Spring 4-10-2014

Well-posedness and stability of a semilinear Mindlin-Timoshenko plate model

Pei Pei
University of Nebraska – Lincoln, ppeiunl@gmail.com

WELL-POSEDNESS AND STABILITY OF A SEMILINEAR MINDLIN-TIMOSHENKO PLATE MODEL

by

Pei Pei

A DISSERTATION

Presented to the Faculty of

The Graduate College at the University of Nebraska

In Partial Fulfilment of Requirements

For the Degree of Doctor of Philosophy

Major: Mathematics

Under the Supervision of Professors Mohammad A. Rammaha and Daniel Toundykov

Lincoln, Nebraska

May, 2014

WELL-POSEDNESS AND STABILITY OF A SEMILINEAR
MINDLIN-TIMOSHENKO PLATE MODEL

Pei Pei, Ph.D.

University of Nebraska, 2014

Adviser: Mohammad A. Rammaha and Daniel Toundykov

I will discuss well-posedness and long-time behavior of Mindlin-Timoshenko plate equations that describe vibrations of thin plates. This system of partial differential equations was derived by R. Mindlin in 1951 (though E. Reissner also considered an analogous model earlier in 1945). It can be regarded as a generalization of the Timoshenko beam model (1937) to flat plates, and is more accurate than the classical Kirchhoff-Love plate theory (1888) because it accounts for shear deformations.

I will present a semilinear version of the Mindlin-Timoshenko system. The primary feature of this model is the interplay between nonlinear frictional forces (“damping) and nonlinear *source terms*. The sources may represent restoring forces, such as (nonlinear refinement on) Hooke’s law, but may also have a destabilizing effect amplifying the total energy of the system, which is the primary scenario of interest.

The dissertation verifies local-in-time existence of solutions to this PDE system, as well as their continuous dependence on the initial data in appropriate function spaces. The global-in-time existence follows when the dissipative frictional effects dominate the sources. In addition, a *potential well* theory is developed for this problem. It allows us to identify sets of initial conditions for which global existence follows without balancing of the damping and sources, and sets of initial conditions for which solutions can be proven to develop singularities in finite time.

ACKNOWLEDGMENTS

I would like to express my special appreciation and thanks to my advisors: Professor Mohammad A. Rammaha and Professor Daniel Toundykov, who have been tremendous mentors for me. I would like to thank them for encouraging my research and for allowing me to grow as a research scientist. Their advice on both research as well as on my career has been priceless. I would also like to thank Professors George Avalos, Mathew Dwyer, and Mikil Foss for serving as my committee members and dissertation readers. I also want to thank them for letting my defense be an enjoyable moment, and for their brilliant comments and suggestions.

A special thanks to my family. Words cannot express how grateful I am to my parents and grandparents for all of the sacrifices that you've made on my behalf. Your support for me was what sustained me thus far. I would also like to thank my best friend Mu-wan Huang who supported me and encouraged me to strive towards my goal during the past five years.

Contents

Chapter 1

Introduction

This discussion is devoted to the Mindlin-Timoshenko equations for plate dynamics which could also be credited as the Reissner-Mindlin plate model and henceforth will be abbreviated as RMT. At the principal level the system is an elastic wave equation and is comprised of a wave equation quantifying transversal oscillations, coupled with a 2D system of dynamic elasticity describing the evolution of the filament angles. The version of the RMT model considered below bears the influence of nonlinear amplitude-modulated forcing terms that could either act as energy "sinks" with a restoring effect (e.g., as a nonlinear refinement on Hooke's law) or in the more interesting case as "sources" that contribute to the build-up of energy and potentially lead to a finite time blow-up of solution. In addition, to counterbalance the effects of potentially destabilizing strong sources, the system incorporates internal viscous (frictional) damping. Besides the Hadamard well-posedness of this problem, the influence of the source-damping interaction on the behavior of solutions is of the main interest in this work.

1.1 Literature overview

Both the Euler-Bernoulli beam (1750) and Kirchhoff-Love plate (1888) theories have limited accuracy when it comes to high-frequency vibrations or when deflections are relatively large with respect to the thickness of the cross-sections. The first attempt to address this issue in beam models was by Rayleigh [45] who accounted for the rotational inertia of the cross-section of the beam. Subsequently, Timoshenko (see e.g. [50]) extended this approach by incorporating shear deformations into the beam model.

Extensions of these efforts to plate theory were subsequently developed by Reissner [46] and Mindlin [37], resulting in what is now often cited as Mindlin-Timoshenko or Reissner-Mindlin equations.

This remarkable theory has drawn a lot of attention and research efforts. It offers higher modeling accuracy while still using a principally linear model, which in addition is of second order in space and time. Such a setting is much more amenable to analytic and numerical investigations than for instance the nonlinear von Karman model.

A vast body of numerical results for the RMT model is currently available. Besides [27, 28], let us also briefly recount some of the more recent analytic developments. Global attractors for the RMT system with full interior damping were studied by Chueshov and Lasiecka in [12]. A very interesting result by Fernández Sare [13] proves non-exponential stability when the plate is subject to linear boundary feedbacks that act only on the filament angles of the state vector. Giorgi and Vegni in [16] investigated the nonlinear RMT plate with memory. For recent results on coupled PDE dynamics with interface on RMT plate equations see the many works by Grobbelaar-van Dalsen [17, 18, 19, 20, 21] (also see [22] and references therein), Giorgi and Naso [15], and Avalos and Toundykov [4, 3]. Other results can be found in [28, 39, 40, 42, 43, 52]. Despite the above mentioned ample work and many references therein, there has been less focus on the interaction of nonlinear sources and damping terms within the RMT framework. This topic is well-understood for scalar wave equations [1], but not for the RMT vectorial system.

1.2 The model

In the RMT model the state of the system is represented by a vector-valued function (w, ψ, ϕ) which depends on position vector $\mathbf{x} = (x, y)$ and time $t \geq 0$. The component $w = w(\mathbf{x}, t)$ corresponds to the vertical displacement of the plate's mid-surface at point $\mathbf{x}$ time t, whereas ψ and ϕ are proportional to the angles of the plate filaments transversal to the mid-surface. Formally, the model can be thought of as a lower-order coupling of a wave equation for w and a 2D system of dynamic elasticity for (ψ, ϕ).

Throughout the dissertation we assume that the mid-surface of the plate $\Omega \subset \mathbb{R}^2$ is a bounded open domain with a C^2 boundary Γ. The RMT system reads [28, pp. 25–26] as follows:

$$\begin{cases} w_{tt} - \Delta w - (\psi_x + \phi_y) + g_1(w_t) = f_1(w, \psi, \phi), \text{ in } Q_T := \Omega \times (0, T), \\ \psi_{tt} - (\psi_{xx} + \frac{1-\mu}{2}\psi_{yy}) - \frac{1+\mu}{2}\phi_{xy} + (\psi + w_x) + g_2(\psi_t) = f_2(w, \psi, \phi), \text{ in } Q_T, \\ \phi_{tt} - (\frac{1-\mu}{2}\phi_{xx} + \phi_{yy}) - \frac{1+\mu}{2}\psi_{xy} + (\phi + w_y) + g_3(\phi_t) = f_3(w, \psi, \phi), \text{ in } Q_T, \\ w = \psi = \phi = 0 \text{ on } \Gamma \times (0, T), \end{cases} \tag{1.2.1}$$

with initial data in the associated *finite energy space* $(H_0^1(\Omega))^3 \times (L^2(\Omega))^3$:

$$\begin{cases} (w(0), \psi(0), \phi(0)) = (w_0, \psi_0, \phi_0) \in (H_0^1(\Omega))^3, \\ (w_t(0), \psi_t(0), \phi_t(0)) = (w_1, \psi_1, \phi_1) \in (L^2(\Omega))^3. \end{cases} \tag{1.2.2}$$

Here, $0 < \mu < 1/2$ is the Poisson ratio, the nonlinear feedbacks f_1, f_2, f_3 model interior sources, while g_1, g_2, g_3 are continuous monotone feedback maps vanishing at the origin and modeling viscous damping.

1.3 Goals and challenges

1.3.1 Existence and uniqueness

The first purpose of the dissertation is to give an essential background for the investigation of RMT plates with source terms. The questions of local and global well-posedness addressed here provide the necessary foundation for subsequent treatment of related problems, e.g., the potential well framework for this model which is the second purpose of this work.

An extensive body of work has been conducted in this direction for scalar equations. While the analysis below relies on well-known tools, many of them cannot be cited directly without verifying a number of additional technical steps in order to accommodate the vectorial structure of the problem:

- In Section 2.1, we give a detailed proof of the local existence statement from Theorem 1.5.4. The first step of the proof establishes the existence of strong solutions which is achieved by recasting the system as an evolution equation whose generator is the negative of a nonlinear m-accretive operator. Proving the properties of the generator requires a combination of the arguments for scalar wave equation and for a 2D system of isotropic elasticity, with the appropriate definition of the suitable inner product on the state space. In addition, the damping has to be interpreted as a single diagonal sub-gradient operator on a product space.

- Section 2.2 is devoted to the derivation of the energy identity (1.5.4) for weak solutions by means of time-difference quotients. This approach avoids the calculus that would have otherwise required strong regularity of solutions.

- In Section 2.3 we demonstrate the continuous dependence of solutions on the initial data, which also confirms uniqueness.

- Section 2.4 contains the proof of global existence stated in Theorem 1.5.6 for the case where the damping exponents dominate those of the sources. It should be remarked that because the sources are coupled (in the sense that each depends on the entire solution vector) then no single damping term alone can "stabilize" any of the sources, regardless of the relation between the their exponents.

- Theorem 1.5.9 in Section 2.5 verifies a finite time blow-up whenever the initial "total" energy of the system is negative. This argument is sensitive to the vectorial nature of the system and the source terms in the model have to satisfy additional structural conditions outlined in Assumption 1.5.7, which would have held automatically in the scalar case.

1.3.2 Decay of energy

The second goal of the dissertation is to describe the dynamics of the RMT model from the perspective of the *potential well theory*. The potential well approach to stability of hyperbolic equations was originally developed by Payne and Sattinger [41]. See the papers by Levine and Smith [33, 34] for applications to heat equations and systems with nonlinear boundary conditions. Since then the technique has been employed by many authors to analyze hyperbolic and Petrowski-type PDE's, e.g. (in chronological order) [38, 51, 54, 55, 14, 31, 10, 35, 24]. With the exception of the last two papers which address systems of coupled plate and wave equations respectively, the cited articles focus on scalar equations.

The present treatment of a vectorial RMT system has to resolve a number of technical issues that are not present in the scalar case, and unlike [35, 24] where the coupled equations have identical principal parts, the RMT model incorporates three coupled PDE's with a more complex structure: one equation being the scalar wave, and the other two comprising a system of isotropic elasticity.

The dissertation achieves this goal as follows:

- Subsection 1.5.3 formulates the potential well framework. The term "potential well" refers to a subset of the state space where one can quantify the relationship between the nonlinear source terms and the principal linear part of the system. This description is based on the properties of the nonlinear functional whose critical points provide solutions to the associated steady-state problem: in this case it is a semilinear coupling between the Poisson's equation and a 2D system of isotropic elasticity (see (1.5.12) below).

  The corresponding state space, where each vector describes the out-of-plane displacement and the two shear variables, is equipped with a compatible inner product in terms of which we define the *Nehari manifold* that separates the "stable" and "unstable" parts of the potential well.

  Yet another complication arising in this vectorial setting is that the nonlinear sources are coupled in the sense that each of the three scalar sources depends on the entire state vector. Thus the analysis does not merely reduce to the study of three independent scalar nonlinearities and additional structural assumptions have to be considered (see Assumption 1.5.10 below).

- Section 3.1 contains a proof of global existence for potential well solutions.

- Section 3.2 derives uniform stabilization estimates by using the standard energy methods, the compactness-uniqueness argument, and the strategy of Lasiecka and Tataru [30] adapted to the system in question.

- Finally, Section 3.3 gives the proof of the blow-up result for positive total initial energy. The idea of the approach goes back to [32], however, a number of adjustments have to be made in a vectorial case. A similar analysis for a coupled system of two wave equations was first carried out in [1].

1.4 Notation and function spaces

We begin by introducing basic notation that will be used throughout the dissertation. For scalar functions on Ω, we will use the following norms and scalar products:

$$||w||_s = ||w||_{L^s(\Omega)}, \qquad ||w||_{1,\Omega} = ||w||_{H^1(\Omega)},$$

$$(v,w)_\Omega = (v,w)_{L^2(\Omega)}, \qquad (v,w)_{1,\Omega} = (v,w)_{H^1(\Omega)}.$$

Similarly, for vector-valued functions $u = (w,\psi,\phi)$ and $\tilde{u} = (\tilde{w},\tilde{\psi},\tilde{\phi})$:

$$(u,\tilde{u})_\Omega = (w,\tilde{w})_\Omega + (\psi,\tilde{\psi})_\Omega + (\phi,\tilde{\phi})_\Omega,$$

$$(u,\tilde{u})_{1,\Omega} = (w,\tilde{w})_{1,\Omega} + (\psi,\tilde{\psi})_{1,\Omega} + (\phi,\tilde{\phi})_{1,\Omega},$$

$$||u||_s = \left(||w||_s^s + ||\psi||_s^s + ||\phi||_s^s\right)^{1/s},$$

$$||u||_{1,\Omega} = \left(\,||w||_{1,\Omega}^2 + ||\psi||_{1,\Omega}^2 + ||\phi||_{1,\Omega}^2\,\right)^{1/2}.$$

Here $H^1(\Omega)$ is the Sobolev space $W^{1,2}(\Omega)$, and $H^1_0(\Omega)$ is the closure of $C_c^\infty(\Omega)$ functions with respect to the H^1-norm. The standard duality pairing between $[H^1(\Omega)]'$ and $H^1(\Omega)$

will be denoted by $\langle\cdot,\cdot\rangle$. Throughout the dissertation, we put:

$$V = (H_0^1(\Omega))^3, \text{ and } H = (H_0^1(\Omega))^3 \times (L^2(\Omega))^3, \tag{1.4.1}$$

and we endow the Hilbert spaces V and H with the following inner products: If $u = (w, \psi, \phi,)$, $\tilde{u} = (\tilde{w}, \tilde{\psi}, \tilde{\phi})$, $u_1 = (w_1, \psi_1, \phi_1)$, $\tilde{u}_1 = (\tilde{w}_1, \tilde{\psi}_1, \tilde{\phi}_1)$, and $U = (u, u_1)$, $\tilde{U} = (\tilde{u}, \tilde{u}_1)$, then

$$\begin{aligned}(u, \tilde{u})_V = &\int_\Omega \Big((1-\mu)\left(\psi_x\tilde{\psi}_x + \phi_y\tilde{\phi}_y\right) + \mu\left(\psi_x + \phi_y\right)\left(\tilde{\psi}_x + \tilde{\phi}_y\right) \\ &+ \frac{1-\mu}{2}\left(\psi_y + \phi_x\right)\left(\tilde{\psi}_y + \tilde{\phi}_x\right)\Big)d\mathbf{x} + (w_x + \psi, \tilde{w}_x + \tilde{\psi})_\Omega + (w_y + \phi, \tilde{w}_y + \tilde{\phi})_\Omega, \end{aligned} \tag{1.4.2}$$

$$(U, \tilde{U})_H = (u, \tilde{u})_V + (u_1, \tilde{u}_1)_\Omega. \tag{1.4.3}$$

As shown in the Appendix (Proposition A.0.2), the corresponding norms $\|u\|_V$, $\|U\|_H$ are equivalent to the standard norms on V and H, where

$$\begin{aligned}\|u\|_V^2 = &\int_\Omega \Big((1-\mu)\left(\psi_x^2 + \phi_y^2\right) + \mu\left(\psi_x + \phi_y\right)^2 + \frac{1-\mu}{2}\left(\psi_y + \phi_x\right)^2\Big)d\mathbf{x} \\ &+ \|w_x + \psi\|_2^2 + \left\|w_y + \phi\right\|_2^2, \end{aligned} \tag{1.4.4}$$

and

$$\|U\|_H^2 = \|u\|_V^2 + \|u_1\|_2^2. \tag{1.4.5}$$

1.5 Preliminaries and main results

1.5.1 Assumption on the nonlinear terms

The most interesting aspect of the system (1.2.1) is the source-damping interaction. The corresponding nonlinear terms satisfy the following assumption:

Assumption 1.5.1 (Damping and sources)**.**

- **Damping:** $g_i : \mathbb{R} \to \mathbb{R}$ *are continuous, monotone increasing functions with* $g_i(0) = 0,\ i = 1, 2, 3$. *In addition, the following growth conditions at infinity hold: there exist positive constant* α *and* β *such that for all* $|s| \geq 1$,

$$\alpha|s|^{p_i+1} \leq g_i(s)s \leq \beta|s|^{p_i+1}, \tag{1.5.1}$$

  with $p_i \geq 1, i = 1, 2, 3$ *where* $p_1 = m, p_2 = r, p_3 = q$.

- **Sources:** $f_j(w, \psi, \phi) \in C^1(\mathbb{R}^3)$ *and there is a positive constant* C *such that*

$$|\nabla f_j(w, \psi, \phi)| \leq C(|w|^{p-1} + |\psi|^{p-1} + |\phi|^{p-1} + 1), \quad j = 1, 2, 3; \ \textit{with} \ \ p \geq 1.$$

The following terminology will be occasionally used when working with functions g_i:

Definition 1.5.2 (Linearly bounded)**.** A function $\gamma(s) : \mathbb{R} \to \mathbb{R}$ will be said to be *linearly bounded near the origin* if there exist slopes $c_1, c_2 > 0$ such that

$$c_1|s| \leq |\gamma(s)| \leq c_2|s| \ \text{ for all } \ |s| < 1\,.$$

1.5.2 Main results for existence and uniqueness

To formulate the results we begin by giving the definition of a weak solution to (1.2.1).

Definition 1.5.3 (Weak solution)**.** A vector-valued function $u = (w, \psi, \phi)$ is said to be a weak solution to (1.2.1) on $[0, T]$ if:

- $u \in C\left([0, T]; V\right), \quad (u(0), u_t(0)) \in H;$
  $u_t \in C\Big([0, T]; (L^2(\Omega))^3\Big) \cap \Big(L^{m+1}(\Omega \times (0, T)) \times L^{r+1}(\Omega \times (0, T)) \times L^{q+1}(\Omega \times (0, T))\Big);$

- $u = (w, \psi, \phi)$ verifies the following identity

$$(u_t(t), \theta(t))_\Omega - (u_t(0), \theta(0))_\Omega + \int_0^t \Big(-(u_t(\tau), \theta_t(\tau))_\Omega + (u(\tau), \theta(\tau))_V\Big)\, d\tau + \int_0^t (\mathscr{G}(u_t), \theta)_\Omega \, d\tau = \int_0^t (\mathscr{F}(u), \theta)_\Omega \, d\tau \tag{1.5.2}$$

for all $t \in [0, T]$ and test functions θ in

$$\Theta := \Big\{\theta = (\theta_1, \theta_2, \theta_3) : \theta \in C\left([0, T]; V\right), \quad \theta_t \in L^1([0, T]; (L^2(\Omega))^3)\Big\},$$

and where,

$$\mathscr{G}(u_t) = \big(g_1(w_t), g_2(\psi_t), g_3(\phi_t)\big), \quad \mathscr{F}(u) = \big(f_1(u), f_2(u), f_3(u)\big).$$

We start with the local well-posdeness result for (1.2.1).

Theorem 1.5.4 (Local existence of weak solutions)**.** *With the validity of Assumption 1.5.1, there exists a local weak solution $u = (w, \psi, \phi)$ to (1.2.1) defined on $[0, T_0]$ for some $T_0 > 0$ which depends only on $\|u(0)\|_V^2$ and $\|u_t(0)\|_2^2$. Moreover, if we define*

$$E(t) = \frac{1}{2}\left(\|u(t)\|_V^2 + \|u_t(t)\|_2^2\right), \tag{1.5.3}$$

then the following energy identity holds for all $t \in [0, T_0]$*:*

$$E(t) + \int_0^t (\mathscr{G}(u_t), u_t)_\Omega \, d\tau = E(0) + \int_0^t (\mathscr{F}(u), u_t)_\Omega \, d\tau. \tag{1.5.4}$$

Theorem 1.5.5 (Uniqueness and continuous dependence)**.** *Under Assumption 1.5.1 weak solutions in* $C([0,T]; H)$ *to* (1.2.1) *furnished by Theorem 1.5.4 depend continuously on their initial data in the state space* H*. In particular, such solutions are unique.*

Our next theorem shows that weak solutions furnished by Theorem 1.5.4 are global solutions, provided the exponents of the damping terms dominate those of the sources.

Theorem 1.5.6 (Global weak solutions)**.** *In addition to Assumption 1.5.1, further assume that* $p \leq \min\{m, r, q\}$*. Then the said weak solution in Theorem 1.5.4 is a global solution and* T_0 *can be taken arbitrarily large.*

To state the blow-up result we impose additional assumptions on damping and sources.

Assumption 1.5.7 (For blow-up)**.** *Suppose the following:*

- *There exist positive constants* α *and* β *such that for all* $s \in \mathbb{R}$ *and* $i = 1, 2, 3$*,*

$$\alpha |s|^{e_i+1} \leq g_i(s)s \leq \beta |s|^{e_i+1} \quad with \quad e_1 = m, \; e_2 = p, \; e_3 = q \geq 1$$

- *There exists a positive function* $F \in C^2(\mathbb{R}^3)$ *such that*

$$f_1(w,\psi,\phi) = \partial_w F(w,\psi,\phi), \quad f_2(w,\psi,\phi) = \partial_\psi F(w,\psi,\phi), \quad f_3(w,\psi,\phi) = \partial_\phi F(w,\psi,\phi).$$

- *There exist* $c_0 > 0, c_1 > 2$ *such that, for all* $u = (w,\psi,\phi) \in \mathbb{R}^3$*,*

$$F(w,\psi,\phi) \geq c_0 \left(|w|^{p+1} + |\psi|^{p+1} + |\phi|^{p+1} \right),$$

and

$$w f_1(w,\psi,\phi) + \psi f_2(w,\psi,\phi) + \phi f_3(w,\psi,\phi) \geq c_1 F(w,\psi,\phi).$$

Remark 1.5.8. It is important to note here that the restrictions on sources in Assumption 1.5.7 are natural and quite reasonable. There is a large class of functions satisfying it. For instance functions of the form

$$F(w,\psi,\phi) = a\,|w+\psi|^{p+1} + b\,|w\psi|^{\frac{p+1}{2}} + c\,|\phi|^{p+1},$$

where a, b, c are positive constants, satisfy Assumption 1.5.7 with $p \geq 3$. Indeed, a quick calculation shows that there exists $c_0 > 0$ such that $F(w,\psi,\phi) \geq c_0\Big(|w|^{p+1} + |\psi|^{p+1} + |\phi|^{p+1}\Big)$, provided b is chosen large enough. Moreover, it is easy to compute and find that $w f_1(w,\psi,\phi) + \psi f_2(w,\psi,\phi) + \phi f_3(w,\psi,\phi) = (p+1)F(w,\psi,\phi)$. Since the blow-up theorems below require $p > m \geq 1$, then $p + 1 > 2$, and so, the assumption $c_1 > 2$ is reasonable.

Theorem 1.5.9 (**B**low up in finite time)**.** *Assume the validity of Assumptions 1.5.1 and 1.5.7. If* $p > \max\{m, r, q\}$ *and* $\mathscr{E}(0) < 0$*, then any weak solution* u *to (1.2.1) furnished by Theorem 1.5.4 blows up in finite time, in the sense that* $\lim_{t\to T} E(t) = +\infty$*, for some* $0 < T < \infty$*, where*

$$\mathscr{E}(t) = E(t) - \int_\Omega F(u(t))d\boldsymbol{x}.$$

1.5.3 Potential well

Here, we introduce the *potential energy functional* J and highlight its relevance with system (1.2.1) and the Mountain Pass Theorem. The potential well framework developed by Payne and Sattinger (see, e.g., [41, 55]) will then be formulated for the problem in question. First, some additional assumptions on sources will be needed.

Assumption 1.5.10. *There exists a nonnegative function* $F(u) \in C^2(\mathbb{R}^3)$ *such that*

$$f_1(u) = \partial_w F(u), \quad f_2(u) = \partial_\psi F(u), \quad f_3(u) = \partial_\phi F(u)$$

where $u = (w, \psi, \phi) \in \mathbb{R}^3$. *Further, assume that* F *is homogeneous of order* $p + 1$*:*

$$F(\lambda u) = \lambda^{p+1} F(u), \quad \textit{for all} \quad \lambda > 0, u \in \mathbb{R}^3.$$

Since F is homogeneous, the Euler Homogeneous Function Theorem yields the following useful identity:

$$f_1(u)w + f_2(u)\psi + f_3(u)\phi = \nabla F(u) \cdot u = (p+1)F(u). \tag{1.5.5}$$

We note that the inequalities

$$\left|\nabla f_j(u)\right| \leq C\left(|w|^{p-1} + |\psi|^{p-1} + |\phi|^{p-1} + 1\right), \quad j = 1, 2, 3,$$

as required by Assumption 1.5.1, imply that there exists a constant $M > 0$ such that $F(u) \leq M\left(|w|^{p+1} + |\psi|^{p+1} + |\phi|^{p+1} + 1\right)$, for all $u \in \mathbb{R}^3$. Therefore, the homogeneity of F implies

$$F(u) \leq M\left(|w|^{p+1} + |\psi|^{p+1} + |\phi|^{p+1}\right). \tag{1.5.6}$$

Moreover, it is easy to see that f_1, f_2, and f_3 are also homogeneous functions of degree p and there exists a positive constant C such that:

$$|f_j(u)| \leq C(|w|^p + |\psi|^p + |\phi|^p), \qquad j = 1, 2, 3. \tag{1.5.7}$$

Remark 1.5.11. There is a large class of functions that satisfies Assumption 1.5.10. For instance, functions of the form (with an appropriate range of values for p)

$$F(w,\psi,\phi) = a\,|w+\psi|^{p+1} + b\,|w\psi|^{\frac{p+1}{2}} + c\,|\phi|^{p+1}$$

satisfies Assumption 1.5.10.

The following notation will be invoked throughout the subsequent discussion. Recall $V := (H_0^1(\Omega))^3$, and define the **potential energy** functional $J : V \to \mathbb{R}$ as

$$J(u) := \frac{1}{2}\|u\|_V^2 - \int_\Omega F(u)d\mathbf{x}. \tag{1.5.8}$$

Thus, the *total energy* of the system (1.2.1) will be defined as follows

$$\mathscr{E}(t) := \frac{1}{2}\left(\|u(t)\|_V^2 + \|u_t(t)\|_2^2\right) - \int_\Omega F(u(t))d\mathbf{x}, \tag{1.5.9}$$

and therefore,

$$\mathscr{E}(t) = \frac{1}{2}\|u_t(t)\|_2^2 + J(u(t)). \tag{1.5.10}$$

In fact, Lemma A.0.5 (in the Appendix) shows that the Fréchet derivative of J at $u \in V$ is given by:

$$D_uJ(\theta) = (u,\theta)_V - \int_\Omega \mathscr{F}(u)\cdot\theta d\mathbf{x}, \quad \text{for all } \theta \in V, \tag{1.5.11}$$

which implies that the critical points of the functional J are weak solutions to the elliptic problem:

$$\begin{cases} -\Delta w - (\psi_x + \phi_y) = f_1(w,\psi,\phi), & \text{in } \Omega \times (0,T), \\ -(\psi_{xx} + \frac{1-\mu}{2}\psi_{yy}) - \frac{1+\mu}{2}\phi_{xy} + (\psi + w_x) = f_2(w,\psi,\phi), & \text{in } \Omega \times (0,T), \\ -(\frac{1-\mu}{2}\phi_{xx} + \phi_{yy}) - \frac{1+\mu}{2}\psi_{xy} + (\phi + w_y) = f_3(w,\psi,\phi), & \text{in } \Omega \times (0,T). \end{cases} \tag{1.5.12}$$

Associated with the functional $J(u)$ is the well-known *Nehari manifold*, namely

$$\mathscr{N} := \{u \in V \setminus \{0\} : D_u J(u) = 0\}.$$

More precisely, it follows from (A.0.10) that the Nehari manifold can be represented as

$$\mathscr{N} = \left\{ u \in V \setminus \{0\} : \|u\|_V^2 = (p+1) \int_\Omega F(u) d\mathbf{x} \right\}. \tag{1.5.13}$$

According to [24, Lemma 2.7] (or see, for example, [2, 9, 23, 44]) the functional J satisfies the hypothesis of the Mountain Pass Theorem and the *mountain pass level d* satisfies

$$d := \inf_{u \in \mathscr{N}} J(u) = \inf_{u \in V \setminus \{0\}} \sup_{\lambda \geq 0} J(\lambda u) . \tag{1.5.14}$$

The following result is needed:

Lemma 1.5.12. *In addition to Assumptions* (1.5.1) *and* (1.5.10)*, further assume that* $p > 1$*, then*

$$d := \inf_{u \in \mathscr{N}} J(u) > 0. \tag{1.5.15}$$

Proof. Fix $u \in \mathscr{N}$, then it follows from (1.5.13) that

$$J(u) = \frac{1}{2}\|u\|_V^2 - \int_\Omega F(u)d\mathbf{x} = \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u\|_V^2 > 0.$$

However, it follows from the Sobolev embedding theorem in 2D: $H_0^1(\Omega) \hookrightarrow L^s(\Omega)$, $1 \leq s < \infty$, the bounds in (1.5.6), and Proposition A.0.2 (in the Appendix) that

$$\|u\|_V^2 = (p+1)\int_\Omega F(u)d\mathbf{x} \leq (p+1)M\|u\|_{p+1}^{p+1} \leq C\|u\|_{1,\Omega}^{p+1} \leq C\|u\|_V^{p+1}.$$

Since $p > 1$ we infer $\|u\|_V \geq C^{-\frac{1}{p-1}} > 0$, and hence $d \geq \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\frac{1}{C}\right)^{\frac{2}{p-1}} > 0$ as desired. □

In addition to the Nehari manifold $\mathscr{N}$, we introduce the following sets:

$$\begin{aligned} \mathscr{W} &:= \{u \in V : J(u) < d\}; \\ \mathscr{W}_1 &:= \left\{u \in \mathscr{W} : \|u\|_V^2 > (p+1)\int_\Omega F(u)d\mathbf{x}\right\} \cup \{0\}; \\ \mathscr{W}_2 &:= \left\{u \in \mathscr{W} : \|u\|_V^2 < (p+1)\int_\Omega F(u)d\mathbf{x}\right\}. \end{aligned} \tag{1.5.16}$$

Clearly, $\mathscr{W}_1 \cap \mathscr{W}_2 = \emptyset$, and $\mathscr{W}_1 \cup \mathscr{W}_2 = \mathscr{W}$. We refer to $\mathscr{W}$ as the **potential well** and d as the *depth* of the well. The set $\mathscr{W}_1$ can be formally regarded as the "good" part of the well, as it will be shown that every weak solution starting therein exists globally provided initial energy is under the level d. On the other hand, if the initial data are taken from $\mathscr{W}_2$ and the source exponents dominate those of the damping, then solutions with nonnegative initial energy $\mathscr{E}(0)$ may blow-up in finite time.

Explicit approximation of the "good" part $\mathscr{W}_1$ of the potential well.

Although the *Nehari manifold* gives a sharp characterization of the potential well, it is important (from computational point of view, especially when deriving uniform decay rates of energy), to approximate the "good" part of the potential well $\mathscr{W}_1$ by a smaller closed set.

The argument we employ (which is by now classical, see [41, 53, 54, 55] as well as [9]) will produce an approximation (as a subset) of the potential well defined in (1.5.16). Let

$$\mathcal{G}(s) := \frac{1}{2}s^2 - MRs^{p+1}, \tag{1.5.17}$$

where the constant $M > 0$ comes from (1.5.6) and

$$R := \sup_{u \in V \setminus \{0\}} \frac{\|u\|_{p+1}^{p+1}}{\|u\|_V^{p+1}}. \tag{1.5.18}$$

Since $p > 1$, due to the Sobolev embedding theorem in 2D: $H_0^1(\Omega) \hookrightarrow L^s(\Omega)$, $1 \leq s < \infty$, we know $0 < R < \infty$. From the definition of potential energy (1.5.8) and the bounds in (1.5.6) it follows that

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|_V^2 - \int_\Omega F(u)d\mathbf{x} \geq \frac{1}{2}\|u\|_V^2 - M\|u\|_{p+1}^{p+1} \\ &\geq \frac{1}{2}\|u\|_V^2 - MR\|u\|_V^{p+1} = \mathcal{G}(\|u\|_V). \end{aligned} \tag{1.5.19}$$

If $p > 1$, a straightforward calculation shows that $\mathcal{G}$ attains its absolute maximum on $[0, \infty)$ at the unique critical point:

$$s_0 = ((p+1)MR)^{-\frac{1}{p-1}}. \tag{1.5.20}$$

Now plug s_0 into $\mathcal{G}$ to find the exact maximum value:

$$\tilde{d} := \sup_{s \in [0,\infty)} \mathcal{G}(s) = \mathcal{G}(s_0) = \frac{p-1}{2(p+1)}((p+1)MR)^{-\frac{2}{p-1}}, \tag{1.5.21}$$

to which we will refer as the "approximate depth" of the potential well. Now define

$$\tilde{\mathscr{W}}_1 := \{u \in V : \|u\|_V < s_0,\ J(u) < \mathcal{G}(s_0)\}. \tag{1.5.22}$$

It is important to note that $\tilde{\mathscr{W}}_1 \neq \{0\}$. In fact, for any $u \in V$, there exists a scalar $\epsilon > 0$ such that $\epsilon u \in \tilde{\mathscr{W}}_1$. Moreover, we have the following result:

Proposition 1.5.13. *$\mathcal{G}(s_0) \leq d$ and $\tilde{\mathscr{W}}_1$ is a subset of $\mathscr{W}_1$.*

Proof. We first show that $\mathcal{G}(s_0) \leq d$. Fix any $u \in V \setminus \{0\}$. Inequality (1.5.19) yields $J(\lambda u) \geq \mathcal{G}(\lambda \|u\|_V)$ for all $\lambda \geq 0$. It follows that

$$\sup_{\lambda \geq 0} J(\lambda u) \geq \mathcal{G}(s_0).$$

Therefore, from (1.5.14) one has

$$d = \inf_{u \in V \setminus \{0\}} \sup_{\lambda \geq 0} J(\lambda u) \geq \mathcal{G}(s_0). \tag{1.5.23}$$

By employing the bounds in (1.5.6) and (1.5.18), we obtain for all $\|u\|_V < s_0$,

$$\begin{aligned}(p+1)\int_\Omega F(u)d\mathbf{x} &\leq (p+1)M\|u\|_{p+1}^{p+1} \leq (p+1)MR\|u\|_V^{p+1} \\ &= \|u\|_V^2\left[(p+1)MR\|u\|_V^{p-1}\right] < \|u\|_V^2\left[(p+1)MRs_0^{p-1}\right] = \|u\|_V^2.\end{aligned} \tag{1.5.24}$$

Therefore, by definition of $\mathscr{W}_1$ it follows that $\tilde{\mathscr{W}}_1 \subset \mathscr{W}_1$. □

For sufficiently small $\delta > 0$, we can define a closed subset of $\tilde{\mathscr{W}}_1$, namely

$$\tilde{\mathscr{W}}_1^\delta := \{u \in V \ :\ \|u\|_V \leq s_0 - \delta,\ J(u) \leq \mathcal{G}(s_0 - \delta)\}. \tag{1.5.25}$$

It is clear from Proposition 1.5.13 that $\tilde{\mathscr{W}}_1^{\delta} \subset \mathscr{W}_1$.

1.5.4 Main results for decay of energy

Our main contributions are summarized in the next three theorems.

Theorem 1.5.14 (Global Solution)**.** *In addition to Assumption 1.5.1 and Assumption 1.5.10, further assume* $u(0) \in \mathscr{W}_1$ *and* $\mathscr{E}(0) < d$*. If* $p > 1$*, then the unique weak solution* u *provided by Theorems 1.5.4 and 1.5.5 is a global solution, i.e., it can be extended to* $[0, \infty)$*. Furthermore, we have:*

(I) $J(u(t)) \leq \mathscr{E}(t) \leq \mathscr{E}(0)$, *(II)* $u(t) \in \mathscr{W}_1$,

(III) $E(t) < d \cdot \rho$, *(IV)* $\frac{1}{\rho} E(t) \leq \mathscr{E}(t) \leq E(t)$,

for all $t \geq 0$*, where* $\rho = \frac{p+1}{p-1}$.

Since the weak solution furnished by Theorem 1.5.14 is a global solution and the total energy $\mathscr{E}(t)$ remains positive for all $t \geq 0$, the next result states the uniform decay rates of the energy. In fact, the decay rates are given as a soution to a certain nonlinear ODE.

Theorem 1.5.15 (Uniform decay rates)**.** *In addition to Assumption 1.5.1 and Assumption 1.5.10, further assume:* $p > 1, u_0 \in \tilde{\mathscr{W}}_1^{\delta}$*, as defined in* (1.5.25)*, and* $\mathscr{E}(0) < \mathcal{G}(s_0 - \delta)$ *for some* $\delta > 0$*. Let* $\varphi_j : [0, \infty) \mapsto [0, \infty)$ *be continuous, strictly increasing, concave functions vanishing at the origin such that*

$$\varphi_j(g_j(s)s) \geq |g_j(s)|^2 + s^2 \quad \textit{for} \quad |s| < 1, \;\; j = 1, 2, 3.$$

Define the function $\Phi : [0,\infty) \mapsto [0,\infty)$ *by*

$$\Phi(s) := \varphi_1(s) + \varphi_2(s) + \varphi_3(s) + s, \quad s \geq 0. \tag{1.5.26}$$

Then, for any $T > 0$ *there exists a concave increasing map* $H = \left(I + \tilde{C}\Phi\right)^{-1}$*, where* $\tilde{C} = \tilde{C}(T, \mathscr{E}(0))$ *(instead of the dependence on* $\mathscr{E}(0)$*, one may use a dependence on* $d \cdot \rho$*) such that*

$$\rho^{-1}E(t) \leq \mathscr{E}(t) \leq S\left(\frac{t}{T} - 1\right) \text{ for all } t \geq T,$$

where S *satisfies the ODE*

$$S'(t) + H(S(t)) = 0,\ S(0) = \mathscr{E}(0). \tag{1.5.27}$$

In order to obtain a quantitative description of decay rates, one needs more information about the behavior of the damping feedbacks $g_i(s)$, $i = 1,2,3$ near the origin. The next two corollaries are examples which illustrate Theorem 1.5.15 by exhibiting exponential and algebraic decay rates for the energy functional.

Corollary 1.5.16 (Exponential decay rate)**.** *Under the hypotheses of Theorem 1.5.15, if* g_1*,* g_2*, and* g_3 *are linearly bounded near the origin, then* $H(s) = \omega s$ *for some* ω *dependent on* $\mathscr{E}(0)$ *and* T*. The total energy* $\mathscr{E}(t)$ *and the quadratic energy* $E(t)$ *decay exponentially:*

$$\rho^{-1}E(t) \leq \mathscr{E}(t) \leq e^{\omega}\mathscr{E}(0)e^{-(\omega/T)t}, \text{ for all } t \geq 0. \tag{1.5.28}$$

Corollary 1.5.17 (Algebraic decay rate)**.** *Under the hypotheses of Theorem 1.5.15, if at least one of the feedback mappings* g_i*,* $i = 1,2,3$ *is not linearly bounded (Definition 1.5.2)* *near the origin and g_1, g_2, and g_3 satisfy for all $|s| < 1$*

$$c_1|s|^{\tilde{m}} \leq |g_1(s)| \leq c_2|s|^{\tilde{m}},\ c_3|s|^{\tilde{r}} \leq |g_2(s)| \leq c_4|s|^{\tilde{r}},\ c_5|s|^{\tilde{q}} \leq |g_3(s)| \leq c_6|s|^{\tilde{q}}, \tag{1.5.29}$$

where $\tilde{m}, \tilde{r}, \tilde{q} > 0$ and $c_j > 0$, $j = 1, \ldots, 6$, then the ODE given by (1.5.27) *can be approximated by*

$$\hat{S}' + C_0\hat{S}^a(t) = 0, \quad \hat{S}(t_0) = S(t_0), \quad t \geq t_0 > 0, \quad \textit{for some} \quad t_0 > 0,$$

and the energy decays as follows:

$$\rho^{-1}E(t) \leq \mathscr{E}(t) \leq C(1+t)^{-b}, \quad \textit{for all} \quad t \geq t_0\,. \tag{1.5.30}$$

The exponent $b > 0$ can be computed explicitly as $(a-1)^{-1}$ for $a > 1$ specified by Examples 3.2.1 and 3.2.2 and the formula (3.2.10)*. In particular, it depends on the damping exponents $\tilde{m}, \tilde{r}, \tilde{q}$. The constants C_0, C depend on T and $\mathscr{E}(0)$.*

The final result addresses the blow-up of potential well solutions with *non-negative* initial energy $\mathscr{E}(0)$. It is important to point out that the blow-up result in Theorem 1.5.9 deals with the case of *negative* initial energy for general weak solutions (not necessarily potential well solutions). In order to prove this result we impose additional assumptions on the damping and sources. Instead of restricting $|s| \geq 1$ as in Assumption 1.5.1, we now require the inequalities to hold for all $s \in \mathbb{R}$.

Assumption 1.5.18 (For blow-up)**.**

- *Damping: Suppose there exist $\alpha, \beta > 0$ such that for all $s \in \mathbb{R}$*

$$\alpha|s|^{p_i+1} \leq g_i(s)s \leq \beta|s|^{p_i+1},$$

with $p_i \geq 1, i = 1, 2, 3$ *where* $p_1 = m, p_2 = r, p_3 = q$.

- *Sources:*

$$F(u) \geq \alpha_0(|w|^{p+1} + |\psi|^{p+1} + |\phi|^{p+1}), \quad \textit{for some} \quad a_0 > 0.$$

Theorem 1.5.19 (Blow-up in finite time)**.** *Assume the validity of Assumptions 1.5.1, 1.5.10 and 1.5.18. In addition, suppose* $p > \max\{m, r, q\}$*,* $0 \leq \mathscr{E}(0) < d$*, and* $u(0) \in \mathscr{W}_2$*, then any weak solution* u *provided by Theorem 1.5.4 blows-up in finite time in the sense that* $\limsup_{t \to T-} E(t) = \infty$ *for some* $T < \infty$.

Remark 1.5.20*.* The blow-up result in Theorem 1.5.19 relies on the blow-up result in Theorem 1.5.9 for negative initial energy, in the sense that we may assume here $\mathscr{E}(t) \geq 0$ for all t, for otherwise the hypotheses of the theorem 1.5.9 would be satisfied again implying the blow-up.

Chapter 2

Existence and Uniqueness

2.1 Local solutions

Here we prove the existence statement of Theorem 1.5.4, which will be carried out in the following four sub-sections.

2.1.1 Operator theoretic formulation

Our first goal is to put problem (1.2.1) in an operator theoretic form. In order to do so, we define the nonlinear operator

$$\mathscr{A} : \mathcal{D}(\mathscr{A}) \subset H \to H$$

$$\mathscr{A} U = \begin{pmatrix} -u_1^{tr} \\ -\Delta w - (\psi_x + \phi_y) + g_1(w_1) - f_1(u) \\ -(\psi_{xx} + \frac{1-\mu}{2}\psi_{yy}) - \frac{1+\mu}{2}\phi_{xy} + (\psi + w_x) + g_2(\psi_1) - f_2(u) \\ -(\frac{1-\mu}{2}\phi_{xx} + \phi_{yy}) - \frac{1+\mu}{2}\psi_{xy} + (\phi + w_y) + g_3(\phi_1) - f_3(u) \end{pmatrix}^{tr},$$

where space H is defined in (1.4.1) and

$$\mathcal{D}(\mathscr{A}) = \left\{U = (w, \psi, \phi, w_1, \psi_1, \phi_1) \in (H_0^1(\Omega))^6 : \mathscr{A}U \in (L^2(\Omega))^6\right\}.$$

With this notation, then system (1.2.1) is equivalent to the Cauchy problem:

$$U_t + \mathscr{A}U = 0 \text{ with } U(0) \in H. \tag{2.1.1}$$

2.1.2 Globally Lipschitz sources

Our first result states the global existence of the Cauchy problem (2.1.1) when the sources $f_i : V \to L^2(\Omega)$, $i = 1, 2, 3$, are globally Lipschitz.

Proposition 2.1.1. *Assume that*

- g_1, g_2, *and* g_3 *satisfy the conditions in Assumption 1.5.1.*
- f_1, f_2, *and* f_3 *are Lipschitz continuous from* $(H_0^1(\Omega))^3 \to L^2(\Omega)$ *with Lipschitz constants* L_{f_i}, $i = 1, 2, 3$.

Then system (2.1.1) *has a unique global strong solution* $U \in W^{1,\infty}(0, T; H)$*, where* $T > 0$ *is arbitrary; provided the initial datum* $U(0) \in \mathcal{D}(\mathscr{A})$.

Proof. In order to prove Proposition 2.1.1, it suffices to show that the operator $\mathscr{A} + \omega I$ is m-accretive for some $\omega > 0$. We say an operator $\mathscr{A} : \mathcal{D}(\mathscr{A}) \subset H \to H$ is *accretive* if $(\mathscr{A}x_1 - \mathscr{A}x_2, x_1 - x_2)_H \geq 0$, for all $x_1, x_2 \in \mathcal{D}(\mathscr{A})$, and it is *m-accretive* if, in addition, $\mathscr{A} + I$ maps $\mathcal{D}(\mathscr{A})$ onto H. It follows from Kato's Theorem (see [48] for instance) that, if $\mathscr{A} + \omega I$ is m-accretive for some $\omega > 0$, then for each $U_0 \in \mathcal{D}(\mathscr{A})$ there is a unique strong solution U of (2.1.1), i.e., $U \in W^{1,\infty}(0, T; H)$ such that $U(0) = U_0$, $U(t) \in \mathcal{D}(\mathscr{A})$ for all

$t \in [0,T]$, and equation (1.5.2) is satisfied a.e. $[0,T]$, where $T > 0$ is arbitrary.

Step 1: Proof of $\mathscr{A} + \omega I$ is accretive for some positive ω. We aim to find $\omega > 0$ such that:

$$((\mathscr{A} + \omega I)\, U - (\mathscr{A} + \omega I)\, \tilde{U}, U - \tilde{U})_H \geq 0, \quad \text{for all } \; U, \tilde{U} \in \mathcal{D}(\mathscr{A}).$$

Recall $V = (H_0^1(\Omega))^3$, $H = (H_0^1(\Omega))^3 \times (L^2(\Omega))^3$. In order to simplify our notation, let $u = (w, \psi, \phi)$, $\tilde{u} = (\tilde{w}, \tilde{\psi}, \tilde{\phi}) \in V$, $u_1 = (w_1, \psi_1, \phi_1,)$, $\tilde{u}_1 = (\tilde{w}_1, \tilde{\psi}_1, \tilde{\phi}_1) \in (L^2(\Omega))^3$. Thus, $U = (u, u_1)$, and $\tilde{U} = (\tilde{u}, \tilde{u}_1) \in H$.

With this notation, we can express $\mathscr{A}$ as follows:

$$\mathscr{A}(u, u_1) = (-u_1,\; \mathscr{B}(u) + \mathscr{G}(u_1) - \mathscr{F}(u)) \tag{2.1.2}$$

where $\mathscr{B} : V \to V'$, $\mathscr{G} : V \to V'$, and $\mathscr{F} : V \to (L^2(\Omega))^3$ are given by:

$$\mathscr{B}(u) = \begin{pmatrix} -\Delta w - (\psi_x + \phi_y) \\ \\ -(\psi_{xx} + \frac{1-\mu}{2}\psi_{yy}) - \frac{1+\mu}{2}\phi_{xy} + (\psi + w_x) \\ \\ -(\frac{1-\mu}{2}\phi_{xx} + \phi_{yy}) - \frac{1+\mu}{2}\psi_{xy} + (\phi + w_y) \end{pmatrix}^{tr}, \tag{2.1.3}$$

$$\mathscr{G}(u_1) = \big(g_1(w_1), g_2(\psi_1), g_3(\phi_1)\big), \quad \mathscr{F}(u) = \big(f_1(u), f_2(u), f_3(u)\big), \tag{2.1.4}$$

Straighforward calculation and Proposition A.0.4 in the Appendix give

$$\begin{aligned}\Big(\mathscr{A}(U) - \mathscr{A}(\tilde{U}), U - \tilde{U}\Big)_H = &-(u_1 - \tilde{u}_1, u - \tilde{u})_V + (u - \tilde{u}, u_1 - \tilde{u}_1)_V \\ &+\Big(\mathscr{G}(u_1) - \mathscr{G}(\tilde{u}_1), u_1 - \tilde{u}_1\Big)_\Omega - \Big(\mathscr{F}(u) - \mathscr{F}(\tilde{u}), u_1 - \tilde{u}_1\Big)_\Omega.\end{aligned} \tag{2.1.5}$$

By using the fact that each g_i is monotone and each f_i is globally Lipschitz from V to $L^2(\Omega)$, the last two terms in (2.1.5) yield

$$\left(\mathscr{A}(U) - \mathscr{A}(\tilde{U}), U - \tilde{U}\right)_H \geq -\frac{3L}{2}\|u - \tilde{u}\|_V^2 - \frac{L}{2}\|u_1 - \tilde{u}_1\|_2^2 \geq -2L\|U - \tilde{U}\|_H^2, \qquad (2.1.6)$$

where $L = \max\{L_{f_1}, L_{f_2}, L_{f_3}\}$. Therefore $\mathscr{A} + \omega I$ is accretive when $\omega > 2L$.

Step 2: Proof of $\mathscr{A} + \lambda I$ is m-accretive, for some $\lambda > 0$. To this end, it suffices to show that the range of $\mathscr{A} + \lambda I$ is all of H, for some $\lambda > 0$.

Let $\tilde{V} = (v, v_1) \in H$. Let's show that there exists $U = (u, u_1) \in \mathcal{D}(\mathscr{A})$ such that $(\mathscr{A} + \lambda I)\, U = \tilde{V}$ for some $\lambda > 0$. It is equivalent to finding $(u, u_1) \in \mathcal{D}(\mathscr{A})$ such that:

$$\begin{cases} \lambda u - u_1 = v, \\ \lambda u_1 + \mathscr{B}(u) + \mathscr{G}(u_1) - \mathscr{F}(u) = v_1. \end{cases} \qquad (2.1.7)$$

Note that (2.1.7) is in turn equivalent to

$$\lambda u_1 + \frac{1}{\lambda}\mathscr{B}(u_1) + \mathscr{G}(u_1) - \mathscr{F}\left(\frac{u_1 + v}{\lambda}\right) = v_1 - \frac{1}{\lambda}\mathscr{B}(v). \qquad (2.1.8)$$

Since $v \in V = (H_0^1(\Omega))^3$, then the right-hand side of (2.1.8) belongs to $V' = (H^{-1}(\Omega))^3$. Thus, we define the operator $S : V \to V'$ by

$$S(u_1) = \lambda u_1 + \frac{1}{\lambda}\mathscr{B}(u_1) - \mathscr{F}\left(\frac{u_1 + v}{\lambda}\right) + \mathscr{G}(u_1).$$

It is clear that domain of S is all of V. Therefore, the issue reduces to proving that the mapping $S : V \to V'$ is surjective. By Corollary 1.2 (p.45) in [5] , it is enough to show that S is maximal monotone and coercive. In order to do so, we split S as the sum of two operators. Let $T, \mathscr{G} : V \to V'$ be given by:

$$T(u_1) = \lambda u_1 + \frac{1}{\lambda}\mathscr{B}(u_1) - \mathscr{F}\left(\frac{u_1 + v}{\lambda}\right), \tag{2.1.9}$$

We first show that T is maximal monotone and coercive. To see that T is maximal monotone $V \to V'$ it is enough to verify that T is monotone and hemicontinuous (see [5] for instance). To check the monotonicity of T, let $u = (w, \psi, \phi),\ \tilde{u} = (\tilde{w}, \tilde{\psi}, \tilde{\phi}) \in V$. Then by straightforward calculations, we have

$$\begin{aligned}\langle T(u) - T(\tilde{u}), u - \tilde{u}\rangle = \lambda\langle u - \tilde{u}, u - \tilde{u}\rangle + \frac{1}{\lambda}\langle B(u - \tilde{u}), u - \tilde{u}\rangle \\ -\left\langle \mathscr{F}\left(\frac{u + v}{\lambda}\right) - \mathscr{F}\left(\frac{\tilde{u} + v}{\lambda}\right), u - \tilde{u}\right\rangle.\end{aligned}$$

Applying Proposition A.0.4 (see the Appendix) yields

$$\langle T(u) - T(\tilde{u}), u - \tilde{u}\rangle \geq \lambda \,\|u - \tilde{u})\|_2^2 + \frac{1}{\lambda}\,\|u - \tilde{u}\|_V^2 - \left(\mathscr{F}\left(\frac{u + v}{\lambda}\right) - \mathscr{F}\left(\frac{\tilde{u} + v}{\lambda}\right), u - \tilde{u}\right)_\Omega.$$

Since each f_i is globally Lipschitz continuous from V to $L^2(\Omega)$, we have

$$\begin{aligned}\langle T(u) - T(\tilde{u}), u - \tilde{u}\rangle &\geq \lambda \,\|u - \tilde{u}\|_2^2 + \frac{1}{\lambda}\,\|u - \tilde{u}\|_V^2 - \frac{3L}{2}\left\|\frac{u - \tilde{u}}{\lambda}\right\|_V^2 - \frac{L}{2}\,\|u - \tilde{u}\|_2^2 \\ &\geq \left(\lambda - \frac{L}{2}\right)\|u - \tilde{u}\|_2^2 + \left(\frac{1}{\lambda} - \frac{3L}{2\lambda^2}\right)\|u - \tilde{u}\|_V^2 \\ &\geq \frac{L}{2\lambda^2}\,\|u - \tilde{u}\|_V^2\,; \end{aligned} \tag{2.1.10}$$

provided $\lambda > 2L$. Thus, T is strongly monotone, which also implies that T is coercive.

Next, to check that T is hemicontinuous we need to prove that: $\operatorname{w-lim}_{\eta\to 0} T(u+\eta u_1) = T(u)$

for every $u, u_1 \in V$. Let $\tilde{u} \in V$, and $\lambda > 2L$, then

$$\langle T(u+\eta u_1), \tilde{u}\rangle - \langle T(u), \tilde{u}\rangle = \langle \lambda(u+\eta u_1), \tilde{u}\rangle - \langle \lambda u, \tilde{u}\rangle$$
$$+ \langle \mathscr{B}(u+\eta u_1), \tilde{u}\rangle - \langle \mathscr{B}(u), \tilde{u}\rangle - (\langle \mathscr{F}(u+\eta u_1), \tilde{u}\rangle - \langle \mathscr{F}(u), \tilde{u}\rangle). \quad (2.1.11)$$

We estimate the right-hand side of (2.1.11) as follows. Indeed, we have

$$|\langle \lambda(u+\eta u_1), \tilde{u}\rangle - \langle \lambda u, \tilde{u}\rangle| = |\lambda\eta|\,|\langle u_1, \tilde{u}\rangle| \le |\lambda\eta|\,\|u\|_V\,\|\tilde{u}\|_V \to 0, \quad (2.1.12)$$

as $\eta \to 0$. Also, since each f_i is globally Lipschitz continuous from V to $L^2(\Omega)$, one has

$$\begin{aligned} |\langle \mathscr{F}(u+\eta u_1), \tilde{u}\rangle - \langle \mathscr{F}(u), \tilde{u}\rangle| &= \left|(\mathscr{F}(u+\eta u_1) - \mathscr{F}(u), \tilde{u})_\Omega\right| \\ &\le \|\mathscr{F}(u+\eta u_1) - \mathscr{F}(u)\|_2\,\|\tilde{u}\|_2 \le L\,|\eta|\,\|u_1\|_V\,\|\tilde{u}\|_2 \\ &\to 0, \end{aligned} \quad (2.1.13)$$

as $\eta \to 0$. By using Proposition A.0.4 in the Appendix, we have

$$|\langle \mathscr{B}(u+\eta u_1), \tilde{u}\rangle - \langle \mathscr{B}(u), \tilde{u}\rangle| = |\eta|\,|(u_1, \tilde{u})_V| \quad \to \quad 0, \quad (2.1.14)$$

as $\eta \to 0$. Therefore, combining (2.1.11)-(2.1.14) yields

$$|\langle T(u+\eta u_1), \tilde{u}\rangle - \langle T(u), \tilde{u}\rangle| \to 0, \quad (2.1.15)$$

as $\eta \to 0$. It follows that $T : V \to V'$ is hemicontinuous and along with the strong monotonicity and coercivity of T, we conclude by Theorem 1.3 (p.45) in [5] that T is maximal monotone.

Next, we show that $\mathscr{G}$ is maximal monotone. Note here, since each g_i, $i = 1, 2, 3$ is polynomially bounded, then by the Sobolev embedding (in 2D) $H_0^1(\Omega) \hookrightarrow L^s(\Omega)$ for all $1 \leq s < \infty$, it follows that $\mathcal{D}(\mathscr{G}) = V$. We define the functional $\Phi_i : H_0^1(\Omega) \to [0, \infty]$ by:

$$\Phi_i(z) = \int_\Omega \varphi_i(z(x))d\mathbf{x}, \quad i = 1, 2, 3,$$

where $\varphi_i : \mathbb{R} \to [0, \infty)$ is the convex function defined by:

$$\varphi_i(s) = \int_0^s g_i(\tau)d\tau, \quad i = 1, 2, 3.$$

Clearly each Φ_i is proper, convex, and lower semi-continuous. Moreover, by Corollary 2.3 in [6] we know that $\partial\Phi_i : H_0^1(\Omega) \to H^{-1}(\Omega)$ satisfies

$$\partial\Phi_i(z) = \left\{u \in H^{-1}(\Omega) \cap L^1(\Omega) : u = g_i(z) \quad \text{a.e. in} \quad \Omega\right\}, \quad i = 1, 2, 3. \tag{2.1.16}$$

It is clear that $\mathcal{D}(\partial\Phi_i) = H_0^1(\Omega)$, by the Sobolev embeddings (in 2D) . In addition, we know that for all $z \in H_0^1(\Omega)$, $\partial\Phi_i(z)$ is a singleton such that $\partial\Phi_i(z) = \{g_i(z)\}$. Since any sub-differential is maximal monotone, by Theorem 2.1 (p.62) in [5], we conclude that each $g_i(\cdot) : H_0^1(\Omega) \to H^{-1}(\Omega)$ is maximal monotone. Therefore, by Proposition 2.6.1 in [23], it follows that $\mathscr{G} : V \to V'$ is maximal monotone. Since T and $\mathscr{G}$ are both maximal monotone and $\mathcal{D}(\mathscr{G}) = V = \mathcal{D}(T)$ by Theorem 1.5 in (p.54) [5], we infer that $S = T + \mathscr{G}$ is maximal monotone.

Finally, since $\mathscr{G}$ is monotone and $\mathscr{G}(0) = 0$, it follows that $\langle \mathscr{G}(u), u \rangle \geq 0$ for all $u \in \mathcal{D}(S) = V$, and with T the operator $S = T + \mathscr{G}$ must be coercive as well. Then the surjectivity of S follows immediately by [5, Corollary 1.2 (p.45)]. Thus, given any $\tilde{V} = (v, v_1) \in H$, there exists $u_1 \in D(S) = V$ that satisfies equation (2.1.8). Hence, $u = \frac{u_1+v}{\lambda} \in V$. In addition, one can easily see that $(u, u_1) \in \mathcal{D}(\mathscr{A})$ since $\mathscr{B}(u) + \mathscr{G}(u_1) - \mathscr{F}(u) \in (L^2(\Omega))^3$. Thus $\mathscr{A} + \lambda I : \mathcal{D}(\mathscr{A}) \subset H \to H$ is surjective, completing the proof of Proposition 2.1.1. □

2.1.3 Locally Lipschitz sources

In this subsection, we loosen the restrictions on sources and allow each f_i, $i = 1, 2, 3$ to be locally Lipschitz continuous from $(H_0^1(\Omega))^3$ to $L^2(\Omega)$. The following lemma is needed.

Lemma 2.1.2. *Assume* $p, m, r, q \geq 1$, *and* $p \cdot \max\{\frac{m+1}{m}, \frac{r+1}{r}, \frac{q+1}{q}\} \leq \frac{2}{\epsilon}$ *for some* $\epsilon > 0$. *Further suppose that* f_j, $j = 1, 2, 3$ *are in* $C^1(\mathbb{R}^3)$ *and satisfy*

$$\nabla f_j(w, \psi, \phi) \leq C(|w|^{p-1} + |\psi|^{p-1} + |\phi|^{p-1} + 1), \tag{2.1.17}$$

for all $w, \psi, \phi \in \mathbb{R}$. *Then:*

- $f_j : (H^{1-\epsilon}(\Omega))^3 \to L^{\sigma_j}(\Omega)$ *are locally Lipschitz mappings for*

$$\sigma_1 = \frac{m+1}{m}, \sigma_2 = \frac{r+1}{r} \text{ and } \sigma_3 = \frac{q+1}{q}.$$

- $f_j : (H_0^1(\Omega))^3 \to L^2(\Omega)$ *are locally Lipschitz.*

Proof. It suffices to prove these statements first statement for f_1. In particular, for the first statement we will show that $f_1 : (H^{1-\epsilon}(\Omega))^3 \to L^{\tilde{m}}(\Omega)$ is locally Lipschitz continuous, where $\tilde{m} = (m+1)m^{-1}$. Let $u = (w, \psi, \phi)$, $\hat{u} = (\hat{w}, \hat{\psi}, \hat{\phi})$, and u, $\hat{u} \in \tilde{V} := (H^{1-\epsilon}(\Omega))^3$ such that $\|u\|_{\tilde{V}}, \|\hat{u}\|_{\tilde{V}} \leq R$, where $R > 0$. By (2.1.17)) and the Mean Value Theorem, we have

$$|f_1(u) - f_1(\hat{u})| \leq C|u - \hat{u}| \cdot (|w|^{p-1} + |\hat{w}|^{p-1} + |\psi|^{p-1} + |\hat{\psi}|^{p-1} + |\phi|^{p-1} + |\hat{\phi}|^{p-1} + 1). \tag{2.1.18}$$

Therefore,

$$\|f_1(u) - f_1(\hat{u})\|_{\tilde{m}}^{\tilde{m}} = \int_\Omega |f_1(u) - f_1(\hat{u})|^{\tilde{m}} d\mathbf{x}$$
$$\leq C \int_\Omega (|w - \hat{w}|^{\tilde{m}} + |\psi - \hat{\psi}|^{\tilde{m}} + |\phi - \hat{\phi}|^{\tilde{m}})$$
$$(|w|^{(p-1)\tilde{m}} + |\hat{w}|^{(p-1)\tilde{m}} + |\psi|^{(p-1)\tilde{m}} + |\hat{\psi}|^{(p-1)\tilde{m}} + |\phi|^{(p-1)\tilde{m}} + |\hat{\phi}|^{(p-1)\tilde{m}} + 1) d\mathbf{x}. \quad (2.1.19)$$

All terms in (2.1.19) are estimated in the same manner: use Hölder's inequality and the Sobolev embedding (in 2D) $H^{1-\epsilon}(\Omega) \hookrightarrow L^{2/\epsilon}(\Omega)$ along with the assumption $p\tilde{m} \leq \frac{2}{\epsilon}$ and the estimate $\|w\|_{H^{1-\epsilon}(\Omega)} \leq \|u\|_{\tilde{V}} \leq R$. For instance,

$$\int_\Omega |w - \hat{w}|^{\tilde{m}} |w|^{(p-1)\tilde{m}} d\mathbf{x} \leq \Big(\int_\Omega |w - \hat{w}|^{p\tilde{m}} d\mathbf{x} \Big)^{\frac{1}{p}} \Big(\int_\Omega |w|^{p\tilde{m}} d\mathbf{x} \Big)^{\frac{p-1}{p}}$$
$$\leq C \|w - \hat{w}\|_{H^{1-\epsilon}(\Omega)}^{\tilde{m}} \|w\|_{H^{1-\epsilon}(\Omega)}^{(p-1)\tilde{m}} \leq C R^{(p-1)\tilde{m}} \|w - \hat{w}\|_{H^{1-\epsilon}(\Omega)}^{\tilde{m}}$$
$$\leq C R^{(p-1)\tilde{m}} \|u - \hat{u}\|_{\tilde{V}}^{\tilde{m}}. \quad (2.1.20)$$

Therefore, we conclude

$$\|f_1(u) - f_1(\hat{u})\|_{\tilde{m}} \leq C(R, p, m) \|u - \hat{u}\|_{\tilde{V}}, \quad (2.1.21)$$

completing the proof of the first statement of the Lemma.

In order to prove the second statement again let $u = (w, \psi, \phi)$, $\hat{u} = (\hat{w}, \hat{\psi}, \hat{\phi})$ and $u, \hat{u} \in V = (H_0^1(\Omega))^3$ such that $\|u\|_{1,\Omega}, \|\hat{u}\|_{1,\Omega} \leq R/2$, some $R > 0$. It follows from Proposition

A.0.2 in the Appendix that $\|u\|_V, \|\hat{u}\|_V \le R$. By (2.1.18), we have

$$\|f_1(u) - f_1(\hat{u})\|_2^2 = \int_\Omega |f_1(u) - f_1(\hat{u})|^2 d\mathbf{x}$$
$$\le C \int_\Omega (|w - \hat{w}|^2 + |\psi - \hat{\psi}|^2 + |\phi - \hat{\phi}|^2)$$
$$(|w|^{2(p-1)} + |\hat{w}|^{2(p-1)} + |\psi|^{2(p-1)} + |\hat{\psi}|^{2(p-1)} + |\phi|^{2(p-1)} + |\hat{\phi}|^{2(p-1)} + 1) d\mathbf{x}. \quad (2.1.22)$$

By Hölder's inequality and Sobolev embedding (in 2D) $H_0^1(\Omega) \hookrightarrow L^s(\Omega)$ for all $1 \le s < \infty$, along with the assumptions $1 \le p$ and $\|w\|_{1,\Omega} \le \|u\|_{1,\Omega} \le R/2$ we obtain

$$\int_\Omega |w - \hat{w}|^2 |w|^{2(p-1)} d\mathbf{x} \le \Big(\int_\Omega |w - \hat{w}|^{2p} d\mathbf{x} \Big)^{\frac{1}{p}} \Big(\int_\Omega |w|^{2p} d\mathbf{x} \Big)^{\frac{p-1}{p}}$$
$$\le C \|w - \hat{w}\|_{1,\Omega}^2 \|w\|_{1,\Omega}^{2(p-1)} \le C R^{2(p-1)} \|w - \hat{w}\|_{1,\Omega}^2. \quad (2.1.23)$$

Hence, $\|f_1(u) - f_1(\hat{u})\|_2^2 \le C(R, p) \|u - \hat{u}\|_{1,\Omega}^2$. By Proposition A.0.2 (in the Appendix), we conclude

$$\|f_1(u) - f_1(\hat{u})\|_2 \le C(R, p) \|u - \hat{u}\|_V.$$

completing the proof of the second statement of the lemma. □

Lemma 2.1.3. *Under Assumption 1.5.1, system* (2.1.1) *has a unique local strong solution* $U \in W^{1,\infty}(0, T_0; H)$ *for some* $T_0 > 0$; *provided the initial datum* $U(0) \in \mathcal{D}(A)$.

Proof. As in [8, 11], we use standard truncation of the sources. Recall that $V = (H_0^1(\Omega))^3$. Let $u = (w, \psi, \phi) \in V$ and define:

$$f_i^K(u) = \begin{cases} f_i(u), & \text{if } \|u\|_V \le K, \\ f_i\Big(\frac{Ku}{\|u\|_V}\Big), & \text{if } \|u\|_V > K, \end{cases} \quad (2.1.24)$$

where $i = 1, 2, 3$ and K is a positive constant such that $K^2 \ge 4E(0) + 1$, where the energy

$E(t)$ is given by $E(t) = \frac{1}{2}\left(\|u\|_V^2 + \|u_t\|_2^2\right)$.

For truncation parameter K consider the corresponding "K-problem": sources above, we consider the following K problem:

$$\begin{cases} w_{tt} - \Delta w - (\psi_x + \phi_y) + g_1(w_t) &= f_1^K(u) \quad \text{in } \Omega \times (0,\infty) \\ \psi_{tt} - (\psi_{xx} + \frac{1-\mu}{2}\psi_{yy}) - \frac{1+\mu}{2}\phi_{xy} + (\psi + w_x) + g_2(\psi_t) &= f_2^K(u) \quad \text{in } \Omega \times (0,\infty) \\ \phi_{tt} - (\frac{1-\mu}{2}\phi_{xx} + \phi_{yy}) - \frac{1+\mu}{2}\psi_{xy} + (\phi + w_y) + g_3(\phi_t) &= f_3^K(u) \quad \text{in } \Omega \times (0,\infty) \\ U(0) = \left(w(0), \psi(0), \phi(0), w_t(0), \psi_t(0), \phi_t(0)\right) &\in H, \end{cases} \tag{K}$$

where $H = (H_0^1(\Omega))^3 \times (L^2(\Omega))^3$. We note here that for each such K, the operators f_1^K, f_2^K, and f_3^K are globally Lipschitz continuous from V to $L^2(\Omega)$ (see [11]). Therefore, by Proposition 2.1.1 the (K) problem has a unique global strong solution $U_K = (u_K, u_K') \in W^{1,\infty}(0,T;H)$ for any $T > 0$ provided $U(0) \in \mathcal{D}(\mathscr{A})$.

To keep notation concise we will temporarily use $u(t)$ instead of $u_K(t)$ though dependence on K is understood. Since each g_i is polynomially bounded, then by the Sobolev embedding (in 2D) $H_0^1(\Omega) \hookrightarrow L^s(\Omega)$ for all $1 \leq s < \infty$, it follows from the definition of $\mathcal{D}(\mathscr{A})$ that $u_t \in V$ and hence $g_1(w_t), g_2(\psi_t), g_3(\phi_t) \in L^2(\Omega)$, then we may use the multiplier $(w_t, \psi_t \phi_t)$ on the (K) problem and obtain the following energy identity:

$$E(t) + \int_0^t \int_\Omega \mathscr{G}(u_t) \cdot u_t d\mathbf{x} d\tau = E(0) + \int_0^t \int_\Omega \mathscr{F}(u) \cdot u_t d\mathbf{x} d\tau. \tag{2.1.25}$$

In addition, since $m, r, q \geq 1$, we know $\tilde{m} = \frac{m+1}{m}$, $\tilde{r} = \frac{r+1}{r}$, $\tilde{q} = \frac{q+1}{q} \leq 2$. Hence, by our assumptions on the sources, it follows that $f_1 : V \to L^{\tilde{m}}(\Omega)$, $f_2 : V \to L^{\tilde{r}}(\Omega)$, and $f_3 : V \to L^{\tilde{q}}(\Omega)$, are Lipschitz on the ball $\{u \in V : \|u\|_V \leq K\}$ with some Lipschitz constants $L_{f_i}(K), i = 1, 2, 3$. Put

$$L_K = \max\{L_{f_i}(K), i = 1, 2, 3\}.$$

Using calculations as in [11, p. 1946], we deduce $f_1^K : V \to L^{\tilde{m}}(\Omega)$, $f_2^K : V \to L^{\tilde{r}}(\Omega)$, and $f_3^K : V \to L^{\tilde{q}}(\Omega)$ are globally Lipschitz continuous with Lipschitz constant L_K.

Next, estimate the source dependent terms in (2.1.25) using Hölder's and Young's inequalities. For any $\epsilon > 0$ we have

$$\begin{aligned}\int_0^t \int_\Omega f_1^K(u) w_t d\mathbf{x} d\tau &\leq \int_0^t \left\| f_1^K(u) \right\|_{\tilde{m}} \|w_t\|_{m+1} \, d\tau \leq C_\epsilon \int_0^t \left\| f_1^K(u) \right\|_{\tilde{m}}^{\tilde{m}} d\tau + \epsilon \int_0^t \|w_t\|_{m+1}^{m+1} \, d\tau \\ &\leq C_\epsilon \int_0^t \left\| f_1^K(u) - f_1^K(0) \right\|_{\tilde{m}}^{\tilde{m}} + \left\| f_1^K(0) \right\|_{\tilde{m}}^{\tilde{m}} d\tau + \epsilon \int_0^t \|w_t\|_{m+1}^{m+1} \, d\tau \\ &\leq C_\epsilon L_K^{\tilde{m}} \int_0^t \|u\|_V^{\tilde{m}} \, d\tau + C_\epsilon t |f_1^K(0)|^{\tilde{m}} |\Omega| + \epsilon \int_0^t \|w_t\|_{m+1}^{m+1} \, d\tau \\ &\text{(analogously for } f_1^K, w_t, \tilde{m} \text{ or } f_2^K, \psi_t, \tilde{r}, r \text{ or } f_3^K, \phi_t, \tilde{q}, q).\end{aligned} \tag{2.1.26}$$

By the assumptions on damping, it follows that, for all $s \in \mathbb{R}$,

$$g_1(s)s \geq \alpha(|s|^{m+1} - 1), \quad g_2(s)s \geq \alpha(|s|^{r+1} - 1), \quad g_3(s)s \geq \alpha(|s|^{q+1} - 1). \tag{2.1.27}$$

Therefore,

$$\begin{aligned}\int_0^t \int_\Omega g_1(w_t) w_t d\mathbf{x} d\tau \geq \alpha \int_0^t \|w_t\|_{m+1}^{m+1} \, d\tau - \alpha t |\Omega| \\ \text{(and similarly for } g_2, \psi_t, r \text{ and } g_3, \phi_t, q) \, .\end{aligned} \tag{2.1.28}$$

For convenience let $\mathbf{D}(t) := \|w_t(t)\|_{m+1}^{m+1} + \|\psi_t(t)\|_{r+1}^{r+1} + \|\phi_t(t)\|_{q+1}^{q+1}$. Using (2.1.26), (2.1.28) in the energy identity (2.1.25) gives

$$\begin{aligned}E(t) + \alpha \int_0^t \mathbf{D}(\tau) d\tau - 3\alpha|\Omega| t \leq E(0) + \epsilon \int_0^t \mathbf{D}(\tau) d\tau + C_\epsilon L_K^{\tilde{m}} \int_0^t \|u(\tau)\|_V^{\tilde{m}} \, d\tau \\ + C_\epsilon L_K^{\tilde{r}} \int_0^t \|u(\tau)\|_V^{\tilde{r}} \, d\tau + C_\epsilon L_K^{\tilde{q}} \int_0^t \|u(\tau)\|_V^{\tilde{q}} \, d\tau + \; t \, |\Omega| C_\epsilon \left(|f_1(0)|^{\tilde{m}} + |f_2(0)|^{\tilde{r}} + |f_3(0)|^{\tilde{q}} \right).\end{aligned} \tag{2.1.29}$$

If $\epsilon \leq \alpha$, then (2.1.29) implies

$$E(t) \leq E(0) + C_\epsilon L_K^{\tilde{m}} \int_0^t \|u\|_V^{\tilde{m}}\, d\tau + C_\epsilon L_K^{\tilde{r}} \int_0^t \|u\|_V^{\tilde{r}}\, d\tau + C_\epsilon L_K^{\tilde{q}} \int_0^t \|u\|_V^{\tilde{q}}\, d\tau$$
$$+ t|\Omega| C_\epsilon \left(|f_1(0)|^{\tilde{m}} + |f_2(0)|^{\tilde{r}} + |f_3(0)|^{\tilde{q}} \right) + 3\alpha t|\Omega|. \qquad (2.1.30)$$

Since $1 \leq \tilde{m}, \tilde{r}, \tilde{q} \leq 2$, then by Young's inequality

$$\int_0^t \|u(\tau)\|_V^\sigma\, d\tau \leq \int_0^t \left(\|u(\tau)\|_V^2 + \tilde{C} \right) d\tau \leq 2 \int_0^t E(\tau) d\tau + \tilde{C} t, \quad \sigma = \tilde{m}, \tilde{r}, \tilde{q},$$

where $\tilde{C}$ is positive constant that depends on m, r, and q. Therefore, if $t \leq T_0$ and we set $C_1 = 2C_\epsilon(L_K^{\tilde{m}} + L_K^{\tilde{r}} + L_K^{\tilde{q}})$, $C_2 = E(0) + C_0 T_0$ where $C_0 := |\Omega| C_\epsilon \left(|f_1(0)|^{\tilde{m}} + |f_2(0)|^{\tilde{r}} + |f_3(0)|^{\tilde{q}} \right) + 3\alpha|\Omega| + \tilde{C} C_\epsilon (L_K^{\tilde{m}} + L_K^{\tilde{r}} + L_K^{\tilde{q}})$, then it follows from (2.1.30) that

$$E(t) \leq C_2 + C_1 \int_0^t E(\tau) d\tau, \quad \text{for all} \quad t \in [0, T_0]. \qquad (2.1.31)$$

The value for T_0 will be chosen below. By Gronwall's inequality, one has

$$E(t) \leq C_2(1 + C_1 t e^{C_1 t}), \quad \text{for all} \quad t \in [0, T_0]. \qquad (2.1.32)$$

The constants C_0 and C_1 depend only on sources f_i, exponents m, r, q, $|\Omega|$ parameter K and ϵ, which in turn depends only on parameter α from (2.1.27). Hence we can select

$$T_0 = \min\left\{ \frac{1}{4C_0}, \frac{\alpha_1}{C_1} \right\}, \quad \text{where} \;\; \alpha_1 > 0 \text{ such that } \alpha_1 e^{\alpha_1} = 1\,. \qquad (2.1.33)$$

Then it follows from (2.1.32) that

$$E(t) \leq 2C_2 = 2E(0) + 2C_0 T_0 \leq 2E(0) + 1/2 \;\; \text{for} \;\; t \in [0, T_0]\,.$$

Recall our assumption $K^2 \geq 4E(0) + 1$. Thus

$$E(t) \leq 2E(0) + 1/2 \leq \frac{K^2}{2}, \text{ for all } t \in [0, T_0]. \tag{2.1.34}$$

This implies that $\|u(t)\|_V \leq K$, for all $t \in [0, T_0]$, and therefore, each $f_i^K(u) = f_i(u)$, $i = 1, 2, 3$ on the time interval $[0, T_0]$. By the uniqueness of solutions to the (K) problem, the solution to the truncated problem (K) coincides with the solution to the system (2.1) for $t \in [0, T_0]$, completing the proof of the Lemma 2.1.3. □

2.1.4 Weak initial data

In this subsection, we relax the restriction on initial datum from $\mathcal{D}(\mathscr{A})$ to $(H_0^1(\Omega))^3 \times (L^2(\Omega))^3$ and complete the proof of the local existence statement in Theorem 1.5.4 in the following four steps.

Step 1: Approximate system. Recall that $H = (H_0^1(\Omega))^3 \times (L^2(\Omega))^3$, so the space $(H_0^1(\Omega) \cap H^2(\Omega))^6 \subseteq \mathcal{D}(\mathscr{A})$, and hence $\mathcal{D}(\mathscr{A})$ is dense in H. Then, for each $U_0 = (u_0, u_1) \in H$, there exists a sequence of functions $U_0^n \in \mathcal{D}(\mathscr{A})$ such that $U_0^n \to U_0$ strongly in H. Put $U = (u, u_t) = (w, \psi, \phi, w_t, \psi_t, \phi_t)$ and consider the approximate system

$$U_t^n + \mathscr{A} U^n = 0 \text{ with } U^n(0) = U_0^n \in \mathcal{D}(\mathscr{A}). \tag{2.1.35}$$

Step 2: Approximate solutions. Since each f_i, $i = 1, 2, 3$ satisfies the assumptions of Lemma 2.1.3, then for each n, the approximate problem (2.1.35) has a strong local solution $U^n = (u^n, u_t^n) = (w^n, \psi^n, \phi^n, w_t^n, \psi_t^n, \phi_t^n) \in W^{1,\infty}(0, T_0; H)$ such that $U^n(t) \in \mathcal{D}(\mathscr{A})$ for all $t \in [0, T_0]$. Let $E^n(t)$ denote the energy E for the solution U^n.

We claim that parameter T_0 from (2.1.33) can be made independent of n. Specifically, T_0 depends on the choice of constant K from (2.1.24), which in turn only has to be large enough to dominate $\sqrt{4E(0)+1}$. Since $U_0^n \to U_0$ strongly in H, we can choose K sufficiently large depending on $E(0)$ such that $K \geq \sqrt{4E^n(0)+1}$ for all n.

Now by (2.1.34), we know $E^n(t) \leq K^2/2$ for all $t \in [0, T_0]$, which implies

$$\|U^n(t)\|_H^2 = \|u^n\|_V^2 + \left\|u_t^n\right\|_2^2 \leq K^2 \text{ for all } t \in [0, T_0]. \tag{2.1.36}$$

Recall our notation $\mathbf{D}_n(t) = \left\|w_t^n(t)\right\|_{m+1}^{m+1} + \left\|\psi_t^n(t)\right\|_{r+1}^{r+1} + \left\|\phi_t^n(t)\right\|_{q+1}^{q+1}$. Letting $0 < \epsilon < \frac{\alpha}{2}$ in (2.1.29), by the fact $\tilde{m}, \tilde{q}, \tilde{r} \leq 2$ and the bound (2.1.36), we deduce that

$$\int_0^{T_0} \mathbf{D}_n(t)dt < C(K), \text{ for some constant } C(K) > 0. \tag{2.1.37}$$

Therefore,

$$u_t^n \in L^{m+1}(\Omega \times (0, T_0)) \times L^{r+1}(\Omega \times (0, T_0)) \times L^{q+1}(\Omega \times (0, T_0)). \tag{2.1.38}$$

Recall that $U^n = (u^n, u_t^n) \in \mathcal{D}(\mathscr{A})$ is a strong solution of (2.1.35). If θ satisfies the conditions imposed on test functions in Definition 1.5.3, then we can test the approximate system (2.1.35) against θ to obtain for all $t \in [0, T_0]$:

$$\begin{aligned}(u_t^n(t), \theta(t))_\Omega - (u_t^n(0), \theta(0))_\Omega + \int_0^t \Big(-(u_t^n(\tau), \theta_t(\tau))_\Omega + (u^n(\tau), \theta(\tau))_V\Big) d\tau \\ + \int_0^t (\mathscr{G}(u_t^n(\tau)), \theta(\tau))_\Omega d\tau = \int_0^t (\mathscr{F}(u^n(\tau)), \theta(\tau))_\Omega d\tau\,. \end{aligned} \tag{2.1.39}$$

Step 3: Passage to the limit. We will prove that there exists a subsequence of $\{U^n\}$ that converges to a solution of the original problem (1.2.1). Before passing to the limit in (2.1.39), we need some preparation.

First, we note that (2.1.36) implies that $\{U^n\}$ is bounded in $L^\infty(0, T_0; H)$. Hence by Alaoglu's theorem, there exists a subsequence, reindexed again by n, such that

$$U^n \to U = (u, u_t) \quad \text{weakly}^* \text{ in } L^\infty(0, T_0; H). \tag{2.1.40}$$

Also, by (2.1.36), we know $\{u^n\}$ is bounded in $L^\infty(0, T_0; V)$, where $V = (H_0^1(\Omega))^3$. Thus, $\{u^n\}$ is bounded in $L^s(0, T_0; V)$ for any $s > 1$, in particular in $L^{\tilde{m}}(0, T_0; L^{\tilde{m}}(\Omega)) \times L^{\tilde{r}}(0, T_0; L^{\tilde{r}}(\Omega)) \times L^{\tilde{q}}(0, T_0; L^{\tilde{q}}(\Omega))$, where $\tilde{m} = \frac{m+1}{m}$, $\tilde{r} = \frac{r+1}{r}$, and $\tilde{q} = \frac{q+1}{q}$. In addition, for any $0 < \epsilon < 1$, the embedding $H_0^1(\Omega) \hookrightarrow H^{1-\epsilon}(\Omega)$ is compact, and likewise

$$(H^{1-\epsilon}(\Omega))^3 \hookrightarrow L^{\tilde{m}}(\Omega) \times L^{\tilde{r}}(\Omega) \times L^{\tilde{q}}(\Omega)\,.$$

For any $s > 1$ by Aubin's Compactness Theorem, there is a subsequence (that we again reindex by n)

$$u^n \to u \quad \text{strongly in} \quad L^s(0, T_0; (H^{1-\epsilon}(\Omega))^3). \tag{2.1.41}$$

Having identified a limit of u^n we will now show that $\{U^n\}$ forms a Cauchy sequence in $C([0, T]; H)$. Consider solutions to two approximate problems (2.1.35): U^n and U^j. For a shorthand put $\tilde{u} = u^n - u^j$. Since $U^n, U^j \in W^{1,\infty}(0, T_0; H)$ and $U^n, U^j \in \mathcal{D}(\mathscr{A})$, then $\tilde{u}_t \in W^{1,\infty}(0, T_0; (L^2(\Omega))^3)$ and $\tilde{u}_t \in V$. Moreover, (2.1.38) gives

$$\tilde{u}_t^n \in L^{m+1}(\Omega \times (0, T_0)) \times L^{r+1}(\Omega \times (0, T_0)) \times L^{q+1}(\Omega \times (0, T_0)). \tag{2.1.42}$$

Hence, we may consider the difference of the approximate problems corresponding to the parameters n and j, and then use the multiplier $\tilde{u}_t$ on equation (2.1.39). By performing integration by parts on equation (2.1.39), one has the following energy identity:

$$\frac{1}{2}(\|\tilde{u}(t)\|_V^2 + \|\tilde{u}_t(t)\|_2^2) + \int_0^t \int_\Omega \Big(\mathscr{G}(u_t^n(\tau)) - \mathscr{G}(u_t^j(\tau))\Big) \cdot \widetilde{u(\tau)}_t d\mathbf{x}d\tau$$
$$= \frac{1}{2}(\|\tilde{u}(0)\|_V^2 + \|\tilde{u}_t(0)\|_2^2) + \int_0^t \int_\Omega \Big(\mathscr{F}(u^n(\tau)) - \mathscr{F}(u^j(\tau))\Big) \cdot \widetilde{u(\tau)}_t d\mathbf{x}d\tau. \quad (2.1.43)$$

We will show that each term on the right-hand side of (2.1.43) vanishes as $n, j \to 0$. First, by assumptions on the initial conditions $\{u_0^n\}$ and $\{u_1^n\}$

$$\lim_{n,j\to\infty} \|\tilde{u}(0)\|_V = \lim_{n,j\to\infty} \|u_0^n - u_0^j\|_V = 0 \;\text{ and }\; \lim_{n,j\to\infty} \|\tilde{u}_t(0)\|_2 = \lim_{n,j\to\infty} \|u_1^n - u_1^j\|_2 = 0. \quad (2.1.44)$$

Next, consider the last term on the right-hand side of (2.1.43). We have

$$\Big|\int_0^t \int_\Omega (\mathscr{F}(u^n) - \mathscr{F}(u^j)) \cdot \tilde{u}_t d\mathbf{x}d\tau\Big|$$
$$\leq \int_0^t \int_\Omega \Big|f_1(u^n) - f_1(u^j)\Big| \, |\tilde{w}_t| d\mathbf{x}d\tau + \int_0^t \int_\Omega \Big|f_2(u^n) - f_2(u^j)\Big| \, |\tilde{\psi}_t| d\mathbf{x}d\tau \quad (2.1.45)$$
$$+ \int_0^t \int_\Omega \Big|f_3(u^n) - f_3(u^j)\Big| \, |\tilde{\phi}_t| d\mathbf{x}d\tau\,.$$

Each term on the right-hand side of (2.1.45) as follows:

$$\int_0^t \int_\Omega \Big|f_1(u^n) - f_1(u^j)\Big| \, |\tilde{w}_t| d\mathbf{x}d\tau \leq \int_0^t \int_\Omega |f_1(u^n) - f_1(u)| \, |\tilde{w}_t| d\mathbf{x}d\tau$$
$$+ \int_0^t \int_\Omega \Big|f_1(u) - f_1(u^j)\Big| \, |\tilde{w}_t| d\mathbf{x}d\tau. \quad (2.1.46)$$

Lemma 2.1.2 states that $f_1 : (H^{1-\epsilon}(\Omega))^3 \to L^{\tilde{m}}(\Omega)$ is locally Lipschitz continuous, hence, because solutions have energy bounded in terms of K we have

$$\begin{aligned}
&\int_0^t \int_\Omega |f_1(u^n) - f_1(u)|\, |\tilde{w}_t| d\mathbf{x} d\tau \\
&\qquad \leq \left(\int_0^t \int_\Omega |f_1(u^n) - f_1(u)|^{\tilde{m}} \, d\mathbf{x} d\tau \right)^{\frac{m}{m+1}} \left(\int_0^t \int_\Omega |\tilde{w}_t|^{m+1} \, d\mathbf{x} d\tau \right)^{\frac{1}{m+1}} \\
&\qquad \leq C(K) \left(\int_0^{T_0} \|u^n - u\|^{\tilde{m}}_{(H^{1-\epsilon}(\Omega))^3} d\tau \right)^{\frac{m}{m+1}} \to 0 \ \text{ uniformly on } \ [0, T_0],
\end{aligned}$$

as $n \to \infty$, where we have used the convergence (2.1.41) and the uniform bound in (2.1.37). Analogous estimate holds if we replace n with j. Consequently, from (2.1.46) we obtain,

$$\int_0^t \int_\Omega \left| f_1(u^n) - f_1(u^j) \right| |\tilde{w}_t| d\mathbf{x} d\tau \ \to 0 \ \text{ as } \ n \to \infty.$$

Similarly, the last two terms on the right-hand side of (2.1.45) also converge to zero as $n \to \infty$. Hence, we obtain

$$\left| \int_0^t \int_\Omega \left(\mathscr{F}(u^n) - \mathscr{F}(u^j) \right) \cdot \tilde{u}_t d\mathbf{x} d\tau \right| \ \to 0 \ \text{ as } \ n \to \infty. \tag{2.1.47}$$

Now, by using the fact that g_1, g_2 and g_3 are monotone increasing and using (2.1.44) and (2.1.47), we can take limit as $n, j \to \infty$ in (2.1.43) to deduce

$$\begin{aligned}
&\lim_{n,j\to\infty} \|\tilde{u}(t)\|_V = \lim_{n,j\to\infty} \|u^n(t) - u^j(t)\|_V = 0 \ \text{ uniformly on } \ [0, T_0], \\
&\lim_{n,j\to\infty} \|\tilde{u}_t(t)\|_2 = \lim_{n,j\to\infty} \|u^n(t) - u^j(t)\|_2 = 0 \ \text{ uniformly on } \ [0, T_0],
\end{aligned} \tag{2.1.48}$$

and

$$\lim_{n,j\to\infty} \int_0^t \int_\Omega (g(u_t^n(t) - g(u_t^j(t))) \cdot (u_t^n(t) - u_t^j(t)) d\mathbf{x} d\tau = 0 \ \text{ uniformly on } \ [0, T_0]. \tag{2.1.49}$$

Therefore,

$$U^n \to U \ \text{ in } \ H \ \text{ uniformly on } \ [0, T_0]. \tag{2.1.50}$$

Since $U^n \in W^{1,\infty}([0, T_0]; H)$, by (2.1.50), we conclude that

$$U = (u, u_t) \in C([0, T_0], H). \tag{2.1.51}$$

It remains to prove that u, u_t satisfy (1.5.2) as stated in Definition 1.5.3, i.e., we focus on passing to the limit in (2.1.39).

Since $\theta \in C([0, t]; V)$ and $\theta_t \in C([0, t]; (L^2(\Omega))^3)$ then by (2.1.40), (2.1.50) and $U_0^n \to U_0$ strongly in H, we can pass to the limit on the first line of (2.1.39) and get

$$\begin{aligned}
\lim_{n\to\infty} (u_t^n(t), \theta(t))_\Omega = (u_t(t), \theta(t))_\Omega, \quad \lim_{n\to\infty} (u_t^n(0), \theta(0))_\Omega = (u_t(0), \theta(0))_\Omega \\
\lim_{n\to\infty} \int_0^t (u^n(\tau), \theta(\tau))_V \, d\tau = \int_0^t (u(\tau), \theta(\tau))_V, \\
\lim_{n\to\infty} \int_0^t (u_t^n(\tau), \theta_t(\tau))_\Omega \, d\tau = \int_0^t (u_t(\tau), \theta_t(\tau))_\Omega.
\end{aligned} \tag{2.1.52}$$

Since $|g_1(s)| \leq \beta(|s|^m + 1)$ then

$$\int_0^{T_0} \int_\Omega |g_1(w_t^n)|^{\tilde{m}} d\mathbf{x}dt \leq C\beta^{\tilde{m}} \int_0^{T_0} \int_\Omega (|w_t^n|^{m+1} + 1) d\mathbf{x}dt < C(K). \tag{2.1.53}$$

It follows that on a subsequence

$$g_1(w_t^n) \to g_1^* \ \text{ weakly in } \ L^{\tilde{m}}(\Omega \times (0, t)), \tag{2.1.54}$$

for some $g_1^* \in L^{\tilde{m}}(\Omega \times (0, t))$. In addition, from (2.1.37), on a (reindexed) subsequence, we have $w_t^n \to w_t$ weakly in $L^{m+1}(\Omega \times (0, T_0))$. Note that it has to be the same limit as follows from (2.1.40). Therefore [5, Lemma 1.3, p.49] along with (2.1.49) and (2.1.54) asserts that $g_1^* = g_1(w_t)$; provided we show that

$$g_1 : L^{m+1}(\Omega \times (0,t)) \to L^{\tilde{m}}(\Omega \times (0,t))$$

is maximal monotone. Indeed, since g_1 is monotone increasing it is easy to see g_1 induces a monotone operator. Thus, we need to verify that g_1 is hemi-continuous, i.e. $\text{w-}\lim_{\lambda\to 0} g_1(w_1 + \lambda\tilde{w}) = g_1(w_1)$ or, specifically:

$$\lim_{\lambda\to 0}\int_0^t\int_\Omega g_1(w_1+\lambda\tilde{w})\hat{w}d\mathbf{x}d\tau = \int_0^t g_1(w_1)\hat{w}d\mathbf{x}d\tau, \tag{2.1.55}$$

for all $w_1, \tilde{w}, \hat{w} \in L^{m+1}(\Omega\times(0,t))$. By continuity $g_1(w_1+\lambda\tilde{w})\hat{w} \to g_1(w_1)\hat{w}$ pointwise as $\lambda \to 0$. Moreover, since $|g_1(s)| \leq \beta(|s|^m + 1)$ for all $s \in \mathbb{R}$, we know if $|\lambda| \leq 1$, then

$$|g_1(w_1+\lambda\tilde{w})\hat{w}| \leq \beta(|w_1+\lambda\tilde{w}|^m+1)|\hat{w}| \leq C(|w_1|^m|\hat{w}| + |\tilde{w}|^m\hat{w}| + |\hat{w}|) \in L^1(\Omega\times(0,t)),$$

by Hölder's inequality. Thus (2.1.55) follows form Dominated Convergence Theorem. Hence, g_1 defines a maximal monotone operator from $L^{m+1}(\Omega\times(0,t))$ to $L^{\tilde{m}}(\Omega\times(0,t))$ and we conclude $g_1^* = g_1(w_t)$, i.e.,

$$g_1(w_t^n) \to g_1(w_t) \quad \text{weakly in} \quad L^{\tilde{m}}(\Omega\times(0,t)) \ \text{ and } \ w_t \in L^{m+1}(\Omega\times(0,t)). \tag{2.1.56}$$

Similarly, we have

$$g_2(\psi_t^n) \to g_2(\psi_t) \quad \text{weakly in} \quad L^{\tilde{r}}(\Omega\times(0,t)) \ \text{ and } \ \psi_t \in L^{r+1}(\Omega\times(0,t)) \tag{2.1.57}$$

$$g_3(\phi_t^n) \to g_3(\phi_t) \ \text{ weakly in } \ L^{\tilde{q}}(\Omega\times(0,t)) \ \text{ and } \ \phi_t \in L^{q+1}(\Omega\times(0,t)). \tag{2.1.58}$$

Hence, it follows that

$$\lim_{n\to\infty}\int_0^t\int_\Omega \mathscr{G}(u_t^n)\cdot\theta d\mathbf{x}d\tau = \int_0^t\int_\Omega \mathscr{G}(u_t)\cdot\theta d\mathbf{x}d\tau. \tag{2.1.59}$$

The fact that

$$\lim_{n\to\infty}\int_0^t\int_\Omega \mathscr{F}(u^n)\cdot\theta d\mathbf{x}d\tau = \int_0^t\int_\Omega \mathscr{F}(u)\cdot\theta d\mathbf{x}d\tau. \tag{2.1.60}$$

Follows by replacing $\tilde{w}_t$ with θ_1, and similarly repeating for $f_{2,3}$

Finally, using (2.1.52), (2.1.59) and (2.1.60) we can pass to the limit in (2.1.39) to obtain (1.5.2). In addition, by (2.1.51) and (2.1.56)–(2.1.58), U satisfies the regularity as stated in Definition 1.5.3, completing the proof.

2.2 Energy identity

In this section we verify the energy identity (1.5.4) of Theorem 1.5.4 for weak solutions. Formally the argument follows if we test (1.2.1) with (w_t, ψ_t, ϕ_t). However, the calculus is not justified in this procedure since (w_t, ψ_t, ϕ_t) are not regular enough to be used as test functions in (1.5.2). In order to overcome this difficulty, we shall use the difference quotients of the solution in time.

2.2.1 Properties of the difference quotient

Let X be a Banach space, for any function $u \in C([0,T];X)$ and $h > 0$, we define the *symmetric difference quotient* by

$$D_{T,h}u(t) = \frac{\operatorname{ext}_T u(t+h) - \operatorname{ext}_T u(t-h)}{2h}, \tag{2.2.1}$$

where $\operatorname{ext}_T u(t)$ denotes the extension of $u(t)$ to $\mathbb{R}$ given by:

$$\operatorname{ext}_T u(t) = \begin{cases} u(0) & \text{for } t \leq 0, \\ u(t) & \text{for } t \in (0, T), \\ u(T) & \text{for } t \geq T. \end{cases} \tag{2.2.2}$$

The following proposition was established by Koch and Lasiecka in [26].

Proposition 2.2.1 ([26])**.** *Let $u \in C([0, T]; X)$ where X is a Hilbert space with inner product $(\cdot, \cdot)_X$ and norm $\|\cdot\|_X$. Then*

$$\lim_{h \to 0} \int_0^T (u(t), D_{T,h}u(t))_X dt = \frac{1}{2}\left(\|u(T)\|_X^2 - \|u(0)\|_X^2\right). \tag{2.2.3}$$

If, in addition, $u_t \in C([0, T]; X)$, then

$$\int_0^T (u_t(t), (D_{T,h}u(t))_t)_X \, dt = 0, \quad \text{for each} \quad h > 0, \tag{2.2.4}$$

and, as $h \to 0$,

$$D_{T,h}u(t) \to u_t(t) \text{ weakly in } X, \text{ for every } t \in (0, T), \tag{2.2.5}$$

$$D_{T,h}u(0) \to \frac{1}{2}u_t(0) \quad \text{and} \quad D_{T,h}u(T) \to \frac{1}{2}u_t(T) \text{ weakly in } X. \tag{2.2.6}$$

The following proposition is essential for the proof of the energy identity (1.5.4).

Proposition 2.2.2 ([23])**.** *Let X and Y be Banach spaces. Assume $u \in C([0, T]; Y)$ and $u_t \in L^1([0, T]; Y) \cap L^p([0, T]; X)$, where $1 \leq p < \infty$. Then $D_{T,h}u \in L^p([0, T]; X)$ and $\|D_{T,h}u\|_{L^p([0,T];X)} \leq \|u_t\|_{L^p([0,T];X)}$. Moreover, $D_{T,h}u \to u_t$ in $L^p([0, T]; X)$, as $h \to 0$.*

2.2.2 Proof of the energy identity

Throughout the proof, we fix $t \in [0, T_0]$ and let $u = (w, \psi, \phi)$ be a weak solution of system (1.2.1) in the sense of Definition 1.5.3. Recall the regularity of w, ψ, and ϕ, in particular, $w \in C([0,t], H_0^1(\Omega))$, $w_t \in C([0,t], L^2(\Omega))$ and $w_t \in L^{m+1}(\Omega \times (0,t)) = L^{m+1}(0,t; L^{m+1}(\Omega))$. We can define the difference quotient $D_{t,h}w(\tau)$ on $[0,t]$ as (2.2.1), i.e.,

$$D_{t,h}w(\tau) = \frac{1}{2h}[\operatorname{ext}_t w(\tau + h) - \operatorname{ext}_t w(\tau - h)],$$

where $\operatorname{ext}_t w(\tau)$ extends $w(\tau)$ from $[0,t]$ to $\mathbb{R}$ as in (2.2.2). By Proposition 2.2.2 with $X = L^{m+1}(\Omega)$, $Y = L^2(\Omega)$ and $p = m+1$, we have

$$D_{t,h}w \in L^{m+1}(\Omega \times (0,t)) \text{ and } D_{t,h}w \to w_t \text{ in } L^{m+1}(\Omega \times (0,t)). \tag{2.2.7}$$

Similar arguments yield:

$$D_{t,h}\psi \in L^{r+1}(\Omega \times (0,t)) \text{ and } D_{t,h}\psi \to \psi_t \text{ in } L^{r+1}(\Omega \times (0,t)), \tag{2.2.8}$$

$$D_{t,h}\phi \in L^{q+1}(\Omega \times (0,t)) \text{ and } D_{t,h}\phi \to \phi_t \text{ in } L^{q+1}(\Omega \times (0,t)). \tag{2.2.9}$$

Moreover, since $u \in C([0,t], V)$ then $D_{t,h}u \in C([0,t], V)$, where $V = (H_0^1(\Omega))^3$.

We now show that $(D_{t,h}u)_t \in L^1(0,t; (L^2(\Omega))^3)$. Indeed, for $0 < h < \frac{t}{2}$, we note that

$$(D_{t,h}u)_t(\tau) = \begin{cases} \frac{1}{2h}u_t(\tau + h) & 0 < \tau < h, \\ \frac{1}{2h}[u_t(\tau + h) - u_t(\tau - h)] & h < \tau < t - h, \\ -\frac{1}{2h}u_t(\tau - h), & t - h < \tau < t, \end{cases} \tag{2.2.10}$$

and since $u_t \in C([0,t];(L^2(\Omega))^3)$, we conclude

$$(D_{t,h}u)_t \in L^1(0,t;(L^2(\Omega))^3). \tag{2.2.11}$$

Thus (2.2.7)–(2.2.11) show that $D_{t,h}u$ possesses the regularity suitable for test functions in Definition 1.5.3. Therefore, by taking $\theta = D_{t,h}u$ in (1.5.2), we obtain

$$\begin{aligned}
&(u_t(t), D_{t,h}u(t))_\Omega - (u_t(0), D_{t,h}u(0))_\Omega \\
&+ \int_0^t [-(u_t(\tau),(D_{t,h}u)_t(\tau))_\Omega + (u(\tau), D_{t,h}u(\tau))_V]d\tau + \int_0^t (\mathscr{G}(u_t(\tau)), D_{t,h}u(\tau))_\Omega d\tau \\
&= \int_0^t (\mathscr{F}(u(\tau)), D_{t,h}u(\tau))_\Omega d\tau.
\end{aligned} \tag{2.2.12}$$

Now it remains to let $h \to 0$ in (2.2.12). Since $u, u_t \in C([0,t];(L^2(\Omega))^3)$ then (2.2.6) shows

$$D_{t,h}u(0) \to \tfrac{1}{2}u_t(0) \;\text{ and }\; D_{t,h}u(t) \to \tfrac{1}{2}u_t(t) \quad \text{weakly in} \quad (L^2(\Omega))^3.$$

Consequently

$$\lim_{h\to 0}(u_t(t), D_{t,h}u(t))_\Omega - \lim_{h\to 0}(u_t(0), D_{t,h}u(0))_\Omega = \frac{1}{2}\|u_t(t)\|_2^2 - \frac{1}{2}\|u_t(0)\|_2^2. \tag{2.2.13}$$

Also, by (2.2.4)

$$\int_0^t (u_t(\tau),(D_{t,h}u)_t(\tau))_\Omega d\tau = 0 \;\text{ for each }\; h > 0. \tag{2.2.14}$$

In addition, since $u \in C([0,t];V)$ then (2.2.3) yields

$$\lim_{h\to 0}\int_0^t (u(\tau), D_{t,h}u(\tau))_V d\tau = \frac{1}{2}\|u(t)\|_V^2 - \frac{1}{2}\|u(0)\|_V^2. \tag{2.2.15}$$

By (2.1.56)-(2.1.58), it is clear that $\mathscr{G}(u_t) \in L^{\tilde{m}}(\Omega\times(0,t)) \times L^{\tilde{r}}(\Omega\times(0,t)) \times L^{\tilde{q}}(\Omega\times(0,t))$, where $\tilde{m} = \frac{m+1}{m}$, $\tilde{r} = \frac{r+1}{r}$ and $\tilde{q} = \frac{q+1}{q}$. Hence, by (2.2.7)-(2.2.9)

$$\lim_{h\to 0}\int_0^t (\mathscr{G}(u_t(\tau)), D_{t,h}u(\tau))_\Omega d\tau = \int_0^t (\mathscr{G}(u_t(\tau)), u_t(\tau))_\Omega d\tau. \tag{2.2.16}$$

In order to handle the interior source it suffices to note that since $u \in C([0,t];V)$, then, by the Sobolev embedding (in 2D) $H_0^1(\Omega) \hookrightarrow L^s(\Omega)$ for all $1 \leq s < \infty$. Hence $\mathscr{F}(u) \in (L^2(\Omega))^3$ whence by Proposition 2.2.2

$$\lim_{h\to 0}\int_0^t (\mathscr{F}(u(\tau)), D_{t,h}u(\tau))_\Omega d\tau = \int_0^t (\mathscr{F}(u(\tau)), u_t(\tau))_\Omega d\tau. \tag{2.2.17}$$

Combining (2.2.13)–(2.2.17) gives the desired identity

$$\begin{aligned} \frac{1}{2}\left(\|u_t(t)\|_2^2 + \|u(t)\|_V^2\right) &+ \int_0^t (\mathscr{G}(u_t(\tau)), u_t(\tau))_\Omega d\tau \\ &= \frac{1}{2}\left(\|u_t(0)\|_2^2 + \|u(0)\|_V^2\right) + \int_0^t (\mathscr{F}(u(\tau)), u_t(\tau))_\Omega d\tau. \end{aligned} \tag{2.2.18}$$

2.3 Continuous dependence of solutions on the initial data

The proof of Theorem 1.5.5 will be carried out by employing an energy identity for a difference of two solutions and Gronwall's inequality.

Proof. **Step 1. Energy identity for a difference of two solutions.** Let $u = (w, \psi, \phi)$ and $\tilde{u} = (\tilde{w}, \tilde{\psi}, \tilde{\phi})$ be two weak solutions on $[0,T]$ in the sense of Definition 1.5.3. Put

$$z = u - \tilde{u} = (z_1, z_2, z_3).$$

The energy corresponding to z is defined by

$$\hat{E}(t) = \frac{1}{2}\left(\|z\|_V^2 + \|z_t\|_2^2\right) \quad \text{for all} \quad t \in [0,T]. \tag{2.3.1}$$

From the regularity of weak solutions stated in Definition 1.5.3, there is $R > 0$ such that

$$\begin{cases} \|u(t)\|_V,\ \|\tilde{u}(t)\|_V,\ \|u_t(t)\|_2,\ \|\tilde{u}_t(t)\|_2 \le R, \\ \int_0^T \|w_t\|_{m+1}^{m+1}dt,\ \int_0^T \|\psi_t\|_{r+1}^{r+1}dt,\ \int_0^T \|\phi_t\|_{q+1}^{q+1}dt \le R, \\ \int_0^T \|\tilde{w}_t\|_{m+1}^{m+1}dt,\ \int_0^T \|\tilde{\psi}_t\|_{r+1}^{r+1}dt,\ \int_0^T \|\tilde{\phi}_t\|_{q+1}^{q+1}dt \le R \end{cases} \tag{2.3.2}$$

for all $t \in [0,T]$. By Definition 1.5.3, z satisfies

$$\begin{aligned} &(z_t(t), \theta(t))_\Omega - (z_t(0), \theta(0))_\Omega + \int_0^t [-(z_t(\tau), \theta_t(\tau))_\Omega + (z(\tau), \theta(\tau))_V]d\tau \\ &+ \int_0^t \int_\Omega (\mathscr{G}(u_t(\tau)) - \mathscr{G}(\tilde{u}_t(\tau))) \cdot \theta(\tau) d\mathbf{x} d\tau = \int_0^t \int_\Omega (\mathscr{F}(u(\tau)) - \mathscr{F}(\tilde{u}(\tau))) \cdot \theta(\tau) d\mathbf{x} d\tau, \end{aligned} \tag{2.3.3}$$

for all $t \in [0,T]$ and for all test functions θ as described in Definition 1.5.3.

Let $\theta(\tau) = D_{t,h}z(\tau)$ in (2.3.3) for $\tau \in [0,t]$, where the difference quotient $D_{t,h}z$ is defined in (2.2.1). Using exactly the same argument as in the proof of the energy identity (1.5.4), we can pass to the limit as $h \searrow 0$ and deduce

$$\hat{E}(t) + \int_0^t \int_\Omega (\mathscr{G}(u_t) - \mathscr{G}(\tilde{u}_t)) \cdot z_t d\mathbf{x} d\tau = \hat{E}(0) + \int_0^t \int_\Omega (\mathscr{F}(u) - \mathscr{F}(\tilde{u})) \cdot z_t d\mathbf{x} d\tau. \tag{2.3.4}$$

Step 2: Estimates. The monotonicity properties of g_i yield

$$\hat{E}(t) \le \hat{E}(0) + \int_0^t \int_\Omega (\mathscr{F}(u(\tau)) - \mathscr{F}(\tilde{u}(\tau))) \cdot z_t(\tau) d\mathbf{x} d\tau, \tag{2.3.5}$$

for all $t \in [0, T]$ where $\hat{E}(t)$ is defined in (2.3.1). Now introduce

$$R_f = \int_0^t \int_\Omega (\mathscr{F}(u) - \mathscr{F}(\tilde{u})) \cdot z_t d\mathbf{x} d\tau. \tag{2.3.6}$$

By Lemma 2.1.2, we know that f_i are locally Lipschitz continuous $(H_0^1(\Omega))^3 \to L^2(\Omega)$. By Hölder's inequality, we have

$$\begin{aligned} R_f &\leq \left(\int_0^t \int_\Omega |\mathscr{F}(u) - \mathscr{F}(\tilde{u})|^2 \, d\mathbf{x} d\tau \right)^{\frac{1}{2}} \left(\int_0^t \int_\Omega |z_t|^2 \, d\mathbf{x} d\tau \right)^{\frac{1}{2}} \\ &\leq C(R) \left(\int_0^t \|z\|_V^2 \, d\tau \right)^{\frac{1}{2}} \left(\int_0^t \|z_t\|_2^2 \, d\tau \right)^{\frac{1}{2}} \leq C(R) \int_0^t \hat{E}(\tau) d\tau. \end{aligned} \tag{2.3.7}$$

Therefore, for the constant R as in (2.3.2) we have

$$\hat{E}(t) \leq \hat{E}(0) + R_f \leq \hat{E}(0) + C(R) \int_0^t \hat{E}(\tau) d\tau. \tag{2.3.8}$$

By Gronwall's inequality we conclude

$$\hat{E}(t) \leq \hat{E}(0) e^{C(R)t} \quad \text{for all } t \in [0, T]. \tag{2.3.9}$$

Since the energy functional is equivalent to the squared norm of the solution on the state space, the latter estimate verifies the dependence of the difference (z, z_t) between trajectories in $C([0, T]; H)$ on the proximity $\|(z(0), z_t(0)\|_H$ of their initial data in H. In addition, (2.3.9) readily implies the uniqueness property, i.e., if $(u(0), u_t(0)) = (\tilde{u}(0), \tilde{u}_t(0))$ in H, then $\hat{E}(t) \equiv \hat{E}(0) = 0$ so $(u, u_t) = (\tilde{u}, \tilde{u}_t)$ in $C([0, T]; H)$. □

2.4 Global existence

Now let's verify Theorem 1.5.6. Let $E_1(t)$ be the modified energy defined by

$$E_1(t) := E(t) + \frac{1}{p+1}\|u(t)\|_{p+1}^{p+1}, \quad \text{where } E(t) = \frac{1}{2}\left(\|u(t)\|_V^2 + \|u_t(t)\|_2^2\right). \tag{2.4.1}$$

By using a standard continuation procedure, one can show that if the energy E_1 doesn't blow up on $[0, T)$, then by Lemma 8.1 (p. 275) in [36] and Lemma 2.1.10 in [47] one can continuously (in the sense of time) extend (u, u_t) to time T in the state space $H = (H_0^1(\Omega))^3 \times (L^2(\Omega))^3$. Therefore, we conclude that either the weak solution $u = (w, \psi, \phi)$ is global or there exists $0 < T < \infty$ such that $\limsup_{t\to T^-} E_1(t) = \infty$.

We aim to show that the latter cannot happen under the assumption of Theorem 1.5.6. Indeed, this assertion is justified by the following proposition.

Proposition 2.4.1. *Let $u = (w, \psi, \phi)$ be a weak solution to* (1.2.1) *on* $[0, T_0]$ *as furnished by Theorem 1.5.4. We have :*

- *If $p \le \min\{m, r, q\}$, then for all $t \in [0, T_0]$, u satisfies*

$$E_1(t) + \int_0^t \left(\|w_t\|_{m+1}^{m+1} + \|\psi_t\|_{r+1}^{r+1} + \|\phi_t\|_{q+1}^{q+1}\right) d\tau \le C(T_0, E_1(0)), \tag{2.4.2}$$

  where $C(T_0, E_1(0))$ is continuous in T_0 and defined for arbitrary $T_0 > 0$.

- *If $p > \max\{m, r, q\}$, then the bound in (2.4.2) holds for $0 < t < T'$, for some $T' \le T_0$, where T' is a continuous and decreasing function with respect to $E_1(0)$.*

Proof. Recall the shorthand $\mathbf{D}(t) := \|w_t(t)\|_{m+1}^{m+1} + \|\psi_t(t)\|_{r+1}^{r+1} + \|\phi_t(t)\|_{q+1}^{q+1}$. With the modified energy as given in (2.4.1) the energy identity (1.5.4) yields

$$
\begin{aligned}
&E_1(t) + \int_0^t \int_\Omega \mathscr{G}(u_t) \cdot u_t d\mathbf{x} d\tau \\
&= E_1(0) + \int_0^t \int_\Omega \mathscr{F}(u) \cdot u_t d\mathbf{x} d\tau + \frac{1}{p+1} \left(\|u(t)\|_{p+1}^{p+1} - \|u(0)\|_{p+1}^{p+1} \right) \\
&= E_1(0) + \int_0^t \int_\Omega \mathscr{F}(u) \cdot u_t d\mathbf{x} d\tau + \int_0^t \int_\Omega \left(|w|^{p-1} w w_t + |\psi|^{p-1} \psi \psi_t + |\phi|^{p-1} \phi \phi_t \right) d\mathbf{x} d\tau.
\end{aligned} \tag{2.4.3}
$$

To estimate the source terms on the right-hand side of (2.4.3) we recall the assumptions: $f_j(u) \leq C\left(|w|^p + |\psi|^p + |\phi|^p + 1\right), j = 1, 2, 3$. By employing Hölder's and Young's inequalities, we obtain

$$
\begin{aligned}
\left| \int_0^t \int_\Omega f_1(u) w_t d\mathbf{x} d\tau \right| &\leq C \int_0^t \left(\|w\|_{p+1}^p + \|\psi\|_{p+1}^p + \|\phi\|_{p+1}^p + |\Omega|^{\frac{p}{p+1}} \right) \|w_t\|_{p+1} d\tau \\
&\leq \epsilon \int_0^t \|w_t\|_{p+1}^{p+1} d\tau + C_\epsilon \int_0^t \left(\|u\|_{p+1}^{p+1} + |\Omega| \right) d\tau \\
&\leq \epsilon \int_0^t \|w_t\|_{p+1}^{p+1} d\tau + C_\epsilon \int_0^t E_1(\tau) d\tau + C_\epsilon T_0 |\Omega|.
\end{aligned} \tag{2.4.4}
$$

In a similar manner, one can replace $f_1(u)w_t$ in (2.4.4) by $f_2(u)\psi_t$ and $f_3(u)\phi_t$ to deduce

$$
\begin{aligned}
\left| \int_0^t (\mathscr{F}(u), u_t)_\Omega d\tau \right| &\leq \left| \int_0^t \int_\Omega f_1(u) w_t d\mathbf{x} d\tau \right| + \left| \int_0^t \int_\Omega f_2(u) \psi_t d\mathbf{x} d\tau \right| + \left| \int_0^t \int_\Omega f_3(u) \phi_t d\mathbf{x} d\tau \right| \\
&\leq \epsilon \int_0^t \|u_t\|_{p+1}^{p+1} d\tau + 3C_\epsilon \int_0^t E_1(\tau) d\tau + 3C_\epsilon T_0 |\Omega|.
\end{aligned} \tag{2.4.5}
$$

By adopting similar estimates as in (2.4.4), we obtain

$$
\begin{aligned}
&\int_0^t \int_\Omega |w|^{p-1} w w_t + |\psi|^{p-1} \psi \psi_t + |\phi|^{p-1} \phi \phi_t d\mathbf{x} d\tau \leq \int_0^t \int_\Omega |w|^p |w_t| + |\psi|^p |\psi_t| + |\phi|^p |\phi_t| d\mathbf{x} d\tau \\
&\leq \epsilon \int_0^t \|u_t\|_{p+1}^{p+1} d\tau + C_\epsilon \int_0^t \|u\|_{p+1}^{p+1} d\tau \leq \epsilon \int_0^t \|u_t\|_{p+1}^{p+1} + C_\epsilon \int_0^t E_1(\tau) d\tau.
\end{aligned} \tag{2.4.6}
$$

By recalling (2.1.28), one has

$$\int_0^t \int_\Omega g_1(w_t)w_t + g_2(\psi)\psi_t + g_3(\phi)\phi_t d\mathbf{x}d\tau \geq \alpha \int_0^t \|u_t\|_{\mathscr{G}} d\tau - 3\alpha T_0|\Omega|. \tag{2.4.7}$$

Now, if $p \leq \min\{m, r, q\}$, it follows form (2.4.4)–(2.4.7) and energy identity (2.4.3) that, for $t \in [0, T_0]$,

$$\begin{aligned} E_1(t) + \alpha \int_0^t \|u_t\|_{\mathscr{G}} d\tau &\leq E_1(0) + 2\epsilon \int_0^t \|u_t\|_{p+1}^{p+1} d\tau + C_\epsilon \int_0^t E_1(\tau)d\tau + C_{T_0,\epsilon} \\ &\leq E_1(0) + 2\epsilon C \int_0^t \|u_t\|_{\mathscr{G}} d\tau + C_\epsilon \int_0^t E_1(\tau)d\tau + C_{T_0,\epsilon}, \end{aligned} \tag{2.4.8}$$

Choosing $0 < 2\epsilon C \leq \alpha/2$, then (2.4.8) yields

$$E_1(t) + \frac{\alpha}{2} \int_0^t \left(\|w_t\|_{m+1}^{m+1} + \|\psi_t\|_{r+1}^{r+1} + \|\phi_t\|_{q+1}^{q+1} \right) d\tau \leq E_1(0) + C_\epsilon \int_0^t E_1(\tau)d\tau + C_{T_0,\epsilon}. \tag{2.4.9}$$

In particular,

$$E_1(t) \leq E_1(0) + C_\epsilon \int_0^t E_1(\tau)d\tau + C_{T_0,\epsilon}. \tag{2.4.10}$$

By Gronwall's inequality, we conclude that

$$E_1(t) \leq (E_1(0) + C_{T_0,\epsilon}) \left(1 + C_\epsilon T_0 e^{C_\epsilon T_0}\right) \quad \text{for} \quad t \in [0, T_0] \tag{2.4.11}$$

where $T_0 > 0$ can be arbitrary. Combining (2.4.9) and (2.4.11), gives the desired result in (2.4.2).

Now if $p > \max\{m, r, q\}$, then we slightly modify (2.4.4) by using different Hölder's conjugates. Specifically, we apply Hölder's inequality with $m+1$ and $\tilde{m} = \frac{m+1}{m}$ followed by

Young's to obtain

$$
\begin{aligned}
&\left|\int_0^t\int_\Omega f_1(u)w_t d\mathbf{x}d\tau\right| \le C\int_0^t\int_\Omega \left(|w|^p+|\psi|^p+|\phi|^p+1\right)|w_t|d\mathbf{x}d\tau \\
&\le \epsilon\int_0^t \|w_t\|_{m+1}^{m+1}d\tau + C_\epsilon\int_0^t \left(\|u\|_{p\tilde{m}}^{p\tilde{m}}+|\Omega|\right)d\tau.
\end{aligned}
\tag{2.4.12}
$$

Since $1 \le m < p$, we have $p\tilde{m} > 2$. From the 2D embedding $H^1(\Omega) \hookrightarrow L^s(\Omega), 1 \le s < \infty$ it then fillows

$$
\left|\int_0^t\int_\Omega f_1(u)w_t d\mathbf{x}d\tau\right| \le \epsilon\int_0^t \|w_t\|_{m+1}^{m+1}d\tau + C_\epsilon\int_0^t \left(\left(\|w\|_{1,\Omega}^2+\|\psi\|_{1,\Omega}^2+\|\phi\|_{1,\Omega}^2\right)^{\frac{p\tilde{m}}{2}}+|\Omega|\right)d\tau.
\tag{2.4.13}
$$

In turn the Proposition (A.0.2) in the Appendix gives

$$
\begin{aligned}
\left|\int_0^t\int_\Omega f_1(u(\tau))w_t(\tau)d\mathbf{x}d\tau\right| &\le \epsilon\int_0^t \|w_t(\tau)\|_{m+1}^{m+1}d\tau + C_\epsilon\int_0^t \left(\|u(\tau)\|_V^{p\tilde{m}}+|\Omega|\right)d\tau \\
&\le \epsilon\int_0^t \|w_t(\tau)\|_{m+1}^{m+1}d\tau + C_\epsilon\int_0^t E_1(\tau)^{\frac{p\tilde{m}}{2}}d\tau + C_\epsilon T_0|\Omega|.
\end{aligned}
\tag{2.4.14}
$$

analogously for $f_2, \psi_t, r, \tilde{r}$ and $f_3, \phi_t, q, \tilde{q}$.

Then employing similar estimates as in (2.4.12)–(2.4.14), we have

$$
\begin{aligned}
&\int_0^t\int_\Omega |w|^{p-1}ww_t + |\psi|^{p-1}\psi\psi_t + |\phi|^{p-1}\phi\phi_t d\mathbf{x}d\tau \\
&\le \epsilon\int_0^t \|w_t\|_{m+1}^{m+1} + \|\psi_t\|_{r+1}^{r+1} + \|\phi_t\|_{q+1}^{q+1}d\tau + C_\epsilon\int_0^t E_1(\tau)^{\frac{p\tilde{m}}{2}} + E_1(\tau)^{\frac{p\tilde{r}}{2}} + E_1(\tau)^{\frac{p\tilde{q}}{2}}d\tau.
\end{aligned}
\tag{2.4.15}
$$

Now (2.4.14)–(2.4.15) along with (2.4.7) gives from the energy identidy (2.4.3)

$$
E_1(t) + \alpha\int_0^t \|u_t(\tau)\|_{\mathscr{G}}d\tau \le E_1(0) + 2\epsilon\int_0^t \|u_t(\tau)\|_{\mathscr{G}}d\tau + C_\epsilon\int_0^t E_1(\tau)^\sigma d\tau + C_{T_0,\epsilon},
\tag{2.4.16}
$$

where $\sigma = \max\{\frac{p\tilde{m}}{2}, \frac{p\tilde{r}}{2}, \frac{p\tilde{q}}{2}\} > 1$. Choosing choosing $0 < 2\epsilon < \alpha/2$, then it follows that for all $t \in [0, T_0]$

$$E_1(t) + \frac{\alpha}{2}\int_0^t \|u_t(\tau)\|_{\mathscr{G}} d\tau \leq E_1(0) + C_\epsilon \int_0^t E_1(\tau)^\sigma d\tau + C_{T_0,\epsilon}. \tag{2.4.17}$$

In particular,

$$E_1(t) \leq E_1(0) + C_\epsilon \int_0^t E_1(\tau)^\sigma d\tau + C_{T_0,\epsilon} \text{ for } t \in [0, T_0]. \tag{2.4.18}$$

By using a standard comparison theorem (see [29] for instance) (2.4.18) yields that $E_1(t) \leq z(t)$, where $z(t) = [(E_1(0) + C_{T_0,\epsilon})^{1-\sigma} - C_\epsilon(\sigma - 1)t]^{-\frac{1}{\sigma-1}}$ solves

$$z(t) = C_\epsilon \int_0^t z(s)^\sigma ds + E_1(0) + C_{T_0,\epsilon}.$$

Since $\sigma > 1$, then $z(t)$ blows up at $T_1 = \frac{1}{C_\epsilon(\sigma-1)}(E_1(0) + C_{T_0,\epsilon})^{1-\sigma}$, i,e., $z(t) \to \infty$, as $t \to T_1^-$. Note that T_1 depends on the initial energy $E_1(0)$ and the original existence time T_0. Nonetheless, if we choose $T' = \min\{T_0, \frac{1}{2}T_1\}$, then

$$E_1(t) \leq z(t) \leq C_0 := [(E_1(0) + C_{T_0,\epsilon})^{1-\sigma} - C_\epsilon(\sigma - 1)T']^{-\frac{1}{\sigma-1}}, \tag{2.4.19}$$

for all $t \in [0, T']$. Finally, we may combine (2.4.17) and (2.4.19) to obtain

$$E_1(t) + \frac{\alpha}{2}\int_0^t \|u_t(\tau)\|_{\mathscr{G}} d\tau \leq E_1(0) + C_\epsilon T' C_0^\sigma + C_{T_0,\epsilon} \text{ for all } t \in [0, T_0], \tag{2.4.20}$$

which completes the proof of the proposition. □

2.5 Blow-up with negative total initial energy

In this section, we provide the proof of Theorem 1.5.9. Let $u = (w, \psi, \phi)$ be a weak solution to (1.2.1) in the sense of Definition 1.5.3. Throughout the proof, we shall assume the validity of Assumption 1.5.1 and Assumption 1.5.7 with $p > \max\{m, r, q\}$. We define the life span T of such a solution $u = (w, \psi, \phi)$ to be the supremum of all $T^* > 0$ such that u is a solution to (1.2.1) in the sense of Definition 1.5.3 on $[0, T^*]$. Our goal is to show that T is necessarily finite, and obtain an upper bound for T.

As in [1, 7], for $t \in [0, T]$, we define:

$$G(t) = -\mathscr{E}(t), \quad N(t) = \|u(t)\|_2^2, \quad S(t) = \int_\Omega F(u(t))d\mathbf{x},$$

where the total energy $\mathscr{E}(t) = \frac{1}{2}(\|u(t)\|_V^2 + \|u_t(t)\|_2^2) - \int_\Omega F(u(t))d\mathbf{x}$, where $\|u(t)\|_V^2$ is defined in (1.4.4) . It follows that,

$$G(t) = -\frac{1}{2}\left(\|u\|_V^2 + \|u_t\|_2^2\right) + S(t), \text{ and } N'(t) = 2\int_\Omega u(t) \cdot u_t(t)d\mathbf{x}. \tag{2.5.1}$$

Moreover, by the assumption $F(w, \psi, \phi) \geq c_0\left(|w|^{p+1} + |\psi|^{p+1} + |\phi|^{p+1}\right)$, one has

$$S(t) \geq c_0\|u(t)\|_{p+1}^{p+1}. \tag{2.5.2}$$

Let

$$0 < a < \min\left\{\frac{1}{m+1} - \frac{1}{p+1}, \frac{1}{r+1} - \frac{1}{p+1}, \frac{1}{q+1} - \frac{1}{p+1}, \frac{p-1}{2(p+1)}\right\}. \tag{2.5.3}$$

In particular, $a < \frac{1}{2}$. To simplify the subsequent notation we introduce the following constants:

$$K_1 = \beta|\Omega|^{\frac{p-m}{(p+1)(m+1)}} c_0^{-\frac{1}{p+1}}, \;\; K_2 = \beta|\Omega|^{\frac{p-r}{(p+1)(r+1)}} c_0^{-\frac{1}{p+1}}, \;\; K_3 = \beta|\Omega|^{\frac{p-q}{(p+1)(q+1)}} c_0^{-\frac{1}{p+1}},$$

$$\delta_1 = \frac{\lambda}{6} G(0)^{\frac{1}{m+1}-\frac{1}{p+1}}, \quad \delta_2 = \frac{\lambda}{6} G(0)^{\frac{1}{r+1}-\frac{1}{p+1}}, \quad \delta_3 = \frac{\lambda}{6} G(0)^{\frac{1}{q+1}-\frac{1}{p+1}}, \tag{2.5.4}$$

where $\lambda = c_1 - 2 > 0, \beta,\ c_1$ are given in Assumption 1.5.7, and let $|\Omega|$ denote the Lebesgue measures of Ω.

We first note that the energy identity (1.5.4) is equivalent to

$$G(t) = G(0) + \int_0^t \int_\Omega \mathscr{G}(u_t(\tau)) \cdot u_t(\tau) d\mathbf{x} d\tau. \tag{2.5.5}$$

By Assumption 1.5.1, and the regularity of u_t as stated in Definition 1.5.3, we conclude that $G(t)$ is absolutely continuous and

$$G'(t) = \int_\Omega \mathscr{G}(u_t(t)) \cdot u_t(t) d\mathbf{x} \geq \alpha \, \mathbf{D}(t) \geq 0, \;\; \text{a.e.} \;\; [0, T), \tag{2.5.6}$$

where $\mathbf{D}(t) = \|w_t(t)\|_{m+1}^{m+1} + \|\psi_t(t)\|_{r+1}^{r+1} + \|\phi_t(t)\|_{q+1}^{q+1}$. Thus, $G(t)$ is non-decreasing, and since $G(0) = -\mathscr{E}(0) > 0$, then it follows that

$$0 < G(0) \leq G(t) \leq S(t) \;\; \text{for} \;\; 0 \leq t < T. \tag{2.5.7}$$

Now define

$$Y(t) = G(t)^{1-a} + \epsilon N'(t), \tag{2.5.8}$$

where $0 < \epsilon \leq G(0)$. Later in the proof we further adjust the requirements on ϵ. We aim to show that

$$Y'(t) = (1-a)G(t)^{-a}G'(t) + \epsilon N''(t), \tag{2.5.9}$$

where

$$N''(t) = \quad 2\|u_t\|_2^2 - 2\|u\|_V^2 - 2\int_\Omega \mathscr{G}(u_t)\cdot u d\mathbf{x} + 2\int_\Omega \mathscr{F}(u)\cdot u d\mathbf{x}, \text{ a.e. } [0,T). \tag{2.5.10}$$

In order to prove (2.5.10), we fist notice that the regularity of u and the Sobolev embedding (in 2D), $H_0^1(\Omega) \hookrightarrow L^s(\Omega),\ 1 \leq s < \infty$ imply,

$$u = (w, \psi, \phi) \in \Big(L^{m+1}(\Omega\times(0,t)) \times L^{r+1}(\Omega\times(0,t)) \times L^{q+1}(\Omega\times(0,t))\Big), \tag{2.5.11}$$

for all $t \in [0,T)$. This shows that u satisfies the regularity restrictions on the test function θ, as stated in Definition 1.5.3. Therefore, by replacing θ in (1.5.2) by u and by using (2.5.1), we obtain

$$\begin{aligned}\frac{1}{2}N'(t) = &\int_\Omega u_t(0)\cdot u(0)d\mathbf{x} + \int_0^t \|u_t(\tau)\|_2^2 - \|u(\tau)\|_V^2 d\tau \\ &- \int_0^t\int_\Omega \mathscr{G}(u_t(\tau))\cdot u(\tau)d\mathbf{x}d\tau + \int_0^t\int_\Omega \mathscr{F}(u(\tau))\cdot u(\tau)d\mathbf{x}d\tau, \text{ a.e. } [0,T).\end{aligned} \tag{2.5.12}$$

From Assumption 1.5.1 and the Mean Value Theorem $|f_j(w,\psi,\phi)| \leq C\,(|w|^p + |\psi|^p + |\phi|^p + 1)$, $j = 1,2,3$. Thus

$$\int_0^t \left|\int_\Omega \mathscr{F}(u)\cdot u\right| d\mathbf{x}d\tau \leq C\int_0^t\int_\Omega (|w|^p + |\psi|^p + |\phi|^p + 1)\,(|w| + |\psi| + |\phi|)d\mathbf{x}\tau. \tag{2.5.13}$$

A typical term on the right-hand side of (2.5.13) can be estimated by using Hölder's inequality and the embedding (in 2D) $H_0^1(\Omega) \hookrightarrow L^s(\Omega), 1 \leq s < \infty$

$$\int_0^t \int_\Omega |w|^p |w| d\mathbf{x} d\tau \leq C \left(\int_0^t \|w\|_{1,\Omega}^{p\frac{m+1}{m}} d\tau \right)^{\frac{m}{m+1}} \|w\|_{L^{m+1}(\Omega\times(0,t))} < \infty, \tag{2.5.14}$$

for all $t \in [0, T)$, where we have used the facts $w \in C([0, t]; H_0^1(\Omega))$ and $w \in L^{m+1}(\Omega\times(0, t))$, as shown in (2.5.11). The other terms on the right-hand side of (2.5.13) can be estimated in the same manner. Thus, we conclude that

$$\int_0^t \left| \int_\Omega \mathscr{F}(u) \cdot u d\mathbf{x} \right| d\tau < \infty, \quad \text{for all} \quad t \in [0, T). \tag{2.5.15}$$

In addition, by using the regularity of the solution u and the assumptions on the parameters, we infer

$$\int_0^t \left| \int_\Omega \mathscr{G}(u_t) \cdot u d\mathbf{x} \right| d\tau < \infty, \quad \text{for all} \quad t \in [0, T). \tag{2.5.16}$$

Hence, it follows from (2.5.12), (2.5.15)–(2.5.16), and the regularity of u that $N'(t)$ is absolutely continuous, and thus (2.5.10) follows immediately.

Now, let us note that (2.5.1) yields

$$\|u(t)\|_V^2 = -\|u_t(t)\|_2^2 + 2S(t) - 2G(t). \tag{2.5.17}$$

By employing (2.5.9), (2.5.10), (2.5.17), and the assumptions $\mathscr{F}(u) \cdot u \geq c_1 F(u)$, one has

$$\begin{aligned} Y'(t) &= (1-a)G(t)^{-a}G'(t) + 4\epsilon\Big(\|u_t(t)\|_2^2 - S(t) + G(t) \Big) \\ &\qquad - 2\epsilon \int_\Omega \mathscr{G}(u_t(t)) \cdot u(t) d\mathbf{x} + 2\epsilon \int_\Omega \mathscr{F}(u(t)) \cdot u(t) d\mathbf{x} \\ &\geq (1-a)G(t)^{-a}G'(t) + 4\epsilon \|u_t(t)\|_2^2 + 4\epsilon G(t) \\ &\qquad - 2\epsilon \int_\Omega \mathscr{G}(u_t(t)) \cdot u(t) d\mathbf{x} + 2\epsilon(c_1 - 2) \int_\Omega F(u(t)) d\mathbf{x}. \end{aligned} \tag{2.5.18}$$

Now, we estimate the term due to damping in (2.5.18). By recalling the assumption $g_1(s)s \le \beta|s|^{m+1}$, the fact $p > m$, and inequality (2.5.2), we have

$$\left|\int_\Omega g_1(w_t(t))w(t)d\mathbf{x}\right| \le \beta\int_\Omega |w(t)||w_t(t)|^m d\mathbf{x} \le \beta\,\|w(t)\|_{m+1}\,\|w_t(t)\|_{m+1}^m$$
$$\le \beta|\Omega|^{\frac{p-m}{(p+1)(m+1)}}\,\|w(t)\|_{p+1}\,\|w_t(t)\|_{m+1}^m \le K_1 S(t)^{\frac{1}{p+1}}\,\|w_t(t)\|_{m+1}^m\,, \tag{2.5.19}$$

where K_1 is defined in (2.5.4). Observe, the definition of a implies $\frac{1}{p+1} - \frac{1}{m+1} + a < 0$. Therefore, by using (2.5.6)-(2.5.7), Young's inequality, and recalling the definition of $\delta_1, \delta_2, \delta_3$ in (2.5.4), we obtain from (2.5.19) that

$$\left|\int_\Omega g_1(w_t(t))w(t)d\mathbf{x}\right| \le K_1 S(t)^{\frac{1}{p+1}-\frac{1}{m+1}} S(t)^{\frac{1}{m+1}}\,\|w_t(t)\|_{m+1}^m$$
$$\le G(t)^{\frac{1}{p+1}-\frac{1}{m+1}}\left(\delta_1 S(t) + C_{\delta_1} K_1^{\frac{m+1}{m}}\,\|w_t(t)\|_{m+1}^{m+1}\right)$$
$$\le \delta_1 G(t)^{\frac{1}{p+1}-\frac{1}{m+1}} S(t) + C_{\delta_1} K_1^{\frac{m+1}{m}}\,\alpha^{-1} G'(t) G(t)^{-a} G(t)^{\frac{1}{p+1}-\frac{1}{m+1}+a}$$
$$\le \delta_1 G(0)^{\frac{1}{p+1}-\frac{1}{m+1}} S(t) + C_{\delta_1} K_1^{\frac{m+1}{m}}\,\alpha^{-1} G'(t) G(t)^{-a} G(0)^{\frac{1}{p+1}-\frac{1}{m+1}+a}$$
$$= \frac{\lambda}{6} S(t) + C_{\delta_1} K_1^{\frac{m+1}{m}}\,\alpha^{-1} G'(t) G(t)^{-a} G(0)^{\frac{1}{p+1}-\frac{1}{m+1}+a}. \tag{2.5.20}$$

By repeating the estimates (2.5.19)-(2.5.20), replacing $w(t)$ by $\psi(t)$ and $\phi(t)$, replacing m by r and q respectively, we deduce

$$\left|\int_\Omega \mathcal{G}(u_t(t))u(t)d\mathbf{x}\right| \le \frac{\lambda}{2} S(t) + C_{\delta_1} K_1^{\frac{m+1}{m}}\,\alpha^{-1} G'(t) G(t)^{-a} G(0)^{\frac{1}{p+1}-\frac{1}{m+1}+a}$$
$$+ C_{\delta_2} K_2^{\frac{r+1}{r}}\,\alpha^{-1} G'(t) G(t)^{-a} G(0)^{\frac{1}{p+1}-\frac{1}{r+1}+a} + C_{\delta_3} K_3^{\frac{q+1}{q}}\,\alpha^{-1} G'(t) G(t)^{-a} G(0)^{\frac{1}{p+1}-\frac{1}{q+1}+a}. \tag{2.5.21}$$

Now, since $0 < a < \frac{1}{2}$, we may choose $0 < \epsilon < 1$ small enough such that

$$\begin{aligned} L := & 1 - a - 2\epsilon\Big(C_{\delta_1} K_1^{\frac{m+1}{m}} \alpha^{-1} G(0)^{\frac{1}{p+1} - \frac{1}{m+1} + a} \\ & + C_{\delta_2} K_2^{\frac{r+1}{r}} \alpha^{-1} G(0)^{\frac{1}{p+1} - \frac{1}{r+1} + a} + C_{\delta_3} K_3^{\frac{q+1}{q}} \alpha^{-1} G(0)^{\frac{1}{p+1} - \frac{1}{q+1} + a}\Big) \geq 0. \end{aligned} \tag{2.5.22}$$

In addition, since $\lambda = c_1 - 2 > 0$, then

$$(c_1 - 2) \int_\Omega F(u(t)) d\mathbf{x} = \lambda S(t). \tag{2.5.23}$$

Hence, by inserting (2.5.21) into (2.5.18) and using (2.5.6), (2.5.7), (2.5.22), and (2.5.23), we conclude

$$\begin{aligned} Y'(t) & \geq L G(t)^{-a} G'(t) + 4\epsilon \|u_t(t)\|_2^2 + 4\epsilon G(t) + 2\lambda\epsilon S(t) \\ & \geq 4\epsilon\Big(\|u_t(t)\|_2^2 + G(t)\Big) + 2\lambda\epsilon S(t). \end{aligned} \tag{2.5.24}$$

By recalling (2.5.7) which implies $S(t) > 0$ and $G(t) > 0$, then inequality (2.5.24) shows that $Y(t)$ is increasing on $[0, T)$. Thus,

$$Y(t) = G(t)^{1-a} + \epsilon N'(t) \geq G(0)^{1-a} + \epsilon N'(0). \tag{2.5.25}$$

If $N'(0) \geq 0$, then no further adjustment on ϵ is needed. However, if $N'(0) < 0$, then we further adjust ϵ by requiring $0 < \epsilon \leq -\frac{G(0)^{1-a}}{2N'(0)}$. In any case, one has

$$Y(t) \geq \frac{1}{2} G(0)^{1-a} > 0 \quad \text{for} \quad t \in [0, T). \tag{2.5.26}$$

Finally, we show that

$$Y'(t) \geq C\epsilon^{1+\sigma} Y(t)^{\eta} \ \text{ for } \ t \in [0,T), \tag{2.5.27}$$

where

$$1 < \eta = \frac{1}{1-a} < 2, \ \ \sigma = 1 - \frac{2}{(1-2a)(p+1)} > 0,$$

and $C > 0$ is a generic constant independent of ϵ. Notice that $\sigma > 0$ follows from the assumption $a < \frac{p-1}{2(p+1)}$.

Now, if $N'(t) \leq 0$ for some $t \in [0,T)$, then for such value of t we have

$$Y(t)^{\eta} = [G(t)^{1-a} + \epsilon N'(t)]^{\eta} \leq G(t), \tag{2.5.28}$$

and in this case, (2.5.24) and (2.5.28) with $0 < \epsilon < 1$, yield

$$Y'(t) \geq 4\epsilon G(t) \geq 4\epsilon^{1+\sigma} G(t) \geq 4\epsilon^{1+\sigma} Y(t)^{\eta}.$$

Hence, (2.5.27) holds for all $t \in [0,T)$ such that $N'(t) \leq 0$. However, if $t \in [0,T)$ is such that $N'(t) > 0$, then showing the validity of (2.5.27) requires a little more effort. First, we note that $Y(t) = G(t)^{1-a} + \epsilon N'(t) \leq G(t)^{1-a} + N'(t)$, and since $1 < \eta < 2$, the one variable function x^{η} is convex for $x > 0$, then

$$Y(t)^{\eta} \leq 2^{\eta-1}[G(t) + N'(t)^{\eta}]. \tag{2.5.29}$$

We estimate $N'(t)^{\eta}$ as follows. Via Hölder's and Young's inequalities and noting that 1 <

$\eta < 2$ we obtain from (2.5.1) that

$$\begin{aligned} N'(t)^\eta &\le 2^\eta\Big(\|w_t(t)\|_2 \, \|w(t)\|_2 + \|\psi_t(t)\|_2 \, \|\psi(t)\|_2 + \|\phi_t(t)\|_2 \, \|\phi(t)\|_2 \Big)^\eta \\ &\le C_{\eta,|\Omega|}\Big(\|w_t(t)\|_2^\eta \, \|w(t)\|_{p+1}^\eta + \|\psi_t(t)\|_2^\eta \, \|\psi(t)\|_{p+1}^\eta + \|\phi_t(t)\|_2^\eta \, \|\phi(t)\|_{p+1}^\eta \Big) \\ &\le C_{\eta,|\Omega|}\Big(\|w_t(t)\|_2^2 + \|w(t)\|_{p+1}^{\frac{2\eta}{2-\eta}} + \|\psi_t(t)\|_2^2 + \|\psi(t)\|_{p+1}^{\frac{2\eta}{2-\eta}} + \|\phi_t(t)\|_2^2 + \|\phi(t)\|_{p+1}^{\frac{2\eta}{2-\eta}} \Big). \quad (2.5.30) \end{aligned}$$

Since $\eta = \frac{1}{1-a}$ and $\sigma > 0$, it is easy to see that

$$\frac{2\eta}{(2-\eta)(p+1)} - 1 = \frac{2}{(1-2a)(p+1)} - 1 = -\sigma < 0. \quad (2.5.31)$$

Therefore, by (2.5.2), (2.5.7), (2.5.31), and by $0 < \epsilon \le G(0)$ as defined in (2.5.8), we have

$$\begin{aligned} \|w(t)\|_{p+1}^{\frac{2\eta}{2-\eta}} &= (\|w(t)\|_{p+1}^{p+1})^{\frac{2\eta}{(2-\eta)(p+1)}} \le CS(t)^{\frac{2\eta}{(2-\eta)(p+1)}} \\ &\le CS(t)^{\frac{2\eta}{(2-\eta)(p+1)}-1} S(t) \le CG(0)^{-\sigma} S(t) \le C\epsilon^{-\sigma} S(t). \quad (2.5.32) \end{aligned}$$

Similarly, by replacing w in (2.5.32) by ψ and ϕ, we obtain

$$\|\psi(t)\|_{p+1}^{\frac{2\eta}{2-\eta}} \le C\epsilon^{-\sigma} S(t), \quad \text{and} \quad \|\phi(t)\|_{p+1}^{\frac{2\eta}{2-\eta}} \le C\epsilon^{-\sigma} S(t). \quad (2.5.33)$$

By (2.5.30) and (2.5.32)-(2.5.33) and noting $\epsilon^{-\sigma} > 1$, we obtain

$$\begin{aligned} N'(t)^\eta &\le C\Big(\|w_t(t)\|_2^2 + \|\psi_t(t)\|_2^2 + \|\phi_t(t)\|_2^2 + \epsilon^{-\sigma} S(t)\Big) \\ &\le C\epsilon^{-\sigma}\Big(\|w_t(t)\|_2^2 + \|\psi_t(t)\|_2^2 + \|\phi_t(t)\|_2^2 + S(t)\Big) \\ &= C\epsilon^{-\sigma}\Big(\|u_t(t)\|_2^2 + S(t)\Big), \quad (2.5.34) \end{aligned}$$

where $C > 0$ is a constant independent of ϵ. Finally, (2.5.24), (2.5.29) and (2.5.34) allow us to conclude that

$$Y'(t) \geq C\epsilon[G(t) + \|u_t(t)\|_2^2 + S(t)] \geq C\epsilon[G(t) + \epsilon^{\sigma} N'(t)^{\eta}]$$
$$\geq C\epsilon^{1+\sigma}[G(t) + N'(t)^{\eta}] \geq C\epsilon^{1+\sigma} Y(t)^{\eta},$$

for all values of $t \in [0, T)$ such that $N'(t) > 0$. Hence, (2.5.27) is valid. By simple calculations, it follows from (2.5.26)-(2.5.27) that T is necessarily finite and

$$T < C\epsilon^{-(1+\sigma)} Y(0)^{-\frac{a}{1-a}} \leq C\epsilon^{-(1+\sigma)} G(0)^{-a}. \tag{2.5.35}$$

This completes the proof of Theorem 1.5.9.

Chapter 3

Asymptotic stability

In this chapter, we show that certain potential well solutions are asymptotically stable, i.e., converge to the zero equilibrium as $t \to \infty$. Moreover, the energy decay rate is uniform for all solutions in the given class.

3.1 Global existence for potential well solutions

This section is devoted to the proof of Theorem 1.5.14. As in [2] or [9], we proceed in two steps. Recall the definitions of the functional J in (1.5.8) and the subdivisions of the potential well $\mathscr{W}, \mathscr{W}_1, \mathscr{W}_2$ from (1.5.16).

Proof. Step 1: $\mathscr{W}_1$ is invariant with respect to the flow associated with (1.2.1), i.e., $u(t) \in \mathscr{W}_1$ for all $t \in [0, T)$, where $[0, T)$ is the maximal interval of existence provided by Theorem 1.5.4. Notice that the energy identity (1.5.4) is equivalent to

$$\mathscr{E}(t) + \int_0^t (\mathscr{G}(u_t), u_t)_\Omega \, d\tau = \mathscr{E}(0). \tag{3.1.1}$$

Since g_1, g_2 and g_3 are all monotone increasing, it follows from the regularity of the velocity component u_t that

$$\mathscr{E}'(t) = -(\mathscr{G}(u_t), u_t)_\Omega \leq 0. \tag{3.1.2}$$

Thus,

$$J(u(t)) \leq \mathscr{E}(t) \leq \mathscr{E}(0) < d, \quad \text{for all } \ t \in [0, T). \tag{3.1.3}$$

It follows that inequality (I) of Theorem 1.5.14 holds, and $u(t) \in \mathscr{W}$ for all $t \in [0, T)$.

To show that $u(t) \in \mathscr{W}_1$ on $[0, T)$, we argue by contradiction. Assume that there exists $t_1 \in (0, T)$ such that $u(t_1) \notin \mathscr{W}_1$. Since $\mathscr{W} = \mathscr{W}_1 \cup \mathscr{W}_2$ and $\mathscr{W}_1 \cap \mathscr{W}_2 = \emptyset$, $u(t_1) \in \mathscr{W}_2$ must be the case.

From the property $|\nabla f_j(u)| \leq C(|w|^{p-1} + |\psi|^{p-1} + |\phi|^{p-1} + 1)$ and the fact that F is homogeneous of order $p+1$, it can be shown that the function $t \mapsto \int_\Omega F(u(t))d\mathbf{x}$ is continuous on $[0, T)$. Therefore, since $u(0) \in \mathscr{W}_1$ and $u(t_1) \in \mathscr{W}_2$, from the definition of $\mathscr{W}_1$ and $\mathscr{W}_2$ we conclude that there exists $s \in (0, t_1)$ such that

$$\|u(s)\|_V^2 = (p+1) \int_\Omega F(u(s))d\mathbf{x}. \tag{3.1.4}$$

Define t^* as the supremum over all $s \in (0, t_1)$ satisfying (3.1.4). Clearly, $t^* \in (0, t_1)$, t^* satisfies (3.1.4) and $u(t) \in \mathscr{W}_2$ for all $t \in (t^*, t_1]$. Now we have two cases to consider:

Case 1: Suppose that $u(t^*) \neq 0$. Since t^* satisfies (3.1.4), then $u(t^*) \in \mathscr{N}$, where $\mathscr{N}$ is the Nehari manifold given in (1.5.13). Thus, by Lemma 1.5.12, it follows that $J(u(t^*)) \geq d$. Since $\mathscr{E}(t) \geq J(u(t))$ for all $t \in [0, T)$, one has $\mathscr{E}(t^*) \geq d$, which contradicts (3.1.3).

Case 2: Suppose that $u(t^*) = 0$. Since $u(t) \in \mathscr{W}_2$ for all $t \in (t^*, t_1]$, then by inequality

(1.5.6) and the definition of $\mathscr{W}_2$, we obtain

$$\|u(t)\|_V^2 < (p+1)M\|u(t)\|_{p+1}^{p+1} \leq C\|u(t)\|_{1,\Omega}^{p+1} \leq C\|u(t)\|_V^{p+1}, \tag{3.1.5}$$

for all $t \in (t^*, t_1]$. Therefore,

$$\|u(t)\|_V > s_1 \quad \text{for all} \quad t \in (t^*, t_1], \ \text{where} \ s_1 = C^{-\frac{1}{p-1}} > 0. \tag{3.1.6}$$

Employing the continuity of weak solution $u \in C([0,T); V)$, we obtain that $\|u(t^*)\|_V \geq s_1 > 0$ which contradicts the assumption $u(t^*) = 0$. Hence, $u(t) \in \mathscr{W}_1$ for all $t \in [0,T)$ which yields Theorem 1.5.14 (II) verifying that $\mathscr{W}_1$ is invariant under the dynamics of (1.2.1).

Step 2: The weak solution u is a global solution. By (3.1.3) and Step 1, we know $J(u(t)) < d$ and $u(t) \in \mathscr{W}_1$ for all $t \in [0,T)$, consequently,

$$d > J(u(t)) = \frac{1}{2}\|u(t)\|_V^2 - \int_\Omega F(u(t))d\mathbf{x} \overset{(1.5.16)}{>} \frac{1}{2}\|u(t)\|_V^2 - \frac{1}{p+1}\|u(t)\|_V^2. \tag{3.1.7}$$

Therefore,

$$\int_\Omega F(u(t))d\mathbf{x} \overset{(1.5.16)}{<} \frac{1}{p+1}\|u(t)\|_V^2 < \frac{2d}{p-1} \quad \text{for all} \quad t \in [0,T). \tag{3.1.8}$$

Combining (3.1.1) and (3.1.8) yields

$$E(t) + \int_0^t (\mathscr{G}(u_t), u_t)_\Omega \, d\tau = \mathscr{E}(0) + \int_\Omega F(u(t))d\mathbf{x} \leq d + \frac{2d}{p-1} = d \cdot \rho, \tag{3.1.9}$$

for all $t \in [0,T)$, where $\rho = \frac{p+1}{p-1}$. By the virtue of monotonicity of g_i, $i = 1,2,3$, the inequality (III) of Theorem 1.5.14 follows. Because the quadratic energy $E(t)$ is bounded uniformly in t, the local existence result from Theorem 1.5.4 (for which the time of existence only depends on the upper energy bound) can be exploited to verify that the local solution u can be extended to $[0,\infty)$.

It remains to check the inequality (IV) of Theorem 1.5.14. Since $F(u)$ is a non-negative function, it is clear that $\mathscr{E}(t) < E(t)$, for all $t \in [0,\infty)$. On the other hand, by the fact $u(t) \in \mathscr{W}_1$ for all $t \in [0,\infty)$ and the definition of $\mathscr{E}(t)$ one has

$$\mathscr{E}(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|u(t)\|_V^2 - \int_\Omega F(u(t))d\mathbf{x} \geq E(t) - \frac{1}{p+1}\|u(t)\|_V^2 \geq \frac{1}{\rho}E(t), \tag{3.1.10}$$

which completes the proof of Theorem 1.5.14. □

The following modified version of the invariance result for the potential well $\mathscr{W}_1$ will be useful for the subsequent analysis of energy decay:

Proposition 3.1.1 (Invariance of the approximate well $\tilde{\mathscr{W}}_1^\delta$)**.** *Let* $(s_0, \tilde{d})$*, be the (unique) global maximum of the function* $\mathcal{G}(s)$ *in* (1.5.17) *(thus,* s_0 *is given by* (1.5.20) *and* $\tilde{d}$ *by* (1.5.21)*). Besides Assumption 1.5.1 and Assumption 1.5.10 with* $(p > 1)$*, suppose* $u_0 \in \tilde{\mathscr{W}}_1^\delta$ *(as defined in* (1.5.25)*) for sufficiently small* $\delta > 0$ *that* $\mathscr{E}(0) \leq \mathcal{G}(s_0 - \delta)$*. Then the global solution* $\{u, u_t\}$ *of* (1.2.1) *furnished by Theorem 1.5.14 satisfies* $u(t) \in \tilde{\mathscr{W}}_1^\delta$*, for all* $t \geq 0$*.*

Proof. By the fact $J(u(t)) \leq \mathscr{E}(t) \leq \mathscr{E}(0)$ and the Assumption $\mathscr{E}(0) \leq \mathcal{G}(s_0 - \delta)$, we obtain $J(u(t)) \leq \mathcal{G}(s_0 - \delta)$ for all $t \geq 0$. To show $\|u(t)\|_V \leq s_0 - \delta$ for all $t \geq 0$, argue again by contradiction. Since $u_0 \in \tilde{\mathscr{W}}_1^\delta$ we know $\|u_0\|_V \leq s_0 - \delta$. Recall that $u \in C(\mathbb{R}^+; V)$. Therefore, if the invariance fails, there must exist $t_1 > 0$ such that $\|u(t_1)\|_V = s_0 - \delta + \epsilon$ for $\epsilon \in (0, \delta)$. Therefore, taking (1.5.19) into account and the fact that $\mathcal{G}$ is strictly increasing on $(0, s_0)$, we obtain that $J(u(t_1)) \geq \mathcal{G}(s_0 - \delta + \epsilon) > \mathcal{G}(s_0 - \delta)$. However, this contradicts the fact $J(u(t)) \leq \mathcal{G}(s_0 - \delta)$ for all $t \geq 0$. □

3.2 Uniform decay rates of the energy

In this section, we study the uniform decay rate of the energy of the global solutions furnished by Theorem 1.5.14. The latter result was presented in Theorem 1.5.15 whose proof is given below. For a shorthand let us define the function

$$G(t) := \int_0^t (\mathscr{G}(u_t), u_t)_\Omega d\tau, \tag{3.2.1}$$

which is non-negative by the monotonicity of g_i. The energy identity (3.1.1) then reads

$$\mathscr{E}(t) + G(t) = \mathscr{E}(0). \tag{3.2.2}$$

We will show that $\mathscr{E}(t)$ decays as the solution to a monotonic ODE of the form

$$S'(t) + H(S(t)) = 0, \quad S(0) = \mathscr{E}(0),$$

with the map H given by $H = \left(I + \hat{C}(1 + C_T)\Phi)\right)^{-1}$ for a certain concave increasing function Φ that vanishes at 0. The next subsection shows how H and Φ are constructed.

3.2.1 Constructing concave maps that quantify the behavior of the damping

The map Φ will be a sum of concave maps φ_j each characterizing the growth of the corresponding damping term g_j. Proceed as in [30, 31]: let $\varphi_j : [0, \infty) \mapsto [0, \infty)$ be continuous, increasing, concave functions vanishing at the origin, such that

$$\varphi_j(g_j(s)s) \geq |g_j(s)|^2 + s^2, \quad \text{for } |s| < 1, \ j = 1, 2, 3. \tag{3.2.3}$$

We then define $\Phi : [0,\infty) \mapsto [0,\infty)$ by

$$\Phi(s) := \varphi_1(s) + \varphi_2(s) + \varphi_3(s) + s, \quad s \geq 0. \tag{3.2.4}$$

Note that φ_1, φ_2, and φ_3 with such properties can always be constructed since g_j are monotone continuous increasing functions passing through the origin. We give several examples here:

Example 3.2.1 (Constructing g_i for a linearly bonded or superlinear feedback)**.** Suppose g_i, for some $i \in \{1,2,3\}$, grows linearly or super linearly near the origin, i.e.,

$$c_{2i-1}|s|^{\xi_i} \leq |g_i(s)| \leq c_{2i}|s|^{\xi_i}, \quad \text{for all} \quad |s| < 1, \tag{3.2.5}$$

where $c_{2i-1} > 0$, $c_{2i} > 0$, and $\xi_i \geq 1$. Define

$$\varphi_i(s) = c_{2i-1}^{-\frac{2}{\xi_i+1}}(1 + c_{2i}^2)s^{\frac{2}{\xi_i+1}}. \tag{3.2.6}$$

It is easy to see the function (3.2.6) satisfis (3.2.3). In particular, we note that, if g_i is linearly bounded near the origin (Definition 1.5.2), φ_i can be chosen to be linear functions.

Example 3.2.2 (Sublinear feedback)**.** If feedback g_i, for some $i \in \{1,2,3\}$, is bounded by sublinear functions near the origin, namely, for all $|s| < 1$,

$$c_{2i-1}|s|^{\frac{1}{\xi_i}} \leq |g_i(s)| \leq c_{2i}|s|^{\frac{1}{\xi_i}}, \tag{3.2.7}$$

where $0 < \xi_i^{-1} < 1$ (i.e., $\xi_i > 1$), $c_{2i-1} > 0$, and $c_{2i} > 0$, then we can let

$$\varphi_i(s) = c_{2i-1}^{-\frac{2}{\xi_i+1}}\left(1 + c_{2i}^2\right)s^{\frac{2}{\xi_i+1}}. \tag{3.2.8}$$

Summarizing the two examples from (3.2.6) and (3.2.8), we see that there exist constants $C_i > 0$, $i = 1, 2, 3$ such that

$$\varphi_i(s) = C_i s^{z_i}, \text{ where } z_i = \frac{2}{\theta_i + 1} \text{ for } \theta_i \geq 1\,. \tag{3.2.9}$$

where constants θ_i depend on whether the g_i's grow super-linearly or sub-linearly near the origin, as specified by (3.2.5) and (3.2.7), respectively. Define

$$a := \max_{i=1,2,3} \{1/z_i\} = (\max_{i=1,2,3} \theta_i + 1)/2. \tag{3.2.10}$$

Note that *if at least one* of g_i, $i = 1, 2, 3$, is not linearly bounded at infinity, then $a > 1$ and in this case we define

$$b := \frac{1}{a-1} > 0. \tag{3.2.11}$$

3.2.2 Perturbed stabilization estimate

The main result of this section is to establish the following "stabilization" inequality.

Proposition 3.2.3. *In addition to Assumption 1.5.1 and Assumption 1.5.10, further assume that $p > 1$, $u_0 \in \mathscr{W}_1$ and $\mathscr{E}(0) < d$. Then the global solution u of* (1.2.1) *furnished by Theorem 1.5.14 satisfies:*

$$\mathscr{E}(T) \leq \hat{C}_T \Big[\Phi(G(T)) + \sup_{s\in[0,T]} \|u(s)\|_2^2 \Big], \text{ for all } T > 0, \tag{3.2.12}$$

where Φ is given in (3.2.4)*, and $\hat{C}_T = C_{\rho,|\Omega|,T} \cdot (1 + (\mathscr{E}(0))^{p-1})$*

Proof. Let $T > 0$ be fixed. Exploiting the fact that $u = (w, \psi, \phi) \in C([0, T]; (H_0^1(\Omega))^3)$ and $\dim \Omega = 2$, the Sobolev embedding implies that u possesses the requisite test function regularity as stated in Definition 1.5.3. Consequently, replacing θ by u in (1.5.2) yields

$$(u_t(t), u(t))_\Omega\Big|_0^T + \int_0^T (\|u\|_V^2 - \|u_t\|_2^2)dt + \int_0^T (\mathscr{G}(u_t), u)_\Omega dt = \int_0^T (\mathscr{F}(u), u)_\Omega dt. \tag{3.2.13}$$

Recalling (1.5.5) and (1.5.6), one obtains

$$\int_0^T E(t)dt \leq \frac{1}{2}\left|-(u_t(T), u(T))_\Omega + (u_t(0), u(0))_\Omega\right| + \int_0^T \|u_t\|_2^2 dt + \frac{1}{2}\int_0^T \int_\Omega |\mathscr{G}(u_t)\cdot u|\, d\mathbf{x}dt + M(p+1)\int_0^T \|u\|_{p+1}^{p+1}dt. \tag{3.2.14}$$

Now, we will estimate each term on the right-hand side of (3.2.14).

Step 1. Estimate for $\left|-(u_t(T), u(T))_\Omega + (u_t(0), u(0))_\Omega\right|$. We have

$$\begin{aligned} &\left| - (u_t(T), u(T))_\Omega + (u_t(0), u(0))_\Omega\right| \\ &\quad\leq \frac{\epsilon}{2}(\|u_t(T)\|_2^2 + \|u_t(0)\|_2^2) + \frac{2}{\epsilon}(\|u(T)\|_2^2 + \|u(0)\|_2^2) \\ &\quad\leq \epsilon(E(T) + E(0)) + \frac{1}{\epsilon}\sup_{s\in[0,T]} \|u(s)\|_2^2 . \end{aligned} \tag{3.2.15}$$

Thus, by inequality (IV) of Theorem 1.5.14 and (3.2.2), one has for all $T \geq 0$ that

$$\begin{aligned} \left|-(u_t(T), u(T))_\Omega + (u_t(0), u(0))_\Omega\right| &\leq \epsilon\rho\,(\mathscr{E}(T) + \mathscr{E}(0)) + \frac{1}{\epsilon}\sup_{s\in[0,T]} \|u(s)\|_2^2 . \\ &\leq \epsilon\rho(2\mathscr{E}(T) + G(T)) + \frac{1}{\epsilon}\sup_{s\in[0,T]} \|u(s)\|_2^2 . \end{aligned} \tag{3.2.16}$$

Step 2. Estimate for

$$\int_0^T \|u\|_{p+1}^{p+1}dt.$$

Since $p > 1$, the Sobolev embedding theorem in 2D: $H_0^1(\Omega) \hookrightarrow L^s(\Omega)$, $1 \leq s < \infty$ and

Proposition A.0.2 (in the Appendix) yield

$$\|w\|_{2p}^{2p} \leq C\|w\|_{1,\Omega}^{2p} \leq C\|u\|_V^{2p}. \tag{3.2.17}$$

Thus,

$$\|w\|_{p+1}^{p+1} = \int_\Omega |w|^p |w| d\mathbf{x} \leq \|w\|_{2p}^p \|w\|_2 \leq \epsilon_0 \|u\|_V^{2p} + \frac{1}{4\epsilon_0}\|w\|_2^2, \tag{3.2.18}$$

where we exploited Hölder's and Young's inequalities. Applying Theorem 1.5.14's (I) and (IV), we find

$$\|u\|_V^2 \leq 2E(t) \leq 2\rho\mathscr{E}(t) \leq 2\rho\mathscr{E}(0), \quad \text{where } \rho = \frac{p+1}{p-1}. \tag{3.2.19}$$

Since $p > 1$, combining (3.2.18) and (3.2.19) yields

$$\|w\|_{p+1}^{p+1} \leq C \cdot \epsilon_0 (2E(t))^p + \frac{1}{4\epsilon_0}\|w\|_2^2 \leq C \cdot \epsilon_0 (2\rho\mathscr{E}(0))^{p-1} E(t) + \frac{1}{4\epsilon_0}\|w\|_2^2. \tag{3.2.20}$$

For each $\epsilon > 0$, if we choose $\epsilon_0 = \frac{\epsilon}{3C\cdot(2\rho\mathscr{E}(0))^{p-1}}$, (3.2.20) gives

$$\|w\|_{p+1}^{p+1} \leq \frac{\epsilon}{3}E(t) + \frac{3C\cdot(2\rho\mathscr{E}(0))^{p-1}}{4\epsilon}\|w\|_2^2 = \frac{\epsilon}{3}E(t) + C_{\epsilon,\rho}\cdot(\mathscr{E}(0))^{p-1}\|w\|_2^2, \tag{3.2.21}$$

(and similarly for ψ and ϕ).

Therefore,

$$\int_0^T \|u\|_{p+1}^{p+1} dt \leq \epsilon \int_0^T E(t)dt + C_{\epsilon,\rho} \cdot (\mathscr{E}(0))^{p-1} \int_0^T \|u\|_2^2 dt$$
$$\leq \epsilon \int_0^T E(t)dt + C_{\epsilon,\rho} \cdot (\mathscr{E}(0))^{p-1} \cdot T \cdot \sup_{s\in[0,T]} \|u(s)\|_2^2. \tag{3.2.22}$$

Step 3. Estimate for

$$\int_0^T \|u_t\|_2^2 dt.$$

Introduce the sets:

$$A := \{(x,t) \in Q_T \ : \ |u_t(x,t)| < 1\}, \text{ and } B := \{(x,t) \in Q_T \ : \ |u_t(x,t)| \geq 1\}. \tag{3.2.23}$$

By Assumption 1.5.1, we know $g_1(s)s \geq \alpha |s|^{m+1}$ for $|s| \geq 1$. Therefore, apply (3.2.3) as well as the fact that φ_1 is concave and increasing to obtain via ("reversed") Jensen's inequality

$$\int_0^T \|w_t\|_2^2 dt = \int_A |w_t|_2^2 d\mathbf{x}dt + \int_B |w_t|_2^2 d\mathbf{x}dt$$
$$\leq \int_A \varphi_1(g_1(w_t)w_t) d\mathbf{x}dt + \frac{1}{\alpha}\int_B g_1(w_t)w_t d\mathbf{x}dt$$
$$\leq \max\{1, T|\Omega|\}\varphi_1\left(\int_0^T \int_\Omega g_1(w_t)w_t d\mathbf{x}dt\right) + \frac{1}{\alpha}\int_0^T \int_\Omega g_1(w_t)w_t d\mathbf{x}dt, \tag{3.2.24}$$

(analogously for ψ_t, φ_2, g_2, or ϕ_t, φ_3, g_3),

In (3.2.24) we therefore set $C(T, |\Omega|) = 1 + T|\Omega| + 1/\alpha$ and get

$$\int_0^T \|u_t\|_2^2 dt \leq \max\{1, T|\Omega|\}\Phi(G(T)) + \frac{1}{\alpha}G(T) \leq C(T, |\Omega|)\Phi(G(T)). \tag{3.2.25}$$

Step 4. Estimate for

$$\int_0^T \int_\Omega |\mathscr{G}(u_t) \cdot u| \, d\mathbf{x}dt.$$

Let us now focus on $\int_0^T \int_\Omega |g_1(w_t)w|\, d\mathbf{x}dt$. Recall A and B from (3.2.23). We have

$$\begin{aligned}\int_0^T \int_\Omega |g_1(w_t)w|\, d\mathbf{x}dt &= \int_A |g_1(w_t)w|\, d\mathbf{x}dt + \int_B |g_1(w_t)w|\, d\mathbf{x}dt \\ &\leq \left(\int_0^T \|w\|_2^2 dt\right)^{\frac{1}{2}} \left(\int_A |g_1(w_t)|^2 d\mathbf{x}dt\right)^{\frac{1}{2}} + \int_B |g_1(w_t)w|\, d\mathbf{x}dt \\ &\leq \epsilon \int_0^T E(t)dt + C_\epsilon \int_A |g_1(w_t)|^2 d\mathbf{x}dt + \int_B |g_1(w_t)w|\, d\mathbf{x}dt, \qquad (3.2.26)\end{aligned}$$

where we have used Hölder's and Young's inequalities. Fromx inequality (3.2.3) and Jensen's inequality (as in (3.2.24)), we have

$$\begin{aligned}\int_A |g_1(w_t)|^2\, d\mathbf{x}dt &\leq \int_A \varphi_1(g_1(w_t)w_t) d\mathbf{x}dt \\ &\leq \max\{1, T|\Omega|\}\varphi_1\left(\int_0^T \int_\Omega g_1(w_t)w_t d\mathbf{x}dt\right). \qquad (3.2.27)\end{aligned}$$

Next, we estimate the last term on the right-hand side of (3.2.26). By Assumption 1.5.1, we know $g_1(s) \leq \beta|s|^m$ for $|s| \geq 1$. Therefore, from Hölder's inequality, we deduce

$$\begin{aligned}\int_B |g_1(w_t)w| d\mathbf{x}dt &\leq \left(\int_B |w|^{m+1}\, d\mathbf{x}dt\right)^{\frac{1}{m+1}} \left(\int_B |g_1(w_t)|^{\frac{m+1}{m}}\, d\mathbf{x}dt\right)^{\frac{m}{m+1}} \\ &\leq \beta^{\frac{1}{m}} \left(\int_0^T \|w\|_{m+1}^{m+1} d\mathbf{x}dt\right)^{\frac{1}{m+1}} \left(\int_B |g_1(w_t)|\, |w_t|\, d\mathbf{x}dt\right)^{\frac{m}{m+1}}. \qquad (3.2.28)\end{aligned}$$

By recalling $m \geq 1$ and the fact that $E(t) \leq d\rho$, for $t \geq 0$ in Theorem 1.5.14, we have

$$\int_0^T \|w\|_{m+1}^{m+1} d\mathbf{x}dt \leq C \int_0^T E(t)^{\frac{m+1}{2}} dt \leq C_{d,\rho} \int_0^T E(t)dt. \qquad (3.2.29)$$

Since g_1 is monotone increasing, combining (3.2.28) and (3.2.29) yields

$$\int_B |g_1(w_t)w| d\mathbf{x}dt \leq \beta^{\frac{1}{m}} C_{d,\rho} \left(\int_0^T E(t)dt\right)^{\frac{1}{m+1}} \left(\int_0^T \int_\Omega g_1(w_t)w_t d\mathbf{x}dt\right)^{\frac{m}{m+1}}$$
$$\leq \epsilon \int_0^T E(t)dt + C_\epsilon \int_0^T \int_\Omega g_1(w_t)w_t d\mathbf{x}dt. \tag{3.2.30}$$

Apply (3.2.27) and (3.2.30) to (3.2.26) to arrive at

$$\int_0^T \int_\Omega |g_1(w_t)w| \, d\mathbf{x}dt$$
$$\leq 2\epsilon \int_0^T E(t)dt + C_\epsilon \max\{1, T|\Omega|\}\varphi_1 \left(\int_0^T \int_\Omega g_1(w_t)w_t d\mathbf{x}dt\right)$$
$$+ C_\epsilon \int_0^T \int_\Omega g_1(w_t)w_t d\mathbf{x}dt. \tag{3.2.31}$$

Similarly, we can obtain analogous estimates for g_2 and g_3. Hence, from the fact that $s \leq \Phi(s)$ for all $s \geq 0$, we have

$$\int_0^T \int_\Omega |\mathscr{G}(u_t) \cdot u| \, d\mathbf{x}dt \leq 6\epsilon \int_0^T E(t)dt + C_\epsilon \max\{1, T|\Omega|\}\Phi(G(T)) + C_\epsilon G(T)$$
$$\leq 6\epsilon \int_0^T E(t)dt + C(\epsilon, |\Omega|, T)\Phi(G(T)). \tag{3.2.32}$$

Now, if we apply the estimates (3.2.16), (3.2.22), (3.2.25) and (3.2.32) to (3.2.14), we conclude

$$\int_0^T E(t)dt \leq 4\epsilon \int_0^T E(t)dt + \frac{\epsilon\rho}{2}\Big(2\mathscr{E}(T) + G(T)\Big) + C(\epsilon, |\Omega|, T)\Phi(G(t))$$
$$+ \left(\frac{1}{2\epsilon} + C_{\epsilon,\rho} \cdot (\mathscr{E}(0))^{p-1}\right) \sup_{s\in[0,T]} \|u(s)\|_2^2. \tag{3.2.33}$$

Hence, for any fixed T by selecting $\epsilon \le \min\{\frac{1}{8}, \frac{T}{4\rho}\}$, we have

$$\frac{1}{2}\int_0^T E(t)dt \le \frac{T}{8}\Big(2\mathscr{E}(T) + G(T)\Big) + C(\rho, |\Omega|, T)\Phi(G(t)) + C_{T,\rho} \cdot \big(1 + (\mathscr{E}(0))^{p-1}\big) \sup_{s\in[0,T]} \|u(s)\|_2^2. \tag{3.2.34}$$

Since $E(t) \ge \mathscr{E}(t)$ for all $t \ge 0$ and $\mathscr{E}(t)$ is non-increasing, one gets

$$\int_0^T E(t)dt \ge \int_0^T \mathscr{E}(t)dt \ge T\mathscr{E}(T). \tag{3.2.35}$$

Thus, (3.2.34) and (3.2.35) yield

$$\frac{T}{2}\mathscr{E}(T) \le \frac{T}{4}T\mathscr{E}(T) + \frac{T}{8}G(T) + C(\rho, |\Omega|, T)\Phi(G(t)) + C_{T,\rho} \cdot \big(1 + (\mathscr{E}(0))^{p-1}\big) \sup_{s\in[0,T]} \|u(s)\|_2^2. \tag{3.2.36}$$

Dividing by $T > 0$, we obtain the inequality

$$\frac{1}{4}\mathscr{E}(T) \le \frac{1}{8}G(T) + C(\rho, |\Omega|, T)\Phi(G(t)) + C_{\rho,T} \cdot \big(1 + (\mathscr{E}(0))^{p-1}\big) \sup_{s\in[0,T]} \|u(s)\|_2^2. \tag{3.2.37}$$

Finally, since $G(T) \le \Phi(G(t))$, if putting

$$\hat{C}_T = 4(\frac{1}{8} + C(\rho, |\Omega|, T) + C_{\rho,T} \cdot (1 + (\mathscr{E}(0))^{p-1})) \equiv C_{\rho,|\Omega|,T} \cdot \big(1 + (\mathscr{E}(0))^{p-1}\big),$$

we use (3.2.37) to estimate

$$\mathscr{E}(T) \le \hat{C}_T\Big(\Phi(G(t)) + \sup_{s\in[0,T]} \|u(s)\|_2^2\Big), \tag{3.2.38}$$

for all $T > 0$. This completes the proof of Proposition 3.2.3. □

3.2.3 Compactness-uniqueness argument

The next step eliminates the "lower order terms" — semi-norms of the solution in topologies coarser than that of the finite energy space — present in (3.2.12). This is accomplished by a standard compactness-uniqueness argument.

Proposition 3.2.4 (Absorption of the lower order terms)**.** *In addition to Assumption 1.5.1 and Assumption 1.5.10 with* $p > 1$ *, further assume* $u_0 \in \tilde{\mathscr{W}}_1^\delta$ *(defined in* (1.5.25)*) and* $\mathscr{E}(0) \leq \mathcal{G}(s_0 - \delta)$ *(*$\mathcal{G}$ *as defined in* (1.5.17)*) for some* $0 < \delta < s_0$*. Then, for any* $T > 0$ *there exists a constant* $C_T > 0$ *such solution u of the system* (1.2.1) *furnished by Theorem 1.5.14 satisfies the inequality*

$$\sup_{s\in[0,T]} \|u(s)\|_2^2 \leq C_T \Phi(G(T)). \tag{3.2.39}$$

Proof. We follow the standard compactness-uniqueness approach (see for instance [25, 30]) and argue by contradiction. For this proof fix any $T > 0$.

Step 1. Constructing a sequence of solutions from the contradiction hypothesis. To argue by contradiction, we assume that we can find a sequence of initial data

$$\{(u_n(0), u_n'(0))\}_{n=1}^\infty \subset \tilde{\mathscr{W}}_1^\delta \times \left(L^2(\Omega)\right)^3$$

such that

$$\mathscr{E}_n(0) \leq \mathscr{G}(s_0 - \delta) < d \tag{3.2.40}$$

and the corresponding weak solutions $\{u_n = (w_n, \psi_n, \phi_n)\}$ satisfy

$$\sup_{s\in[0,T]} \|u_n(s)\|_2^2 > n\Phi(G_n(T)), \quad \text{for all} \quad n \in \mathbb{N},$$

and hence verify

$$\lim_{n\to\infty} \frac{\Phi(G_n(T))}{\sup\limits_{s\in[0,T]} \|u_n(s)\|_2^2} = 0, \tag{3.2.41}$$

where

$$G_n(t) := \int_0^t \int_\Omega g_1(w_{nt})w_{nt} + g_2(\psi_{nt})\psi_{nt} + g_3(\phi_{nt})\phi_{nt} d\mathbf{x}d\tau\,. \tag{3.2.42}$$

Step 1A. Find a convergent subsequence. Because the solutions come from the "good" part of the potential well and satisfy (3.2.40), the energy estimate in Theorem 1.5.14 implies

$$0 \le \sup_{s\in[0,T]} \|u_n(s)\|_2^2 \le C \sup_{s\in[0,T]} E_n(s) \le Cd\rho \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.2.43}$$

Estimate (3.2.43) shows that $\{(u_n, u_{nt})\}$ is a bounded sequence in $L^\infty(0,T;H)$, where $H = V \times \left(L^2(\Omega)\right)^3$ and $V = (H_0^1(\Omega))^3$. Hence, by Alaoglu's theorem, there exists a subsequence, reindexed again by n, such that

$$(u_n, u_{nt}) \longrightarrow (u, u_t) \quad \text{weakly* in} \quad L^\infty(0,T;H)\,. \tag{3.2.44}$$

In addition, since for any $0 < \epsilon < 1$, the embedding $H_0^1(\Omega) \hookrightarrow H^{1-\epsilon}(\Omega)$ is compact, then by Simon's compactness theorem [49] , there is a subsequence (again reindexed by n)

$$u_n \to u \;\text{ strongly in }\; L^\infty(0,T;(H^{1-\epsilon}(\Omega))^3). \tag{3.2.45}$$

Moreover, $u_n \in C([0,T];(H^{1-\epsilon}(\Omega))^3)$, thus the sequence is Cauchy in $C([0,T];(H^{1-\epsilon}(\Omega))^3)$ and

$$u \in C([0,T];(H^{1-\epsilon}(\Omega))^3)\,. \tag{3.2.46}$$

In the next two steps we shall show $u(t) = 0$ on $[0,T]$.

Step 1B. We start by showing $u(t) \in \mathscr{N} \cup \{0\}$ on $[0, T]$. By selecting a test function

$$\theta \in \left(C(\overline{Q_t}) \cap C([0,t]; H_0^1(\Omega))\right)^3$$

such that $\theta(0) = \theta(t) = 0$, and $\theta_t \in (L^2(Q_t))^3$, then equation (1.5.2) gives us

$$\int_0^t [(u_n, \theta)_V - (u_{nt}, \theta_t)_\Omega] d\tau + \int_0^t \int_\Omega \mathscr{G}(u_{nt}) \cdot \theta d\mathbf{x} d\tau = \int_0^t \int_\Omega \mathscr{F}(u_n) \cdot \theta d\mathbf{x} d\tau. \qquad (3.2.47)$$

First we look at the limit of

$$\int_0^t -(u_{nt}, \theta_t)_\Omega d\tau.$$

It follows from (3.2.41) and (3.2.43) that

$$\lim_{n\to\infty} \Phi(G_n(T)) = 0. \qquad (3.2.48)$$

Now (3.2.25) yields

$$\lim_{n\to\infty} \int_0^T \|u_{nt}(t)\|_2^2 dt = 0.$$

Therefore,

$$\lim_{n\to\infty} \int_0^t -(u_{nt}, \theta_t)_\Omega d\tau = 0, \quad \text{for all } t \in [0, T]. \qquad (3.2.49)$$

Next let's consider the limit of

$$\int_0^t \int_\Omega \mathscr{G}(u_{nt}) \cdot \theta d\mathbf{x} d\tau.$$

If we define

$$A_n := \{(x,t) \in Q_T : |w_{nt}(x,t)| < 1\}, \quad B_n := \{(x,t) \in Q_T : |w_{nt}(x,t)| \geq 1\}, \qquad (3.2.50)$$

then from

$$\int_0^T \int_\Omega |g_1(w_{nt})|^{\frac{m+1}{m}}\, d\mathbf{x}dt = \int_{A_n} |g_1(w_{nt})|^{\frac{m+1}{m}}\, d\mathbf{x}dt + \int_{B_n} |g_1(w_{nt})|^{\frac{m+1}{m}}\, d\mathbf{x}dt$$
$$\leq \beta^{\frac{m+1}{m}}|\Omega|T + \beta^{\frac{m+1}{m}}\frac{1}{\alpha}\int_0^T \int_\Omega g_1(w_{nt})w_{nt}d\mathbf{x}dt \tag{3.2.51}$$

and (3.2.48) (recall that Φ is monotonically increasing , vanishing at 0) we conclude that on a subsequence (re-indexed again by n), $w_{nt} \to 0$ a.e. in Q_T. Thus, from the continuity of g_1 we know that $g_1(w_{nt}) \to 0$ a.e. in Q_T. Hence,

$$g_1(w_{nt}) \to 0 \quad \text{weakly in} \quad L^{\frac{m+1}{m}}(Q_T) \qquad \text{and} \qquad \text{strongly in } L^1(Q_T)$$
$$\text{(analogously for } g_2, \psi_{nt}, r, \text{ and } g_3, \phi_{nt}, q). \tag{3.2.52}$$

In particular,

$$\lim_{n\to\infty}\int_0^t \int_\Omega \mathscr{G}(u_{nt})\cdot\theta d\mathbf{x}d\tau = 0, \quad \text{for all } t \in [0,T]. \tag{3.2.53}$$

Finally, we pass to the limit on

$$\int_0^t \int_\Omega \mathscr{F}(u_n)\cdot\theta d\mathbf{x}d\tau.$$

To this end, recall the definition (2.1.4) of $\mathscr{F}$. Exploiting for $j = 1,2,3$ the estimate

$$\left|\nabla f_j(u)\right| \leq C\left(|w|^{p-1} + |\psi|^{p-1} + |\phi|^{p-1} + 1\right),$$

we obtain

$$\int_{Q_t}(\mathscr{F}(u_n) - \mathscr{F}(u))\cdot\theta d\mathbf{x}d\tau = \int_{Q_t}\int_0^1 D\mathscr{F}(\lambda u + (1-\lambda u_n))d\lambda(u-u_n)\cdot\theta d\mathbf{x}d\tau$$
$$\leq C_\theta\left(\|u\|^{p-1}_{(L^p(\Omega))^3} + \|u_n\|^{p-1}_{(L^p(\Omega))^3}\right)\|u-u_n\|_{L^p(Q_t)},$$

where $D\mathscr{F}$ denotes the Jacobi matrix of $\mathscr{F}$. Therefore,

$$\lim_{n\to\infty}\int_0^t\int_\Omega \mathscr{F}(u_n)\cdot\theta d\mathbf{x}d\tau = \int_0^t\int_\Omega \mathscr{F}(u)\cdot\theta d\mathbf{x}d\tau, \quad \text{for all } \ t\in[0,T]. \tag{3.2.54}$$

Now combine the above limits. Using (3.2.44), (3.2.49), (3.2.53), and (3.2.54), we can pass to the limit in (3.2.47) to obtain

$$\int_0^t (u,\theta)_V d\tau = \int_0^t\int_\Omega \mathscr{F}(u)\cdot\theta d\mathbf{x}d\tau. \tag{3.2.55}$$

Fix an arbitrary $\tilde{\theta}\in\left(H_0^1(\Omega)\bigcap C(\overline{\Omega})\right)^3$ and substitute $\theta(x,y,\tau) := \tau(t-\tau)\tilde{\theta}(x,y)$ into (3.2.55). Since u is continuous by the virtue of (3.2.46), a two-fold differentiation with respect to t yields for every $t\in[0,T]$,

$$(u(t),\tilde{\theta})_V = \int_\Omega \mathscr{F}(u(t))\cdot\tilde{\theta}d\mathbf{x}. \tag{3.2.56}$$

Thus, for every $t\in[0,T]$ the function $u(t)$ is the distributional solution to the elliptic problem (1.5.12).

Pick a sequence $\tilde{\theta}_n\in(H_0^1(\Omega)\cap C(\overline{\Omega}))^3$ such that $\tilde{\theta}_n\to u(t)$ in $\left(H_0^1(\Omega)\right)^3$ for a fixed t. Taking $n\to\infty$ and using the continuity of $\mathscr{F}$, there follows

$$\|u(t)\|_V^2 = \int_\Omega \nabla F(u(t))\cdot u(t)d\mathbf{x} = (p+1)\int_\Omega F(u(t))d\mathbf{x} \ \text{ for } \ t\in[0,T]. \tag{3.2.57}$$

Thus, either $u(t)=0$ or $u\in\mathscr{N}$ for $t\in[0,T]$.

Step 1C. To prove $u(t)=0$ on $[0,T]$, it suffices to show that $u(t)\in\tilde{\mathscr{W}}_1^\delta\subset\mathscr{W}_1$ on $[0,T]$ (since $\mathscr{W}_1\cap\mathscr{N}=\emptyset$). Let us remind the reader that we already have: $\{u_n\}$ is bounded in $C([0,T];V)$ and $u_n\to u$ strongly in $C([0,T];(H^{1-\epsilon}(\Omega))^3)$. Since $u_n(0)\in\tilde{\mathscr{W}}_1^\delta$ and $E_n(0)<\mathscr{G}(s_0-\delta)$, Proposition 3.1.1 states that $u_n(t)\in\tilde{\mathscr{W}}_1^\delta$ for all $t\geq 0$. By the definition of $\tilde{\mathscr{W}}_1^\delta$ in

(1.5.25),

$$\|u_n(t)\|_V \leq s_0 - \delta \;\text{ and }\; J(u_n(t)) \leq \mathcal{G}(s_0 - \delta) \;\text{ for all }\; t \geq 0.$$

First, we aim to show that $\|u(t)\| \leq s_0 - \delta$. Note that, for each fixed t there exists a subsequence $u_{n_k}(t)$ convergent weakly to some χ in V. Moreover χ must coincide with $u(t)$. Thus,

$$\|u(t)\|_V \leq \liminf_{k\to\infty} \|u_{n_k}(t)\|_V \leq s_0 - \delta\,.$$

Moreover, since F is continuous, then (1.5.6) and the Lebesgue's dominated convergence theorem yield:

$$\lim_{k\to\infty} \int_\Omega F(u_{n_k}(t))d\mathbf{x} = \int_\Omega F(u(t))d\mathbf{x}.$$

Consequently, by taking $\liminf$ as $k \to \infty$ in

$$\mathcal{G}(s_0 - \delta) \geq J(u_{n_k}(t)) = \frac{1}{2}\left\|u_{n_k}(t)\right\|_V - \int_\Omega F(u_{n_k}(t))d\mathbf{x}, \tag{3.2.58}$$

one has

$$\mathcal{G}(s_0 - \delta) \geq J(u(t)) \text{ on } [0, T]. \tag{3.2.59}$$

Hence, $u(t) \in \tilde{\mathscr{W}}_1^{\delta} \subset \mathscr{W}_1$ on $[0, T]$. Thus, it must be the case that

$$u(t) = 0 \;\text{ on }\; [0, T]. \tag{3.2.60}$$

Step 2: Construct a re-normalized sequence of solutions converging to 0 from the contradiction hypothesis. Define

$$N_n := \sup_{s\in[0,T]} \|u_n(s)\|_2\,.$$

(3.2.46) and (3.2.60) imply

$$N_n \to 0, \;\text{ as }\; n \to \infty. \tag{3.2.61}$$

Set $y_n := u_n/N_n$, whence

$$\sup_{s\in[0,T]} \|y_n(s)\|_2^2 \equiv 1. \tag{3.2.62}$$

Each y_n satisfies the variational identity

$$\int_0^t \left[(y_n,\theta)_V - (y_{nt},\theta_t)_\Omega\right] d\tau + \int_0^t \int_\Omega \frac{\mathscr{G}(u_{nt})}{N_n} \cdot \theta d\mathbf{x} d\tau = \int_0^t \int_\Omega \frac{\mathscr{F}(u_n)}{N_n} \cdot \theta d\mathbf{x} d\tau, \tag{3.2.63}$$

where $\theta \in \left(C(\overline{Q_t}) \cap C([0,t];H_0^1(\Omega))\right)^3$ so that $\theta(0) = \theta(t) = 0$, and $\theta_t \in (L^2(Q_t))^3$.

Step 2A. Identify the limit of the variational formulation (3.2.63). By the contradiction hypothesis (3.2.41) and (3.2.62), we have

$$\lim_{n\to\infty} \frac{\Phi(G_n(T))}{N_n^2} = 0, \tag{3.2.64}$$

and along with (3.2.25), we obtain

$$\lim_{n\to\infty} \frac{1}{N_n^2} \sup_{s\in[0,T]} \|u_{nt}(s)\|_2^2 = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} \sup_{s\in[0,T]} \|y_{nt}(s)\|_2^2 = 0. \tag{3.2.65}$$

Let $\mathscr{E}_n$ be the total energy (1.5.9) corresponding to the solution $\{u_n\}$. The inequalities (III) and (IV) of Theorem 1.5.14 show that $0 \leq \mathscr{E}_n(t) \leq d\rho$ for all $t \geq 0$. Also by the intermediate stabilization estimate (3.2.12) as well as equations (3.2.62), (3.2.64), we obtain

$$\limsup_{n\to\infty} \frac{\mathscr{E}_n(T)}{N_n^2} \leq \hat{C}\,.$$

The energy identity (3.2.2) yields $\mathscr{E}_n(t) + G_n(t) = \mathscr{E}_n(0)$, in particular $\mathscr{E}_n(t)/N_n^2 \leq \mathscr{E}_n(0)/N_n^2$. Then, inequality (IV) in Theorem 1.5.14 guarantees that

$$\left\{ \frac{E_n(t)}{N_n^2} = \frac{1}{2}\left(\|y_n\|_V^2 + \|y_{nt}\|_2^2\right) \right\}$$

is uniformly bounded on $[0, T]$, where E_n is the quadratic energy given in (1.5.3) corresponding to u_n. Therefore, $\{(y_n, y_{nt})\}$ is a bounded sequence in $L^\infty(0, T; H)$ where $H = (H_0^1(\Omega))^3 \times (L^2(\Omega))^3$. In particular, by Alaoglu's theorem, on a subsequence,

$$y_n \to y \text{ weakly* in } L^\infty(0, T; V) . \tag{3.2.66}$$

As in the case with u_n, Simon's compactness result now yields

$$y_n \to y \text{ strongly in } L^\infty(0, T; (H^{1-\epsilon}(\Omega))^3), \tag{3.2.67}$$

Note (3.2.62) and (3.2.67) show that

$$\lim_{n\to\infty} \sup_{s\in[0,T]} \|y_n(s)\|_2^2 = \sup_{s\in[0,T]} \|y(s)\|_2^2 \, dt = 1. \tag{3.2.68}$$

Thus, the limiting function y is non-trivial.

Step 2B. Get a contradiction by showing $y(t) = 0$ for all $t \in [0, T]$.

First we claim that

$$\lim_{n\to\infty} \int_0^t \int_\Omega \frac{\mathscr{G}(u_{nt})}{N_n} \cdot \theta d\mathbf{x} d\tau = 0 \text{ for all } t \in [0, T]. \tag{3.2.69}$$

Since $\theta \in C(\overline{Q_T})$, it suffices to show $g_1(w_{nt})/N_n \to 0$ in $L^1(Q_T)$. We will prove

$$\frac{g_1(w_{nt})}{N_n} \to 0 \text{ strongly in } L^{\frac{m+1}{m}}(Q_T), \tag{3.2.70}$$

(analogously for g_2, ψ_{nt}, r, or g_3, ϕ_{nt}, q).

Recall the definition of the sets A_n and B_n in (3.2.50). We may assume $N_n < 1$, so

$$\begin{aligned}
&\int_0^T \int_\Omega \left|\frac{g_1(w_{nt})}{N_n}\right|^{\frac{m+1}{m}} d\mathbf{x}dt \\
&\leq |Q_T|^{\frac{m-1}{2m}} \left(\int_{A_n} \left|\frac{g_1(w_{nt})}{N_n}\right|^2 d\mathbf{x}dt\right)^{\frac{m+1}{2m}} + \frac{1}{N_n^2}\int_{B_n} |g_1(w_{nt})|^{\frac{m+1}{m}} d\mathbf{x}dt.
\end{aligned}$$

From (3.2.27) (which is based upon Jensen's inequality) and the bound on the g_i's (1.5.1), one has

$$\begin{aligned}
&\int_0^T \int_\Omega \left|\frac{g_1(w_{nt})}{N_n}\right|^{\frac{m+1}{m}} d\mathbf{x}dt \\
&\leq |Q_T|^{\frac{m-1}{2m}} \left(\frac{1}{N_n^2}\int_{A_n} \varphi_1\left(g_1(w_{nt})w_{nt}\right) d\mathbf{x}dt\right)^{\frac{m+1}{2m}} + \frac{\beta^{\frac{m+1}{m}}}{\alpha N_n^2}\int_{B_n} g_1(w_{nt})w_{nt} d\mathbf{x}dt \\
&\leq C(T, |\Omega|)\left(\frac{\Phi(G_n(T))}{N_n^2}\right)^{\frac{m+1}{2m}} + \frac{\beta^{\frac{m+1}{m}}}{\alpha}\left(\frac{\Phi(G_n(T))}{N_n^2}\right) \to 0, \text{ as } n \to \infty,
\end{aligned}$$

where $G_n(T)$ as defined in (3.2.42). Thus, (3.2.70) follows.

Next we'll show

$$\lim_{n\to\infty}\int_0^t \int_\Omega \frac{\mathscr{F}(u_n)}{N_n} \cdot \theta d\mathbf{x}d\tau = 0. \tag{3.2.71}$$

Recall that $u_n = (w_n, \psi_n, \phi_n)$ and $y_n = u_n/N_n = (y_{n1}, y_{n2}, y_{n3})$. To estimate the terms in (3.2.71) we use (1.5.7) and obtain

$$\begin{aligned}
&\int_0^t \int_\Omega \left|\frac{f_j(u_n)}{N_n}\theta_j\right| d\mathbf{x}d\tau \\
&\quad\leq C_\theta \int_0^t \int_\Omega \left(|y_{n1}|\,|w_n|^{p-1} + |y_{n2}|\,|\psi_n|^{p-1} + |y_{n3}|\,|\phi_n|^{p-1}\right) d\mathbf{x}d\tau\,.
\end{aligned}\tag{3.2.72}$$

By the virtue of (3.2.45), (3.2.60) and (3.2.67), it follows that

$$\int_0^T \int_\Omega |y_{n1}|\,|w_n|^{p-1}\, d\mathbf{x}dt \le \|y_{n1}\|_{L^p(Q_T)}\,\|w_n\|^{p-1}_{L^p(Q_T)}$$

$$\le \|y_n\|_{L^p(Q_T)}\,\|u_n\|^{p-1}_{L^p(Q_T)} \longrightarrow 0, \tag{3.2.73}$$

$$\text{(analogously for } y_{n2}, \psi_n, \text{ or } \; y_{n3}, \phi_n)$$

which verifies (3.2.71).

Finally we with (3.2.69) and (3.2.71) at hand we pass to the limit in normalized variational identity (3.2.63). Applying (3.2.65), (3.2.69), (3.2.66) (3.2.73) to (3.2.63) yields

$$\int_0^t (y(\tau), \theta(\tau))_V d\tau = 0, \quad \text{for all } \; t \in (0, T). \tag{3.2.74}$$

As before, fix an arbitrary $\tilde{\theta} \in \left(H_0^1(\Omega) \bigcap C(\overline{\Omega})\right)^3$ and substitute $\theta(x, y, \tau) := \tau(t - \tau)\tilde{\theta}(x, y)$ into (3.2.74). Differentiating the result twice with respect to t yields

$$(y, \tilde{\theta})_V = 0, \quad \text{for all } \; t \in (0, T), \tag{3.2.75}$$

which by density implies $y(t) = 0$ in V for all $t \in (0, T)$. Essentially, this is the uniqueness statement for the linearized elliptic problem (1.5.12). However, this conclusion contradicts the fact (3.2.68) that y is nonzero. Hence, we have finished the proof of Proposition 3.2.4.

□

3.2.4 Completing the proof of Theorem 1.5.15

With the perturbed "stabilization estimate" (3.2.12) and the lower-order estimate (3.2.39) at hand we can adopt the now-classical approach [30, Lemma 3.3] to construct an ODE whose solution quantifies the decay of the total energy $\mathscr{E}$ (and therefore via Theorem 1.5.14 (IV) the one of the finite energy E) as $t \to \infty$.

Proof of Theorem 1.5.15. Combining Proposition 3.2.3 and Proposition 3.2.4 yields

$$\mathscr{E}(T) \leq \hat{C}_T(1 + C_T)\Phi(G(T)) \text{ for all } T > 0,$$

where $\hat{C}_T$ and C_T are given in (3.2.12) and (3.2.39). Define $\Phi_T = \hat{C}_T(1 + C_T)\Phi$, to get

$$\mathscr{E}(T) \leq \Phi_T(G(T)) = \Phi_T(\mathscr{E}(0) - \mathscr{E}(T)),$$

which implies

$$\mathscr{E}(T) + \Phi_T^{-1}(\mathscr{E}(T)) \leq \mathscr{E}(0).$$

By iterating the estimate on intervals $[mT, (m+1)T]$, $m = 0, 1, 2, \ldots$, we have

$$\mathscr{E}((m+1)T) + \Phi_T^{-1}(\mathscr{E}((m+1)T)) \leq \mathscr{E}(mT), m = 0, 1, 2, \ldots.$$

Note that Φ_T does not depend on m here because the system is autonomous. Therefore, by [30, Lemma 3.3], one has

$$\mathscr{E}(mT) \leq S(m) \text{ for all } m = 0, 1, 2, \ldots, \tag{3.2.76}$$

where S is the solution to the ODE:

$$S' + \left[I - \left(I + \Phi_T^{-1}\right)^{-1}\right](S) = 0, \quad S(0) = \mathscr{E}(0). \tag{3.2.77}$$

Note also that

$$I - \left(I + \Phi_T^{-1}\right)^{-1} = (I + \Phi_T)^{-1},$$

allowing for the ODE (3.2.77) to be reduced to:

$$S' + (I + \Phi_T)^{-1}(S) = 0, \quad S(0) = \mathscr{E}(0), \tag{3.2.78}$$

where (3.2.78) has a unique solution defined on $[0, \infty)$. Since Φ_T is increasing and passing through the origin, $(I + \Phi_T)^{-1}$ is also increasing and vanishing at zero. Hence, the solution of the monotone autonomous ODE satisfies $S(t) \to 0$ as $t \to \infty$.

For any $t > T > 0$, there exists $m \in \mathbb{N}$ such that $t = mT + \delta$ with $0 \leq \delta < T$, and thus $m = \frac{t}{T} - \frac{\delta}{T} > \frac{t}{T} - 1$. By (3.2.76) and the fact that $\mathscr{E}(t)$ and $S(t)$ are monotone decreasing, we obtain

$$\mathscr{E}(t) = \mathscr{E}(mT + \delta) \leq \mathscr{E}(mT) \leq S(m) \leq S\left(\frac{t}{T} - 1\right), \text{ for any } t > T. \tag{3.2.79}$$

Thus, the proof of Theorem 1.5.15 is completed. □

3.2.5 Proof of Corollary 1.5.16 (exponential decay)

If g_i are linearly bounded near the origin, then (3.2.6) shows that φ_i are linear, and it follows that Φ_T is linear, implying $(I + \Phi_T)^{-1}$ is also linear. Therefore, the ODE (3.2.78) is of the form $S' + \omega S = 0, \quad S(0) = \mathscr{E}(0)$ (for some positive constant $\omega = \omega(T, \mathscr{E}(0))$) with the unique solution $S(t) = \mathscr{E}(0)e^{-\omega t}$. Thus, from (3.2.79) we know

$$\mathscr{E}(t) \leq \mathscr{E}(0)e^{-\omega\left(\frac{t}{T}-1\right)} = \left(e^{\omega}\mathscr{E}(0)\right)e^{(-\omega/T)t}, \text{ for all } t > T,$$

which provides the exponential estimate (1.5.28), where $\omega = \dfrac{1}{1 + C_T(1 + \mathscr{E}(0)^{p-1})}$

3.2.6 Proof of Corollary 1.5.17 (algebraic decay)

Recall that $\varphi_j(s) = C_j s^{z_j}$, where $0 < z_j \leq 1, j = 1, 2, 3$ are given in (3.2.9), as well as $a := 1/\min\{z_j\}$ according to (3.2.10).

Let $h = \varphi_1 + \varphi_2 + \varphi_3$. Obviously, we can find $h_b = C_h s^{\min\{z_i\}}$ and h_s such that $h = h_b + h_s$ satisfies the hypothesis of [31, Corollary 1 (p. 1770)], where C_h depends on C_1, C_2 and C_3. By in [31, Corollary 1 (p.1770)], we conclude that there exists $t_0 > 0$ such that

$$\mathscr{E}(t) \leq \hat{S}\left(\frac{t}{T} - 1\right) \quad \text{for all} \quad t \geq t_0, \tag{3.2.80}$$

where $\hat{S}$ is the solution of the ODE

$$\hat{S}'(t) + C_0 \hat{S}(t)^a = 0, \quad \hat{S}(t_0) = S(t_0). \tag{3.2.81}$$

Since the solution of (3.2.81) is given by

$$\hat{S}(t) = \left[C_0(a-1)(t-t_0) + S(t_0)^{1-a}\right]^{-\frac{1}{a-1}}, \quad \text{for all} \quad t \geq t_0,$$

the proof of Corollary 1.5.17 is completed with $b = \frac{1}{a-1}$.

3.3 Blow-up for small positive initial total energy

The proof of blow-up result of Theorem 1.5.19 relies on the following lemma which states that $\mathscr{W}_2$ as defined in (1.5.16) is an invariant set under the flow.

Lemma 3.3.1. *Under the validity of Assumption 1.5.1 and Assumption 1.5.10 with $p > 1$, we further assume that $u(0) \in \mathscr{W}_2$ and $\mathscr{E}(0) < d$. Then, the unique weak solution u*

furnished by Theorems 1.5.4 and 1.5.5 satisfies

$$u(t) \in \mathscr{W}_2 \ \text{for all} \ \ t \in [0, T), \tag{3.3.1}$$

$$\|u(t)\|_V^2 > 2\rho \cdot d \ \text{for all} \ \ t \in [0, T), \tag{3.3.2}$$

where $\rho := \frac{p+1}{p-1}$ *and* $[0, T)$ *is the maximal interval of existence.*

Proof. **Step 1.** Let $u(0) \in \mathscr{W}_2$, we first show that $u(t) \in \mathscr{W}_2$ for all $t \in [0, T)$. Arguing by contradiction, we assume that there exists $t_0 \in [0, T)$ such that $u(t_0) \notin \mathscr{W}_2$, which this implies

$$\|u(t_0)\|_V^2 \geq (p+1) \int_\Omega F(u(t_0)) d\mathbf{x}.$$

By recalling $u \in C([0, T); V)$, where $V = (H_0^1(\Omega))^3$, with the assumption $u(0) \in \mathscr{W}_2$, we conclude that there exists at least one $s \in (0, t_0]$ such that

$$\|u(s)\|_V^2 = (p+1) \int_\Omega F(u(s)) d\mathbf{x}.$$

Put

$$t^* := \inf \left\{ s \in (0, t_0]; \|u(s)\|_V^2 = (p+1) \int_\Omega F(u(s)) d\mathbf{x} \right\}.$$

The fact $u \in C([0, T), V)$ guarantees the existence of $t^* \in (0, t_0]$ such that

$$\|u(t^*)\|_V^2 = (p+1) \int_\Omega F(u(t^*)) d\mathbf{x}, \tag{3.3.3}$$

while $u(t) \in \mathscr{W}_2$ for all $t \in (0, t^*)$. Now, we have two cases to consider:

Case I : Suppose that $\|u(t^*)\|_V^2 \neq 0$. In this case, equation (3.3.3) implies that $u(t^*) \in \mathscr{N}$ (by the definition of the Nehari manifold $\mathscr{N}$ in (1.5.13)), and by Lemma 1.5.12, we know $J(u(t^*)) \geq d$. Thus, $\mathscr{E}(t^*) = \frac{1}{2}\|u_t(t^*)\|_2^2 + J(u(t^*)) > d$, contradicting the fact that total energy is decreasing, i.e., $\mathscr{E}(t) \leq \mathscr{E}(0) < d$, for all $t \in [0, T)$.

Case II : Suppose that $\|u(t^*)\|_V^2 = 0$. Then $u \in C([0, T), V)$ implies that $\lim_{t \to t^{*-}} \|u(t)\|_V^2 = 0$. Since $u(t) \in \mathscr{W}_2$ for all $t \in (0, t^*)$, utilizing a similar argument as in (3.1.5) and (3.1.6), we obtain $\|u(t)\|_V > s_1$, for all $t \in [0, t^*)$ and some $s_1 > 0$. By using the fact $u \in C([0, T), V)$, we obtain that $\|u(t^*)\|_V \geq s_1 > 0$, contradicting the assumption $\|u(t^*)\|_V^2 = 0$.

Combining Case I and Case II, we conclude that $u(t) \in \mathscr{W}_2$ for all $t \in [0, T)$.

Step 2. It remains to show inequality (3.3.2). Let $u \in \mathscr{W}_2$ be fixed. Recall $V = (H_0^1(\Omega))^3$. By the definition of $\mathscr{W}_2$ in (1.5.16) we have $u \in V \setminus \{0\}$. By Assumption 1.5.10

$$J(\lambda u) = \frac{1}{2}\lambda^2 \|u\|_V^2 - \int_\Omega \lambda^{p+1} F(u) d\mathbf{x}, \quad \text{for} \ \ \lambda \geq 0, \tag{3.3.4}$$

we get

$$\frac{d}{d\lambda} J(\lambda u) = \lambda \|u\|_V^2 - (p+1)\lambda^p \int_\Omega F(u) d\mathbf{x}. \tag{3.3.5}$$

Hence, the map $\lambda \mapsto J(\lambda u)$ has only one critical point $\lambda_0 > 0$ which satisfies

$$\|u\|_V^2 = (p+1)\lambda_0^{p-1} \int_\Omega F(u) d\mathbf{x}. \tag{3.3.6}$$

Since $u \in \mathscr{W}_2$, $\lambda_0 < 1$. In addition, since the function $\lambda \mapsto J(\lambda u)$ attains its absolute maximum over the positive axis at its critical point $\lambda = \lambda_0$, then by (1.5.14) and (3.3.6),

$$d \leq \sup_{\lambda \geq 0} J(\lambda u) = J(\lambda_0 u) = \lambda_0^2 \frac{1}{2} \|u\|_V^2 - \lambda_0^{p+1} \int_\Omega F(u) d\mathbf{x} = \lambda_0^2 \frac{p-1}{2(p+1)} \|u\|_V^2 . \tag{3.3.7}$$

Since $\lambda_0 < 1$, one has $\|u\|_V^2 > 2d\frac{p+1}{p-1} = 2\rho d$, completing the proof of Lemma 3.3.1. □

3.3.1 Proof of Theorem 1.5.19.

In order to show that the maximal existence time T is necessarily finite, we argue by contradiction. Assume that the weak solution provided by Theorem 1.5.4 can be extended to $[0,\infty)$, then Lemma 3.3.1 asserts $u(t)\in\mathscr{W}_2$ for all $t\in[0,\infty)$, i.e.,

$$\|u(t)\|_V^2<(p+1)\int_\Omega F(u(t))d\mathbf{x},\quad \text{for all }\ t\in[0,\infty). \tag{3.3.8}$$

In addition, by the assumption $0\le\mathscr{E}(0)<d$, we first show that the total energy $\mathscr{E}(t)$ remains nonnegative and satisfies

$$0\le\mathscr{E}(t)\le\mathscr{E}(0)<d\ \text{ for all }\ t\in[0,\infty). \tag{3.3.9}$$

In order to check $\mathscr{E}(t)\ge0$ for all $t\ge0$, we argue by contradiction and assume that $\mathscr{E}(t_0)<0$ for some $t_0\in[0,\infty)$. Then, the blow-up result in Theorem 1.5.9 asserts that $\|u\|_V\to\infty$ as $t\to T^-$, for some $0<T<\infty$, i.e., the weak solution $u(t)$ blows up in finite time, which is contrary to our assumption at the beginning of the proof. Therefore, we conclude that the total energy $\mathscr{E}(t)$ remains nonnegative for all $t\ge0$. To this end, put

$$N(t)=\|u(t)\|_2^2,\quad S(t)=\int_\Omega F(u(t))d\mathbf{x}\ge0,\quad t\in[0,\infty). \tag{3.3.10}$$

Step 1. We first show that $N(t)$ has a quadratic growth as $t\to\infty$. As in the proof of the blow-up result in Theorem 1.5.9, here we also have:

$$N''(t)=2\|u_t\|_2^2-2\|u\|_V^2+2(p+1)\int_\Omega F(u)d\mathbf{x}-2\int_\Omega \mathscr{G}(u_t(t))\cdot u(t)d\mathbf{x}, \tag{3.3.11}$$

for almost all $t\in[0,T)$.

We estimate the last term (due to the damping) on the right-hand side of (3.3.11) as follows. First, the assumption $|g_1(s)| \leq \beta |s|^m$ for all $s \in \mathbb{R}$ implies

$$\left| \int_\Omega g_1(w_t(t)) w(t) d\mathbf{x} \right| \leq \beta \, \|w(t)\|_{m+1} \, \|w_t(t)\|_{m+1}^m . \tag{3.3.12}$$

Since $1 \leq \max\{m, r, s\} < p$ and by using a standard interpolation estimate, we have

$$\|w(t)\|_{m+1} \leq \|w(t)\|_2^\lambda \|w(t)\|_{p+1}^{1-\lambda}, \tag{3.3.13}$$

where λ satisfies $\frac{\lambda}{2} + \frac{1-\lambda}{p+1} = \frac{1}{m+1}$, i.e., $\lambda = \frac{(m+1)^{-1} - (p+1)^{-1}}{2^{-1} - (p+1)^{-1}}$. By using (3.3.8), the fact $F(u) \geq \alpha_0 \left(|w|^{p+1} + |\psi|^{p+1} + |\phi|^{p+1} \right)$ and the Sobolev embedding theorem in 2D: $H_0^1(\Omega) \hookrightarrow L^s(\Omega)$, $1 \leq s < \infty$, we obtain

$$\|w(t)\|_2^2 \leq c\|w(t)\|_{1,\Omega}^2 \leq c\|u(t)\|_V^2 \leq CS(t), \tag{3.3.14}$$

$$\|w(t)\|_{p+1}^{p+1} \leq \|u(t)\|_{p+1}^{p+1} \leq C \int_\Omega F(u(t)) d\mathbf{x} = CS(t), \tag{3.3.15}$$

where C is some positive constant. It follows from (3.3.12)-(3.3.15) that

$$\begin{aligned} \left| 2 \int_\Omega g_1(w_t(t)) w(t) d\mathbf{x} \right| &\leq C\|w(t)\|_2^\lambda \|w(t)\|_{p+1}^{1-\lambda} \, \|w_t(t)\|_{m+1}^m \\ &\leq CS(t)^{\frac{\lambda}{2} + \frac{1-\lambda}{p+1}} \, \|w_t(t)\|_{m+1}^m = CS(t)^{\frac{1}{m+1}} \, \|w_t(t)\|_{m+1}^m , \end{aligned} \tag{3.3.16}$$

where we used the fact that $\frac{\lambda}{2} + \frac{1-\lambda}{p+1} = \frac{1}{m+1}$. By Young's inequality, one has

$$\left| 2 \int_\Omega g_1(w_t(t)) w(t) d\mathbf{x} \right| \leq \frac{2\epsilon}{3} S(t) + C_\epsilon \, \|w_t(t)\|_{m+1}^{m+1} , \tag{3.3.17}$$

(analogously for g_2, ψ, r, or g_3, ϕ, q).

Now we define

$$K(t) := \|u(t)\|_V^2 - (p+1)\int_\Omega F(u(t))d\mathbf{x}. \tag{3.3.18}$$

Then, it follows from (3.3.11) and (3.3.17) that

$$\begin{aligned} N''(t) \geq 2\|u_t(t)\|_2^2 - 2K(t) - 2\epsilon S(t) \\ - C_\epsilon\left[\|w_t(t)\|_{m+1}^{m+1} + \|\psi_t(t)\|_{r+1}^{r+1} + \|\phi_t(t)\|_{q+1}^{q+1}\right], \end{aligned} \tag{3.3.19}$$

where $\epsilon > 0$ is to be chosen later. Now, let $\delta > 2$ to be specified below. Since the total energy $\mathscr{E}(t)$ in (1.5.9) satisfies $0 \leq \mathscr{E}(t) \leq \mathscr{E}(0) < d$, we have $K(t) \leq K(t) + \delta(\mathscr{E}(0) - \mathscr{E}(t))$. Then, by (3.3.10), we have

$$\begin{aligned} K(t) \leq& \|u(t)\|_V^2 - (p+1)\int_\Omega F(u(t))d\mathbf{x} + \delta\mathscr{E}(0) - \delta\left(\frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|u(t)\|_V^2 - S(t)\right) \\ =& \|u(t)\|_V^2 - (p+1)S(t) + \delta\mathscr{E}(0) - \delta\left(\frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|u(t)\|_V^2 - S(t)\right) \\ =& \left(1 - \frac{\delta}{2}\right)\|u(t)\|_V^2 + (\delta - p - 1)S(t) + \delta\mathscr{E}(0) - \frac{\delta}{2}\|u_t(t)\|_2^2. \end{aligned} \tag{3.3.20}$$

The estimates in (3.3.19) and (3.3.20) yield

$$\begin{aligned} N''(t) \geq& (2+\delta)\|u_t(t)\|_2^2 + 2(p+1-\delta-\epsilon)S(t) - 2\delta\mathscr{E}(0) \\ &+ (\delta-2)\|u(t)\|_V^2 - C_\epsilon\left[\|w_t(t)\|_{m+1}^{m+1} + \|\psi_t(t)\|_{r+1}^{r+1} + \|\phi_t(t)\|_{q+1}^{q+1}\right]. \end{aligned} \tag{3.3.21}$$

At this end, we select and fix δ such that

$$2 < \frac{2d(p+1)}{d(p+1) - (p-1)\mathscr{E}(0)} < \delta < p+1. \tag{3.3.22}$$

This choice of δ is possible because of the assumption $\mathscr{E}(0) < d$. By recalling Lemma

3.3.1, in particular (3.3.2), and by the choice of δ, we have

$$\begin{aligned}(\delta-2)\|u(t)\|_V^2 - 2\delta\mathscr{E}(0) >& 2d(\delta-2)\left(\frac{p+1}{p-1}\right) - 2\delta\mathscr{E}(0) \\ =& \frac{2\delta\left[d(p+1)-\mathscr{E}(0)(p-1)\right]-4d(p+1)}{p-1} > 0. \end{aligned} \tag{3.3.23}$$

After having fixed δ satisfying (3.3.22), we select $\epsilon > 0$ sufficiently small such that

$$A := 2(p+1-\delta-\epsilon) > 0.$$

Hence, it follows from (3.3.2), (3.3.8), (3.3.21), and (3.3.23) that

$$\begin{aligned} N''(t) + C_\epsilon &\left[\|w_t(t)\|_{m+1}^{m+1} + \|\psi_t(t)\|_{r+1}^{r+1} + \|\phi_t(t)\|_{q+1}^{q+1}\right] \\ &> AS(t) > \frac{A}{p+1}\|u(t)\|_V^2 > \frac{2dA}{p-1} := 2B > 0. \end{aligned} \tag{3.3.24}$$

By integrating (3.3.24) from 0 to t, we obtain

$$N'(t) + C_\epsilon \int_0^t \left[\|w_t(t)\|_{m+1}^{m+1} + \|\psi_t(t)\|_{r+1}^{r+1} + \|\phi_t(t)\|_{q+1}^{q+1}\right] d\tau > 2Bt + N'(0). \tag{3.3.25}$$

However, the energy identity (3.1.1), (3.1.3) and Assumption 1.5.18 on damping yield that

$$C_\epsilon \int_0^t \left[\|w_t(t)\|_{m+1}^{m+1} + \|\psi_t(t)\|_{r+1}^{r+1} + \|\phi_t(t)\|_{q+1}^{q+1}\right] d\tau \le C(\mathscr{E}(0) - \mathscr{E}(t)) < Cd. \tag{3.3.26}$$

for all $t \in [0,\infty)$ and some constant $C > 0$. Therefore, (3.3.25) and (3.3.26) imply

$$N'(t) > 2Bt + N'(0) - Cd. \tag{3.3.27}$$

By integrating (3.3.27) we obtain

$$N(t) > Bt^2 + (N'(0) - Cd)t + N(0) \;\text{ for all }\; t \in [0, \infty). \tag{3.3.28}$$

That is, $N(t)$ has a quadratic growth as $t \to \infty$.

Step 2. Get contradiction by estimating $N(t) = \|u(t)\|_2^2$ directly and showing $N(t)$ grows sub-quadratically as $t \to \infty$. The regularity of u allows us to write

$$|w(t)|^2 = \left|w(0) + \int_0^t w_t(\tau)d\tau\right|^2 \leq 2|w(0)|^2 + 2t\left(\int_0^t |w_t(\tau)|^2 d\tau\right),$$

(analogously for ψ or ϕ).

Therefore,

$$N(t) = \|u(t)\|_2^2 \leq 2\|u(0)\|_2^2 + 2t\int_0^t \int_\Omega \left(|w_t(\tau)|^2 + |\psi_t(\tau)|^2 + |\phi_t(\tau)|^2\right) d\mathbf{x}d\tau. \tag{3.3.29}$$

From Hölder's inequality and (3.3.26), we have

$$\int_0^t \int_\Omega |w_t(\tau)|^2 d\mathbf{x}d\tau \leq (|\Omega|t)^{\frac{m-1}{m+1}} \left(\int_0^t \int_\Omega |w_t(\tau)|^{m+1} d\mathbf{x}d\tau\right)^{\frac{2}{m+1}} \leq C_\epsilon d^{\frac{2}{m+1}} t^{\frac{m-1}{m+1}}, \tag{3.3.30}$$

(analogously for ψ_t, r, or ϕ_t, q).

It follows from (3.3.30) and (3.3.29) that

$$N(t) = \|u(t)\|_2^2 \leq 2\|u(0)\|_2^2 + 2C_\epsilon \left(d^{\frac{2}{m+1}} t^{\frac{2m}{m+1}} + d^{\frac{2}{r+1}} t^{\frac{2r}{r+1}} + d^{\frac{2}{q+1}} t^{\frac{2q}{q+1}}\right), \tag{3.3.31}$$

for all $t \in [0, \infty)$. Since $\frac{2m}{m+1}, \frac{2r}{r+1}, \frac{2q}{q+1} < 2$ then (3.3.31) contradicts the quadratic growth $N(t)$ as $t \to \infty$ as shown in Step 1. Therefore we conclude that the weak solution $u(t)$ can not be extended to the whole interval $[0, \infty)$, completing the proof of Theorem 1.5.19.

Appendix A

Proposition A.0.2. *Let V denote the Hilbert space $(H_0^1(\Omega))^3$ which is endowed with the standard inner product $(\cdot,\cdot)_{1,\Omega}$ and corresponding norm $\|\cdot\|_{1,\Omega}$. Then, $(\cdot,\cdot)_V$ is an inner product on V (defined in (1.4.2)) and the corresponding norm $\|\cdot\|_V$ defined in (1.4.4) is an equivalent norm on V. In particular, there exists $\alpha > 0$ such that for all $u = (w, \psi, \phi) \in V$*

$$\alpha \, \|u\|_{1,\Omega}^2 \leq \|u\|_V^2 \leq 2 \, \|u\|_{1,\Omega}^2 \; . \tag{A.0.1}$$

Proof. We begin by recalling (1.4.2):

$$\begin{aligned}(u, \tilde{u})_V = &\int_\Omega \Big((1-\mu)\left(\psi_x \tilde{\psi}_x + \phi_y \tilde{\phi}_y\right) + \mu \left(\psi_x + \phi_y\right)\left(\tilde{\psi}_x + \tilde{\phi}_y\right) \\ &+ \frac{1-\mu}{2}\left(\psi_y + \phi_x\right)\left(\tilde{\psi}_y + \tilde{\phi}_x\right)\Big) d\mathbf{x} + (w_x + \psi, \tilde{w}_x + \tilde{\psi})_\Omega + (w_y + \phi, \tilde{w}_y + \tilde{\phi})_\Omega.\end{aligned}$$

Direct calculation shows that $(\cdot,\cdot)_V$ is symmetric, bilinear, positive-definite. To prove that $(\cdot,\cdot)_V$ defines an inner product on V that indices the norm, recall Korn's inequality:

$$\|\psi\|_{1,\Omega}^2 + \|\phi\|_{1,\Omega}^2 \leq C \int_\Omega \Big(|\psi|^2 + |\phi|^2 + |\psi_x|^2 + |\phi_y|^2 + \frac{1}{2}|\psi_y + \phi_x|^2 + \frac{1}{2}|\psi_x + \phi_y|^2 \Big) d\mathbf{x}.$$

From here Via Poincaré's estimate we get

$$\begin{aligned}\|\psi\|_{1,\Omega}^2 + \|\phi\|_{1,\Omega}^2 \le & C_1 \int_\Omega \left(|\psi_x|^2 + |\psi_y|^2 + |\phi_x|^2 + |\phi_y|^2 + 2\psi_y\phi_x + |\psi_x + \phi_y|^2\right) d\mathbf{x} \\ = & C_1 \int_\Omega \left(|\psi_x|^2 + |\phi_y|^2 + |\psi_y + \phi_x|^2 + |\psi_x + \phi_y|^2\right) d\mathbf{x} \\ \le & C_2 \int_\Omega \left((1-\mu)\left(|\psi_x|^2 + |\phi_y|^2\right) + \frac{1-\mu}{2}|\psi_y + \phi_x|^2 + \mu|\psi_x + \phi_y|^2\right) d\mathbf{x}.\end{aligned}$$

Choosing $\alpha_0 = 1/C_2$ gives

$$\alpha_0 \left(\|\psi\|_{1,\Omega}^2 + \|\phi\|_{1,\Omega}^2\right) \le \int_\Omega \left((1-\mu))\left(|\psi_x|^2 + |\phi_y|^2\right) + \frac{1-\mu}{2}|\psi_y + \phi_x|^2 + \mu|\psi_x + \phi_y|^2\right) d\mathbf{x}. \quad \text{(A.0.2)}$$

Next, with $\epsilon = (2 + \alpha_0)^{-1} < 1/2$ we derive

$$\begin{aligned}\|w_x + \psi\|_2^2 + \left\|w_y + \phi\right\|_2^2 &\ge (1 - 2\epsilon)\|\nabla w\|_2^2 + \left(1 - \frac{1}{2\epsilon}\right)\left(\|\psi\|_2^2 + \|\phi\|_2^2\right) \\ &\ge \left(1 - \frac{2}{2+\alpha_0}\right)\alpha_1 \|w\|_{1,\Omega}^2 - \frac{\alpha_0}{2}\left(\|\psi\|_{1,\Omega}^2 + \|\phi\|_{1,\Omega}^2\right),\end{aligned} \quad \text{(A.0.3)}$$

where α_1 again denotes a Poincaré constant $\alpha_1 \|w\|_{1,\Omega}^2 \le \|\nabla w\|_2^2$ in the last line of (A.0.3). Letting $\alpha = \left(1 - \frac{2}{2+\alpha_0}\right)\alpha_1$, we conclude with the help of (A.0.2) that

$$\begin{aligned}\alpha \|u\|_{1,\Omega}^2 \le & \int_\Omega \left((1-\mu)\left(|\psi_x|^2 + |\phi_y|^2\right) + \frac{1-\mu}{2}|\psi_y + \phi_x|^2 + \mu|\psi_x + \phi_y|^2\right) d\mathbf{x} \\ & + \|w_x + \psi\|_2^2 + \left\|w_y + \phi\right\|_2^2 = \|u\|_V^2,\end{aligned}$$

completing the proof of the left-hand of inequality (A.0.1).

The right-hand side of (A.0.1) is straightforward. Indeed,

$$\begin{aligned}
&\int_\Omega \Big((1-\mu)\left(|\psi_x|^2+|\phi_y|^2\right)+\frac{1-\mu}{2}|\psi_y+\phi_x|^2+\mu|\psi_x+\phi_y|^2\Big)d\mathbf{x} \\
&\quad\le \int_\Omega \Big((1-\mu))\left(|\psi_x|^2+|\phi_y|^2\right)+(1-\mu)(|\psi_y|^2+|\phi_x|^2)+2\mu(|\psi_x|^2+|\phi_y|^2)\Big)d\mathbf{x} \qquad \text{(A.0.4)}\\
&\quad\le 2(\|\nabla\psi\|_2^2+\|\nabla\phi\|_2^2).
\end{aligned}$$

Therefore,

$$\|u\|_V^2 \le 2\,\|\nabla\psi\|_2^2 + 2\,\|\nabla\phi\|_2^2 + 2\,\|\nabla w\|_2^2 + 2\,\|\psi\|_2^2 + 2\,\|\phi\|_2^2 \le 2\,\|u\|_{1,\Omega}^2\,,$$

completing the proof of Proposition A.0.2. □

Remark A.0.3. An immediate consequence of Propostion A.0.2 is that $\|\cdot\|_H$ defined in (1.4.5) is equivalent to the standard norm on H. More precisely, there exists $\alpha > 0$ such that

$$\alpha\,\|U\|^2_{(H_0^1(\Omega))^3\times(L^2(\Omega))^3} \le \|U\|_H^2 \le 2\,\|U\|^2_{(H_0^1(\Omega))^3\times(L^2(\Omega))^3}\,; \qquad \text{(A.0.5)}$$

for all $U = (w, \psi, \phi, w_1, \psi_1, \phi_1) \in H$.

Proposition A.0.4. *Let* $\mathscr{B} : (H_0^1(\Omega))^3 \to (H^{-1}(\Omega))^3$ *be given by:*

$$\mathscr{B}\begin{pmatrix} w \\ \psi \\ \phi \end{pmatrix}^{tr} = \begin{pmatrix} -\Delta w - (\psi_x + \phi_y) \\ -(\psi_{xx} + \frac{1-\mu}{2}\psi_{yy}) \;-\; \frac{1+\mu}{2}\phi_{xy} + (\psi + w_x) \\ -(\frac{1-\mu}{2}\phi_{xx} + \phi_{yy}) \;-\; \frac{1+\mu}{2}\psi_{xy} + (\phi + w_y) \end{pmatrix}^{tr}. \qquad \text{(A.0.6)}$$

(a) If $u = (w, \psi, \phi) \in (H^2(\Omega) \cap H_0^1(\Omega))^3$ *and* $\tilde u = (\tilde w, \tilde\psi, \tilde\phi) \in (L^2(\Omega))^3$*, then*

$$(\mathscr{B}(u), \tilde u)_\Omega = (u, \tilde u)_V\,. \qquad \text{(A.0.7)}$$

(b) If $u = (w, \psi, \phi) \in (H_0^1(\Omega))^3$ and $\tilde{u} = (\tilde{w}, \tilde{\psi}, \tilde{\phi}) \in (H_0^1(\Omega))^3$, then

$$\langle \mathscr{B}(u), \tilde{u} \rangle = (u, \tilde{u})_V \tag{A.0.8}$$

(where $\langle \cdot, \cdot \rangle$ is the duality pairing of V' and V).

Proof. A straightforward computation shows that

$$\begin{aligned}
(\mathscr{B}(u), \tilde{u})_\Omega = &- (\Delta w, \tilde{w})_\Omega - \left(\psi_x + \phi_y, \tilde{w}\right)_\Omega \\
&+ \left(w_x, \tilde{\psi}\right)_\Omega + \left(\psi, \tilde{\psi}\right)_\Omega - \left(\psi_{xx}, \tilde{\psi}\right)_\Omega - \frac{1-\mu}{2}\left(\psi_{yy}, \tilde{\psi}\right)_\Omega - \frac{1+\mu}{2}\left(\phi_{xy}, \tilde{\psi}\right)_\Omega \\
&+ \left(w_y, \tilde{\phi}\right)_\Omega + \left(\phi, \tilde{\phi}\right)_\Omega - \frac{1-\mu}{2}\left(\phi_{xx}, \tilde{\phi}\right)_\Omega - \left(\phi_{yy}, \tilde{\phi}\right)_\Omega - \frac{1+\mu}{2}\left(\psi_{xy}, \tilde{\phi}\right)_\Omega .
\end{aligned}$$

By the strong regularity of u, it easy to see

$$\begin{aligned}
(\mathscr{B}(u), \tilde{u})_\Omega = &- \left(\psi_x + \phi_y, \tilde{w}\right)_\Omega + \left(w_x, \tilde{\psi}\right)_\Omega + \left(\psi, \tilde{\psi}\right)_\Omega + \left(w_y, \tilde{\phi}\right)_\Omega + \left(\phi, \tilde{\phi}\right)_\Omega \\
&+ \left(\psi_x, \tilde{\psi}_x\right)_\Omega + \frac{1-\mu}{2}\left(\psi_y, \tilde{\psi}_y\right)_\Omega + \frac{1-\mu}{2}\left(\phi_x, \tilde{\psi}_y\right)_\Omega + \mu\left(\phi_y, \tilde{\psi}_x\right)_\Omega \\
&+ \frac{1-\mu}{2}\left(\phi_x, \tilde{\phi}_x\right)_\Omega + \left(\phi_y, \tilde{\phi}_y\right)_\Omega + \frac{1-\mu}{2}\left(\psi_y, \tilde{\phi}_x\right)_\Omega + \mu\left(\psi_x, \tilde{\phi}_y\right)_\Omega + (\nabla w, \nabla \tilde{w})_\Omega .
\end{aligned}$$

Rearranging yields

$$\begin{aligned}
(\mathscr{B}(u), \tilde{u})_\Omega = &- \left(\psi_x + \phi_y, \tilde{w}\right)_\Omega + \left(w_x, \tilde{\psi}\right)_\Omega + \left(\psi, \tilde{\psi}\right)_\Omega + \left(w_y, \tilde{\phi}\right)_\Omega + \left(\phi, \tilde{\phi}\right)_\Omega \\
&+ (1-\mu)\left[(\psi_x, \tilde{\psi}_x)_\Omega + (\phi_y, \tilde{\phi}_y)_\Omega\right] + \frac{1-\mu}{2}\left(\psi_y + \phi_x,\ \tilde{\psi}_y + \tilde{\phi}_x\right)_\Omega \\
&+ \mu\left(\psi_x + \phi_y, \tilde{\psi}_x + \tilde{\phi}_y\right)_\Omega + (\nabla w, \nabla \tilde{w})_\Omega .
\end{aligned}$$

By noting that

$$-(\psi_x + \phi_y, \tilde{w})_\Omega + (w_x, \tilde{\psi})_\Omega + (w_y, \tilde{\phi})_\Omega + (\psi, \tilde{\psi})_\Omega + (\phi, \tilde{\phi})_\Omega + (\nabla w, \nabla \tilde{w})_\Omega$$
$$= (w_x + \psi, \tilde{w}_x + \tilde{\psi})_\Omega + (w_y + \phi, \tilde{w}_y + \tilde{\phi})_\Omega, \tag{A.0.9}$$

part (a) follows. Part (b) readily follows from (a) by density: using the right-hand side of (A.0.7) we extend $\mathscr{B}$ to a bounded linear operator $V \to V'$. □

Lemma A.0.5. *Under Assumption 1.5.1 and 1.5.10, the Fréchet derivative of the functional J (as defined in* (1.5.8)*) at $u \in V$ is given by:*

$$D_u J(\theta) = (u, \theta)_V - \int_\Omega \mathscr{F}(u) \cdot \theta d\mathbf{x}, \quad \textit{for all} \quad \theta = (\theta_1, \theta_2, \theta_3) \in V, \tag{A.0.10}$$

where $(\cdot, \cdot)_V$ is given by (1.4.2) *and $\mathscr{F}(u) = (f_1(u), f_2(u), f_3(u)) = \nabla F(u)$.*

Proof. For each fixed $u \in V$, it is clear that the right-hand side in (A.0.10) defines a bounded linear map on V. By a direct calculation, we have

$$\frac{1}{\|\theta\|_V} \left| J(u + \theta) - J(u) - \left((u, \theta)_V - \int_\Omega \mathscr{F}(u) \cdot \theta d\mathbf{x} \right) \right|$$
$$= \frac{1}{\|\theta\|_V} \left| \frac{1}{2} \|\theta\|_V^2 - \int_\Omega F(u + \theta) d\mathbf{x} + \int_\Omega F(u) d\mathbf{x} + \int_\Omega \nabla F(u) \cdot \theta d\mathbf{x} \right|. \tag{A.0.11}$$

Since $F \in C^2(\mathbb{R}^3)$, we know that for each $\mathbf{x} \in \Omega$

$$F(u + \theta) - F(u) = \nabla F(\xi) \cdot \theta,$$

where $\xi = u + (1-\lambda)\theta$ for some $0 < \lambda < 1$ depending on $u(\mathbf{x})$ and $\theta(\mathbf{x})$. Hence,

$$\frac{1}{\|\theta\|_V}\left|J(u+\theta) - J(u) - \left((u,\theta)_V - \int_\Omega \mathscr{F}(u)\cdot\theta d\mathbf{x}\right)\right| \\ \leq \frac{1}{\|\theta\|_V}\left(\frac{1}{2}\|\theta\|_V^2 + \int_\Omega |(\nabla F(u) - \nabla F(\xi))\cdot\theta|\, d\mathbf{x}\right). \tag{A.0.12}$$

The right-hand side of (A.0.12) can be estimated as follows:

$$\int_\Omega |(\nabla F(u) - \nabla F(\xi))\cdot\theta|\, d\mathbf{x} \leq \sum_j \int_\Omega \left|(f_j(u) - f_j(\xi))\right| |\theta_j| d\mathbf{x} \\ \leq \sum_j \|f_j(u) - f_j(\xi)\|_2 \|\theta_j\|_2. \tag{A.0.13}$$

Now, for fixed $u \in V$ put $R = \|u\|_V + 1$. Then,

$$\|\xi\|_V = \|u + \lambda\theta\|_V \leq R, \quad \text{for all } \lambda \in (0,1) \text{ and all } \|\theta\|_V \leq 1.$$

By Lemma 2.1.2, we know that

$f_j : V \to L^2(\Omega)$ are locally Lipschitz continuous and

$$\int_\Omega \left|(f_j(u) - f_j(\xi))\right| |\theta_j| d\mathbf{x} \leq C(R)\|u-\xi\|_V \leq C(R)\|\theta\|_V\|\theta_j\|_2 \leq C(R)\|\theta\|_V^2, \tag{A.0.14}$$

for $j = 1,2,3$. Hence, combining (A.0.12) and (A.0.14) yields

$$\frac{\left|J(u+\theta) - J(u) - \left((u,\theta)_V - \int_\Omega \mathscr{F}(u)\cdot\theta d\mathbf{x}\right)\right|}{\|\theta\|_V} \leq \frac{\frac{1}{2}\|\theta\|_V^2 + C(R)\|\theta\|_V^2}{\|\theta\|_V} \\ = \frac{1}{2}\|\theta\|_V + C(R)\|\theta\|_V \longrightarrow 0, \quad \text{as } \|\theta\|_V \to 0. \tag{A.0.15}$$

Therefore, J is Fréchet differentiable at every $u \in V$ and its Fréchet derivative at u is given by (A.0.10), completing the proof. □

Bibliography

[1] K. Agre and M. A. Rammaha. Systems of nonlinear wave equations with damping and source terms. *Differential Integral Equations*, 19(11):1235–1270, 2006.

[2] C. O. Alves, M. M. Cavalcanti, V. N. Domingos Cavalcanti, M. A. Rammaha, and D. Toundykov. On existence, uniform decay rates and blow up for solutions of systems of nonlinear wave equations with damping and source terms. *Discrete Contin. Dyn. Syst. Ser. S*, 2(3):583–608, 2009.

[3] George Avalos and Daniel Toundykov, *On stability and trace regularity of solutions to Reissner-Mindlin-Timoshenko equations*, Modern aspects of the theory of partial differential equations, Oper. Theory Adv. Appl., vol. 216, Birkhäuser/Springer Basel AG, Basel, 2011, pp. 79–91.

[4] George Avalos and Daniel Toundykov, *Boundary stabilization of structural acoustic interactions with interface on a Reissner-Mindlin plate*, Nonlinear Anal. Real World Appl. **12** (2011), no. 6, 2985–3013.

[5] Viorel Barbu, *Analysis and control of nonlinear infinite-dimensional systems*, Mathematics in Science and Engineering, vol. 190, Academic Press Inc., Boston, MA, 1993.

[6] Viorel Barbu, Yanqiu Guo, Mohammad A. Rammaha, and Daniel Toundykov, *Convex integrals on Sobolev spaces*, J. Convex Anal. **19** (2012), no. 3, 837–852.

[7] Lorena Bociu and Irena Lasiecka, *Blow-up of weak solutions for the semilinear wave equations with nonlinear boundary and interior sources and damping*, Appl. Math. (Warsaw) **35** (2008), no. 3, 281–304.

[8] Lorena Bociu and Irena Lasiecka, *Local Hadamard well-posedness for nonlinear wave equations with supercritical sources and damping*, J. Differential Equations **249** (2010), no. 3, 654–683.

[9] L. Bociu, M. A. Rammaha, and D. Toundykov. On a wave equation with supercritical interior and boundary sources and damping terms. *Mathematische Nachrichten*, 284(16):2032–2064, 2011.

[10] M. M. Cavalcanti, V. N. Domingos Cavalcanti, and I. Lasiecka. Well-posedness and optimal decay rates for the wave equation with nonlinear boundary damping-source interaction. *J. Differential Equations*, 236(2):407–459, 2007.

[11] Igor Chueshov, Matthias Eller, and Irena Lasiecka, *On the attractor for a semilinear wave equation with critical exponent and nonlinear boundary dissipation*, Comm. Partial Differential Equations **27** (2002), no. 9-10, 1901–1951.

[12] Igor Chueshov and Irena Lasiecka, *Global attractors for Mindlin-Timoshenko plates and for their Kirchhoff limits*, Milan J. Math. **74** (2006), 117–138.

[13] Hugo D. Fernández Sare, *On the stability of Mindlin-Timoshenko plates*, Quart. Appl. Math. **67** (2009), no. 2, 249–263.

[14] F. Gazzola. Finite-time blow-up and global solutions for some nonlinear parabolic equations. *Differential Integral Equations*, 17(9-10):983–1012, 2004.

[15] C. Giorgi and M. G. Naso. Mathematical models of Reissner-Mindlin thermoviscoelastic plates. *J. Thermal Stresses*, 29(7):699–716, 2006.

[16] Claudio Giorgi and Federico M. Vegni, *The longtime behavior of a nonlinear Reissner-Mindlin plate with exponentially decreasing memory kernels*, J. Math. Anal. Appl. **326** (2007), no. 2, 754–771.

[17] M. Grobbelaar-Van Dalsen. On a structural acoustic model with interface a Reissner-Mindlin plate or a Timoshenko beam. *J. Math. Anal. Appl.*, 320:121–144, 2006.

[18] M. Grobbelaar-Van Dalsen. On a structural acoustic model which incorporates shear and thermal effects in the structural component. *J. Math. Anal. Appl.*, 341(2):1253–1270, 2008.

[19] Marié Grobbelaar-Van Dalsen, *On a fluid-structure model in which the dynamics of the structure involves the shear stress due to the fluid*, J. Math. Fluid Mech. **10** (2008), no. 3, 388–401.

[20] M. Grobbelaar-Van Dalsen. Strong stabilization of models incorporating the thermoelastic Reissner-Mindlin plate equations with second sound. *Appl. Anal.*, 90(9):1419–1449, 2011.

[21] Marié Grobbelaar-Van Dalsen, *On the dissipative effect of a magnetic field in a Mindlin-Timoshenko plate model*, Z. Angew. Math. Phys. **63** (2012), no. 6, 1047–1065.

[22] Marié Grobbelaar-Van Dalsen, *The role of magnetic fields in the strong stabilization of a hybrid magneto-elastic structure*, Math. Methods Appl. Sci. **35** (2012), no. 2, 228–237.

[23] Y. Guo. *Systems of nonlinear wave equations with damping and supercritical sources*. ProQuest LLC, Ann Arbor, MI, 2012. Thesis (Ph.D.)–The University of Nebraska - Lincoln.

[24] Y. Guo and M. A. Rammaha. Global existence and decay of energy to systems of wave equations with damping and supercritical sources. *Z. Angew. Math. Phys.*, 64(3):621–658, 2013.

[25] M. A. Horn and I. Lasiecka. Asymptotic behavior with respect to thickness of boundary stabilizing feedback for the Kirchhoff plate. *J. Differential Equations*, 114(2):396–433, 1994.

[26] Herbert Koch and Irena Lasiecka, *Hadamard well-posedness of weak solutions in nonlinear dynamic elasticity-full von Karman systems*, Evolution equations, semigroups and functional analysis (Milano, 2000), Progr. Nonlinear Differential Equations Appl., vol. 50, Birkhäuser, Basel, 2002, pp. 197–216.

[27] J. Lagnese and J.-L. Lions. *Modelling analysis and control of thin plates*, volume 6 of *Recherches en Mathématiques Appliquées [Research in Applied Mathematics]*. Masson, Paris, 1988.

[28] J. E. Lagnese. *Boundary stabilization of thin plates*, volume 10. Society for Industrial and Applied Mathematics, SIAM; Philadelphia, PA, 1989.

[29] V. Lakshmikantham and S. Leela, *Differential and integral inequalities: Theory and applications. Vol. I: Ordinary differential equations*, Academic Press, New York, 1969, Mathematics in Science and Engineering, Vol. 55-I.

[30] I. Lasiecka and D. Tataru. Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping. *Differential Integral Equations*, 6(3):507–533, 1993.

[31] I. Lasiecka and D. Toundykov. Energy decay rates for the semilinear wave equation with nonlinear localized damping and source terms. *Nonlinear Anal.*, 64(8):1757–1797, 2006.

[32] H. A. Levine. Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = -Au + \mathcal{F}(u)$. *Trans. Amer. Math. Soc.*, 192:1–21, 1974.

[33] H. A. Levine and R. A. Smith. A potential well theory for the heat equation with a nonlinear boundary condition. *Math. Methods Appl. Sci.*, 9(2):127–136, 1987.

[34] H. A. Levine and R. A. Smith. A potential well theory for the wave equation with a nonlinear boundary condition. *J. Reine Angew. Math.*, 374:1–23, 1987.

[35] G. Li, Y. Sun, and W. Liu. Global existence, uniform decay and blow-up of solutions for a system of Petrovsky equations. *Nonlinear Anal.*, 74(4):1523–1538, 2011.

[36] J.-L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications. Vol. I*, Springer-Verlag, New York, 1972.

[37] R. D. Mindlin. Thickness-shear and flexural vibrations of crystal plates. *J. Appl. Physics*, 22:316–323, 1951.

[38] K. Ono. On global existence, asymptotic stability and blowing up of solutions for some degenerate non-linear wave equations of Kirchhoff type with a strong dissipation. *Math. Methods Appl. Sci.*, 20(2):151–177, 1997.

[39] R. Paroni, P. Podio-Guidugli, and G. Tomassetti. The Reissner-Mindlin plate theory via Γ-convergence. *C. R. Math. Acad. Sci. Paris*, 343(6):437–440, 2006.

[40] R. Paroni, P. Podio-Guidugli, and G. Tomassetti. A justification of the Reissner-Mindline plate theory through variational convergence. *Anal. Appl. (Singap.)*, 5(2):165–182, 2007.

[41] L. E. Payne and D. H. Sattinger. Saddle points and instability of nonlinear hyperbolic equations. *Israel J. Math.*, 22(3-4):273–303, 1975.

[42] M. Pedersen. Well posedness and regularity of the controlled Mindlin-Timoshenko plate model. *Int. J. Pure Appl. Math.*, 40(2):273–289, 2007.

[43] M. Pedersen. A variational formulation of the controlled plate model. *Int. J. Pure Appl. Math.*, 43(1):27–42, 2008.

[44] M. A. Rammaha, D. Toundykov, and W. Zahava Global existence and decay of energy for a nonlinear wave equation with p-Laplacian damping. *Discrete Contin. Dyn. Syst.*, 32(12):4361–4390, 2012.

[45] John William Strutt Rayleigh, Baron, *The Theory of Sound*, Dover Publications, New York, N. Y., 1945, 2d ed.

[46] E. Reissner. The effect of transverse shear deformation on the bending of elastic plates. *J. Appl. Mech.*, 12:A–69–A–77, 1945.

[47] Sawanya Sakuntasathien, *Global well-posedness for systems of nonlinear wave equations*, ProQuest LLC, Ann Arbor, MI, 2008, Thesis (Ph.D.)–The University of Nebraska-Lincoln.

[48] R. E. Showalter, *Monotone operators in Banach space and nonlinear partial differential equations*, Mathematical Surveys and Monographs, vol. 49, American Mathematical Society, Providence, RI, 1997.

[49] J. Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4)*, 146:65–96, 1987.

[50] S. P. Timoshenko. *The collected papers of Stephen P. Timoshenko*. McGraw-Hill Book Company, Inc, New York-Toronto-London, 1953.

[51] G. Todorova. Stable and unstable sets for the Cauchy problem for a nonlinear wave equation with nonlinear damping and source terms. *J. Math. Anal. Appl.*, 239(2):213–226, 1999.

[52] N. F. J. van Rensburg, L. Zietsman, and A. J. van der Merwe. Solvability of a Reissner-Mindlin-Timoshenko plate-beam vibration model. *IMA J. Appl. Math.*, 74(1):149–162, 2009.

[53] E. Vitillaro. Some new results on global nonexistence and blow-up for evolution problems with positive initial energy. *Rend. Istit. Mat. Univ. Trieste*, 31(suppl. 2):245–275, 2000. Workshop on Blow-up and Global Existence of Solutions for Parabolic and Hyperbolic Problems (Trieste, 1999).

[54] E. Vitillaro. Global existence for the wave equation with nonlinear boundary damping and source terms. *J. Differential Equations*, 186(1):259–298, 2002.

[55] E. Vitillaro. A potential well theory for the wave equation with nonlinear source and boundary damping terms. *Glasg. Math. J.*, 44(3):375–395, 2002.