= 1 + \frac{G(c)}{F(c)}. \end{aligned} \tag{5.35}$$

As above, the fact that $\gamma_2(0) = 1$ implies that (5.16) holds for a constant $k_2 = 1$ only if $G(c) = 0$ for all c. Taking the same x_0 as defined in (5.28), we observe that the ball $\mathrm{B}(x_0; \frac{c}{2})$ is contained in B'_c and then using (5.30) we can demonstrate that

$$0 < \mathfrak{m}_y(\mathrm{B}\left(x_0; \tfrac{c}{2}\right)) \leq \mathfrak{m}_y(B'_c) = G(c)$$

for $c > 0$, and so (5.16) cannot hold for any constant k_2.

Easily extending results from [43], in [47] we obtain the explicit formula

$$\begin{aligned} F(c) &= \mathfrak{m}_y \left\{ x : \sup_{\partial Q} x(s,t) \geq c \right\} \\ &= \frac{3}{\sqrt{2\pi ST}} \int_c^\infty \exp\left(-\frac{u^2}{2ST}\right) \mathrm{d}u - \frac{\exp\left(\frac{4c^2}{ST}\right)}{\sqrt{2\pi ST}} \int_{3c}^\infty \exp\left(-\frac{u^2}{2ST}\right) \mathrm{d}u. \end{aligned} \tag{5.36}$$

In the same way as Lemma 22 above, from (5.33), (5.34), and (5.36) we now easily have the following properties of F, G, and H.

Lemma 23.

1. *The function* $F(c) = \mathfrak{m}_y(A'_c)$ *is smooth and strictly decreasing, with* $F(0) = 1$ *and* $\lim_{c\to\infty} F(c) = 0$*, and* F *has a fixed point in* $(0,1)$*.*

2. *The function* $H(c) = \mathfrak{m}_y(D'_c)$ *is continuous and strictly increasing on* $[0,\infty)$*, with* $H(0) = 0$*,* $\lim_{c\to\infty} H(c) = 1$*, and* $H(c) = 1 - c$ *for some* c *in* $(0,1)$*.*

3. *The function* $G(c) = \mathfrak{m}_y(B'_c)$ *is continuous on* $[0,\infty)$ *with* $g(0) = 0$*,* $\lim_{c\to\infty} G(c) = 0$*, and* $F(c) + G(c) = c$ *for some* c *in* $(0,1)$*.*

While we are unable to obtain γ_1 and γ_2 explicitly for all c, we can say a few things about their behavior. We collect these additional observations below.

Lemma 24. $\lim_{c\to\infty} \gamma_1(c) = 4$ *and* γ_1 *has a fixed point in the interval* $(2,4)$*.*

Proof. In [26], for the special case $S = T = 1$, Goodman showed (see [47] for our setting) that

$$\lim_{c\to\infty} \frac{\mathfrak{m}_y \left\{ x : \sup_Q x(s,t) \geq c \right\}}{\frac{4}{\sqrt{2\pi ST}} \int_c^\infty \exp\left(-\frac{u^2}{2ST}\right) \mathrm{d}u} = 1. \tag{5.37}$$

Also, by (5.23) and (5.26) we see that

$$\begin{aligned}\frac{\mathfrak{m}_y\left\{x : \sup_Q x(s,t) \geq c\right\}}{\frac{4}{\sqrt{2\pi ST}}\int_c^\infty \exp\left(-\frac{u^2}{2ST}\right)\mathrm{d}u} &= \frac{1-h(c)}{2f(c)}\\ &= \frac{f(c)+g(c)}{2f(c)}\\ &= \frac{1}{2}+\frac{g(c)}{2f(c)}.\end{aligned} \tag{5.38}$$

From (5.37) and (5.38) we determine that $\lim_{c\to\infty}\frac{g(c)}{f(c)} = 1$, and then using (5.26) it is easy to see that

$$\lim_{c\to\infty}\gamma_1(c) = \lim_{c\to\infty}\left(2+\frac{2g(c)}{f(c)}\right) = 4. \tag{5.39}$$

The existence of the fixed point now follows immediately from the continuity of γ_1 and the fact that $2 < \gamma_1(2) < \gamma_1(4) < 4$. □

Along the same line, the next lemma follows directly from [47].

Lemma 25. $\lim_{c\to\infty}\gamma_2(c) = \frac{3}{2}$ *and* γ_2 *has a fixed point in the interval* $(1, \frac{3}{2})$*.*

We collect our various gleanings about γ_1 and γ_2 in the following theorem.

Theorem 39. *The functions* γ_1 *and* γ_2 *satisfy*

$$\begin{aligned}\mathfrak{m}_y\left\{x : \sup_Q x(s,t) \geq c\right\} &= \gamma_1(c)\mathfrak{m}_y\left\{x : x(S,T) \geq c\right\}\\ &= \gamma_2(c)\mathfrak{m}_y\left\{x : \sup_{\partial Q} x(s,t) \geq c\right\}\end{aligned} \tag{5.40}$$

and have the following properties:

1. γ_1 *and* γ_2 *are smooth functions,*

2. $\lim_{c\to\infty}\gamma_1(c) = 4$ *and* $\lim_{c\to\infty}\gamma_2(c) = \frac{3}{2}$*,*

3. for $-\infty < c \le 0$, $\gamma_2(c) = 1$ *and*

$$\gamma_1(c) = \frac{\sqrt{2\pi ST}}{\int_c^\infty \exp\left(-\frac{u^2}{2ST}\right) \mathrm{d}u}.$$

Proof. The smoothness of γ_1 and γ_2 follows from (5.23), (5.26), (5.34), (5.35), the smoothness of f and F, and the remarkable result from [40] by Nualart showing that the cumulative distribution of the random variable $\sup_Q x(s,t)$ is smooth, *i.e.* the function $M(c) = \mathfrak{m}_y \left\{x : \sup_Q x(s,t) \ge c\right\}$ is smooth.

The second property follows directly from Lemmas 24 and 25. We obtain the third property by recalling that $\gamma_1(0) = 2$ and $\gamma_2(0) = 1$, that

$$\mathfrak{m}_y \{x : x(S,T) \ge c\} = \frac{1}{\sqrt{2\pi ST}} \int_c^\infty \exp\left(-\frac{u^2}{2ST}\right) \mathrm{d}u,$$

and that $\mathfrak{m}_y \left\{x : \sup_Q x(s,t) \ge c\right\} = \mathfrak{m}_y \left\{x : \sup_{\partial Q} x(s,t) \ge c\right\} = 1$ for $c < 0$. □

5.4 A positive reflection result for $C_2(Q)$

In light of (5.25), we see a way in which to formulate a reflection principle which will hold for $C_2(Q)$. We have a partial result in Proposition 4, but we can quickly extend this in a very natural manner.

Let $\le$ be a partial order on Q such that $(s_1,t_1) \le (s_2,t_2)$ if and only if $s_1 \le s_2$ and $t_1 \le t_2$. We will say that a differentiable function $\phi : [0,S] \to Q$ is a smooth increasing path in Q if it satisfies $\phi(s_1) \le \phi(s_2)$ whenever $s_1 \le s_2$, and $0 < ||\phi'(s)|| < M$ for some positive M (here ϕ' is the derivative vector for ϕ and $||\cdot||$ is the Euclidean norm on Q).

We begin with a very basic lemma.

Lemma 26. *Let* $0 \le s_1 \le s_2 \le s_3 \le s_4 \le S$ *and* $0 \le t_1 \le t_2 \le t_3 \le t_4 \le T$ *with each*

(s_i, t_i) *distinct for* $i = 1, 2, 3, 4$. *Then the random variables* $X = x(s_4, t_4) - x(s_3, t_3)$ *and* $X' = x(s_2, t_2) - x(s_1, t_1)$ *are independent and symmetrically distributed.*

Proof. The proof of independence is essentially a calculation, as

$$\begin{aligned}\mathbb{E}[XX'] &= \int_{C_2(Q)} [x(s_4, t_4) - x(s_3, t_3)][x(s_2, t_2) - x(s_1, t_1)]\mathfrak{m}_y(dx) \\ &= \min(s_4, s_2)\min(t_4, t_2) - \cdots + \min(s_3, s_1)\min(t_3, t_1) \\ &= s_2t_2 - s_1t_1 - s_2t_2 + s_1t_1 \\ &= 0.\end{aligned}$$

Now, the fact that X and X' are Gaussian (being the sum of Gaussian random variables), independence and symmetry follow immediately. □

Using this and Lemma 21 we can prove the following theorem in essentially the same manner as Theorem 37. It establishes a reflection principle on $C_2(Q)$ when our attention is restricted to the behavior of the space only over an increasing path ϕ in Q.

Theorem 40. *Let* $\phi : [0, S] \to Q$ *be a smooth increasing path in* Q *with* $\phi(0) = (0, t_0)$ *or* $\phi(0) = (s_0, 0)$ *and let* $c \geq 0$. *Then*

$$\mathfrak{m}_y \left\{ x : \sup_{[0,S]} x(\phi(s)) \geq c \right\} = 2\mathfrak{m}_y \left\{ x : x(\phi(S)) \geq c \right\}. \tag{5.41}$$

Proof. Note that the condition on $\phi(0)$ guarantees that $x(\phi(0)) = 0$ and the fact that $0 < ||\phi'||$ both prevents the potential pathologies of a constant path, and combined with the increasing property of ϕ ensures that for any $s_1 < s_2 < s_3 < s_4$, the points $\{\phi(s_i)\}$ will satisfy the hypotheses of Lemma 26.

Now we can use the independence and symmetry guaranteed by Lemma 26 to assure that for $X_0 = 0$ and $X_k = x(s_k) - x(s_{k-1})$ satisfy the hypotheses of Lemma 21. Then we essentially mimic the proof of Theorem 37, taking a to be the zero function and taking $b(s) = \phi_1(s)\phi_2(s)$, where ϕ_1 and ϕ_2 are the coordinate functions of ϕ.

The only point of concern might be the estimate in (5.7). However, note that $b'(s) = \phi_1(s)\phi_2'(s) + \phi_2(s)\phi_1'(s)$ and the condition that $||\phi'|| \leq M$ is certainly sufficient to bound $||b'||_2$, so this poses no difficulties. □

Note that the theorem certainly holds for any vertical or horizontal path in Q, as Proposition 4 would indicate. The restrictions on the path ϕ above are fairly strong and can certainly be relaxed, as the following corollary shows.

Corollary 22. *Let $\phi : [0, S] \to Q$ be any continuous function with $\phi(0) = (0, t_0)$ or $\phi(0) = (s_0, 0)$ and let $c \geq 0$. Then*

$$\mathfrak{m}_y \left\{ x : \sup_{[0,S]} x(\phi(s)) \geq c \right\} = 2\mathfrak{m}_y \left\{ x : x(\phi(S)) \geq c \right\}. \tag{5.42}$$

Proof. Observe that there is a sequence of increasing paths $\{\phi_n\} \subseteq C^1([0, S], Q)$ converging uniformly to ϕ. Now, note that

$$\lim_{n\to\infty} \chi_{\left\{x:\sup_{[0,S]} x(\phi_n(s))\geq c\right\}}(x) = \chi_{\left\{x:\sup_{[0,S]} x(\phi(s))\geq c\right\}}(x)$$

and also that

$$\lim_{n\to\infty} \chi_{\{x:x(\phi_n(S))\geq c\}}(x) = \chi_{\{x:x(\phi(S))\geq c\}}(x)$$

pointwise in x. From this we conclude that

$$\begin{aligned} \mathfrak{m}_y \left\{ x : \sup_{[0,S]} x(\phi(s)) \geq c \right\} &= \lim_{n\to\infty} \mathfrak{m}_y \left\{ x : \sup_{[0,S]} x(\phi_n(s)) \geq c \right\} \\ &= \lim_{n\to\infty} 2\mathfrak{m}_y \left\{ x : x(\phi_n(S)) \geq c \right\} = 2\mathfrak{m}_y \left\{ x : x(\phi(S)) \geq c \right\}, \end{aligned}$$

by dominated covergence. □

We conclude by remarking that the condition $\phi(0) = (s_0, 0)$ or $\phi(0) = (0, t_0)$ can also be relaxed by taking $b(s) = \phi_1(s)\phi_2(s) - \phi_1(0)\phi_2(0)$ in the proof of Theorem 40; this results in no difference to the proof or results of the theorem or its corollary.

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